304470_Bello_ch07_sec8_web 11/8/06 7:04 PM Page L1

7.8 Linear Programming

GETTING STARTED

"You Want a Coke with That?"

Are you on a steady diet of hamburgers and fries? Are you also watching your fat and protein intake? A regular hamburger contains about 11 g of fat and 12 g of protein. Your daily consumption of fat and protein should be about 56 g and 51 g, respectively. (You can check this out in the *Fast Food Guide* by Michael Jacobson and Sarah Fritschner.) Suppose that a regular hamburger costs 39¢ and a regular order of fries is 79¢. How many of each could you eat so that you met the recommended fat and protein intake and, at the same time, minimized the cost? The given information is shown in Table 1, where h and f represent the number of hamburgers and orders of fries, respectively.

TABLE 1

	Hamburgers	*French Fries*	*Total*
Fat (g)	11	12	$11h + 12f$
Protein (g)	12	3	$12h + 3f$
Cost (¢)	39	79	$39h + 79f$

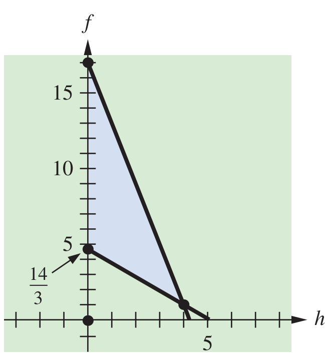

FIGURE 1

Since the recommended amounts of fat and protein are 56 and 51 g, respectively, we have the system of equations

$$11h + 12f = 56$$
$$12h + 3f = 51$$

In the first equation when $h = 0, f = \frac{56}{12} = \frac{14}{3}$. Graph $\left(0, \frac{14}{3}\right)$. When $f = 0$, $h = \frac{56}{11}$. Graph $\left(\frac{56}{11}, 0\right)$. Join the two points with a line. Similarly, graph the second equation.

Now, look at the graph in Figure 1. The shaded region is called the **feasible region,** and the minimum or maximum for a linear function (such as the cost function given by $C = 39h + 79f$) occurs at one of the corner points. First, approximate $\left(0, \frac{14}{3}\right)$ to (0, 5) and then try each of the points.

For (0, 5), $C = 39 \cdot 0 + 79 \cdot 5 = \$3.95$
For (0, 17), $C = 39 \cdot 0 + 79 \cdot 17 = \$13.43$
For (4, 1), $C = 39 \cdot 4 + 1 \cdot 79 = \$2.35$

Thus, the most economical way to meet your fat and protein intake by eating hamburgers and fries is to eat 4 hamburgers and 1 order of fries for \$2.35 (not necessarily all at once!). ►

Many problems and situations require better or more efficient ways of accomplishing a given objective. When studying how to conserve energy, recycle materials, or manufacture certain products, we usually want to minimize the cost. If the equations and inequalities used to model the problem are linear, we have a **linear programming problem.** These problems are usually done with a computer using the **simplex algorithm** developed by George Dantzig in the late 1940s.

Although we shall not consider the simplex method, we shall solve some simple problems that illustrate the variety of applications of linear programming methods. The concepts we employ are basic even for the larger problems that are beyond the scope of this book.

As an example of a linear programming problem, let us assume that Sew & Sew, Inc., manufactures pants and vests. The profit on each pair of pants is $6 and on each vest is $5. The pants use 2 yd of material and the vests use 1.5 yd of material each. Because of production limitations, Sew & Sew cannot manufacture more than 10 of these garments per day and cannot use more than 18 yd of material per day. If Sew & Sew can sell all the pants and vests it makes, find the number of each garment it should produce per day to **maximize its profit.**

One way of presenting a linear programming problem so that it is easier for our minds to grasp is to put the data into tabular form. For the Sew & Sew problem, we tabulate the data and make some minor calculations as shown in Table 2. We can now see that the total profit P is

TABLE 2

	Pants	*Vests*
Number produced	x	y
Yards used	$2x$	$1.5y$
Profit $	$6x$	$5y$

$$P = 6x + 5y \text{ dollars}$$

Thus, P is a linear function of x and y, and we wish to determine x and y so that P has its maximum value.

Next, we must express the restrictions in the problem in terms of x and y. The first restriction is that Sew & Sew produce not more than 10 garments per day. Since it produces x pants and y vests, the total number of garments is $x + y$. Thus,

$$x + y \le 10$$

A second restriction is that it use not more than 18 yd of material per day. This means that

$$2x + 1.5y \le 18$$

Another restriction is that x and y cannot be negative. This gives the **positivity conditions:**

$$x \ge 0 \qquad y \ge 0$$

We can summarize our problem as follows: We want to maximize the linear function P given by

$$P = 6x + 5y$$

subject to the **constraints** (that is, the restrictions)

$$x + y \le 10 \tag{1}$$

$$2x + 1.5y \le 18 \tag{2}$$

$$x \ge 0 \qquad y \ge 0 \tag{3}$$

To make a start on the solution of this problem, we graph the solution set of the system of inequalities (1), (2), and (3), just as we did in the preceding section. This gives the region shown in Figure 2. Each point (x, y) in this region represents a combination of garments that satisfies all the constraints. For this reason, the region is called the **feasible region.**

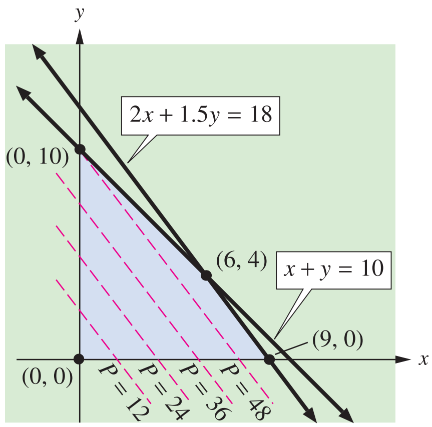

FIGURE 2

We need to select a point from the feasible region that will maximize the expression $6x + 5y$. In order to do this, let us examine the equation $6x + 5y = P$. For a given value of P, this is an equation of a straight line, which in slope-intercept form is

$$y = -\frac{6}{5}x + \frac{P}{5}$$

This equation shows that the slope is $-\frac{6}{5}$ regardless of the value of P. Hence, for a set of values of P, we get a set of parallel straight lines. Furthermore, for positive increasing values of P, the lines move out away from the origin. (This follows because the y intercept is $P/5$.) The dashed lines in Figure 2 are the graphs of $6x + 5y = P$ for the values $P = 12$, 24, 36, and 48. The smaller the value of P, the closer the line is to the origin.

These considerations make plausible the following basic theorem for feasible regions that are **convex** (non-reentrant) polygons, that is, regions such that the points of the line segment joining any two points on the boundary lie entirely inside the region or else on the boundary. See Figure 3.

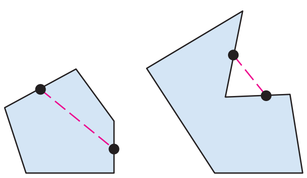

FIGURE 3

THEOREM 7.1 Maximum or Minimum of a Linear Equation

If the feasible region is a convex (non-reentrant) polygon, then the desired maximum (or minimum) value of a linear function occurs at a corner point (vertex) of the region.

TABLE 3

Vertex	$P = 6x + 5y$	
(0, 0)	0	
(9, 0)	54	
(6, 4)	56	← Maximum
(0, 10)	50	

To make use of this theorem, we need only check the values of P at the vertices of the polygon. These vertices are indicated in Figure 2 and can, of course, be found by solving the appropriate pairs of linear equations. By direct calculation, we find the values in Table 3. Thus, Sew & Sew should produce 6 pants and 4 vests per day to maximize its profit.

304470_Bello_ch07_sec8_web 11/8/06 7:04 PM Page L4

EXAMPLE 1 ▶ The Ducks and the Geese

Little Abner raises ducks and geese. He is too lazy to take care of more than 30 birds altogether but wants to make as much profit as possible (naturally). It costs him $1 to raise a duck and $1.50 to raise a goose, and he has only $40 to cover this cost. If Little Abner makes a profit of $1.50 on each duck and $2 on each goose, what is his maximum profit?

Solution

Letting x and y be the number of ducks and geese, respectively, that Little Abner should raise, we place the given information in Table 4.

TABLE 4

	Ducks	*Geese*
Number	x	y
Cost	\$1.00 each	\$1.50 each
Profit	\$1.50 each	\$2.00 each
Total cost	\$$x$	\$$1.5y$
Total profit	\$$1.5x$	\$$2y$

It appears that Little Abner's total profit from both ducks and geese is \$$P$, where

$$P = 1.5x + 2y$$

The constraints are (in the order stated in the problem)

$$x + \quad y \le 30 \qquad \text{Too lazy to raise more than 30 birds} \qquad (4)$$

$$x + 1.5y \le 40 \qquad \text{Has only \$40 to cover his costs} \qquad (5)$$

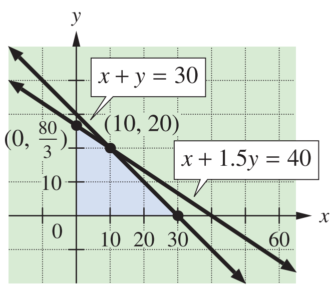

FIGURE 4

Although it is not stated, we must also obey the positivity conditions:

$$x \ge 0 \qquad y \ge 0 \qquad (6)$$

We proceed as before to find the feasible region by graphing the system of inequalities (4), (5), and (6), as shown in Figure 4. We then find the vertices and check them in the profit function P in Table 5.

TABLE 5

Vertex	$P = 1.5x + 2y$	
(0, 0)	0	
(30, 0)	45	
(10, 20)	55	← Maximum
$(0, \frac{80}{3})$	$53\frac{1}{3}$	

By raising 10 ducks and 20 geese, Little Abner will make the maximum possible profit, \$55. ■

304470_Bello_ch07_sec8_web 11/8/06 7:04 PM Page L5

The ideas we have studied are used to make decisions involving production, distribution, and advertising of many products including computers. Do you have a laptop computer or a desktop? How do companies decide which type to produce? How much profit should they try to make? Here is an example adapted from a quantitative methods course taught by Gerard Cornuejols and Michael Trick.

EXAMPLE 2 ► Producing Laptops and Desktops

A computer company wishes to produce laptop and desktop computers and must decide its product mix for the next quarter. It has the following constraints:

1. Each computer requires a processing chip. The supplier has allocated 10(000) chips to the company.
2. Each computer requires memory which comes in 16-MB chip sets. Laptops require 16 MB, so they take one chip, whereas desktops need 32 MB (two chips). These chips are cheaper when bought in great quantities, and 15(000) of them are available.
3. Each computer has to be assembled. A laptop computer takes 4 min versus 3 min for the desktop. There are 25(000) min of assembly time available.

Under current market conditions, material cost, and production time, each laptop produced and sold generates \$500 profit, and each desktop \$600. How many of each type computer should the company manufacture next quarter?

Solution

Suppose the company wishes to manufacture x laptop and y desktop computers. It wants to maximize its total profits P. Since the company generates \$500 for each laptop and \$600 for each desktop, $P = 500x = 600y$. The constraints are as follows:

1.	$x + y \le 10$	The company only has 10(000) chips, 1 for each computer.
2.	$x + 2y \le 15$	It only has 15(000) memory chips, 1 for laptops, 2 for desktops
3.	$4x + 3y \le 25(000)$	It only has 25(000) min, 4 for laptops, 3 for desktops.

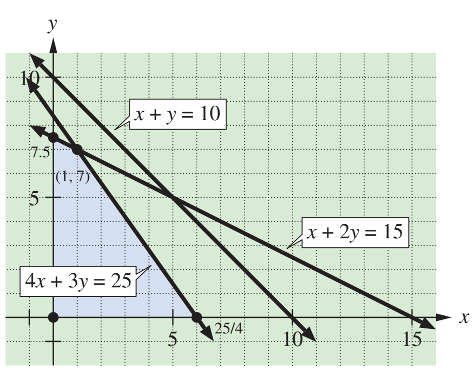

FIGURE 5

304470_Bello_ch07_sec8_web 11/8/06 7:04 PM Page L6

Graph $x + y = 10$, $x + 2y = 15$, and $4x + 3y = 25$, as shown in Figure 5. The feasible region is shown shaded. To find the point of intersection of $4x + 3y = 25$ and $x + 2y = 15$, multiply the second equation by -4. Then,

$$\begin{aligned} 4x + 3y &= 25 \\ -4x - 8y &= -60 \\ -5y &= -35, \quad \text{or} \quad y = 7 \end{aligned}$$

Since $x + 2(7) = 15$, $x = 1$. Thus, the point of intersection is at (1, 7). The vertices of the feasible region and the profit P for each vertex are given in Table 6. As you can see, the maximum profit ($4700) occurs at (1, 7), so the company should produce 1(000) laptop and 7(000) desktop computers.

TABLE 6

Vertex	$P = 500x + 600y$	
(0, 0)	0	
(0, 7.5)	$4500	
(1, 7)	$4700	← Maximum
$\left(\frac{25}{4}, 0\right)$	$3125	

Online Study Center
To further explore linear programming problems, access links 7.8.1 and 7.8.2 on this textbook's Online Study Center. To further explore simplex problem solvers, access link 7.8.3.

You have probably been wondering what to do if the desired maximum (or minimum) occurs at a vertex with noninteger coordinates. In the case of two-variable problems, if the solution must be in integers, you can try the integer points inside the feasible region that are nearest to this vertex (see Getting Started) and select the point that gives the desired maximum (or minimum). For many-variable problems, more complicated techniques, which are beyond the scope of this book, must be used.

EXERCISES 7.8

1. Find the minimum value of $C = 2x + y$ subject to the constraints

$$\begin{aligned} x &\geq 1 \\ x &\leq 4 \\ y &\leq 4 \\ x - 3y &\leq -2 \end{aligned}$$

2. Find the maximum value of $P = x + 4y$ subject to the constraints

$$\begin{aligned} y - x &\leq 0 \\ x &\leq 4 \\ y &\geq 0 \\ x + 2y &\leq 6 \end{aligned}$$

3. Find the minimum value of $W = 4x + y$ subject to the constraints

$$\begin{aligned} x + y &\geq 1 \\ 2y - x &\leq 1 \\ x &\leq 1 \end{aligned}$$

4. Find the minimum value of $C = 2x + 3y$ subject to the constraints

$$\begin{aligned} 2x + y &\geq 18 \\ x + y &\geq 12 \\ 3x + 2y &\leq 34 \end{aligned}$$

5. Find the minimum value of $C = x + 2y$ subject to the constraints

$$8 \le 3x + y \le 10$$
$$x \ge 1$$
$$y \ge 2$$

6. Find the maximum value of $P = 2x + 3y$ subject to the constraints

$$y - x \le 2$$
$$x + y \le 4$$
$$0 \le x \le 3$$
$$y \ge 0$$

7. Find the maximum value of $P = x + 2y$ subject to the constraints

$$2x + y \ge 6$$
$$0 \le y \le 4$$
$$0 \le x \le 2$$

8. Find the maximum value of $P = 4x + 5y$ subject to the constraints

$$y - x \le 2 \quad x \ge 0$$
$$x - y \le 2 \quad y \ge 0$$
$$x + y \le 6$$

9. The E-Z-Park storage lot can hold at most 100 cars and trucks. A car occupies 100 ft^2 and a truck 200 ft^2, and the lot has a usable area of 12,000 ft^2. The storage charge is \$20 per month for a car and \$35 per month for a truck. How many of each should be stored to bring E-Z-Park the maximum revenue?

10. The Zig-Zag Manufacturing Company produces two products, zigs and zags. Each of these products has to be processed through all three of its machines, as shown in the table. If Zig-Zag makes \$12 profit on each zig and \$8 profit on each zag, find the number of each that the company should make in order to maximize its profit.

Machine	*Hours Available*	*Hours/Piece*	
		Zigs	*Zags*
1	Up to 100	4	12
2	Up to 120	8	8
3	Up to 84	6	0

11. The Kwik-Pep Vitamin Company wishes to prepare the following two types of vitamin tablets:

The first type tablet contains 1 mg of vitamin B_1 and 1 mg of vitamin B_2.

The second type tablet contains 1 mg of vitamin B_1 and 2 mg of vitamin B_2.

The profit on the two types of tablets is as follows:

2¢ for each tablet of the first type
3¢ for each tablet of the second type

In manufacturing two bottles of tablets, one of each type, but with the same number of tablets, Kwik-Pep wants to use no more than 100 mg of vitamin B_1 and 150 mg of vitamin B_2. How many tablets should be packed in each bottle to obtain the largest profit?

12. A nutritionist is designing a meal for one of her patients. The meal must include two vegetables, A and B, but not more than 100 g of each. Suppose that each 10-g portion of A contains 2 units of iron and 2 units of vitamin B_{12}, and each 10-g portion of B contains 1 unit of iron and 5 units of vitamin B_{12}. The number of calories in each 10-g portion of these vegetables is 5 for A and 3 for B. If the patient needs at least 20 units of iron and 36 units of vitamin B_{12} in the meal, how many grams of each vegetable should the nutritionist include to satisfy the iron and vitamin requirements while minimizing the number of calories in the meal?

13. The Jeri Tonic Company wishes to manufacture Jeri Tonic so that each bottle contains at least 32 units of vitamin A, 10 units of vitamin B, and 40 units of vitamin C. To supply the vitamins, the company uses additive X, which costs 20¢ per ounce and contains 16 units of vitamin A, 2 units of B, and 4 of C; and additive Y, which costs 40¢ per ounce and contains 4 units of vitamin A, 2 units of B, and 14 of C. If the total amount of additives is not to exceed 10 oz, how many ounces of each additive should the company put into each bottle to minimize its cost?

14. The Write-Right Paper Company operates two factories that manufacture three different grades of paper. There is a demand for each grade, and the

304470_Bello_ch07_sec8_web 11/8/06 7:04 PM Page L8

company has contracts to supply 16 tons of low-grade, 5 tons of medium-grade, and 20 tons of high-grade paper, all in not more than 8 working days. It costs \$1000 per day to operate the first factory and \$2000 per day to operate the second factory. In 1 day's operation, factory number 1 produces 8 tons of low-grade, 1 ton of medium-grade, and 2 tons of high-grade paper, while factory number 2 produces 2 tons of low-grade, 1 ton of medium-grade, and 7 tons of high-grade paper. For how many days should Write-Right operate each factory in order to minimize its cost of filling these contracts?

15. Two oil refineries produce three grades of gasoline, A, B, and C. The refineries operate so that the various grades they produce are in a fixed proportion. Refinery I produces 1 unit of A, 3 units of B, and 1 unit of C per batch, and refinery II produces 1 unit of A, 4 units of B, and 5 units of C per batch. The price per batch is \$300 from refinery I and \$500 from refinery II. A dealer needs 100 units of A, 340 units of B, and 150 units of C. If the maximum number of batches he can get from either refinery is 100, how should he place his orders to minimize his cost?

16. A local television station is faced with a problem. It found that program A with 20 min of music and 1 min of commercials draws 30,000 viewers, whereas program B with 10 min of music and 1 min of commercials draws 10,000 viewers. The sponsor insists that at least 6 min per week be devoted to his commercials, and the station can afford no more than 80 min of music per week. How many times per week should each program be run to obtain the maximum number of viewers?

17. A fruit dealer ships her fruit north on a truck that holds 800 boxes of fruit. She must ship at least 200 boxes of oranges, which net her 20¢ profit per box; at least 100 boxes of grapefruit, which net her 10¢ profit per box; and at most 200 boxes of tangerines, which net her 30¢ profit per box. How should she load the truck for maximum profit? (*Hint:* If she ships x boxes of oranges and y boxes of grapefruit, then she ships $800 - x - y$ boxes of tangerines.)

18. Ms. Jones has a maximum of \$15,000 to invest in two types of bonds. Bond A returns 8% and bond B returns 10% per year. Because bond B is not as safe as bond A, Ms. Jones decides that her investment in bond B will not exceed 40% of her investment in bond A by more than \$1000. How much should she invest at each rate to obtain the maximum number of dollars in interest per year?

19. Growfast Nursery is adding imported fruit trees and oriental shrubs to its existing line of nursery products. The trees yield a profit of \$6 each, and the shrubs a profit of \$7 each. The trees require 2 ft^2 of display space per tree, and the shrubs require 3 ft^2 per shrub. In addition, it takes 2 min to prepare a tree for display, and 1 min to prepare a shrub. The space and time constraints are as follows:

At most 12 ft^2 of display space is available.
At most 8 min of preparation time is available.

If Growfast can sell all the trees and shrubs it displays, how many trees and how many shrubs should it display each day to maximize its profit? (Assume that it is possible to arrange a display only once per day.)

20. The Excelsior Mining Company operates two mines, EMC 1 and EMC 2. EMC 1 produces 20 tons of lead ore and 30 tons of low-grade silver ore per day of operation. EMC 2 produces 15 tons of lead ore and 35 tons of low-grade silver ore per day of operation. Lead ore sells for \$14 per ton and low-grade silver ore sells for \$34 per ton. The company can sell at most 630 tons of the low-grade silver ore per month, but it can sell all the lead ore it produces. However, there is no space available for stockpiling any silver ore. The company employs one crew and operates only one of the mines at a time. Furthermore, union regulations stipulate that the crew not be worked in excess of 20 days per month. How many days per month should Excelsior schedule for each mine so that the income from the sale of the ore is a maximum?

21. The ABC Fruit Juice Company wants to make an orange-grapefruit drink and is concerned with the vitamin content. The company plans to use orange juice that has 2 units of vitamin A, 3 units of vitamin C, and 1 unit of vitamin D per ounce, and grapefruit juice that has 3 units of vitamin A, 2 units of vitamin C, and 1 unit of vitamin D per ounce. Each can of the orange-grapefruit drink is to contain not more than 15 oz and is to have at least 26 units of vitamin A, 30 units of vitamin C, and 12 units of vitamin D. The per-ounce cost of the orange juice is 4¢, and of the grapefruit juice 3¢.

a. How many ounces of each should be put into a can if the total cost of the can is as low as possible?
b. What is the minimum cost per can?
c. What is the vitamin content per can?

22. Joey likes a mixture of Grape-nuts, Product 19, and Raisin Bran for his breakfast. Here is some information about these cereals (each quantity in the table is per ounce).

Nutritional Value	*Grape-nuts*	*Product 19*	*Raisin Bran*
Calories	100	110	90
Fat	1 g	0 g	1 g
Sodium	195 mg	325 mg	170 mg

Joey is on a low-sodium diet and tries to make up 12 oz of the mixture so that the number of calories is at least 1200 but not over 1500, the total amount of fat is not more than 10 g, and the sodium content is minimized. Can he do it? If so, what are the per-ounce quantities of calories, fat, and sodium in his mixture? (*Hint:* If he uses x oz of Grape-nuts and y oz of Product 19, then he must use $12 - x - y$ oz of Raisin Bran.)

In Other Words

In a linear programming problem, describe in your own words what is meant by the following:

23. **a.** Constraint **b.** Feasible region

24. **a.** Positivity condition **b.** Convex polygon

Discovery

25. Suppose there is a championship prize fight. Gary the Gambler is tired of losing money and wants to hedge his bets so as to win at least \$100 on the fight. Gary finds two gambling establishments: A, where the odds are 5 to 3 in favor of the champion, and B, where the odds are 2 to 1 in favor of the champion. What is the least total amount of money that Gary can bet, and how should he place it to be sure of winning at least \$100? You can do this as a linear programming problem. First, let x dollars be placed on the champion to win with A, and y dollars on the challenger to win with B. Then, verify the following:

a. If the champion wins, Gary wins $\frac{3}{5}x$ dollars and loses y dollars, for a net gain of $\left(\frac{3}{5}x - y\right)$ dollars.
b. If the challenger wins, then Gary loses x dollars and wins $2y$ dollars, for a net gain of $(-x + 2y)$ dollars.
c. The problem now is to minimize $x + y$ subject to the constraints

$$\begin{aligned} \tfrac{3}{5}x - y &\geq 100 \\ -x + 2y &\geq 100 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

Solve this problem.

26. Show that there is no feasible region if Gary reverses his bets and places x dollars on the challenger to win with A, and y dollars on the champion to win with B.