Supporting Information for "Efficient inversion and uncertainty quantification of a tephra fallout model"

Contents of this file

1. Text S1 - Summary of Tephra2 formulation

Additional Supporting Information (Files uploaded separately)

1. Datasets S1 and S2. Tephra thickness and grainsize data used in the analysis of the 2011 Kirishima eruption and the 1992 Cerro Negro eruption.

Introduction

This supplementary information contains both a summary of the theortical formulation of the Tephra2 model and the tephra data used for the Kirishima and Cerro Negro analyses. The tephra data are provided as comman-seperated value files.

Text S1 - Summary of Tephra2 formulation.

The variable $f_{i,j}(x, y)$ in equation (1) in the main text describes the atmospheric diffusion of tephra particles of size j that are released from column layer i and then advected by the wind away from the erupting column. Tephra particles of varying sizes have varying settling velocities in the atmosphere, which is ultimately reflected in their total fall time; fall time is measured from a particle's release point in the eruption column to deposition on the ground surface. For example, large particles with faster settling velocities will be less dispersed in the deposit than small particles with slower settling velocities, if both are released from the same height. Similarly, particles of a given settling velocity will be more dispersed when released from higher in the eruption column than when released lower in the column. The distribution of mass in the deposit is strongly affected by the total fall time of particles, which in turn, depends on their settling velocities. Modeling particles within a limited range of settling velocities (particles of similar size and density) allows values of $f_{i,j}(x,y)$, and hence $M^o_{i,j}$, to be better constrained for a given mass fraction $M(x,y)$ mapped in a specific deposit.

The settling velocity of a particle is a function of multiple parameters including particle density, particle volume (here characterized as an equivalent sphere diameter based on sieve measurements), particle shape, atmospheric properties, gravitational acceleration, and local Reynold's number. Tephra2 employs a formulation of settling velocity proposed that changes based on whether a particle settles via a laminar, intermediate or turbulent regime at different heights in the atmosphere,

$$v_j = \begin{cases} \frac{\rho_j g d_j^2}{18\mu} & \text{if laminar, } Re < 6, \\ d_j \left[\frac{4g^2\rho_j^2}{225\mu\rho_a}\right]^{1/3} & \text{if intermediate, } 6 \leq Re < 500, \\ \left[\frac{3.1\rho_j g d_j}{\rho_a}\right]^{1/2} & \text{if turbulent, } Re \geq 500, \end{cases} \quad (1)$$

where ρ_j is the density and d_j the diameter of particles of size class j, $\mu = 0.000018325\,\mathrm{Pa\,s}$ is the air viscosity, $g = 9.81$ is gravity, ρ_a is the air density of layer k in the atmosphere, with

$$\rho_a = \rho_{std} e^{-h_k/8200} \quad (2)$$

where $\rho_{std} = 1.293\,\mathrm{kg\,m^3}$ at sea-level, and h_k is the height above mean sea level (AMSL) of the center of atmospheric layer k . Note the difference in notation, i represents layers in the eruption column and k represents layers in the atmosphere, because these layers need not represent the same elevation or have the same total number of divisions.

Tephra particle density also varies as a function of particle size given that coarse particles contain numerous vesicles and fine particles are more likely to be made up of individual crystals. Tephra2 uses a linear model to approximate this change in particle density as a function of particle size,

$$\rho_j = \begin{cases} \rho_{min} & \text{if coarse, } \phi_j < \phi_C, \\ \rho_{max} - \frac{(\rho_{max}-\rho_{min})(\phi_j-\phi_F)}{(\phi_C-\phi_F)} & \text{if intermediate, } \phi_C \leq \phi_j < \phi_F, \\ \rho_{max} & \text{if fine, } \phi_j \geq \phi_F, \end{cases} \quad (3)$$

where ρ_{min} and ρ_{max} are the mean densities of the coarsest and finest fractions of the deposit, respectively. The coarse particle threshold is defined as ϕ_C, the fine particle threshold is defined as ϕ_F, with particle density varying linearly between the two. Tephra2 assumes fixed densities for fine and coarse particles, so the only factor affecting settling velocity is, d_j, the particle diameter.

The total fall time of a particle is thus estimated by summing particle fall times through multiple atmospheric layers, starting from a particle's initial release height from the eruption column, H_i, to its final resting place on the ground,

$$t_{i,j} = \sum_{k=0}^{H_i} \frac{z_k}{v_j} \quad (4)$$

where z_k is the thickness of layer k. The settling velocity is assumed to be constant within a given atmospheric layer, k, so the summation must be done in sufficiently small steps to avoid large settling velocity errors. Also, it is assumed that a particle instantaneously reaches its settling velocity within layer k. This assumption is reasonable for low levels in the atmosphere, but not necessarily true for high levels in the atmosphere (*e.g.*, $> 40\,\mathrm{km}$), where low atmospheric density creates very high settling velocities.

The total diffusion of particles of size j that are released from height H_i with a total fall time, $t_{i,j}$, is estimated using a linear, Fickian, model for coarse particles and a non-linear model for fine particles, with the switch between the two models determined by a parameter, τ, known in the Tephra2 code as the fall time threshold (FTT),

$$\sigma^2_{i,j} = \begin{cases} 4K(t_{i,j} + t_i^{'}) & \text{if } t_{i,j} < \tau, \\ \frac{8C}{5}(t_{i,j} + t_i^{'})^{5/2} & \text{if } t_{i,j} \geq \tau, \end{cases} \qquad (5)$$

where K is the Fickian diffusion coefficient, C is a turbulent eddy constant, $t_i^{'}$ is the increased horizontal diffusion time accounting for the broadening of the eruptive column with height. For relatively coarse particles, $t_{i,j} < \tau$, the diffusion is assumed to be linear with respect to total fall time. For small particles found in tephra fallout deposits, $t_{i,j} \geq \tau$, diffusion is nonlinear with respect to total particle fall time. Analysis of deposits suggests that a value for τ of approximately 1 hr is often appropriate. In this formulation diffusion is assumed to be isotropic in the horizontal (X, Y) direction; vertical diffusion is neglected.

The variable $t_i^{'}$ reflects a change in width of the eruption column with height, a feature of eruption columns which effectively broadens the area impacted by the resulting tephra fallout deposit:

$$t_i^{'} = \left[0.2 h_i^2\right]^{2/5} \qquad (6)$$

and this formula is used here, acknowledging that more complex formulations associated with laterally spreading plumes are possible.

Given these important, albeit complex, details, it is possible to estimate diffusion and advection away from the eruptive column,

$$f_{i,j}(x,y) = \frac{1}{2\pi\sigma^2_{i,j}} \exp\left[-\frac{(x-\bar{x}^2_{i,j}) + (y-\bar{y}^2_{i,j})}{2\sigma^2_{i,j}}\right] \tag{7}$$

where

$$\bar{x}_{i,j} = x_0 + \sum_{k=0}^{H_i} \frac{w_{x,k} z_k}{v_j, k} \tag{8}$$

and

$$\bar{y}_{i,j} = y_0 + \sum_{k=0}^{H_i} \frac{w_{y,k} z_k}{v_j, k} \tag{9}$$

Particles originate from a point-release source in the eruption column (x_0, y_0, H_i) and are advected by the wind (velocity components $w_{x,k}$ and $w_{y,k}$). Thus, the wind direction and speed may vary as a function of height, but for simplicity do not vary as a function of horizontal distance.

Equation 7 shows that $\sigma^2_{i,j}$ represents the variance of a Gaussian distribution, and equations 4 and 5 indicate that this dispersion term is directly controlled by a diffusion model that depends on particle settling velocity, and hence upon particle size.

Tephra2 assumes that the total particle size distribution (for the entire erupted mass of tephra) follows a lognormal distribution (*i.e.,* is normally distributed in units of ϕ) . Therefore the fraction of particles in size class j is

$$f_j(\phi) = \frac{1}{\sqrt{2\pi\bar{\sigma}_\phi}} \exp\left[-\frac{\phi - \bar{\mu}_\phi}{2\bar{\sigma}^2_\phi}\right] \tag{10}$$

where $\bar{\mu}_\phi$ and $\bar{\sigma}^2_\phi$ are the estimated mean and variance of the particle size distribution, respectively. A simple plume model:

$$f_i(z) = U\left[H_{min}, H_t\right] \tag{11}$$

assumes that all particles are uniformly distributed within the eruptive column between H_{min} and H_t, regardless of particle size; for strong plumes, H_{min} and H_t represent the base and top of the umbrella region, respectively. In this case, tephra fallout associated with the development or waning stages of the plume are not accounted for, nor is sedimentation from the margins of the plume during steady eruption or ballistic transport . Alternatively, a probability density function may be used to describe the distribution of tephra released from the erupting column as a function of height above the vent. Tephra2 employs a plume model using the beta distribution,

$$f_i(z) = \frac{1}{B(\alpha, \beta)} z^{\alpha-1}(1-z)^{\beta-1} \tag{12}$$

where

$$B(\alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}, \qquad \alpha > 0, \beta > 0 \tag{13}$$

and $\Gamma()$ is the gamma function. If $\alpha = \beta = 1$, then the beta distribution is the same as a uniform random distribution (equation 11). For cases where $\beta = 1$, if $\alpha > 1$ then particle release heights are skewed toward the top of the erupting column; when $\alpha < 1$, particle release heights are skewed toward the lower part of the column. In our inversion, we fixed $\beta = 1$ and only varied α. Thus the equation,

$$M^o_{i,j} = M^{\phi_{max}}_{\phi_{min}} f_j(\phi) f_i(z) \quad (14)$$

describes the total mass of tephra, $M^{\phi_{max}}_{\phi_{min}}$, between size classes ϕ_{min} and ϕ_{max}.

Given these model assumptions, everything is in place to calculate tephra mass deposited per unit area ($\mathrm{kg\,m^{-2}}$), $M(x, y)$, using the model parameters just described.

Data Set S1. Tephra thickness and grainsize data used for analysis of the 2011 Kirishima eruption.

Data Set S2. Tephra thickness and grainsize data used for analysis of the 1992 Cerro Negro eruption.