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On Some Problems on Polynomial Interpolation in Several Variables

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by

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Table of Contents

Abstract

Polynomial approximation is a long studied process, with a history dating back to the 1700s, At which time Lagrange, Newton and Taylor developed their famed approximation methods. At that time, it was discovered that every Taylor projection (projector) is the pointwise limit of Lagrange projections. This leaves open a rather large and intriguing question, What happens in several variables?

To this end we define a linear idempotent operator to be an ideal projector whenever its kernel is and ideal. No matter the number of variables, Taylor projections and Lagrange projections are always ideal projectors, and it is well known that in one variable, that, not only Taylor projections, but every ideal projector, is the pointwise limit of Lagrange projections. This is also true in two variables, but false in three or more variables. We call the projectors which are the pointwise limits of Lagrange projectors, Hermite projectors. As it turns out, the Hermite projectors happen to be exactly those projectors whose kernels are something algebraic geometers refer to as smoothable. The question of which ideals are smoothable is also an open question in algebraic geometry. This correlation, provides the humble researcher with a whole new slew of tools to apply to problems.

It is the aim of this dissertation to provide a field map to this interesting environment, in which some problems, previously intractable, can be approached with renewed vigor. One such problem, unstudied except for some very specific cases, is, given sets $\mathcal{V}_1, \ldots, \mathcal{V}_n$, is it possible to find a polynomial p which interpolates each polynomial p_i on $\mathcal{V}_i$. We present the results of a paper which was submitted for publication providing a generalized extension of a theorem by W.K. Hayman and Z. G. Shandze. For the second part of the dissertation we present a result of a second paper that was submitted for publication, in which we make a useful contribution to a question of Carl de Boor, which ideal projectors are Hermite.

In the first part of this dissertation we find that the answer to the question about interpolation on sets is: sometimes. We will provide some conditions under which it is, and is not, possible to do this. One of these conditions is the aforementioned extension of a result of W.K. Hayman and Z. G. Shandze. In the second part we make a contribution to the question of which projectors are Hermite. The Laskar-Noether theorem shows that every ideal has a unique minimal decomposition into primary ideals, we prove that if $J_1, \ldots, J_k$

is the minimal primary decomposition of J, and $P, P_1, \ldots, P_k$ are ideal projectors with kernels $J, J_1, \ldots, J_k$ respectively, then P is Hermite if and only if each P_i is Hermite. In the language of algebraic geometry, an equivalent statement, is that the ideal J is smoothable, if and only if each J_i is smoothable.

0 Introduction

In this dissertation, we will study polynomial interpolation in several variables. In one variable, the solutions too interpolation problems are the well known Taylor polynomial, Lagrange projection and classical Hermite interpolation. The main idea that will extend throughout this entire document, is the use of ideals and concepts from algebraic geometry to simplify these problems. Whenever possible, we will attempt to provide translations between the languages of analysis and algebraic geometry.

We will Let $\mathbb{K}$ be algebraically closed field and we will use $\mathbb{K}[\mathbf{x}] = \mathbb{K}[x_1, \ldots, x_d]$ to mean the ring of polynomials in d variables over $\mathbb{K}$. It is important to establish this, as this will allow us to use our most important tool, namely, Hilbert's Nullstellensatz. The Nullstellensatz tells us that if J is a non-trivial proper ideal in $\mathbb{K}[\mathbf{x}]$ then there exists at least one point $\mathbf{z} \in \mathbb{K}^d$ such that $f(\mathbf{z}) = 0$ for all $f \in \mathbb{K}[\mathbf{x}]$.

Definition 0.1. *Let $J \subset \mathbb{K}[\mathbf{x}]$ be an ideal. Then $\mathcal{V}(J) = \{\mathbf{x} : f(\mathbf{x}) = 0 \text{ for all } f \in J\}$. We call the set $\mathcal{V}(J)$ the* ***variety*** *of J.*

It is easy to see that for any ideal $J \subset \mathbb{K}[\mathbf{x}]$ if $f - g \in J$ then then it must be that $f(\mathbf{x}) = g(\mathbf{x})$ for any $\mathbf{x} \in \mathcal{V}(J)$. Let $A \subset \mathbb{K}^d$, whenever $f(\mathbf{x}) = g(\mathbf{x})$ agree for all $\mathbf{x} \in A$ we say f **interpolates** g on A. Moreover, given any set $A \subset \mathbb{K}^d$, the functions $f \in \mathbb{K}[\mathbf{x}]$ such that $f(\mathbf{x}) = 0$ for all $\mathbf{x} \in A$ form an ideal J since if $f(\mathbf{x}) = 0$ then $gf(\mathbf{x}) = 0$. It must be the case that $A \subset \mathcal{V}(J)$. It is, of course possible to interpolate derivatives as well, as we will see in the following example.

Example 0.2. *Let us consider the space of polynomials such that $f(0) = 0$ and $\frac{\partial f}{\partial x_1}(0)$, since $\frac{\partial}{\partial x_1}(gf) = f\frac{\partial}{\partial x_1}g + g\frac{\partial}{\partial x_1}f$ it is clear that this is an ideal, let's call it J. Now if $p - q \in J$ then it will immediately follow that $p(0) = q(0)$ and that $\frac{\partial p}{\partial x_1}(0) = \frac{\partial q}{\partial x_1}(0)$. Hence, p interpolates q at zero and $\frac{\partial p}{\partial x_1}(0)$ interpolates $\frac{\partial q}{\partial x_1}(0)$.*

Since this will never cause confusion, we will freely abuse notation and simply say that, the function p in our example interpolates q both at the origin, and at the partial derivative with respect to x_1 at the origin. It seems that membership in an ideal can show more than simply that two functions agree on a set of points. Of course, as mentioned above, the condition that f interpolates g on some set $A \subset \mathbb{K}^d$ actually implies that f interpolates g on the variety of some ideal. In lieu of this fact, henceforth, we shall only refer to interpolation on varieties.

We will refer to the polynomial we wish to interpolate on a variety $\mathcal{V}$ as the **data** given on $\mathcal{V}$. We will use $< f_1, \ldots, f_k >$ to mean the ideal $\{h : h = gf_i \text{ for some } 1 \leq i \leq k\}$, and we will use $I(\mathcal{V})$ to mean the ideal $\{f(\mathbf{x}) = 0 : \text{for all } \mathbf{x} \in \mathcal{V}\}$. Finally for an ideal J we will use $\sqrt{J}$ to mean the ideal $\{f : f^n \in J\}$, this notation is standard in algebraic geometry.

Whenever $J = \sqrt{J}$, J is called a **radical** ideal. One of the many formulations of Hilbert's Nullstellensatz states that $I(\mathcal{V}(J)) = \sqrt{J}$. In other words, a radical ideal is one such that if $f - g \in J$ this ensures only that $f(\mathbf{x}) = g(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{V}(J)$.

Having presented enough of the basic concepts used in this dissertation, let us now discuss some of our results, we will remark that this dissertation is broken into two main parts. For the first part, we we present a first attempt to study polynomial interpolation on general varieties. To the best knowledge of the author, up until now, except for a couple notable exceptions, the study of polynomial interpolation has been confined to the interpolation of data given at points. The notable exceptions being a study of interpolation on "flats" by Carl de Boor, Nira Dyn, and Amos Ron [4], and a second study of the same, referenced therein, by Hakopian and Sahakian.

In this first part, we address the question: given varieties $\mathcal{V}_1, \ldots, \mathcal{V}_k$ and functions $p_1, \ldots, p_k$ when is it possible to find a function f that interpolates the data given by p_i on $\mathcal{V}_i$ for each $1 \leq i \leq k$. We find that, ultimately for this to be possible, one must either impose restrictions on the varieties $\mathcal{V}_i$, or impose restrictions on the data given by the polynomials p_i. In closing of this first part, we will seek solutions to the interpolation problem among polynomial solutions to PDEs with constant coefficients, thus approaching the subject of very general boundary problems for PDEs. In particular we will extends some results of W.K. Hayman and Z. G. Shanidze of [11] who considered the problem in two variables and for homogeneous PDEs of degree 2.

For an explanation of the second part of this dissertation, we will be forced to present a few more definitions in order to explain the question. We will use the term **projector** (projection) to mean an Idempotent linear operator.

Definition 0.3. *A linear projector $P : \mathbb{K}[\mathbf{x}] \to \mathbb{K}[\mathbf{x}]$ is called an **Ideal Projector** if the kernel of P is an ideal in $\mathbb{K}[\mathbf{x}]$.*

The idea of the ideal projector was first described by Birkoff [2], and was brought to public attention by Carl de Boor [3]. Since it is true that for any projector P, ideal or not, that $P(P(f) - f) = Pf - Pf = 0$, it is clear that in the case where P is ideal, that $P(f) - f \in J = \ker P$. Since J is an ideal, this tells us that $P(f)$ interpolates certain data from f on $\mathcal{V}(J)$. In-fact, it follows from example 0.2 that the operator $T_n(f)$

giving the nth Taylor polynomial of f, is in-fact, an ideal projector, as is the classical Hermite projector in one variable which interpolates at points and at various derivatives at those points. Finally, it follows from that fact that $I(\mathcal{V})$ is an ideal, that the operator $L_{x_1,\ldots,x_k}(f)$ giving the Lagrange polynomial interpolating at points $\{x_1, \ldots, x_k\}$ is also an ideal projector.

Definition 0.4. *We call an ideal projector P a* ***Lagrange projector*** *whenever* $\ker P$ *is a radical ideal in* $\mathbb{K}[\mathbf{x}]$

It is easy to see that any classical Lagrange projection in one variable is a Lagrange projector, by this definition. Furthermore, in one variable it is the well known result of Newton, that for any Taylor projector T there exists a sequence of Lagrange projectors L_n such that $L_n(f) \to T(f)$ pointwise for all $f \in \mathbb{C}[\mathbf{x}]$. It is in-fact known that that this holds for any classical Hermite projector. It turns out, that every ideal projector in two variables is also the pointwise limit of Lagrange projectors [5], however, this turns out to be false in three or more variables.

Definition 0.5. *Let H be an ideal projector onto a finite dimensional space $G \subset \mathbb{K}[\mathbf{x}]$, if there exist Lagrange projectors L_n onto G, such that $L_n(f) \to H(f)$ pointwise for all $f \in \mathbb{K}[\mathbf{x}]$ then we call H a* ***Hermite*** *projector.*

Remark 0.6. *Notice that since G in the definition above is finite dimensional, this implies that all convergences on G are the same, so in the case of the Hermite projector, this convergence is in-fact uniform. As such, when referring to such projectors, we will drop the pointwise, and simply say $L_n \to H$*

It is a question of Carl de Boor [3], to find out which ideal projectors are Hermite. This turns out to be the same as an open question in algebraic geometry, which ideals are smoothable. In the second part of this dissertation, we provide a great deal of necessary background in ideal projection, culminating in a contribution to this question. The Laskar-Noether Theorem tells us that every ideal J has a unique minimal decomposition into primary ideals. If $J_1, \ldots, J_k$ is this unique minimal primary decomposition for some ideal J, what we show, is that if $P_1, \ldots, P_k, P$ are ideal projectors with kernels $J_1, \ldots, J_k, J$ respectively, then P is Hermite, if and only if each P_i is Hermite, or in the language of algebraic geometry, that J is smoothable, if and only if each J_i is smoothable.

1 Interpolation on Varieties

In this, the first part of our dissertation, we will concern ourselves with interpolating polynomials $p_1 \ldots, p_k$ on arbitrary sets $\mathcal{V}_1, \ldots, \mathcal{V}_k \subset \mathbb{K}^d$. The motivation for studying this, comes from a paper by Carl de Boor, Nira Dyn, and Amos Ron [4]. In this paper, the authors provide a method and conditions for interpolation on **flats**, linear subspaces of dimension $s < d$ of $\mathbb{R}^d$. The proofs therein are quite technical and tedious, so the question becomes, can this be done more easily in the language of ideals?

First of all, what do we mean by interpolation on varieties? Let $g \in \mathbb{K}[\mathbf{x}]$ then for a given set $\mathcal{V}'$ the space of functions p such that $p(\mathcal{V}') = 0$ is an ideal in $\mathbb{K}[\mathbf{x}]$ let us call this ideal J. The Nullstellensatz tells us that J has a variety $\mathcal{V}$ and it is clear that $\mathcal{V}' \subset \mathcal{V}$. So for every polynomial $f = g + p, f \in J$, it follows that $f - g$ is zero on $\mathcal{V}'$, but it is also true that $f - g$ is zero on $/V$. evidently f interpolates not only on $\mathcal{V}'$ but also on $\mathcal{V}$. In short, all functions that interpolate g on $\mathcal{V}'$ also interpolate g on $\mathcal{V}$.

Remark 1.1. *Algebraic geometers construct a topology on arbitrary $\mathbb{K}^d$ by defining closed sets to be varieties in $\mathbb{K}^d$, this topology is known as the **Zariski Topology**. For a set $A \subset \mathbb{K}^d$ the **Zariski closure** of A is the smallest variety containing A. This means that we can restate the above discussion as any function that interpolates g on $\mathcal{V}'$ also interpolates g on the Zariski closure of $\mathcal{V}'$. We will talk more about Zariski closures in the second part of this dissertation.*

So far, so good, but what if we want to interpolate on more than one variety? Formally, what if we have varieties $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_n$ on which polynomials $p_1, p_2, \ldots, p_n$ are defined respectively? Is it possible to find a polynomial $f(\mathbf{x})$ such that $(f - p_i)|_{\mathcal{V}_i} = 0$? Well this is where things get interesting.

Example 1.2. *Let $J_1 =< x >$ and $J_2 =< y >$. Let $p_1 = 0$ and $p_2 = 1$. Since $g(0)$ cannot equal both one and zero, it is clear that in this case the interpolation is not possible.*

And so, the question becomes, what are necessary and sufficient conditions for the interpolation to be possible?

1.1 *Disjoint Varieties*

Let us consider the case in which the varieties are pairwise disjoint. In this case the obstacle we encountered in Example 1.2, cannot occur. This is the most similar to the well known case of interpolation on points, as points are varieties. Obviously, distinct points do not intersect. So one might guess that the same thing would happen in the case of general distinct varieties.

It turns out that this is true, and the algebraist may have already realized that this is an immediate result of the Nullstellensatz and the Chinese remainder theorem. We will not present this proof, and will instead prove this fact in a much more analytic way, that is, perhaps, more illuminating.

Lemma 1.3. *[16] Let J_1, J_2 be ideals such that $\mathcal{V}(J_1) \cap \mathcal{V}(J_2) = \emptyset$, then there exists a polynomial g such that $g - 1 \in J_1$ and $g \in J_2$*

Proof. Since $\mathcal{V}(J_1 + J_2) = \mathcal{V}(J_1) \cap \mathcal{V}(J_2) = \emptyset$, the Nullstellensatz tells us that $1 \in (J_1 + J_2)$. This implies that there exist polynomials $g_1 \in J_1$ and $g_2 \in J_2$ such that $g_1 + g_2 = 1$. Setting $g = g_2$ yields the desired polynomial. □

Lemma 1.3 is basically a corollary of the Nullstellensatz, but it bears a striking resemblance to Hahn-Banach Theorem, only for algebraic sets. Let us continue our line of reasoning that this should work in the same way it works for points, the following corollary constructs polynomials that are similar to the well known Lagrange fundamental polynomials in one variable.

Corollary 1.4. *[16] Let $J_1, \ldots, J_n$ be a collection of ideals with pairwise disjoint varieties. Then there exist polynomials $g_1, \ldots, g_n$ such that $g_i - \delta_{i,j} \in J_j$, where $\delta_{i,j}$ is the Kronecker delta, i.e $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ otherwise.*

Proof. If the ideals $J_1, \ldots, J_n$ have pairwise disjoint varieties, then since $\mathcal{V}(\cap_{j=1}^{n-1} J_j) = \cup j = 1^{n-1}(\mathcal{V} J_j)$, then $\mathcal{V}(\cap_{j=1}^{n-1} J_j)$ and $\mathcal{V}(J_n)$ are disjoint. It follows from Lemma 1.3 that there exists a polynomial $g_n \in \mathcal{V}(\cap_{j=1}^{n-1} J_j)$ such that $g_n - 1 \in J_n$ □

In the case where each J_i is a radical ideal whose variety is a distinct point in one variable, it is easy to see that the Lagrange fundamental polynomials actually satisfy the corollary. Following our model of the Lagrange polynomials, it is now a simple matter to prove our claim.

Theorem 1.5. *[16] Let $J_1, \ldots, J_n$ be ideals in $\mathbb{K}[\mathbf{x}]$ such that for all i, j, $\mathcal{V}(J_i) \cap \mathcal{V}(J_j) = \emptyset$. Let $p_1, \ldots, p_n \in \mathbb{K}_x$, then there exists a function $f \in \mathbb{K}[\mathbf{x}]$ such that $f - p_i \in J_i$*

Proof.

$$f = \sum_{j=1}^{n} p_j g_j$$

where g_j is given by Corollary 1.4 $\square$

In the paper of de Boor, Dyn, and Ron, the case of disjoint varieties (flats in their paper) was handled as a special circumstance. It is interesting to note that, using our approach this case turned out to be simple to handle in general. Theorem 1.5 shows that it is always possible to interpolate polynomials on disjoint varieties. This is what we expected, since this case is the most like the finite dimensional case. When the varieties do intersect, it will not always be possible to interpolate arbitrary functions on arbitrary varieties. We will be forced to either restrict the varieties, or the data to be interpolated, as seen in Example 1.2.

1.2 *Interpolation on Two Varieties*

It is clear from Example 1.2 that we must require the data to agree wherever the varieties intersect. This actually seems like a rather obvious requirement, but is it enough? One might think that since we wish to interpolate the values of the polynomials p_i when evaluated at the points in $\mathcal{V}_i$, i.e. we wish to require $f - p_i \in I(\mathcal{V}_i)$ that it might be enough that $p_i = p_j$ on $\mathcal{V}_i \cap \mathcal{V}_j$. Unfortunately, not even this turns out to be true in general.

Example 1.6. *Consider the Ideals $J_1 =< x^2 + y >$ and $J_2 =< y >$. Set $p_1 = x, p_2 = 0$. One can clearly see that both J_1 and J_2 are radical ideals. Inspection shows that the varieties intersect only at the origin, and certainly $p_1 = p_2$ at the origin. Now suppose there exists a polynomial g such that $g - p_1 \in J_1$ and $g - p_2 \in J_2$. It follows that there exists a polynomial h such that*

$$g - p_1 = g - x = (x^2 + y)h$$

this implies that $g = (x^2 + y)h + x$, However, there must also exist a polynomial f such that

$$yf = g - p_2 = g - 0 = (x^2 + y)h + x$$

but this polynomial has a freestanding x that cannot possibly be canceled, so this is impossible!

So, what went wrong? Close inspection of Example 1.6 will reveal some interesting issues. For one thing, one might expect that since J_1 and J_2 were radical, and since $p_1 - p_2 = 0$ on $\mathcal{V}_1 \cap \mathcal{V}_2$, it should follow that we could interpolate on $\mathcal{V}_1 \setminus (\mathcal{V}_1 \cap \mathcal{V}_2)$ and $\mathcal{V}_2 \setminus (\mathcal{V}_1 \cap \mathcal{V}_2)$ and everything would work fine. Unfortunately, these are not varieties, so what we actually need is for $p_1 - p_2$ to be in the ideal $J_1 + J_2$. But, J_1 and J_2 were radical so $J_1 + J_2$ should also be radical and everything should still work right?

Unfortunately, $J_1 + J_2$ is NOT radical. Simple computations will show that $J_1 + J_2 =< x^2 + y > + < y >=< x^2, x^2 + y, y >$ an ideal which contains x^2, but not x. This means, that there is some other condition the interpolation polynomial must satisfy. Notice that $p_1 - p_2 = x$, which is exactly the missing polynomial from $J_1 + J_2$. x is, of course, zero at 0, and obviously $x \in \sqrt{J_1 + J_2}$, but this wasn't enough to guarantee interpolation was possible.

So now, two paths diverge in the wood, and we must decide, do we restrict the varieties on which interpolation can occur? or do we restrict the data which can be interpolated? In the paper motivating this discussion [4], authors de Boor, Dyn, and Ron imposed restrictions on both. The varieties were limited to flats all of the same dimension, and restrictions the authors named *consistent* and *X-compatible* were placed upon the data. In this dissertation we will separately explore these choices.

The first approach we will discuss, is what happens when one chooses to restrict the varieties themselves. This approach has the advantage that we will always be able to interpolate polynomials $p_1, \ldots, p_n$ on varieties $\mathcal{V}_1, \ldots, \mathcal{V}_n$ as long as those varieties satisfy certain conditions. We will begin with two varieties, it turns out simply preventing the obstacle we encountered in Example 1.6 will be sufficient in this case.

Lemma 1.7. *[16] Let $\mathcal{V}_1, \mathcal{V}_2$ be varieties in $\mathbb{K}^d$, then, the following are equivalent:*

i) For EVERY pair of polynomials $p_1, p_2 \in \mathbb{K}[\mathbf{x}]$ such that

$$p_1|_{\mathcal{V}_1 \cap \mathcal{V}_2} = p_2|\mathcal{V}_1 \cap \mathcal{V}_2$$

there exists a polynomial p such that

$$p|_{\mathcal{V}_j} = p_j|_{\mathcal{V}_j} \tag{1.1}$$

ii) $I(\mathcal{V}_1) + I(\mathcal{V}_2) = \sqrt{I(\mathcal{V}_1) + I(\mathcal{V}_2)}$

Proof. ii)→i) It follows from assumption that $p_1 - p_2 \in I(\mathcal{V}_1) + I(\mathcal{V}_2)$. Therefore there exist polynomials g_1, g_2 in $I(\mathcal{V}_1), I(\mathcal{V}_2)$ respectively, such that

$$p_1 - p_2 = g_1 + g_2$$

Since g_1 vanishes on $\mathcal{V}_1$, it must be that $g_2|_{\mathcal{V}_1} = p_1 - p_2$. Since g_2 vanishes on $\mathcal{V}_2$, the polynomial

$$f = g_2 + p_2$$

satisfies equation (1.1).

i)→ii) Suppose $I(\mathcal{V}_1)+I(\mathcal{V}_2)$ is not radical. In this case, there exists a polynomial $g^n \in I(\mathcal{V}_1)+I(\mathcal{V}_2)$ but g is not in $I(\mathcal{V}_1)+I(\mathcal{V}_2)$. Choose $p_1 = g$ and $p_2 = 0$ (*as we did in Example 1.6*). To reach a contradiction, suppose there exists an f satisfying equation (1.1) then $f|_{\mathcal{V}_2} = 0$ so $f \in I(\mathcal{V}_2)$, but it also must be so that $g - f \in I(\mathcal{V}_1)$. However, this implies that

$$I(\mathcal{V}_1) + I(\mathcal{V}_2) \ni (g - f) + f = g$$

but this is a contradiction since $g \notin I(\mathcal{V}_1) + I(\mathcal{V}_2)$ □

Now, let us look at what happens if we choose to restrict the data. Again, at least for two varieties it turns out to be sufficient to prevent the obstacle we encountered in Example 1.6. The general idea of the Lemma is that if we ensure $p_1 - p_2 \in J_1 + J_2$ then we actually can copy what we did to get Theorem 1.5. In the case where the varieties satisfy Lemma 2.27, the conditions are, of course, exactly the same.

Lemma 1.8. *[16]Let $I_1, I_2 \subset \mathbb{K}[\mathbf{x}]$ be ideals, Let $p_1, p_2 \in \mathbb{K}[\mathbf{x}]$ then, the following are equivalent:*

i) There exists a polynomial p such that $p - p_k \in I_k$ for all $0 < k \leq 2$

ii) $p_1 - p_2 \in (I_1 + I_2)$

Proof. ii)→i) By assumption:

$$(p_1 - p_2) \in (I_1 + I_2)$$

so there exist polynomials q_1, q_2 in I_1, I_2 respectively, such that $q_1+q_2 = p_1-p_2$. This means $q_1+q_2-p_1+p_2 \in I_1$, which since $q_1 \in I_1$ implies that

$$(q_2 + p_2) - p_1 \in I_1 \tag{1.2}$$

so if we let

$$g = q_2 + p_2$$

then by equation (1.2), $g - p_1 \in I_1$, and since $q_2 \in I_2$ it follows that $g - p_2 \in I_2$, so, evidently, g satisfies condition *i).*

i)$\to$*ii)* Since $p - p_i \in I_i$, let $h_i : \mathbb{K}[\mathbf{x}] \to \mathbb{K}[\mathbf{x}]/I_i$ be a homomorphism. Clearly, $h_i(p - p_i) = 0$. Now, by the third Isomorphism Theorem, since $I_i \subset (I_1 + I_2)$ it follows that $(I_1 + I_2)/(I_i)$ is an ideal in $\mathbb{K}[\mathbf{x}]/I_i$, and it also follows that:

$$\frac{\mathbb{K}[\mathbf{x}]/I_i}{(I_1 + I_2)/I_i} \cong \frac{\mathbb{K}[\mathbf{x}]}{I_1 + I_2}$$

so, let $\rho : \frac{\mathbb{K}[\mathbf{x}]/I_i}{(I_1+I_2)/I_i} \to \frac{\mathbb{K}[\mathbf{x}]}{I_1+I_2}$ be this isomorphism of rings, and

$$\text{let } h_0 : \mathbb{K}[\mathbf{x}]/I_i \to \frac{\mathbb{K}[\mathbf{x}]/I_i}{(I_1 + I_2)/I_i}$$

be a homomorphism, then the composition is a homomorphism $h = \rho(h_0(h_i))$. Which maps $\mathbb{K}[\mathbf{x}]$ to $\mathbb{K}[\mathbf{x}]/(I_1 + I_2)$. Furthermore, $h(p - p_i) = 0$ thus $(p - p_i)$ is in the ideal $(I_1 + I_2)$, and, since i was arbitrary, *ii)* holds. □

Remark 1.9. *Observe that the ideals in Lemma 1.8 need not be radical. Recall Example 0.2 where we showed that a set of interpolation conditions determine an ideal. This means Lemma 1.8 also works if one wishes to interpolate at various derivatives. Note that although we interpolate data on varieties, we could if we wanted too, use this lemma to interpolate derivatives on varieties as well.*

These two lemmas cover the cases in which there are only two varieties. In the case of only two varieties things worked out well enough. We only needed to impose some very simple conditions in order to guarantee interpolation was possible, however, the results of de Boor, Dyn, and Ron, were far more complicated than this. As we mentioned, this is to the best knowledge of the author, new research, so something is bound to go wrong if we try to extend these results to n varieties. For now, though, let us naively soldier on.

1.3 *The General Case*

We learned from the case of two varieties, that if we wish to interpolate on varieties $\mathcal{V}(J_1)$ and $\mathcal{V}(J_2)$, then the polynomials p_1, p_2, which we intend to interpolate must agree on $\mathcal{V}(J_1) \cap \mathcal{V}(J_2)$, moreover, it must be that $p_1 - p_2 \in J_1 + J_2$. Given this, the hope would be, that for ideals $J_1, \ldots, J_n$, all that was needed was for $p_i - p_j \in J_i + J_j$.

Example 1.10. *Let* $J_1 =< x >$, $J_2 =< x + y^2 >$, *and* $J_3 =< x - y^2 >$. *Let* $p_1 = 0$, $p_2 = y^2$, *and* $p_3 = y^2$.

then:

$$0 \in \text{ any Ideal}$$
$$y^2 - 0 = x + y^2 - x, \text{ so: } y^2 \in J_1 + J_2$$
$$y^2 - 0 = x - (x - y^2), \text{ so: } y^2 \in J_1 + J_3$$
$$y^2 - y^2 \in J_2 + J_3$$

It follows that $p_i - p_j \in J_i + J_j$ for all i, j, furthermore it is clear that $y^2 - p_2 \in J_2$ and $y^2 - p_3 \in J_3$. Moreover any polynomial f that interpolates these three polynomials, must satisfy $f - y^2 \in J_2 \cap J_3$ and $f - 0 \in J_1$. Now, the question becomes: is $(y^2 - 0) \in (J_1 + J_2 \cap J_3)$? It turns out, that it is not. If we can show that $y^2 \notin (J_1 + J_2 \cap J_3)$, it will follow from Lemma 1.8, that, it is not possible to interpolate p_1, p_2, p_3 on $\mathcal{V}(J_1), \mathcal{V}(J_2), \mathcal{V}(J_3)$

Proof. To reach a contradiction, let's assume

$$y^2 \in (< x > + < x - y^2 > \cap < x + y^2 >)$$

This implies that there exist $px \in < x >$ and $q \in < x - y^2 > \cap < x + y^2 >$ such that

$$px + q = y^2 \tag{1.3}$$

For a moment, let us look at what sort of polynomials are in $< x - y^2 > \cap < x + y^2 >$, since these ideals have a single generator (are principle ideals), there must exist polynomials f, g such that

$$f(x - y^2) = q = g(x + y^2)$$

it follows that

$$fx - fy^2 - gx - gy^2 = 0$$
$$\implies x(f - g) = y^2(f + g)$$

This means $f - g$ must have y^2 as a factor and $f + g$ must have x as a factor. W.L.O.G:

$$f - g = y^2$$
$$f + g = x$$
$$\implies 2f = y^2 + x$$

However by equation (1.3) this means that there exists a polynomial h

$$y^2 = px + \frac{(y^2 + x)(y^2 - x)}{2} = h(y^4 - x^2) + px$$

Certainly, this relation must hold when $x = 0$, but this would mean that

$$y^2 = h(0, y)y^4$$

but this is impossible, since h is a polynomial, $h \neq \frac{1}{y^2}$, contradiction. $\square$

The problem here, is that the existence of an interpolation polynomial p for p_1, p_2, p_3 that interpolates the data on varieties $\mathcal{V}(J_1), \mathcal{V}(J_2), \mathcal{V}(J_3)$, also implies the existence of an interpolation polynomial p_0 for p_2, p_3 interpolating the data on $\mathcal{V}(J_2), \mathcal{V}(J_3)$. The reason this ultimately failed, however, is that p_0 by itself, gives data on $\mathcal{V}(J_2) \cup \mathcal{V}(J_3)$. This means that p needs to interpolate p_1 on $\mathcal{V}(J_1)$ and p_0 on $\mathcal{V}(J_2 \cap J_3)$. Herein, lies the problem.

Using the approach where we restrict the varieties allowed for interpolation, this is simple enough to solve, and we have a complete solution. Unfortunately this means that there are certain collections of varieties that simply cannot be interpolated on. We present that theorem here.

Theorem 1.11. *[16] Let $\mathcal{V}_1, \ldots, \mathcal{V}_n$ be a collection of varieties. The following are equivalent:*

i) For every collection of polynomial $p_1, \ldots, p_n$ such that

$$p_k - p_j \in I(\mathcal{V}_k \cap \mathcal{V}_j) \textit{ for all } k$$

there exists a polynomial p such that $p - p_k$ is zero on $\mathcal{V}_k$

ii) For every $m < n$ the ideal $I(\cup_{j=1}^{m} \mathcal{V}_j) + I(\mathcal{V}_{m+1})$ is radical.

Proof. ii)→i) We proceed by induction, the claim is clear when $n = 1$. Choose varieties $\mathcal{V}_1, \ldots, \mathcal{V}_n$, by inductive hypothesis there exists a polynomial p_0 such that $p_0 - p_k$ is zero on $\mathcal{V}_k$ whenever $k < n$. If we treat p_0 as a function defined on $\cup_{j=1}^{n-1}\mathcal{V}_j$ then p_0, p_n, and $\mathcal{V}_n, \cup_{j=1}^{n-1}\mathcal{V}_j$ satisfy the hypothesis of Lemma 2.27, and so *i)* holds.

i)$\rightarrow$*ii)* This follows immediately from Lemma 2.27. Supposing $I(\cup_{j=1}^{m}\mathcal{V}_j) + I(\mathcal{V}_{m+1})$ is not radical, there exist polynomials p_0, p_{m+1} defined on varieties $\cup_{j=1}^{m}\mathcal{V}_j$ and $\mathcal{V}_m$ respectively that cannot be interpolated on these varieties. $\square$

This ends the description of varieties where interpolation is possible for arbitrary polynomials. Any other collections of varieties, invariably, have functions it is not possible to interpolate, and for reasons that are not immediately obvious.

Things get sort of sticky at this point, since more and more difficulties continue to arise. We will present a pair of theorems which answer these questions, if not in an entirely satisfactory manner. The first of these provides a completely usable set of conditions under which interpolation is always possible.

Theorem 1.12. *[16] Let $\mathcal{J} = \{J_1, \ldots, J_n\}$ be a finite collection of ideals in $\mathbb{K}[\mathbf{x}]$, Let $p_1, \ldots, p_n$ be a family of polynomials. If, for any i, j*

$$(p_i - p_j) \in (J_i + \cap_{k\neq i} J_k)$$

Then, there exists a polynomial p, such that $p - p_i \in J_i$

Proof. Observe that for all i, j, $(p_i - p_j) \in (J_i + \cap_{k\neq i} J_k)$, this implies that if we let

$$f_i = p_i - p_1$$

then $f_i \in (J_i + \cap_{k\neq i} J_k)$ it is clear that $0 \in (J_i + \cap_{k\neq i} J_k)$ so by Lemma 1.8 there exists a polynomial g_i such that $g_i - f_i \in J_i$ and $g_i \in J_k$ whenever $k \neq i$. Since i was arbitrary it, follows that

$$p = p_1 + \sum_{i=1}^{n} g_i$$

is the desired polynomial.

$\square$

Unfortunately It seems the converse is not true.

Example 1.13. *Consider the ideals* $J_1 =< z - x >, J_2 =< z - y >$ *and* $J_3 =< xy >$*. Let* $p_1 = x, p_2 = y$ *and* $p_3 = z$*. Notice that* $p_3 - p_1 \in J_1$ *and* $p_3 - p_2 \in J_2$*, so evidently* $p = z$ *satisfies* $p - p_i \in J_i$*.*

However, $J_1 \cap J_3 =< yz - xy >$*, and therefore does not contain* $p_1 - p_2 = x - y$*, and* $p_1 - p_2$ *is obviously not in* J_3*.It is obvious that it also cannot be in the sum* $< zx - x^2, yz - xy > + < z - y >$*, since* $J_1 \cap J_2$ *does not contain any terms of degree sum 1.*

So Theorem 1.12 gives us a condition on the data under which, it is always possible to find a interpolating polynomial. However example 1.13 shows that it may still be possible to find an interpolating polynomial, even if the conditions of Theorem 1.12 are not met. We will present another theorem which is if and only if, but the conditions of this theorem are not really feasible to check, so we merely state it for completion.

Proposition 1.14. *Let* $\mathcal{J} = \{J_1, \ldots, J_n\}$ *be a finite collection of ideals in* $\mathbb{K}[\mathbf{x}]$*, Let* $p_1, \ldots, p_n$ *be a family of polynomials. Then the following are equivalent*

i) There exists a function p *such that* $p - p_i \in J_i$ *for all* $1 \le i \le n$

ii) for all $A \subset_{\neq} \{1, \ldots, n\}$ *there exists a* p_0 *such that* $p_0 - p_i \in J_i : i \in A$*, and for any such* p_0 *whenever* $j \notin A$ *then* $p_0 - p_j \in (J_j + \cap_{i \in A} J_i)$

Proof. i)→ii) Choose any set A according to *ii)*, then p satisfies the conditions for p_0 so at least one such p_0 exists. for any such p_0 and any $j \notin A$, $p - p_0 \in \cap_{i \in A}$ and $p - p_j \in J_j$ so by Lemma (1.8) $p_0 - p_j \in (J_j + \cap_{i \in A} J_i)$.

ii)→i) Choose $1 \le j \le n$, and let $A = \{1, \ldots, n\} \setminus \{j\}$. By assumption there exists p_0 such that $p_0 - p_i \in J_i$ for all $i \in A$. Since $p_j - p_0 \in J_j + \cap_{i \in A} J_i$, Lemma 1.8 provides a p such that $p - p_j \in J_j$ and $p - p_0 \in \cap_{i \in A} J_i$. It follows that $p - p_k \in J_k$ for all $1 \le k \le n$ □

The conditions of Proposition 1.14 are not particularly useful, and what it claims is not particularly startling, namely that if you can interpolate polynomials $p_1, \ldots, p_k$ on the varieties of the ideals $\mathcal{J}$ if and only if there is an interpolation polynomial on every proper subcollection of the varieties of $\mathcal{J}$. What it does tell us, however, is that something strange happens when we try to interpolate on these varieties. Somehow it isn't necessary that the data actually match where the data "overlaps" just that it is good enough.

Question 1.15. *Is it possible to improve upon the conditions of Theorem 1.12, in a way that gives useful conditions, only on the data?*

Remark 1.16. *Note that as Theorem 1.12 actually promises $p - p_i \in J_i$, it is actually possible to interpolate various derivatives on $\mathcal{V}(J_i)$ as well. However, at the moment there is no formal definition for Hermite type interpolation on general varieties.*

1.4 *Some Applications to Partial Differential Equations*

In this section, we will address a different type of restriction possible in these problems. We will consider interpolation on subspaces of polynomials that form solutions of linear homogeneous differential equations. In this case it is possible to prove some additional, and independently interesting results. Of particular note, is an extension of a result of W. K. Hayman and Z. G. Shanidze in [11], who considered this problem for the case of Homogeneous quadratic equations in two variables.

We will use $\mathbb{C}_{\leq\alpha}[\mathbf{x}]$ to mean the space of polynomials of degree less than or equal to α, We will use $\mathbf{D}$ to mean $\partial_{x_1}, \ldots, \partial_{x_d}$ and we will need the following Theorem of Matsuura:

Theorem 1.17. *[15] Let L be a polynomial in d variables such that the lowest degree of monomial in L is l then,*

$$L(\mathbf{D})\mathbb{C}_{\leq n}[\mathbf{x}] = \mathbb{C}_{\leq n-l}[\mathbf{x}]$$

Theorem 1.18. *[16] Let $J =< q >$ be an ideal, where $q \in \mathbb{C}[\mathbf{x}]$ is a polynomial of degree l. Let $L \in \mathbb{C}[\mathbf{x}]$ such that the lowest degree of monomial in L is l. If $\ker L(\mathbf{D}) \cap J = \{0\}$, then for any $p \in \mathbb{C}[\mathbf{x}]$, there exists a unique polynomial $f \in \ker L(\mathbf{D})$ such that $(f - p) \in J$.*

Moreover f is a polynomial with minimal degree such that $(f - p)|_{\mathcal{V}(J)} = 0$.

Proof. Consider $L(\mathbf{D}) : \mathbb{C}_{\leq n}[\mathbf{x}] \to \mathbb{C}_{\leq n}[\mathbf{x}]$, then

$$\dim(\ker L(\mathbf{D}) \cap \mathbb{C}_{\leq n}[\mathbf{x}]) + \dim(\mathrm{ran} L(\mathbf{D}) \cap \mathbb{C}_{\leq n}[\mathbf{x}]) = \dim(\mathbb{C}_{\leq n}[\mathbf{x}])$$

Note that by Theorem 1.17

$$\mathrm{ran} L(\mathbf{D}) \cap \mathbb{C}_{\leq n}[\mathbf{x}] = \mathbb{C}_{\leq n-l}[\mathbf{x}]$$

Hence,

$$\dim(\ker L(\mathbf{D}) \cap \mathbb{C}_{\leq n}[\mathbf{x}]) = \dim \mathbb{C}_{\leq n}[\mathbf{x}] - \dim \mathbb{C}_{\leq n-l}[\mathbf{x}] \tag{1.4}$$

Since J is generated by q whose degree is l, it follows $J \cap \mathbb{C}_{\leq n}[\mathbf{x}] = q\mathbb{C}_{\leq n-l}[\mathbf{x}]$ and therefore, $\dim(J \cap$

$\mathbb{C}_{\leq n}[\mathbf{x}]) = \dim \mathbb{C}_{\leq n-l}[\mathbf{x}]$, combining with (1.4) yields

$$\dim(\ker L(\mathbf{D}) \cap \mathbb{C}_{\leq n}[\mathbf{x}]) + \dim(J \cap \mathbb{C}_{\leq n}[\mathbf{x}]) = \dim \mathbb{C}_{\leq n}[\mathbf{x}]$$

since, it was our assumption that $\ker L(\mathbf{D}) \cap J = \emptyset$, it follows that

$$(\ker L(\mathbf{D}) \cap \mathbb{C}_{\leq n}[\mathbf{x}]) \oplus (J \cap \mathbb{C}_{\leq n}[\mathbf{x}]) = \mathbb{C}_{\leq n}[\mathbf{x}]$$

since this is true for all n, it must be that $\mathbb{C}[\mathbf{x}] = J \oplus \ker L(\mathbf{D})$. Therefore, for any $p \in \mathbb{C}[\mathbf{x}]$ there exists a unique $f \in \ker L(\mathbf{D})$ such that $f - p \in J$, proving the first part of our claim.

For the second part, let f_1 be a polynomial of minimal degree such that $p - f \in J$, and let the degree of f be m, then by our previous discussion, there exists a polynomial $f \in \ker L(\mathbf{D}) \cap \mathbb{C}_{\leq m}[\mathbf{x}]$ such that $f - f_1 \in J$, and therefore $f - p \in J$, and evidently the degree of f is m. $\square$

Notice that, in the case where J is radical, the resulting f is a polynomial of minimal degree that interpolates p on $\mathcal{V}(J)$. In the language of PDE, this translates to the following: If $I(\mathcal{V}) \cap L(\mathbf{D}) = \{0\}$, then the boundary value problem

$$\begin{cases} L(\mathbf{D})f = 0 \\ f|_{\mathcal{V}} = p \end{cases}$$

Has a polynomial solution, in fact, the solution is even unique.

Remark 1.19. *Theorem 1.18 only provides for interpolation on a single curve, namely the zero set of a given polynomial q, However, notice that if $J =< q_1, q_2, \ldots, q_k >$, since $J \subset < q_i >$ it follows that there exists an f that interpolates on $\mathcal{V}(J)$, however, in this case we have no guarantee that this polynomial is unique.*

Remark 1.20. *Given a sequence of boundary conditions $p_1, \ldots, p_k$ on $\mathcal{V}_i, \ldots, \mathcal{V}_k$. It is simply a matter of finding a p such that $p - p_i \in I(\mathcal{V}_i)$, and then Theorem 1.18 can be applied to $p, I(\cup \mathcal{V}_i)$, providing existence of a solution to the problem*

$$\begin{cases} L(\mathbf{D})f = 0 \\ f|_{\mathcal{V}_i} = p_i \end{cases}$$

Remark 1.21. *It is an interesting observation that though the condition $L(\mathbf{D}) \cap J = \{0\}$ is clearly necessary and sufficient for the interpolation polynomial to be unique, if it exists, this condition actually guarantees the existence of an interpolation polynomial. This is a perfect analogue of an elementary result in linear algebra: The equation $Ax = b$ has a solution for all b if and only if, that solution, if it exists, is unique.*

Our next theorem will be to show that for any ideal of the form $< q >$ there exists a homogeneous polynomial L such that $\ker \bar{L}(\mathbf{D}) \cap < q >= \{0\}$, and hence, $\bar{L}$ satisfies the conditions of Theorem 1.18. In fact the polynomial that will work is $L = q^\dagger$ where $q^\dagger$ is the homogeneous component of q of maximum degree. This theorem will require a few preliminaries of it's own, which we present below:

We will let $\mathbb{H}_k[\mathbf{x}] \subset \mathbb{C}[\mathbf{x}]$ denote the space of homogeneous polynomials of degree k. We introduce on $\mathbb{H}_k[\mathbf{x}]$ the Hermitian inner-product

$$< f, L >:= \bar{L}(\mathbf{D})f$$

In the discussion preceding equation (2.7) we will provide a discussion showing in detail that this inner-product has some immediate properties, first of all it is clear that

$$< \mathbf{x}^\alpha, \mathbf{x}^\beta >= \begin{cases} 0 & \text{if } \alpha \neq \beta \\ \alpha! & \text{if } \alpha = \beta \end{cases}$$

secondly it will follow from equation (2.7) that

$$< x_j f, L >=< f, D_j L > \tag{1.5}$$

of course, in this expression, the LHS is an inner-product in $\mathbb{H}_{k+1}[\mathbf{x}]$ while the RHS is an inner-product in $\mathbb{H}_k[\mathbf{x}]$, so these are not exactly adjoints.

Finally we let $J^\dagger = f^\dagger, f \in J$. Now we are ready for the theorem.

Theorem 1.22. *[16] Let $J =< q >$ and let $L = q^\dagger$, then $\ker \bar{L}(\mathbf{D}) \cap J = \{0\}$ and $\bar{L}, J$ satisfy the conditions of Theorem 1.18*

Proof. Let $\mathbb{F}_k := \{F \in \mathbb{H}_k[\mathbf{x}] :< f, F >= 0 \text{ for all } f \in J^\dagger \cap \mathbb{H}_k[\mathbf{x}]\}$, then it follows from properties of the inner product, that

$$\mathbb{F}_k \oplus (J^\dagger \cap \mathbb{H}_k[\mathbf{x}]) = \mathbb{H}_k[\mathbf{x}] \tag{1.6}$$

For any $F \in \mathbb{F}_{k+1}$ it is true by definition of F_k that $< x_i f, F >= 0$ for all $f \in J^\dagger \cap \mathbb{H}_k[\mathbf{x}]$. It follows from equation (1.5) that $< f, D_i F >= 0$, hence

$$F \in \mathbb{F}_{k+1} \implies D_i F \in \mathbb{F}_k \text{ for all } 1 \leq i \leq d \tag{1.7}$$

Now, we would like to show that $\mathbb{F}_k \subset \ker \bar{L}(\mathbf{D})$ for all k. Let l be the degree of q (and therefore the degree of L), then for every $F \in \mathbb{F}_l$, (since $\bar{0} = 0$) we have

$$0 =< q^\dagger, F >=< F, q^\dagger >$$

By inductive hypothesis, assume $\mathbb{F}_k \subset \ker \bar{L}(\mathbf{D})$, then by (1.7) we have $D_i F \in \mathbb{F}_k$ for every $1 \leq i \leq d$, and every $F \in \mathbb{F}_{k+1}$. Therefore,

$$0 = \bar{L}(\mathbf{D}) D_j F = D_j \bar{L}(\mathbf{D}) F$$

However, this means, since $\bar{L}(\mathbf{D})F$ is a homogeneous polynomial, whose partial derivatives are all zero, that $\bar{L}(\mathbf{D})F = 0$. And so, $\mathbb{F}_k \subset \ker \bar{L}(\mathbf{D})$ for all k.

Observe that by Theorem 1.17 since the degree of $q^\dagger$ is l, then $\mathbb{C}_{<l}[\mathbf{x}] \subset \bar{L}(\mathbf{D})$. In light of (1.6) and the fact that $\mathbb{F}_k \subset \ker \bar{L}(\mathbf{D})$ it is sufficient to show that for all k

$$\dim \mathbb{F}_k = \dim(\ker \bar{L}(\mathbf{D}) \cap \mathbb{H}_k[\mathbf{x}]) \tag{1.8}$$

It follows from (1.6) that $\dim \mathbb{F}_k = \dim \mathbb{H}_k[\mathbf{x}] - \dim(J^\dagger \cap \mathbb{H}_k[\mathbf{x}])$, furthermore, since $J \cap \mathbb{C}_{<k}[\mathbf{x}] = q \cdot \mathbb{C}_{k-l}[\mathbf{x}]$ it follows that $\dim(J^\dagger \cap \mathbb{H}_k[\mathbf{x}]) = \dim \mathbb{H}_{k-l}$ and thus

$$\dim \mathbb{F}_k = \dim \mathbb{H}_k[\mathbf{x}] - \dim \mathbb{H}_{k-l}[\mathbf{x}]$$

Now, Theorem 1.17 gives us that the range of

$$\bar{L}(\mathbf{D})(\mathbb{H}_k[\mathbf{x}])$$

is $\mathbb{H}_{k-l}[\mathbf{x}]$ and hence,

$$\dim(\ker \bar{L}(\mathbf{D}) \cap \mathbb{H}_k[\mathbf{x}]) = \dim \mathbb{H}_k[\mathbf{x}] - \dim \mathbb{H}_{k-l}[\mathbf{x}]$$

It follows that (1.8) holds, and our claim follows from (1.6)

$\square$

2 Ideal Projectors

In this, the second part of this dissertation, we will present some necessary background on ideal projectors, about which, a fair amount is known already. It will be the ultimate goal of this section to prove our claim that given an ideal projector whose kernel is the ideal J, if the minimal primary decomposition of J is $J_1, \ldots, J_k$ and $P_1, \ldots, P_k$ are projectors with kernels $J_1, \ldots, J_k$ respectively, then, P is Hermite if and only if each P_i is Hermite. We will begin with some observations about ideal projectors.

It is well known to algebraists, that the kernel of any ring homomorphism is an ideal, and that any ideal J defines a ring homomorphism, $H : \mathbb{K}[\mathbf{x}] \to \mathbb{K}[\mathbf{x}]/J$. Note, however, that the members of $\mathbb{K}[\mathbf{x}]/J$ are equivalence classes in $\mathbb{K}[\mathbf{x}]$ with J being the equivalence class $[0]$. It is a simple matter to construct a group isomorphism P' from $\mathbb{K}[\mathbf{x}]$ to a well chosen additive group $G \subset \mathbb{K}[\mathbf{x}]$ simply by mapping each $f \in G$ to $[f] \in \mathbb{K}[\mathbf{x}]/J$. If H is a projection, then $P'(H)$ is also a projection, and it is easy to see that $\mathbb{K}[\mathbf{x}] = G \oplus J$.

Remark 2.1. *In algebra, such homomorphisms are called **split**, since the slitting lemma implies that given the short exact sequence*

$$0 \to J \to \mathbb{K}[\mathbf{x}] \xrightarrow{P} G \to 0$$

If $\mathbb{K}[\mathbf{x}] = J \oplus G$ then P is a projection. (the kernel of any homomorphism is an ideal)

This is all simple enough, but since our goal is interpolation, dealing in equivalence classes is a bit unwieldy. Though it is interesting to note that if the homomorphism H is split then $H(H(f) - f) = 0$ implying that $H(f)$ and f are always in the same equivalence class. In other words, the equivalence class $[f]$ is precisely all those polynomials that interpolate f on $\mathcal{V}(\ker H)$. For convenience, in order to avoid explicitly discussing equivalence classes in $\mathbb{K}[\mathbf{x}]/(\ker H)$, we will make use of the following observation of Carl de Boor [3].

Lemma 2.2. *A linear projector P is an ideal projector if and only if*

$$P(fg) = P(f * P(g)) \tag{2.1}$$

Proof. Suppose P is an ideal projector, then $g - P(g)$ is in the kernel of P since P is a projection, since it is an ideal projector, $fg - fP(g)$ is also in the kernel of P, and so $P(fg - fP(g)) = 0$ and it follows that $P(fg) = P(f * P(g))$.
Conversely, suppose $P(fg) = P(f * P(g))$, then if g is in the kernel of P, then $P(fg) = P(f * P(g)) = P(f * 0) = P(0) = 0$ so P is an ideal projector. □

Although it is possible to define an ideal projector interpolating on any variety, the research that has been done on them so far, is restricted to the case in which these varieties are finite sets of points. In this second part of the dissertation we will likewise restrict ourselves to this case. When the varieties are restricted to finite sets of points, certain additional properties come into play, that are useful for circumventing the obstacles we encountered in first part of the dissertation. To begin, let us observe that any finite set $A \subset \mathbb{K}^d$ is a variety. Certainly in one variable, any such set defines a Lagrange projector. In one variable, we have already established that such projectors are ideal, and that the point set on which they interpolate is a variety. In order to see that this works exactly the same way in d variables, we present a simple and to the knowledge of the author, new, corollary of Lagrange interpolation.

Remark 2.3. *[6] Let $Y \subset \mathbb{K}^d$ be a finite set, and let $< \mathbf{u}, \mathbf{v} >$ denote a Hermitian inner product on $\mathbb{K}^d$. In the case where $\mathbb{K} = \mathbb{C}$, or $\mathbb{K} = \mathbb{R}$ the ordinary and well known standard inner product will, of course, work. Now choose $\mathbf{z} \in Y$ consider the polynomial:*

$$\omega_{\mathbf{z}}(\mathbf{x}) = \frac{\prod_{\mathbf{y} \neq \mathbf{z}} < \mathbf{x} - \mathbf{y}, \mathbf{z} - \mathbf{y} >}{\prod_{\mathbf{y} \neq \mathbf{z}} ||\mathbf{z} - \mathbf{y}||} \tag{2.2}$$

It is clear that this polynomial is zero whenever $\mathbf{x} \in Y$ but $\mathbf{x} \neq \mathbf{z}$ and is one when $\mathbf{x} = \mathbf{z}$. It is also obvious that the expression

$$\sum_{\mathbf{z} \in Y} f(z) \omega_{\mathbf{z}}(\mathbf{x})$$

will give a Lagrange interpolation polynomial for f with respect to the set Y

It is clear that the above defines a Lagrange projector, and that the kernel of this projector is precisely those polynomials which are zero when evaluated at the points in Y, hence, Y is a variety. It is also clear that if we call this projector L, then $\#Y = \dim(\mathrm{ran} L)$. In general for a projector P whose kernel is a finite dimensional ideal J, $\#\mathcal{V}(J) \leq \dim(\mathbb{K}[\mathbf{x}]/J) = \dim(\mathrm{ran} P)$, equality holds if and only if J is radical, hence, if and only if P is Lagrange.

Another nice thing about working with finite varieties is that we can easily translate our problem into linear algebra. Formally, given an ideal projector P we will set $M_i() := P(x_i())$. We will refer to these operators M_i as the **multiplication operators** for P, a term that comes from approximation theory. Now let us make some observations about the multiplication operators.

First of all by Equation (2.1), $M_iM_j() = P(x_iP(x_j())) = P(x_ix_j()) = P(x_jx_i()) = M_jM_i()$ so these operators commute. The next thing to notice, is that for any polynomial $f = \sum \beta_\alpha \mathbf{x}^\alpha \in \mathbb{K}[\mathbf{x}]$,

$$f(M_1, \ldots, M_d)(P(1)) = \left[\sum \beta_\alpha \mathbf{M}^\alpha\right](P(1)) = P(f) \tag{2.3}$$

by equation (2.1). Finally notice that whenever $f \in \mathrm{ran}P$, equation (2.3) becomes

$$f(M_1, \ldots, M_d)(P(1)) = \left[\sum \beta_\alpha \mathbf{M}^\alpha\right](P(1)) = P(f) = f$$

so evidently $\{M_1, \ldots, M_d\}$ are cyclic for $\mathrm{ran}P$.

The next thing to note is that in the case were our varieties are finite, the multiplication operators are actually easily expressed as $n \times n$ matrices, where $n = \dim(\mathrm{ran}P)$. It turns out that any family of d commuting $n \times n$ matrices cyclic for $\mathbb{K}^n$, also define an ideal projector, given an appropriate n dimensional linear space $G \subset \mathbb{K}[\mathbf{x}]$. For now we state this without proof, but we will prove this formally in Theorem 2.19.

Definition 2.4. *The* ***dual*** *(*algebraic dual*) of J is the collection of all linear functionals which vanish on J. We will denote the dual of J as $J^\perp$.* This space differs from the dual space known to analysts in that the functionals in $J^\perp$ need not be bounded.

Another advantage of restricting ourselves to finite varieties, is, that it is well known that in the case where $\mathrm{ran}P$ is finite dimensional $\dim J^\perp = \dim(\mathrm{ran}P)$. It is also well known that the number of interpolation conditions precisely determines the dimension of the image space. Notice that if $f - g \in J$ then for any $\lambda \in J^\perp$ it follows that $\lambda(f) = \lambda(g)$. It is easy to see that the functionals in $J^\perp$ are precisely the interpolation conditions on $\mathcal{V}(J)$. Any projection onto a finite dimensional linear space with basis $\{g_1, \ldots, g_n\}$ can be written in the form

$$P = \sum_{i=1}^{n} g_i \lambda_i$$

where $\lambda_i g_j = \delta_{i,j}$. Here we use $\delta_{i,j}$ to mean the Kronecker delta, i.e. $\delta_{i,j} = 1$ if $i = j$ and zero otherwise. It is clear that in the case where P is an ideal projector, that the functionals λ_i form a basis for $\ker^\perp P$.

2.1 *Linear Algebra*

In our search for a solution to our problem, it is natural that we turn to linear algebra. We have already shown that the multiplication operators that describe an ideal projector take the form of matrices, and the ideal projector itself is a linear operator. One of the most useful things about linear algebra is that it is a language, both approximation theorists and algebraists can understand. We will use $\mathbf{M}$ to denote the family of matrices $\{M_1, \ldots, M_d\}$.

From this point on, as we will make heavy use of topological concepts, such as convergence, we will limit ourselves to the spaces $\mathbb{C}$ and $\mathbb{C}[\mathbf{x}]$. It may be possible to extend some, or perhaps all of these results to general algebraically closed fields, but as our main purpose in this dissertation is to produce results to problems in approximation theory, we will be content to limit ourselves to $\mathbb{C}$. We will also assume all varieties to be finite (all ideals to be zero dimensional), unless otherwise stated, as ideal projectors are used to interpolate at finitely many points. Let us begin by developing some simple theorems in linear algebra useful for dealing with ideal projectors.

Lemma 2.5. *[18] For an ideal projector P with kernel J, the following are equivalent:*

i) P is Lagrange ($\ker P$ *is a radical ideal)*

ii) $\dim(\mathrm{ran}P) = \#\mathcal{V}(J)$

iii) $\#\mathcal{V}(J) = \dim(J^\perp)$

iv) The family $\mathbf{M}$*, corresponding to P is simultaneously diagonalizable*

Proof. i) $\Longleftrightarrow$ *iv)* this will be the work of Theorem 2.7.

i)$\rightarrow$*ii)* Consider the primary ideals $J_1 \cap \ldots \cap J_m$, each of their varieties contain only one point, and it is easy to see they must also be radical. If $f^k \in J_i$ then $f \in J_i$ so this is the set of all functions that are zero at a given point so $J_i^\perp = \mathrm{span}\{\delta_{\mathcal{V}(J_i)}\}$ which is one functional. Since $J^\perp = J_1^\perp + \ldots + J_m^\perp$ and $J^\perp$ must be the span of $\dim(\mathrm{ran}P)$ functionals it follows that $\dim(\mathrm{ran}P) = \#\mathcal{V}(J)$.

ii)$\rightarrow$*iii)* As $\dim(\mathrm{ran}P) = \dim(J^\perp)$, this is immediate.

iii)$\rightarrow$*i)* Since the Nullstellensatz tells us $J^\perp$ must contain a point evaluation functional for each point in $\mathcal{V}(J)$, these can be the only functionals in $J^\perp$ and since it is obvious that if $f^k(\mathbf{z}) = 0$ then $f(\mathbf{z}) = 0$, J is radical. □

As our first application of linear algebra, we give an important piece of Theorem 2.7 as a Lemma, and note this clever insight, as well as the original proof of the following result are due to Hans Stetter [3]. We will use the notation $f(\mathbf{M})$ to mean we take f to be the polynomial in the matrices $\mathbf{M} = \{M_1, \ldots, M_d\}$ formed by replacing each of $x_1, \ldots, x_d$ with $M_1, \ldots, M_d$ respectively.

Lemma 2.6. *Given an ideal projector P with multiplication matrices $\mathbf{M}$, Choose any $f \in \mathbb{C}[\mathbf{x}]$, then the eigenvalues of $f(\mathbf{M})$ are the same as the values of $f(\mathcal{V}(\ker P))$*

Proof. let $f \in \mathbb{C}[\mathbf{x}]$ and $\mu \in \mathbb{C}$, let

$$h := f - \mu$$

notice that if $\mu \neq f(\mathbf{z})$ for some $\mathbf{z} \in \mathcal{V}(\ker P) = \mathcal{V}$ then $h(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in \mathcal{V}(\ker P)$. Therefore, there exists a polynomial r such that $1 - hr$ vanishes everywhere on $\mathcal{V}$. By the Nullstellensatz, for some k, $(1 - hr)^k \in \ker P$, so

$$0 = (1 - hr)^k(\mathbf{M}) = (I_d - h(\mathbf{M})r(\mathbf{M}))^k = I_d - h(\mathbf{M})B$$

for some matrix B. This implies that $h(\mathbf{M}) = f(\mathbf{M}) - \mu I_d$ is invertible and so μ is not an eigenvalue of $f(\mathbf{M})$.

If, $\mu = f(\mathbf{z})$ for some $\mathbf{z} \in \mathcal{V}(\ker P)$ then if $g \in \mathrm{ran}P$

$$f(\mathbf{M})g(\mathbf{z}) = P(fg)(\mathbf{z}) = fg(\mathbf{z}) = \mu g(\mathbf{z})$$

and since $\mathcal{V}(\ker P)$ are precisely the interpolation nodes of P, the set of eigenvalues of $f(\mathbf{M})$ is the set $f(\mathcal{V})$. $\square$

Our next theorem will finish the proof of Lemma 2.5.

Theorem 2.7. *[3] An ideal projector P is Lagrange if and only if it's multiplication matrices $\mathbf{M}$ are simultaneously diagonalizable.*

Proof. Suppose P is Lagrange, and $\mathcal{V}(\ker P) = \{\mathbf{z}_1, \ldots, \mathbf{z}_n\}$ then (based on Remark 2.3 if not for other obvious reasons) there exist n polynomials f_i, such that $f_i(\mathbf{z}_j) = \delta_{i,j}$. Let $\{g_i = P(f_i)\}$, we know $g_i(\mathbf{z}_j) = \delta_{i,j}$ and since there are exactly n of them, $\{g_i\}$ forms a linear basis for $\mathrm{ran}P$. When $\mathbf{M}$ is given in this basis, we get $M_j(g_i) = P(x_j g_i)$, and since $(x_j g_i)(\mathbf{z}_k) = z_i^{(j)}\delta_{i,k}$. Then $M_j(g_i) = z_i^{(j)} g_i$ and it follows the matrices $\mathbf{M}$ are in-fact diagonal, and thus, in any other basis, simultaneously diagonalizable.

Suppose $\mathbf{M}$ is simultaneously diagonalizable. Let A be the matrix such that AM_iA^{-1} is diagonal. Then the map

$$\mathbb{C}[\mathbf{x}] \to \mathbb{C}^{nxn} : p \to Ap(\mathbf{M})A^{-1}$$

has $\ker P$ as it's kernel, furthermore, if we set $\lambda_{i,j}$ to be the map that gives the (i,j) entry of $Ap(\mathbf{M})A^{-1}$, we get

$$\ker P = \cap_{i,j} \ker \lambda_{i,j} = \cap_{i=1}^{n} \lambda_{i,i}$$

since the matrices $Ap(M)A^{-1}$ are all diagonal. It follows since F maps $\mathbb{C}[\mathbf{x}]$ to the space of diagonal $n \times n$ matrices, a space which has n dimensions, and there are n dimensions to $\text{ran}P$, the polynomials for which $Ap(M)A^{-1}$ does not vanish, there must exist a polynomial p such that none of the diagonal entries of $Ap(M)A^{-1}$ are zero.

Since $p(\mathbf{M})$ had precisely n eigenvalues, our previous lemma shows $\mathcal{V}(\ker P)$ has n entries, and since $\dim(\text{ran}P) = \#\mathcal{V}(\ker P)$, P is lagrange. □

In order to make the most use of this theorem, we will present a definition, but first we need to give a couple somewhat well known facts.

Lemma 2.8. *[14] If M is a matrix, and If H is a nonzero, M invariant subspace, then H contains an eigenvector of M.*

Lemma 2.9. *[17] If $\mathbf{M}$ is a family commuting matrices, $\mathbf{M}$ has a common eigenvector.*

Proof. This is obviously true for one matrix, so we proceed by induction on d, let H be the space of common eigenvectors for $M_1, \dots, M_{d-1}$.If $f \in H$, then

$$0 = M_d(M_i - \lambda_i I_d)f = (M_i - \lambda_i I_d)M_d f$$

so $M_d f \in H$. Hence, H is M_d invariant, and therefore contains an eigenvector of M_d. □

Now that we have established that the contents of the following definition are non-trivial:

Definition 2.10. *An element of $\mathbb{K}^d$, $\mathbf{z} = (z_1, \dots, z_d)$, is called an **eigen-tuple** for $\{M_1, \dots, M_d\}$ if there exists a non zero vector $\mathbf{f}$ in the domain of $\{M_1, \dots, M_d\}$ such that*

$$M_i\mathbf{f} = z_i\mathbf{f} : \forall 1 \leq i \leq d$$

We shall use $\sigma(M_1, \dots, M_d)$ to denote the set of all eigen-tuples for $\{M_1, \dots, M_d\}$

When one looks at the proof of Lemma 2.5, it is easy to notice that, it works because the coordinates of $\mathbf{z} \in \mathcal{V}(J)$ are the eigen-tuples. Based again on the proof, one might suspect that we can find some sort of extension of this when P is not Lagrange, and the matrices are not simultaneously diagonalizable. This intuition turns out to be correct, and we have the following theorem.

Theorem 2.11. *[17] Let $\mathbf{M}$ be a family of pairwise commuting matrices onto and cyclic for G, then $\sigma(\mathbf{M}) = \mathcal{V}(J)$, where J is the associated ideal.*

Proof. Let $\mathbf{z}$ be an eigen-tuple of $\mathbf{M}$, with eigenvector f, then for any $p \in J$, $p(\mathbf{z})f = p(\mathbf{M})f = P(pf) = 0$ so $\mathbf{z} \in \mathcal{V}(J)$.

To reach a contradiction, let us assume that $z \in \mathcal{V}(J)$ but $z \notin \sigma(\mathbf{M})$. For each $\mathbf{y} \in \sigma(\mathbf{M})$, let X_y be the set of vectors orthogonal to $\mathbf{z} - \mathbf{y}$. Note that X_y is a $d-1$ dimensional subspace, therefore either all the X_y are the same, or their union is not a subspace. In either case $\cup X_y \neq \mathbb{C}^d$, it follows that there exists $\boldsymbol{\omega} \in \mathbb{C}^d$ such that $\boldsymbol{\omega} \cdot (\mathbf{z} - \mathbf{y}) \neq 0$ for all $\mathbf{y} \in \sigma(\mathbf{M})$. Consider the matrix $\sum \omega_i M_i$, let p be it's characteristic polynomial, then $0 = p(\sum \omega_i M_i) = p(\sum \omega_i x_i)$. by Lemma 2.6. However $\sum \omega_i z_i \neq \sum \omega y_i$ for any $\mathbf{y} \in \sigma(\mathbf{M})$ so $\mathbf{z}$ is not a root of $p' = p(\sum \omega_i x_i)$, however $p'(\mathbf{M})() = 0()$ so p' is in J, contradiction. □

It is actually possible to prove an even stronger result, however, it takes some doing, and so we will omit the proof. The full extension of Theorem 2.7 is the main result of [17], we state it below.

Theorem 2.12. *Let P be an ideal projector onto the N-dimensional subspace G and let*

$$P = P_1 + P_2 + \ldots + P_m$$

be the projectors corresponding to the primary decomposition of $\ker P$*. Then*

i) $\mathbf{M}$ has a unique (up to order of blocks) block diagonalization $\mathbf{M}_P = diag(\mathbf{M}_P^{(j)})$ consisting of m blocks where m is the maximal number of blocks in any block-diagonalization of $\mathbf{M}$

ii) Each block $\mathbf{M}^{(j)}$ defines a distinct primary ideal

$$\ker P_j = \{p \in \mathbb{C}[\mathbf{x}] : p(\mathbf{M}^{(J)}) = 0\}$$

As the statement of this theorem is hard to follow, let us explain for clarification, this theorem states that there is a basis for which the matrices $\mathbf{M}$ take the form:

$$M_j = \begin{pmatrix} M_j^{(1)} & 0 & \dots & 0 \\ 0 & M_j^{(2)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & M_j^{(m)} \end{pmatrix}$$

and $\mathbf{M}_P^{(j)}$ defines a sequence of cyclic commuting matrices similar to $\mathbf{M}_{P_j}$.

So now we know that the matrices $\mathbf{M}$ are similar to Jordan block form matrices, the individual blocks of which describe projectors $P_1 + \ldots + P_m = P$ such that $\ker P_i$ is primary. We also know that the Jordan blocks are all 1×1 if and only if $\ker P$ is a radical ideal. This all makes perfect sense since the primary decomposition of such an ideal would be radical ideals which "interpolate" at a single point, and hence, would be maps to the space of constants in $\mathbb{C}^d$. So the projection would just be $P(f) = \delta_{\mathbf{z}} f = f(\mathbf{z})$. Where $\delta_{\mathbf{z}}$ is the point evaluation functional at $\mathbf{z}$.

One might naturally intuit that the varieties of $\ker P_n$ converge to the variety of $\ker P$, if $P_n \to P$. However this might not be so obvious as it seems, since some of the points might wander off to infinity or simply not converge at all. It is possible that this is well be known to algebraic geometers, however, proving it in such a direct way would require some rather heavy machinery from algebraic geometry. A simple way to reach this conclusion is to recall that the if a sequence of diagonalizable matrices M_n converges to a matrix M then the eigenvalues of M_n converge to the eigenvalues of M [1]. A much cuter way, is to extend that theorem to eigen-tuples, and so we provide an extension of Theorem 5.2.1 in Artin's Algebra [1].

Theorem 2.13. *[6] Let $(M_1^{(n)}, \ldots, M_d^{(n)})$ and $(M_1, \ldots, M_d)$ be d-tuples of commuting operators onto an N-dimensional space G. Then the sets $\sigma(M_1^{(n)}, \ldots, M_d^{(n)})$ are uniformly bounded and all accumulation points of the sequences from $\sigma(M_1^{(n)}, \ldots, M_d^{(n)})$ are the points in $\sigma(M_1, \ldots, M_d)$*

Proof. Let $\mathbf{z}^{(n)} \in \sigma(M_1^{(n)}, \ldots, M_d^{(n)})$, then there exists $\mathbf{u}_n$ such that

$$M_j^{(n)} \mathbf{u}_n = z_j^{(n)} \mathbf{u}_n : \ 1 \le j \le d$$

we may assume without loss of generality that $||\mathbf{u}_n|| = 1$, giving us

$$|z_j^{(n)}| = ||z_j^{(n)} \mathbf{u}_n|| = ||M_j^{(n)} \mathbf{u}_n|| \le ||M_j^{(n)}||$$

by assumption $M_j^{(n)}$ converges and therefore $||M_j^{(n)}||$ are uniformly bounded. It is left to prove that $\sigma(M_1, \ldots, M_d)$ are the accumulation points.

Let $\mathbf{z}$ be an accumulation point of

$$\sigma(M_1^{(n)}, \ldots, M_d^{(n)})$$

then there exists a subsequence $\mathbb{N}_1$ such that $z_j^{(n)} \to z_j$ for $n \in \mathbb{N}_1$. In-fact, whenever $n \in \mathbb{N}_1$ we have $M_j^{(n)}\mathbf{u}_n = z_j^{(n)}\mathbf{u}_n$. Since $||\mathbf{u}_n|| = 1$, $\mathbf{u}_n$ is a uniformly bounded sequence in the finite dimensional space G, hence, it is compact, and there exists a subsequence $\mathbb{N}_2 \subset \mathbb{N}_1$ such that $\mathbf{u}_n \to \mathbf{u} \in G$ and $||\mathbf{u}|| = 1$. Since

$$(M_1^{(n)}, \ldots, M_d^{(n)})$$

is a sequence of finite dimensional operators, the convergence is uniform, and we have for $n \in \mathbb{N}_2$

$$z_j^{(n)}\mathbf{u}_n = M_j^{(n)}\mathbf{u}_n \to M_j\mathbf{u}$$

In conclusion, since $z_j^{(n)}\mathbf{u}_n \to z_j\mathbf{u}$ and $\mathbf{z}$ was an arbitrary accumulation point, we have $\mathbf{z} \in \sigma(M_1, \ldots, M_d)$, and have proven our claim. $\square$

Theorem 2.11 tells us that if L_n is Lagrange projector with multiplication matrices $\mathbf{M}_{L_n}$ then $\sigma\mathbf{M}_{L_n} = \mathcal{V}(\ker L_n)$. If P is a Hermite projector, there exists a sequence $L_n \to P$, and therefore $\mathbf{M}_{L_n} \to \mathbf{M}_P$. Combining this with Theorem 2.13 gives the following Corollary:

Corollary 2.14. *[6] Let $P : \mathbb{K}[\mathbf{x}] \to G$ be a Hermite projector. Then, for any Lagrange projectors P_n : $\mathbb{K}[\mathbf{x}] \to G$ such that $P_n \to P$, if $\mathcal{V}(\ker P) = \{\mathbf{x}_1, \ldots, \mathbf{x}_K\}$ and $\mathcal{V}(\ker P_n) = \{\mathbf{x}_1^{(n)}, \ldots, \mathbf{x}_N^{(n)}\}$ then there exists a constant C such that*
$||\mathbf{x}_j^{(n)}|| \leq C$ for all n and all j, and $\{\mathbf{x}_1, \ldots, \mathbf{x}_K\}$ are the only limit points of $\{\mathcal{V}(\ker P_n)\}$.

Now that we have a tool box from linear algebra, Let us introduce a tool from algebraic geometry that will grant us some deep insight into our objects of study, the *border scheme*.

2.2 *The Border Scheme*

Algebraic geometers understand our problem in terms of schemes, which are a way of describing the set of all ideals that complement a given fixed linear subspace $G \supset \mathbb{K}[\mathbf{x}]$, or perhaps the space of all ideals that are complemented by linear subspaces of dimension n. The *border scheme* is one of these methods of describing all ideals that complement G.

Definition 2.15. *Let $G \subset \mathbb{K}[\mathbf{x}]$ be a fixed linear subspace, Then $\mathfrak{J}_G$ is the collection of all ideals J such that*

$$\mathbb{K}[\mathbf{x}] = G \oplus J$$

In order to describe the border basis, let us look at the action of the multiplication operators. Recall that $M_i(f) = P(x_i f)$, since $M_i M_j f = P(x_i P(x_j f))$ Equation 2.1 gives $P(x_i x_j f) = M_i M_j f$. If $\{g_1, \ldots, g_n\}$ is a basis for ran P then the polynomials $M_j g_i$ entirely determine the action of P, moreover, we can show that if $\text{ran}P = \text{span}\{g_1, \ldots, g_n\}$, the ideal $\ker P$ is determined by the polynomials $x_j g_i - M_j(g_i)$.

Theorem 2.16. *[3] If P is an ideal projector onto $G = \text{span}\{g_1, \ldots, g_n\}$, with $1 \in G$ then*

$$\ker P = I(\{x_j g_i - M_j(g_i) : 1 \leq i \leq n, 1 \leq j \leq d\} = J$$

Proof. Since $x_j g_i - M_J = x_j g_i - P(x_j g_i) \in \ker P$ it is immediate that $J \subset \ker P$.

Now we must show that $\ker P \subset J$. Set $G_0 := \{g_1, \ldots, g_n\}$ and, Construct sets

$$G_i = \bigcup_{k=1}^{i-1} (\sum_{j=1}^{d} x_j G_k)$$

it is clear that $\cup G_i = \mathbb{K}[\mathbf{x}]$ since G contains 1. So if we can show that whenever $f \in G_i \cap \ker P$ then $f \in J$ we are done. We proceed by induction, by assumption, this is true when $i = 1$. Now, choose $f \in G_k \cap \ker P$, then

$$f = \sum_{j=1}^{d} x_j f_j$$

where $f_j \in G_i : i < k$ which implies $f_j - Pf_j$ is in $G_i + G_0 = G_l : l < k$ as well as in $\ker P$. Since it is in $G_l \cap \ker P$, this means $f_j - Pf_j \in J$, by induction hypothesis. Therefore,

$$f \in \sum_{j=1}^{d} x_j (Pf_j + J) = \sum_{j=1}^{d} x_j P f_j + J \tag{2.4}$$

Equation (2.1) gives us

$$0 = Pf = P(\sum x_j f_j) = P(\sum x_j P f_j)$$

So again by induction $\sum x_j P f_j \in J$ and by Equation (2.4) it follows that $f \in J$, and so we have $\ker P \subset J$, and therefore: $\ker P = J$ ◻

Definition 2.17. *Given a linear space* $1 \in G \subset \mathbb{K}[\mathbf{x}]$ *with basis* $\mathfrak{g} = (g_1, \ldots, g_n)$ *we will call the set* $\{x_i g_j : 1 \le i \le d, 1 \le j \le n\}$ *the* ***border*** *of* G*. we will denote this set* $\partial\mathfrak{g}$

Notice that the border of G is determined only by G and our choice of basis for G. This means that every ideal in $\mathfrak{J}_G$ can be expressed in terms of the same border. Also notice that our requirement that $1 \in G$ is not particularly limiting, since for any proper ideal $J \subset \mathbb{K}[\mathbf{x}]$, the nullstellensatz tells us that $1 \notin J$, hence it is always possible to choose a valid G such that $1 \in G$

Definition 2.18. *Given an ideal projector* P *onto* $\operatorname{span}(\mathfrak{g})$ *we will call the set* $\{f - Pf : f \in \partial\mathfrak{g}\}$ *the* ***border basis*** *for* $\ker P$*.*

Looking at the multiplication operators, it is easy to see that the border basis is completely determined by multiplication operators. In our case where the multiplication operators are in-fact families of d $n \times n$ commuting matrices, the entries of those matrices completely describe $\partial\mathfrak{g}$. If only we knew which multiplication operators defined an ideal projector onto G, or equivalently which sets of the form $\{f - Pf : f \in \partial\mathfrak{g}\}$ describe an ideal that complements G. (recall that equivalently $< f - Pf : f \in \partial\mathfrak{g} > \in \mathfrak{J}_G$)

For the moment (these will turn out to be equivalent to what we have defined already), Let us associate with each $b \in \partial\mathfrak{g}$ a polynomial $p_b \in G$ and define multiplication operators:

$$M_i g_k = \begin{cases} x_i g_k & \text{if } x_i g_k \in G \\ p_{x_i g_k} & \text{if } x_i g_k \notin G \end{cases}$$

Some papers and texts refer to the set $\{p_b : b \in \partial\mathfrak{g}\} \subset G$ *as a border pre-basis*

Theorem 2.19. *[18] Let* $\{p_b : b \in \partial\mathfrak{g}\} \subset G$*, then* $< b - p_b : b \in \partial\mathfrak{g} > \in \mathfrak{J}_G$ *if and only if*

i) $M_i M_k - M_k M_i = 0$

ii) $g(M_1, \ldots, M_d)p_1 = g$ *for all* $g \in G$

Proof. If $< b - p_b : b \in \partial\mathfrak{g} > \in \mathfrak{J}_G$, It is clear that $\{M_i\}$ are the multiplication matrices for an ideal projector P and we have shown in previous discussions that these are cyclic and that they commute, so *i)* and *ii)* hold. Now suppose *i)* and *ii)* hold, then the mapping $\phi : \mathbb{K}[\mathbf{x}] \to \mathbb{K}[\mathbf{x}]$ defined by

$$\phi f = f(M_1, \ldots, M_d)p_1$$

is a homomorphism of rings, and its kernel K is an ideal in $\mathbb{K}[\mathbf{x}]$. Furthermore $G \simeq \mathbb{K}[\mathbf{x}]/K$. We just need to show that $K =< b - p_b : b \in \partial\mathfrak{g} >$. Let $h_b : b \in \mathfrak{g}$ be the border basis for K, then since $b - h_b \in K$ we have

$$0 = (b(M_1, \ldots, M_d) - h_b(M_1, \ldots, M_d))p_1 = b(M_1, \ldots, M_d) - h_b$$

but by definition of M_i, $b(M_1, \ldots, M_d)p_1 = p_b$ which implies that $p_b - h_b = 0$, and so $K =< b - p_b : b \in \partial\mathfrak{g} >$ as desired. $\square$

What this tells us, is that $\mathfrak{J}_G$ is entirely defined by those sets $\{p_b : b \in \mathfrak{g}\} \subset G$ whose multiplication matrices are cyclic and commuting. In otherwords, if G has dimension n, any family of d cyclic commuting $n \times n$ matrices define an ideal, and hence as multiplication operators an ideal projector. This means that given a linear subspace G with a fixed basis $\mathfrak{g}$, the space of families of d $n \times n$ commuting matrices, parameterizes $\mathfrak{J}_G$. Imposing the Zariski topology on this space leads to the following definition.

Definition 2.20. *The collection $\mathcal{B}_{\mathfrak{g}}$ of those sets $\{p_b : b \in \partial\mathfrak{g}\} \subset G$ which satisfy the conditions of Theorem 2.19, is an affine scheme, called the **border scheme**.*

It is known to algebraic geometers that all radical ideals are contained in the same irreducible component of the border scheme (in fact they know this to hold for any scheme) [8]. As such they define the following:

Definition 2.21. *An ideal is called **smoothable** if its representation is in the Zariski closure of the subset of the border scheme consisting of the representations of all radical ideals in $\mathfrak{J}_G$*

One of the claims we made in the introduction is an equivalence between smoothable and Hermite, to show this we will need the following result from algebraic geometry, we present it without proof, as proving results in algebraic geometry is beyond the scope of this dissertation.

Theorem 2.22. *[20][8] If $U \subset \mathcal{V}$ is Zariski open and, $\mathcal{V}$ is an irreducible variety, then the Euclidean closure of U is the same as the Zariski closure of U*

Since we know it's Zariski closure to be irreducible, if we can show the space of all multiplication matrices representing Lagrange projectors to be Zariski open, it is clear by definition of the multiplication operators that the above theorem will immediately guarantee that the kernel of any Hermite projector is smoothable.

Theorem 2.23. *The set of all $\mathfrak{L}$ Lagrange projectors onto G is a Zariski open set*

Proof. Consider the set $\mathfrak{L}_1$ of all ideal projectors onto G such that M_1 has $n = \dim G$ distinct eigenvalues. We know $\mathfrak{L}_1 \subset \mathfrak{L}$ by Theorem 2.11. If p is the characteristic polynomial for some $M_1 \in \mathfrak{L}_1$ then p has degree n, it is known ([7] exercise 8a, p340) that for any n there exists a polynomial ϕ_n such that $\phi_n(a_1, \ldots, a_n) \neq 0$ if and only if the polynomial $a_1 + a_2x + a_3x^2 + \ldots + a_nx^{n-1} + x^n$ has n distinct roots. Note that the coefficients of p are all polynomials in the entries of M_1, hence the set of entries of matrices M_1 that do not have n distinct eigenvalues are the zeros of ϕ_n, and therefore these matrices not having n eigenvalues form a variety, hence, a Zariski closed set. Therefore, its complement, $\mathfrak{L}_1$ is Zariski open.

Note that for any Lagrange projector, there exists a change of variables such that M_1 has n distinct eigenvalues. Thus, $\mathfrak{L}$, as the union of Zariski open sets is a Zariski open set. □

2.3 *Ideal Annihilators*

So far we have learned that given a fixed linear subspace G with basis $\mathfrak{g}$ an ideal projector defines multiplication operators $\mathbf{M}$ and that cyclic commuting matrices define an ideal projector. From algebraic geometry we have learned that given the same space G and basis $\mathfrak{g}$ an ideal that complements G defines an ideal projector. In this section we will explore yet another equivalence, namely the space $J^{\perp}$ for a given ideal J.

Example 2.24. *Consider the power series $e^{\mathbf{z}\cdot\mathbf{x}} = \sum_{k=0}^{\infty} \frac{(\mathbf{z}\cdot\mathbf{x})^k}{k!}$. Notice that if we replace $\mathbf{x}$ with the differential operator $\mathbf{D}$ we get $\sum_{k=0}^{\infty} \frac{(\mathbf{z}\cdot\mathbf{D})^k}{k!}$ a differential operator. Notice that for $p \in \mathbb{C}[\mathbf{x}]$ any term whose degree sum is greater than that of p will produce zero when applied to p.*

Let us consider the one dimensional case: $e^{\alpha\frac{dy}{dx}}(x^n)(0) = \sum_{k=0}^{\infty} \frac{(\alpha\frac{dy}{dx})^k}{k!}(x^n)(0)$

$$\sum_{k=0}^{\infty} \frac{(\alpha\frac{dy}{dx})^k}{k!}(x^n)(0) = \sum_{k=0}^{n} \frac{(\alpha\frac{dy}{dx})^k}{k!}(x^n)(0) = \sum_{k=0}^{n} \frac{\alpha^k}{k!}\frac{n!}{(n-k)!}x^{n-k}(0)$$

if the degree of a given term is less than n, than we will evaluate x^{n-k} at zero and get zero, if it's degree is greater than n, the derivative will be zero and hence we will still get zero. Only the n-th term will produce anything of interest, leaving us with exactly α^k. In the multi-variate case, this works exactly the same way, though it is more cumbersome to check.

It is simple to see that the powerseries of $D_i e^{\mathbf{z}\cdot\mathbf{D}}$ will evaluate a given polynomials derivative with respect to x_i at $\mathbf{z}$, and the same can then be said for any polynomial g, $g(\mathbf{D})e^{\mathbf{z}\cdot\mathbf{D}}$ will evaluate $g(\mathbf{D})f$ at $\mathbf{z}$. For a given ideal J of finite co-dimension, in-fact, any element of $J^{\perp}$ can be written in this way and so, we identify every functional dual to $\mathbb{C}[\mathbf{x}]$ with a member of $\mathbb{C}[[\mathbf{x}]]$ according to the following rule:

$$\mathbb{C}[[\mathbf{x}]] \to \mathbb{C}[\mathbf{x}]^{\perp} : \sum_{k=0}^{\infty} a_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} \to \sum_{k=0}^{\infty} a_{\boldsymbol{\alpha}}\mathbf{D}^{\boldsymbol{\alpha}}()(\mathbf{0})$$

Whenever it does not cause confusion, we will use $\lambda \in \mathbb{C}[[\mathbf{x}]]$ to mean either a formal power series or a functional in $\mathbb{C}[\mathbf{x}]^{\perp}$ interchangeably.

Let us take a moment to look at which spaces of functional complement ideals, and therefore define ideal projectors. We will present a theorem of Macualay [13] and use it's proof to shed some light on what is happening. As we have identified functionals with elements of $\mathbb{C}[[\mathbf{x}]]$ we will use the term D-invariant to mean that $F \subset \mathbb{C}[[x]]$ is closed under differentiation.

Let us, for now, look at how the operator D, behaves in a single variable. We have:

$$D^{k+1}x^{m+1}(0) = D^k(m+1)x^m(0) = (m+1)!\delta_{m,k} \tag{2.5}$$

and we have:

$$(k+1)D^k x^m(0) = (k+1)!\delta_{m,k} \tag{2.6}$$

Now let's look at this in the context of $\mathcal{J}^\perp$, if $\lambda \in \mathcal{J}^\perp$ then λ has the form $\sum_{j=1}^{\infty} \alpha_j \mathbf{x}^{\boldsymbol{\alpha}_j}(\mathbf{D})$. Again for our example, we will only consider a single variable:

$$(D\lambda)(D)(f)(0) = \lambda'(D)(f)(0)$$

of course, by linearity, it suffices to consider what happens to a single term of λ so the relevant part is:

$$D(x^{k+1})(D)(f)(0) = (k+1)D^k f(0)$$

Which is exactly the situation of equation (2.6), now let's consider $\lambda(xf)(0)$:

$$(x^{k+1})(D)(xf)(0) = D^{k+1}(xf)(0)$$

Which is the same as equation (2.6). And so looking at the right hand sides of (2.5) and (2.6), we have:

$$D\lambda(f)(0) = \lambda(xf)(0) \tag{2.7}$$

using the product rule for differentiation, one can easily compute this same result in the multivariate case, though it is more tedious and messier. Accepting this, the above equation yields the following revelation:

$$D_j P(f) = P(x_j f) = M_j f$$

In other-words, the operators D_j and M_j function like adjoints, notice that this is exactly the same relationship we obtained in the first part of this dissertation with our Hermitian inner product in Equation 1.5.

Definition 2.25. *We say a subspace $A \subset \mathbb{C}[[\mathbf{x}]]$ is* **D***-invariant if $f \in A$ implies $D_j f \in A$ for all $1 \leq j \leq d$*

The following theorem of Macualay will show that the subspaces of $\mathbb{C}[[\mathbf{x}]]$ that are dual to ideals in $\mathbb{C}[\mathbf{x}]$, are precisely those subspaces that are **D**-invariant.

Theorem 2.26. *[13] A subspace $\mathcal{J} \subset \mathbb{C}[\mathbf{x}]$ is ideal if and only if $\mathcal{J}^{\perp} \subset \mathbb{C}[[\mathbf{x}]]$ is* **D***-invariant.*

Proof. Suppose $F \subset \mathbb{C}[[x]]$ is a D invariant subspace, then if f vanishes for all $\lambda \in F$ then since, by equation (2.7), $\lambda(x_j f) = D_j \lambda f$, $x_j f$ vanishes for all λ. It follows that $\ker F$ is an ideal.
To prove the converse, choose $\lambda \in \mathcal{J}^{\perp}$, $f \in \mathcal{J}$, then again by (2.7), $0 = \lambda(x_j f) = D_j \lambda f$, since f was arbitrary, it follows that $D_j \lambda \in \mathcal{J}^{\perp}$ $\square$

We now have a way of identifying those spaces dual to ideals, next we will show convergence of those spaces implies convergence of the associated ideal projectors in the finite dimensional case. To this end we present another definition that will prove useful.

Definition 2.27. *If J_n is a collection of ideals, we will say $J_n \to J$ if whenever $\lambda \in J^{\perp}$, there exists a sequence $\lambda_n \in J_n^{\perp}$ such that $\lambda_n(f) \to \lambda(f)$ for all $f \in \mathbb{C}[\mathbf{x}]$*

Our final theorem of this section will use this definition to show the equivalence of convergence of the spaces $J^{\perp}$ and convergence of ideal projectors.

Theorem 2.28. *(cf. [19]) Let P_m and P be ideal projectors onto a finite-dimensional space $G \subset \mathbb{C}[\mathbf{x}]$. Then $P_m \to P$ if and only if $\ker P_m \to \ker P$ for every functional $F \in \ker^{\perp} P$, i.e. whenever $F \in \ker^{\perp} P$ there exist $F_n \in \ker^{\perp} P_m$ such that:*

$$F_m f \to F f, \, \forall f \in \mathbb{C}[\mathbf{x}]. \tag{2.8}$$

As the complete proof found in [19] has many tedious details, we will only provide a sketch of the proof below:

Proof. Suppose $\ker P_m \to \ker P$, In this case we simply need to show that $P_m(\mathbf{x}^{\alpha}) \to P(\mathbf{x}^{\alpha})$ for all $\alpha \in \mathbb{C}^d$, since the point evaluation functional $1 \in \mathbb{C}[[\mathbf{x}]]$ is in $\ker^{\perp} P_m$ and in $\ker^{\perp} P$ it follows that $P_m(1) \to P(1)$. We proceed by induction, since $P_m \mathbf{x}^{\alpha} \to P\mathbf{x}^{\alpha}$ (where $\alpha \cdot \mathbf{1} < k$) and $x_j P_m \mathbf{x}^{\alpha}, x_j P \mathbf{x}^{\alpha} \in x_j G$, it follows from the fact that P_m converges uniformly in the unit ball of G, that $P_m x_j \mathbf{x}^{\alpha} \to P x_j \mathbf{x}^{\alpha}$.
To prove the converse, suppose $P_m \to P$, and choose $\lambda \in \ker^{\perp} P$ then $P^* \lambda = \lambda$, and since the restriction of a functional onto the finite dimensional space G is continuous, the functionals $P_m^* \lambda \to P^* \lambda = \lambda$. $\square$

It follows from this theorem that P is Hermite if and only if every $F \in \ker^{\perp} P$ is the limit of a sequence of sums of point evaluation functionals. More specifically, if $n = \dim(\text{ran}P)$ then P is Lagrange if and only if there exist sequences of points $\{\mathbf{x}_1^{(k)}, \ldots, \mathbf{x}_N^{(k)}\}$ and constants $\{\alpha_1^{(k)}, \ldots, \alpha_N^{(k)}\}$ such that

$$F = \lim_{k \to \infty} \big(\sum_{i=1}^{n} \alpha_i^{(k)} \delta_{\mathbf{x}_i^{(k)}} \big)$$

for all $F \in \ker^{\perp} P$

2.4 *Equivalences*

Let us examine what we have learned so far in the form of a (relatively) simple example. We will examine the space of ideal projectors onto

$$G = span\{\mathfrak{g} = \{1, x, y\}\}$$

any ideal projector P onto G, has multiplication matrices

$$M_x = \begin{array}{c c} \begin{array}{ccc} x & x^2 & xy \end{array} & \\ \begin{bmatrix} 0 & a_0 & a_1 \\ 1 & a_2 & a_3 \\ 0 & a_4 & a_5 \end{bmatrix} & \begin{array}{c} 1 \\ x \\ y \end{array} \end{array}, \quad M_y = \begin{array}{c c} \begin{array}{ccc} y & xy & y^2 \end{array} & \\ \begin{bmatrix} 0 & b_0 & b_1 \\ 0 & b_2 & b_3 \\ 1 & b_4 & b_5 \end{bmatrix} & \begin{array}{c} 1 \\ x \\ y \end{array} \end{array}$$

It is obvious that 1 is the cyclic vector for $\mathbf{M}$, but these matrices also must satisfy

$$M_x M_y - M_y M_x = 0$$

Solving this, immediately reveals that

$$a_1 = b_0,$$

$$a_3 = b_2,$$

$$a_5 = b_4$$

Some more computation illuminates the situation further, giving

$$a_0 = -a_2b_4 + a_3a_4 - a_4b_5 + b_4^2$$
$$b_0 = -a_3b_4 + a_4b_3$$
$$b_1 = -a_2b_3 + a_3^2 - a_3b_5 + b_3b_4$$

leaving us with six dimensions. The variables $a_2, b_4, a_3, a_4, b_5, b_3$ define a 6-dimensional manifold in $\mathbb{C}^12$ that is the zero set of the polynomial equations $M_xM_y - M_yM_x = 0$. Thus, this manifold is itself a variety and hence is a Zariski closed set.

Any $a_2, b_4, a_3, a_4, b_5, b_3$ define an ideal projector onto G according to the matrices M_x, M_y, and an ideal $J =< \{b - P(b) : b \in \partial\mathfrak{g}\} >$. They also define a **D**-invariant subspace of $\mathbb{C}[[\mathbf{x}]]$, $J^\perp$. In order to see this in action, let us look at a particular projector onto G, we will set all 6 variables equal to one and see what happens, the matrices become:

$$M_x = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, M_y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

so this projector has the map

$$P : Px^2 = Pxy = Py^2 = x + y$$

It's kernel is the ideal generated by $x^2 = x + y, xy = x + y, y^2 = x + y$. Solving the ideal reveals that this projector has variety $\mathcal{V} = \{(0,0),(2,2)\}$. $J^\perp$ is composed of the span of functionals $\{\delta_0, \delta_{(2,2)}, (D_x + D_y)\delta_{(2,2)}\}$. So, evidently P interpolates at $(0,0)$, $(2,2)$, and the $D_x + D_y$ derivative at $(2,2)$.

It is clear, that though tedious, we could compute these for any dimension and any set G. Let us now restate what we have shown so far, in the form of some convenient equivalence theorems. It is the hope of the author that at some point, a language be developed that would make it easier to discuss our subject formally.

Theorem 2.29. *Given a fixed n dimensional linear subspace $G \subset \mathbb{C}[\mathbf{x}]$, with fixed basis $\mathfrak{g}$ there is a one to one coorespondence between each of the following:*

i) ideal projectors onto G

ii) ideals that compliment G

iii) families of d $n \times n$ cyclic commuting matrices mapping $x_j G \to G$ (up to a change of basis for G)

Proof. *i)* $\iff$ *ii)* By definition, an ideal projector defines an ideal, on the other hand given an ideal, the fundamental homomorphism defines an ideal projector.

i) $\iff$ *iii)* given a basis for G an ideal projector uniquely defines its multiplication operators, on the other hand, given a basis for G a family of d $n \times n$ cyclic commuting matrices defines a unique ideal projector onto G $\square$

We also showed that every D-invariant subspace of $\mathbb{C}[[\mathbf{x}]]$ defines an ideal, and that every ideal defines a D-invariant subspace of $\mathbb{C}[[x]]$. Unfortunately, it is not obvious exactly which D-invariant spaces define ideals that compliment a given linear subspace G.

Proposition 2.30. *Let $G \subset \mathbb{C}[\mathbf{x}]$ be a finite dimensional linear space with basis $\mathfrak{g}$, and let $\mathbf{M} = (M_1, \dots, M_k)$ be multiplication operators onto G in basis $\mathfrak{g}$ defining ideal projector P and ideal $J = \ker P$, then, if $\{M_1^{(n)}, \dots, M_d^{(n)}\}$ is a sequence of families of d $n \times n$ cyclic commuting matrices defining projectors P_n and ideals J_n the following are equivalent:*

i) $P_n \to P$

ii) $(M_1^{(n)}, \dots, M_d^{(n)}) \to (M_1, \dots, M_d)$

iii) $J_n \to J$

Proof. The equivalence of *i)* and *ii)* is obvious by definition of the multiplication operators. The equivalence of *i)* and *iii)* is Theorem 2.28. $\square$

Finally we will discuss Hermite projectors and establish the equivalence of Hermite and smoothable.

Corollary 2.31. *Let $G \subset \mathbb{C}[\mathbf{x}]$ be a finite dimensional linear subspace with basis $\mathfrak{g}$, and let P be an ideal projector with kernel J and multiplication operator $(M_1, \dots, M_\beta)$ respectively, then, the following are equivalent:*

i) P is Hermite

ii) J is smoothable

iii) $(M_1, \dots, M_d)$ is the limit of a sequence of families of d cyclic commuting $n \times n$ simultaneously diagonalizable matrices *iv) for all* $F \in J^\perp$ *there exists a sequence* $F_n = \sum_{i=1}^N \alpha_i^{(n)} \delta_{\mathbf{x}_i^{(n)}}$ *such that* $\mathbf{x}_i^{(n)} \neq \mathbf{x}_j^{(n)}$ *when* $i \neq j$ *and* $F_n f \to F f$ *for all* $f \in \mathbb{C}[\mathbf{x}]$

Proof. The equivalence of *i)* and *ii)* follows from Theorem 2.23 and Theorem 2.22. Since the closure of the space of Lagrange projectors is the space of Hermite projectors, Theorem 2.22 implies that so is the Zariski closure.The equivalence of *i)* and *iv)* is given by Theorem 2.28, since for any Lagrange projector L, $\ker^\perp L$ is the span of point evaluation functionals.

The equivalence of *i)* and *iii)* follows from Theorem 2.11 and proposition 2.30

□

Question 2.32. *A big open question in any of the forms provided by Corollary 2.31, is, What classes of ideal projectors are Hermite?*

A partial answer to this question is given in [5], and we will give another partial answer in the following section.

2.5 *Primary Decomposition Theorem*

It is a result of the well known Laskar-Noether theorem, that every ideal in a Noetherian ring can be decomposed into finitely many primary ideals. Before we go into the proof, let us take a moment to look at what primary means in terms of zero dimensional ideal projectors.

An ideal J is said to be *primary* if whenever $fg \in J$ this implies that either $f \in J$ or $g^n \in J$ for some n.

Lemma 2.33. *If* $J \subset \mathbb{K}[\mathbf{x}]$ *is a zero dimensional ideal, then* J *is primary if and only if* $\mathcal{V}(J)$ *contains only a single element.*

Proof. We will proceed by contradiction, Let $J \subset \mathbb{K}[\mathbf{x}]$ be primary and suppose without loss of generality that $\mathcal{V}(J) = \{\mathbf{z}_1, \mathbf{z}_2\}$. The nulstellensatzt garauntees that $f = [(\mathbf{x} - \mathbf{z}_1)(\mathbf{x} - \mathbf{z}_2)]^k \in J$ for some k. Since J is primary, $(\mathbf{x} - \mathbf{z}_1)^m \in J$ for some $m \geq k$. However, the only root of this polynomial is $\mathbf{z}_1$, contradiction.

To prove the converse, suppose $\mathcal{V}(J)$ contains a single element $\mathbf{z}$, then if $fg \in J$, $\mathbf{z}$ is a zero of either f or g, Without loss of generality suppose it to be g, then the nulstellensatz garauntees $g^k \in J$ for some k, and hence, J is primary. □

Remark 2.34. *We use here, for simplicity, in defining the above polynomials, as it does not truly cause confusion:* $(\mathbf{x} - \mathbf{z})$ *to mean* $(x_1 - z_1) * \ldots * (x_d - z_d)$

It will be the main result of this section to prove that for a zero dimensional ideal projector P with $J = \ker P$, if P_j are ideal projectors with kernels J_j, J_j are the minimal primary decomposition of J, then P is Hermite, if and only if all the P_j are Hermite. Or, in the language of algebraic geometry, J is smoothable, if and only if each J_j is smoothable.

It is a simple matter to show that if each P_j is Hermite, then P is Hermite. Since every functional in $\ker^{\perp} P$ is the sum of functionals in $\ker^{\perp} P_j$ it follows from Theorem 2.28 that every functional in $\ker^{\perp} P_j$ is the limit of point evaluation functionals, and hence, so are their sums in $\ker^{\perp} P$. We present a more formal proof of this fact below.

Lemma 2.35. *[6] Let $P_1 \dots P_m$ be Hermite projectors such that $\mathcal{V}(\ker P_i)$ each consist of a distinct point. then the projector $P = P_1 \oplus \dots \oplus P_m$ is Hermite*

Proof. P is an N-dimensional ideal projector and $\ker P$ has minimal primary decomposition

$$\ker P = \cap_{j=1}^{m} \ker P_j$$

where the ideals $\ker P_j$ have codimensions N_j. Since

$$\ker^{\perp} P = \oplus \ker P_j^{\perp}$$

and the $\ker P_j$ are distinct, we have $\sum_{j=1}^{M} N_j = N$. If each J_j is a kernel of Hermite projector then, by Corollary 2.14 there exists a set $\mathcal{X}_j^{(n)} \subset \mathbb{C}^d$ of N_j distinct points such that for every $F \in J_j^{\perp}$, the Functional F is the limit (weak-$\star$) of linear combinations of the functionals $\{\delta_{\mathbf{x}} : \mathbf{x} \in \mathcal{X}_j^{(n)}\}$. By Corollary 2.31 every $F \in \ker^{\perp} P$ can be written as $\sum_{j=1}^{m} F_j$ and

$$F_j = \lim_{n\to\infty} \sum_{\mathbf{x}\in\mathcal{X}_j^{(n)}} a_{\mathbf{x}}^{(n)} \delta_{\mathbf{x}} \tag{2.9}$$

it follows that $F = \lim\limits_{n\to\infty} \sum_{j=1}^{m} \sum_{\mathbf{x}\in\mathcal{X}_j^{(n)}} a(\mathbf{x})\delta_{\mathbf{x}} = \lim\limits_{n\to\infty} \sum_{\mathbf{x}\in\cup\mathcal{X}_j^{(n)}} b_{\mathbf{x}}^{(n)} \delta_{\mathbf{x}}$ and hence every $F \in \ker^{\perp} P$ is the limit of a linear combination of N point evaluations, which, by the Theorem 2.28, implies that P is Hermite.

□

The final result of this dissertation, is that the converse of this lemma is also true. Namely that if P is Hermite then if $P_1, \dots, P_m$ are ideal projectors such that $(\ker P_1) \cap \dots \cap (\ker P_m) = \ker P$ are the minimal primary decomposition of $\ker P$ then each P_i is also Hermite. We will begin with a sketch of the proof.[6]

We know that if P is Hermite, then every functional in $\ker^\perp P$ is the limit of the sums of point evaluation functionals. i.e. if $F \in \ker^\perp P$ then there exist points $\{\mathbf{x}_1^{(k)}, \ldots, \mathbf{x}_n^{(k)}\}$ and constants $\alpha_j^{(k)}$ such that

$$\sum_{j-1}^{n} \alpha_j^{(k)} \delta_{\mathbf{x}_j^{(k)}} \to F \text{ as } k \to \infty \tag{2.10}$$

Recall that if J_i is in the unique minimal primary decomposition of $\ker P$, then $\ker P \subset J_i$, hence $J_i^\perp \subset \ker^\perp P$. The main idea of our proof is to show that whenever $F_i^\perp \subset \ker^\perp P$ then the sum in (2.10) depends only on those points which converge to the single $\mathbf{x} \in \mathcal{V}(J_i)$.

To this end we will do something like this: Let us assume that $\ker P = \cap_{j=1}^m \ker P^{(j)}$ and $\mathcal{V}(\ker P^{(1)}) = \{\mathbf{y}\}$. Next we will decompose $\mathcal{X}_n = \mathcal{Y}_n \cup \mathcal{Z}_n$ so that all accumulation points of $\{\mathbf{x} : \mathbf{x} \in \mathcal{Z}_n, n \in \mathbb{N}\}$ will be away from $\mathbf{y}$.

If P is Hermite then every functional $F \in \ker^\perp P$ satisfies (2.10) and, in particular, every functional $F \in \ker^\perp P_1$ satisfies (2.10). That is, for every $f \in \mathbb{C}[\mathbf{x}]$

$$F(f) = \lim_{n\to\infty} (\sum_{\mathbf{x}\in\mathcal{Y}_n} a_{\mathbf{x}}^{(n)} (\delta_{\mathbf{x}}) f + \sum_{\mathbf{x}\in\mathcal{Z}_n} a_{\mathbf{x}}^{(n)} (\delta_{\mathbf{x}}) f. \tag{2.11}$$

Now we just have to show that this implies

$$F(f) = \lim_{n\to\infty} \sum_{\mathbf{x}\in\mathcal{Y}_n} a_{\mathbf{x}}^{(n)} (\delta_{\mathbf{x}}) f \tag{2.12}$$

Unfortunately, in actual practice, proving this statement in such a direct manner turns out to be nigh on impossible, so instead, we will choose $F \in (\cap J_{j=2}^m)^\perp$. This is still fine, since it is true that $J \subset \cap J_{j=2}^m$, hence, equation (2.11) holds. In this case the computations will turn out to be sufficiently tamable and we will be able to show that:

$$F = \lim_{n\to\infty} \sum_{\mathbf{x}\in\mathcal{Z}_n} a_{\mathbf{x}}^{(n)} \delta_{\mathbf{x}}$$

Having shown this, the assertion we made in (2.12) will follow from induction on m. Once this is established our claim will follow simply from Theorem 2.28. In order to prove this we will, of course, need some tricks, first among which will be to produce some polynomials which vanish on those points which converge to the points in $\mathcal{V}(\cap J_{j=2}^m)$. We will use our very simple multivariate analog of Lagrange fundamental polynomials described in Remark 2.3 in a brief proposition which establishes the properties we need of them.

Proposition 2.36. *[6] Let $\mathcal{Y}$ be a finite set of m points in $\mathbb{C}^d$ and let $\mathbf{z} \in \mathbb{C}^d$ such that $\mathcal{Y}$ and $\mathbf{z}$ lie in the* *interior of a ball $B \subset \mathbb{C}^d$ of radius R. Let*

$$r = \min\{\|\mathbf{y} - \mathbf{z}\| : \mathbf{y} \in \mathcal{Y}\} > 0$$

Then there exists a constant $C(R, r)$ and polynomial $\omega(\mathbf{x}) = \omega_{\mathcal{Y},\mathbf{z}} \in \mathbb{C}[\mathbf{x}]$ of degree at most m such that

$$\omega(\mathbf{z}) = 1, \omega(\mathbf{y}) = 0, \forall \mathbf{y} \in \mathcal{Y}$$

and

$$\|\omega\|_B \leq C(R, r) = (\frac{2R}{r})^{2m}.$$

(here $\|\omega\|_B$ denote the supremum of the polynomial ω over the ball $B \subset \mathbb{C}^d$)

Proof. Let $< \mathbf{u}, \mathbf{v} >$ denote the Hermitian inner product in the space $\mathbb{C}^d$ and consider the polynomial in $\mathbf{x}$:

$$\omega(\mathbf{x}) = \frac{\prod_{\mathbf{y} \in \mathcal{Y}} < \mathbf{x} - \mathbf{y}, \mathbf{z} - \mathbf{y} >}{\prod_{\mathbf{y} \in \mathcal{Y}} \|\mathbf{z} - \mathbf{y}\|^2}$$

Since $< \mathbf{x} - \mathbf{y}, \mathbf{z} >$ is a linear polynomial in $\mathbb{C}[\mathbf{x}]$, hence $\omega(\mathbf{x})$ is a polynomial of degree at most m. Clearly $\omega(\mathbf{y}) = 0$ for all $\mathbf{y} \in \mathcal{Y}$ and $\omega(\mathbf{z}) = 1$.

The norm of the product

$$\left\| \prod_{\mathbf{y} \in \mathcal{Y}} < \mathbf{x} - \mathbf{y}, \mathbf{z} - \mathbf{y} > \right\| \leq \prod_{\mathbf{y} \in \mathcal{Y}} |< \mathbf{x} - \mathbf{y}, \mathbf{z} - \mathbf{y} >| \leq (2R)^{2m}$$

while $\prod_{\mathbf{y} \in \mathcal{Y}} \|z - \mathbf{y}\|^2 \geq r^{2m}$. □

We will evaluate our functional on these polynomials, and thus obtain a sum which must vanish, regardless of the remaining terms. Of course, as we need to show that this is true for all $f \in \mathbb{C}[\mathbf{x}]$, and not just multiples of these polynomials, the polynomials are not enough to prove the claim by themselves. And so, we will also need a somewhat tedious computational lemma, that will ultimately be used to show the remainder of the sum, does, in-fact vanish for all $f \in \mathbb{C}[\mathbf{x}]$.

Lemma 2.37. *[6] Let $(u_1^{(n)}, ..., u_m^{(n)})$ and $(\gamma_1^{(n)}, ..., \gamma_m^{(n)})$ be sequences in $\mathbb{C}^m$ be such that $\gamma_j^{(n)} \to 1$ as $n \to \infty$ and*

$$\sum_{j=1}^{m} u_j^{(n)} (\gamma_j^{(n)})^k \to 0 \text{ as } n \to \infty \tag{2.13}$$

for all $k = 1, ..., m$. Then $\sum_{j=1}^{m} u_j^{(n)} \to 0$ as $n \to \infty$.

Proof. By induction on m. If $m = 1$ then $u_1^{(n)}\gamma_1^{(n)} \to 0$ and $\gamma_1^{(n)} \to 1$ immediately implies that $u_1^{(n)} \to 0$.

Assume that the statement is true for a fixed m and

$$\sum_{j=1}^{m+1} u_j^{(n)}(\gamma_j^{(n)})^k \to 0 \text{ as } n \to \infty$$

Then, for every $k \le m$

$$\gamma_{m+1}^{(n)} \sum_{j=1}^{m+1} u_j^{(n)}(\gamma_j^{(n)})^k = \sum_{j=1}^{m+1} u_j^{(n)}(\gamma_j^{(n)})^k \gamma_{m+1}^{(n)} \to 0$$

since $\gamma_{m+1}^{(n)} \to 1$. Hence

$$\sum_{j=1}^{m+1} u_j^{(n)}(\gamma_j^{(n)})^k \gamma_{m+1}^{(n)} - \sum_{j=1}^{m+1} u_j^{(n)}(\gamma_j^{(n)})^{k+1} = \sum_{j=1}^{m} u_j^{(n)}(\gamma_{m+1}^{(n)} - \gamma_j^{(n)})(\gamma_j^{(n)})^k \to 0$$

for all $k \le m$. Letting $\tilde{u}_j^{(n)} = u_j^{(n)}(\gamma_{m+1}^{(n)} - \gamma_j^{(n)})$ we have $\sum_{j=1}^{m} \tilde{u}_j^{(n)}(\gamma_j^{(n)})^k \to 0$ for all $k \le m$ and using the inductive assumption we conclude that

$$\sum_{j=1}^{m} u_j^{(n)}(\gamma_j^{(n)} - \gamma_{m+1}^{(n)}) = \sum_{j=1}^{m} u_j^{(n)}\gamma_j^{(n)} - \gamma_{m+1}^{(n)} \sum_{j=1}^{m} u_j^{(n)} \to 0$$

Taking into account $\sum_{j=1}^{m} u_j^{(n)}\gamma_j^{(n)} + \gamma_{m+1}^{(n)} u_{m+1}^{(n)} \to 0$ we have

$$\sum_{j=1}^{m} u_j^{(n)}\gamma_j^{(n)} - \gamma_{m+1}^{(n)} \sum_{j=1}^{m} u_j^{(n)} - (\sum_{j=1}^{m} u_j^{(n)}\gamma_j^{(n)} + \gamma_{m+1}^{(n)} u_{m+1}^{(n)}) = -\gamma_{m+1}^{(n)} \sum_{j=1}^{m+1} u_j^{(n)} \to 0$$

and, since $-\gamma_{m+1}^{(n)} \to 1$ we conclude that $(\sum_{j=1}^{m+1} u_j^{(n)}) \to 0$. $\square$

With all of the necessities established, we are finally ready to prove our main result. Let us reveal how this all fits together to form our long awaited theorem:

Theorem 2.38. *[6] Let P be a Hermite projector onto an N-dimensional space $G \subset \mathbb{C}[\mathbf{x}]$. Suppose that*

$$P = P^{(1)} \oplus P^{(2)} \oplus \dots P^{(m)} \tag{2.14}$$

where $P^{(k)}$ are ideal projectors with the property that the ideals $\ker P^{(k)}$ form a the primary decomposition of the ideal $\ker P$:

$$\ker P = \cap_{k=1}^{m} \ker P^{(k)}. \tag{2.15}$$

Then each $P^{(j)}$ is Hermite.

Proof. Without loss of generality we will start with $P^{(1)}$. Assume that $\mathcal{V}(\ker P) = \{\mathbf{u}_1, \dots \mathbf{u}_m\}$ and $\ker P^{(j)} = \{\mathbf{u}_j\}$. Since P is Hermite $P = \lim P_n$ where P_n are Lagrange. Thus, $\mathcal{X}^{(n)} = \mathcal{V}(\ker P_n)$ consist of exactly N points and, by Corollary 2.31, for every functional $F \in \ker^{\perp} P$

$$F(f) = \lim_{n\to\infty} \sum_{\mathbf{x}\in\mathcal{X}^{(n)}} a_{\mathbf{x}}^{(n)} \delta_{\mathbf{x}}(f) = \lim_{n\to\infty} \sum_{\mathbf{x}\in\mathcal{X}^{(n)}} a_{\mathbf{x}}^{(n)} f(\mathbf{x}) \tag{2.16}$$

for every $f \in \mathbb{C}[\mathbf{x}]$. In particular if $F \in \cap_{j=2}^{m} \ker^{\perp} P^{(j)}$. Then $F \in \ker^{\perp} P$ and hence (2.16) holds for this F. By corollary 2.14 the sets $\mathcal{X}^{(n)}$ lie in some ball in $\mathbb{C}^d$ of, say, radius R and $\{\mathbf{u}_1, \dots \mathbf{u}_m\}$ are the only accumulation points of $\mathcal{X}^{(n)}$. Partition the points $\mathcal{X}^{(n)} = \mathcal{Y}^{(n)} \cup \mathcal{Z}^{(n)}$ so that every $\mathbf{x}_n \in \mathcal{Z}^{(n)}$ we have

$$\mathbf{x}_n \to \mathbf{u}_1 \tag{2.17}$$

and for the points $\mathbf{x}_n \in \mathcal{Y}^{(n)}$ are, for sufficiently large n, arbitrary close to the set $\{\mathbf{u}_2, \dots \mathbf{u}_m\}$ and in particular

$$\|\mathbf{x}_n - \mathbf{u}_1\| \geq r > 0 \tag{2.18}$$

for all $\mathbf{x}_n \in \mathcal{Y}^{(n)}$. Then (2.16) can be rewritten as

$$F(f) = \lim_{n\to\infty} \Bigg(\sum_{\mathbf{x}\in\mathcal{Y}_n} a_{\mathbf{x}}^{(n)} f(\mathbf{x}) + \sum_{\mathbf{x}\in\mathcal{Z}_n} a_{\mathbf{x}}^{(n)} f(\mathbf{x}) \Bigg). \tag{2.19}$$

where the points in $\mathcal{Y}_n$ and $\mathcal{Z}_n$ satisfy (2.18) and (2.17). Now let p be a polynomial in $\cap_{j=2}^{m} \ker P^{(j)}$ such that $p(\mathbf{u}_1) = 1$. Such polynomial exists since, otherwise every polynomial in $\cap_{j=2}^{m} \ker P^{(j)}$ would vanish at $\mathbf{u}_1$ and hence $\mathbf{u}_1 \in \mathcal{V}(\cap_{j=2}^{m} \ker P^{(j)}) = \{\mathbf{u}_2, \dots, \mathbf{u}_m\}$. Next we look at polynomials

$$h_{k,n} = (p \cdot \omega_{\mathcal{Y}_n, \mathbf{u}_1})^k f \tag{2.20}$$

for $k = 1, \dots, m$ where $\omega_{\mathcal{Y}_n, \mathbf{u}_1}$ is defined as in Proposition 2.36, and f is arbitrary. Since p is in the ideal $\cap_{j=2}^{m} ker P^{(j)}$ so are $h_{k,n}$ hence $F(h_{k,n}) = 0$. By the same proposition and by (2.17) these polynomials are uniformly bounded and belong to a finite-dimensional space of polynomials of degree $\leq (mm + \deg p) + \deg f$. Thus the convergence (2.19) on this space is uniform and (2.19) gives

$$F(h_{k,n}) = 0 = \lim_{n\to\infty} \Bigg(\sum_{\mathbf{x}\in\mathcal{Y}_n} a_{\mathbf{x}}^{(n)} h_{k,n}(\mathbf{x}) + \sum_{\mathbf{x}\in\mathcal{Z}_n} a_{\mathbf{x}}^{(n)} h_{k,n}(\mathbf{x}) \Bigg). \tag{2.21}$$

Furthermore, since $\omega_{\mathcal{Y}_n,\mathbf{u}_1}$ vanishes on $\mathcal{Y}_n$ it follows that

$$0 = \lim_{n\to\infty} \sum_{\mathbf{x}\in\mathcal{Z}_n} a_{\mathbf{x}}^{(n)} h_{k,n}(\mathbf{x}) = \lim_{n\to\infty} \sum_{\mathbf{x}\in\mathcal{Z}_n} a_{\mathbf{x}}^{(n)} (p(\mathbf{x}) \cdot \omega_{\mathcal{Y}_n,\mathbf{u}_1}(\mathbf{x}))^k f(\mathbf{x}) \tag{2.22}$$

Finally observe that since $\mathbf{u}_1$ is the limit point of $\mathcal{Z}_n$

$$\lim_{n\to\infty} \left(p(\mathbf{x}_n) \cdot \omega_{\mathcal{Y}_n,\mathbf{u}_m}(\mathbf{x}_n)\right) = 1 \quad \text{whenever} \quad \mathbf{x}_n \in \mathcal{Z}_n.$$

Setting $\gamma_n = p(\mathbf{x}_n) \cdot \omega_{\mathcal{Y}_n,\mathbf{u}_1}(\mathbf{x}_n)$ for $x_n \in \mathcal{Z}_n$ and applying Lemma 2.37 we conclude that

$$\lim_{n\to\infty} \sum_{\mathbf{x}\in\mathcal{Z}_n} a_{\mathbf{x}}^{(n)} f(\mathbf{x}) = 0$$

Thus eliminating the points that accumulate at $\mathbf{u}_1$ from the sum in (2.19). Since, this implies (2.16) holds for $P^{(1)} \oplus \ldots \oplus P^{(m-1)}$, we can repeat this procedure inductively, eliminating all points from $\mathcal{X}^{(n)}$ in the sum (2.16) that accumulate at $\mathbf{u}_{m-1}, \ldots, \mathbf{u}_2$ and conclude that there are points $x_1^{(n)}, \ldots, x_{N_n^{(1)}}^{(n)} \in \mathcal{X}_n$ such that every $F \in \ker^{\perp} P^{(1)}$ is in the weak-$\star$ closure of the space $span\{\delta_{\mathbf{x}}, \mathbf{x} \in \mathcal{X}_n^{(1)}\}$ for some $\mathcal{X}_n^{(1)} \subset \mathcal{X}_n$ that have accumulation point at $\mathbf{u}_m$. Thus, for sufficiently large n, the dimension of this space must be greater or equal then the dimension of the space $\ker^{\perp} P^{(1)}$. Hence $\left|\mathcal{X}_n^{(1)}\right| \geq \dim\ker^{\perp} P^{(1)}$. Repeating this procedure for the rest of the points $\mathbf{u}_j \in \mathcal{V}(\ker P)$ we will obtain disjoint partition of $\mathcal{X}_n$:

$$\mathcal{X}_n = \cup_{j=1}^m \mathcal{X}_n^{(j)}$$

such that every $F \in \ker^{\perp} P^{(j)}$ is in the weak-$\star$ closure of $span\{\delta_{\mathbf{x}}, \mathbf{x} \in \mathcal{X}_n^{(j)}\}$ and

$$\left|\mathcal{X}_n^{(j)}\right| \geq \dim\ker^{\perp} P^{(j)}.$$

But for every n

$$\sum_{j=1}^m \left|\mathcal{X}_n^{(j)}\right| = N = \sum_{j=1}^m \dim\ker^{\perp} P^{(j)}.$$

Hence, for sufficiently large n we have $\left|\mathcal{X}_n^{(j)}\right| = \dim\ker^{\perp} P^{(j)}$ and hence, by the Theorem 2.28 every $P^{(j)}$ is Hermite. $\square$

3 Some Applications of Our Results

Our theorems have some immediate and interesting consequences. First among which are some corollaries. Seeing as how we have established that ideal projectors, ideals, ideal compliments and families of cyclic commuting matrices, are all in a sense, essentially the same thing, let us go over what this means in terms of our primary decomposition theorem.

Observe that the quality of being Hermite, is in-fact a property of the ideals themselves:

Corollary 3.1. *If P is Hermite and $\ker P' = \ker P$ then P' is also Hermite*

Proof. Our claim follows from the fact that $\ker^{\perp} P' = \ker^{\perp} P$ □

And so, we have the following corollary of 2.31 and of our primary decomposition theorem:

Corollary 3.2. *Let $J \subset \mathbb{C}[\mathbf{x}]$ be an ideal, Let $J_1, \ldots, J_m$ be the primary decomposition of J, then, J is smoothable if and only if each J_i is smoothable*

We found it necessary, in our case, to limit ourselves to $\mathbb{C}[\mathbf{x}]$. This was done to make use of the topology of $\mathbb{C}[\mathbf{x}]$, as well as the inner product defined on it. However, we have mentioned that it seems known to algebraic geometers [8], that the Zariski topology induces some topological properties on any polynomial ring over an algebraically closed field. This leaves us with a question:

Question 3.3. *Is it possible to extend Corollary 3.2 to ideals over any polynomial ring over an algebraically closed field?*

For a moment, allow us to discuss what our result means in $\mathbb{C}[\mathbf{x}]^{\perp}$. This is actually the core of the proof of Theorem 2.38. Careful reading of the proof, reveals that what this theorem actually says is that if $F_n \to F$ where each F_n is the sum point evaluation functionals, then only points that converge to the correct things matter. Recall that by Theorem 2.26, for any functional $F \in \mathbb{C}[\mathbf{x}]^{\perp} \simeq \mathbb{C}[[\mathbf{x}]]$, the *deflation* of F, defines an ideal. We use the term *deflation* to mean all the derivatives of F, this of course, produces a D-invariant linear subspace.

Let us re-state this information, formally, in the form of a Corollary to Theorem 2.28:

Corollary 3.4. *[6] Let $F \in \mathbb{C}[[\mathbf{x}]]$ then if $\mathcal{F}$ is the linear space generated by the deflation of F, Theorem 2.26 gives that $\ker \mathcal{F} = \mathcal{J}$ is an ideal. $\mathcal{J}$ is the kernel of a Hermite projector, if and only if every $F \in \mathcal{F}$ is the limit of sequences $\sum_{j=1}^{n} \alpha_j^{(m)} \delta_{\mathbf{z}_j^{(m)}}$ where $\alpha_j^{(m)} \in \mathbb{C}$, $n = \dim(\mathbb{C}[\mathbf{x}]/\mathcal{J})$, and each sequence $\mathbf{z}_j^{(m)}$ converges to a point in $\mathcal{V}(\mathcal{J})$ as $m \to \infty$.*

Now, we will address what our results mean in terms of matrices. Commuting Matrices that are the limits of simultaneously diagonalizable matrices, have received a fair amount of attention in linear algebra [9], [10], [12], and our result allows us to make a contribution to this problem, when combined with Theorem 2.12. It is clear that if each family of blocks in the maximal block diagonalization of a family of commuting matrices $\mathbf{M}$ is simultaneously diagonalizable then $\mathbf{M}$ is as well, however, our result allows us to make an assertion about the converse:

Corollary 3.5. *[6] Let $\mathbf{M}$ be a family of cyclic commuting matrices, with maximal block diagonalization $\mathbf{M}_j$ where $\mathbf{M} = diag(\mathbf{M}_j)$. Then, $\mathbf{M}$ is approximable by simultaneously diagonalizable matrices, if and only if each $\mathbf{M}_j$ is approximable by simultaneously diagonalizable matrices.*

Of course, it is still an open question when the matrices are not cyclic. At this point we will remark upon the problem that actually motivated our study into our primary decomposition theorem. Recall that a it is a question posed by Carl de Boor [3] to classify all Hermite Projectors. Well, one such class of projectors considered by Boris Shekhtman is those whose kernels are *symmetric.*

Definition 3.6. *An ideal $J \in \mathbb{C}[\mathbf{x}]$ is called **symmetric** if for any $p(x_1, \ldots, x_d) \in J$, the polynomial $p(x_{\sigma(1)}, \ldots, x_{\sigma(d)}) \in J$ for every permutation σ on $\{1, \ldots, d\}$. An ideal projector P is called **symmetric** if $\ker P$ is symmetric.*

The author was given the task of determining whether or not this class of projectors was Hermite. So we provide the following theorem as proof that symmetric projectors are not, in general, Hermite.

Theorem 3.7. *[6] In three or more variables, there exists a finite dimensional symmetric projector which is not Hermite.*

Proof. There exists a finite dimensional projector Q whose kernel is primary and is not Hermite. Without loss of generality, let $\mathcal{V}(\ker Q) = \{(1, 2, \ldots, d)\}$ now for every permutation σ let $Q_\sigma f = Qf(x_{\sigma(1)}, \ldots, x_{\sigma(d)})$, then clearly the ideal

$$J = \cap_\sigma \ker Q_\sigma$$

is a symmetric ideal. since none of the ideals in its primary decomposition are Hermite, it follows that any projector P with kernel J is not Hermite. $\square$

Of course, the counter example in our proof only seems to show that there exists a non-Hermite symmetric ideal projector. By construction, and out of necessity due to the nature of our theorem, this non-Hermite symmetric ideal projector is not primary. And so we present an open question:

Question 3.8. *Does there exist a non-Hermite ideal projector whose kernel is both symmetric and primary?*

Now, we shall turn the discussion to our Theorems about interpolation on curves. Recall that the motivation for studying this came from a paper by Carl de Boor, Nira Dyn, and Amos Ron, in which they provided a systematic way to go about interpolating data on "flats" in $\mathbb{R}^d$ [4]. The authors of this paper state that this is a natural problem that arises in box-spline theory [4]. Recall that by "flats" we mean hyperplanes, so in one sense our theorems are in-fact an extension of the results in de Boor, Dyn, and Ron's paper.

In order to explain their result we will need some definitions, as they work in a very different language than we do in this dissertation:

Definition 3.9. *Let X be a collection of pairs $(\mathbf{x}, \lambda_{\mathbf{x}})$, where $\mathbf{x} \in \mathbb{R}^d, \lambda_{\mathbf{x}} \in \mathbb{R}$. We call X a direction set.*

Each vector in X and scalar $\lambda_{\mathbf{x}}$ defines a hyperplane $H_{\mathbf{x}}$ that is the zero set of the polynomial in $\mathbf{z}$, $\mathbf{x} \cdot \mathbf{z} - \lambda_{\mathbf{x}}$

Definition 3.10. *We denote by $M_s(X)$ the collection of s dimensional intersections of the $H_{\mathbf{x}}$ defined by X*

In the main result from the de Boor, Dyn, and Ron paper [4], conditions are provided for interpolation on the varieties, $M_s(X)$, and in some cases ideals with these varieties.

$$\mathcal{P}(X) = \text{span}\{\prod_{u \in L}(u \cdot \mathbf{z}) : L \subset X \text{ such that } \text{span}(X \setminus L) = \text{span}X\}$$

Definition 3.11. *Let $\mathcal{P}_s(X) := \mathcal{P}(X)\mathbb{C}_{\leq s-d+\dim(span(X))}[\mathbf{x}]$.*

Definition 3.12. *Let Ξ be a direction set, let $\{\varphi\}$ be the set of polynomials in $\mathbb{C}[\mathbf{x} \cdot \mathbf{z} - \lambda_{\mathbf{x}} : \mathbf{x} \in \Xi]$, such that $\left[\prod_{\mathbf{x} \in K}(\mathbf{x} \cdot \mathbf{D})\right]\varphi = 0$ for all $K \subset \Xi$ such that $\dim(\text{span}(\Xi \setminus K)) < \dim(\text{span}\Xi)$. Then:*

$$\mathcal{D}(\Xi) := \text{span}\{\varphi\}$$

de Boor, Dyn, and Ron require the polynomials p_i to be interpolated on $M_s(X)$ to be *consistent* and X*-compatible.*

Definition 3.13. *We call the functions $p_M : M \in M_s(X)$, X**-compatible** if $\varphi(D)p_m \in \mathcal{P}_s(X)$ for all $\varphi \in \mathcal{D}(X_M)$, where $X_M \subset X$ such that for each $\mathbf{x} \in X_M$, $M \subset H_{\mathbf{x}}$*

Definition 3.14. *The functions $\varphi(D)p_m$ are said to be **consistent** if for some Y such that $\mathcal{P}(X \cup Y) = \mathcal{P}_s(X)$, whenever $\theta \in M_0(X \cup Y)$, then the statement:*

$$0 = \sum_{M \in M_s(X_\theta)} \sum_i \varphi_i q_i : \varphi_i \in \mathcal{D}(X_M), q_i \in \mathbb{C}[\mathbf{x} \cdot \mathbf{z} - \lambda_{\mathbf{x}} : \mathbf{x} \in X_M]$$

implies that:

$$\sum_{M \in M_s(X_\theta)} \sum_i \varphi_i(\mathbf{D}) q_i(\mathbf{D}) p_m(\theta) = 0$$

Finally, we can state the main Theorem of [4]:

Theorem 3.15. *Let X be a direction set, Let $s \in \{1, \ldots, d-1\}$, Let $\{p_M\}$ be a collection of consistent and X-compatible functions, then there exists a unique $p \in \mathcal{P}_s(X)$ such that*

$$\varphi(\mathbf{D})(p - p_M)|_M = 0, \quad \textit{for all } M \in M_s(X), \varphi \in \mathcal{D}(X)$$

Certainly, our theorems in Section 2 continues the spirit of this result, providing conditions under which we can find a function that interpolates on any collection of varieties, and not just flats. However, the question of equivalence between these theorems in the specific case of flats of like dimension remains open.

If the flats described by $M_s(X)$ are all disjoint (they do not intersect) then Theorem 1.5 promises the existence of an interpolant for ANY functions p_i on ideals J_i. Our theorem certainly provides an interpolating function in this case. Our function is not unique however, though it does define an equivalence class which is unique as an element of $\mathbb{C}[\mathbf{x}]/\cap J_i$.

In the case where the ideals defined by the flats defined by $M_s(X)$ satisfy the conditions of Theorem 1.11 again interpolation is always possible.

Remark 3.16. *Note that if $< \prod_{j=1}^{\mu}(m_j x_j - \alpha_j) >$, $< \prod_{j=1}^{\nu}(n_j x_j - \beta_j) >$ are ideals, their sum: $< \prod_{j=1}^{\mu}(m_j x_j - \alpha_j), \prod_{j=1}^{\nu}(n_j x_j - \beta_j) >$ is also radical, since this is the ideal of functions whose zero set passes through the intersection of these planes*

Conjecture 3.17. *If the direction set X defines a collection of distinct flats of dimension s then $I(\cap M : M \in M_s(X))$ is radical.*

It is also clear that if there exist distinct subsets X_M, X'_M of X both defining the same s dimensional flat M, then the definitions in [4] do not define a radical ideal. In this case we have provided our own set of interpolation conditions, very different looking from those in [4], namely that $p_i - p_j \in J_i + \cap_{k \neq i} J_k$. Although this condition again ensures interpolation is possible, a question remains:

Question 3.18. *If p_i are consistent and X-compatible, does this imply $p_i - p_j \in J_i + \cap_{k \neq i} J_k$ where J_i are given by a direction set X as in [4]? What about the converse statement?*

Another question:

Question 3.19. *Given functions p_i and ideals J_i, what is the minimum degree of an interpolating function p, $p - p_i \in J_i$?*

In closing, we Have shown that an ideal projector P is Hermite if and only if ideal projectors $P_1, \ldots, P_m$ such that $\ker P_1, \ldots, \ker P_m$ form the minimal primary decomposition of $\ker P$ are also Hermite. Obviously this means the same can be said for those strange beasts which are non-Hermite projectors. We have presented some consequences of this and left the reader with some remaining questions.

With regards to our results in the area of interpolation on curves, in which we gave a variety of different circumstances in which interpolation is assured to be possible, we are also left with questions. We have mentioned some of these in our comparison to the work of de Boor, Dyn, and Ron, but we would like to leave the reader with a final question tying together the entire discourse of this dissertation. After all, the best thing to a mathematician, is an open question.

Question 3.20. *If J is not a zero dimensional ideal, how should we define **Hermite** for J, in order that it be meaningful with regards to interpolation on curves? In other-words is it possible to cast an interesting convergence problem for curves, and how should this be done?*

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