

March 2020

## On the $p(x)$ -Laplace equation in Carnot groups

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On the  $p(x)$ -Laplace equation in Carnot groups

by

Robert D. Freeman

A dissertation submitted in partial fulfillment  
of the requirements for the degree of  
Doctor of Philosophy  
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Date of Approval:  
March 6, 2020

Keywords: Sub-Riemannian geometry, Non-linear potential theory, Viscosity solutions,  $p(x)$ -Laplace equation, Removability

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## **Acknowledgments**

I would like to acknowledge several people for guiding/mentoring me through my years of graduate study. First, I would like to acknowledge my advisor, Thomas Bieske. I would not be the mathematician I am today if it weren't for his guidance and wisdom. His mentoring and encouragement have helped me become a part of the mathematics community, which is something I could not have achieved independently. I am truly thankful for this and all the help he has given me. I would also like to acknowledge many people for their encouragement and advice: Kathie and Lou Panichelli, S. Matt Shriver, Brian Curtin, Sherwin Kouchekian, Yuncheng You, Razvan Teodorescu, Leslaw Skrzypek, Zachary Forrest, Boris Shekhtman, Robert Connelly, and Brendan Nagle. Next, I would like to acknowledge the mathematics department at the University of South Florida and thank them for the many opportunities they have given me during my time in graduate school. Moreover, I would like to thank the department for providing me with a strong foundation in advanced mathematics and giving me the opportunity to work with Thomas Bieske. Finally, I would like to acknowledge my fiancé, Becky Gorman, for her never-ending patience, her continuous confidence in me, and her enduring support.

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<sup>1</sup>A Note to Reader: This section has been reproduced from [BF1].

<sup>2</sup>A Note to Reader: This chapter has been reproduced from [BF2].

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## **On the $p(x)$ -Laplace equation in Carnot groups**

**Robert D. Freeman**

### **Abstract**

In this thesis, we examine the  $p(x)$ -Laplace equation in the context of Carnot groups. The  $p(x)$ -Laplace equation is the prototype equation for a class of nonlinear elliptic partial differential equations having so-called *nonstandard growth conditions*. An important and useful tool in studying these types of equations is *viscosity theory*. We prove a  $p(\cdot)$ -Poincaré-type inequality and use it to prove the equivalence of potential theoretic weak solutions and viscosity solutions to the  $p(x)$ -Laplace equation. We exploit this equivalence to prove a Radó-type removability result for solutions to the  $p$ -Laplace equation in the Heisenberg group. Then we extend this result to the  $p(x)$ -Laplace equation in the Heisenberg group.

# Chapter 1

## Introduction

The focus of this dissertation concerns analytic and geometric properties of solutions to partial differential equations in sub-Riemannian spaces. Sub-Riemannian spaces are manifolds in which tangent vectors to curves can lie only in certain restricted directions. Thus, the (topological) dimension of the tangent space is less than the (topological) dimension of the manifold. Therefore, sub-Riemannian spaces are a class of metric spaces whose underlying geometry behaves unlike standard Euclidean  $\mathbb{R}^n$ . Sub-Riemannian spaces are used to model phenomena in which motion is restricted, such as driving a four-wheeled vehicle or travel through mountainous terrain. In order to mimic the algebraic structure of standard Euclidean  $\mathbb{R}^n$ , we will focus on Carnot groups, a subset of sub-Riemannian spaces that have an algebraic (non-abelian) group law.

One key partial differential equation under consideration is the  $p$ -Laplace equation, which is the standard prototype equation of potential theory. One can replace the constant  $p$  with an appropriate function  $p(x)$  to produce another key partial differential equation, the  $p(x)$ -Laplace equation. The  $p(x)$ -Laplace equation is the prototype equation modeling nonstandard growth. Equations exhibiting nonstandard growth appear frequently in various applications. For instance, electrorheological fluids are viscous fluids defined by their capability to drastically change mechanical properties with dependence on an applied electric field. The model for treating the electric field as a variable is characterized by nonstandard growth and is utilized in many technological applications. (See [RR1] and [RR2].) Nonstandard growth conditions also model image enhancement and restoration. For instance, given an observed noisy image, a model exhibiting nonstandard growth can be constructed to exploit general anisotropic diffusion in a way that the speed and diffusion at each location depend on the local behavior. The advantages of this type of model are that it accommodates the local image information. (See [CLR].)

We will explore algebraic and geometric properties of pointwise weak solutions, called viscosity solutions, to the  $p$ -Laplace equation and the  $p(x)$ -Laplace equation. Our main focus will be the  $p(x)$ -Laplace equation in Carnot groups, including specifically in the well-known Heisenberg group  $\mathbb{H}_n$ . In order to

achieve our goal, we must first establish existence-uniqueness of viscosity solutions to the  $p(x)$ -Laplace equation.

Our objective is to expand the well-known Euclidean results into a sub-Riemannian environment. Because of the differing geometric structure, the Euclidean proofs do not directly apply, and so new proofs must be constructed.

We first recall the Laplacian in the Euclidean setting. Let  $\Omega \subset \mathbb{R}^n$  be a (bounded and connected) domain and  $v : \partial\Omega \rightarrow \mathbb{R}$  be a continuous function. The classical Dirichlet boundary value problem concerns finding a function  $u : \bar{\Omega} \rightarrow \mathbb{R}$  of appropriate regularity such that

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = v & \text{on } \partial\Omega. \end{cases}$$

Recall the Laplace equation  $-\Delta u = -\operatorname{div}(\nabla u) = 0$  is linear in that given solutions  $u$  and  $v$  and real numbers  $\alpha$  and  $\beta$ , then  $\alpha u + \beta v$  is also a solution.

We may extend the Dirichlet problem to a fixed  $p$  where  $1 < p < \infty$ . The  $p$ -Dirichlet boundary value problem involves finding a function  $u : \bar{\Omega} \rightarrow \mathbb{R}$  of appropriate regularity such that

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 & \text{in } \Omega \\ u = v & \text{on } \partial\Omega. \end{cases}$$

The above  $p$ -Laplace equation, namely

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0, \tag{1.1}$$

is the Euler-Lagrange equation for the  $p$ -Dirichlet energy integral on  $\Omega$ , which is given by:

$$\frac{1}{p} \int_{\Omega} |\nabla u|^p dx.$$

Note that the Laplacian corresponds to  $p = 2$ .

Observe that the  $p$ -Laplace equation is not linear like the classic Laplace equation. However, it is known that solutions of Equation (1.2) can be scaled. That is, if  $u$  is a solution to the  $p$ -Laplace equation and  $\alpha$  and  $\beta$  are real numbers, then  $\alpha u + \beta$  is also a solution.

The Dirichlet problem can also be extended from the fixed exponent case to the variable exponent case.

Let  $\Omega \in \mathbb{R}^n$  be a domain and assume  $1 < p(x) < \infty$  where  $p(x) \in C^1(\Omega)$  and  $x \in \Omega$ . The  $p(x)$ -Dirichlet boundary value problem involves finding a function  $u : \bar{\Omega} \rightarrow \mathbb{R}$  of appropriate regularity such that

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = 0 & \text{in } \Omega \\ u = v & \text{on } \partial\Omega. \end{cases}$$

The above  $p(x)$ -Laplace equation, namely

$$-\Delta_{p(x)}u = -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = 0, \quad (1.2)$$

is the Euler-Lagrange equation for the  $p(x)$ -Dirichlet energy integral on  $\Omega$ , which is given by:

$$\int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx.$$

Note that the  $p(x)$ -Laplace equation is not linear or in general scalable. Indeed, if  $u$  is a solution to the  $p(x)$ -Laplace equation, then in general,  $\alpha u + \beta$  is not a solution when  $\alpha = \pm 1$ . This leads to major differences among the structure of the classic Laplace, the  $p$ -Laplace, and the  $p(x)$ -Laplace equations. When considering solutions to the classic Laplace equation, representation formulas play a fundamental role. For solutions to the  $p$ -Laplace or the  $p(x)$ -Laplace equation, these representation formulas cannot be employed, but rather, are replaced by estimates. (See [Se] and [Tr].) In the constant exponent case, the standard estimates employed are independent of the solution, whereas in the variable exponent case, these estimates do depend upon the solution itself.

Another major difference between the constant  $p$ -Laplace equation and the variable  $p(x)$ -Laplace equation can be observed when considering each in nondivergent form. Recall the  $p$ -Laplace equation in nondivergent form can be found by formally computing the divergence. This process produces:

$$-\Delta_p u = -(\|\nabla u\|^{p-2} \operatorname{trace}((D^2 u)) + (p-2)\|\nabla u\|^{p-4} \langle (D^2 u) \nabla u, \nabla u \rangle) = 0, \quad (1.3)$$

where  $(D^2 u)$  is the standard Hessian matrix. On the other hand, the same process applied to the  $p(x)$ -



Laplace equation produces the divergence form, which is given by:

$$\begin{aligned}
& - \left( \|\nabla u\|^{p(x)-2} \text{tr}((D^2u)) + (p(x) - 2) \|\nabla u\|^{p(x)-4} \langle (D^2u) \nabla u, \nabla u \rangle \right. \\
& \quad \left. + \|\nabla u\|^{p(x)-2} \log(\|\nabla u\|) \langle \nabla p(x), \nabla u \rangle \right) = 0.
\end{aligned} \tag{1.4}$$

Obviously, in the variable exponent case, there is a log term that does not appear in the constant exponent case.

The Euclidean variable exponent Dirichlet problem can therefore be extended to Carnot groups, which is a major focus of this dissertation. Let  $\Omega \subseteq \mathbb{G}$  be a domain, where  $\mathbb{G}$  is a Carnot group. Also assume  $1 < p(x) < \infty$  for  $x \in \Omega$ . This dissertation is concerned with the  $p(x)$ -Laplace equation in Carnot groups, which is given by

$$-\Delta_{p(x)} u = -\text{div}(\|\nabla_0 u\|^{p(x)-2} \nabla_0 u) = 0. \tag{1.5}$$

Here  $u \in C^1_{\text{sub}}(\Omega)$  and  $\nabla_0$  is the horizontal gradient. (See Chapter 2 for relevant definitions.) Observe that Equation (1.5) is the Euler-Lagrange equation for the  $p(x)$ -Dirichlet energy integral on  $\Omega$ , or

$$\int_{\Omega} \frac{1}{p(x)} |\nabla_0 u|^{p(x)} dx,$$

where  $u : \bar{\Omega} \rightarrow \mathbb{G}$  is of appropriate regularity. In nondivergent form, the  $p(x)$ -Laplace equation in Carnot groups can be formally computed to produce the equation

$$\begin{aligned}
& - \left( \|\nabla_0 u\|^{p(x)-2} \text{trace}((D^2u)^*) + (p(x) - 2) \|\nabla_0 u\|^{p(x)-4} \langle (D^2u)^* \nabla_0 u, \nabla_0 u \rangle \right. \\
& \quad \left. + \|\nabla_0 u\|^{p(x)-2} \log(\|\nabla_0 u\|) \langle \nabla_0 p(x), \nabla_0 u \rangle \right) = 0,
\end{aligned} \tag{1.6}$$

where  $(D^2u)^*$  is the symmetrized horizontal second derivative matrix. (Again, see Section 2.2 for the definitions.) The geometric structure of Carnot groups presents even more difficulties pertaining to the estimates on the solutions to the  $p(x)$ -Laplace equation, as seen in Chapters 3 and 4.

In Chapter 2, we review some definitions and key properties of the Heisenberg group, Carnot groups, and variable exponent Lebesgue and Sobolev spaces. We also present a  $p(x)$ -Poincaré-type inequality in Section 2.3.2 that is necessary in Chapter 3 to achieve the equivalence of potential theoretic weak solutions

and viscosity solutions to the  $p(x)$ -Laplace equation in general Carnot groups, under reasonable restrictions. Then, in Chapter 4, as an application of both the equivalence from Chapter 3 and viscosity theory, we obtain a Radó-type removability theorem for solutions to the  $p(x)$ -Laplace equation in the Heisenberg group.

## Chapter 2

### Background and Motivating Results

#### 2.1 The Heisenberg Group $\mathbb{H}_n$

We first recall some fundamental definitions and key properties of the Heisenberg group  $\mathbb{H}_n$ . We begin with  $\mathbb{R}^{2n+1}$  using the coordinates  $(x_1, x_2, \dots, x_{2n}, x_{2n+1})$ . We consider the vector fields  $\{X_i, X_j, X_{2n+1}\}$ , where the index  $i$  ranges from 1 to  $n$  and the index  $j$  ranges from  $n+1$  to  $2n$ , defined by

$$\begin{aligned} X_1 &:= \frac{\partial}{\partial x_1} - \frac{x_{n+1}}{2} \frac{\partial}{\partial x_{2n+1}} \\ &\vdots \\ X_i &:= \frac{\partial}{\partial x_i} - \frac{x_{n+i}}{2} \frac{\partial}{\partial x_{2n+1}} \\ &\vdots \\ X_n &:= \frac{\partial}{\partial x_n} - \frac{x_{2n}}{2} \frac{\partial}{\partial x_{2n+1}} \\ X_{n+1} &:= \frac{\partial}{\partial x_{n+1}} + \frac{x_1}{2} \frac{\partial}{\partial x_{2n+1}} \\ &\vdots \\ X_j &:= \frac{\partial}{\partial x_j} + \frac{x_{j-n}}{2} \frac{\partial}{\partial x_{2n+1}} \\ &\vdots \\ X_{2n} &:= \frac{\partial}{\partial x_{2n}} + \frac{x_n}{2} \frac{\partial}{\partial x_{2n+1}} \end{aligned}$$

and

$$X_{2n+1} := \frac{\partial}{\partial x_{2n+1}}.$$

These vector fields obey the relations

$$[X_i, X_j] = \begin{cases} X_{2n+1} & j = i + n \\ 0 & \text{otherwise.} \end{cases}$$

For all  $i$  and  $j$ , we also have

$$[X_i, X_{2n+1}] = 0 \quad \text{and} \quad [X_j, X_{2n+1}] = 0.$$

These relations generate a Lie Algebra denoted  $h_n$  that decomposes as a direct sum

$$h_n = V_1 \oplus V_2,$$

where  $V_1$  is spanned by the  $X_i$ 's and  $X_j$ 's, and  $V_2$  is spanned by  $X_{2n+1}$ . We endow  $h_n$  with an inner product  $\langle \cdot, \cdot \rangle$  and related norm  $\| \cdot \|$  so that this basis is orthonormal. The corresponding Lie Group is called the general Heisenberg group of dimension  $n$  and is denoted by  $\mathbb{H}_n$ . With this choice of vector fields the exponential map can be used to identify elements of  $h_n$  and  $\mathbb{H}_n$  with each other via

$$\sum_{k=1}^{2n} x_k X_k + x_{2n+1} X_{2n+1} \in h_n \leftrightarrow (x_1, x_2, \dots, x_{2n}, x_{2n+1}) \in \mathbb{H}_n.$$

In particular, for any  $x, y$  in  $\mathbb{H}_n$ , written as

$$x = (x_1, x_2, \dots, x_{2n}, x_{2n+1}) \quad \text{and} \quad y = (y_1, y_2, \dots, y_{2n}, y_{2n+1}),$$

the group multiplication law is given by

$$\begin{aligned} x \cdot y &= \left( x_1 + y_1, x_2 + y_2, \dots, x_{2n} + y_{2n}, x_{2n+1} + y_{2n+1} + \frac{1}{2} \sum_{i=1}^n (x_i y_{n+i} - x_{n+i} y_i) \right) \\ &= x + y + \left( 0, 0, \frac{1}{2} \sum_{l=1}^n (x_l y_{n+l} - x_{n+l} y_l) \right). \end{aligned}$$

The natural metric on  $\mathbb{H}_n$  is the Carnot-Carathéodory metric given by

$$d_{CC}(x, y) := \inf_{\Gamma} \int_0^1 \|\gamma'(t)\| dt$$

where  $\Gamma$  is the set of all curves  $\gamma$  such that  $\gamma(0) = x, \gamma(1) = y$  and  $\gamma'(t) \in V_1$ . By Chow's theorem, (See, for example, [Be].) any two points can be connected by such a curve, which makes  $d_{CC}(x, y)$  a left-invariant metric on  $\mathbb{H}_n$ . This metric induces a homogeneous norm on  $\mathbb{H}_n$ , denoted  $|\cdot|$ , by

$$|x| = d_{CC}(0, x)$$

and we have the estimate

$$|x| \sim \sum_{k=1}^{2n} |x_k| + |x_{2n+1}|^{\frac{1}{2}}.$$

This estimate leads us to define the left-invariant gauge  $\mathcal{N}$  which is bi-Lipschitz equivalent to the Carnot-Carathéodory metric and is given by

$$\mathcal{N}(x) := \left( \left( \sum_{k=1}^{2n} x_k^2 \right)^2 + 16x_{2n+1}^2 \right)^{\frac{1}{4}}.$$

We define the Heisenberg balls  $B(x, r)$  and the Heisenberg gauge balls  $B_{\mathcal{N}}(x, r)$  in the obvious way.

Given a smooth function  $u : \mathbb{H}_n \rightarrow \mathbb{R}$ , we define the horizontal gradient by

$$\nabla_0 u := (X_1 u, X_2 u, \dots, X_{2n} u),$$

the full gradient by

$$\nabla u := (X_1 u, X_2 u, \dots, X_{2n} u, X_{2n+1} u),$$

and the symmetrized horizontal second derivative matrix  $(D^2 u)^*$  by

$$((D^2 u)^*)_{ab} := \frac{1}{2}(X_a X_b u + X_b X_a u).$$

The main operator we are concerned with is the horizontal  $p(x)$ -Laplacian for  $1 < p(x) < \infty$  defined by

$$\Delta_{p(x)} u := \operatorname{div}(\|\nabla_0 u\|^{p(x)-2} \nabla_0 u),$$

which is a specific type of operator in an important class of operators in potential theory as detailed in [HH] and [HKM].

A function  $u$  is  $C_{\text{sub}}^1(\Omega)$  if  $X_i u, X_j u$  are continuous in  $\Omega$  for all  $i$  and  $j$ ; and  $u$  is  $C_{\text{sub}}^2(\Omega)$  if  $X_a X_b u$  is continuous in  $\Omega$  for all  $1 \leq a \leq 2n$  and  $1 \leq b \leq 2n$ .

We remark that when  $n = 1$ , we have the first Heisenberg group,  $\mathbb{H}_1$ . Using the classical coordinates  $(x, y, z)$ , we consider the vector fields  $\{X, Y, Z\}$  defined by

$$\begin{aligned} X &:= \frac{\partial}{\partial x} - \frac{y}{2} \frac{\partial}{\partial z} \\ Y &:= \frac{\partial}{\partial y} + \frac{x}{2} \frac{\partial}{\partial z} \\ Z &:= \frac{\partial}{\partial z} = [X, Y]. \end{aligned}$$

Note for these vector fields, we have

$$[X, Z] = [Y, Z] = 0.$$

For any two points  $p = (x_1, y_1, z_1)$  and  $q = (x_2, y_2, z_2)$ , the group multiplication law is given by

$$p \cdot q = \left( x_1 + x_2, y_1 + y_2, z_1 + z_2 + \frac{1}{2}(x_1 y_2 - x_2 y_1) \right).$$

Let  $u : \mathbb{H}_1 \rightarrow \mathbb{R}$  be a smooth function. The horizontal gradient of  $u$  is given by

$$\nabla_0 u = (Xu, Yu),$$

the full gradient by

$$\nabla u = (Xu, Yu, Zu),$$

and the symmetrized second derivative matrix  $(D^2 u)^*$  by

$$(D^2 u)^* = \begin{pmatrix} XXu & \frac{1}{2}(XYu + YXu) \\ \frac{1}{2}(XYu + YXu) & YYu \end{pmatrix}.$$

Moreover,  $p(x)$ -Laplace equation in  $\mathbb{H}_n$  for  $1 < p(x) < \infty$  is defined by

$$-\Delta_{p(x)}u := -\operatorname{div} \left( \|\nabla_0 u\|^{p(x)-2} \nabla_0 u \right) = X \left( \|\nabla_0 u\|^{p(x)-2} Xu \right) + Y \left( \|\nabla_0 u\|^{p(x)-2} Yu \right). \quad (2.1)$$

For a more complete treatment of the Heisenberg group, the interested reader is directed to [Be], [B3], [F], [FS] [G], [He], [K], [St] and the references therein.

## 2.2 Carnot Groups

The Heisenberg group  $\mathbb{H}_n$  is the simplest nontrivial Carnot group. We therefore turn our focus to some fundamental definitions and key properties of general Carnot groups. We begin by denoting an arbitrary Carnot group in  $\mathbb{R}^N$  by  $\mathbb{G}$  and its corresponding Lie Algebra by  $g$ . Recall that  $g$  is nilpotent and stratified, resulting in the decomposition

$$g = V_1 \oplus V_2 \oplus \cdots \oplus V_l$$

for appropriate vector spaces that satisfy the Lie bracket relation  $[V_1, V_j] = V_{1+j}$ . The Lie Algebra  $g$  is associated with the group  $\mathbb{G}$  via the exponential map  $\exp : g \rightarrow \mathbb{G}$ . Since this map is a diffeomorphism, we can choose a basis for  $g$  so that it is the identity map. Denote this basis by

$$X_1, X_2, \dots, X_{n_1}, Y_1, Y_2, \dots, Y_{n_2}, Z_1, Z_2, \dots, Z_{n_3}$$

so that

$$\begin{aligned} V_1 &= \operatorname{span}\{X_1, X_2, \dots, X_{n_1}\} \\ V_2 &= \operatorname{span}\{Y_1, Y_2, \dots, Y_{n_2}\} \\ V_3 \oplus V_4 \oplus \cdots \oplus V_l &= \operatorname{span}\{Z_1, Z_2, \dots, Z_{n_3}\}. \end{aligned}$$

We endow  $g$  with an inner product  $\langle \cdot, \cdot \rangle$  and related norm  $\|\cdot\|$  so that this basis is orthonormal. Clearly, the Riemannian dimension of  $g$  (and so  $\mathbb{G}$ ) is  $N = n_1 + n_2 + n_3$ . However, we will also consider the homogeneous dimension of  $\mathbb{G}$ , denoted  $\mathcal{Q}$ , which is given by

$$\mathcal{Q} = \sum_{i=1}^l i \cdot \dim V_i.$$

We also recall that vectors  $X_i$  at the point  $x \in \mathbb{G}$  can be written as

$$X_i(x) = \sum_{j=1}^N a_{ij}(x) \frac{\partial}{\partial x_j}$$

forming the  $n_1 \times N$  matrix  $\mathcal{A}$  with smooth entries  $\mathcal{A}_{ij} = a_{ij}(x)$ .

Before proceeding with the calculus, we recall the group and metric space properties. Since the exponential map is the identity, the group law is the Baker-Campbell-Hausdorff formula. (See, for example, [Bo].)

For our purposes, this formula is given by

$$x \cdot y = x + y + \frac{1}{2}[x, y] + R(x, y) \quad (2.2)$$

where  $R(x, y)$  are terms of order 3 or higher. The identity element of  $\mathbb{G}$  will be denoted by 0 and called the origin. There is also a natural metric on  $\mathbb{G}$ , which is the Carnot-Carathéodory distance, defined for the points  $x$  and  $y$  as

$$d_C(x, y) = \inf_{\Gamma} \int_0^1 \|\gamma'(t)\| dt,$$

where  $\Gamma$  is the set of all curves  $\gamma$  such that  $\gamma(0) = x, \gamma(1) = y$  and  $\gamma'(t) \in V_1$ . By Chow's theorem, (See, for example, [Be].) any two points can be connected by such a curve, which means  $d_C(x, y)$  is an honest metric. Define a Carnot-Carathéodory ball of radius  $r$  centered at a point  $x_0$  by

$$B(x_0, r) = \{x \in \mathbb{G} : d_C(x, x_0) < r\}.$$

In addition to the Carnot-Carathéodory metric, there is a smooth (off the origin) gauge. This gauge is defined for a point  $x = (\zeta_1, \zeta_2, \dots, \zeta_l)$  with  $\zeta_i \in V_i$  by

$$\mathcal{N}(x) = \left( \sum_{i=1}^l \|\zeta_i\|^{\frac{2l}{i}} \right)^{\frac{1}{2l}} \quad (2.3)$$

and it induces a distance  $d_{\mathcal{N}}$  given by

$$d_{\mathcal{N}}(x, y) = \mathcal{N}(x^{-1} \cdot y).$$

This distance is bi-Lipschitz equivalent to the Carnot-Carathéodory metric. We define a gauge ball of radius



$r$  centered at a point  $x_0$  by

$$B_{\mathcal{N}}(x_0, r) = \{x \in \mathbb{G} : d_{\mathcal{N}}(x, x_0) < r\}.$$

In this environment, a smooth function  $u : \mathbb{G} \rightarrow \mathbb{R}$  has the horizontal derivative given by

$$\nabla_0 u = (X_1 u, X_2 u, \dots, X_{n_1} u)$$

and the symmetrized horizontal second derivative matrix, denoted by  $(D^2 u)^*$ , with entries

$$((D^2 u)^*)_{ab} = \frac{1}{2}(X_a X_b u + X_b X_a u)$$

for  $a, b = 1, 2, \dots, n_1$ . We also consider the semi-horizontal derivative given by

$$\nabla_1 u = (X_1 u, X_2 u, \dots, X_{n_1} u, Y_1 u, Y_2 u, \dots, Y_{n_2} u).$$

With these derivatives, we have the following natural definition:

**Definition 2.2.1.** A function  $f : \mathbb{G} \rightarrow \mathbb{R}$  is  $C_{\text{sub}}^1(\mathbb{G})$  if  $\nabla_0 f$  is continuous. A function  $f : \mathbb{G} \rightarrow \mathbb{R}$  is  $C_{\text{sub}}^2(\mathbb{G})$  if  $\nabla_1 f$  and  $X_i X_j f$  is continuous for all  $i, j = 1, 2, \dots, n_1$ .

**Remark 2.2.2.**  $C_{\text{sub}}^2$  is different from Euclidean  $C^2$ . Consider the function  $u : \mathbb{H}_1 \rightarrow \mathbb{R}$  defined by  $u(x, y, z) = z^{\frac{3}{2}}$ . Then all Euclidean second derivatives are 0, except

$$\frac{\partial^2}{\partial z^2} = \frac{3}{4z^{\frac{1}{2}}},$$

which clearly does not exist at the origin. However, the  $Z$  vector field in  $\mathbb{H}_1$  is a second derivative, and the second partial derivative of  $u$  with respect to  $z$  in  $\mathbb{H}_1$  is

$$Zu = \frac{3}{2}z^{\frac{1}{2}},$$

which clearly exists at the origin.

We next let  $p : \mathbb{G} \rightarrow (1, \infty)$ , called a variable exponent, be in  $C(\mathbb{G})$  and let  $\Omega$  be a bounded domain in  $\mathbb{G}$ . Using the variable exponent  $p(x)$ , we define the  $p(x)$ -Laplacian of a smooth function  $u$  for  $1 < p(x) < \infty$

by

$$\begin{aligned}
\Delta_{p(x)}u &= \operatorname{div}(\|\nabla_0 u\|^{p(x)-2}\nabla_0 u) = \sum_{i=1}^{n_1} X_i(\|\nabla_0 u\|^{p(x)-2}\nabla_0 u) \\
&= \|\nabla_0 u\|^{p(x)-2} \operatorname{trace}((D^2 u)^*) + (p(x) - 2)\|\nabla_0 u\|^{p(x)-4} \langle (D^2 u)^* \nabla_0 u, \nabla_0 u \rangle \\
&\quad + \log(\|\nabla_0 u\|)\|\nabla_0 u\|^{p(x)-2} \langle \nabla_0 p(x) u, \nabla_0 u \rangle.
\end{aligned}$$

Note that just as in the Euclidean case, there is a first-order term involving  $\log(\|\nabla_0 u\|)$  that does not appear in the case when  $p(x)$  is constant. Also note that if  $p(x)$  is constant, then we have the standard  $p$ -Laplacian in Carnot groups. (See [B1].)

## 2.3 Variable Exponent Lebesgue and Sobolev Spaces

### 2.3.1 Variable Exponent Lebesgue and Sobolev Spaces in General Carnot Groups

In this section, we review some key properties of variable exponent Lebesgue spaces and Sobolev spaces employing the variable exponent. Let  $\Omega$  be a bounded domain in a Carnot group  $\mathbb{G}$ . (Note that  $\mathbb{G}$  could be the Heisenberg group  $\mathbb{H}_n$ .) Let the variable exponent  $p : \Omega \rightarrow [1, \infty]$  be a measurable function. We denote the following

$$p^+ = \sup_{x \in \Omega} p(x) \quad \text{and} \quad p^- = \inf_{x \in \Omega} p(x)$$

and we will assume throughout this section that

$$1 < p^- \leq p^+ < \infty$$

holds in compact subsets of  $\Omega$ .

We define the variable exponent Lebesgue space as in [Lu1]:  $L^{p(x)}(\Omega)$  is the space of measurable functions  $u$  on  $\Omega$  such that the modular  $\varrho_{p(x)}$  satisfies

$$\varrho_{p(x)}(u) = \int_{\Omega} |u(x)|^{p(x)} dx < \infty.$$

Moreover, we use the Luxemburg norm:

$$\|u\|_{L^{p(x)}(\Omega)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}.$$

Note that because  $p^+ < \infty$ ,  $L^{p(x)}(\Omega)$  equipped with this norm is a Banach space. Also note that if  $p(x)$  is constant, then  $L^{p(x)}(\Omega)$  reduces to the standard Lebesgue space  $L^p(\Omega)$ .

The definition of the norm produces the following relationship between the modular and the norm:

$$\min\{\|u\|_{L^{p(x)}(\Omega)}^{p^+}, \|u\|_{L^{p(x)}(\Omega)}^{p^-}\} \leq \varrho_{p(x)}(u) \leq \max\{\|u\|_{L^{p(x)}(\Omega)}^{p^+}, \|u\|_{L^{p(x)}(\Omega)}^{p^-}\}. \quad (2.4)$$

**Lemma 2.3.1** ([BF1], Propostion 3.1). *These inequalities (2.4) directly imply for any sequence  $\{u_k\}_{k \in \mathbb{N}} \xrightarrow{k \rightarrow \infty} u$ , we have:*

$$\varrho_{p(x)}(u - u_k) \rightarrow 0 \iff \|u - u_k\|_{L^{p(x)}(\Omega)} \rightarrow 0 \quad (2.5)$$

as  $k \rightarrow \infty$ .

*Proof.* Assume  $\|u - u_k\|_{L^{p(x)}(\Omega)} \rightarrow 0$  as  $k \rightarrow \infty$ . We then have

$$\|u - u_k\|_{L^{p(x)}(\Omega)}^{p^-} \rightarrow 0 \text{ and } \|u - u_k\|_{L^{p(x)}(\Omega)}^{p^+} \rightarrow 0.$$

By inequality (2.4) we have  $\varrho_{p(x)}(u - u_k) \rightarrow 0$ .

Now assume  $\varrho_{p(x)}(u - u_k) \rightarrow 0$  as  $k \rightarrow \infty$ . By definition we have

$$\int_{\Omega} |u(x) - u_k(x)|^{p(x)} dx \rightarrow 0.$$

Given  $\varepsilon > 0$ , choose  $k_0$  so that

$$\int_{\Omega} |u(x) - u_{k_0}(x)|^{p(x)} dx < \varepsilon.$$

Because for  $\lambda > 0$ , we have

$$\frac{1}{\lambda^{p(x)}} \leq \frac{1}{\lambda^{p^-}}$$

we then conclude

$$\int_{\Omega} \left| \frac{u(x) - u_{k_0}(x)}{\lambda} \right|^{p(x)} dx \leq \varepsilon \frac{1}{\lambda^{p^-}}.$$

Therefore,

$$\|u - u_{k_0}\|_{L^{p(x)}(\Omega)} = \varepsilon^{\frac{1}{p^-}}.$$

The result follows since  $\varepsilon$  was arbitrary. □

The following Lemma gives key properties of the modular.

**Lemma 2.3.2** ([BF1], Lemma 3.2). *Let  $\varrho_{\mathbb{P}(x)}(u)$  be defined as above. Then:*

- a)  $\varrho_{\mathbb{P}(x)}(u)$  is convex,
- b)  $\varrho_{\mathbb{P}(x)}(u) = 0$  if and only if  $u = 0$ ,
- c) if  $0 < \varrho_{\mathbb{P}(x)}(u) < \infty$ , then  $\lambda \mapsto \varrho_{\mathbb{P}(x)}\left(\frac{u}{\lambda}\right)$  is continuous and decreasing on the interval  $[1, \infty)$ ,
- d)  $\varrho_{\mathbb{P}(x)}\left(\frac{u}{\|u\|_{L^{\mathbb{P}(x)}(\Omega)}}\right) \leq 1$  for every  $u$  with  $0 < \|u\|_{L^{\mathbb{P}(x)}(\Omega)} < \infty$ .

*Proof.* Let  $u, v \in L^{\mathbb{P}(x)}(\Omega)$ . Then for all  $t \in [0, 1]$ , we have

$$\begin{aligned} \varrho_{\mathbb{P}(x)}(tu(x) + (1-t)v(x)) &= \int_{\Omega} |tu + (1-t)v|^{\mathbb{P}(x)} dx \\ &\leq \int_{\Omega} |tu|^{\mathbb{P}(x)} dx + \int_{\Omega} |(1-t)v|^{\mathbb{P}(x)} dx = \int_{\Omega} t^{\mathbb{P}(x)} |u|^{\mathbb{P}(x)} dx + \int_{\Omega} (1-t)^{\mathbb{P}(x)} |v|^{\mathbb{P}(x)} dx \\ &\leq \int_{\Omega} t |u|^{\mathbb{P}(x)} dx + \int_{\Omega} (1-t) |v|^{\mathbb{P}(x)} dx = t\varrho_{\mathbb{P}(x)}(u) + (1-t)\varrho_{\mathbb{P}(x)}(v) \end{aligned}$$

so Property a) holds. Properties b) and c) are straightforward and omitted. Property d) is proved in [KR, Lemma 2.9]. □

We then have the following corollary of Properties a), b), and d) of Lemma 2.3.2:

**Corollary 2.3.3** ([BF1], Corollary 3.3). *If  $\|u\|_{L^{\mathbb{P}(x)}(\Omega)} \leq 1$ , then  $\varrho_{\mathbb{P}(x)}(u) \leq \|u\|_{L^{\mathbb{P}(x)}(\Omega)}$ .*

*Proof.* Assume  $\|u\|_{L^{\mathbb{P}(x)}(\Omega)} \leq 1$ . If  $\|u\|_{L^{\mathbb{P}(x)}(\Omega)} = 0$ , then Equation (2.4) implies  $\varrho_{\mathbb{P}(x)}(u) = 0$ , which implies  $u = 0$  by Property b) of Lemma 2.3.2, and the claim is true trivially. We therefore assume that  $0 < \|u\|_{L^{\mathbb{P}(x)}(\Omega)} \leq 1$ . Then by Property d) of Lemma 2.3.2,

$$\varrho_{\mathbb{P}(x)}\left(\frac{u}{\|u\|_{L^{\mathbb{P}(x)}(\Omega)}}\right) \leq 1.$$

Since  $\|u\|_{L^{\mathbb{P}(x)}(\Omega)} \leq 1$  and also since  $\varrho$  is convex by Property a) of Lemma 2.3.2, we have, writing  $\|u\|$  for  $\|u\|_{L^{\mathbb{P}(x)}(\Omega)}$ ,

$$\varrho_{\mathbb{P}(x)}(u) = \varrho_{\mathbb{P}(x)}\left(\frac{\|u\|u}{\|u\|}\right) \leq \|u\| \varrho_{\mathbb{P}(x)}\left(\frac{u}{\|u\|}\right) \leq \|u\|.$$

□

Given functions  $u \in L^{p(x)}(\Omega)$  and  $v \in L^{q(x)}(\Omega)$ , where the conjugate exponent  $q(x)$  of  $p(x)$  is defined pointwise, we have a form of Hölder's inequality (cf. [KR, Theorem 2.1], [DHHR, Lemma 3.2.20]):

$$\int_{\Omega} |u| |v| dx \leq 2 \|u\|_{L^{p(x)}(\Omega)} \|v\|_{L^{q(x)}(\Omega)}. \quad (2.6)$$

Additionally, if  $1 < p(x)^- \leq p(x)^+ < \infty$ , then the dual of  $L^{p(x)}(\Omega)$  is  $L^{q(x)}(\Omega)$  and  $L^{p(x)}(\Omega)$  is reflexive.

We finish this section by recalling some key properties of variable exponent Sobolev spaces for Carnot groups. Let  $\Omega \subset \mathbb{G}$  be a domain in  $\mathbb{G}$ . We will use the following notation and definition for the variable exponent Sobolev space  $W^{1,p(x)}(\mathbb{G})$  with  $p^+ < \infty$  (cf. [HHP1]):

$$W^{1,p(x)}(\mathbb{G}) = \left\{ f \in L^{p(\cdot)}(\mathbb{G}), |\nabla_0 f| \in L^{p(\cdot)}(\mathbb{G}) : \int_{\mathbb{G}} |f(x)|^{p(x)} + |\nabla_0 f(x)|^{p(x)} dx < \infty \right\},$$

where we use the norm:

$$\|f\|_{W^{1,p(x)}(\mathbb{G})} = \|f\|_{L^{p(x)}(\mathbb{G})} + \|\nabla_0 f\|_{L^{p(x)}(\mathbb{G})},$$

which makes  $W^{1,p(x)}(\mathbb{G})$  a Banach space ([HHP1, Theorem 3.4]). Similarly, we define the variable exponent Sobolev space  $W^{1,p(x)}(\Omega)$  for  $p^+ < \infty$  as:

$$W^{1,p(x)}(\Omega) = \left\{ f \in L^{p(\cdot)}(\Omega), |\nabla_0 f| \in L^{p(\cdot)}(\Omega) : \int_{\Omega} |f(x)|^{p(x)} + |\nabla_0 f(x)|^{p(x)} dx < \infty \right\}.$$

Replacing  $L^{p(x)}(\Omega)$  by  $L_{\text{loc}}^{p(x)}(\Omega)$ , we define the space  $W_{\text{loc}}^{1,p(x)}(\Omega)$ , which consists of functions  $f$  that belong to  $W_{\text{loc}}^{1,p(x)}(\Omega')$  for all open sets  $\Omega' \Subset \Omega$ , in the natural way.

Lastly, we define the function  $\varrho_{1,p(\cdot)} : W^{1,p(\cdot)}(\Omega) \rightarrow [0, \infty)$  by

$$\varrho_{1,p(\cdot)}(f) = \varrho_{p(\cdot)}(f) + \varrho_{p(\cdot)}(|\nabla_0 f|).$$

### 2.3.2 A $p(\cdot)$ -Poincaré-type Inequality for Variable Exponent Sobolev Spaces with Zero Boundary Values in Carnot Groups <sup>1</sup>

We will need a  $p(\cdot)$ -Poincaré-type inequality to achieve some of our results, namely for the equivalence of potential theoretic weak solutions and viscosity solutions to the  $p(x)$ -Laplace equation. We begin by defining Sobolev  $p(\cdot)$ -capacity and quasicontinuity in the Carnot group setting. These definitions are adopted

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<sup>1</sup>A Note to Reader: This section has been reproduced from [BF1].

from the metric space version of the variable exponent case. (See [HHP1].) For the metric space version of the fixed exponent case, see [KM].

**Definition 2.3.4.** For  $\mathcal{U} \subset \mathbb{G}$ , denote

$$S_{p(\cdot)}(\mathcal{U}) = \{u \in W^{1,p(\cdot)}(\mathbb{G}) : u \geq 1 \text{ in an open set containing } \mathcal{U}\}.$$

Functions in  $S_{p(\cdot)}(\mathcal{U})$  are said to be  $p(\cdot)$ -admissible for  $\mathcal{U}$ . We note that since the norm in  $W^{1,p(\cdot)}(\mathbb{G})$  decreases under truncation, we can choose those  $u \in \mathcal{U}$  such that  $0 \leq u \leq 1$ . The Sobolev  $p(\cdot)$ -capacity of  $\mathcal{U}$  is then defined as:

$$C_{p(\cdot)}(\mathcal{U}) = \inf_{u \in S_{p(\cdot)}(\mathcal{U})} \varrho_{1,p(\cdot)}(u) = \inf_{u \in S_{p(\cdot)}(\mathcal{U})} \int_{\mathbb{G}} (|u(x)|^{p(x)} + |\nabla_0 u(x)|^{p(x)}) dx.$$

In the case that  $S_{p(\cdot)}(\mathcal{U}) = \emptyset$ , we set  $C_{p(\cdot)}(\mathcal{U}) = \infty$ . Furthermore, by standard arguments, the set function  $\mathcal{U} \mapsto C_{p(\cdot)}(\mathcal{U})$  is an outer measure. The proof of the next lemma follows the Euclidean case ([HHKV1, Theorem 3.1]) and is omitted.

**Lemma 2.3.5.** *The set function  $\mathcal{U} \mapsto C_{p(\cdot)}(\mathcal{U})$  is an outer measure. In other words,*

- i)  $C_{p(\cdot)}(\emptyset) = 0$ .
- ii) [Monotonicity] If  $\mathcal{U}_1 \subset \mathcal{U}_2$ , then  $C_{p(\cdot)}(\mathcal{U}_1) \leq C_{p(\cdot)}(\mathcal{U}_2)$ .
- iii) [Subadditivity] If  $\mathcal{U}_i \subset \mathbb{G}$  for  $i = 1, 2, \dots$ , then

$$C_{p(\cdot)}\left(\bigcup_{i=1}^{\infty} \mathcal{U}_i\right) \leq \sum_{i=1}^{\infty} C_{p(\cdot)}(\mathcal{U}_i).$$

**Lemma 2.3.6** ([BF1, Lemma 3.5]). *The set function  $\mathcal{U} \mapsto C_{p(\cdot)}(\mathcal{U})$  is an outer capacity.*

*Proof.* To prove we have an outer capacity, we must show

- i)  $C_{p(\cdot)}(\emptyset) = 0$ .
- ii) If  $\mathcal{U}_1 \subset \mathcal{U}_2$ , then  $C_{p(\cdot)}(\mathcal{U}_1) \leq C_{p(\cdot)}(\mathcal{U}_2)$ .
- iii) Let  $\mathcal{U} \subset \mathbb{G}$ . Then for all open subsets  $\Omega$  and  $\mathcal{V}$  such that  $\mathcal{U} \subset \mathcal{V} \subset \Omega \subset \mathbb{G}$ , we have

$$C_{p(\cdot)}(\mathcal{U}) = \inf_{\mathcal{U} \subset \mathcal{V} \subset \Omega} C_{p(\cdot)}(\mathcal{V}).$$

Items i) and ii) are proved in Lemma 2.3.5 so we show iii). Let  $\mathcal{U} \subset \mathbb{G}$  and  $\mathcal{V} \subset \Omega$  be open such that  $\mathcal{U} \subset \mathcal{V} \subset \Omega$ . By ii) we have  $C_{p(\cdot)}(\mathcal{U}) \leq C_{p(\cdot)}(\mathcal{V})$ , which implies

$$C_{p(\cdot)}(\mathcal{U}) \leq \inf_{\mathcal{U} \subset \mathcal{V} \subset \Omega} C_{p(\cdot)}(\mathcal{V}).$$

On the other hand, fix  $\varepsilon > 0$  and let  $\mathcal{U} \subset \mathbb{G}$  and  $\mathcal{V} \subset \Omega$  be open sets such that  $\mathcal{U} \subset \mathcal{V} \subset \Omega$ . Then by ii) again, we have  $C_{p(\cdot)}(\mathcal{U}) \leq C_{p(\cdot)}(\mathcal{V})$  so we can find a function  $u \in S_{p(\cdot)}(\mathcal{U})$  such that  $\mathcal{U} \subset \mathcal{V}$  and

$$C_{p(\cdot)}(\mathcal{U}) \leq \inf_{\mathcal{U} \subset \mathcal{V} \subset \Omega} C_{p(\cdot)}(\mathcal{V}) \leq C_{p(\cdot)}(\mathcal{V}) \leq \int_{\mathbb{G}} (|u(x)|^{p(x)} + |\nabla_0 u(x)|^{p(x)}) dx \leq C_{p(\cdot)}(\mathcal{U}) + \varepsilon,$$

so iii) follows by letting  $\varepsilon \rightarrow 0$ . □

**Definition 2.3.7.** A function  $u : \mathbb{G} \rightarrow \mathbb{R}$  is said to be  $p(\cdot)$ -quasicontinuous in  $\mathbb{G}$  if for every  $\varepsilon > 0$  there exists an open set  $\Omega$  with  $C_{p(\cdot)}(\Omega) < \varepsilon$  such that  $u$  is continuous on  $\mathbb{G} \setminus \Omega$ . Moreover, for a subset  $\mathcal{U}$  of  $\mathbb{G}$ , we say that a claim holds  $p(\cdot)$ -quasieverywhere in  $\mathcal{U}$  if it holds everywhere except possibly in a set  $K \subset \mathcal{U}$  where  $K$  has zero  $p(\cdot)$ -capacity.

We will need the following lemma in order to show the uniqueness result of the minimizer of the  $p(\cdot)$ -Dirichlet energy integral. Kilpeläinen [Ki] gives a more general topological proof of Statement (i) for any outer capacity. Statement (ii) is well-known in the fixed exponent case [KKM, Remark 3.3]. The proof in the variable exponent case is identical and omitted.

**Lemma 2.3.8.** *Let  $1 < p^- \leq p^+ < \infty$ , and let  $u, v$  be  $p(\cdot)$ -quasicontinuous functions in  $\mathbb{G}$ . Suppose that  $\Omega \subset \mathbb{G}$  is open. Then we have the following:*

(i) *If  $u = v$  a.e. in  $\Omega$ , then  $u = v$   $p(\cdot)$ -quasieverywhere in  $\Omega$ .*

(ii) *If  $u \leq v$  a.e. in  $\Omega$ , then  $u \leq v$   $p(\cdot)$ -quasieverywhere in  $\Omega$ .*

Now we consider a Sobolev  $p(\cdot)$ -capacity based on  $p(\cdot)$ -quasicontinuous functions. For  $\mathcal{U} \subset \mathbb{G}$  and  $1 < p^- \leq p^+ < \infty$ , we let

$$\tilde{S}_{p(\cdot)}(\mathcal{U}) = \{u \in W^{1,p(\cdot)}(\mathbb{G}) : u \text{ is } p(\cdot)\text{-quasicontinuous and } u \geq 1 \text{ } p(\cdot)\text{-quasieverywhere in } \mathcal{U}\},$$

and then define

$$\tilde{C}_{\mathfrak{p}(\cdot)}(\mathcal{U}) = \inf_{u \in \tilde{S}_{\mathfrak{p}(\cdot)}(\mathcal{U})} \int_{\mathbb{G}} (|u(x)|^{\mathfrak{p}(x)} + |\nabla_0 u(x)|^{\mathfrak{p}(x)}) dx.$$

As above, in the case that  $\tilde{S}_{\mathfrak{p}(\cdot)}(\mathcal{U}) = \emptyset$ , we set  $\tilde{C}_{\mathfrak{p}(\cdot)}(\mathcal{U}) = \infty$ .

Because Carnot groups are locally compact doubling spaces, they satisfy the so-called density condition that continuous functions with compact support are dense in  $L^{\mathfrak{p}(\cdot)}(\mathbb{G})$ . [HHP2, Theorem 3.3]. (Recall that  $\mathfrak{p}^+ < \infty$ .) This fact gives us the next theorem. To prove it, we will need some lemmas. The Euclidean case for each of the lemmas is given in the citations. The proofs are identical and omitted.

**Lemma 2.3.9.** [HHKV1, Lemma 5.1] *Let  $1 < \mathfrak{p}^- \leq \mathfrak{p}^+ < \infty$ . For every Cauchy sequence of functions  $\{u_i\}_{i \in \mathbb{N}}$  such that for all  $i \in \mathbb{N}$ ,  $u_i$  is continuous and  $u_i \in W^{1, \mathfrak{p}(\cdot)}(\mathbb{G})$ , there is a subsequence of  $\{u_i\}$  converging to  $u$  pointwise  $\mathfrak{p}(\cdot)$ -quasieverywhere in  $\mathbb{G}$ . Additionally, outside a set of arbitrary small  $\mathfrak{p}(\cdot)$ -capacity, the convergence is uniform.*

**Lemma 2.3.10.** [HHKV1, Theorem 5.2] *Let  $\mathfrak{p}$  satisfy the density condition with  $1 < \mathfrak{p}^- \leq \mathfrak{p}^+ < \infty$ . For every  $u \in W^{1, \mathfrak{p}(\cdot)}(\mathbb{G})$ , there exists a  $\mathfrak{p}(\cdot)$ -quasicontinuous function  $v \in W^{1, \mathfrak{p}(\cdot)}(\mathbb{G})$  such that  $u = v$  almost everywhere in  $\mathbb{G}$ .*

**Theorem 2.3.11** ([BF1], Theorem 3.9). *If  $1 < \mathfrak{p}^- \leq \mathfrak{p}^+ < \infty$  and  $\mathcal{U} \in \mathbb{G}$ , then  $C_{\mathfrak{p}(\cdot)}(\mathcal{U}) \leq \tilde{C}_{\mathfrak{p}(\cdot)}(\mathcal{U})$ . Moreover, in Carnot groups, we have equality.*

*Proof.* The proof parallels the proof of Theorem 2.2 (a) in [HHKV1], which proves this condition in Euclidean space and follows the proof of Theorem 3.4 in [KKM] for the fixed exponent case in metric measure spaces. We will need the following standard inequality (see, for instance, [MZ, Lemma 1.1]) for arbitrary  $\zeta, \eta \in \mathbb{R}$  and every  $\delta > 0$ :

$$|\zeta + \eta|^m \leq (1 + \delta)^{m-1} |\zeta|^m + \left(1 + \frac{1}{\delta}\right)^{m-1} |\eta|^m \quad (2.7)$$

for  $1 \leq m < \infty$ . Let  $v \in \tilde{S}_{\mathfrak{p}(\cdot)}(\mathcal{U})$  and, by truncation, assume that  $0 \leq v \leq 1$ . Fix  $0 < \varepsilon < 1$  and choose open set  $\mathcal{V}$  with  $C_{\mathfrak{p}(\cdot)}(\mathcal{V}) < \varepsilon$  so that  $v = 1$  on  $\mathcal{U} \setminus \mathcal{V}$  and so that  $v$  restricted to  $\mathbb{G} \setminus \mathcal{V}$  is continuous. Also define the set

$$\mathcal{W} = \{x \in \mathbb{G} \setminus \mathcal{V} \mid v(x) > 1 - \varepsilon\} \cup \mathcal{V}.$$



Then  $\mathcal{U} \setminus \mathcal{V} \subset \mathcal{W} \setminus \mathcal{V}$  by definition. Next, choose  $u \in S_{p(\cdot)}(\mathcal{V})$  such that

$$\int_{\mathcal{V}} (|u(x)|^{p(x)} + |\nabla_0 u(x)|^{p(x)}) dx < \varepsilon$$

and such that  $0 \leq u \leq 1$ . Therefore, by definition of  $S_{p(\cdot)}(\mathcal{V})$ , we must have  $u = 1$  in an open set containing  $\mathcal{V}$ . Defining  $z = \frac{v}{1-\varepsilon} + u$  implies  $z \geq 1$  almost everywhere in  $(\mathcal{W} \setminus \mathcal{V}) \cup \mathcal{V} = \mathcal{W}$ . Since  $\mathcal{W}$  is an open neighborhood of  $\mathcal{U}$ , then  $z \in S_{p(\cdot)}(\mathcal{U})$ . Then we have by Equation (2.7), for every  $\delta > 0$ ,

$$\begin{aligned} \varrho_{p(\cdot)}(z) &= \int_{\mathbb{G}} \left| \frac{v(x)}{1-\varepsilon} + u(x) \right|^{p(x)} dx \\ &\leq \int_{\mathbb{G}} \left| \frac{v(x)}{1-\varepsilon} \right|^{p(x)} (1+\delta)^{p(x)-1} dx + \int_{\mathbb{G}} |u(x)|^{p(x)} \left(1 + \frac{1}{\delta}\right)^{p(x)-1} dx \\ &\leq (1+\delta)^{p^+-1} \int_{\mathbb{G}} \left| \frac{v(x)}{1-\varepsilon} \right|^{p(x)} dx + \left(1 + \frac{1}{\delta}\right)^{p^+-1} \int_{\mathbb{G}} |u(x)|^{p(x)} dx \\ &\leq (1+\delta)^{p^+-1} \int_{\mathbb{G}} \frac{|v(x)|^{p(x)}}{(1-\varepsilon)^{p(x)}} dx + \left(1 + \frac{1}{\delta}\right)^{p^+-1} \int_{\mathbb{G}} |u(x)|^{p(x)} dx \\ &\leq (1+\delta)^{p^+-1} \int_{\mathbb{G}} \frac{|v(x)|^{p(x)}}{(1-\varepsilon)^{p^+}} dx + \left(1 + \frac{1}{\delta}\right)^{p^+-1} \int_{\mathbb{G}} |u(x)|^{p(x)} dx \\ &< \frac{(1+\delta)^{p^+-1}}{(1-\varepsilon)^{p^+}} \int_{\mathbb{G}} |v(x)|^{p(x)} dx + \left(1 + \frac{1}{\delta}\right)^{p^+-1} \varepsilon \\ &= \frac{(1+\delta)^{p^+}}{(1-\varepsilon)^{p^+}} \frac{1}{1+\delta} \int_{\mathbb{G}} |v(x)|^{p(x)} dx + \left(1 + \frac{1}{\delta}\right)^{p^+} \frac{\delta}{1+\delta} \varepsilon \\ &\leq \left(\frac{1+\delta}{1-\varepsilon}\right)^{p^+} \int_{\mathbb{G}} |v(x)|^{p(x)} dx + \left(1 + \frac{1}{\delta}\right)^{p^+} \varepsilon, \end{aligned}$$

where the strict inequality follows from the choice of  $u$  such that  $\varrho_{1,p(\cdot)}(u) < \varepsilon$  and so

$$\int_{\mathbb{G}} |u(x)|^{p(x)} dx < \varepsilon.$$

Now choosing  $\delta = \varepsilon^{\frac{1}{2p^+}}$  yields

$$\left(\frac{1+\delta}{1-\varepsilon}\right)^{p^+} = \left(\frac{1+\varepsilon^{\frac{1}{2p^+}}}{1-\varepsilon}\right)^{p^+} \rightarrow 1$$

and

$$\left(1 + \frac{1}{\delta}\right)^{p^+} \varepsilon = \left(1 + \frac{1}{\varepsilon^{\frac{1}{2p^+}}}\right)^{p^+} \varepsilon^{\left(\frac{1}{p^+}\right)p^+} = \left(\varepsilon^{\frac{1}{p^+}} + \varepsilon^{\frac{1}{2p^+}}\right)^{p^+} \rightarrow 0$$

as  $\varepsilon \rightarrow 0$ . Therefore,

$$\varrho_{p(\cdot)}(z) \leq \int_{\mathbb{G}} |v(x)|^{p(x)} dx = \varrho_{p(\cdot)}(v). \quad (2.8)$$

Similarly, we can show that

$$\varrho_{p(\cdot)}(|\nabla_0 z|) \leq \int_{\mathbb{G}} |\nabla_0 v(x)|^{p(x)} dx = \varrho_{p(\cdot)}(|\nabla_0 v|), \quad (2.9)$$

where the strict inequality comes from the choice of  $u$  such that  $\varrho_{1,p(\cdot)}(u) < \varepsilon$  and so

$$\int_{\mathbb{G}} |\nabla_0 u(x)|^{p(x)} dx < \varepsilon.$$

Equations (2.8) and (2.9) imply  $\varrho_{1,p(\cdot)}(z) \leq \varrho_{1,p(\cdot)}(v)$ , and since  $v$  was chosen arbitrarily, then we have  $C_{p(\cdot)}(\mathcal{U}) \leq \tilde{C}_{p(\cdot)}(\mathcal{U})$ .

Now we assume that  $p$  satisfies the density condition and we finish the proof of the theorem by using Lemma 2.3.10 to show the reverse inequality, namely  $\tilde{C}_{p(\cdot)}(\mathcal{U}) \leq C_{p(\cdot)}(\mathcal{U})$ . Let  $\mathcal{U} \subset \mathbb{G}$ . Choose  $u \in S_{p(\cdot)}(\mathcal{U})$  and let  $\Omega$  be open with  $\mathcal{U} \subset \Omega$  and such that  $u \geq 1$  on  $\Omega$ . Then Lemma 2.3.10 gives us the existence of a  $p(\cdot)$ -quasicontinuous function  $\tilde{u} \in \mathbb{G}$  such that  $\tilde{u} = u$  a.e in  $\Omega$ . Hence,  $\tilde{u} \geq 1$  a.e. in  $\Omega$ . By Lemma 2.3.8 we have  $\tilde{u} \geq 1$   $p(\cdot)$ -quasieverywhere in  $\Omega$ . It follows that  $\tilde{u} \geq 1$   $p(\cdot)$ -quasieverywhere in  $\mathcal{U}$  so  $\tilde{u} \in \tilde{S}_{p(\cdot)}(\mathcal{U})$ . Therefore,  $\tilde{C}_{p(\cdot)}(\mathcal{U}) \leq C_{p(\cdot)}(\mathcal{U})$  and equality follows.  $\square$

The next lemma is an extension of Lemma 2.3.9 in that the regularity of the functions  $u_i$  is relaxed. The fixed exponent metric measure space case corresponds to [KKM, Lemma 3.5] and a sharpening of that statement in variable exponent Euclidean case corresponds to [HHKV1, Lemma 2.3]. We will use the latter case since the additional result that  $u$  is  $p(\cdot)$ -quasicontinuous is needed to prove Theorem 2.3.14.

**Lemma 2.3.12** ([BF1], Lemma 3.10). *Let  $1 < p^- \leq p^+ < \infty$ . Suppose that  $\{u_i\}_{i \in \mathbb{N}} \subset W^{1,p(\cdot)}(\mathbb{G})$  is a sequence of  $p(\cdot)$ -quasicontinuous functions that converge in  $W^{1,p(\cdot)}(\mathbb{G})$  to the function  $u$ . Then  $u$  is  $p(\cdot)$ -quasicontinuous and there is a subsequence  $\{u_{i_k}\}_{k \in \mathbb{N}}$  that converges pointwise to  $u$   $p(\cdot)$ -quasieverywhere in  $\mathbb{G}$ .*

*Proof.* There exists a subsequence of  $\{u_i\}_{i \in \mathbb{N}}$ , which we also denote  $\{u_i\}_{i \in \mathbb{N}}$ , such that

$$\sum_{i=1}^{\infty} 2^{i p^+} \|u - u_i\|_{W^{1,p(x)}} < 1.$$

For  $i = 1, 2, \dots$ , we denote  $\mathcal{U}_i = \{x \in G : |u_i(x) - u_{i+1}(x)| > 2^{-i}\}$  and  $\mathcal{V}_j = \bigcup_{i=j}^{\infty} \mathcal{U}_i$ .

Then clearly  $2^i |u_i - u_{i+1}| \in \tilde{S}_{p(\cdot)}(\mathcal{U}_i)$ . By Theorem 2.3.11, we have

$$C_{p(\cdot)}(\mathcal{U}_i) \leq \int_{\mathbb{G}} (2^i |u_i - u_{i+1}|)^{p(x)} + |\nabla_0(2^i(u_i - u_{i+1}))|^{p(x)} dx \leq 2^{i p^+} \|u_i - u_{i+1}\|_{W^{1,p(x)}}.$$

By the subadditivity of the Sobolev  $p(\cdot)$ -capacity, we then have

$$C_{p(\cdot)}(\mathcal{V}_j) \leq \sum_{i=j}^{\infty} C_{p(\cdot)}(\mathcal{U}_i) \leq \sum_{i=j}^{\infty} 2^{i p^+} \|u_i - u_{i+1}\|_{W^{1,p(x)}}.$$

Since  $\bigcap_{j=1}^{\infty} \mathcal{V}_j \subset \mathcal{V}_j$  for every  $j$ , the monotonicity of the Sobolev  $p(\cdot)$ -capacity implies

$$C_{p(\cdot)}\left(\bigcap_{j=1}^{\infty} \mathcal{V}_j\right) \leq \lim_{j \rightarrow \infty} C_{p(\cdot)}(\mathcal{V}_j) = 0.$$

Furthermore, we have  $u_i \rightarrow u$  pointwise in  $\mathbb{G} \setminus \bigcap_{j=1}^{\infty} \mathcal{V}_j$ , and so the convergence  $p(\cdot)$ -quasieverywhere in  $\mathbb{G}$  follows.

It remains to show that  $u$  is  $p(\cdot)$ -quasicontinuous. This means we must show that for every  $\varepsilon > 0$  there exists an open set with Sobolev  $p(\cdot)$ -capacity less than  $\varepsilon$  such that  $u$  is continuous in the complement. Let  $\varepsilon > 0$ . Using the first half of this proof, there is a set  $\mathcal{V}_j \subset \mathbb{G}$  such that  $C_{p(\cdot)}(\mathcal{V}_j) < \frac{\varepsilon}{2}$  and such that  $u_i \rightarrow u$  pointwise in  $\mathbb{G} \setminus \mathcal{V}_j$ . Since each  $u_i$  is  $p(\cdot)$ -quasicontinuous in  $\mathbb{G}$  by assumption, for each  $i \in \mathbb{N}$  we can choose open sets  $\mathcal{W}_i \subset \mathbb{G}$ , such that  $C_{p(\cdot)}(\mathcal{W}_i) < \frac{\varepsilon}{2^{i+1}}$  and such that  $u_i$  restricted to  $\mathbb{G} \setminus \mathcal{W}_i$  is continuous. Let  $\mathcal{W} = \bigcup_i \mathcal{W}_i$ . Then we have, via subadditivity,

$$C_{p(\cdot)}(\mathcal{W}) = C_{p(\cdot)}\left(\bigcup_{i=1}^{\infty} \mathcal{W}_i\right) < \frac{\varepsilon}{2}.$$

We use the subadditivity of the the Sobolev  $p(\cdot)$ -capacity to obtain

$$C_{p(\cdot)}(\mathcal{V}_j \cup \mathcal{W}) \leq C_{p(\cdot)}(\mathcal{V}_j) + C_{p(\cdot)}(\mathcal{W}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Furthermore,

$$|u_r(x) - u_k(x)| \leq \sum_{m=r}^{k-1} |u_m(x) - u_{m+1}(x)| \leq \sum_{m=r}^{k-1} 2^{-m} < 2^{r-l}$$

for all  $x \in \mathbb{G} \setminus (\mathcal{V}_j \cup \mathcal{W})$  and every  $k > r > j$ . This implies we have uniform convergence in  $\mathbb{G} \setminus (\mathcal{V}_j \cup \mathcal{W})$ , and thus  $u$  is continuous in  $\mathbb{G} \setminus (\mathcal{V}_j \cup \mathcal{W})$ .  $\square$

Now we are ready to define variable exponent Sobolev spaces with zero boundary values, denoted  $W_0^{1,p(x)}(\Omega)$ , as in [KKM]:

**Definition 2.3.13.** The function  $u$  belongs to  $W_0^{1,p(x)}(\Omega)$  if there exists a  $p(\cdot)$ -quasicontinuous function  $\tilde{u} \in W^{1,p(x)}(\mathbb{G})$  such that  $u = \tilde{u}$  almost everywhere in  $\Omega$  and  $\tilde{u} = 0$  quasieverywhere in  $\mathbb{G} \setminus \Omega$ . With this definition, we have the norm

$$\|u\|_{W_0^{1,p(x)}(\Omega)} = \|\tilde{u}\|_{W^{1,p(x)}(\mathbb{G})}.$$

Furthermore, we say that the  $p(\cdot)$ -quasicontinuous function  $\tilde{u} \in W^{1,p(x)}(\mathbb{G})$  is a *canonical representative* of the function  $u \in W_0^{1,p(x)}(\Omega)$  if  $u = \tilde{u}$  almost everywhere in  $\Omega$  and  $\tilde{u} = 0$   $p(\cdot)$ -quasieverywhere in  $\mathbb{G} \setminus \Omega$ .

Note that the norm does not depend on the choice of the quasicontinuous representative since  $C_{p(\cdot)}(\Omega) = 0$  means the measure of  $\Omega$  is 0.

The Euclidean version of the next theorem is Theorem 3.1 in [HHKV1]. The metric space version using Newtonian spaces is Theorem 3.4 in [HHP1]. The proof is standard and omitted.

**Theorem 2.3.14.** Assume  $1 < p^- \leq p^+ < \infty$ . Then  $W_0^{1,p(x)}(\Omega)$  is a Banach space.

In addition, we have the following key identification, whose proof follows that of [HHKV1, Theorem 3.3]. (Cf. [AH, Section 9.2].)

**Theorem 2.3.15.** Let  $\Omega \subset \mathbb{G}$  be open. Then

$$\overline{(C_0^\infty(\Omega))} = W_0^{1,p(x)}(\Omega),$$

where  $C_0^\infty(\Omega)$  is the set of continuous, infinitely differentiable functions with compact support in  $\Omega$  and  $\overline{(C_0^\infty(\Omega))}$  denotes the closure of  $C_0^\infty(\Omega)$  in  $W^{1,p(x)}(\Omega)$ .

The next theorem holds in any metric measure space. [HHP1, Section 2.2] and [KR, Theorem 2.8].

**Theorem 2.3.16.** *Let  $\Omega \in \mathbb{G}$  be open. Suppose  $0 < |\Omega| < \infty$  and  $q(x) \leq p(x)$  for a.e.  $x \in \Omega$ . Then*

$$L^{p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega).$$

Moreover, we have

$$\|f\|_{L^{q(\cdot)}} \leq (1 + |\Omega|) \|f\|_{L^{p(\cdot)}}.$$

We use Theorem 2.3.16 in variable exponent Lebesgue spaces to show this next theorem in variable exponent Sobolev spaces with zero boundary values. The Carnot proof is similar to the Euclidean proof of [HHKV1, Theorem 3.4] with the obvious modifications.

**Theorem 2.3.17.** *Let  $1 < q^-, p^+ < \infty$  and  $q(x) \leq p(x)$  for a.e.  $x \in \mathbb{G}$ . Assume  $\Omega \subset \mathbb{G}$  is a bounded, open set. Then*

$$W_0^{1,p(\cdot)}(\Omega) \hookrightarrow W_0^{1,q(\cdot)}(\Omega).$$

Moreover, the norm of the embedding operator does not exceed  $1 + |\Omega|$ .

In order to use the direct method for calculus of variations, the functional must be defined on a reflexive space. To that end, we have the next theorem.

**Theorem 2.3.18** ([BF1], Theorem 4.5). *Assume  $1 < p^- \leq p^+ < \infty$ . Then  $W_0^{1,p(\cdot)}(\Omega)$  is reflexive.*

*Proof.* A closed subspace of a reflexive space is also reflexive. We know that the space  $L^{p(\cdot)}(\mathbb{G})$  is reflexive and thus  $\hat{K} := L^{p(\cdot)}(\mathbb{G}) \times L^{p(\cdot)}(\mathbb{G})$  is a reflexive space. Since  $W_0^{1,p(\cdot)}(\Omega)$  is isomorphic to a closed subspace of  $\hat{K}$  by the isomorphism  $\Phi : W_0^{1,p(\cdot)}(\Omega) \rightarrow \hat{K}$  defined by  $u \mapsto (u, \nabla_0 u)$ , we are done.  $\square$

Now we are ready to prove a  $p(\cdot)$ -Poincaré type inequality for variable exponent Sobolev spaces with zero boundary values, stated as Theorem 2.3.19 below. Assume  $\Omega \subset \mathbb{G}$  is open. For any open set  $A$  in  $\mathbb{G}$ , we use the following notation:

$$p_A^+ = \operatorname{ess\,sup}_{x \in A \cap \Omega} p(x) \quad \text{and} \quad p_A^- = \operatorname{ess\,inf}_{x \in A \cap \Omega} p(x)$$

and assume that

$$1 < p_A^- \leq p_A^+ < \infty$$

holds in compact subsets of  $\Omega$ . Recall that we use  $Q$  to denote the homogeneous dimension of  $\mathbb{G}$ . Let  $B_{\hat{N}}(x, \delta)$  be a gauge ball of radius  $\delta$ , centered at  $x$  (recall Equation (2.3)). If  $p_\Omega^+ < \infty$  and if there exists

$\delta > 0$  such that for every point  $x \in \Omega$  either

$$p_{B_{\hat{N}}(x,\delta)}^- \geq Q \quad \text{or} \quad p_{B_{\hat{N}}(x,\delta)}^+ \leq \frac{Q p_{B_{\hat{N}}(x,\delta)}^-}{Q - p_{B_{\hat{N}}(x,\delta)}^-}$$

holds, then the variable exponent  $p$  is said to satisfy the jump condition in  $\Omega$  with constant  $\delta$ . Observe that if  $\Omega$  is bounded and  $p$  is continuous in  $\bar{\Omega}$ , then there is some  $\delta > 0$  such that  $p$  satisfies the jump condition in  $\Omega$ . We will also use the following notation in the next proof:

$$p_{B_{\hat{N}}(x,\delta)}^*(x) = \begin{cases} \frac{Q \cdot p_{B_{\hat{N}}(x,\delta)}^-}{Q - p_{B_{\hat{N}}(x,\delta)}^-} & \text{if } p_{B_{\hat{N}}(x,\delta)}^- < Q, \\ p_{B_{\hat{N}}(x,\delta)}^+ & \text{if } p_{B_{\hat{N}}(x,\delta)}^- \geq Q. \end{cases}$$

Note that

$$\frac{1}{p_{B_{\hat{N}}(x,\delta)}^*} = \frac{1}{p_{B_{\hat{N}}(x,\delta)}^-} - \frac{1}{Q} \quad \text{when } p_{B_{\hat{N}}(x,\delta)}^- < Q \quad \text{and we always have } p^* > p_{B_{\hat{N}}(x,\delta)}^-.$$

The Euclidean version for the variable exponent  $p(\cdot)$ -Poincaré-type inequality is Theorem 4.1 in [HHKV1].

**Theorem 2.3.19** ([BF1], Theorem 5.1, A  $p(\cdot)$ -Poincaré-Type Inequality). *Let  $\Omega \in \mathbb{G}$  be a bounded open set and assume that  $p$  satisfies the jump condition in  $\Omega$  with  $\delta > 0$ . Then for every  $u \in W_0^{1,p(x)}(\Omega)$ , we have*

$$\|u\|_{L^{p(x)}(\Omega)} \leq C \|\nabla_0 u\|_{L^{p(x)}(\Omega)},$$

where  $C$  is independent of  $u$ .

*Proof.* Since  $\bar{\Omega}$  is compact, then there exist  $x_1, \dots, x_j$  such that for any set  $D \subset \Omega$ , we have

$$D \subset \bigcup_{i=1}^j B_{\hat{N}}(x_i, \delta),$$

where  $B_{\hat{N}}(x_i, \delta)$  is a gauge ball of radius  $\delta$ , centered at  $x_i$ , for all  $i$  such that  $1 \leq i \leq j$ . Let  $B^i = B_{\hat{N}}(x_i, \delta)$  and let  $\tilde{u}$  be the canonical representative of  $u$ . Then  $u = \tilde{u}$  almost everywhere in  $\Omega$  and  $\tilde{u} = 0$   $p(\cdot)$ -

quasieverywhere in  $\mathbb{G} \setminus \Omega$ . Since  $\tilde{u}$  is a canonical representative of  $u$ , by Theorem 2.3.16 we obtain

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \|\tilde{u}\|_{L^{p(\cdot)}(\mathbb{G})} \leq \sum_{i=1}^j \|\tilde{u}\|_{L^{p(\cdot)}(B^i)} \leq (1 + |\Omega|) \sum_{i=1}^j \|\tilde{u}\|_{L^{p_{B^i}^*}(B^i)}.$$

Then by the triangle inequality, where  $\tilde{u}_{B^i}$  is the average of  $\tilde{u}$  over the balls  $B^i$ ,

$$\begin{aligned} (1 + |\Omega|) \sum_{i=1}^j \|\tilde{u}\|_{L^{p_{B^i}^*}(B^i)} &\leq (1 + |\Omega|) \sum_{i=1}^j \left( \|\tilde{u} - \tilde{u}_{B^i}\|_{L^{p_{B^i}^*}(B^i)} + \|\tilde{u}_{B^i}\|_{L^{p_{B^i}^*}(B^i)} \right) \\ &\leq (1 + |\Omega|) \sum_{i=1}^j \left( \|\tilde{u} - \tilde{u}_{B^i}\|_{L^{p_{B^i}^*}(B^i)} + |\tilde{u}_{B^i}| \|\chi_{B^i}\|_{L^{p_{B^i}^*}(B^i)} \right), \end{aligned}$$

where  $\chi_{B^i}$  is the characteristic function of the ball  $B^i$ .

Next, we estimate  $\|\tilde{u} - \tilde{u}_{B^i}\|_{L^{p_{B^i}^*}(B^i)}$  and then  $|\tilde{u}_{B^i}|$ , both in terms of  $\|\nabla_0 u\|_{L^{p(\cdot)}(\Omega)}$ . To accomplish this, we will apply a Sobolev-Poincaré inequality for  $\|\tilde{u} - \tilde{u}_{B^i}\|_{L^{p_{B^i}^*}(B^i)}$  and a classical type Poincaré inequality for  $|\tilde{u}_{B^i}|$  (see, for example, [Je], [DGP, Theorem 2.1]) By the global Poincaré inequality on metric balls, for the fixed exponent case presented by Jerison [Je] (and restated in [DGP, Theorem 2.1]), and Theorem 2.3.16 we have a constant  $C$  independent of  $\tilde{u}$  such that for every  $i = 1, \dots, j$ ,

$$\|\tilde{u} - \tilde{u}_{B^i}\|_{L^{p_{B^i}^*}(B^i)} \leq C \|\nabla_0 \tilde{u}\|_{L^{p_{B^i}^-}(B^i)} \leq C(1 + |B^i|) \|\nabla_0 \tilde{u}\|_{L^{p(\cdot)}(B^i)}.$$

The doubling property for gauge balls, namely  $|B(x, \delta)| \leq C\delta^Q$  where  $C = C(Q)$ , gives us

$$\|\tilde{u} - \tilde{u}_{B^i}\|_{L^{p_{B^i}^*}(B^i)} \leq C(Q, \delta) \|\nabla_0 u\|_{L^{p(\cdot)}(\Omega)}.$$

Next, the Poincaré inequality in Carnot groups [Je] implies

$$|\tilde{u}_{B^i}| = AVG \int_{B^i} |u| dx \leq C(Q, \delta, B^i) \int_{B^i} |\nabla_0 u| dx \leq C(Q, \delta)(1 + |\Omega|) \|\nabla_0 u\|_{L^{p(\cdot)}(\Omega)}.$$

It follows that

$$\begin{aligned}
\|u\|_{L^{p(\cdot)}(\Omega)} &\leq (1 + |\Omega|) \sum_{i=1}^j \left( \|\tilde{u} - \tilde{u}_{B^i}\|_{L^{p_{B^i}^*}(B^i)} + |\tilde{u}_{B^i}| \|\chi_{B^i}\|_{L^{p_{B^i}^*}(B^i)} \right) \\
&\leq C(Q, \delta, \Omega) \sum_{i=1}^j \left( \|\nabla_0 u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla_0 u\|_{L^{p(\cdot)}(\Omega)} \|\chi_{B^i}\|_{L^{p_{B^i}^*}(B^i)} \right) \\
&\leq C(Q, \delta, \Omega) \|\nabla_0 u\|_{L^{p(\cdot)}(\Omega)}.
\end{aligned}$$

The proof is complete. □

We finish this section with a brief discussion about  $p(\cdot)$ -Dirichlet energy integral minimizers in Carnot groups. Let  $\mathcal{O} \subset G$  be an open set and let  $w \in W^{1,p(\cdot)}(\mathcal{O})$ . The energy operator corresponding to the boundary value function  $w$ , acting on the space  $W_0^{1,p(\cdot)}(\mathcal{O})$  is defined by

$$I_{\mathcal{O},w}^{p(\cdot)}(u) = \int_{\mathcal{O}} |\nabla_0 u(x) + \nabla_0 w(x)|^{p(x)} dx. \quad (2.10)$$

We want to find a function that minimizes the values of  $I_{\mathcal{O},w}^{p(\cdot)}$  on  $W_0^{1,p(\cdot)}(\mathcal{O})$ . This task is equivalent to finding a  $p(\cdot)$ -Dirichlet energy minimizing function. To show that a minimizer exists, we follow the same path as the fixed exponent case in [Sh] and the variable exponent case in [HHKV2]. We will need the next lemma from functional analysis, but first we recall some definitions. Let  $\mathcal{B}$  be a reflexive Banach space. An operator  $I$  is convex if for all  $t \in [0, 1]$  and each pair  $u, v \in \mathcal{B}$ , we have

$$I(tu + (1-t)v) \leq tI(u) + (1-t)I(v).$$

Also,  $I$  is said to be lower semicontinuous if  $I(u) \leq \liminf_{i \rightarrow \infty} I(u_i)$  whenever  $u_i$  is a sequence of elements in  $\mathcal{B}$  such that  $u_i \rightarrow u$ . Finally,  $I$  is coercive if  $I(u_i) \rightarrow \infty$  whenever  $\|u_i\|_{\mathcal{B}} \rightarrow \infty$ .

**Lemma 2.3.20.** *Let  $\mathcal{B}$  be a reflexive Banach space. If  $I : \mathcal{B} \rightarrow \mathbb{R}$  is a convex, lower semicontinuous, and coercive operator, then there exists an element in  $\mathcal{B}$  that minimizes  $I$ .*

Now we are ready to show the existence of the minimizer.

**Theorem 2.3.21** ([BF1], Theorem 6.2). *Let  $\mathcal{O} \subset G$  be a bounded open set. Assume that  $p$  satisfies the jump*



condition in  $\mathcal{O}$  and  $1 < p^- \leq p^+ < \infty$ . Then there exists a function  $u \in W_0^{1,p(\cdot)}(\mathcal{O})$  such that

$$I_{\mathcal{O},\omega}^{p(\cdot)}(u) = \inf_{v \in W_0^{1,p(\cdot)}(\mathcal{O})} I_{\mathcal{O},\omega}^{p(\cdot)}(v). \quad (2.11)$$

*Proof.* We know that  $W_0^{1,p(\cdot)}(\mathcal{O})$  is a reflexive Banach space by Theorems 2.3.14 and 2.3.18. We will show that the operator  $I_{\mathcal{O},\omega}^{p(\cdot)}$  is convex, lower semicontinuous, and coercive. Then by Lemma 2.3.20, we will have the existence of a minimizer. For every fixed  $1 < p < \infty$ ,  $x \rightarrow x^p$  is convex so that

$$(t|u(x)| + (1-t)|v(x)|)^{p(x)} \leq t|u(x)|^{p(x)} + (1-t)|v(x)|^{p(x)} \quad (2.12)$$

for every  $0 < t < 1$ , every  $x \in \mathcal{O}$ , and for every  $u, v \in W_0^{1,p(\cdot)}(\mathcal{O})$ . Therefore it follows that  $I_{\mathcal{O},\omega}^{p(\cdot)}$  is convex. Next, we show that  $I_{\mathcal{O},\omega}^{p(\cdot)}$  is lower semicontinuous. Let  $\{u_i\}$  be a sequence of functions in  $W_0^{1,p(\cdot)}(\mathcal{O})$  that converge to  $u \in W_0^{1,p(\cdot)}(\mathcal{O})$ . Then  $\nabla_0(u_i + w) \rightarrow \nabla_0(u + w)$  in  $L^{p(\cdot)}(\mathcal{O})$ . That is,  $\|\nabla_0(u_i + w) - \nabla_0(u + w)\|_{L^{p(\cdot)}(\mathcal{O})} \rightarrow 0$  as  $i \rightarrow \infty$ . By Equation (2.5), we have

$$\varrho_{p(x)}(\nabla_0(u_i + w) - \nabla_0(u + w)) \rightarrow 0 \quad \text{as } i \rightarrow \infty.$$

By [HHKV1, Lemma 2.6], this produces

$$\varrho_{p(x)}(\nabla_0(u_i + w)) \xrightarrow{i \rightarrow \infty} \varrho_{p(x)}(\nabla_0(u + w)).$$

Because Carnot groups are a metric space, we have that the sequential lower semicontinuity of the operator  $I_{\mathcal{O},\omega}^{p(\cdot)}$  implies it is lower semicontinuous.

It remains to show that the operator  $I_{\mathcal{O},\omega}^{p(\cdot)}$  is coercive. Assume that  $\|u_i\|_{W_0^{1,p(\cdot)}(\mathcal{O})} \rightarrow \infty$ . Then by Theorem 2.3.19 ( $p(\cdot)$ -Poincaré Inequality),  $\|\nabla_0 u_i\|_{L^{p(\cdot)}(\mathcal{O})} \rightarrow \infty$  so then  $\|\nabla_0 u_i + \nabla_0 w\|_{L^{p(\cdot)}(\mathcal{O})} \rightarrow \infty$  as  $i \rightarrow \infty$ . It follows that  $I_{\mathcal{O},\omega}^{p(\cdot)} \rightarrow \infty$  as  $i \rightarrow \infty$  since  $p^+ < \infty$ . Therefore  $I_{\mathcal{O},\omega}^{p(\cdot)}$  is coercive and the proof is complete.  $\square$

We also need the following theorem concerning uniqueness of the  $p(\cdot)$ -quasicontinuous representative. The proof is identical to that of [HHKV2, Theorem 5.3] and omitted.

**Theorem 2.3.22** ([BF1], Theorem 6.3). *The  $p(\cdot)$ -quasicontinuous representative  $\tilde{u}$  of the minimizing function  $u$  in Equation (2.11) is unique up to a set of zero  $p(\cdot)$ -capacity.*

We conclude with a theorem whose proof matches that of [HHKV2, Theorem 5.4]:

**Theorem 2.3.23** ([BF1], Theorem 6.4). *Let  $1 < p^- \leq p^+ < \infty$  and  $u \in W_0^{1,p(\cdot)}(\mathcal{O})$ . Then  $u$  minimizes*

$$I_{\mathcal{O},\omega}^{p(\cdot)}(u)$$

*if and only if*

$$\int_{\mathcal{O}} p(x) |\nabla_0 u(x) + \nabla_0 w(x)|^{p(x)-2} (\nabla_0 u(x) + \nabla_0 w(x)) \cdot \nabla_0 (v(x) - u(x)) dx \geq 0, \quad (2.13)$$

*for every  $v \in W_0^{1,p(\cdot)}(\mathcal{O})$  and  $w \in W_0^{1,p(\cdot)}(\mathcal{O})$  such that  $u - w \in W_0^{1,p(\cdot)}(\mathcal{O})$ .*

## 2.4 Notions of Solutions to the $p(x)$ -Laplace Equation and Some Preliminary Results

We now turn our attention to a few different notions of solutions to the  $p(x)$ -Laplacian where we assume that  $1 < p(x) < \infty$  and  $\Omega \subset \mathbb{G}$ . Note that all of the definitions and results in this section apply in the Heisenberg group; in other words, we can take  $\Omega \subset \mathbb{H}_n$ . The main goal of Chapter 3 is achieved by relating three different notions of solutions to the  $p(x)$ -Laplace equation, namely

$$-\Delta_{p(x)} u = -\operatorname{div}(\|\nabla_0 u\|^{p(x)-2} \nabla_0 u) = 0 \quad (2.14)$$

in a bounded domain  $\Omega$ .

We begin by considering weak solutions to Equation (2.14).

**Definition 2.4.1.** The function  $u \in W_{\text{loc}}^{1,p(x)}(\Omega)$  is a *weak solution* to Equation (2.14) if

$$\int_{\Omega} \|\nabla_0 u\|^{p(x)-2} \langle \nabla_0 u, \nabla_0 \phi \rangle dx = 0$$

for all  $\phi \in C_0^\infty(\Omega)$ . A weak solution to Equation (2.14) is also called  *$p(x)$ -harmonic*.

In addition to weak solutions, we define weak supersolutions and weak subsolutions to Equation (2.14).

**Definition 2.4.2.** The function  $u \in W_{\text{loc}}^{1,p(x)}(\Omega)$  is a *weak supersolution* to Equation (2.14) if

$$\int_{\Omega} \|\nabla_0 u\|^{p(x)-2} \langle \nabla_0 u, \nabla_0 \phi \rangle dx \geq 0$$

for all nonnegative  $\phi \in C_0^\infty(\Omega)$ . The function  $u \in W_{\text{loc}}^{1,p(x)}(\Omega)$  is a *weak subsolution* to Equation (2.14) if

$-u$  is a weak supersolution. That is, the function  $u \in W_{loc}^{1,p(x)}(\Omega)$  is a *weak subsolution* to Equation (2.14) if

$$\int_{\Omega} \|\nabla_0 u\|^{p(x)-2} \langle \nabla_0 u \nabla_0 \phi \rangle dx \leq 0$$

for all nonnegative  $\phi \in C_0^\infty(\Omega)$ .

For some of our results, namely the equivalence of potential theoretic weak solutions and viscosity solutions to the  $p(x)$ -Laplace equation in Chapter 3, we will need to do more. In turn, we must consider weak solutions to a wider class of equations. Letting  $\varepsilon \geq 0$  be a real parameter, we consider equations of the form

$$-\Delta_{p(x)} u = -\operatorname{div}(\|\nabla_0 u\|^{p(x)-2} \nabla_0 u) = \varepsilon \quad (2.15)$$

in a bounded domain  $\Omega$ . Note that Equation (2.14) corresponds to Equation (2.15) with  $\varepsilon = 0$ . We define  $\varepsilon$ -weak solutions to Equation (2.15) and then  $\varepsilon$ -weak super and subsolutions to Equation (2.15).

**Definition 2.4.3.** The function  $u \in W_{loc}^{1,p(x)}(\Omega)$  is an  $\varepsilon$ -weak solution to Equation (2.15) if

$$\int_{\Omega} \|\nabla_0 u\|^{p(x)-2} \langle \nabla_0 u \nabla_0 \phi \rangle dx = \varepsilon \int_{\Omega} \phi dx$$

for all  $\phi \in C_0^\infty(\Omega)$ . A weak solution to Equation (2.14) (or 0-weak solution to Equation (2.15)) is also called  $p(x)$ -harmonic.

In addition to  $\varepsilon$ -weak solutions, we define  $\varepsilon$ -weak supersolutions and  $\varepsilon$ -weak subsolutions in the natural way.

**Definition 2.4.4.** The function  $u \in W_{loc}^{1,p(x)}(\Omega)$  is an  $\varepsilon$ -weak supersolution to Equation (2.15) if

$$\int_{\Omega} \|\nabla_0 u\|^{p(x)-2} \langle \nabla_0 u \nabla_0 \phi \rangle dx \geq \varepsilon \int_{\Omega} \phi dx$$

for all nonnegative  $\phi \in C_0^\infty(\Omega)$ . The function  $u \in W_{loc}^{1,p(x)}(\Omega)$  is an  $\varepsilon$ -weak subsolution to Equation (2.15) if  $-u$  is an  $\varepsilon$ -weak supersolution.

**Remark 2.4.5.** 1. Using these definitions when  $\varepsilon_1 > \varepsilon_2 \geq 0$ , we observe that an  $\varepsilon_1$ -weak solution is an  $\varepsilon_2$ -weak supersolution and an  $\varepsilon_2$ -weak solution is an  $\varepsilon_1$ -weak subsolution.

2. If  $u \in W^{1,p(x)}(\Omega)$ , we may use test functions in  $W_0^{1,p(x)}(\Omega)$  via standard approximation arguments.

Next, we have the following comparison principle, whose proof is identical to the Euclidean version and omitted ([JLP, Lemma 5.1]).

**Lemma 2.4.6.** *Let  $u$  and  $v$  be functions in  $W^{1,p(x)}(\Omega)$  such that  $(u - v)_+ \in W^{1,p(x)}(\Omega)$ . If*

$$\int_{\Omega} |\nabla_0 u|^{p(x)-2} \langle \nabla_0 u, \nabla_0 \phi \rangle dx \leq \int_{\Omega} |\nabla_0 v|^{p(x)-2} \langle \nabla_0 v, \nabla_0 \phi \rangle dx$$

for all positive test functions  $\phi \in W^{1,p(x)}(\Omega)$ , then  $u \leq v$  almost everywhere in  $\Omega$ .

**Corollary 2.4.7.** *Let  $u \in W^{1,p(x)}(\Omega)$  be a  $\varepsilon$ -weak subsolution to Equation (2.15) and let*

*$v \in W^{1,p(x)}(\Omega)$  be a  $\varepsilon$ -weak supersolution to Equation (2.15) in  $\Omega$ . If*

*$\gamma \equiv \min\{v - u, 0\} \in W_0^{1,p(x)}(\Omega)$ , then  $u \leq v$  almost everywhere in  $\Omega$ .*

We can now formulate the existence-uniqueness of  $p(x)$ -harmonic functions. For the case of the  $p$ -Laplacian in Carnot groups, see [HKM, Lemma 3.17] and [HH, Section 4.10].

**Theorem 2.4.8.** *Given a bounded domain  $\Omega$  with boundary data  $\Theta \in W^{1,p(x)}(\Omega)$ , there is a unique  $p(x)$ -harmonic function  $u$  that satisfies  $u - \Theta \in W_0^{1,p(x)}(\Omega)$ .*

Next, we define  $p(x)$ -superharmonic functions:

**Definition 2.4.9.** The function  $u : \Omega \rightarrow \mathbb{R} \cup \{\infty\}$  is  $p(x)$ -superharmonic if the following hold:

1.  $u$  is lower semicontinuous,
2.  $u$  is finite almost everywhere, and
3. the comparison principle holds: For each subdomain  $D \subset\subset \Omega$ , a  $p(x)$ -harmonic function  $g$  in  $D$  that is continuous in  $\bar{D}$  with  $g \leq u$  on  $\partial D$  implies  $g \leq u$  in  $D$ .

A function  $u$  is  $p(x)$ -subharmonic if  $-u$  is  $p(x)$ -superharmonic. That is, the function  $u : \Omega \rightarrow \mathbb{R} \cup \{-\infty\}$  is  $p(x)$ -subharmonic if the following hold:

1.  $u$  is upper semicontinuous,
2.  $u$  is finite almost everywhere, and
3. the comparison principle holds: For each subdomain  $D \subset\subset \Omega$ , a  $p(x)$ -harmonic function  $g$  in  $D$  that is continuous in  $\bar{D}$  with  $g \geq u$  on  $\partial D$  implies  $g \geq u$  in  $D$ .

We remark that in the second condition in both definitions, we have required  $u$  to be finite almost everywhere. This is different from the case in which  $p(x)$  is constant where it is only assumed that a  $p$ -superharmonic function is not identically  $+\infty$  in each component. We need the stronger condition for the characterization of  $p(x)$ -superharmonic functions as pointwise increasing limits of 0-weak supersolutions to Equation (2.15) [HHKLM].

We will need some basic facts about  $p(x)$ -superharmonic functions. We use the notation:

$$u^*(x) = \operatorname{ess\,lim\,inf}_{y \rightarrow x} u(y).$$

First, every weak supersolution has a lower semicontinuous representative which is  $p(x)$ -superharmonic. See [HKL, Theorem 4.1] and [HHLN, pg 18].

**Theorem 2.4.10.** *Let  $u$  be a weak supersolution in  $\Omega$ . Then  $u = u^*$  almost everywhere and  $u^*$  is  $p(x)$ -superharmonic.*

We also have the following converse [HHKLM, Corollary 6.6].

**Theorem 2.4.11.** *A locally bounded  $p(x)$ -superharmonic function is a weak supersolution.*

We then conclude that a function is a weak solution ( $p(x)$ -harmonic) if it is both  $p(x)$ -superharmonic and  $p(x)$ -subharmonic.

Now we turn our attention to viscosity solutions. Consider Equation (2.14) in nondivergence form. Namely,

$$\begin{aligned} - \left( \|\nabla_0 u\|^{p(x)-2} \operatorname{trace}((D^2 u)^*) + (p(x) - 2) \|\nabla_0 u\|^{p(x)-4} \langle (D^2 u)^* \nabla_0 u, \nabla_0 u \rangle \right. \\ \left. + \|\nabla_0 u\|^{p(x)-2} \ln(\|\nabla_0 u\|) \langle \nabla_0 p(x), \nabla_0 u \rangle \right) = 0 \end{aligned} \quad (2.16)$$

in a bounded domain  $\Omega$ . Before we define viscosity solutions, we will need the following definitions.

**Definition 2.4.12.** Given the upper semicontinuous function  $u : \Omega \subset \mathbb{G} \rightarrow \mathbb{R}$ , we may define the set of test functions that touch  $u$  from above at  $x_0$ , denoted  $\mathcal{TA}(u, x_0)$ , and given a lower semicontinuous function  $v$ , we may define the set of test functions that touch  $v$  from below at  $x_0$ , denoted  $\mathcal{TB}(v, x_0)$ . Namely,

$$\mathcal{TA}(u, x_0) = \{ \phi : \Omega \rightarrow \mathbb{R} \mid \phi \in C_{\text{sub}}^2(\Omega), \phi(x_0) = u(x_0), \phi(x) > u(x) \text{ for } x \text{ near } x_0 \}$$

$$\mathcal{TB}(v, x_0) = \{ \phi : \Omega \rightarrow \mathbb{R} \mid \phi \in C_{\text{sub}}^2(\Omega), \phi(x_0) = v(x_0), \phi(x) < v(x) \text{ for } x \text{ near } x_0 \}.$$

**Definition 2.4.13.** The function  $u : \Omega \rightarrow \mathbb{R} \cup \{\infty\}$  is a *viscosity supersolution* to Equation (2.16) if the following hold:

1.  $u$  is lower semicontinuous,
2.  $u$  is finite almost everywhere, and
3. For  $x_0 \in \Omega$ ,  $\phi \in \mathcal{TB}(u, x_0)$  with  $\nabla_0 \phi(x_0) \neq 0$  satisfies

$$-\Delta_{p(x)} \phi(x_0) \geq 0.$$

Note that a function  $u$  is a *viscosity subsolution* to Equation (2.16) if  $-u$  is a viscosity supersolution. That is, the function  $u : \Omega \rightarrow \mathbb{R} \cup \{-\infty\}$  is a *viscosity subsolution* to Equation (2.16) if the following hold:

1.  $u$  is upper semicontinuous,
2.  $u$  is finite almost everywhere, and
3. For  $x_0 \in \Omega$ ,  $\phi \in \mathcal{TA}(u, x_0)$  with  $\nabla_0 \phi(x_0) \neq 0$  satisfies

$$-\Delta_{p(x)} \phi(x_0) \leq 0.$$

A function  $u$  is a *viscosity solution* to Equation (2.16) if it is both a viscosity supersolution and a viscosity subsolution.

**Remark 2.4.14.** The condition that  $\nabla_0 \phi(x_0) \neq 0$  in the definition is irrelevant in the case  $2 \leq p(x) < \infty$ , since then the  $p(x)$ -Laplace equation is well-defined. In fact, when  $p(x) \geq 2$ , whether we include the condition that  $\nabla_0 \phi(x_0) \neq 0$  or not, the same class of solutions is produced (see [JLM, Remark 2.4]). However, in the singular case, the  $p(x)$ -Laplace equation has singularities at points where the gradient is zero. For our purposes, we take this definition for  $1 < p(x) < \infty$ .

In Chapter 3, we will also need to consider viscosity solutions to the following equation: for  $\kappa \in \mathbb{R}^+$ , let

$$F_\kappa(u) = \max\{\|\nabla_1 u\| - \kappa, -\Delta_{p(x)} u\}, \quad (2.17)$$

where we recall that  $\nabla_1$  is the semi-horizontal derivative.

**Definition 2.4.15.** The function  $u : \Omega \rightarrow \mathbb{R} \cup \{-\infty\}$  is a *viscosity subsolution* to  $F_\kappa(u) = 0$  if the following hold:

1.  $u$  is upper semicontinuous,
2.  $u$  is finite almost everywhere, and
3. For  $x_0 \in \Omega$ ,  $\mu \in \mathcal{TA}(u, x_0)$  with  $\nabla_0 \mu(x_0) \neq 0$  satisfies

$$F_\kappa \mu(x_0) \leq 0.$$

The function  $v : \Omega \rightarrow \mathbb{R} \cup \{\infty\}$  is a *viscosity supersolution* to  $F_\kappa(v) = 0$  if the following hold:

1.  $v$  is lower semicontinuous,
2.  $v$  is finite almost everywhere, and
3. For  $x_0 \in \Omega$ ,  $\nu \in \mathcal{TB}(u, x_0)$  with  $\nabla_0 \nu(x_0) \neq 0$  satisfies

$$F_\kappa \nu(x_0) \geq 0.$$

The function  $w : \Omega \rightarrow \mathbb{R} \cup \{\pm\infty\}$  is a viscosity solution to  $F_\kappa(w) = 0$  if it is both a viscosity supersolution and a viscosity subsolution to  $F_\kappa(w) = 0$ .

**Remark 2.4.16.** Note that if  $u$  is a viscosity subsolution to  $F_\kappa(u) = 0$ , then  $u$  is a viscosity subsolution to  $-\Delta_{\mathfrak{p}(x)} u = 0$ . Also, if  $v$  is a viscosity supersolution to  $-\Delta_{\mathfrak{p}(x)} v = 0$ , then  $v$  is a viscosity supersolution to  $F_\kappa(v) = 0$ . However, the converse implications are not true. A viscosity supersolution  $v$  to  $F_\kappa(v) = 0$  has two possible properties: either  $v$  is a viscosity supersolution to  $-\Delta_{\mathfrak{p}(x)} v = 0$  or  $v$  is a viscosity subsolution to  $-\Delta_{\mathfrak{p}(x)} v = 0$  and  $\|\nabla_1 v\| > \kappa$  in the  $C_{\text{sub}}^2$  viscosity sense. An analogous observation holds for subsolutions.

The next lemma relates  $\mathfrak{p}(x)$ -harmonic functions to viscosity solutions. The proof is standard and omitted.

**Lemma 2.4.17.** [B2, Lemma 3.5][JLP, Theorem 4.1] *A  $\mathfrak{p}(x)$ -sub(super-)harmonic function is a viscosity sub(super-)solution to Equation (2.16). It follows that a  $\mathfrak{p}(x)$ -harmonic function is a viscosity solution to Equation (2.16).*

We have the following Corollary due to Remark 2.4.16.

**Corollary 2.4.18.** *A  $\mathfrak{p}(x)$ -superharmonic function is a viscosity supersolution to  $F_\kappa(u) = 0$ .*

### Chapter 3

#### Equivalence of Potential Theoretic Weak and Viscosity Solutions to the $p(x)$ -Laplace Equation <sup>1</sup>

This chapter focuses on the equivalence of potential theoretic weak solutions and viscosity solutions to the  $p(x)$ -Laplace equation in Carnot groups. To achieve our goal, we first need to show that viscosity solutions and  $p(x)$ -harmonic solutions coincide. Then the equivalence is immediate. While a routine argument is used to show that  $p(x)$ -harmonic solutions are viscosity solutions, the converse implication is more involved. We will need to consider viscosity solutions to the nondivergence form of the  $p(x)$ -Laplace equation, which we recall for easier reference in this chapter, is defined by

$$\begin{aligned}
 -\left( \|\nabla_0 u\|^{p(x)-2} \text{trace}((D^2 u)^*) + (p(x) - 2) \|\nabla_0 u\|^{p(x)-4} \langle (D^2 u)^* \nabla_0 u, \nabla_0 u \rangle \right. \\
 \left. + \|\nabla_0 u\|^{p(x)-2} \log(\|\nabla_0 u\|) \langle \nabla_0 p(x), \nabla_0 u \rangle \right) = 0
 \end{aligned} \tag{3.1}$$

in a bounded domain  $\Omega$  in  $\mathbb{G}$ . Moreover, we will need to consider viscosity solutions to

$$F_\kappa(u) = \max\{\|\nabla_1 u\| - \kappa, -\Delta_{p(x)} u\}, \tag{3.2}$$

for  $\kappa \in \mathbb{R}^+$ . We will first show a preliminary comparison principle with respect to weak solutions of

$$-\Delta_{p(x)} u = -\text{div}(\|\nabla_0 u\|^{p(x)-2} \nabla_0 u) = -\varepsilon, \tag{3.3}$$

and viscosity subsolutions to Equation (3.2). Finally, we will also mention the divergence form of the  $p(x)$ -Laplace equation. Namely,

$$-\Delta_{p(x)} u = -\text{div}(\|\nabla_0 u\|^{p(x)-2} \nabla_0 u) = 0. \tag{3.4}$$

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<sup>1</sup>A Note to Reader: This chapter has been reproduced from [BF2].



We need a comparison principle to achieve the equivalence of potential theoretic weak and viscosity solutions. Specifically, we want to prove

**Theorem 3.0.1** ([BF2], Lemma 4.11). *Assume  $p(x)$  is  $C^1(\Omega)$ . Fix  $\varepsilon > 0$ . Let  $\Omega$  be a bounded domain in  $\mathbb{G}$ , let  $v$  be a continuous  $\varepsilon$ -weak solution and let  $u$  be a viscosity subsolution to  $F_\kappa(u) = 0$  so that  $u \leq v$  on  $\partial\Omega$ . Then  $u \leq v$  in  $\Omega$ .*

The proof combines the Euclidean approach of [JLP] along with the monotonicity proof done in [B2]. The proof here will follow the standard argument but need some careful estimates, utilizing the Carnot Group Maximum Principle (See [B1].) to the fullest extent. Here we cannot follow the Euclidean case since that approach relies on the  $C^{1,\alpha}$  estimates of the weak solutions, which is unknown in general Carnot groups. In particular, the lack of regularity theory is the motivation for the restriction  $\|\nabla_1 u\| \leq C$  in the viscosity sense. Under this restriction, we prove a comparison principle for viscosity subsolutions and viscosity supersolutions to the  $p(x)$ -Laplace equation. This result leads to showing that when  $1 < p(x) < \infty$ , weak solutions to Equation (3.3), viscosity solutions to Equation (3.1), and viscosity solutions to Equation (3.2) all coincide. (For the equivalence of all three, see Corollary 3.0.8.)

Before we can prove Theorem 3.0.1, we recall a technical lemma whose Euclidean version is Lemma 5.3 in [JLP]. The  $p$ -Laplacian case in Carnot groups is stated in [B1], and its proof is done in the Heisenberg group as Lemma 4.1 in [B2]. The proof is identical and omitted.

**Lemma 3.0.2.** [B2, Lemma 4.1] *Assume  $1 < p(x) < \infty$  and  $\Omega \subset \mathbb{G}$ . Let  $v \in W_{loc}^{1,p(x)}(\Omega)$  be a continuous  $\varepsilon$ -weak solution to the  $p(x)$ -Laplace equation in  $\Omega$ . Let  $x_0 \in \Omega$  and let  $\phi \in C_{sub}^2(\Omega)$  be a function such that  $v - \phi$  has a strict local minimum at  $x_0$ . Then*

$$\limsup_{x \rightarrow x_0, x \neq x_0} \left( - \left( \Delta_{p(\cdot)} \phi \right) (x) \right) \geq \varepsilon$$

*provided that  $\nabla_0 \phi(x_0) \neq 0$  or  $x_0$  is an isolated critical point.*

Note that in the case when  $p(x) \geq 2$  continuity gives us  $-\Delta_{p(x)} \phi(x_0) \geq \varepsilon$  and so we have  $\nabla_0(x_0) \neq 0$  near  $x_0$ . In the case when  $1 < p(x) < 2$ ,  $-\Delta_{p(x)} \phi(x)$  has a singularity at the critical points.

Finally, we will consider the function  $\varphi : \mathbb{G} \times \mathbb{G} \rightarrow \mathbb{R}$  given by

$$\varphi(x, y) = \frac{1}{m} \sum_{i=1}^N ((x \cdot y^{-1})_i)^m$$

for some large even positive integer  $m > 4$ . We note that 4 is chosen so that  $\varphi$  is  $C_{\text{sub}}^2$ . Here  $(x \cdot y^{-1})_i$  is the  $i$ -th component of  $x \cdot y^{-1}$ . We are now ready to Prove Theorem 3.0.1.

*Proof of Theorem 3.0.1.* We follow the standard argument and assume  $u - v$  has a strict interior maximum in  $\Omega$  and find a contradiction. Now assume that  $u - v > 0$  occurs at the interior point  $x_0 \in \Omega$ . Consider the functions  $\Psi_j : \mathbb{G} \times \mathbb{G} \rightarrow \mathbb{R}$  defined by

$$\Psi_j(x, y) = u(x) - v(y) - j\varphi(x, y)$$

with  $m$  chosen so that  $m > \max\{4, \frac{p^-}{p^- - 1}, p^+ + 2\}$ . Combining the methods in [CIL], [B1], and [JLP], we let the maximum of  $\Psi_j$  occur at the point  $(x_j, y_j) \in u(\bar{\Omega}) \times u(\bar{\Omega})$ . By the Carnot Group Maximum Principle [B1, Lemma 3.6],  $x_j$  and  $y_j$  tend to  $x_0$  as  $j \rightarrow \infty$  and

$$(j\eta_j \oplus j\xi_j, \mathcal{X}_j) \in \bar{\mathcal{J}}^{2,+}u(x_j) \quad \text{and} \quad (j\eta_j \oplus j\xi_j, \mathcal{Y}_j) \in \bar{\mathcal{J}}^{2,-}v(y_j),$$

where  $j\eta_j \oplus j\xi_j \in V_1 \oplus V_2$ , and  $\mathcal{X}_j$  and  $\mathcal{Y}_j$  are defined as in [B1, Lemma 3.6]. Recall that  $\bar{\mathcal{J}}^{2,+}u(x_j), \bar{\mathcal{J}}^{2,-}v(y_j)$  are the set-theoretic closures of the second order superjet and subjet, respectively. Since we only need the horizontal gradient in the  $p(x)$ -Laplacian term and not the semi-horizontal gradient, we will only consider  $j\eta_j$ .

**Claim 3.0.3** ([BF2], Claim 4.12). *By passing to a subsequence if needed, we may assume  $\eta_j(x_j, y_j) \neq 0$ .*

*Proof of Claim 3.0.3.* Fix  $j > 0$ . By definition, we have for any  $x$  and  $y$ ,

$$u(x) - v(y) - j\varphi(x, y) \leq u(x_j) - v(y_j) - j\varphi(x_j, y_j)$$

and so when  $x = x_j$ , we have

$$v(y) \geq v(y_j) + j\varphi(x_j, y_j) - j\varphi(x_j, y).$$

Defining the function  $\beta(y)$  by

$$\beta(y) = v(y_j) + j\varphi(x_j, y_j) - j\varphi(x_j, y) - \varphi(y_j, y)$$

we see that

$$v(y) - \beta(y) = v(y) - v(y_j) - j\varphi(x_j, y_j) + j\varphi(x_j, y) + \varphi(y_j, y)$$

and so  $v - \beta$  has a strict local minimum at the isolated critical point  $y_j$ .

Applying Lemma 3.0.2, we have

$$\limsup_{y \rightarrow y_j} \left( - \left( \Delta_{\mathfrak{p}(\cdot)} \beta \right) (y) \right) \geq \varepsilon. \quad (3.5)$$

Now set  $F(y) = -j\varphi(x_j, y) - \varphi(y_j, y)$ . Then by the definition of  $\beta(y)$  and the non-divergence form of the  $\mathfrak{p}(\cdot)$ -Laplacian, we have

$$\begin{aligned} |(\Delta_{\mathfrak{p}(y)} \beta)(y)| &\lesssim \|\nabla_0 F(y)\|^{\mathfrak{p}(y)-2} \left( \left| \text{trace}(D^2 F(y))^* + \|(D^2 F(y))^*\| \right| \right. \\ &\quad \left. + \log(\|\nabla_0 F(y)\|) \langle \nabla_0 \mathfrak{p}(y), \nabla_0 F(y) \rangle \right). \end{aligned} \quad (3.6)$$

We note that given the standard vectors  $e_k$  with every entry 0 except for the  $k$ -th entry, which is equal to 1, we see that for any matrix  $A$ ,

$$\text{trace}(A) = \sum \langle A e_k, e_k \rangle$$

and so

$$|\text{trace}(D^2 F(y))^*| \lesssim \|(D^2 F(y))^*\|.$$

Then from Inequality (3.6), we have

$$\begin{aligned} |(\Delta_{\mathfrak{p}(y)} \beta)(y)| &\lesssim \|\nabla_0 F(y)\|^{\mathfrak{p}(y)-2} \left( \|(D^2 F(y))^*\| + \log(\|\nabla_0 F(y)\|) \langle \nabla_0 \mathfrak{p}(y), \nabla_0 F(y) \rangle \right) \\ &\lesssim \|\nabla_0 F(y)\|^{\mathfrak{p}(y)-2} \left( \|(D^2 F(y))^*\| + \log(\|\nabla_0 F(y)\|) \|\nabla_0 \mathfrak{p}(y)\| \|\nabla_0 F(y)\| \right) \\ &\lesssim \|\nabla_0 F(y)\|^{\mathfrak{p}(y)-2} \|(D^2 F(y))^*\| + C \|\nabla_0 F(y)\|^{\mathfrak{p}(y)-1} |\log(\|\nabla_0 F(y)\|)|, \end{aligned}$$

where the second inequality follows from the Cauchy-Schwartz inequality. Since  $j$  is fixed, the second derivative term is bounded. Then using the smoothness of  $\varphi(x, y)$  and the fact that we are in a bounded

domain, we have

$$\begin{aligned}
\lim_{y \rightarrow y_j} (-\Delta_{\mathbb{P}(y)} \beta)(y) &\lesssim \|\nabla_0 F(y_j)\|^{\mathbb{P}(y_j)-2} \|(D^2 F(y))^*\| \\
&+ \lim_{y \rightarrow y_j} \left( C \|\nabla_0 F(y)\|^{\mathbb{P}(y)-1} |\log(\|\nabla_0 F(y)\|)| \right) \\
&\lesssim \|j \nabla_0 \varphi(x_j, y_j)\|^{\mathbb{P}(y_j)-2} + \limsup_{y \rightarrow y_j} \left( C \|\nabla_0 F(y)\|^{\mathbb{P}(y)-1} |\log(\|\nabla_0 F(y)\|)| \right) \\
&\sim \| -j \eta(x_j, y_j) \|^{\mathbb{P}(y_j)-2} + \limsup_{y \rightarrow y_j} \left( C \|\nabla_0 F(y)\|^{\mathbb{P}(y)-1} |\log(\|\nabla_0 F(y)\|)| \right).
\end{aligned}$$

We consider the second term. Note that  $\|\nabla_0 F(y)\| \rightarrow 0$  as  $y \rightarrow y_j$ . We therefore conclude

$$\limsup_{y \rightarrow y_j} \left( C \|\nabla_0 F(y)\|^{\mathbb{P}(x)-1} |\log(\|\nabla_0 F(y)\|)| \right) = 0.$$

It follows that if  $x_j$  and  $y_j$  are points so that  $\eta(x_j, y_j) = 0$ , then

$$\limsup_{y \rightarrow y_j} \left( -\left( \Delta_{\mathbb{P}(\cdot)} \beta \right)(y) \right) \leq 0.$$

This contradicts Equation (3.5) since  $\varepsilon > 0$ . □

Now,  $u$  is a viscosity subsolution to  $F_\kappa(u) = 0$ . That is,

$$\max\{\|\nabla_1 u\| - \kappa, -\Delta_{\mathbb{P}(x)} u\} \leq 0.$$

Then  $\|\nabla_1 u\| \leq \kappa$  and we have

$$\begin{aligned}
0 &\geq -\left( \|j \eta_j(x_j, y_j)\|^{\mathbb{P}(x_j)-2} \text{trace}(\mathcal{X}_j)^* + (\mathbb{P}(x_j) - 2) \|j \eta_j(x_j, y_j)\|^{\mathbb{P}(x_j)-4} \langle \mathcal{X}_j j \eta_j(x_j, y_j), j \eta_j(x_j, y_j) \rangle \right. \\
&\quad \left. + \|j \eta_j(x_j, y_j)\|^{\mathbb{P}(x_j)-2} \log(\|j \eta_j(x_j, y_j)\|) \langle j \eta_j(x_j, y_j), \nabla_0 \mathbb{P}(x_j) \rangle \right).
\end{aligned}$$

Using Lemmas 2.4.17 and 3.0.2 along with the definition of  $\bar{J}^{2,-}$ , we have  $\|\nabla_1 u\| \leq \kappa$  and

$$\begin{aligned}
\varepsilon &\leq -\left( \|j \eta_j(x_j, y_j)\|^{\mathbb{P}(y_j)-2} \text{trace}(\mathcal{Y}_j)^* + (\mathbb{P}(y_j) - 2) \|j \eta_j(x_j, y_j)\|^{\mathbb{P}(y_j)-4} \langle \mathcal{Y}_j j \eta_j(x_j, y_j), j \eta_j(x_j, y_j) \rangle \right. \\
&\quad \left. + \|j \eta_j(x_j, y_j)\|^{\mathbb{P}(y_j)-2} \log(\|j \eta_j(x_j, y_j)\|) \langle j \eta_j(x_j, y_j), \nabla_0 \mathbb{P}(y_j) \rangle \right).
\end{aligned}$$

Subtracting these two inequalities yields

$$\begin{aligned}
0 &< \varepsilon < \|j\eta_j(x_j, y_j)\|^{\mathfrak{p}(x_j)-4} \left( \|j\eta_j(x_j, y_j)\|^2 \text{trace}(\mathcal{X}_j)^* + (\mathfrak{p}(x_j) - 2) \langle \mathcal{X}_j j\eta_j(x_j, y_j), j\eta_j(x_j, y_j) \rangle \right. \\
&+ \|j\eta_j(x_j, y_j)\|^2 \log(\|j\eta_j(x_j, y_j)\|) \langle j\eta_j(x_j, y_j), \nabla_0 \mathfrak{p}(x_j) \rangle \Big) \\
&+ \|j\eta_j(x_j, y_j)\|^{\mathfrak{p}(y_j)-4} \left( - \|j\eta_j(x_j, y_j)\|^2 \text{trace}(\mathcal{Y}_j)^* - (\mathfrak{p}(y_j) - 2) \langle \mathcal{Y}_j j\eta_j(x_j, y_j), j\eta_j(x_j, y_j) \rangle \right. \\
&- \|j\eta_j(x_j, y_j)\|^2 \log(\|j\eta_j(x_j, y_j)\|) \langle j\eta_j(x_j, y_j), \nabla_0 \mathfrak{p}(y_j) \rangle \Big). \\
&\leq \|j\eta_j(x_j, y_j)\|^{\mathfrak{p}(x_j)-4} \left( \|j\eta_j(x_j, y_j)\|^2 \text{trace}(\mathcal{X}_j)^* + (\mathfrak{p}(x_j) - 2) \langle \mathcal{X}_j j\eta_j(x_j, y_j), j\eta_j(x_j, y_j) \rangle \right) \\
&- \|j\eta_j(x_j, y_j)\|^{\mathfrak{p}(y_j)-4} \left( \|j\eta_j(x_j, y_j)\|^2 \text{trace}(\mathcal{Y}_j)^* + (\mathfrak{p}(y_j) - 2) \langle \mathcal{Y}_j j\eta_j(x_j, y_j), j\eta_j(x_j, y_j) \rangle \right) \\
&+ \|j\eta_j(x_j, y_j)\|^{\mathfrak{p}(x_j)-2} \log(\|j\eta_j(x_j, y_j)\|) \langle j\eta_j(x_j, y_j), \nabla_0 \mathfrak{p}(x_j) \rangle \\
&- \|j\eta_j(x_j, y_j)\|^{\mathfrak{p}(y_j)-2} \log(\|j\eta_j(x_j, y_j)\|) \langle j\eta_j(x_j, y_j), \nabla_0 \mathfrak{p}(y_j) \rangle \\
&:= \omega_{x_j} - \omega_{y_j} + \tau_{x_j} - \tau_{y_j}.
\end{aligned} \tag{3.7}$$

We know  $\|j\eta_j(x_j, y_j)\| \leq \kappa < \infty$  and by the claim, we also know that  $\|j\eta_j(x_j, y_j)\| \neq 0$ .

First, let's consider the terms  $\tau_{x_j} - \tau_{y_j}$ . We continue as in [JLP]. We first note that for some  $r \in [\mathfrak{p}(x_j), \mathfrak{p}(y_j)]$ ,

$$\begin{aligned}
\left| \|j\eta_j\|^{\mathfrak{p}(x_j)-2} - \|j\eta_j\|^{\mathfrak{p}(y_j)-2} \right| &= \left| e^{\log(\|j\eta_j\|^{\mathfrak{p}(x_j)-2})} - e^{\log(\|j\eta_j\|^{\mathfrak{p}(y_j)-2})} \right| \\
&\leq \left| \frac{\partial e^{(r-2) \log(\|j\eta_j\|)}}{\partial r} \right| \left| \mathfrak{p}(x_j) - \mathfrak{p}(y_j) \right| \\
&= \left| \log(\|j\eta_j\|) \right| \|j\eta_j\|^{r-2} \left| \mathfrak{p}(x_j) - \mathfrak{p}(y_j) \right|.
\end{aligned}$$

Since  $\mathfrak{p}(x) \in C^1(G)$ , we have

$$\begin{aligned}
\tau_{x_j} - \tau_{y_j} &= \langle \|j\eta_j\|^{\mathfrak{p}(x_j)-2} \log \|j\eta_j\| j\eta_j, \nabla_0 \mathfrak{p}(x_j) \rangle - \langle \|j\eta_j\|^{\mathfrak{p}(y_j)-2} \log \|j\eta_j\| j\eta_j, \nabla_0 \mathfrak{p}(y_j) \rangle \\
&\leq \|j\eta_j\|^{\mathfrak{p}(x_j)-2} \log \|j\eta_j\| j\eta_j \cdot \nabla_0 \mathfrak{p}(x_j) - \|j\eta_j\|^{\mathfrak{p}(y_j)-2} \log \|j\eta_j\| j\eta_j \cdot \nabla_0 \mathfrak{p}(y_j) \\
&+ \|j\eta_j\|^{\mathfrak{p}(x_j)-1} \left| \log \|j\eta_j\| \right| \left| \nabla_0 \mathfrak{p}(y_j) - \nabla_0 \mathfrak{p}(x_j) \right| \\
&\leq \|j\eta_j\|^{\mathfrak{p}(x_j)-1} \left| \log \|j\eta_j\| \right| \left| \nabla_0 \mathfrak{p}(x_j) - \nabla_0 \mathfrak{p}(y_j) \right| \\
&+ \|j\eta_j\| \left| \log \|j\eta_j\| \right| \left| \nabla_0 \mathfrak{p}(y_j) \right| \left| \|j\eta_j\|^{\mathfrak{p}(x_j)-2} - \|j\eta_j\|^{\mathfrak{p}(y_j)-2} \right| \\
&\leq \|j\eta_j\|^{\mathfrak{p}(x_j)-1} \left| \log \|j\eta_j\| \right| \left| \nabla_0 \mathfrak{p}(x_j) - \nabla_0 \mathfrak{p}(y_j) \right| \\
&+ \|j\eta_j\| \left| \log \|j\eta_j\| \right| \left| \nabla_0 \mathfrak{p}(y_j) \right| \left( \left| \log(\|j\eta_j\|) \right| \|j\eta_j\|^{r-2} \left| \mathfrak{p}(x_j) - \mathfrak{p}(y_j) \right| \right) \\
&\leq \|j\eta_j\|^{\mathfrak{p}(x_j)-1} \left| \log \|j\eta_j\| \right| \left| \nabla_0 \mathfrak{p}(x_j) - \nabla_0 \mathfrak{p}(y_j) \right| \\
&+ \widehat{C} \|j\eta_j\|^{r-1} \left| \log \|j\eta_j\| \right|^2 \left| \mathfrak{p}(x_j) - \mathfrak{p}(y_j) \right|,
\end{aligned}$$

where the constant  $\widehat{C}$  comes from the fact that  $|\nabla_0 \mathfrak{p}(y_j)| \leq \widehat{C}$ . Since

$$\mathfrak{p}(x_j) > 1 \text{ and } \|j\eta_j(x_j, y_j)\| \leq \kappa$$

by assumption, and  $x_j \rightarrow x_0$  as  $j \rightarrow \infty$ , then we have  $\|j\eta_j\|^{\mathfrak{p}(x_j)-1} \log \|j\eta_j\|$  is bounded. Therefore, due to the continuity of  $x \mapsto \nabla_0 \mathfrak{p}(x)$ ,

$$\|j\eta_j\|^{\mathfrak{p}(x_j)-1} \left( \log \|j\eta_j\| \right) \left| \nabla_0 \mathfrak{p}(x_j) - \nabla_0 \mathfrak{p}(y_j) \right| \rightarrow 0$$

as  $j \rightarrow \infty$ . Similarly, we have

$$\widehat{C} \|j\eta_j\|^{r-1} \left| \log \|j\eta_j\| \right|^2 \left| \mathfrak{p}(x_j) - \mathfrak{p}(y_j) \right| \rightarrow 0$$

as  $j \rightarrow \infty$ . It follows that  $\tau_{x_j} - \tau_{y_j} \rightarrow 0$  as  $j \rightarrow \infty$ .

In order to proceed, we will need some calculations. We first consider a more convenient way to write  $\omega_{x_j} - \omega_{y_j}$ . We use the following notation, which is similar to [JLP]. For any vector  $\xi \neq 0$ , we say  $\xi \otimes \xi$  is

the matrix with entries  $\xi_i \xi_j$ . Let

$$A(x, \xi) = |\xi|^{\mathfrak{p}(x)-2} \left( I + (\mathfrak{p}(x) - 2) \frac{\xi}{|\xi|} \otimes \frac{\xi}{|\xi|} \right),$$

$$B(x, \xi) = \langle |\xi|^{\mathfrak{p}(x)-2} \log |\xi| \xi, \nabla_0 \mathfrak{p}(x) \rangle, \text{ and}$$

$$\gamma(x, \xi, \mathcal{X}) = \text{trace}(A(x, \xi) \mathcal{X}) + B(x, \xi),$$

where  $x \in \Omega, \xi \in \mathbb{R}^N$ , and  $\mathcal{X}$  is a symmetric  $N \times N$  matrix. Then

$$\gamma\left(x, \nabla_0 f(x), \left(D^2 f(x)\right)^*\right) = \text{trace}\left(A(x, \nabla_0 f(x)) \left(D^2 f(x)\right)^*\right) + B(x, \nabla_0 f(x)) = \Delta_{\mathfrak{p}(x)} f(x),$$

provided  $\nabla_0 f \neq 0$ .

We also observe that

$$A^{\frac{1}{2}}(x, \xi) = |\xi|^{\frac{\mathfrak{p}(x)}{2}-1} \left( I + C(x) \frac{\xi}{|\xi|} \otimes \frac{\xi}{|\xi|} \right),$$

where

$$C(x) = (\mathfrak{p}(x) - 1)^{\frac{1}{2}} - 1. \tag{3.8}$$

So now since  $u$  is a viscosity subsolution to  $F_\kappa(u) = 0$ , then

$$-\text{trace}\left(A(x_j, j\eta_j(x_j, y_j)) \mathcal{X}_j\right) - B(x_j, j\eta_j(x_j, y_j)) \leq 0, \tag{3.9}$$

and again using Lemmas 2.4.17 and 3.0.2 along with the definition of  $\bar{J}^{2,-}$ , we have

$$-\text{trace}\left(A(y_j, j\eta_j(x_j, y_j)) \mathcal{Y}_j\right) - B(y_j, j\eta_j(x_j, y_j)) \geq \varepsilon. \tag{3.10}$$

Therefore taking the difference of the Equations (3.9) and (3.10) is an analogous form of Equation (3.7),

where  $B(x_j, j\eta_j(x_j, y_j)) - B(y_j, j\eta_j(x_j, y_j))$  is analogous to  $\tau_{x_j} - \tau_{y_j}$ .

Moreover, since

$$\begin{aligned}
\text{trace}\left(A(x_j, j\eta_j(x_j, y_j))\mathcal{X}_j\right) &= \text{trace}\left(A^{\frac{1}{2}}(x_j, j\eta_j(x_j, y_j))A^{\frac{1}{2}}(x_j, j\eta_j(x_j, y_j))\mathcal{X}_j\right) \\
&= \text{trace}\left(\left(A^{\frac{1}{2}}(x_j, j\eta_j(x_j, y_j))\right)^T \mathcal{X}_j A^{\frac{1}{2}}(x_j, j\eta_j(x_j, y_j))\right) \\
&= \sum_{k=1}^{n_1} \mathcal{X}_j A_k^{\frac{1}{2}}(x_j, j\eta_j(x_j, y_j)) \cdot A_k^{\frac{1}{2}}(x_j, j\eta_j(x_j, y_j)),
\end{aligned}$$

where  $A_k^{\frac{1}{2}}(x_j, j\eta_j(x_j, y_j))$  is the  $k$ -th column of  $A^{\frac{1}{2}}(x_j, j\eta_j(x_j, y_j))$  and  $A_k^{\frac{1}{2}}(y_j, j\eta_j(x_j, y_j))$  is the  $k$ -th column of  $A^{\frac{1}{2}}(y_j, j\eta_j(x_j, y_j))$ . Then

$$\sum_{k=1}^{n_1} \mathcal{X}_j A_k^{\frac{1}{2}}(x_j, j\eta_j) \cdot A_k^{\frac{1}{2}}(x_j, j\eta_j) - \sum_{k=1}^{n_1} \mathcal{Y}_j A_k^{\frac{1}{2}}(y_j, j\eta_j) \cdot A_k^{\frac{1}{2}}(y_j, j\eta_j)$$

is an analogous form of  $\omega_{x_j} - \omega_{y_j}$ . By [B1, Lemma 3.6],

$$\begin{aligned}
\omega_{x_j} - \omega_{y_j} &:= \sum_{k=1}^{n_1} \mathcal{X}_j A_k^{\frac{1}{2}}(x_j, j\eta_j) \cdot A_k^{\frac{1}{2}}(x_j, j\eta_j) - \sum_{k=1}^{n_1} \mathcal{Y}_j A_k^{\frac{1}{2}}(y_j, j\eta_j) \cdot A_k^{\frac{1}{2}}(y_j, j\eta_j) \\
&\leq j \langle (D_x^2 \varphi)^*(x_j \cdot y_j^{-1}) \left( A_k^{\frac{1}{2}}(x_j, j\eta_j) - A_k^{\frac{1}{2}}(y_j, j\eta_j) \right), \left( A_k^{\frac{1}{2}}(x_j, j\eta_j) - A_k^{\frac{1}{2}}(y_j, j\eta_j) \right) \rangle \\
&\quad + j \langle \mathfrak{M} \left( A_k^{\frac{1}{2}}(x_j, j\eta_j) \oplus A_k^{\frac{1}{2}}(y_j, j\eta_j) \right), \left( A_k^{\frac{1}{2}}(x_j, j\eta_j) \oplus A_k^{\frac{1}{2}}(y_j, j\eta_j) \right) \rangle \\
&\quad + j \|\mathcal{M}\|^2 \left( \|\mathcal{A}(x_j)^T A_k^{\frac{1}{2}}(x_j, j\eta_j) \oplus \mathcal{A}(y_j)^T A_k^{\frac{1}{2}}(y_j, j\eta_j)\|^2 \right),
\end{aligned}$$

where  $\mathcal{M}$  is the  $2N \times 2N$  matrix

$$\begin{pmatrix} D_{x_j x_j}^2 \varphi(x_j, y_j) & D_{x_j y_j}^2 \varphi(x_j, y_j) \\ D_{y_j x_j}^2 \varphi(x_j, y_j) & D_{y_j y_j}^2 \varphi(x_j, y_j) \end{pmatrix},$$

and  $\mathfrak{M}$  is the  $2n_1 \times 2n_1$  matrix

$$\begin{pmatrix} 0 & \frac{1}{2}(W - W^T) \\ \frac{1}{2}(W^T - W) & 0 \end{pmatrix},$$



where  $W$  is the  $n_1 \times n_1$  matrix with entries

$$W_{ab} = X_a(x)X_b(y)\varphi(x_j, y_j).$$

We first consider the second term of Equation (3.11):

$$j\langle \mathfrak{M}\left(A_k^{\frac{1}{2}}(x_j, j\eta_j) \oplus A_k^{\frac{1}{2}}(y_j, j\eta_j)\right), \left(A_k^{\frac{1}{2}}(x_j, j\eta_j) \oplus A_k^{\frac{1}{2}}(y_j, j\eta_j)\right) \rangle.$$

Before we estimate, we note from [B1, Lemma 3.6] that we have

$$X_b(y)\varphi(x_j, y_j) = -X_b(x)\varphi(x_j, y_j).$$

We write  ${}_\alpha A_k^{\frac{1}{2}}(x_j, j\eta_j)$  and  ${}_\alpha A_k^{\frac{1}{2}}(y_j, j\eta_j)$  to denote the  $\alpha$ -th entry of the column vector  $A_k^{\frac{1}{2}}(x_j, j\eta_j)$  and  $A_k^{\frac{1}{2}}(y_j, j\eta_j)$ , respectively. Similarly, we write  ${}_\alpha \eta_j$  to denote the  $\alpha$ -th entry of the vector  $\eta_j$ . We note that:

$$\begin{aligned} & \langle \mathfrak{M}\left(A_k^{\frac{1}{2}}(x_j, j\eta_j) \oplus A_k^{\frac{1}{2}}(y_j, j\eta_j)\right), \left(A_k^{\frac{1}{2}}(x_j, j\eta_j) \oplus A_k^{\frac{1}{2}}(y_j, j\eta_j)\right) \rangle \\ &= \frac{1}{2} \langle (W - W^T)\left(A_k^{\frac{1}{2}}(y_j, j\eta_j)\right), \left(A_k^{\frac{1}{2}}(x_j, j\eta_j)\right) \rangle \\ &+ \frac{1}{2} \langle (W^T - W)\left(A_k^{\frac{1}{2}}(x_j, j\eta_j)\right), \left(A_k^{\frac{1}{2}}(y_j, j\eta_j)\right) \rangle \\ &= \frac{1}{2} \left[ \langle W\left(A_k^{\frac{1}{2}}(y_j, j\eta_j)\right), \left(A_k^{\frac{1}{2}}(x_j, j\eta_j)\right) \rangle + \langle W^T\left(A_k^{\frac{1}{2}}(x_j, j\eta_j)\right), \left(A_k^{\frac{1}{2}}(y_j, j\eta_j)\right) \rangle \right] \\ &+ \frac{1}{2} \left[ -\langle W^T\left(A_k^{\frac{1}{2}}(y_j, j\eta_j)\right), \left(A_k^{\frac{1}{2}}(x_j, j\eta_j)\right) \rangle - \langle W\left(A_k^{\frac{1}{2}}(x_j, j\eta_j)\right), \left(A_k^{\frac{1}{2}}(y_j, j\eta_j)\right) \rangle \right] \\ &= \langle (W^T - W)\left(A_k^{\frac{1}{2}}(x_j, j\eta_j)\right), \left(A_k^{\frac{1}{2}}(y_j, j\eta_j)\right) \rangle \\ &= \sum_{a,b}^N (W^T - W)_{ab} {}_b A_k^{\frac{1}{2}}(x_j, j\eta_j) {}_a A_k^{\frac{1}{2}}(y_j, j\eta_j). \end{aligned}$$

But we know

$$\begin{aligned} & \left( W^T - W \right)_{ab} = \left( X_b(x)X_a(y)\varphi(x_j, y_j) - X_a(x)X_b(y)\varphi(x_j, y_j) \right) \\ &= \left( X_a(x)X_b(x)\varphi(x_j, y_j) - X_b(x)X_a(x)\varphi(x_j, y_j) \right) \\ &= [X_a, X_b](x)\varphi(x_j, y_j) \\ &= \sum_{l=1}^{n_2} d_l^{ab} Y_l(x)\varphi(x_j, y_j), \end{aligned}$$

where  $\{Y_l(x)\}$  is a basis for  $V_2$ . It follows that

$$\begin{aligned}
& \sum_{a,b}^N (W^T - W)_{ab} {}_b A_k^{\frac{1}{2}}(x_j, j\eta_j) {}_a A_k^{\frac{1}{2}}(y_j, j\eta_j) \\
&= \sum_{a,b}^N [X_a, X_b](x) \varphi(x_j, y_j) {}_b A_k^{\frac{1}{2}}(x_j, j\eta_j) {}_a A_k^{\frac{1}{2}}(y_j, j\eta_j) \\
&= \sum_{a < b}^N [X_a, X_b](x) \varphi(x_j, y_j) {}_b A_k^{\frac{1}{2}}(x_j, j\eta_j) {}_a A_k^{\frac{1}{2}}(y_j, j\eta_j) \\
&+ \sum_{a > b}^N [X_a, X_b](x) \varphi(x_j, y_j) {}_b A_k^{\frac{1}{2}}(x_j, j\eta_j) {}_a A_k^{\frac{1}{2}}(y_j, j\eta_j) \\
&= \sum_{a < b}^N [X_a, X_b](x) \varphi(x_j, y_j) \left( {}_b A_k^{\frac{1}{2}}(x_j, j\eta_j) {}_a A_k^{\frac{1}{2}}(y_j, j\eta_j) - {}_a A_k^{\frac{1}{2}}(y_j, j\eta_j) {}_b A_k^{\frac{1}{2}}(x_j, j\eta_j) \right) \\
&= \sum_{a < b}^N \sum_{l=1}^{n_2} d_l^{ab} Y_l(x) \varphi(x_j, y_j) \|j\eta_j\|^{\frac{p(x_j)}{2}-1} \|j\eta_j\|^{\frac{p(y_j)}{2}-1} \left[ \left( \delta_{bk} + C(x_j) \frac{j^2 {}_b \eta_j {}_k \eta_j}{\|j\eta_j\|^2} \right) \right. \\
&\times \left. \left( \delta_{ak} + C(y_j) \frac{j^2 {}_k \eta_j {}_a \eta_j}{\|j\eta_j\|^2} \right) - \left( \delta_{ak} + C(x_j) \frac{j^2 {}_a \eta_j {}_k \eta_j}{\|j\eta_j\|^2} \right) \left( \delta_{bk} + C(y_j) \frac{j^2 {}_k \eta_j {}_b \eta_j}{\|j\eta_j\|^2} \right) \right] \\
&= \sum_{a < b}^N \sum_{l=1}^{n_2} d_l^{ab} Y_l(x) \varphi(x_j, y_j) \|j\eta_j\|^{\frac{p(x_j)}{2}-1} \|j\eta_j\|^{\frac{p(y_j)}{2}-1} \left[ \delta_{bk} \delta_{ak} + C(x_j) \frac{{}_b \eta_j {}_k \eta_j}{\|\eta_j\|^2} \delta_{ak} \right. \\
&+ C(y_j) \frac{{}_a \eta_j {}_k \eta_j}{\|\eta_j\|^2} \delta_{bk} + C(x_j) C(y_j) \frac{{}_b \eta_j {}_a \eta_j {}_k \eta_j^2}{\|\eta_j\|^4} \\
&- \delta_{ak} \delta_{bk} + C(x_j) \frac{{}_a \eta_j {}_k \eta_j}{\|\eta_j\|^2} \delta_{bk} \\
&- \left. C(y_j) \frac{{}_b \eta_j {}_k \eta_j}{\|\eta_j\|^2} \delta_{ak} - C(x_j) C(y_j) \frac{{}_a \eta_j {}_b \eta_j {}_k \eta_j^2}{\|\eta_j\|^4} \right] \\
&= \sum_{a < b}^N \sum_{l=1}^{n_2} d_l^{ab} Y_l(x) \varphi(x_j, y_j) \|j\eta_j\|^{\frac{p(x_j)}{2}-1} \|j\eta_j\|^{\frac{p(y_j)}{2}-1} \left( C(x_j) - C(y_j) \right) \left( \frac{{}_b \eta_j {}_k \eta_j}{\|\eta_j\|^2} \delta_{ak} - \frac{{}_a \eta_j {}_k \eta_j}{\|\eta_j\|^2} \delta_{bk} \right) \\
&\leq 2\widehat{\kappa} \kappa^{\frac{p(x_j)+p(y_j)}{2}-2} \left( C(x_j) - C(y_j) \right) \\
&= \widetilde{\kappa} \left( C(x_j) - C(y_j) \right),
\end{aligned}$$

where  $C$  is defined in Equation (3.8). Moreover,  $\widetilde{\kappa} < \infty$  since  $p^+ < \infty$ ,  $\kappa < \infty$  and  $\sum_{a < b}^N \sum_{l=1}^{n_2} d_l^{ab} Y_l(x) \varphi(x_j, y_j) < \kappa$ . Note that here we explicitly use the full implication of the constraint

$\nabla_1 u \leq \kappa$ .

Now let  $j \rightarrow \infty$ . Then then by the continuity of  $x \mapsto p(x)$ , we have

$$\begin{aligned} & j \langle \mathfrak{M} \left( A_k^{\frac{1}{2}}(x_j, j\eta_j) \oplus A_k^{\frac{1}{2}}(y_j, j\eta_j) \right), \left( A_k^{\frac{1}{2}}(x_j, j\eta_j) \oplus A_k^{\frac{1}{2}}(y_j, j\eta_j) \right) \rangle \\ & \lesssim \tilde{\kappa} \left( C(x_j) - C(y_j) \right) \\ & \rightarrow \tilde{\kappa} \left( C(x_0) - C(y_0) \right) = 0. \end{aligned}$$

Next, we control the third term of Equation (3.11), namely

$$j \|\mathcal{M}\|^2 \left( \|\mathcal{A}(x_j)^T A_k^{\frac{1}{2}}(x_j, j\eta_j) \oplus \mathcal{A}(y_j)^T A_k^{\frac{1}{2}}(y_j, j\eta_j)\|^2 \right).$$

We have an estimate for the matrix  $\|\mathcal{M}\|^2$ , so we only consider

$$\|\mathcal{A}(x_j)^T A_k^{\frac{1}{2}}(x_j, j\eta_j) \oplus \mathcal{A}(y_j)^T A_k^{\frac{1}{2}}(y_j, j\eta_j)\|^2.$$

Since every entry of the vector  $A_k^{\frac{1}{2}}(x_j, j\eta_j)$  is bounded and since  $\mathcal{A}(x_j)$  has (finite) smooth entries  $\mathcal{A}_{kl}(x_j)$ , then every entry in the vector  $\mathcal{A}(x_j)^T A_k^{\frac{1}{2}}(x_j, j\eta_j)$  is bounded. A similar argument applies for every entry in the vector  $\mathcal{A}(y_j)^T A_k^{\frac{1}{2}}(y_j, j\eta_j)$ . It follows that

$$\|\mathcal{A}(x_j)^T A_k^{\frac{1}{2}}(x_j, j\eta_j) \oplus \mathcal{A}(y_j)^T A_k^{\frac{1}{2}}(y_j, j\eta_j)\|^2 < \mathcal{H} < \infty.$$

Therefore, by [B1, Lemma 3.6],

$$j \|\mathcal{M}\|^2 \left( \|\mathcal{A}(x_j)^T A_k^{\frac{1}{2}}(x_j, j\eta_j) \oplus \mathcal{A}(y_j)^T A_k^{\frac{1}{2}}(y_j, j\eta_j)\|^2 \right) \lesssim Cj(\varphi(x_j, y_j))^{\frac{2m-4}{m}}.$$

Since  $m > 4$  then  $\frac{2m-4}{m} = 2 - \frac{4}{m} > 1$  and so

$$j \|\mathcal{M}\|^2 \left( \|\mathcal{A}(x_j)^T A_k^{\frac{1}{2}}(x_j, j\eta_j) \oplus \mathcal{A}(y_j)^T A_k^{\frac{1}{2}}(y_j, j\eta_j)\|^2 \right) \rightarrow 0$$

as  $j \rightarrow \infty$ .

Finally, we estimate the first term of Equation (3.11), or

$$j \langle \left( D_x^2 \varphi \right)^* (x_j \cdot y_j^{-1}) \left( A_k^{\frac{1}{2}}(x_j, j\eta_j) - A_k^{\frac{1}{2}}(y_j, j\eta_j) \right), \left( A_k^{\frac{1}{2}}(x_j, j\eta_j) - A_k^{\frac{1}{2}}(y_j, j\eta_j) \right) \rangle.$$

Arguing as above, we have  $(D_x^2)_i^* \in V_2$  and thus  $(D_x^2)_i^* \leq H < \infty$ . We then have

$$\begin{aligned} j \langle (D_x^2 \varphi)^* (x_j \cdot y_j^{-1}) (A_k^{\frac{1}{2}}(x_j, j\eta_j) - A_k^{\frac{1}{2}}(y_j, j\eta_j)), (A_k^{\frac{1}{2}}(x_j, j\eta_j) - A_k^{\frac{1}{2}}(y_j, j\eta_j)) \rangle \\ \lesssim \|A^{\frac{1}{2}}(x_j, j\eta_j) - A^{\frac{1}{2}}(y_j, j\eta_j)\|_2^2. \end{aligned}$$

We first consider

$$\begin{aligned} & \|A(x_j, j\eta_j) - A(y_j, j\eta_j)\|_2 \tag{3.12} \\ & \leq \left| \|j\eta_j\|^{\mathfrak{p}(x_j)-2} - \|j\eta_j\|^{\mathfrak{p}(y_j)-2} \right| + \left| \|j\eta_j\|^{\mathfrak{p}(x_j)-4}(\mathfrak{p}(x_j) - 2) - \|j\eta_j\|^{\mathfrak{p}(y_j)-4}(\mathfrak{p}(y_j) - 2) \right|. \end{aligned}$$

Since  $\|j\eta_j\| \leq \kappa < \infty$  for all  $j$  by assumption, then there is a convergent subsequence such that  $\|j\eta_j\| \rightarrow \vartheta \in V_1$  as  $j \rightarrow \infty$ . Using the continuity of  $x \mapsto \mathfrak{p}(x)$ , we then have

$$\left| \|j\eta_j\|^{\mathfrak{p}(x_j)-2} - \|j\eta_j\|^{\mathfrak{p}(y_j)-2} \right| \rightarrow \left| \|\vartheta\|^{\mathfrak{p}(x_0)-2} - \|\vartheta\|^{\mathfrak{p}(x_0)-2} \right| = 0$$

as  $j \rightarrow \infty$ , and

$$\begin{aligned} & \left| \|j\eta_j\|^{\mathfrak{p}(x_j)-4}(\mathfrak{p}(x_j) - 2) - \|j\eta_j\|^{\mathfrak{p}(y_j)-4}(\mathfrak{p}(y_j) - 2) \right| \\ & \rightarrow \left| \|\vartheta\|^{\mathfrak{p}(x_0)-4}(\mathfrak{p}(x_0) - 2) - \|\vartheta\|^{\mathfrak{p}(x_0)-4}(\mathfrak{p}(x_0) - 2) \right| = 0 \end{aligned}$$

as  $j \rightarrow \infty$ .

It follows that

$$\|A(x_j, j\eta_j) - A(y_j, j\eta_j)\|_2^2 \rightarrow 0 \tag{3.13}$$

as  $j \rightarrow \infty$ .

Thus, we have shown that  $\omega_{x_j} - \omega_{y_j} \rightarrow 0$  as  $j \rightarrow \infty$ . It follows that Equation (3.7) implies

$$0 < \varepsilon \leq \omega_{x_j} - \omega_{y_j} + \tau_{x_j} - \tau_{y_j} \rightarrow 0$$

as  $j \rightarrow \infty$ , and we have our contradiction. □

The next Lemma extends the Euclidean case for the  $p(x)$ -Laplacian ([JLP, Lemma 5.2]) and the Carnot case of the  $p$ -Laplacian ([B1, Lemma 5.9]). Here we cannot follow the Euclidean case in [JLP] since the proof depends on  $C^{1,\alpha}$  regularity of the solutions, and this is not known for Carnot groups.

**Lemma 3.0.4** ([BF2], Lemma 4.13). *Suppose  $\Omega$  is a bounded and open set and  $p(x)$  is continuous. Let  $v$  be a  $p(x)$ -harmonic function in  $\Omega$ . For each  $\varepsilon \geq 0$ , let  $v_\varepsilon$  be the continuous  $\varepsilon$ -weak solution to  $v$  on the boundary. Then  $v_\varepsilon \rightarrow v$  pointwise as  $\varepsilon \rightarrow 0$ .*

*Proof.* First, we show that  $v_\varepsilon \rightarrow v$  in  $L^{p(x)}$ , by following the argument in [JLP]. Since  $v_\varepsilon$  minimizes the functional (see [BF1, Theorem 6.2])

$$f \mapsto \int_{\Omega} \left( \frac{1}{p(x)} \|\nabla_0 f\|^{p(x)} - \varepsilon f \right) dx,$$

then

$$\int_{\Omega} \|\nabla_0 v_\varepsilon\|^{p(x)} dx \leq C \int_{\Omega} \left( \|\nabla_0 v\|^{p(x)} + \varepsilon |v_\varepsilon - v| \right) dx,$$

and so by Hölder's Inequality (2.6),

$$\int_{\Omega} \|\nabla_0 v_\varepsilon\|^{p(x)} dx \leq C \left( \int_{\Omega} \|\nabla_0 v\|^{p(x)} dx + \varepsilon \|1\|_{L^{q(x)}(\Omega)} \|v_\varepsilon - v\|_{L^{p(x)}(\Omega)} \right).$$

By the  $p(\cdot)$ -Poincaré inequality (See Section 2.3.2), we have

$$\int_{\Omega} \|\nabla_0 v_\varepsilon\|^{p(x)} dx \leq C \left( \int_{\Omega} \|\nabla_0 v\|^{p(x)} dx + \varepsilon \|\nabla_0 v_\varepsilon - \nabla_0 v\|_{L^{p(x)}(\Omega)} \right)$$

and so using the modular inequalities (2.4), we see

$$\int_{\Omega} \|\nabla_0 v_\varepsilon\|^{p(x)} dx \leq C \left( 1 + \|\nabla_0 v\|_{L^{p(x)}(\Omega)}^{p^+} + \varepsilon (\|\nabla_0 v\|_{L^{p(x)}(\Omega)} + \|\nabla_0 v_\varepsilon\|_{L^{p(x)}(\Omega)}) \right).$$

Using the modular inequalities once more, we have:

$$\|\nabla_0 v_\varepsilon\|_{L^{p(x)}(\Omega)}^{p^-} \leq \int_{\Omega} \|\nabla_0 v_\varepsilon\|^{p(x)} dx \leq C \left( 1 + \|\nabla_0 v\|_{L^{p(x)}(\Omega)}^{p^+} + \varepsilon (\|\nabla_0 v\|_{L^{p(x)}(\Omega)} + \|\nabla_0 v_\varepsilon\|_{L^{p(x)}(\Omega)}) \right),$$

which implies that

$$\|\nabla_0 v_\varepsilon\|_{L^{p(x)}(\Omega)}^{p^-} - C\varepsilon \|\nabla_0 v_\varepsilon\|_{L^{p(x)}(\Omega)} \leq C \left( 1 + \|\nabla_0 v\|_{L^{p(x)}(\Omega)}^{p^+} + \varepsilon \|\nabla_0 v\|_{L^{p(x)}(\Omega)} \right)$$

so that for all  $\varepsilon > 0$  small enough, we have:

$$\|\nabla_0 v_\varepsilon\|_{L^{p(x)}(\Omega)} \leq C \left( 1 + \|\nabla_0 v\|_{L^{p(x)}(\Omega)}^{p^+} \right)^{\frac{1}{p^-}} \quad (3.14)$$

where  $C$  is a constant independent of  $\varepsilon$ . Since  $1 < p^-$  and  $1 + \|\nabla_0 v\|_{L^{p(x)}(\Omega)}^{p^+} \geq 1$ , it follows that

$$\|\nabla_0 v - \nabla_0 v_\varepsilon\|_{L^{p(x)}(\Omega)} \leq C \left( 1 + \|\nabla_0 v\|_{L^{p(x)}(\Omega)}^{p^+} \right). \quad (3.15)$$

Let  $v - v_\varepsilon \in W^{1,p(x)}(\Omega)$  be a test function in Equations (3.3), where  $\varepsilon > 0$  and (3.4). Subtracting these equations yields

$$\int_{\Omega} \langle \|\nabla_0 v\|^{p(x)} v - \|\nabla_0 v_\varepsilon\|^{p(x)} v_\varepsilon, \nabla_0 v - \nabla_0 v_\varepsilon \rangle dx = \varepsilon \int_{\Omega} (v_\varepsilon - v) dx. \quad (3.16)$$

Since

$$\varepsilon \int_{\Omega} (v_\varepsilon - v) dx \leq C\varepsilon \|1\|_{L^q(\Omega)} \|v_\varepsilon - v\|_{L^{p(x)}(\Omega)} \leq C\varepsilon \|\nabla_0 v_\varepsilon - \nabla_0 v\|_{L^{p(x)}(\Omega)},$$

where we have used Hölder's Inequality (2.6) and a  $p(\cdot)$ -Poincaré Inequality, [BF2, Theorem 5.1]. Here, since  $C$  is independent of the function  $v_\varepsilon - v$ , we have an upper bound by Equation (3.15):

$$\varepsilon \int_{\Omega} (v_\varepsilon - v) dx \leq C\varepsilon \left( 1 + \|\nabla_0 v\|_{L^{p(x)}(\Omega)}^{p^+} \right). \quad (3.17)$$

Before continuing, we will need the following well-known vector inequalities:

$$\langle |\eta|^{m-2} \eta - |\zeta|^{m-2} \zeta, \eta - \zeta \rangle \geq \begin{cases} 2^{2-m} |\eta - \zeta|^m & \text{if } m \geq 2, \\ (m-1) \frac{|\eta - \zeta|^2}{(|\eta| + |\zeta|)^{2-m}} & \text{if } 1 < m < 2 \end{cases} \quad (3.18)$$

for all  $\eta, \zeta \in \mathbb{R}^N$ . We now find a lower bound for the left-hand side of Equation (3.16). Following the

vector inequalities above, we split the set  $\Omega$  into the subsets:

$$\Omega_1 := \{x \in \Omega : 1 < p(x) < 2\} \quad \text{and} \quad \Omega_2 := \{x \in \Omega : p(x) \geq 2\}.$$

For  $\Omega_2$ , it follows from the vector inequality (3.18) that

$$\begin{aligned} \int_{\Omega_2} \|\nabla_0 v - \nabla_0 v_\varepsilon\|^{p(x)} dx &\leq \int_{\Omega_2} 2^{p(x)-2} \langle \|\nabla_0 v\|^{p(x)-2} \nabla_0 v - \|\nabla_0 v_\varepsilon\|^{p(x)-2} \nabla_0 v_\varepsilon, \nabla_0 v - \nabla_0 v_\varepsilon \rangle dx \\ &\leq C \int_{\Omega_2} \langle \|\nabla_0 v\|^{p(x)-2} \nabla_0 v - \|\nabla_0 v_\varepsilon\|^{p(x)-2} \nabla_0 v_\varepsilon, \nabla_0 v - \nabla_0 v_\varepsilon \rangle dx. \end{aligned} \quad (3.19)$$

For  $\Omega_1$ , we introduce the notation:

$$\hat{p}^- := \inf_{\Omega_1} p(x) \quad \text{and} \quad \hat{p}^+ := \sup_{\Omega_1} p(x).$$

By Hölder's Inequality (2.6), we have:

$$\begin{aligned} \int_{\Omega_1} \|\nabla_0 v - \nabla_0 v_\varepsilon\|^{p(x)} dx &\leq C \left\| \frac{\|\nabla_0 v - \nabla_0 v_\varepsilon\|^{p(x)}}{(\|\nabla_0 v\| + \|\nabla_0 v_\varepsilon\|)^{\frac{p(x)}{2}(2-p(x))}} \right\|_{L^{\frac{2}{p(x)}}(\Omega_1)} \\ &\quad \times \left\| (|\nabla_0 v| + |\nabla_0 v_\varepsilon|)^{\frac{p(x)}{2}(2-p(x))} \right\|_{L^{\frac{2}{2-p(x)}}(\Omega_1)} \\ &\leq C \max_{p \in \{\hat{p}^-, \hat{p}^+\}} \left( \int_{\Omega_1} \frac{\|\nabla_0 v - \nabla_0 v_\varepsilon\|^2}{(\|\nabla_0 v\| + \|\nabla_0 v_\varepsilon\|)^{2-p(x)}} dx \right)^{\frac{p}{2}} \times \left( 1 + \int_{\Omega_1} (\|\nabla_0 v\| + \|\nabla_0 v_\varepsilon\|)^{p(x)} dx \right)^{\frac{1}{2}}, \end{aligned}$$

where the second inequality follows from the modular inequalities (2.4). By the vector inequality (3.18) and the fact that  $1 < \hat{p}^-, \hat{p}^+ \leq 2$ , we have

$$\begin{aligned} &\max_{p \in \{\hat{p}^-, \hat{p}^+\}} \left( \int_{\Omega_1} \frac{\|\nabla_0 v - \nabla_0 v_\varepsilon\|^2}{(\|\nabla_0 v\| + \|\nabla_0 v_\varepsilon\|)^{2-p(x)}} dx \right)^{\frac{p}{2}} \\ &\leq \max_{p \in \{\hat{p}^-, \hat{p}^+\}} C \left( \int_{\Omega_1} \langle \|\nabla_0 v\|^{p(x)-2} \nabla_0 v - \|\nabla_0 v_\varepsilon\|^{p(x)-2} \nabla_0 v_\varepsilon, \nabla_0 v - \nabla_0 v_\varepsilon \rangle dx \right)^{\frac{p}{2}}. \end{aligned}$$

Young's inequality then implies

$$\begin{aligned} & \max_{p \in \{\hat{p}^-, \hat{p}^+\}} \left( \int_{\Omega_1} \frac{\|\nabla_0 v - \nabla_0 v_\varepsilon\|^2}{(\|\nabla_0 v\| + \|\nabla_0 v_\varepsilon\|)^{2-p(x)}} dx \right)^{\frac{p}{2}} \\ & \leq C \left( \delta^{\frac{2}{2-p^-}} + \delta^{-\frac{2}{p^-}} \int_{\Omega_1} \langle \|\nabla_0 v\|^{p(x)-2} \nabla_0 v - \|\nabla_0 v_\varepsilon\|^{p(x)-2} \nabla_0 v_\varepsilon, \nabla_0 v - \nabla_0 v_\varepsilon \rangle dx \right) \end{aligned} \quad (3.20)$$

for any  $0 < \delta < 1$ , to be chosen below. Now, using Inequality (3.14), we can bound the term  $(1 + \int_{\Omega_1} (\|\nabla_0 v\| + \|\nabla_0 v_\varepsilon\|)^{p(x)} dx)$ . Indeed, using the same argument as in the beginning of the proof, we see since  $v + v_\varepsilon$  minimizes the functional

$$u \rightarrow \int_{\Omega_1} \left( \frac{1}{p(x)} \|\nabla_0(u)\|^{p(x)} - \varepsilon u \right) dx$$

and so we have, by (3.14),

$$1 + \int_{\Omega_1} (\|\nabla_0 v\| + \|\nabla_0 v_\varepsilon\|)^{p(x)} dx \leq 1 + C(1 + \|\nabla_0 v\|_{L^{p(x)}(\Omega)}^{p^+})^{\frac{1}{p^-}},$$

which implies

$$\left( 1 + \int_{\Omega_1} (\|\nabla_0 v\| + \|\nabla_0 v_\varepsilon\|)^{p(x)} dx \right)^{\frac{1}{2}} \leq \left( 1 + C(1 + \|\nabla_0 v\|_{L^{p(x)}(\Omega)}^{p^+})^{\frac{1}{p^-}} \right)^{\frac{1}{2}}.$$

Therefore, by adjusting the constant  $C$ , we have by inequality (3.20):

$$\begin{aligned} & \int_{\Omega_1} \|\nabla_0 v - \nabla_0 v_\varepsilon\|^{p(x)} dx \\ & \leq C \left( \delta^{\frac{2}{2-p^-}} + \delta^{-\frac{2}{p^-}} \int_{\Omega_1} \langle \|\nabla_0 v\|^{p(x)-2} \nabla_0 v - \|\nabla_0 v_\varepsilon\|^{p(x)-2} \nabla_0 v_\varepsilon, \nabla_0 v - \nabla_0 v_\varepsilon \rangle dx \right), \end{aligned} \quad (3.21)$$

where  $C$  depends on the functions  $v$  and  $p$ ,  $|\Omega|$ ,  $\text{diam}(\Omega)$ , and the dimension  $\mathcal{Q}$ , but not on  $\varepsilon$  or  $\delta$ . We combine the estimates (3.19) and (3.21) to obtain

$$\begin{aligned} & \int_{\Omega} \|\nabla_0 v - \nabla_0 v_\varepsilon\|^{p(x)} dx \\ & \leq C \left( \delta^{\frac{2}{2-p^-}} + \delta^{-\frac{2}{p^-}} \int_{\Omega} \langle \|\nabla_0 v\|^{p(x)-2} \nabla_0 v - \|\nabla_0 v_\varepsilon\|^{p(x)-2} \nabla_0 v_\varepsilon, \nabla_0 v - \nabla_0 v_\varepsilon \rangle dx \right). \end{aligned}$$



Using (3.16) and (3.17) and then choosing  $\delta = \varepsilon^{\frac{(2-p^-)p^-}{4}}$ , we have

$$\begin{aligned} \int_{\Omega} \|\nabla_0 v - \nabla_0 v_\varepsilon\|^{p(x)} dx & \leq C(\delta^{\frac{2}{2-p^-}} + \delta^{-\frac{2}{p^-}} \varepsilon \|\nabla_0 v - \nabla_0 v_\varepsilon\|_{L^{p(x)}(\Omega)}) \\ & \leq C\varepsilon^{\frac{p^-}{2}} (1 + \|\nabla_0 v - \nabla_0 v_\varepsilon\|_{L^{p(x)}(\Omega)}). \end{aligned}$$

By estimate (3.15), we see

$$\int_{\Omega} \|\nabla_0 v - \nabla_0 v_\varepsilon\|^{p(x)} dx \leq C\varepsilon^{\frac{p^-}{2}} (1 + \hat{C}(1 + \|\nabla_0 v\|_{L^{p(x)}(\Omega)}^{p^+})) \leq C\varepsilon^{\frac{p^-}{2}} (1 + \|\nabla_0 v\|_{L^{p(x)}(\Omega)}^{p^+}).$$

It follows from the modular inequalities (Inequality (2.4)) that

$$\|\nabla_0 v - \nabla_0 v_\varepsilon\|_{L^{p(x)}(\Omega)} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

Therefore, employing Equation (3.16) and, once again, the modular inequalities, we have

$$\|v - v_\varepsilon\|_{L^{p(x)}(\Omega)} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0,$$

and so  $v_\varepsilon \rightarrow v$  in  $L^{p(x)}(\Omega)$ .

Now, to complete the proof, we follow the argument in [B2]. We may assume that  $\varepsilon \leq 1$  and recall, as mentioned above, that if  $\varepsilon_1 > \varepsilon_2$ , then  $v_{\varepsilon_2}$  is a weak subsolution to Equation (3.3) for  $\varepsilon_1$ . By the comparison principle, Corollary 2.4.7, we have that  $v_{\varepsilon_2} \leq v_{\varepsilon_1}$  when  $\varepsilon_1 > \varepsilon_2$ . Specifically, for all  $\varepsilon > 0$ , we have  $v \leq v_\varepsilon$ . Therefore, it follows that

$$\omega = \lim_{\varepsilon \rightarrow 0} v_\varepsilon = \inf_{\varepsilon > 0} \{v_\varepsilon\}$$

exists and  $v \leq \omega$ . Since  $v_\varepsilon \rightarrow \omega$  pointwise, then  $|v_\varepsilon|^{p(x)} \rightarrow |\omega|^{p(x)}$ . We apply the Lebesgue Dominated Convergence Theorem with  $v_1$  as the dominator and conclude that  $v_\varepsilon \rightarrow \omega$  in  $L^{p(x)}(\Omega)$ , so that actually we have  $v = \omega$ , and the proof is complete.  $\square$

Combining the previous lemmas, we obtain the following consequence.

**Lemma 3.0.5** ([BF2], Lemma 4.14). *Suppose  $\Omega$  is a bounded domain and  $p(x)$  is  $C^1$ . Then viscosity subsolutions to  $F_\kappa(u) = 0$  are  $p(x)$ -subharmonic.*

*Proof.* Fix  $\kappa \in \mathbb{R}^+$ . We let  $u$  be a viscosity subsolution to  $F_\kappa(u) = 0$  that is not  $p(x)$ -subharmonic. Then there is a  $p(x)$ -harmonic function  $v$  so that  $u \leq v$  on  $\partial\Omega$  but for some  $x_0 \in \Omega$ , we have  $u(x_0) > v(x_0)$ . For  $\varepsilon \leq 1$ , we let  $v_\varepsilon$  be  $\varepsilon$ -weak solutions equal to  $v$  on  $\partial\Omega$  so that  $u \leq v_\varepsilon$  on  $\partial\Omega$ . By Lemma 3.0.4 we conclude for some  $\varepsilon$  near 0,  $u(x_0) > v_\varepsilon(x_0)$ , contradicting Theorem 3.0.1.  $\square$

By Lemma 2.4.17, Lemma 3.0.5, and Remark 2.4.16 we have the following corollary:

**Corollary 3.0.6** ([BF2], Corollary 4.15). *Suppose  $\Omega$  is a bounded domain and  $p(x)$  is  $C^1$ . Then for  $\kappa \in \mathbb{R}^+$ ,*

$$\begin{aligned} u \text{ is a viscosity subsolution to } F_\kappa(u) = 0 \\ \implies u \text{ is } p(x)\text{-subharmonic} \implies u \text{ is a viscosity subsolution to } -\Delta_{p(x)}u = 0, \end{aligned}$$

and

$$\begin{aligned} v \text{ is } p(x)\text{-superharmonic} \implies v \text{ is a viscosity supersolution to } -\Delta_{p(x)}v = 0 \\ \implies v \text{ is a viscosity supersolution to } F_\kappa(v) = 0. \end{aligned}$$

As a consequence of the corollary, we obtain the following comparison principle. The proof is similar to [JLP, Theorem 4.4].

**Theorem 3.0.7** ([BF2], Theorem 4.16). *Let  $\kappa \in \mathbb{R}^+$ . Let  $v$  be a viscosity supersolution to  $-\Delta_{p(x)}v = 0$  and let  $u$  be a viscosity subsolution to  $-\Delta_{p(x)}u = 0$  such that  $\|\nabla_1 u\| \leq C$ , in the viscosity sense, and such that  $u \leq v$  on  $\partial\Omega$ . Then  $u \leq v$  in  $\Omega$ .*

*Proof.* Without loss of generality, assume  $\|\nabla_1 u\| \leq C$ . Since  $u$  is a viscosity subsolution to  $-\Delta_{p(x)}u = 0$ , then the condition that  $\|\nabla_1 u\| \leq C$  implies there is a  $\kappa \in \mathbb{R}^+$  such that  $u$  is a viscosity subsolution to  $F_\kappa(u) = 0$ . By Lemma 3.0.5,  $u$  is  $p(x)$ -subharmonic so that for any  $\delta > 0$ , there exists a smooth subdomain  $D \subset\subset \Omega$  such that

$$u < v + \delta \text{ in } \Omega \setminus D.$$

By semicontinuity, there exists a smooth function  $\varphi$  such that

$$u < \varphi < v + \delta \text{ on } \partial D.$$

Let  $h$  be the unique weak solution to  $-\Delta_{\mathfrak{p}(x)}h = 0$  with boundary values  $\varphi$ . Then

$$u < h < v + \delta \text{ on } \partial D.$$

For  $\varepsilon > 0$ , let  $h_\varepsilon$  be the unique  $\varepsilon$ -weak continuous solution to Equation 3.3 such that  $h_\varepsilon - h \in W_0^{1,\mathfrak{p}(x)}(D)$ . Since  $u$  is a viscosity subsolution to  $F_\kappa(u) = 0$ , then Theorem 3.0.1 implies  $u \leq h_\varepsilon$  in  $D$ . Now, Lemma 3.0.4 implies  $u \leq h$  in  $D$ .

Using the same argument in a symmetric way, we have  $h \leq v + \delta$  in  $D$ , and therefore  $u \leq v + \delta$  in  $D$ . This means we have  $u \leq v + \delta$  in  $D$  and in  $\Omega \setminus D$ , and it follows that  $u \leq v + \delta$  in  $\Omega$ . We complete the argument by letting  $\delta \rightarrow 0$ .  $\square$

We have the following corollary to Theorem 3.0.7, which is the main result of this chapter:

**Corollary 3.0.8** ([BF2], Corollary 4.17). *Suppose  $\Omega$  is a bounded domain and  $\mathfrak{p}(x)$  is  $C^1$  with  $1 < \mathfrak{p}(x) < \infty$  and assume that  $\|\nabla_1 u\| \leq C$  in the viscosity sense. Let  $\kappa > C$ . Then viscosity sub(super)solutions to  $F_\kappa(u) = 0$  and  $\mathfrak{p}(x)$ -sub(super)harmonic functions coincide. In particular,  $u$  is  $\mathfrak{p}(x)$ -harmonic if and only if  $u$  is a 0-viscosity solution to Equation (3.1) if and only if  $u$  is a viscosity solution to  $F_\kappa(u) = 0$ .*

*Proof.* In light of Corollary 3.0.6 we only need to show

$$\begin{aligned} u \text{ is a viscosity subsolution to } -\Delta_{\mathfrak{p}(x)}u = 0 &\implies u \text{ is } \mathfrak{p}(x)\text{-subharmonic} \\ &\implies u \text{ is a viscosity subsolution to } F_\kappa(u) = 0 \end{aligned} \quad (3.22)$$

and

$$\begin{aligned} v \text{ is a viscosity supersolution to } F_\kappa(v) = 0 \\ &\implies v \text{ is a viscosity supersolution to } -\Delta_{\mathfrak{p}(x)}v = 0 \implies v \text{ is } \mathfrak{p}(x)\text{-superharmonic.} \end{aligned} \quad (3.23)$$

We show Equation (3.22) first. Since  $\|\nabla_1 u\| \leq C < \kappa$  in the viscosity sense, it is clear that  $u$  is a viscosity subsolution to  $-\Delta_{\mathfrak{p}(x)}u = 0$  implies that  $u$  is a viscosity subsolution to  $F_\kappa(u) = 0$ . Then Equation (3.22) follows from Corollary 3.0.6.

Now we show Equation (3.23). Assume  $v$  is a viscosity supersolution to  $F_\kappa(v) = 0$ . Then again our condition that  $\|\nabla_1 v\| \leq C < \kappa$  in the viscosity sense implies that  $v$  is a viscosity supersolution to  $-\Delta_{\mathfrak{p}(x)}v = 0$ .

Now since  $v$  is a viscosity supersolution, then  $-v$  is a viscosity subsolution. Then by Equation (3.22),  $-v$  is  $p(x)$ -subharmonic, and it follows that  $v$  is  $p(x)$ -superharmonic.  $\square$

## Chapter 4

### Removability of a Level Set in the Heisenberg Group

#### 4.1 The Case of the $p$ -Laplace Equation <sup>1</sup>

In [JL1], Juutinen and Lindqvist produce a proof in the Euclidean environment for removability of level sets for weak solutions to the  $p$ -Laplace Equation

$$-\Delta_p u := -\operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0 \tag{4.1}$$

in  $\Omega \subseteq \mathbb{R}^n$  is a domain and  $1 < p < \infty$ . They employ arguments presented in [Kr] and [M] and rely on the equivalence of weak and viscosity solutions to achieve a Radó type result. Specifically, the authors prove that if  $u$  is a  $C^1(\Omega)$  viscosity solution to Equation (4.1) in the set  $\Omega \setminus \{x \in \Omega : u(x) = 0\}$ , then it is a viscosity solution to Equation (4.1) in  $\Omega$ .

The proof in [JL1] relies heavily on the Euclidean geometry of  $\mathbb{R}^n$ . It is natural to ask if this argument can be extended to sub-Riemannian spaces, which possess a different geometric structure. We consider the well-known Heisenberg group and can answer this question in the affirmative. The Euclidean proof invokes properties of (Euclidean) hypersurfaces. Motivated by [JL2], we provide an alternative to this methodology in the Heisenberg group by using results and techniques from [FSS]. We combine this with [B2, Corollary 4.8] which shows the equivalence of  $p$ -harmonic functions and viscosity solutions to Equation (4.1) in order to produce the following theorem:

**Theorem 4.1.1** ([BFF1], Main Theorem). *Assume  $1 < p < \infty$ . Let  $\Omega \subseteq \mathbb{H}_n$  be a domain, and suppose that  $u \in C_{sub}^1(\Omega)$  is  $p$ -harmonic in the set  $\Omega \setminus \{x \in \Omega : u(x) = 0\}$ . Then  $u$  is  $p$ -harmonic in  $\Omega$ .*

The approach to proving Theorem 4.1.1 can be summarized as follows:

- I. We begin by considering Theorem 4.1.1 in terms of viscosity solutions. Identifying  $\Delta_p$  with an appropriate symmetric matrix acting on the horizontal second derivative matrix, we observe that if  $u$  fails

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<sup>1</sup>A Note to Reader: This section has been reproduced from [BFF1]. This work has been submitted for review.

to be a viscosity sub(super)solution at a point  $z$ , then there exists a test function which has a nonzero horizontal gradient at  $z$ . We then employ the Heisenberg implicit function theorem of [FSS] to produce a family of smooth functions whose level set is a gauge-ball tangent to the test function at  $z$ .

II. We choose a compact set  $B$  centered at the critical point to produce key properties.

III. We then compare directional derivatives and produce a contradiction.

A key result needed to prove Theorem 4.1.1 is the following lemma:

**Lemma 4.1.2** ([BFF1], Lemma 1.1). *Assume  $1 < p < \infty$  and let  $u \in C_{\text{sub}}^1(\Omega)$ . If  $u$  is  $p$ -harmonic in*

$$\Omega \setminus \{x \in \Omega : \nabla_0 u(x) = 0\},$$

*then  $u$  is  $p$ -harmonic in  $\Omega$ .*

*Proof.* Let  $u$  be  $p$ -harmonic in  $\Omega \setminus D$ , where  $D := \{x \in \Omega : \nabla_0 u(x) = 0\}$ . Then by Corollary 3.0.8, we know that  $u$  is a viscosity solution to Equation (4.1) in  $\Omega \setminus D$ . If  $\varphi$  is a touching function (from above or below) at the point  $z \in \Omega$ , then by regularity of  $u$  and  $\varphi$  we must have

$$\nabla_0 \varphi(z) = \nabla_0 u(z). \tag{4.2}$$

If  $z \in D$ , then Definition 2.4.13 is satisfied since Equation (4.2) implies that  $\nabla_0 \varphi(z) = 0$ . In other words,  $u$  is a viscosity solution at each point of  $D$ . We conclude that  $u$  is a viscosity solution to Equation (4.1) in all of  $\Omega$ , and therefore  $u$  is  $p$ -harmonic in all of  $\Omega$  by Corollary 3.0.8.  $\square$

To prove Theorem 4.1.1, we follow the argument in [JL2, Theorem 2.2], which is based on a proof similar to that of the classical Hopf maximum principle. Before we prove Theorem 4.1.1, we mention some notation we will use throughout this section. For any vector  $\xi \neq 0$ , we let  $\xi \otimes \xi$  be the matrix with entries  $\xi_i \xi_j$ . Then we define  $A : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \rightarrow \mathcal{S}^{2n}$ :

$$\begin{aligned} A(x, \xi) &:= |\xi|^{p-2} \left( I + (p-2) \frac{\xi}{|\xi|} \otimes \frac{\xi}{|\xi|} \right) \\ &= |\xi|^{p-2} I + (p-2) |\xi|^{p-4} \xi \otimes \xi, \end{aligned} \tag{4.3}$$

and note that  $A_{ij}$  has continuous entries. We have a lemma involving some key properties of  $A$ .

**Lemma 4.1.3** ([BFF1], Lemma 4.1). *Let  $\Omega \subseteq \mathbb{H}_n$  and suppose  $u \in C_{\text{sub}}^1(\Omega)$ . Define the mapping  $A$  as in Equation (4.3). Then for  $1 < p < \infty$ :*

(a)  *$A$  is symmetric.*

(b) *We can write*

$$-\Delta_p u = -A(x, \nabla_0 u(x)) \circ (D^2 u)^*(x),$$

where  $M \circ N$  is the trace of the matrix resulting from the matrix product  $MN$ .

*Proof.* Notice that item (a) is trivially true by the construction of  $A$ . Observe that because  $\text{trace}(\cdot)$  is a linear functional,

$$\begin{aligned} -A(x, \nabla_0 u) \circ ((D^2 u)^*(x)) &= -\|\nabla_0 u\|^{p-2} \left( I + (p-2) \frac{\nabla_0 u}{\|\nabla_0 u\|} \otimes \frac{\nabla_0 u}{\|\nabla_0 u\|} \right) \circ (D^2 u)^*(x) \\ &= -\|\nabla_0 u\|^{p-2} \text{trace} [(D^2 u)^*(x)] \\ &\quad - (p-2) [\|\nabla_0 u\|^{p-4} (\nabla_0 u \otimes \nabla_0 u) \circ (D^2 u)^*(x)] \\ &= -\|\nabla_0 u\|^{p-2} \text{trace} [(D^2 u)^*] - (p-2) \|\nabla_0 u\|^{p-4} \langle (D^2 u)^* \nabla_0 u, \nabla_0 u \rangle \\ &= -\Delta_p u \end{aligned}$$

where we note

$$(\nabla_0 u \otimes \nabla_0 u) \circ (D^2 u)^*(x) = \langle (D^2 u)^* \nabla_0 u, \nabla_0 u \rangle$$

since  $(D^2 u)^*(x)$  is symmetric. □

Now, we are ready to prove Theorem 4.1.1. Owing to Corollary 3.0.8, we may restate Theorem 4.1.1 in terms of viscosity solutions. That is, we seek to prove the following:

**Theorem 4.1.4** ([BFF1], Theorem 4.2). *Assume  $1 < p < \infty$ . Let  $\Omega \subseteq \mathbb{H}_n$  and suppose  $u \in C_{\text{sub}}^1(\Omega)$  is a viscosity solution to*

$$-\Delta_p u = -\text{div}(\|\nabla_0 u\|^{p-2} \nabla_0 u) = 0$$

in  $\Omega \setminus \{x \in \Omega : u(x) = 0\}$ . Then  $u$  is a viscosity solution in  $\Omega$ .

*Proof.* Let  $Z := \{x \in \Omega : u(x) = 0\}$  and suppose  $u \in C_{\text{sub}}^1(\Omega)$  is a viscosity solution to

$$-\Delta_p u = -\text{div}(\|\nabla_0 u\|^{p-2} \nabla_0 u) = 0$$

in  $\Omega \setminus Z$ , but  $u$  is not a viscosity solution in all of  $\Omega$ . That is, we assume that there is some point  $z \in Z$  such that  $u$  is not both a viscosity supersolution and viscosity subsolution to the  $p$ -Laplace equation. Without loss of generality, we may assume  $u$  is not a viscosity subsolution to the  $p$ -Laplace equation. That is, there is some  $z \in Z$  and  $\varphi \in C_{\text{sub}}^2(\Omega)$  such that  $\varphi \in \mathcal{TA}(u, z)$  and

$$-\Delta_p \varphi(z) > 0, \quad (4.4)$$

or equivalently

$$-A(z, \nabla_0 \varphi(z)) \circ (D^2 \varphi)^*(z) > 0$$

by item (c) of Lemma 4.1.3. Recall we may assume that  $\nabla_0 \varphi(z) \neq 0$ .

By our assumptions and the construction of  $A$ , for any  $\varepsilon > 0$  chosen sufficiently small, there exists a constant  $C > 0$  such that for all  $x$  in the Heisenberg gauge ball  $B_{\mathcal{N}}(z, \varepsilon)$ , we have

$$\|\nabla_0 \varphi(x)\| > C, \quad \|\nabla_0 u(x)\| > C, \quad \text{and} \quad -A(x, \nabla_0 \varphi(x)) \circ (D^2 \varphi)^*(x) > C. \quad (4.5)$$

Next, define the operator  $L$ , acting on a  $C_{\text{sub}}^2$  function  $v$ , by

$$Lv(x) := -A(x, \nabla_0 u(x)) \circ (D^2 v)^*(x). \quad (4.6)$$

Since  $A$  is continuous in its entries by construction, then

$$\|A(x, \nabla_0 \varphi(x)) - A(x, \nabla_0 u(x))\| \rightarrow 0 \quad \text{as } x \rightarrow z.$$

Then it follows that

$$|A(x, \nabla_0 \varphi(x)) \circ (D^2 \varphi)^*(x) - A(x, \nabla_0 u(x)) \circ (D^2 \varphi)^*(x)| \rightarrow 0 \quad \text{as } x \rightarrow z. \quad (4.7)$$



By Equation (4.21), we have

$$\begin{aligned}
C &< -A(x, \nabla_0 \varphi(x)) \circ (D^2 \varphi)^*(x) \\
&= \left[ -A(x, \nabla_0 \varphi(x)) \circ (D^2 \varphi)^*(x) + A(x, \nabla_0 u(x)) \circ (D^2 \varphi)^*(x) \right] \\
&\quad - A(x, \nabla_0 u(x)) \circ (D^2 \varphi)^*(x) \\
&=: [T_1] + T_2.
\end{aligned}$$

By Equation (4.22), we see that

$$[T_1] = \left[ -A(x, \nabla_0 \varphi(x)) \circ (D^2 \varphi)^*(x) + A(x, \nabla_0 u(x)) \circ (D^2 \varphi)^*(x) \right] \rightarrow 0$$

as  $x \rightarrow z$ . It follows that

$$[T_1] + T_2 \rightarrow -A(z, \nabla_0 u(z)) \circ (D^2 \varphi)^*(z)$$

as  $x \rightarrow z$ . Therefore we can write

$$L\varphi(x) = -A(x, \nabla_0 u(x)) \circ (D^2 \varphi)^*(x) > \frac{C}{2} \quad (4.8)$$

for  $x \in B_{\mathcal{N}}(z, \varepsilon)$ , replacing  $\varepsilon > 0$  by a smaller value if necessary.

**Claim 4.1.5** ([BFF1], Claim 4.3). *Define  $\theta := \varphi - u$ . Then:*

(A)  $\theta(z) = 0$ .

(B)  $\theta(x) \geq 0$  in  $B_{\mathcal{N}}(z, \varepsilon)$ .

(C)  $L\theta(x) > \frac{C}{2} > 0$  in  $B_{\mathcal{N}}(z, \varepsilon) \setminus Z$  in the viscosity sense.

*Proof.* First observe that items (A) and (B) are trivially true by the definition of  $\theta$  since  $\varphi$  touches  $u$  from above. It remains to show item (C).

We consider  $\psi \in C_{\text{sub}}^2(\Omega)$  touching  $\theta$  from below at  $x \in B_{\mathcal{N}}(z, \varepsilon) \setminus Z$ . Then for  $y$  near  $x$ , we have by definition

$$0 \leq \theta(y) - \psi(y) = \varphi(y) - u(y) - \psi(y).$$

This leads to

$$u(y) \leq \varphi(y) - \psi(y).$$

Since  $\theta(x) = \psi(x)$  implies  $u(x) = \varphi(x) - \psi(x)$ , we have  $\varphi - \psi \in \mathcal{TA}(u, x)$ . Combining this fact with the identity  $-\Delta_{\mathbb{P}}u(x) = Lu(x)$  from Property (b) of Lemma 4.1.3, we have

$$L(\varphi - \psi)(x) \leq 0.$$

It follows from Equation (4.8) that

$$\frac{C}{2} < L\varphi(x) \leq L\psi(x).$$

□

Now since  $\nabla_0\varphi(z) \neq 0$ , there exists  $1 \leq k \leq 2n$  such that  $X_k\varphi(z) \neq 0$ . Then appealing to the Heisenberg Implicit Function Theorem [FSS, Proposition 3.12], there exists a continuous function  $f$  such that, locally at  $z$ , we may write the set

$$\{x \in B_{\mathcal{N}}(z, \varepsilon) : \varphi(x) = 0\}$$

as the graph of  $f$ . The continuity of  $f$  implies that for sufficiently small  $\delta > 0$  we may find an  $\tilde{x} \in B_{\mathcal{N}}(z, \varepsilon)$  such that we have  $\delta = d_{\mathcal{N}}(\tilde{x}, z) := \mathcal{N}(\tilde{x}^{-1} \cdot z)$  and

$$B_{\mathcal{N}}(\tilde{x}, \delta) \subset \{x \in B_{\mathcal{N}}(z, \varepsilon) : \varphi(x) < 0\}.$$

Set

$$\begin{aligned} \omega(x) &:= \frac{\sigma}{2} (\delta^4 - d_{\mathcal{N}}^4(x, \tilde{x})) \\ &= \frac{\sigma}{2} \left( \delta^4 - \left[ \left( \sum_{l=1}^{2n} (x_l - \tilde{x}_l)^2 \right)^2 + 16 \left( x_{2n+1} - \tilde{x}_{2n+1} + \frac{1}{2} \sum_{i=1}^n (x_{n+i}\tilde{x}_i - x_i\tilde{x}_{n+i}) \right)^2 \right] \right) \\ &= \frac{\sigma}{2} \delta^4 - \frac{\sigma}{2} \left( \sum_{l=1}^{2n} (x_l - \tilde{x}_l)^2 \right)^2 - 8\sigma \left( x_{2n+1} - \tilde{x}_{2n+1} + \frac{1}{2} \sum_{i=1}^n (x_{n+i}\tilde{x}_i - x_i\tilde{x}_{n+i}) \right)^2 \\ &=: \frac{\sigma}{2} \delta^4 - \frac{\sigma}{2} \left( \sum_{l=1}^{2n} (x_l - \tilde{x}_l)^2 \right)^2 - 8\sigma \Phi(x, \tilde{x}), \end{aligned}$$

where we assume that  $\sigma > 0$  and we delay our choice of particular  $\sigma$  for the moment. Then we have  $\omega \in C_{\text{sub}}^2(\overline{B_{\mathcal{N}}(\tilde{x}, \delta)})$  and so by definition of  $\omega$  and the vector fields, with  $1 \leq i \leq n < j \leq 2n$ , we compute:

$$X_i \omega(x) = -2\sigma(x_i - \tilde{x}_i) \sum_{l=1}^{2n} ((x_l - \tilde{x}_l)^2) + 8\sigma \tilde{x}_{i+n} \Phi + 8\sigma x_{i+n} \Phi$$

and

$$X_j \omega(x) = -2\sigma(x_j - \tilde{x}_j) \sum_{l=1}^{2n} ((x_l - \tilde{x}_l)^2) - 8\sigma \tilde{x}_i \Phi - 8\sigma x_{j-n} \Phi.$$

**Claim 4.1.6** ([BFF1], Claim 4.4). *For appropriate choice of  $\sigma > 0$ ,  $L\omega \leq \frac{C}{4}$  in  $B_{\mathcal{N}}(\tilde{x}, \delta)$ .*

*Proof.* For convenience, we define a function  $F : \overline{B_{\mathcal{N}}(\tilde{x}, \delta)} \rightarrow \mathbb{R}$  implicitly by the equation  $\omega(x) = \sigma F(x)$ . From the construction of  $A(x, \xi)$  and the definition of  $F$ , it is clear that the entries of  $A(x, \nabla_0 u(x))$ ,  $(D^2 F)^*(x)$  are continuous. From this, we observe that

$$LF(x) = \text{trace} \left[ A(x, \nabla_0 u(x)) \cdot (D^2 F)^*(x) \right]$$

(where  $\cdot$  is the matrix product) is continuous; and by the compactness of  $\overline{B_{\mathcal{N}}(\tilde{x}, \delta)}$  it follows that there exists  $C(\delta) > 0$  such that

$$L\omega(x) \leq \left| \text{trace} \left[ A(x, \nabla_0 u(x)) \cdot (D^2 \omega)^*(x) \right] \right| = \sigma |LF(x)| < \sigma C(\delta)$$

on  $\overline{B_{\mathcal{N}}(\tilde{x}, \delta)}$ . Since  $\delta > 0$  is fixed and  $C(\delta)$  depends only on  $\delta$ ,  $C(\delta)$  is fixed. Choosing  $\sigma$  sufficiently small, we therefore achieve the desired inequality.  $\square$

By definition of  $\omega$ , we have  $\omega \equiv 0$  on  $\partial B_{\mathcal{N}}(\tilde{x}, \delta)$ . Thus by Claim 4.1.5, we have

$$\omega \leq \theta \tag{4.9}$$

on  $\partial B_{\mathcal{N}}(\tilde{x}, \delta)$ .

**Claim 4.1.7** ([BFF1], Claim 4.5). *We have  $\omega \leq \theta$  in  $B_{\mathcal{N}}(\tilde{x}, \delta)$ .*

*Proof.* Suppose there is some point  $\tilde{y} \in \overline{B_{\mathcal{N}}(\tilde{x}, \delta)}$  such that

$$\theta(\tilde{y}) - \omega(\tilde{y}) = \min_{y \in \overline{B_{\mathcal{N}}(\tilde{x}, \delta)}} [\theta(y) - \omega(y)] < 0. \quad (4.10)$$

From Inequality (4.9), we must have  $\tilde{y} \in B_{\mathcal{N}}(\tilde{x}, \delta)$  because  $\tilde{y} \notin \partial B_{\mathcal{N}}(\tilde{x}, \delta)$ . Equation (4.10) therefore implies that we have

$$\omega + [\theta(\tilde{y}) - \omega(\tilde{y})] \in \mathcal{TB}(\theta, \tilde{y}).$$

Claim 4.1.5 then asserts  $L\omega(\tilde{y}) \geq \frac{C}{2}$ . However, this contradicts Claim 4.1.6. We then conclude that  $\omega \leq \theta$  in  $\overline{B_{\mathcal{N}}(\tilde{x}, \delta)}$ .  $\square$

Claim 4.1.7 and the fact that  $\omega$  is 0 on the boundary of the ball motivate the following claim:

**Claim 4.1.8** ([BFF1], Claim 4.6). *Suppose for some  $x \in \partial B_{\mathcal{N}}(\tilde{x}, \delta)$  we have  $\theta(x) = 0 = \omega(x)$ . Then denoting the outward Heisenberg normal to  $\partial B_{\mathcal{N}}(\tilde{x}, \delta)$  at  $x$  by  $\nu$ , we have*

$$\frac{\partial \theta}{\partial \nu}(x) \leq \frac{\partial \omega}{\partial \nu}(x). \quad (4.11)$$

*Proof.* To ensure that we remain in the domain of definition, we will consider only negative multiples of  $\nu$ . Therefore we let  $h < 0$  and observe that

$$\frac{\theta(x + h\nu)}{h} \leq \frac{\omega(x + h\nu)}{h}.$$

This implies at once that

$$\lim_{h \uparrow 0} \frac{\theta(x + h\nu) - \theta(x)}{h} \leq \lim_{h \uparrow 0} \frac{\omega(x + h\nu) - \omega(x)}{h},$$

from which Inequality (4.23) follows.  $\square$

Recalling the definition of  $\omega$ , it follows that

$$\nabla_0 \omega(z) = \langle \alpha_1, \alpha_2, \dots, \alpha_k, \dots, \alpha_{2n} \rangle$$

where

$$\alpha_k = \begin{cases} -2\sigma \left( (z_k - \tilde{x}_k) \sum_{l=1}^{2n} \left( (z_l - \tilde{x}_l)^2 \right) - 4\Phi(z, \tilde{x})(\tilde{x}_{k+n} + z_{k+n}) \right), & 1 \leq k \leq n \\ -2\sigma \left( (z_k - \tilde{x}_k) \sum_{l=1}^{2n} \left( (z_l - \tilde{x}_l)^2 \right) + 4\Phi(z, \tilde{x})(\tilde{x}_{k-n} + z_{k-n}) \right), & n+1 \leq k \leq 2n. \end{cases}$$

Next, the exterior normal at  $z$  to  $B_{\mathcal{N}}(\tilde{x}, \delta)$  is given by [AF] and [DGN] as

$$\nu = \langle \beta_1, \beta_2, \dots, \beta_k, \dots, \beta_{2n} \rangle$$

where we have:

$$\beta_k = \begin{cases} 2 \left( (z_k - \tilde{x}_k) \sum_{l=1}^{2n} \left( (z_l - \tilde{x}_l)^2 \right) - 4\Phi(z, \tilde{x})(\tilde{x}_{k+n} + z_{k+n}) \right), & 1 \leq k \leq n \\ 2 \left( (z_k - \tilde{x}_k) \sum_{l=1}^{2n} \left( (z_l - \tilde{x}_l)^2 \right) + 4\Phi(z, \tilde{x})(\tilde{x}_{k-n} + z_{k-n}) \right), & n+1 \leq k \leq 2n. \end{cases}$$

Then

$$\begin{aligned} \nu \cdot \nabla_0 \omega(z) &= \sum_{k=1}^{2n} \alpha_k \beta_k \\ &= -4\sigma \left[ \sum_{i=1}^n \left( (z_i - \tilde{x}_i) \sum_{l=1}^{2n} \left( (z_l - \tilde{x}_l)^2 \right) - 4\Phi(z, \tilde{x})(\tilde{x}_{i+n} + z_{i+n}) \right)^2 \right. \\ &\quad \left. + \sum_{j=n+1}^{2n} \left( (z_j - \tilde{x}_j) \sum_{l=1}^{2n} \left( (z_l - \tilde{x}_l)^2 \right) + 4\Phi(z, \tilde{x})(\tilde{x}_{j-n} + z_{j-n}) \right)^2 \right] \\ &= -4\sigma \sum_{i=1}^n \left( (z_i - \tilde{x}_i) \sum_{l=1}^{2n} \left( (z_l - \tilde{x}_l)^2 \right) - 4\Phi(z, \tilde{x})(\tilde{x}_{i+n} + z_{i+n}) \right)^2 \\ &\quad - 4\sigma \sum_{j=n+1}^{2n} \left( (z_j - \tilde{x}_j) \sum_{l=1}^{2n} \left( (z_l - \tilde{x}_l)^2 \right) - 4\Phi(z, \tilde{x})(\tilde{x}_{j-n} + z_{j-n}) \right)^2 \\ &< 0 \end{aligned}$$

since  $\sigma > 0$ . Invoking Claim 4.1.8 for  $x = z$ , we have shown that:

$$\frac{d\theta}{d\nu}(z) \leq \frac{d\omega}{d\nu}(z) = \nu \cdot \nabla_0 \omega(z) < 0,$$

which is a contradiction since  $\nabla_0 \theta(z) = 0$ , implying  $\nu \cdot \nabla_0 \theta(z) = 0$ . Therefore,  $u$  is a viscosity solution of  $-\Delta_p u = 0$  in all of  $\Omega$ .  $\square$

By [B2, Corollary 4.8] and Theorem 4.1.4 we conclude that Theorem 4.1.1 holds.

We conclude this section with a brief discussion about our assumption that  $u \in C_{\text{sub}}^1(\Omega)$ . In [JL2] the authors demonstrate that weakened regularity such as  $u \in \text{Lip}(\Omega)$  is insufficient to guarantee the Euclidean cases of the above-proven results; the counterexample that follows is an adaptation of that work to the setting  $\mathbb{H}_n$ . In particular, we will produce a Heisenberg Lipschitz function  $u$  such that:

- (i)  $u$  is a viscosity solution to the  $p(x)$ -Laplace equation for  $2 \leq p < \infty$  in  $B \setminus Z$  where  $B$  is a Carnot-Carathéodory ball centered at 0 and  $Z$  is defined as above.
- (ii)  $0 \in Z$ .
- (iii)  $u$  is not a viscosity solution of the  $p$ -Laplace equation at 0.

Fix  $\xi_0 = \langle \xi_0^1, \dots, \xi_0^{2n}, 0 \rangle \in h_n \setminus \{0\}$ . For  $R > 0$ , define the function  $u : \overline{B(0, R)} \rightarrow \mathbb{R}$  by (abusing notation since the exponential map is the identity)

$$u(x) := 2|\langle \xi_0, x \rangle| = 2 \left| \sum_{k=1}^{2n} \xi_0^k x_k \right|.$$

Then  $u(0) = 0$  and  $u \in C_{\text{sub}}^1(\overline{B(0, R)} \setminus Z)$ .

**Claim 4.1.9** ([BFF1], Claim 5.1). *The function  $u$  is Heisenberg Lipschitz on  $\overline{B(0, R)}$ .*

*Proof.* Let  $x, y \in \overline{B(0, R)}$ . Employing the Cauchy-Schwarz inequality,

$$\begin{aligned} |u(x) - u(y)| &\leq 2 \left| \sum_{k=1}^{2n} \xi_0^k (x_k - y_k) \right| \\ &\leq 2 \max_{1 \leq k \leq 2n} |\xi_0^k| \left( \left[ \sum_{k=1}^{2n} (x_k - y_k)^2 \right] \right)^{\frac{1}{2}} \\ &\leq 2\sqrt{2n} \max_{1 \leq k \leq 2n} |\xi_0^k| \left( \sum_{k=1}^{2n} (x_k - y_k)^2 \right)^{\frac{1}{2}} \\ &=: C(\xi_0, n) \left( \sum_{k=1}^{2n} (x_k - y_k)^2 \right)^{\frac{1}{2}}. \end{aligned} \tag{4.12}$$

Comparing the terminal line of (4.12) to the Euclidean distance  $d_{\text{Eucl}}(\cdot, \cdot)$  and invoking [NSW, Proposition

1.1], there exists a constant  $C_1 = C_1(\overline{B(0, R)}) > 0$  so that

$$|u(x) - u(y)| \leq C(\xi_0, n)d_{\text{Eucl}}(x, y) \leq C_1 \cdot C(\xi_0, n)d_{CC}(x, y).$$

Since  $\overline{B(0, R)}$ ,  $\xi_0$ , and  $n$  are fixed, we conclude that  $u$  is Heisenberg Lipschitz.  $\square$

Fixing  $x_0 \in \overline{B(0, R)} \setminus Z$ , for  $1 \leq i \leq n < j \leq 2n$  we compute

$$\left\{ \begin{array}{l} X_i u(x_0) = \pm \xi_0^i \\ X_j u(x_0) = \pm \xi_0^j \\ X_i X_j u(x_0) = \pm \frac{1}{2} \xi_0^{2n+1} = 0 \\ X_j X_i u(x_0) = \mp \frac{1}{2} \xi_0^{2n+1} = 0 \\ X_i X_i u(x_0) = 0 \\ X_j X_j u(x_0) = 0. \end{array} \right.$$

It follows that  $(D^2 u)^* = 0$  off  $Z$ , and so  $-\Delta_{\mathbb{P}} u = 0$  in  $\overline{B(0, R)} \setminus Z$  in the classical sense.

Again abusing notation, we consider the function

$$\phi(x) := \langle \xi_0, x \rangle + \frac{1}{2} \langle (\xi_0 \otimes \xi_0)x, x \rangle = \langle \xi_0, x \rangle + \frac{1}{2} (\langle \xi_0, x \rangle)^2.$$

Then  $\phi \in C_{\text{sub}}^2(\overline{B(0, R)})$  and  $\phi(0) = 0 = u(0)$ .

**Claim 4.1.10** ([BFF1], Claim 5.2).  $\phi \in \mathcal{TB}(u, 0)$ .

*Proof.* Since  $u$  is Heisenberg Lipschitz, there exists  $\delta > 0$  so that  $x_0 \in B(0, \delta)$  implies

$$u(x_0) = |u(x_0) - u(0)| < 2, \tag{4.13}$$

that is,  $|\langle \xi_0, x_0 \rangle| < 1$ . In particular, we have  $(\langle \xi_0, x_0 \rangle)^2 < |\langle \xi_0, x_0 \rangle| < 1$ . Then we have shown

$$\phi(x_0) < |\langle \xi_0, x_0 \rangle| + \frac{1}{2} |\langle \xi_0, x_0 \rangle| < u(x_0)$$

for  $x_0$  near to 0. Since  $\phi(0) = u(0)$ , our claim is proven.  $\square$

We compute:

$$\left\{ \begin{array}{l} X_i \phi(0) = \xi_0^i \\ X_j \phi(0) = \xi_0^j \\ X_i X_i \phi(0) = (\xi_0^i)^2 \\ X_j X_j \phi(0) = (\xi_0^j)^2 \\ X_i X_j \phi(0) = \xi_0^j \xi_0^i \\ X_j X_i \phi(0) = \xi_0^j \xi_0^i. \end{array} \right.$$

From the above,

$$\nabla_0 \phi(0) = \bar{\xi}_0 := \langle \xi_0^1, \dots, \xi_0^{2n} \rangle \in \mathbb{R}^{2n} \setminus \{0\}$$

and

$$((D^2 \phi)^*(x))_{a,b} = \xi_0^a \xi_0^b \text{ for all } 1 \leq a, b \leq 2n.$$

Then for  $2 \leq p < \infty$ , we have

$$\begin{aligned} -\Delta_p \phi(0) &= -\|\nabla_0 \phi(0)\|^{p-2} \text{trace}[(D^2 \phi)^*(0)] - (p-2)\|\nabla_0 \phi(0)\|^{p-4} \langle (D^2 \phi)^*(0) \nabla_0 \phi(0), \nabla_0 \phi(0) \rangle \\ &= -\|\xi_0\|^{p-2} \sum_{k=1}^{2n} (\xi_0^k)^2 - (p-2)\|\xi_0\|^{p-4} \langle \eta, \bar{\xi}_0 \rangle. \end{aligned}$$

where we define

$$\eta := \left\langle \xi_0^1 \sum_{k=1}^{2n} (\xi_0^k)^2, \dots, \xi_0^{2n} \sum_{k=1}^{2n} (\xi_0^k)^2 \right\rangle \in \mathbb{R}^{2n}.$$

Simplification yields

$$-\Delta_p \phi(0) = -\|\xi_0\|^{p-2} \sum_{k=1}^{2n} (\xi_0^k)^2 - (p-2)\|\xi_0\|^{p-4} \left[ \sum_{k=1}^{2n} (\xi_0^k)^2 \right]^2 = -(p-1)\|\xi_0\|^p < 0.$$

Because  $\phi \in \mathcal{TB}(u, 0)$ , the definition of viscosity solutions implies that  $u$  is not a supersolution to the  $p$ -Laplace equation.

□



## 4.2 The Case of the $p(x)$ -Laplace Equation <sup>2</sup>

In [JLP, Section 7], the authors extended the Radó-type removability result of [JL1] to  $p(x)$ -harmonic functions in  $\mathbb{R}^n$  as an application of the equivalence of potential theoretic weak solutions and viscosity solutions to the  $p(x)$ -Laplace equation. Specifically, they obtain the removability of level sets for viscosity solutions to the  $p(x)$ -Laplace equation, which is given by

$$-\Delta_{p(x)}u(x) := -\operatorname{div} \left( |\nabla u(x)|^{p(x)-2} \nabla u(x) \right) = 0 \quad (4.14)$$

in  $\Omega \subseteq \mathbb{R}^n$  for  $1 < p(x) < \infty$ , where  $u \in C^1(\Omega)$  and  $p \in C^1(\Omega)$ .

The purpose of this section is to extend the constant exponent results in Section 4.1 pertaining to the removability of level sets for viscosity solutions to the  $p$ -Laplace equation in the Heisenberg group to the variable exponent case. Consequently, this also extends the application in Euclidean space [JLP, Section 7] to the Heisenberg group. Recall the  $p(x)$ -Laplace equation in the Heisenberg group is defined by:

$$-\Delta_{p(x)}u(x) := -\operatorname{div} \left( \|\nabla_0 u(x)\|^{p(x)-2} \nabla_0 u(x) \right) = 0 \quad (4.15)$$

in  $\Omega \subseteq \mathbb{H}_n$  for  $1 < p(x) < \infty$  where  $p \in C^1_{\text{sub}}(\Omega)$ . Also recall the nondivergent form of the  $p(x)$ -Laplace equation (for easier reference in this section) is given by:

$$\begin{aligned} - \left( \|\nabla_0 u\|^{p(x)-2} \operatorname{trace}((D^2 u)^*) + (p(x) - 2) \|\nabla_0 u\|^{p(x)-4} \langle (D^2 u)^* \nabla_0 u, \nabla_0 u \rangle \right. \\ \left. + \|\nabla_0 u\|^{p(x)-2} \log(\|\nabla_0 u\|) \langle \nabla_0 p(x), \nabla_0 u \rangle \right) = 0 \end{aligned} \quad (4.16)$$

in a bounded domain  $\Omega \subset \mathbb{H}_n$ , where  $u \in C^1_{\text{sub}}(\Omega)$  and  $p \in C^1_{\text{sub}}(\Omega)$ . (Note that we need  $p \in C^1_{\text{sub}}(\Omega)$  in order to apply the viscosity theory.) The main result of this section is the following theorem:

**Theorem 4.2.1** ([BFF2], Main Theorem). *Let  $\Omega \subseteq \mathbb{H}_n$  be a bounded domain and assume  $1 < p(x) < \infty$  with  $p \in C^1_{\text{sub}}(\Omega)$ . Suppose  $u \in C^1_{\text{sub}}(\Omega)$  is a viscosity solution to Equation (4.16) in  $\Omega \setminus \{x \in \Omega : u(x) = 0\}$ . Then  $u$  is a viscosity solution to Equation (4.16) in  $\Omega$ .*

The outline of the proof of Theorem 4.2.1 follows the proof of the fixed exponent case in Section 4.1, but requires modifications owing ultimately to the variable exponent, and in turn, the logarithmic term which

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<sup>2</sup>A Note to Reader: This section has been reproduced from [BFF2]. This work has been submitted for review.

does not appear when the exponent is constant. A necessary instrument we will use to prove Theorem 4.2.1 is the following theorem:

**Theorem 4.2.2** ([BFF2], Theorem 1.1). *Assume  $1 < p(x) < \infty$  with  $p \in C_{\text{sub}}^1(\Omega)$ . Let  $\Omega \subseteq \mathbb{H}_n$  be a domain, and suppose that  $u \in C_{\text{sub}}^1(\Omega)$  is a viscosity solution to the  $p(x)$ -Laplace equation in  $\Omega \setminus \{x \in \Omega : \nabla_0 u(x) = 0\}$ . Then  $u$  is a viscosity solution to the  $p(x)$ -Laplace equation in  $\Omega$ .*

*Proof.* Let  $u$  be a viscosity solution to Equation (4.16) in  $\Omega \setminus \{x \in \Omega : \nabla_0 u(x) = 0\}$ . If  $\varphi$  is a touching function, touching (from above or below) at the point  $z \in \Omega$ , then by regularity of  $u$  and  $\varphi$ , we must have

$$\nabla_0 \varphi(z) = \nabla_0 u(z).$$

By the definition of viscosity solutions, if  $\nabla_0 \varphi(z) = 0 = \nabla_0 u(z)$ , then we have nothing to prove. If  $\nabla_0 u(z) \neq 0$ , then by the definition of feeble viscosity solutions we have that  $u$  is viscosity solution in  $\Omega$ .  $\square$

Before we prove Theorem 4.2.1, we mention some notation we will use throughout this section. For any vector  $\xi \neq 0$ , we let  $\xi \otimes \xi$  be the matrix with entries  $\xi_i \xi_j$ . Then we define  $A : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \rightarrow S^{2n}$ :

$$\begin{aligned} A(x, \xi) &= |\xi|^{p(x)-2} \left( I + (p(x) - 2) \frac{\xi}{|\xi|} \otimes \frac{\xi}{|\xi|} \right) \\ &= |\xi|^{p(x)-2} I + (p(x) - 2) |\xi|^{p(x)-4} \xi \otimes \xi, \end{aligned} \quad (4.17)$$

and note that  $A_{ij}$  has continuous entries. We also define  $B : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \rightarrow \mathbb{R}$ :

$$B(x, \xi) = |\xi|^{p(x)-2} \log |\xi| \langle \xi, \nabla_0 p(x) \rangle, \quad (4.18)$$

and note that  $B$  is continuous in both  $x$  and  $\xi$ . We have a lemma involving some key properties of  $A$  and  $B$ .

**Lemma 4.2.3** ([BFF2], Lemma 5.1). *Let  $\Omega \subseteq \mathbb{H}_n$  and assume  $1 < p(x) < \infty$  with  $p \in C_{\text{sub}}^1(\Omega)$ . Define the mapping  $A$  as in Equation (4.17). Then:*

(a)  *$A$  is symmetric.*

(b) *We can write*

$$-\Delta_{p(x)} u(x) = -A(x, \nabla_0 u(x)) \circ (D^2 u)^*(x) - B(x, \nabla_0 u(x)), \quad (4.19)$$

where  $M \circ N$  is the trace of the matrix resulting from the matrix product  $MN$ .

*Proof.* Notice that item (a) is trivially true by the construction of  $A$ . Observe that

$$\begin{aligned}
-A(x, \nabla_0 u) \circ ((D^2 u)^*(x)) &= -\|\nabla_0 u\|^{p(x)-2} \left( I + (p(x) - 2) \frac{\nabla_0 u}{\|\nabla_0 u\|} \otimes \frac{\nabla_0 u}{\|\nabla_0 u\|} \right) \circ (D^2 u)^*(x) \\
&= -\|\nabla_0 u\|^{p(x)-2} \text{trace}((D^2 u)^*(x)) \\
&\quad - (p(x) - 2) \|\nabla_0 u\|^{p(x)-4} (\|\nabla_0 u\| \otimes \|\nabla_0 u\|) \circ (D^2 u)^*(x) \\
&= -\|\nabla_0 u\|^{p(x)-2} \text{trace}((D^2 u)^*(x)) \\
&\quad - (p(x) - 2) \|\nabla_0 u\|^{p(x)-4} \langle (D^2 u)^* \nabla_0 u, \nabla_0 u \rangle
\end{aligned}$$

where we write

$$(\|\nabla_0 u\| \otimes \|\nabla_0 u\|) \circ (D^2 u)^*(x) = \langle (D^2 u)^* \nabla_0 u, \nabla_0 u \rangle$$

since  $(D^2 u)^*(x)$  is symmetric. Then we have

$$\begin{aligned}
&-A(x, \nabla_0 u) \circ ((D^2 u)^*(x)) - B(x, \nabla_0 u(x)) \\
&= -\|\nabla_0 u\|^{p(x)-2} \text{trace}((D^2 u)^*(x)) - (p(x) - 2) \|\nabla_0 u\|^{p(x)-4} \langle (D^2 u)^* \nabla_0 u, \nabla_0 u \rangle \\
&\quad - |\nabla_0 u|^{p(x)-2} \log |\nabla_0 u| \langle \nabla_0 u, \nabla_0 p(x) \rangle \\
&= -\Delta_{p(x)} u(x).
\end{aligned}$$

Item (b) follows. □

We now prove our main result. The proof below is the variable exponent version of the fixed exponent case in Section 4.1 and incorporates many necessary modifications.

*Proof of Theorem 4.2.1.* Let  $Z := \{x \in \Omega : u(x) = 0\}$  and suppose  $u \in C_{\text{sub}}^1(\Omega)$  is a viscosity solution to

$$-\Delta_{p(x)} u = -\text{div} \left( \|\nabla_0 u\|^{p(x)-2} \nabla_0 u \right) = 0$$

in  $\Omega \setminus Z$ , but  $u$  is not a viscosity solution in  $\Omega$ . That is, we assume that there is some point  $z \in Z$  such that  $u$  is not both a viscosity supersolution and viscosity subsolution to the  $p(x)$ -Laplace equation. Without loss of generality, we may assume  $u$  is not a viscosity subsolution to the  $p(x)$ -Laplace equation. That is, there is

some  $z \in Z$  and  $\varphi \in \text{sub}^2(\Omega)$  such that  $\varphi \in \mathcal{TA}(u, z)$  and

$$-\Delta_{\mathbb{P}(x)}\varphi(z) > 0, \quad (4.20)$$

or equivalently

$$-A(z, \nabla_0\varphi(z)) \circ (D^2\varphi)^*(z) - B(z, \nabla_0\varphi(z)) > 0$$

by item (b) of Lemma 4.2.3. We may assume without loss of generality that  $\nabla_0\varphi(z) \neq 0$ .

By our assumptions and the construction of  $A$  and  $B$ , for any  $\varepsilon > 0$  chosen sufficiently small, there exists a constant  $C > 0$  such that for all  $x$  in the Heisenberg gauge ball  $B_{\mathcal{N}}(z, \varepsilon)$ , we have

$$\|\nabla_0\varphi(x)\| > C, \quad \|\nabla_0u(x)\| > C, \quad \text{and} \quad -A(x, \nabla_0\varphi(x)) \circ (D^2\varphi)^*(x) - B(x, \nabla_0\varphi(x)) > C. \quad (4.21)$$

We denote

$$\tilde{A}(x) := A(x, \nabla_0u(x))$$

and

$$\tilde{B}(x) := B(x, \nabla_0u(x)).$$

Observe

$$\begin{aligned} & \| [A(x, \nabla_0\varphi(x)) + B(x, \nabla_0\varphi(x))] - [\tilde{A}(x) + \tilde{B}(x)] \| \\ &= \| [A(x, \nabla_0\varphi(x)) - \tilde{A}(x)] + [B(x, \nabla_0\varphi(x)) - \tilde{B}(x)] \| \\ &=: \| \mathcal{T} + \mathcal{S} \| \leq \| \mathcal{T} \| + \| \mathcal{S} \|. \end{aligned}$$

Since  $A$  is continuous in its entries by construction, then

$$\| \mathcal{T} \| \rightarrow 0 \quad \text{as } x \rightarrow z.$$

Moreover, since  $B$  is continuous in both  $x$  and  $\nabla_0u(x)$  by construction, then

$$\| \mathcal{S} \| \rightarrow 0 \quad \text{as } x \rightarrow z.$$

It follows that

$$\|\mathcal{T}\| + \|\mathcal{S}\| \rightarrow 0 \text{ as } x \rightarrow z. \quad (4.22)$$

By Equation (4.21), we have

$$\begin{aligned} C &< -A(x, \nabla_0 \varphi(x)) \circ (D^2 \varphi)^*(x) - B(x, \nabla_0 \varphi(x)) \\ &= -A(x, \nabla_0 \varphi(x)) \circ (D^2 \varphi)^*(x) - B(x, \nabla_0 \varphi(x)) \\ &\quad + \tilde{A}(x) \circ (D^2 \varphi)^*(x) + \tilde{B}(x) \\ &\quad - \tilde{A}(x) \circ (D^2 \varphi)^*(x) - \tilde{B}(x) \\ &= [\tilde{A}(x) \circ (D^2 \varphi)^*(x) - A(x, \nabla_0 \varphi(x)) \circ (D^2 \varphi)^*(x) \\ &\quad + \tilde{B}(x) - B(x, \nabla_0 \varphi(x))] \\ &\quad - \tilde{A}(x) \circ (D^2 \varphi)^*(x) - \tilde{B}(x) \\ &=: [\mathcal{T} + \mathcal{S}] - \tilde{\mathcal{S}}. \end{aligned}$$

By Equation (4.22), we see that

$$[\mathcal{T} + \mathcal{S}] \rightarrow 0$$

as  $x \rightarrow z$ . It follows that

$$[\mathcal{T} + \mathcal{S}] - \tilde{\mathcal{S}} \rightarrow -\tilde{A}(z) \circ (D^2 \varphi)^*(z) - \tilde{B}(z)$$

as  $x \rightarrow z$ . Therefore we can write

$$-\tilde{A}(x) \circ (D^2 \varphi)^*(x) - \tilde{B}(x) > \frac{C}{2}$$

for  $x \in B_{\mathcal{N}}(z, \varepsilon)$ , replacing  $\varepsilon > 0$  by a smaller value if necessary.

**Claim 4.2.4** ([BFF2], Claim 5.2). *Define  $\theta := \varphi - u$ . Then:*

(A)  $\theta(z) = 0$ .

(B)  $\theta(x) \geq 0$  in  $B_{\mathcal{N}}(z, \varepsilon)$ .

(C)  $-\tilde{A}(x) \circ (D^2\theta)^*(x) \geq \frac{C}{2} > 0$  in  $B_{\mathcal{N}}(z, \varepsilon) \setminus Z$  in the viscosity sense.

*Proof of Claim 4.2.4.* First observe that items (A) and (B) are trivially true by the definition of  $\theta$  since  $\varphi$  touches  $u$  from above. It remains to show item (C).

We consider  $\psi \in \text{sub}^2(\Omega)$  touching  $\theta$  from below at  $x \in B_{\mathcal{N}}(z, \varepsilon) \setminus Z$ . Then for  $y$  near  $x$ , we have by definition

$$0 \leq \theta(y) - \psi(y) = \varphi(y) - u(y) - \psi(y).$$

This leads to

$$u(y) \leq \varphi(y) - \psi(y).$$

Since  $\theta(x) = \psi(x)$  implies  $u(x) = \varphi(x) - \psi(x)$ , we have  $\varphi - \psi \in \mathcal{TA}(u, x)$ . Owing to the fact that  $u \in \text{sub}^1(\Omega)$  we have  $\nabla_0 u(x) = \nabla_0(\varphi - \psi)(x)$ , and from Property (b) of Lemma 4.2.3:

$$-\tilde{A}(x) \circ ((D^2[\varphi - \psi])^*(x)) - \tilde{B}(x) \leq 0.$$

Since  $(D^2[\varphi - \psi])^*(x) = (D^2\varphi)^*(x) - (D^2\psi)^*(x)$ , the trace operation on matrices is linear, and the operation of multiplication of matrices is distributive, then we can write

$$-\tilde{A}(x) \circ ((D^2[\varphi - \psi])^*(x)) = -\tilde{A}(x) \circ ((D^2\varphi)^*(x)) + \tilde{A}(x) \circ ((D^2\psi)^*(x)).$$

It follows that

$$0 < \frac{C}{2} \leq -\tilde{A}(x) \circ (D^2\varphi)^*(x) - \tilde{B}(x) \leq -\tilde{A}(x) \circ (D^2\psi)^*(x)$$

and we have the claim. □

Now, since  $\nabla_0\varphi(z) \neq 0$ , there exists  $1 \leq k \leq 2n$  such that  $X_k\varphi(z) \neq 0$ . Then appealing to the Heisenberg Implicit Function Theorem [FSS, Proposition 3.12], there exists a continuous function  $f$  such that, locally at  $z$ , we may write the set

$$\{x \in B_{\mathcal{N}}(z, \varepsilon) : \varphi(x) = 0\}$$

as the graph of  $f$ . The continuity of  $f$  implies that for sufficiently small  $\delta > 0$  we may find an  $\tilde{x} \in B_{\mathcal{N}}(z, \varepsilon)$

such that we have  $\delta = d_{\mathcal{N}}(\tilde{x}, z) := \mathcal{N}(\tilde{x}^{-1} \cdot z)$  and

$$B_{\mathcal{N}}(\tilde{x}, \delta) \subset \{x \in B_{\mathcal{N}}(z, \varepsilon) : \varphi(x) < 0\}.$$

Set

$$\begin{aligned} \omega(x) &:= \frac{\sigma}{2} (\delta^4 - d_{\mathcal{N}}^4(x, \tilde{x})) \\ &= \frac{\sigma}{2} \left( \delta^4 - \left[ \left( \sum_{l=1}^{2n} (x_l - \tilde{x}_l)^2 \right)^2 + 16 \left( x_{2n+1} - \tilde{x}_{2n+1} + \frac{1}{2} \sum_{i=1}^n (x_{n+i} \tilde{x}_i - x_i \tilde{x}_{n+i}) \right)^2 \right] \right) \\ &= \frac{\sigma}{2} \delta^4 - \frac{\sigma}{2} \left( \sum_{l=1}^{2n} (x_l - \tilde{x}_l)^2 \right)^2 - 8\sigma \left( x_{2n+1} - \tilde{x}_{2n+1} + \frac{1}{2} \sum_{i=1}^n (x_{n+i} \tilde{x}_i - x_i \tilde{x}_{n+i}) \right)^2 \\ &=: \frac{\sigma}{2} \delta^4 - \frac{\sigma}{2} \left( \sum_{l=1}^{2n} (x_l - \tilde{x}_l)^2 \right)^2 - 8\sigma \Phi(x, \tilde{x}), \end{aligned}$$

where we assume that  $\sigma > 0$  and we delay our choice of particular  $\sigma$  for the moment. Then we have  $\omega \in \text{sub}^2(\overline{B_{\mathcal{N}}(\tilde{x}, \delta)})$  and so by definition of  $\omega$  and the vector fields, with  $1 \leq i \leq n < j \leq 2n$ , we compute:

$$\begin{aligned} X_i \omega(x) &= -2\sigma (x_i - \tilde{x}_i) \sum_{l=1}^{2n} ((x_l - \tilde{x}_l)^2) + 8\sigma \tilde{x}_{i+n} \Phi(x, \tilde{x}) + 8\sigma x_{i+n} \Phi(x, \tilde{x}) \\ X_j \omega(x) &= -2\sigma (x_j - \tilde{x}_j) \sum_{l=1}^{2n} ((x_l - \tilde{x}_l)^2) - 8\sigma \tilde{x}_i \Phi(x, \tilde{x}) - 8\sigma x_{j-n} \Phi(x, \tilde{x}). \end{aligned}$$

In order to obtain necessary results, we state a property of  $\omega$  which parallels part (C) of Claim 4.2.4:

**Claim 4.2.5** ([BFF1], Claim 4.4). *For appropriate choice of  $\sigma > 0$ ,*

$$\tilde{A}(x) \circ (D^2 \omega)^*(x) \leq \frac{C}{4}$$

in  $B_{\mathcal{N}}(\tilde{x}, \delta)$ .

We now present two claims we need to finish the proof. Each of these claims is proved in Section 4.1 where the variable exponent is constant. The proofs here are identical and omitted.

**Claim 4.2.6** ([BFF1], Claim 4.5).  $\omega \leq \theta$  in  $B_{\mathcal{N}}(\tilde{x}, \delta)$ .

**Claim 4.2.7** ([BFF2], Claim 4.6). *Suppose for some  $x \in \partial B_{\mathcal{N}}(\tilde{x}, \delta)$  we have  $\theta(x) = 0 = \omega(x)$ . Then denoting the outward Heisenberg normal to  $\partial B_{\mathcal{N}}(\tilde{x}, \delta)$  at  $x$  by  $\nu$ , we have*

$$\frac{\partial \theta}{\partial \nu}(x) \leq \frac{\partial \omega}{\partial \nu}(x). \quad (4.23)$$

Recalling the definition of  $\omega$ , it follows that

$$\nabla_0 \omega(z) = \langle \alpha_1, \alpha_2, \dots, \alpha_k, \dots, \alpha_{2n} \rangle$$

where

$$\alpha_k = \begin{cases} -2\sigma \left( (z_i - \tilde{x}_i) \sum_{l=1}^{2n} ((z_l - \tilde{x}_l)^2) - 4\Phi(z, \tilde{x})(\tilde{x}_{i+n} + z_{i+n}) \right), & 1 \leq k \leq n \\ -2\sigma \left( (z_j - \tilde{x}_j) \sum_{l=1}^{2n} ((z_l - \tilde{x}_l)^2) + 4\Phi(z, \tilde{x})(\tilde{x}_{j-n} + z_{j-n}) \right), & n+1 \leq k \leq 2n. \end{cases}$$

Next, referencing [AF] in order to compute the exterior normal at  $z$  to the Heisenberg gauge ball, denoted

$$\nu = \langle \beta_1, \beta_2, \dots, \beta_k, \dots, \beta_{2n} \rangle,$$

we have:

$$\beta_k = \begin{cases} 2 \left( (z_i - \tilde{x}_i) \sum_{l=1}^{2n} ((z_l - \tilde{x}_l)^2) - 4\Phi(z, \tilde{x})(\tilde{x}_{i+n} + z_{i+n}) \right), & 1 \leq k \leq n \\ 2 \left( (z_j - \tilde{x}_j) \sum_{l=1}^{2n} ((z_l - \tilde{x}_l)^2) + 4\Phi(z, \tilde{x})(\tilde{x}_{j-n} + z_{j-n}) \right), & n+1 \leq k \leq 2n. \end{cases}$$



Then

$$\begin{aligned}
\nu \cdot \nabla_0 \omega(z) &= \sum_{k=1}^{2n} \alpha_k \beta_k \\
&= -4\sigma \left[ \sum_{i=1}^n \left( (z_i - \tilde{x}_i) \sum_{l=1}^{2n} ((z_l - \tilde{x}_l)^2) - 4\Phi(z, \tilde{x})(\tilde{x}_{i+n} + z_{i+n}) \right)^2 \right. \\
&\quad \left. + \sum_{j=n+1}^{2n} \left( (z_j - \tilde{x}_j) \sum_{l=1}^{2n} ((z_l - \tilde{x}_l)^2) + 4\Phi(z, \tilde{x})(\tilde{x}_{j-n} + z_{j-n}) \right)^2 \right] \\
&= -4\sigma \sum_{i=1}^n \left( (z_i - \tilde{x}_i) \sum_{l=1}^{2n} ((z_l - \tilde{x}_l)^2) - 4\Phi(z, \tilde{x})(\tilde{x}_{i+n} + z_{i+n}) \right)^2 \\
&\quad - 4\sigma \sum_{j=n+1}^{2n} \left( (z_j - \tilde{x}_j) \sum_{l=1}^{2n} ((z_l - \tilde{x}_l)^2) - 4\Phi(z, \tilde{x})(\tilde{x}_{j-n} + z_{j-n}) \right)^2 \\
&< 0
\end{aligned}$$

since  $\sigma > 0$ . Invoking Claim 4.2.7 for  $x = z$ , we have shown that:

$$\frac{d\theta}{d\nu}(z) \leq \frac{d\omega}{d\nu}(z) = \nu \cdot \nabla_0 \omega(z) < 0,$$

which is a contradiction since  $\nabla_0 \theta(z) = 0$ , implying  $\nu \cdot \nabla_0 \theta(z) = 0$ . Therefore,  $u$  is a viscosity solution of  $-\Delta_{p(x)} u = 0$  in all of  $\Omega$ .  $\square$

**Remark 4.2.8.** We remark about our assumption:  $u \in C_{\text{sub}}^1(\Omega)$ . At the end of Section 4.1, we showed through a counterexample that in the Heisenberg group, weakened regularity such as  $u \in \text{Lip}(\Omega)$ , where  $\Omega \subseteq \mathbb{H}_n$ , is insufficient to guarantee even the fixed exponent cases of the results proved above (see [BFF1, Section 5]).

### 4.3 Equivalence in the Heisenberg Group Revisited <sup>3</sup>

In Chapter 3, we showed that potential theoretic weak solutions and viscosity solutions to the  $p(x)$ -Laplace equation in Carnot groups coincide under reasonable assumptions. We conclude this chapter by revisiting this equivalence in the Heisenberg group. In particular, we mention the following modification specific to the Heisenberg group  $\mathbb{H}_n$ , with an argument also in  $\mathbb{H}_n$ , validating the modification. Additionally, we have a Radó-type result as an immediate application of the equivalence.

<sup>3</sup>A Note to Reader: This section has been reproduced from [BFF2]. This work has been submitted for review.

**Theorem 4.3.1** ([BFF2], Theorem 6.1). *Let  $\Omega \subseteq \mathbb{H}_n$  be a bounded domain and assume  $1 < p(x) < \infty$  with  $p \in C_{\text{sub}}^1(\Omega)$ . Also assume  $u \in C_{\text{sub}}^1(\Omega)$  and that  $X_{2n+1}u$  is bounded in the viscosity sense in  $\Omega$ . Then  $u$  is  $p(x)$ -harmonic if and only if it is a viscosity solution to the  $p(x)$ -Laplace equation.*

*Proof.* Since the Heisenberg group  $\mathbb{H}_n$  is the simplest nontrivial Carnot group, we can apply our proof of Corollary 3.0.8 from Chapter 3 ([BF2, Corollary 4.17]), provided we can show our situation yields an upper bound for the full gradient. Since  $u \in C_{\text{sub}}^1(\Omega)$  there exists  $C_1(\overline{\Omega}) > 0$  such that

$$\|\nabla_0 u\| \leq C_1(\overline{\Omega})$$

in  $\Omega$ . Because  $X_{2n+1}u$  is bounded in the viscosity sense, there also exists  $C_2(\overline{\Omega}) > 0$  such that for all  $y \in \overline{\Omega}$  and all  $\psi \in \mathcal{TA}(u, y)$  we will have

$$\|X_{2n+1}\psi(y)\| \leq C_2(\overline{\Omega}).$$

Defining  $C(\overline{\Omega}) := \max\{C_1(\overline{\Omega}), C_2(\overline{\Omega})\}$  we have that for all  $y \in \overline{\Omega}$  and all  $\varphi \in \mathcal{TA}(u, y)$

$$\|\nabla\varphi\| \leq \|\nabla_0\varphi\| + \|X_{2n+1}\varphi\| \leq 2C(\overline{\Omega}). \quad (4.24)$$

Since  $C(\overline{\Omega})$  depends only upon  $\overline{\Omega}$ , then  $C(\overline{\Omega})$  is fixed and we have an upper bound for the full gradient in the viscosity sense. We now can apply the proof in [BF2, Corollary 4.17] to achieve the equivalence.  $\square$

Under reasonable restrictions and applying Theorem 4.3.1 and Theorem 4.2.1, we have the following Radó-type result as an application:

**Corollary 4.3.2** ([BFF2], Theorem 6.2). *Let  $\Omega \subseteq \mathbb{H}_n$  be a bounded domain and assume  $1 < p(x) < \infty$  with  $p \in C_{\text{sub}}^1(\Omega)$ . Also assume  $u \in C_{\text{sub}}^1(\Omega)$  and that  $X_{2n+1}u$  is bounded in the viscosity sense in  $\Omega$ . If  $u$  is  $p(x)$ -harmonic in  $\Omega \setminus \{x \in \Omega : u(x) = 0\}$ , then  $u$  is  $p(x)$ -harmonic in  $\Omega$ .*

*Proof.* Applying Theorem 4.3.1 and Theorem 4.2.1, we know that  $u$  is a viscosity solution to Equation (4.16) in  $\Omega$ . Since  $u \in C_{\text{sub}}^1(\Omega)$  and  $X_{2n+1}u$  is bounded in the viscosity sense, we may apply Theorem 4.3.1 to conclude that  $u$  is  $p(x)$ -harmonic in  $\Omega$ .  $\square$

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