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Lump Solutions and Riemann-Hilbert Approach to Soliton Equations

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by

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Dedication

I dedicate this dissertation to my husband; Ahmed who has planted the seeds of success on my way and sacrificed a lot for me. I also dedicate this work to my parents who have stood by my side from the beginning and taught me to have hope in God, the most Gracious, the most Knowledgeable. To my daughters; Tala, Rital, and Layal who are just about the best children a mom could hope for.
May the Lord "Allah" protect you all and unduly reward you for all your efforts.

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Abstract

In the first part of this dissertation we introduce two matrix iso-spectral problems, a Kaup-Newell type and a generalization of the Dirac spectral problem, associated with the three-dimensional real Lie algebras $sl(2,\mathbb{R})$ and $so(3,\mathbb{R})$, respectively. Through zero curvature equations, we furnish two soliton hierarchies. Hamiltonian structures for the resulting hierarchies are formulated by adopting the trace identity. In addition, we prove that each of the soliton hierarchies has a bi-Hamiltonian structure which leads to the integrability in the Liouville sense. The motivation of the first part is to construct soliton hierarchies with infinitely many commuting symmetries and conservation laws.

The second part of the dissertation is dedicated to the investigation of exact solutions to some nonlinear evolution equations. We find lump solutions and lump-type solutions to a (2+1)-dimensional 5th-order KdV-like equation and a (3+1)-dimensional Jimbo–Miwa-like equation, respectively. Moreover, we explore interaction solutions of lump-type solutions with kink solutions and resonance stripe solitons solutions for the Jimbo–Miwa-like equation. Finally, we consider a Riemann-Hilbert problem for a coupled complex modified-KdV system and present its N-soliton solutions.

Chapter 1
Introduction

1.1 Background

Soliton theory is interesting and attractive: it relates to many areas of mathematics and has a variety of applications in physics. In 1834, John Scott Russell discovered the first solitary wave on the Edinburgh-Glasgow canal. The theory has developed mainly in the last decade starting with by [53]. According to Russel, he noticed a water wave created when a moving boat suddenly stopped. The wave traveled through the channel at a steady speed and without a change in shape or form. Russel called this wave the "great wave of translation". Unlike low-amplitude dispersive waves, this wave could not be explained by linear partial differential equations (PDEs) which introduced a controversy between scientists. Russel continued studying the wave in the laboratory and concluded that the amount of water in the wave is the same as the amount displaced and the solitary wave speed, s, can be defined as

$$s^2 = g(h + m), \tag{1.1}$$

where h is the water depth, g is the gravitational acceleration and m is the wave amplitude. In 1870's, Boussinesq [7] and Lord Rayleigh [52] also considered the problem. They contributed to the field by deriving the wave profile $z = u(x, t)$ as follows

$$u(x, t) = m \operatorname{sech}^2 \alpha(x - st), \tag{1.2}$$

where $\alpha^{-2} = \frac{4h^2(h+m)}{3m}$, for any $m > 0$. In 1895, Korteweg and de Vries provided an equation for $u(x, t)$ that adopts (1.2) as a solution [30]. They conducted a comprehensive theoretical analysis and derived what is called now the Korteweg-de Vries (KdV) equation:

$$u_t - 6uu_x + u_{xxx} = 0. \tag{1.3}$$

Herein u is the water surface elevation and the subscripts denote partial differentiation. No further substantial investigations were done in this area prior to 1965 when Kruskal and Zabusky used digital simulation to solve the initial value problem for the KdV equation with periodic boundary conditions. They discovered that nonlinear waves solutions of the KdV equation interact with each other elastically, and they called these waves "solitons" [73]. This result paved the way for great studies in the field and in 1967 Gardener, Greene, Kruskal and Miura innovated the inverse scattering method and were able to analytically solve the KdV equation and found all its soliton solutions [15]. Later, the method was applied to solve many other important nonlinear equations like the nonlinear Schrödinger (NLS) equation, the sine-Gordon equation [1, 4, 49], and these equations are known as integrable equations. More work related to the inverse scattering method was considered by Lax in 1968 [31]. Lax introduced a method for associating a pair of linear operators, known as the "Lax pair", with nonlinear evolution equations so that the eigenvalues of the linear operators are constants of the motion for the nonlinear evolution equations. Ablowitz, Kaup, Newell and Segur in 1974 derived from a matrix spectral problem, nonlinear evolution equations which can be solved using inverse scattering method [3]. Moreover, the theory of integrability was spread in different domains such as the Riemann-Hilbert approach [49] and the direct method [22] and until now the developments of the theory continue. Although there is no precise definition of a soliton, it can be defined as a solution of nonlinear evolution equations which possesses the following properties:

1. is localized in the region, i.e., it decays to a constant at infinity,
2. depicts a wave of permanent form,
3. can collide with other solitons but preserve their individual shapes and speeds.

A soliton becomes a solitary wave when it is infinitely separated from any other soliton [11].

1.2 Dissertation Outline

The organization of this dissertation is as follows. Chapter 2 is assigned to consider two matrix isospectral problems associated with the three-dimensional simple linear Lie algebra $sl(2, \mathbb{R})$ and the three-dimensional special orthogonal real Lie algebra $so(3, \mathbb{R})$, respectively. Based on the spectral matrices and via the zero curvature formulation we construct two soliton hierarchies.

In Chapter 3, we use the trace identity, a special case of the variational identity, to formulate Hamiltonian and bi-Hamiltonian structures of the soliton hierarchies and to prove that they are Liouville integrable.

Chapter 4 studies lump and lump-type solutions of two nonlinear evolution equations of mathematical physics. First, we produce lump solutions for a (2+1)-dimensional 5th-order KdV-like equation and lump-type solutions for a Jimbo-Miwa-like equation in (3+1)-dimension. Then we inspect interaction solutions of lump-type solutions and kink solutions and resonance stripe solitons solutions for the Jimbo-Miwa-like equation.

In Chapter 5, we present the Ablowitz-Kaup-Newell soliton hierarchy with two potentials. Then a Riemann-Hilbert problem is built for the third nonlinear system in the hierarchy which is a coupled complex modified Korteweg-de Vries system. At the end, we generate N-soliton solutions for the system through solving the associated non-regular Riemann-Hilbert problem.

Chapter 2

Spectral Problems and Soliton Hierarchies of Integrable Systems

2.1 Introduction

In the past decades, researchers become more interested in studying soliton theory due to its usefulness in understanding nonlinear phenomena [1, 4, 11]. A standout amongst the most essential research areas in soliton theory is discovering soliton hierarchies along with their exact solutions and integrable properties. Solitons hierarchies can be derived from appropriate spectral problems associated with matrix Lie algebras. In 1974, Ablowitz, Kaup, Newell and Segur constructed an infinite hierarchy of integrable equations known as the AKNS hierarchy which contains the nonlinear Schrödinger equation [3]. Other significant hierarchies followed such as the Kaup-Newell (KN) [27], the Wadati Konno Ichikawa [64], and Dirac hierarchies [19].

We start this chapter by giving some basic definitions and notations [51]. The rest of the chapter is organized as follows. In Section 2.3, we introduce some methods for constructing integrable systems and their application to the KdV equation and the nonlinear Schrödinger equation. In Section 2.4, we prove that an isospectral matrix problem of Kaup-Newell type endangers a hierarchy of soliton equations. In Section 2.5, a generalized $so(3, \mathbb{R})$ counterpart spectral problem of the Dirac soliton hierarchy is proposed and its associated soliton hierarchy is generated.

2.2 Preliminaries

Let $x = (x^1, x^2, \ldots, x^p)$ be the independent variables and $u = (u^1, u^2, \ldots, u^q)$ be the dependent variables in the spaces $X = \mathbb{R}^p$ and $U = \mathbb{R}^q$, respectively, where $\mathbb{R}$ is the set of real number. Let O be an open subset of $X \times U$.

DEFINITION 2.2.1. *The collection of smooth functions $L(x, u^{(n)})$ that depend on x, u and derivatives of u till a finite order n is algebra and we denote it by $\mathcal{A}$. Any function $L(x, u^{(n)}) \in \mathcal{A}$ is called a differential* *function. The differential function $L(x, u^{(n)})$ is expressed as $L[u]$ if the number of derivatives of u that L depends on is not important.*

DEFINITION 2.2.2. *The quotient space of $\mathcal{A}$ under the image of the total divergence is the space $\mathcal{F}$ of functionals $\mathcal{L} = \int L\, dx$.*

The vector space of q-tuples of differential functions, $L[u] = (L_1[u], L_2[u], ..., L_q[u])$, where $L_j \in \mathcal{A}, 1 \leq j \leq q$, is denoted by $\mathcal{A}^q$.

DEFINITION 2.2.3. *The Gateaux derivative of an q-tuple of differential functions*

$$L[u] = L(x, u^{(n)}) \in \mathcal{A}^q,$$

is a differential operator $d_L : \mathcal{A}^r \to \mathcal{A}^q$ defined so that

$$d_L(Q) = L'(u)[Q] = \frac{d}{d\varepsilon} L[u + \varepsilon Q[u]]\Big|_{\varepsilon=0}, \tag{2.1}$$

for any $Q \in \mathcal{A}^r$.

EXAMPLE 1. If

$$L[u] = u_x + uu_x,$$

then the Gateaux derivative of L is

$$d_L(Q) = L'[Q] = \frac{d}{d\varepsilon}\big[(u_x + \varepsilon D_x Q) + (u + \varepsilon Q)(u_x + \varepsilon D_x Q)\big]\Big|_{\varepsilon=0}$$

$$= D_x Q + u D_x Q + u_x Q.$$

DEFINITION 2.2.4. *A Lie algebra is a vector space $\mathfrak{g}$ over a field $\mathbb{F}$ together with a bilinear operation called a Lie bracket $[\ ,\ ]_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which satisfies the following properties*

1. *skew-symmetry, i.e., for every $X, Y \in \mathfrak{g}$*

$$[X, Y]_{\mathfrak{g}} = -[Y, X]_{\mathfrak{g}}, \tag{2.2}$$

2. *The Jacobi identity holds,that is, for all $X, Y, Z \in \mathfrak{g}$*

$$[X, [Y, Z]]_{\mathfrak{g}} + [Z, [X, Y]]_{\mathfrak{g}} + [Y, [Z, X]]_{\mathfrak{g}} = 0. \tag{2.3}$$

If a Lie bracket of two elements X and Y of a Lie algebra $\mathfrak{g}$ is zero, i.e, $[X,Y]_{\mathfrak{g}} = 0$, we say that X and Y commute. If for every X, Y in a Lie algebra $\mathfrak{g}$ satisfies $[X,Y]_{\mathfrak{g}} = 0$ then $\mathfrak{g}$ is said to be commutative.

DEFINITION 2.2.5. *The commutator of two q-tuples of differential functions $L, K \in \mathcal{A}^q$ is defined as*

$$[L,K] = L'K - K'L. \tag{2.4}$$

$(\mathcal{A}^q, [.,.])$ forms a Lie algebra over $\mathbb{R}$.

DEFINITION 2.2.6. *The i-th total derivative of*

$$L[u] = L(x, u^{(n)}),$$

is the unique smooth function $D_i L(x, u^{(n+1)})$ defined as

$$D_i L = \frac{\partial L}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_J u^\alpha_{J,i} \frac{\partial L}{\partial u^\alpha_J}, \tag{2.5}$$

where, for $J = (j_1, ..., j_k)$,

$$u^\alpha_{J,i} = \frac{\partial u^\alpha_J}{\partial x^i} = \frac{\partial^{k+1} u^\alpha}{\partial x^i \partial x^{j_1} ... \partial x^{j_k}}. \tag{2.6}$$

The sum in (2.5) is over all J's of order $0 \leq \#J \leq n$, where n is the highest order derivative appearing in L.

EXAMPLE 2. If we take $X = \mathbb{R}^2$ and $U = \mathbb{R}$, then the two total derivatives D_x and D_y are given by

$$D_x L = \frac{\partial L}{\partial x} + u_x \frac{\partial L}{\partial u} + u_{xx} \frac{\partial L}{\partial u_x} + u_{xy} \frac{\partial L}{\partial u_y} + u_{xxx} \frac{\partial L}{\partial u_{xx}} + \cdots, \tag{2.7}$$

$$D_y L = \frac{\partial L}{\partial y} + u_y \frac{\partial L}{\partial u} + u_{xy} \frac{\partial L}{\partial u_x} + u_{yy} \frac{\partial L}{\partial u_y} + u_{xxy} \frac{\partial L}{\partial u_{xx}} + \cdots. \tag{2.8}$$

If we take $L = xyu_{xy}$, then

$$D_x L = yu_{xy} + xyu_{xxy}, \tag{2.9}$$

$$D_y L = xu_{xy} + xyu_{xyy}. \tag{2.10}$$

DEFINITION 2.2.7. *If we can write a partial differential equation as*

$$u_t = K[u] = K(x, t, u, u_x, u_{xx}, \cdots), \tag{2.11}$$

then it is called an evolution equation. Here $K[u]$ is a differential functions and $u(x,t)$ is a column vector of dependent variables. Eq.(2.11) is known as a nonlinear evolution equation (NLEE) when K is a nonlinear function.

2.3 Integrable Systems Derivation Methods

There are many tools for constructing integrable systems in the literature. For example, the Lax pair [31], zero-curvature representation and the Tu-Ma scheme [37, 61, 62]. This section aims to shed light on these methods in details.

2.3.1 Lax Pair

Lax pair concept was originated in 1968 by Peter Lax [31]. He discovered a particular class of nonlinear evolution equations

$$u_t = K[u] = K(x, t, u, u_x, u_{xx}, \cdots), \tag{2.12}$$

corresponds to linear PDEs

$$L\psi = \lambda\psi, \quad \psi_t = A\psi, \tag{2.13}$$

with linear differential operators L and A known as Lax pair. Herein the function ψ is an eigenfunction of L associated with the eigenvalue λ. Taking the derivative of the left equation in (2.13) with respect to t and utilizing the right one leads to

$$\frac{d}{dt}(L\psi) = \frac{d}{dt}(\lambda\psi) = \lambda\psi_t = A(\lambda\psi) = AL\psi, \tag{2.14}$$

and

$$\frac{d}{dt}(L\psi) = \frac{dL}{dt}\psi + L\psi_t = \frac{dL}{dt}\psi + LA\psi, \tag{2.15}$$

which gives rise to

$$\frac{dL}{dt} + LA - AL = 0, \tag{2.16}$$

or equivalently,

$$\frac{dL}{dt} + [L, A] = 0. \tag{2.17}$$

Here $[L, A] = LA - AL$ is the commutator of the operators L and A. Eq.(2.17) is said to be a Lax equation. When the eigenvalues λ are time-independent i.e., $\lambda_t = 0$, the eigenvalue problem (2.13) is known as an isospectral problem.

EXAMPLE 3. Consider the Lax pair

$$\begin{cases} L = -\dfrac{\partial^2}{\partial x^2} + u, \\ \\ A = -4\dfrac{\partial^3}{\partial x^3} + 6u\dfrac{\partial}{\partial x} + 3\dfrac{\partial u}{\partial x}. \end{cases} \tag{2.18}$$

An easy calculation shows that

$$[L, A] = \Big[-\frac{\partial^2}{\partial x^2} + u, -4\frac{\partial^3}{\partial x^3} + 6u\frac{\partial}{\partial x} + 3\frac{\partial u}{\partial x}\Big] = \frac{\partial^3 u}{\partial x^3} - 6u\frac{\partial u}{\partial x}, \tag{2.19}$$

and

$$\frac{dL}{dt} = \frac{\partial u}{\partial t}. \tag{2.20}$$

Inserting Eqs(2.19) and (2.20) into the Lax equation (2.17), we obtain

$$u_t - 6uu_x + u_{xxx} = 0, \tag{2.21}$$

which is the famous KdV equation.

2.3.2 Zero Curvature Representation

Let

$$\begin{cases} \psi_x = U(x,t;\lambda)\psi, \\ \psi_t = V(x,t;\lambda)\psi, \end{cases} \tag{2.22}$$

be a system of linear partial differential equations where U and V are matrix functions of the variables x and t which depend on the spectral parameter λ and the column vector ψ has entries depend on (x,t,λ). Differentiate the first equation in (2.22) with respect to t and the second one with respect to x, respectively, we get

$$\psi_{xt} = U_t(\lambda)\psi + U(\lambda)\psi_t, \tag{2.23}$$

and

$$\psi_{tx} = V_x(\lambda)\psi + V(\lambda)\psi_x. \tag{2.24}$$

Now, the consistency condition $\psi_{xt} = \psi_{tx}$ with (2.22) leads to

$$U_t(\lambda)\psi + U(\lambda)\psi_t - V_x(\lambda)\psi - V(\lambda)\psi_x = (U_t(\lambda) - V_x(\lambda) + [U(\lambda), V(\lambda)])\psi = 0, \tag{2.25}$$

or equivalently,

$$U_t - V_x + [U, V] = 0. \tag{2.26}$$

Eq.(2.26) is called the zero curvature equation and the whole scheme is called the zero curvature representation [61]. Many nonlinear integrable equations admit the zero curvature representation (2.26).

DEFINITION 2.3.1. *A nonlinear evolution equation*

$$u_t = K(u), \tag{2.27}$$

is called Lax integrable if it admits the zero-curvature representation (2.26).

EXAMPLE 4. If

$$U = \begin{bmatrix} -i\lambda & u \\ -\bar{u} & i\lambda \end{bmatrix}, \qquad V = \begin{bmatrix} -2i\lambda^2 + iu\bar{u} & iu_x + 2\lambda u \\ i\bar{u}_x - 2\lambda\bar{u} & 2i\lambda^2 - iu\bar{u} \end{bmatrix} \tag{2.28}$$

where $u = u(x, t)$ and $\bar{u}$ is the complex conjugate of u, then the zero curvature equation (2.26) leads to the nonlinear Schrödinger (NLS) equation

$$iu_t + u_{xx} + 2u^2\bar{u} = 0. \tag{2.29}$$

A soliton hierarchy is a hierarchy of equations of the form

$$u_{t_m} = K_m[u] = K_m(x, t, u, u_x, u_{xx}, ...), \tag{2.30}$$

can be formed from an evolution equation

$$u_t = K[u] = K(x, t, u, u_x, u_{xx}, ...), \tag{2.31}$$

by using the zero curvature equation (2.26). In this hierarchy, each member commutes with any other member and is a symmetry of Eq.(2.31). This indicates that the symmetries of any equation in the hierarchy comprise vector fields $\{K_n : n = 0, 1, 2, \cdots\}$ that are mutually commutative, i.e.,

$$[K_n, K_m] = 0, \quad n, m = 0, 1, 2, \cdots . \tag{2.32}$$

2.3.3 Tu-Ma Scheme

The Tu-Ma scheme [37, 62, 63] is the most practical tool to construct soliton hierarchies. We start with some notations and then present the steps for this scheme.

Let $so(3,\mathbb{R})$ be the collection of all 3×3 trace-free, skew symmetric matrices whose elements are real numbers. If the Lie bracket of two elements $M_1, M_2 \in so(3,\mathbb{R})$ is defined by the matrix commutator

$$[M_1, M_2] = M_1M_2 - M_2M_1 \tag{2.33}$$

then $so(3,\mathbb{R})$ is a Lie algebra of dimension 3. We can choose a basis for this Lie algebra consisting of the elements e_1, e_2, and e_3 that defined as

$$e_1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad e_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.34}$$

then the commutator relations are gives as follows

$$[e_1, e_2] = e_3, \qquad [e_2, e_3] = e_1, \qquad [e_3, e_1] = e_2. \tag{2.35}$$

The three-dimensional special linear Lie algebra, $sl(2,\mathbb{R})$, consists of 2×2 trace-free matrices and has three basis elements

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad e_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad e_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \tag{2.36}$$

with the following commutator relations

$$[e_1, e_2] = 2e_2, \qquad [e_2, e_3] = e_1, \qquad [e_3, e_1] = -2e_3. \tag{2.37}$$

DEFINITION 2.3.2. *If $\mathfrak{g}$ is a finite-dimensional Lie algebra over the complex space $\mathbb{C}$, then their corresponding loop algebra $\tilde{\mathfrak{g}}$ is given by*

$$\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[\lambda, \lambda^{-1}], \tag{2.38}$$

where $\mathbb{C}[\lambda,\lambda^{-1}]$ is the set of Laurent polynomials in λ. Let $\{e_1, ..., e_r\}$ be a basis of $\mathfrak{g}$. Then $\{e_1(n), ..., e_r(n) | n \in \mathbb{Z}\}$, where $e_j(n) = e_j \otimes \lambda^n = e_j\lambda^n$ $(1 \leq j \leq r)$, provides a basis for $\tilde{\mathfrak{g}}$.

In the steps below, we explain how to built hierarchies of soliton equations using the Tu-Ma procedure.

Step 1 Begin with a spatial iso-spectral problem

$$\phi_x = U\phi = U(u,\lambda)\phi, \quad U(u,\lambda) \in \tilde{\mathfrak{g}}. \tag{2.39}$$

The spectral parameter λ is time independent i.e., $\lambda_t = 0$ and u is a column vector of dependent variables, based on a matrix loop algebra $\tilde{\mathfrak{g}}$ associated with a given matrix Lie algebra $\mathfrak{g}$, often being simple or semisimple.

Step 2 Find a solution

$$W = W(u,\lambda) = \sum_{i\geq 0} W_{0,i}\lambda^{-i}, \quad W_{0,i} \in \mathfrak{g}, \quad i \geq 0, \tag{2.40}$$

to the stationary zero curvature representation

$$W_x = [U, W]. \tag{2.41}$$

Step 3 Introduce a sequence of Lax matrices

$$V^{[n]} = V^{[n]}(u,\lambda) = (\lambda^n W)_+ + \Delta_n \in \tilde{\mathfrak{g}}, \quad n \geq 0, \tag{2.42}$$

so that the temporal spectral problems are formulated as

$$\phi_{t_n} = V^{[n]}\phi = V^{[n]}(u,\lambda)\phi, \quad n \geq 0. \tag{2.43}$$

Here P_+ represents the polynomial part of P in λ and the term Δ_n belongs to a Lie algebra $\tilde{\mathfrak{g}}$ is a modification term which aims to ensure that the compatibility conditions of Eq.(2.39) and Eq.(2.43), that is, the zero curvature equations

$$U_{t_n} - V_x^{[n]} + [U, V^{[n]}] = 0, \quad n \geq 0, \tag{2.44}$$

present a hierarchy of soliton equations

$$u_{t_n} = K_n(u), \quad n \geq 0. \tag{2.45}$$

In this dissertation, we apply the Tu-Ma scheme to derive new hierarchies of soliton equations by taking the matrix loop algebra $\tilde{\mathfrak{g}}$ to be

$$\widetilde{sl}(2,\mathbb{R}) = \Big\{ \sum_{j=0}^{\infty} M_j \lambda^{m-j} | M_j \in sl(2,R),\ j \geq 0,\ m \in \mathbb{Z} \Big\} \tag{2.46}$$

and

$$\widetilde{so}(3,\mathbb{R}) = \Big\{ \sum_{j=0}^{\infty} M_j \lambda^{m-j} | M_j \in so(3,R),\ j \geq 0,\ m \in \mathbb{Z} \Big\} \tag{2.47}$$

The matrix loop algebras $\widetilde{sl}(2,\mathbb{R})$ and $\widetilde{so}(3,\mathbb{R})$ are associated with the three-dimensional special linear Lie algebra $sl(2,\mathbb{R})$, and the three-dimensional special orthogonal Lie algebra $so(3,\mathbb{R})$, respectively.

2.4 A Soliton Hierarchy of Kaup-Newell Type

In this section, we are going to generate a hierarchy of soliton equations of Kaup-Newell type from the isospectral problem

$$\phi_x = U\phi = U(u,\lambda)\phi, \quad u = \begin{bmatrix} p \\ q \end{bmatrix}, \quad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}, \tag{2.48}$$

associated with the Lie algebra $sl(2,\mathbb{R})$ with the following spectral matrix

$$U = (\lambda^2 + \alpha q)e_1 + \lambda p e_2 + \lambda e_3 = \begin{bmatrix} \lambda^2 + \alpha q & \lambda p \\ \lambda & -(\lambda^2 + \alpha q) \end{bmatrix} \in \widetilde{sl}(2,\mathbb{R}). \tag{2.49}$$

Here α is an arbitrary nonzero real constant and e_1, e_2, and e_3 are the basis for $sl(2,\mathbb{R})$ given by (2.36). This spectral problem (2.48)-(2.49) is unlike the real form of the Kaup-Newell spectral problem [27] with the spectral matrix

$$U = \begin{bmatrix} \lambda^2 & \lambda p \\ \lambda q & -\lambda^2 \end{bmatrix}. \tag{2.50}$$

THEOREM 2.1. *The spectral problem (2.48) with the spectral matrix (2.49) produces a hierarchy of soliton equations*

$$u_{t_n} = K_n = \begin{bmatrix} -2a_{n+1,x} \\ \frac{1}{\alpha}(a_{n+1,x} - c_{n+1,x}) \end{bmatrix}, \qquad n \geq 0, \tag{2.51}$$

where elements of K_n can be provided from the following recursion relations

$$\begin{cases} a_{i,x} = pc_i - b_i, \\ b_{i+1} = pa_{i+1} - \alpha q b_i + \frac{1}{2} b_{i,x}, \qquad i \geq 0. \\ c_{i+1} = a_{i+1} - \alpha q c_i - \frac{1}{2} c_{i,x}. \end{cases} \tag{2.52}$$

Proof. Let W represented as

$$W = ae_1 + be_2 + ce_3 = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \in \tilde{sl}(2,\mathbb{R}), \tag{2.53}$$

be a solution of the stationary zero curvature representation (2.41). Then Eq.(2.41) leads to

$$\begin{cases} a_x = \lambda pc - \lambda b, \\ b_x = 2\alpha q b + 2\lambda^2 b - 2\lambda pa, \\ c_x = -2\alpha qc - 2\lambda^2 c + 2\lambda a. \end{cases} \tag{2.54}$$

Taking the Laurent series expansions of a, b and c as

$$a = \sum_{i\geq 0} a_i \lambda^{-2i}, \tag{2.55}$$

$$b = \sum_{i\geq 0} b_i \lambda^{-2i-1}, \tag{2.56}$$

$$c = \sum_{i\geq 0} c_i \lambda^{-2i-1}, \tag{2.57}$$

then substituting them into Eq.(2.54) and comparing the coefficients of the same powers of λ, gives rise to

$$\begin{cases} a_{i,x} = pc_i - b_i, \\ b_{i+1} = pa_{i+1} - \alpha q b_i + \frac{1}{2} b_{i,x}, \qquad i \geq 0. \\ c_{i+1} = a_{i+1} - \alpha q c_i - \frac{1}{2} c_{i,x}, \end{cases} \tag{2.58}$$

We take the initial values

$$\begin{cases} a_0 = 1, \\ b_0 = p, \\ c_0 = 1, \end{cases} \tag{2.59}$$

which are obtained by solving the equations

$$\begin{cases} a_{0,x} = pc_0 - b_0, \\ a_0 p - b_0 = 0, \\ a_0 - c_0 = 0. \end{cases} \tag{2.60}$$

From the last two equations in (2.58), we have

$$a_{i+1,x} = -\alpha p q c_i - \frac{1}{2} p c_{i,x} + \alpha q b_i - \frac{1}{2} b_{i,x}. \tag{2.61}$$

The sequence of functions $\{a_i, b_i, c_i \mid i \geq 1\}$ can be uniquely determined from (2.58) by setting constant of integration to zero, that is

$$a_i|_{u=0} = b_i|_{u=0} = c_i|_{u=0} = 0, \quad i \geq 1. \tag{2.62}$$

Through Maple symbolic computations, the first few sets of the sequence are presented as follows

$$\begin{cases} a_1 = -\frac{1}{2} p, \\ b_1 = \frac{1}{2} p_x - \alpha p q - \frac{1}{2} p^2, \\ c_1 = -\alpha q - \frac{1}{2} p; \end{cases} \tag{2.63}$$

$$\begin{cases} a_2 = \alpha pq + \frac{3}{8}p^2 - \frac{1}{4}p_x, \\ b_2 = \alpha^2 pq^2 + \frac{3}{2}\alpha p^2 q + \frac{3}{8}p^3 - \frac{1}{2}\alpha pq_x - \alpha p_x q - \frac{3}{4}pp_x + \frac{1}{4}p_{xx}, \\ c_2 = \alpha^2 q^2 + \frac{3}{2}\alpha pq + \frac{3}{8}p^2 + \frac{1}{2}\alpha q_x; \end{cases} \tag{2.64}$$

and

$$\begin{cases} a_3 = & -\frac{3}{2}\alpha p^2 q - \frac{3}{2}\alpha^2 pq^2 - \frac{5}{16}p^3 - \frac{1}{8}(-6\alpha q - 3p)p_x - \frac{1}{8}p_{xx}, \\ b_3 = & -\alpha^3 pq^3 - 3\alpha^2 p^2 q^2 - \frac{15}{8}\alpha p^3 q + \frac{3}{2}\alpha^2 pqq_x + \frac{3}{2}\alpha^2 p_x q^2 - \frac{5}{16}p^4 + \frac{3}{4}\alpha p^2 q_x \\ & +3\alpha pqp_x + \frac{5}{16}p^2 p_x - \frac{1}{4}\alpha pq_{xx} - \frac{3}{4}\alpha p_{xx}q - \frac{3}{4}\alpha p_x q_x - \frac{1}{2}pp_{xx} - \frac{3}{8}p_x^2 + \frac{1}{8}p_{xxx}, \\ c_3 = & -\alpha^3 q^3 - 3\alpha^2 pq^2 - \frac{15}{8}\alpha p^2 q - \frac{3}{2}\alpha^2 qq_x - \frac{5}{16}p^3 - \frac{3}{4}\alpha pq_x - \dfrac{1}{4}\alpha q_{xx} - \frac{1}{8}p_{xx}. \end{cases} \tag{2.65}$$

Now, based on the form of the matrix U in (2.49) and the recursion relations (2.58), we introduce the lax matrices

$$V^{[n]} = \lambda(\lambda^{2n+1}W)_+ + \Delta_n \in \tilde{sl}(2,\mathbb{R}), \quad n \geq 0, \tag{2.66}$$

with Δ_n selected as

$$\Delta_n = \delta_n e_1 = \begin{bmatrix} \delta_n & 0 \\ 0 & -\delta_n \end{bmatrix}, \quad n \geq 0. \tag{2.67}$$

As a result, we obtain

$$V_x^{[n]} - [U, V^{[n]}] = \lambda(\lambda^{2n+1}W_x)_+ + \delta_{n,x}e_1 - \lambda[U, (\lambda^{2n+1}W)_+] - [U, \delta_n e_1], \quad n \geq 0, \tag{2.68}$$

On one hand, we have

$$(\lambda^{2n+1}W_x)_+ - [U, (\lambda^{2n+1}W)_+] = \begin{bmatrix} 0 & -2pa_{n+1} + 2b_{n+1} \\ 2a_{n+1} - 2c_{n+1} & 0 \end{bmatrix}, \quad n \geq 0, \tag{2.69}$$

and on the other hand, we have

$$[U, \delta_n e_1] = -2\lambda p\delta_n e_2 + 2\lambda\delta_n e_3 = \begin{bmatrix} 0 & -2\lambda p\delta_n \\ 2\lambda\delta_n & 0 \end{bmatrix}, \quad n \geq 0. \tag{2.70}$$

Hence Eq.(2.68), for $n \geq 0$, becomes

$$V_x^{[n]} - [U, V^{[n]}] = \lambda \begin{bmatrix} 0 & -2pa_{n+1} + 2b_{n+1} \\ 2a_{n+1} - 2c_{n+1} & 0 \end{bmatrix} + \begin{bmatrix} \delta_{n,x} & 2\lambda p\delta_n \\ -2\lambda\delta_n & -\delta_{n,x} \end{bmatrix}. \tag{2.71}$$

Consequently, from the zero curvature equations (2.44), we get

$$\begin{cases} p_{t_n} = 2p\delta_n - 2pa_{n+1} + 2b_{n+1}, \\ q_{t_n} = \frac{1}{\alpha}\delta_{n,x}, & n \geq 0, \\ -2\delta_n + 2a_{n+1} - 2c_{n+1} = 0. \end{cases} \tag{2.72}$$

Solving the last equation for δ_n gives

$$\delta_n = a_{n+1} - c_{n+1}, \quad n \geq 0. \tag{2.73}$$

Inserting the value of δ_n into the other equations in (2.72), we reach

$$\begin{cases} p_{t_n} = -2a_{n+1,x}, \\ q_{t_n} = \frac{1}{\alpha}(a_{n+1,x} - c_{n+1,x}), & n \geq 0, \end{cases} \tag{2.74}$$

whose vector form is as follows

$$u_{t_n} = K_n = \begin{bmatrix} -2a_{n+1,x} \\ \frac{1}{\alpha}(a_{n+1,x} - c_{n+1,x}) \end{bmatrix}, \qquad n \geq 0. \tag{2.75}$$

□

PROPOSITION 2.1. *The functions $\{a_i, b_i, c_i \mid i \geq 1\}$ defined by Eq.(2.58), with the initial data (2.59) and under the conditions (2.62) are differential functions in u with respect to x, hence, they are all local.*

Proof. In the view of the stationary zero curvature representations (2.41), we can work out that

$$\frac{d}{dx}\text{tr}(W^2) = 2\text{tr}(WW_x) = 2\text{tr}(W[U, W]) = 0. \tag{2.76}$$

It is clearly to see that

$$a^2 + bc = (a^2 + bc)|_{u=0} = 1, \tag{2.77}$$

since $\text{tr}(W^2) = 2(a^2 + bc)$. The last equality in (2.77) follows from the initial values (2.59). By using the Laurent series expansions of the functions a, b and c in (2.55), we can rewrite Eq. (2.77) as

$$\sum_{k\geq 0}\sum_{l\geq 0} a_k a_l \lambda^{-2(k+l)} + \sum_{k\geq 0}\sum_{l\geq 0} b_k c_l \lambda^{-2(k+l)-2} = 1. \tag{2.78}$$

For each $i \geq 0$, we balance the coefficients of λ^i and this leads to

$$a_i = -\frac{1}{2}\Big(\sum_{\substack{k+l=i,\\ k,l\geq 1}} a_k a_l + \sum_{\substack{k+l=i-1,\\ k,l\geq 0}} b_k c_l\Big), \quad i \geq 2. \tag{2.79}$$

Upon considering the above relation and the first two relations in (2.58), an application of mathematical induction tell us that all the functions $\{a_i, b_i, c_i \mid i \geq 1\}$ are differential functions in u with respect to x, and thus they are all local. $\square$

From the recursion relations (2.58) we have

$$\begin{aligned} a_{n+1,x} &= pc_{n+1} - b_{n+1} \\ &= p(-\tfrac{1}{2}c_{n,x} - \alpha q c_n + a_{n+1}) - (\tfrac{1}{2}b_{n,x} - \alpha q b_n + p a_{n+1}) \\ &= (\tfrac{1}{2}\partial - \alpha q)a_{n,x} + (-\tfrac{1}{2}p - \tfrac{1}{2}\partial p \partial^{-1})c_{n,x}, \end{aligned} \tag{2.80}$$

and

$$\begin{aligned} c_{n+1,x} &= -\tfrac{1}{2}c_{n,xx} - \alpha \partial q c_n + a_{n+1,x} \\ &= (\tfrac{1}{2}\partial - \alpha q)a_{n,x} + (-\tfrac{1}{2}\partial - \alpha\partial q \partial^{-1} - \tfrac{1}{2}p - \tfrac{1}{2}\partial p \partial^{-1})c_{n,x}, \end{aligned} \tag{2.81}$$

where $\partial = \frac{\partial}{\partial x}$ and ∂^{-1} is the inverse operator of ∂. Consequently,

$$-2a_{n+1,x} = (\tfrac{1}{2}\partial - \alpha q - \tfrac{1}{2}p - \tfrac{1}{2}\partial p \partial^{-1})(-2a_{n,x}) + (-\alpha p - \alpha \partial p \partial^{-1})(\tfrac{1}{\alpha}a_{n,x} - \tfrac{1}{\alpha}c_{n,x}), \tag{2.82}$$

and

$$\tfrac{1}{\alpha}a_{n+1,x} - \tfrac{1}{\alpha}c_{n+1,x} = (-\tfrac{1}{4\alpha}\partial - \tfrac{1}{2}\partial q \partial^{-1})(-2a_{n,x}) + (-\tfrac{1}{2}\partial - \alpha\partial q \partial^{-1})(\tfrac{1}{\alpha}a_{n,x} - \tfrac{1}{\alpha}c_{n,x}). \tag{2.83}$$

Therefore, we can write the soliton hierarchy (2.51) as

$$u_{t_n} = K_n = \begin{bmatrix} -2a_{n+1,x} \\ \frac{1}{\alpha}(a_{n+1,x} - c_{n+1,x}) \end{bmatrix} = \Phi \begin{bmatrix} -2a_{n,x} \\ \frac{1}{\alpha}(a_{n,x} - c_{n,x}) \end{bmatrix}, \qquad n \geq 1, \tag{2.84}$$

where Φ is a recursion operator given by

$$\Phi = \begin{bmatrix} \frac{1}{2}\partial - \alpha q - \frac{1}{2}p - \frac{1}{2}\partial p \partial^{-1} & -\alpha p - \alpha \partial p \partial^{-1} \\ -\frac{1}{4\alpha}\partial - \frac{1}{2}\partial q \partial^{-1} & -\frac{1}{2}\partial - \alpha \partial q \partial^{-1} \end{bmatrix}. \tag{2.85}$$

The first nonlinear system in the above hierarchy is

$$u_{t_1} = \begin{bmatrix} p \\ q \end{bmatrix}_{t_1} = K_1 = \begin{bmatrix} -2\alpha p_x q - 2\alpha p q_x - \frac{3}{2}pp_x + \frac{1}{2}p_{xx} \\ -2\alpha q q_x - \frac{1}{2}pq_x - \frac{1}{2}p_x q - \frac{1}{2}q_{xx} - \frac{1}{4\alpha}p_{xx} \end{bmatrix}. \tag{2.86}$$

2.5 Generalization of the Dirac Soliton Hierarchy

The classical Dirac spectral problem [19]

$$\phi_x = U\phi = U(u,\lambda)\phi \in \widetilde{sl}(2,\mathbb{R}), \tag{2.87}$$

where

$$U = pe_1 + (\lambda+q)e_2 + (-\lambda+q)e_3 = \begin{bmatrix} p & \lambda+q \\ -\lambda+q & -p \end{bmatrix} \in \widetilde{sl}(2,\mathbb{R}), \tag{2.88}$$

is associated with the Lie algebra $sl(2,\mathbb{R})$ with the basis elements e_1, e_2, and e_3 defined in (2.36), while its $so(3,\mathbb{R})$ counterpart [77] is given by

$$\phi_x = U\phi = U(u,\lambda)\phi \in \widetilde{so}(3,\mathbb{R}), \tag{2.89}$$

with

$$\begin{aligned} U &= pe_1 + (\lambda+q)e_2 + (-\lambda+q)e_3 \\ &= \begin{bmatrix} 0 & \lambda-q & -p \\ -\lambda+q & 0 & -\lambda-q \\ p & \lambda+q & 0 \end{bmatrix} \in \widetilde{so}(3,\mathbb{R}), \end{aligned} \tag{2.90}$$

where the basis elements e_1, e_2, and e_3 defined in (2.34). In this section, we form a generalization of the spectral problem (2.90) and derive its soliton hierarchy. Introduce the spectral matrix

$$\begin{aligned} U &= pe_1 + (\lambda+q+h)e_2 + (-\lambda+q-h)e_3 \\ &= \begin{bmatrix} 0 & \lambda-q+h & -p \\ -\lambda+q-h & 0 & -\lambda-q-h \\ p & \lambda+q+h & 0 \end{bmatrix} \in \widetilde{so}(3,R), \quad h = \alpha(p^2+2q^2), \end{aligned} \tag{2.91}$$

associated with the isospectral problem

$$\phi_x = U\phi = U(u,\lambda)\phi, \quad u = \begin{bmatrix} p \\ q \end{bmatrix}, \quad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}, \tag{2.92}$$

where α is a real constant.

THEOREM 2.2. *The spectral problem (2.92) with the spectral matrix (2.91) creates a hierarchy of soliton equations*

$$u_{t_n} = K_n = \begin{bmatrix} 2a_{n+1} - 8\alpha q b_{n+1} \\ -c_{n+1} + 4\alpha p b_{n+1} \end{bmatrix}, \qquad n \geq 0, \tag{2.93}$$

with components of K_n can be determined from the recursion relations

$$\begin{cases} a_{i+1} = \frac{1}{2} c_{i,x} + q b_i - a_i h, \\ c_{i+1} = p b_i - c_i h - a_{i,x}, \qquad i \geq 0. \\ b_{i+1,x} = q c_{i+1} - p a_{i+1}, \end{cases} \tag{2.94}$$

Proof. Select the shape of a solution W of the stationary zero curvature equation (2.41) to be

$$W = ce_1 + (a+b)e_2 + (a-b)e_3 = \begin{bmatrix} 0 & -a+b & -c \\ a-b & 0 & -a-b \\ c & a+b & 0 \end{bmatrix} \in \tilde{so}(3, \mathbb{R}).$$

Accordingly, Eq.(2.41) leads to

$$\begin{cases} a_x = -\lambda c - hc + pb \\ b_x = qc - pa \\ c_x = 2\lambda a + 2ha - 2qb. \end{cases} \tag{2.95}$$

Letting

$$a = \sum_{i \geq 0} a_i \lambda^{-i}, \tag{2.96}$$

$$b = \sum_{i \geq 0} b_i \lambda^{-i}, \tag{2.97}$$

$$c = \sum_{i \geq 0} c_i \lambda^{-i}, \tag{2.98}$$

and choosing the initial values

$$\begin{cases} a_0 = 0, \\ b_0 = 1, \\ c_0 = 0, \end{cases} \tag{2.99}$$

we get the recursion relations

$$\begin{cases} a_{i+1} = \frac{1}{2}c_{i,x} + qb_i - ha_i, \\ c_{i+1} = pb_i - hc_i - a_{i,x}, \qquad i \geq 0. \\ b_{i+1,x} = qc_{i+1} - pa_{i+1}, \end{cases} \tag{2.100}$$

The following condition is imposed on the constants of integration

$$a_i|_{u=0} = b_i|_{u=0} = c_i|_{u=0} = 0, \quad i \geq 1, \tag{2.101}$$

to guarantee the uniqueness of the sequence of functions $\{a_i, b_i, c_i \mid i \geq 1\}$.

With the aid of Maple symbolic computation, we can compute the sequence of functions $\{a_i, b_i, c_i \mid i \geq 1\}$ recursively by utilizing the relations in (2.100) with the initial values (2.99) . We list the first three sets

$$\begin{cases} a_1 = q, \\ b_1 = 0, \\ c_1 = p; \end{cases} \tag{2.102}$$

$$\begin{cases} a_2 = \frac{1}{2}p_x - \alpha(p^2 + 2q^2)q, \\ b_2 = -\frac{1}{4}p^2 - \frac{1}{2}q^2, \\ c_2 = -\alpha(p^2 + 2q^2)p - q_x; \end{cases} \tag{2.103}$$

and

$$\begin{cases} a_3 = & -\frac{1}{2}q_{xx} - \frac{1}{2}q^3 + \alpha^2(p^2 + 2q^2)^2 q - \frac{1}{4}qp^2 - \alpha(p^2 + 2q^2)p_x - \alpha(pp_x + 2qq_x)p, \\ b_3 = & \alpha(q^2 + \frac{1}{2}p^2)(p^2 + 2q^2) - \frac{1}{2}p_x q + \frac{1}{2}q_x p, \\ c_3 = & -\frac{1}{2}p_{xx} - \frac{1}{2}pq^2 + \alpha^2(p^2 + 2q^2)^2 p - \frac{1}{4}p^3 + \alpha(2pp_x + 4qq_x)q + 2\alpha(p^2 + 2q^2)q_x. \end{cases} \tag{2.104}$$

Now, we consider the corresponding temporal spectral problems

$$V^{[n]} = (\lambda^n W)_+ + \Delta_n \in \tilde{so}(3, \mathbb{R}), \quad n \geq 0, \tag{2.105}$$

where Δ_n is taking to be

$$\begin{aligned} \Delta_n &= f_n e_2 + g_n e_3 \\ &= \begin{bmatrix} 0 & -g_n & 0 \\ g_n & 0 & -f_n \\ 0 & f_n & 0 \end{bmatrix}, \quad n \geq 0. \end{aligned} \tag{2.106}$$

Consequently, we attain

$$V_x^{[n]} - [U, V^{[n]}] = (\lambda^n W_x)_+ + \Delta_{n,x} - [U, (\lambda^n W)_+] - [U, \Delta_n], \quad n \geq 0, \tag{2.107}$$

On one hand, we have

$$(\lambda^n W_x)_+ - [U, (\lambda^n W)_+] = \begin{bmatrix} 0 & c_{n+1} & -2a_{n+1} \\ -c_{n+1} & 0 & c_{n+1} \\ 2a_{n+1} & -c_{n+1} & 0 \end{bmatrix}$$

$$= 2a_{n+1}e_1 - c_{n+1}e_2 - c_{n+1}e_3, \quad n \geq 0. \tag{2.108}$$

On the other hand, we obtain

$$\Delta_{n,x} - [U, \Delta_n] = \begin{bmatrix} 0 & -g_{n,x} + pf_n & -k_n \\ g_{n,x} - pf_n & 0 & -f_{n,x} - pg_n \\ k_n & f_{n,x} + pg_n & 0 \end{bmatrix}$$

$$= k_n e_1 + (f_{n,x} + pg_n)\, e_2 + (g_{n,x} - pf_n)\, e_3, \quad n \geq 0, \tag{2.109}$$

where

$$k_n = q(f_n - g_n) - (\lambda + h)(f_n + g_n). \tag{2.110}$$

A direct calculation gives

$$U_{t_n} = \begin{bmatrix} 0 & -q_{t_n} + h_{t_n} & -p_{t_n} \\ q_{t_n} - h_{t_n} & 0 & -q_{t_n} - h_{t_n} \\ p_{t_n} & q_{t_n} + h_{t_n} & 0 \end{bmatrix}$$

$$= p_{t_n} e_1 + (q_{t_n} + h_{t_n}) e_2 + (q_{t_n} - h_{t_n}) e_3, \quad n \geq 0. \tag{2.111}$$

Substituting the relations (2.108)-(2.111) into the zero curvature equations (2.44), we see that

$$\begin{cases} g_n = -f_n, \\ p_{t_n} = 2a_{n+1} + 2qf_n, \qquad n \geq 0. \\ q_{t_n} = -c_{n+1} - pf_n, \\ h_{t_n} = f_{n,x}, \end{cases} \tag{2.112}$$

Using the second and the third equations of the above system, we can compute

$$\begin{aligned} f_{n,x} = h_{t_n} &= \alpha(2pp_{t_n} + 4qq_{t_n}) \\ &= 2\alpha p(2a_{n+1} + 2qf_n) + 4\alpha q(-c_{n+1} - pf_n) \\ &= 4\alpha pa_{n+1} - 4\alpha qc_{n+1} \\ &= -4\alpha b_{n+1,x}, \end{aligned} \tag{2.113}$$

where the last equation in (2.100) is utilized. Therefore,

$$f_n = -4\alpha b_{n+1}. \tag{2.114}$$

Inserting the value of f_n into (2.112), we get the following hierarchy of soliton equations

$$\begin{cases} p_{t_n} = 2a_{n+1} - 8\alpha qb_{n+1}, \\ q_{t_n} = -c_{n+1} + 4\alpha pb_{n+1}, \qquad n \geq 0, \end{cases} \tag{2.115}$$

that is,

$$u_{t_n} = K_n = \begin{bmatrix} 2a_{n+1} - 8\alpha qb_{n+1} \\ -c_{n+1} + 4\alpha pb_{n+1} \end{bmatrix}, \qquad n \geq 0. \tag{2.116}$$

□

PROPOSITION 2.2. *The functions $\{a_i, b_i, c_i \mid i \geq 1\}$ defined by Eq.(2.100), with the initial data (2.99) and under the conditions (2.101) are differential functions in u with respect to x, hence, they are all local.*

The proof is analogous to the proof of proposition (2.1) and so we omit the details.

Through lengthy computation involving the relations in (2.100), we can write the hierarchy (2.93) as

$$u_{t_n} = K_n = \begin{bmatrix} 2a_{n+1} - 8\alpha qb_{n+1} \\ -c_{n+1} + 4\alpha pb_{n+1} \end{bmatrix} = \Phi \begin{bmatrix} 2a_n - 8\alpha qb_n \\ -c_n + 4\alpha pb_n \end{bmatrix}, \qquad n \geq 1, \tag{2.117}$$

where the recursion operator Φ is given by

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix}, \tag{2.118}$$

with

$$\Phi_{11} = 4\alpha^2(p^2+2q^2)q\partial^{-1}p - 2\alpha\partial p\partial^{-1}p + \alpha(p^2+2q^2) - q\partial^{-1}p - 16\alpha^2 q\partial^{-1}q\partial q\partial^{-1}p$$
$$+4\alpha q\partial^{-1}q\partial - 8\alpha^2 q\partial^{-1}p\partial p\partial^{-1}p - 4\alpha^2 q\partial^{-1}(p^2+2q^2)p, \tag{2.119a}$$

$$\Phi_{12} = -\partial + 8\alpha^2(p^2+2q^2)q\partial^{-1}q - 4\alpha\partial p\partial^{-1}q - 2q\partial^{-1}q - 16\alpha^2 q\partial^{-1}p\partial p\partial^{-1}q$$
$$-4\alpha q\partial^{-1}p\partial - 32\alpha^2 q\partial^{-1}q\partial q\partial^{-1}q - 8\alpha^2 q\partial^{-1}(p^2+2q^2)q, \tag{2.119b}$$

$$\Phi_{21} = \frac{1}{2}\partial - 2\alpha^2(p^2+2q^2)p\partial^{-1}p - 2\alpha\partial q\partial^{-1}p + \frac{1}{2}p\partial^{-1}p + 8\alpha^2 p\partial^{-1}q\partial q\partial^{-1}p$$
$$-2\alpha p\partial^{-1}q\partial + 4\alpha^2 p\partial^{-1}p\partial p\partial^{-1}p + 2\alpha^2 p\partial^{-1}(p^2+2q^2)p, \tag{2.119c}$$

$$\Phi_{22} = \alpha(p^2+2q^2) - 4\alpha^2(p^2+2q^2)p\partial^{-1}q - 4\alpha\partial q\partial^{-1}q + p\partial^{-1}q + 8\alpha^2 p\partial^{-1}p\partial p\partial^{-1}q$$
$$+2\alpha p\partial^{-1}p\partial + 16\alpha^2 p\partial^{-1}q\partial q\partial^{-1}q + 4\alpha^2 p\partial^{-1}(p^2+2q^2)q. \tag{2.119d}$$

The first nonlinear system in the hierarchy (2.93) is

$$\begin{cases} p_{t_2} = -q_{xx} + 8\alpha q\Big(\frac{1}{2}\alpha(p^2+2q^2)p^2 + \frac{1}{2}q_x p - \frac{1}{2}qp_x + \alpha(p^2+2q^2)q^2\Big) - q^3 - \frac{1}{2}qp^2 \\ \qquad +2\alpha^2(p^2+2q^2)^2 q - 2\alpha(p^2+2q^2)p_x, \\ \\ q_{t_2} = \frac{1}{2}p_{xx} - 4\alpha p\Big(\frac{1}{2}\alpha(p^2+2q^2)p^2 + \frac{1}{2}q_x p - \frac{1}{2}qp_x + \alpha(p^2+2q^2)q^2\Big) + \frac{1}{2}pq^2 + \frac{1}{4}p^3 \\ \qquad -\alpha^2 p(p^2+2q^2)^2 - 2\alpha(p^2+2q^2)q_x. \end{cases} \tag{2.120}$$

Chapter 3

Hamiltonian Structure of the Integrable Soliton Hierarchies

3.1 Introduction

The Hamiltonian formalism method is an attractive topic in the integrable theory. In 1971, Gardner [14] and Faddeev and Zakharov [74] studied the Hamiltonian approach to integrability of nonlinear evolution equations. They independently showed that the KdV equation is the first known equation that constitutes an infinite dimensional completely integrable Hamiltonian system. Their discovery led to more research in this area and Hamiltonian formulations of other interesting models are constructed. Magri [46] developed an important result in the Hamiltonian theory. He found that integrable Hamiltonian systems have an additional structure. They are bi-Hamiltonian systems, that is, they are Hamiltonian with respect to two different compatible Hamiltonian operators. Magri's theory provides the relation between the existence of a bi-Hamiltonian formulation for a system and its integrability in the Liouville sense. The goal of this chapter is to study Hamiltonian structures of the soliton hierarchies introduced in Chapter 2 and to discuss their integrability using bi-Hamiltonian structures. We begin by recalling some basic definitions and results that we need to go further. Most of the facts and notations are from [50].

3.2 Preliminaries

Recall from Chapter 1 that $\mathcal{A}$ is the algebra of smooth functions $L(x,u^{(n)})$ depending on x,u and derivatives of u up to finite order n and its quotient space under the image of the total divergence is the space $\mathcal{F}$ of functional $\mathcal{L} = \int L \ dx$. $\mathcal{A}^q$ is the vector space of q-tuples of differential functions, $L[u] = (L_1[u], L_2[u], ..., L_q[u])$, where $L_j \in \mathcal{A}, 1 \leq j \leq q$.

DEFINITION 3.2.1. *[12] Given an evolution equation*

$$u_t = K(u), \quad K \in \mathcal{A}^q. \tag{3.1}$$

A vector-valued function $S \in \mathcal{A}^q$ is called a symmetry of Eq.(3.1) if the infinitesimal transformation

$$u(t) \to u(t) + \varepsilon S(u(t)), \tag{3.2}$$

leaves (3.1) form invariant. $K(.)$ *is a symmetry.*

If $Q \in \mathcal{A}^q$ is a vector field and u is a solution of (3.1), then one get

$$\frac{dQ}{dt} = \frac{\partial Q}{\partial t} + Q'[u_t] = \frac{\partial Q}{\partial t} + Q'[K] = \frac{\partial Q}{\partial t} + K'[Q] - [K, Q]. \tag{3.3}$$

where $\frac{d}{dt}$ denotes the total t-derivative and $[.,.]$ is defined by Eq.(2.4).

THEOREM 3.1. *[13] A vector field* $S \in \mathcal{A}^q$ *is a symmetry of (3.1) if and only if* S *satisfies*

$$\frac{\partial S}{\partial t} = [K, S]. \tag{3.4}$$

As a result, we have the following corollary.

COROLLARY 3.0.1. *When a vector field* $S \in \mathcal{A}^q$ *is explicitly independent of* t *, that is,* $\frac{\partial S}{\partial t} = 0$*, then* S *is a symmetry of (3.1) if and only if* $[K, S] = 0$.

Generally speaking, a symmetry of an evolution equation generates a transformation that takes solutions to solutions. If we know a symmetry of an evolution equation we can derive new solutions from any known solution.

DEFINITION 3.2.2. *Let*

$$\mathcal{D} = \sum_J L_J[u] D_J, \qquad L_J \in \mathcal{A}, \tag{3.5}$$

be a differential operator. The adjoint operator of $\mathcal{D}$ *is the differential operator* $\mathcal{D}^\dagger$ *which satisfies*

$$\int_\Omega L.\mathcal{D}P dx = \int_\Omega P.(\mathcal{D}^\dagger L) dx, \tag{3.6}$$

for every pair of differential functions $L, P \in \mathcal{A}$ *which vanish when* $u \equiv 0$*, every domain* $\Omega \subset \mathbb{R}^p$ *and every function* $u = f(x)$ *of compact support in* Ω*. Using integration by parts gives rise to*

$$\mathcal{D}^\dagger = \sum_J (-D)_J . L_J, \tag{3.7}$$

which means that for any $P \in \mathcal{A}$

$$\mathcal{D}^\dagger P = \sum_J (-D)_J . [L_J P]. \tag{3.8}$$

In a similar way, if $\mathcal{D} : \mathcal{A}^r \to \mathcal{A}^q$ *is a matrix differential operator with entries* $\mathcal{D}_{jk}$*, then its adjoint operator* $\mathcal{D}^\dagger : \mathcal{A}^q \to \mathcal{A}^r$ *has entries* $\mathcal{D}^\dagger_{jk} = (\mathcal{D}_{kj})^\dagger$*, the adjoint of the transpose components of* $\mathcal{D}$*.*

EXAMPLE 5. If

$$\mathcal{D} = D_x^3 - u^2 D_x, \tag{3.9}$$

then its adjoint $\mathcal{D}^\dagger$ is given by

$$\mathcal{D}^\dagger = (-D_x)^3 - (-D_x).u^2 = -D_x^3 + u^2 D_x + 2uu_x. \tag{3.10}$$

DEFINITION 3.2.3. *An operator $\mathcal{D}$ mapping $\mathcal{A}^q$ into itself is called self-adjoint if*

$$\mathcal{D}^\dagger = \mathcal{D}, \tag{3.11}$$

it is called skew-adjoint if

$$\mathcal{D}^\dagger = -\mathcal{D}. \tag{3.12}$$

EXAMPLE 6. Consider the operators

$$\mathcal{D}_1 = D_x^2 + u, \tag{3.13}$$

and

$$\mathcal{D}_2 = D_x^3 + 2uD_x + u_x. \tag{3.14}$$

The adjoint operators of $\mathcal{D}_1$ and $\mathcal{D}_2$ are

$$\mathcal{D}_1^\dagger = D_x^2 + u = \mathcal{D}_1, \tag{3.15}$$

and

$$\mathcal{D}_2^\dagger = -D_x^3 - 2uD_x - u_x = -\mathcal{D}_2, \tag{3.16}$$

respectively. Thus, $\mathcal{D}_1$ is self adjoint, while $\mathcal{D}_2$ is skew adjoint.

DEFINITION 3.2.4. *A linear differential operator $\Phi : \mathcal{A}^q \to \mathcal{A}^q$ is called a recursion operator for (3.1) if it satisfies the property that whenever $S \in \mathcal{A}^q$ is a symmetry of (3.1), so is ΦS.*

DEFINITION 3.2.5. *[36] Assume that $\Phi : \mathcal{A}^q \to \mathcal{A}^q$ is a differential operator and $K \in \mathcal{A}^q$. The Lie derivative of Φ with respect to K is a differential operator $L_K\Phi : \mathcal{A}^q \to \mathcal{A}^q$ defined as*

$$(L_K\Phi)Q = \Phi[K, Q] - [K, \Phi Q], \quad Q \in \mathcal{A}^q. \tag{3.17}$$

THEOREM 3.2. *[12, 36] A linear differential operator $\Phi : \mathcal{A}^q \to \mathcal{A}^q$ is a recursion operator for $K \in \mathcal{A}^q$ if and only if*

$$L_K\Phi = \Phi'[K] - [K', \Phi] = 0, \tag{3.18}$$

which means that when $\frac{\partial \Phi}{\partial t} = 0$, Φ is invariant under K.

DEFINITION 3.2.6. *[12] A differential operator $\Phi : \mathcal{A}^q \to \mathcal{A}^q$ is said to be a hereditary symmetry if it satisfies the following*

$$\Phi^2[Q, R] + [\Phi Q, \Phi R] - \Phi\{[Q, \Phi R] + [\Phi Q, R\} = 0 \qquad Q, R \in \mathcal{A}^q. \tag{3.19}$$

DEFINITION 3.2.7. *[50] The variational derivative of a functional $\mathcal{P} \in \mathcal{F}$ is defined by*

$$\int \left(\frac{\delta \mathcal{P}}{\delta u}\right)^T \xi dx = \mathcal{P}'(u)[\xi(u)], \qquad \xi \in \mathcal{A}^q, \tag{3.20}$$

where $\mathcal{P}'$ is the Gateaux derivative defined in (2.1).

DEFINITION 3.2.8. *[50] A linear operator $\mathcal{D} : \mathcal{A}^q \to \mathcal{A}^q$ is called Hamiltonian if its Poisson bracket*

$$\{\mathcal{P}, \mathcal{Q}\}_{\mathcal{D}} = \int \left(\frac{\delta \mathcal{P}}{\delta u}\right)^T . \mathcal{D}\left(\frac{\delta \mathcal{Q}}{\delta u}\right) dx, \qquad \mathcal{P}, \mathcal{Q} \in \mathcal{F}, \tag{3.21}$$

satisfies the conditions of skew-symmetry

$$\{\mathcal{P}, \mathcal{Q}\}_{\mathcal{D}} = -\{\mathcal{Q}, \mathcal{P}\}_{\mathcal{D}} \tag{3.22}$$

and for all functionals $\mathcal{P}, \mathcal{Q}, \mathcal{R} \in \mathcal{F}$, the Jacobi identity

$$\{\{\mathcal{P}, \mathcal{Q}\}, \mathcal{R}\}_{\mathcal{D}} + \{\{\mathcal{R}, \mathcal{P}\}, \mathcal{Q}\}_{\mathcal{D}} + \{\{\mathcal{Q}, \mathcal{R}\}, \mathcal{P}\}_{\mathcal{D}} = 0, \tag{3.23}$$

is hold.

DEFINITION 3.2.9. *A system of evolution equations of the form*

$$u_t = \mathcal{D}\frac{\delta \mathcal{H}}{\delta u}, \tag{3.24}$$

is called a Hamiltonian system. Here $\mathcal{D}$ is a Hamiltonian operator, δ is the variational derivative with respect to u and a functional $\mathcal{H} = \int H dx \in \mathcal{F}$ is referred to as a Hamiltonian functional.

LEMMA 3.1. *[50] Let the variational derivatives of the functionals $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ be*

$$\frac{\delta \mathcal{P}}{\delta u} = P, \frac{\delta \mathcal{Q}}{\delta u} = Q, \frac{\delta \mathcal{R}}{\delta u} = R \in \mathcal{A}^q. \tag{3.25}$$

Consequently, the following expression

$$\int \Big[P.\mathcal{D}'[\mathcal{D}R]Q + R.\mathcal{D}'[\mathcal{D}Q]P + Q.\mathcal{D}'[\mathcal{D}P]R\Big] dx = 0, \tag{3.26}$$

is equivalent to the Jacobi identity (3.23) or more frequently,

$$\langle P, \mathcal{D}'[\mathcal{D}R]Q\rangle + \langle R, \mathcal{D}'[\mathcal{D}Q]P\rangle + \langle Q, \mathcal{D}'[\mathcal{D}P]R\rangle = \langle P, \mathcal{D}'[\mathcal{D}R]Q\rangle + cycle(P, Q, R) = 0. \tag{3.27}$$

PROPOSITION 3.1. *[50] Assume that $\mathcal{D}$ is a $q \times q$ matrix differential operator with bracket (3.21) on the space of functionals. Then the bracket is skew-symmetric, i.e., (3.22) holds, if and only if $\mathcal{D}$ is skew-adjoint: $\mathcal{D}^\dagger = -\mathcal{D}$.*

PROPOSITION 3.2. *[50] Let $\mathcal{D}$ be a $q \times q$ skew-adjoint matrix differential operator. Then the bracket (3.21) satisfies the Jacobi identity (3.23) if and only if for all q-tuples $\mathcal{P}, \mathcal{Q}, \mathcal{R} \in \mathcal{A}^q$, the relation in (3.26) vanishes.*

LEMMA 3.2. *[50] Given an $q \times q$ skew-adjoint matrix differential operator $\mathcal{D}$. If all the coefficients of $\mathcal{D}$ are independent of the potential vector u or its derivatives, then $\mathcal{D}$ is a Hamiltonian operator.*

EXAMPLE 7. The KdV equation

$$u_t = u_{xxx} + uu_x, \tag{3.28}$$

is Hamiltonian in two different ways

$$u_t = \mathcal{D}_1 \frac{\delta \mathcal{H}_2}{\delta u} = \mathcal{D}_2 \frac{\delta \mathcal{H}_1}{\delta u}. \tag{3.29}$$

The Hamiltonian functionals are

$$\mathcal{H}_1 = \int \frac{1}{2} u^2 dx, \qquad \mathcal{H}_2 = \int \Big(\frac{1}{6} u^3 - \frac{1}{2} u_x^2\Big) dx, \tag{3.30}$$

with the corresponding operators

$$\mathcal{D}_1 = D_x, \qquad \mathcal{D}_2 = D_x^3 + \frac{2}{3} u D_x + \frac{1}{3} u_x, \tag{3.31}$$

Clearly, $\mathcal{D}_1$ is a Hamiltonian operator since it is skew-adjoint and independent of u. The operator $\mathcal{D}_2$ is also Hamiltonian and the proof can be found in [46].

DEFINITION 3.2.10. • *Any conservation law of a system of evolution equations (3.1) takes the form*

$$D_t T + \textit{Div}\ X = 0, \tag{3.32}$$

where Div denotes spatial divergence and the conserved density $T(x, t, u^{(n)})$ can be assumed to depend only on x-derivatives of u. Equivalently, for $\Omega \subset X$, the functional

$$\mathcal{T}[t; u] = \int_{\Omega} T(x, t, u^{(n)})dx \tag{3.33}$$

is a constant, does not depend on t, for all solutions u such that $T(x; t; u^{(n)}) \to 0$ as $x \to \partial\Omega$.

- *A conserved functional of a Hamiltonian system*

$$u_t = \mathcal{D}\frac{\delta \mathcal{H}}{\delta u}, \tag{3.34}$$

is a functional $\mathcal{T} = \int T dx$ which determines a conservation law (3.32) of (3.34).

THEOREM 3.3. *[62] Assume that $\mathcal{D}, \mathcal{E} : \mathcal{A}^q \to \mathcal{A}^q$ are two linear operators and suppose that*

1. *both $\mathcal{D}$ and $\mathcal{D}\mathcal{E}$ are skew-adjoint, i.e.,*

$$\mathcal{D}^{\dagger} = -\mathcal{D}, \quad (\mathcal{D}\mathcal{E}) = \mathcal{E}^{\dagger}\mathcal{D} \tag{3.35}$$

2. *there exists a series of scalar functions $\{\mathcal{H}_n\}$ that satisfies*

$$\mathcal{E}^n f(u) = \frac{\delta \mathcal{H}_n}{\delta u}, \tag{3.36}$$

for some $f(u) \in \mathcal{A}^q$.

Then $\{\mathcal{H}_n\}$ is a common series of conserved densities for the whole hierarchy of equations

$$u_t = \mathcal{D}\mathcal{E}^n f(u), \tag{3.37}$$

and we have

$$\{\mathcal{H}_n, \mathcal{H}_m\}_{\mathcal{D}} = 0 \quad \forall m, n \geq 0. \tag{3.38}$$

DEFINITION 3.2.11. *[40] A system of evolution equations (3.1) is called Liouville integrable if it can be written as the Hamiltonian system (3.24) with a well defined Poisson bracket $\{.,.\}$ and it possesses an infinite number of conserved functionals $\{\mathcal{H}_n\}$ which are in involution in pairs $\{\mathcal{H}_n, \mathcal{H}_m\}_{\mathcal{D}} = 0$.*

DEFINITION 3.2.12. *Let $\mathcal{D}$ and $\mathcal{E}$ be a pair of $q \times q$ skew-adjoint matrix differential operators. Then $\mathcal{D}$ and $\mathcal{E}$ is said to form a Hamiltonian pair if every linear combination $a\mathcal{D} + b\mathcal{E}$ with $a, b \in \mathbb{R}$ is a Hamiltonian operator.*

DEFINITION 3.2.13. *A system of evolution equations (3.1) is called a bi-Hamiltonian system if there exist two Hamiltonian functionals $\mathcal{H}_1, \mathcal{H}_2$ and a Hamiltonian pair $\mathcal{D}, \mathcal{E}$ so that the system (3.1) can be written as*

$$u_t = K[u] = \mathcal{D}\frac{\delta \mathcal{H}_2}{\delta u} = \mathcal{E}\frac{\delta \mathcal{H}_1}{\delta u} \tag{3.39}$$

EXAMPLE 8. According to example 7, the KdV equation

$$u_t = u_{xxx} + uu_x, \tag{3.40}$$

is bi-Hamiltonian equation with a Hamiltonian pair

$$\mathcal{D} = D_x, \qquad \mathcal{E} = D_x^3 + \frac{2}{3}uD_x + \frac{1}{3}u_x. \tag{3.41}$$

LEMMA 3.3. *[50] Two skew-adjoint operators $\mathcal{D}$ and $\mathcal{E}$ constitute a Hamiltonian pair if and only if all $\mathcal{D}, \mathcal{E}$ and $\mathcal{D} + \mathcal{E}$ are Hamiltonian operators.*

DEFINITION 3.2.14. *If there exists a nonzero differential operator $\widetilde{\mathcal{D}} : \mathcal{A}^r \to \mathcal{A}$ for a differential operator $\mathcal{D} : \mathcal{A}^q \to \mathcal{A}^r$ so that*

$$\widetilde{\mathcal{D}}.\mathcal{D} \equiv 0, \tag{3.42}$$

then $\mathcal{D}$ is said to be degenerate.

EXAMPLE 9. The matrix operator

$$\mathcal{D} = \begin{bmatrix} D_x^3 & -D_x^2 \\ D_x^2 & -D_x \end{bmatrix}, \tag{3.43}$$

is degenerate, since if $\widetilde{\mathcal{D}} = [1, -D_x]$, then $\widetilde{\mathcal{D}}.\mathcal{D} \equiv 0$.

THEOREM 3.4. *[50] Let*

$$u_t = K_1[u] = \mathcal{D}\frac{\delta \mathcal{H}_1}{\delta u} = \mathcal{E}\frac{\delta \mathcal{H}_0}{\delta u}, \tag{3.44}$$

be a bi-Hamiltonian system of evolution equations. Assume that the operator $\mathcal{D}$ of the Hamiltonian pair is non-degenerate Let $\mathcal{R} = \mathcal{E}\mathcal{D}^{-1}$ be the corresponding recursion operator, and let $K_0 = \mathcal{D}\frac{\delta \mathcal{H}_0}{\delta u}$. Assume that for each $n = 1, 2, \cdots$ we can recursively define

$$K_n = \mathcal{R}K_{n-1}, \qquad n \geq 1, \tag{3.45}$$

meaning that for each n, K_{n-1} lies in the image of $\mathcal{D}$. Then there exists a sequence of functionals $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2, \cdots$ such that

1. for each $n \geq 1$, the evolution equation

$$u_t = K_n[u] = \mathcal{D}\frac{\delta \mathcal{H}_n}{\delta u} = \mathcal{E}\frac{\delta \mathcal{H}_{n-1}}{\delta u}, \tag{3.46}$$

is a bi-Hamiltonian system,

2. the symmetries $\{K_n\}$ are all mutually commuting

$$[K_m, K_n] = 0, \qquad m, n \geq 0, \tag{3.47}$$

3. the Hamiltonian functionals $\mathcal{H}_n$ are all in involution with respect to either Poisson bracket:

$$\{\mathcal{H}_m, \mathcal{H}_n\}_{\mathcal{D}} = 0 = \{\mathcal{H}_m, \mathcal{H}_n\}_{\mathcal{E}} \qquad m, n \geq 0, \tag{3.48}$$

and hence provide an infinite collection of conservation laws for each of the bi-Hamiltonian systems in (3.46).

In what follows, we use the trace identity [37, 61, 62]

$$\frac{\delta}{\delta u}\int \text{tr}\Big(W\frac{\partial U}{\partial \lambda}\Big)dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\text{tr}\Big(W\frac{\partial U}{\partial u}\Big), \quad \gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\text{tr}(W^2)|, \tag{3.49}$$

which is a powerful method for presenting the hierarchies (2.51) and (2.93) in Hamiltonian forms.

3.3 Bi-Hamiltonian Structures of the Kaup-Newell Type Hierarchy

A Hamiltonian structure and a bi-Hamiltonian structure of the Kaup-Newell type hierarchy (2.51) are investigated in this section.

3.3.1 Hamiltonian Structure

It is direct to see

$$\frac{\partial U}{\partial \lambda} = \begin{bmatrix} 2\lambda & p \\ 1 & -2\lambda \end{bmatrix}, \quad \frac{\partial U}{\partial p} = \begin{bmatrix} 0 & \lambda \\ 0 & 0 \end{bmatrix}, \quad \frac{\partial U}{\partial q} = \begin{bmatrix} \alpha & 0 \\ 0 & -\alpha \end{bmatrix}. \tag{3.50}$$

Thus, we have

$$\operatorname{tr}\Big(W\frac{\partial U}{\partial \lambda}\Big) = \operatorname{tr}\begin{bmatrix} 2\lambda a + b & ap - 2\lambda b \\ 2\lambda c - a & pc + 2\lambda a \end{bmatrix} = 4\lambda a + b + pc, \tag{3.51}$$

$$\operatorname{tr}\Big(W\frac{\partial U}{\partial p}\Big) = \operatorname{tr}\begin{bmatrix} 0 & \lambda a \\ 0 & \lambda c \end{bmatrix} = \lambda c, \tag{3.52}$$

$$\operatorname{tr}\Big(W\frac{\partial U}{\partial q}\Big) = \operatorname{tr}\begin{bmatrix} \alpha a & -\alpha b \\ \alpha c & \alpha a \end{bmatrix} = 2\alpha a. \tag{3.53}$$

Plugging these quantities into the trace identity (3.49) leads to

$$\frac{\delta}{\delta u}\int (4\lambda a + b + pc)dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\begin{bmatrix} \lambda c \\ 2\alpha a \end{bmatrix}. \tag{3.54}$$

If we balance the coefficients of λ^{-2n-1} in Eq.(3.54), then we obtain

$$\frac{\delta}{\delta u}\int (4a_{n+1} + b_n + pc_n)dx = (\gamma - 2n)\begin{bmatrix} c_n \\ 2\alpha a_n \end{bmatrix}, \quad n \geq 0. \tag{3.55}$$

The identity with $n = 1$ yields $\gamma = 0$, and hence we have

$$\frac{\delta}{\delta u}\mathcal{H}_{n+1} = \begin{bmatrix} c_{n+1} \\ 2\alpha a_{n+1} \end{bmatrix}, \quad n \geq 1, \tag{3.56}$$

with the Hamiltonian functionals

$$\mathcal{H}_0 = \int \frac{1}{2}(pp_x + qq_x)dx, \tag{3.57}$$

$$\mathcal{H}_{n+1} = \int \frac{4a_{n+2} + b_{n+1} + pc_{n+1}}{-2(n+1)}dx, \quad n \geq 1. \tag{3.58}$$

The above calculation is the proof of the coming proposition.

PROPOSITION 3.3. *The Kaup-Newell type soliton hierarchy (2.51) can be written in the Hamiltonian form*

$$u_{t_n} = K_n = \begin{bmatrix} -2a_{n+1,x} \\ -\frac{1}{\alpha}(a_{n+1,x} - c_{n+1,x}) \end{bmatrix} = J\begin{bmatrix} c_{n+1} \\ 2\alpha a_{n+1} \end{bmatrix} = J\frac{\delta \mathcal{H}_{n+1}}{\delta u}, \quad n \geq 1, \tag{3.59}$$

where the Hamiltonian operator J is defined as

$$J = \begin{bmatrix} 0 & -\frac{1}{\alpha}\partial \\ -\frac{1}{\alpha}\partial & \frac{1}{2\alpha^2}\partial \end{bmatrix}, \tag{3.60}$$

and the Hamiltonian functionals $\mathcal{H}_0$ and $\mathcal{H}_{m+1}$ are defined as

$$\mathcal{H}_0 = \int \frac{1}{2}(pp_x + qq_x)dx, \tag{3.61}$$

$$\mathcal{H}_{n+1} = \int \frac{4a_{n+2} + b_{n+1} + pc_{n+1}}{-2(n+1)}dx, \quad n \geq 1. \tag{3.62}$$

It is easy to see that

$$J^\dagger = \begin{bmatrix} 0 & \frac{1}{\alpha}\partial \\ \frac{1}{\alpha}\partial & -\frac{1}{2\alpha^2}\partial \end{bmatrix} = -J, \tag{3.63}$$

which means that the operator J is a skew-adjoint and since its components do not depend on the potentials p and q or their derivatives, lemma 3.2 tells us that J is a Hamiltonian operator.

3.3.2 Bi-Hamiltonian Structure

PROPOSITION 3.4. *The soliton hierarchy of the Kaup-Newell type (2.51) has a bi-Hamiltonian structure*

$$u_{t_n} = K_n = J\frac{\delta\mathcal{H}_{n+1}}{\delta u} = M\frac{\delta\mathcal{H}_n}{\delta u}, \quad n \geq 1, \tag{3.64}$$

with a second Hamiltonian operator M defined by

$$M = \begin{bmatrix} p\partial + \partial p & q\partial - \frac{1}{2\alpha}\partial^2 \\ \partial q + \frac{1}{2\alpha}\partial^2 & 0 \end{bmatrix}, \tag{3.65}$$

J and $\mathcal{H}_{n+1}$ are given by (3.60) and (3.62), respectively.

Proof. First, we need to prove that the operator N given by

$$N = \mu \begin{bmatrix} 0 & -\frac{1}{\alpha}\partial \\ -\frac{1}{\alpha}\partial & \frac{1}{2\alpha^2}\partial \end{bmatrix} + \eta \begin{bmatrix} p\partial + \partial p & q\partial - \frac{1}{2\alpha}\partial^2 \\ \partial q + \frac{1}{2\alpha}\partial^2 & 0 \end{bmatrix}, \tag{3.66}$$

where μ and η are arbitrary real constants, is a Hamiltonian operator, that is, we need to show that

1. N is a skew-adjoint, and

2. N satisfies the Jacobi identity (3.23) or equivalently

$$\langle X, N'[NY]Z\rangle + \langle Y, N'[NZ]X\rangle + \langle Z, N'[NX]Y\rangle = \langle X, N'[NY]Z\rangle + \text{cycle}(X,Y,Z) = 0, \tag{3.67}$$

for any vectors of functions X, Y and Z, where $\langle\cdot,\cdot\rangle$ denotes the inner product and cycle(X, Y, Z) denotes the cyclic permutation of X, Y, Z [44].

We see that

$$N^\dagger = \begin{bmatrix} \eta(-\partial p - p\partial) & \frac{\mu}{\alpha}\partial + \eta(-q\partial + \frac{1}{2\alpha}\partial^2) \\ \frac{\mu}{\alpha}\partial + \eta(-\partial q - \frac{1}{2\alpha}\partial^2) & -\frac{\mu}{2\alpha^2}\partial \end{bmatrix} = -N, \tag{3.68}$$

which implies that N is a skew-adjoint operator. Now, assume that

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \quad Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}, \quad Z = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}, \quad W = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}, \tag{3.69}$$

are two dimensional vectors of functions. From (3.66) we have

$$NY = \begin{bmatrix} \eta P(Y) + \eta\partial p Y_1 - \frac{\mu}{\alpha}Y_{2,x} - \frac{\eta}{2\alpha}Y_{2,xx} \\ \\ -\frac{\mu}{\alpha}Y_{1,x} + \eta\partial q Y_1 + \frac{\eta}{2\alpha}Y_{1,xx} + \frac{\mu}{2\alpha^2}Y_{2,x} \end{bmatrix} := \begin{bmatrix} W_1(Y) \\ \\ W_2(Y) \end{bmatrix} = W, \tag{3.70}$$

where $P(Y) = (pY_{1,x} + qY_{2,x})$. Applying the definition of the Gateaux derivative, $N'[W]$ is computed as follows

$$N'[W] = N'[NY] = \begin{bmatrix} \eta\partial W_1 + \eta W_1\partial & \eta W_2\partial \\ \eta\partial W_2 & 0 \end{bmatrix}. \tag{3.71}$$

Then

$$N'[NY]Z = \begin{bmatrix} \eta\partial W_1 Z_1 + \eta W_1 Z_{1,x} + \eta W_2 Z_{2,x} \\ \eta\partial W_2 Z_1 \end{bmatrix}. \tag{3.72}$$

So, we have

$$\langle X, N'[NY]Z\rangle = \eta\int X_1(\partial W_1 Z_1 + W_1 Z_{1,x} + W_2 Z_{2,x})dx + \eta\int X_2\partial W_2 Z_1 dx. \tag{3.73}$$

Using integration by parts, the above equality becomes

$$\begin{aligned}\langle X, N'[NY]Z\rangle &= -\eta\int X_{1,x}(W_1 Z_1 + \partial^{-1}W_1 Z_{1,x} + \partial^{-1}W_2 Z_{2,x})dx - \eta\int X_{2,x}W_2 Z_1 dx \\ &= -\eta\int (X_{1,x}W_1 + X_{2,x}W_2)Z_1 dx - \eta\int X_{1,x}\partial^{-1}(W_1 Z_{1,x} + W_2 Z_{2,x})dx.\end{aligned} \tag{3.74}$$

Using integration by parts again with the cyclic permutation of X, Y and Z, we obtain

$$\int (X_{1,x}W_1 + X_{2,x}W_2)Z_1 dx + \text{cycle}(X, Y, Z) = 0, \tag{3.75}$$

and

$$\int X_{1,x}\partial^{-1}(W_1 Z_{1,x} + W_2 Z_{2,x})dx + \text{cycle}(X, Y, Z) = 0. \tag{3.76}$$

Consequently,

$$\langle X, N'[NY]Z\rangle + \text{cycle}(X, Y, Z) = 0. \tag{3.77}$$

Therefore, the operator N is a Hamiltonian operator. This implies that the operator M defined by

$$M = \Phi J = \begin{bmatrix} p\partial + \partial p & q\partial - \frac{1}{2\alpha}\partial^2 \\ \partial q + \frac{1}{2\alpha}\partial^2 & 0 \end{bmatrix}, \tag{3.78}$$

and obtained from the operator N by setting $\mu = 0$ and $\eta = 1$ is a Hamiltonian operator where Φ is given by (2.85). Furthermore, the operator $J + M$, found by setting $\mu = 1$ and $\eta = 1$ in the operator N,

$$J + M = \begin{bmatrix} 0 & -\frac{1}{\alpha}\partial \\ -\frac{1}{\alpha}\partial & \frac{1}{2\alpha^2}\partial \end{bmatrix} + \begin{bmatrix} p\partial + \partial p & q\partial - \frac{1}{2\alpha}\partial^2 \\ \partial q + \frac{1}{2\alpha}\partial^2 & 0 \end{bmatrix} \tag{3.79}$$

is also a Hamiltonian operator. Thus, according to lemma 3.3 the operators J and M constitute a Hamiltonian pair. This means that the hierarchy of soliton equations (2.51) possesses a bi-Hamiltonian structure (3.64). The proof is completed. □

Due to the result in [46], the bi-Hamiltonian hierarchy (3.64) is said to be Liouville integrable, i.e., has infinitely many conserved functions in involution

$$\{\mathcal{H}_m, \mathcal{H}_n\}_J = \int \Big(\frac{\delta \mathcal{H}_m}{\delta u}\Big)^T J \frac{\delta \mathcal{H}_n}{\delta u} dx = 0, \quad m, n \geq 0, \tag{3.80}$$

$$\{\mathcal{H}_m, \mathcal{H}_n\}_M = \int \Big(\frac{\delta \mathcal{H}_m}{\delta u}\Big)^T M \frac{\delta \mathcal{H}_n}{\delta u} dx = 0, \quad m, n \geq 0, \tag{3.81}$$

and commuting symmetries

$$[K_m, K_n] = K_m'(u)[K_n] - K_n'(u)[K_m] = 0, \quad m, n \geq 0. \tag{3.82}$$

3.4 Bi-Hamiltonian Structures of the Generalized Dirac Soliton Hierarchy

In this section, we consider a Hamiltonian and a bi-Hamiltonian structures of the generalized Dirac hierarchy (2.93).

3.4.1 Hamiltonian Structure

PROPOSITION 3.5. *The generalized Dirac hierarchy of soliton equations (2.93) possesses the following Hamiltonian structure for* $n \geq 1$:

$$u_{t_n} = K_n = \begin{bmatrix} 2a_{n+1} - 8\alpha q b_{n+1} \\ -c_{n+1} + 4\alpha p b_{n+1} \end{bmatrix} = J \begin{bmatrix} 4\alpha p b_{n+1} + c_{n+1} \\ 2a_{n+1} + 8\alpha q b_{n+1} \end{bmatrix} = J \frac{\delta \mathcal{H}_{n+1}}{\delta u}, \tag{3.83}$$

where the operator J *defined as*

$$J = \begin{bmatrix} -16\alpha q \partial^{-1} q & 1 + 8\alpha q \partial^{-1} p \\ -1 + 8\alpha p \partial^{-1} q & -4\alpha p \partial^{-1} p \end{bmatrix}, \tag{3.84}$$

is a Hamiltonian operator and the Hamiltonian functional $\mathcal{H}_{n+1}$ *is given by*

$$\mathcal{H}_{n+1} = \int \frac{2b_{n+2}}{-(n+1)} dx, \quad n \geq 1. \tag{3.85}$$

Proof. To constitute Hamiltonian structures, we apply the trace identity (3.49). From the partial derivatives

$$\frac{\partial U}{\partial \lambda} = \begin{bmatrix} 0 & \lambda & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \frac{\partial U}{\partial p} = \begin{bmatrix} 0 & 2\alpha p & -1 \\ -2\alpha p & 0 & -2\alpha p \\ 1 & 2\alpha p & 0 \end{bmatrix}, \tag{3.86}$$

and

$$\frac{\partial U}{\partial q} = \begin{bmatrix} 0 & -1 + 4\alpha q & 0 \\ 1 - 4\alpha q & 0 & -1 - 4\alpha q \\ 0 & 1 + 4\alpha q & 0 \end{bmatrix}, \tag{3.87}$$

we have

$$\operatorname{tr}\Big(W\frac{\partial U}{\partial \lambda}\Big) = \operatorname{tr}\begin{bmatrix} a-b & -c & a-b \\ 0 & -2b & 0 \\ -a-b & c & -a-b \end{bmatrix} = -4b, \tag{3.88}$$

$$\operatorname{tr}\Big(W\frac{\partial U}{\partial p}\Big) = \operatorname{tr}\begin{bmatrix} 2\alpha p(a-b)-c & -2\alpha pc & 2\alpha p(a-b) \\ -(a+b) & -4\alpha pb & -a+b \\ -2\alpha p(a+b) & 2\alpha pc & -2(a+b)\alpha p - c \end{bmatrix} = -8\alpha pb - 2c, \tag{3.89}$$

$$\operatorname{tr}\Big(W\frac{\partial U}{\partial q}\Big) = \operatorname{tr}\begin{bmatrix} 4\alpha q(a-b)-a+b & -4\alpha qc - c & 4\alpha q(a-b)+a-b \\ 0 & -2a-8\alpha qb & 0 \\ a+b-4\alpha q(a+b) & -c+4\alpha qc & -(a+b)-4\alpha q(a+b) \end{bmatrix}$$

$$= -4a - 16\alpha qb. \tag{3.90}$$

In this case, the trace identity (3.49) presents

$$\frac{\delta}{\delta u}\int 2b\, dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\begin{bmatrix} 4\alpha pb + c \\ 2a + 8\alpha qb \end{bmatrix}. \tag{3.91}$$

Equating coefficients of λ^{-n-1} in the equality shows

$$\frac{\delta}{\delta u}\int 2b_{n+1} dx = (\gamma - n)\begin{bmatrix} 4\alpha pb_n + c_n \\ 2a_n + 8\alpha qb_n \end{bmatrix}, \quad n \geq 0. \tag{3.92}$$

Checking a special case with $n = 1$ yields $\gamma = 0$, and thus we get

$$\frac{\delta}{\delta u}\mathcal{H}_{n+1} = \begin{bmatrix} 4\alpha pb_{n+1} + c_{n+1} \\ 2a_{n+1} + 8\alpha qb_{n+1} \end{bmatrix}, \quad n \geq 1, \tag{3.93}$$

with the Hamiltonian functional

$$\mathcal{H}_{n+1} = \int \frac{2b_{n+2}}{-(n+1)} dx, \quad n \geq 1. \tag{3.94}$$

Now from the recursion relations (2.100), we have

$$\begin{aligned} 2a_{n+1} - 8\alpha qb_{n+1} &= (2a_{n+1} + 8\alpha qb_{n+1}) - 16\alpha qb_{n+1} \\ &= (2a_{n+1} + 8\alpha qb_{n+1}) - 16\alpha q\partial^{-1}qc_{n+1} + 16\alpha q\partial^{-1}pa_{n+1} \qquad (3.95) \\ &= (1 + 8\alpha q\partial^{-1}p)(2a_{n+1} + 8\alpha qb_{n+1}) - 16\alpha q\partial^{-1}qc_{n+1} - 64\alpha^2 q\partial^{-1}pqb_{n+1} \\ &= (1 + 8\alpha q\partial^{-1}p)(2a_{n+1} + 8\alpha qb_{n+1}) - 16\alpha q\partial^{-1}q(4\alpha pb_{n+1} + c_{n+1}), \end{aligned}$$

and

$$
\begin{aligned}
-c_{n+1}+4\alpha p b_{n+1} &= -(c_{n+1}+4\alpha p b_{n+1})+8\alpha p b_{n+1} \\
&= -(c_{n+1}+4\alpha p b_{n+1})+8\alpha p\partial^{-1}q c_{n+1}-8\alpha p\partial^{-1}p a_{n+1} \qquad (3.96)\\
&= (-1+8\alpha p\partial^{-1}q)(c_{n+1}+4\alpha p b_{n+1})-32\alpha p\partial^{-1}qpb_{n+1}-8\alpha p\partial^{-1}pa_{n+1} \\
&= (-1+8\alpha p\partial^{-1}q)(4\alpha p b_{n+1}+c_{n+1})-4\alpha p\partial^{-1}p(2a_{n+1}+8\alpha p b_{n+1}).
\end{aligned}
$$

The soliton hierarchy (2.93) with the relations (3.95) and (3.96) tell us that

$$
\begin{aligned}
u_{t_n}=K_n &= \begin{bmatrix} 2a_{n+1}-8\alpha q b_{n+1} \\ -c_{n+1}+4\alpha p b_{n+1} \end{bmatrix} \\
&= \begin{bmatrix} -16\alpha q\partial^{-1}q & 1+8\alpha q\partial^{-1}p \\ -1+8\alpha p\partial^{-1}q & -4\alpha p\partial^{-1}p \end{bmatrix} \begin{bmatrix} 4\alpha p b_{n+1}+c_{n+1} \\ 2a_{n+1}+8\alpha q b_{n+1} \end{bmatrix} \\
&= J\frac{\delta \mathcal{H}_{n+1}}{\delta u}, \quad n\geq 1. \qquad (3.97)
\end{aligned}
$$

Finally, we need to show that the operator

$$
J=\begin{bmatrix} -16\alpha q\partial^{-1}q & 1+8\alpha q\partial^{-1}p \\ -1+8\alpha p\partial^{-1}q & -4\alpha p\partial^{-1}p \end{bmatrix}, \qquad (3.98)
$$

is a Hamiltonian operator. It is a skew-adjoint since

$$
J^{\dagger}=\begin{bmatrix} 16\alpha q\partial^{-1}q & -1-8\alpha q\partial^{-1}p \\ 1-8\alpha p\partial^{-1}q & 4\alpha p\partial^{-1}p \end{bmatrix}=-J, \qquad (3.99)
$$

and the Jacobi identity can be easily verified in a similar way as in the proof of proposition 3.4. Thus J is a Hamiltonian operator. This finishes the proof. □

3.4.2 Bi-Hamiltonian Structure

Through heavy and long calculations which involve the recursion relations (2.100), we find that

$$
\frac{\delta \mathcal{H}_{n+1}}{\delta u}=\begin{bmatrix} 4\alpha p b_{n+1}+c_{n+1} \\ 2a_{n+1}+8\alpha q b_{n+1} \end{bmatrix}=\Psi\begin{bmatrix} 4\alpha p b_n+c_n \\ 2a_m+8\alpha q b_m \end{bmatrix}=\Psi\frac{\delta \mathcal{H}_n}{\delta u}, \qquad (3.100)
$$

where

$$
\Psi=\begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{bmatrix}, \qquad (3.101)
$$

with

$$\Psi_{11} = -4\alpha^2 p\partial^{-1}q(p^2+2q^2) - 2\alpha p\partial^{-1}p\partial - \alpha(p^2+2q^2) + p\partial^{-1}q + 16\alpha^2 p\partial^{-1}q\partial q\partial^{-1}q$$
$$+4\alpha\partial q\partial^{-1}q + 8\alpha^2 p\partial^{-1}p\partial p\partial^{-1}q + 4\alpha^2 p(p^2+2q^2)\partial^{-1}q, \tag{3.102}$$

$$\Psi_{12} = -\frac{1}{2}\partial + 2\alpha^2 p\partial^{-1}p(p^2+2q^2) - 2\alpha p\partial^{-1}q\partial - \frac{1}{2}p\partial^{-1}p - 8\alpha^2 p\partial^{-1}q\partial q\partial^{-1}p$$
$$-2\alpha\partial q\partial^{-1}p - 4\alpha^2 p\partial^{-1}p\partial p\partial^{-1}p - 2\alpha^2 p(p^2+2q^2)\partial^{-1}p, \tag{3.103}$$

$$\Psi_{21} = \partial - 8\alpha^2 q\partial^{-1}q(p^2+2q^2) - 4\alpha q\partial^{-1}p\partial + 2q\partial^{-1}q + 16\alpha^2 q\partial^{-1}p\partial p\partial^{-1}q$$
$$-4\alpha\partial p\partial^{-1}q + 32\alpha^2 q\partial^{-1}q\partial q\partial^{-1}q + 8\alpha^2 q(p^2+2q^2)\partial^{-1}q, \tag{3.104}$$

$$\Psi_{22} = -\alpha(p^2+2q^2) + 4\alpha^2 q\partial^{-1}p(p^2+2q^2) - 4\alpha q\partial^{-1}q\partial - q\partial^{-1}p - 8\alpha^2 q\partial^{-1}p\partial p\partial^{-1}p$$
$$+2\alpha\partial p\partial^{-1}p - 16\alpha^2 q\partial^{-1}q\partial q\partial^{-1}p - 4\alpha^2 q(p^2+2q^2)\partial^{-1}p. \tag{3.105}$$

Utilizing a similar but more complicated argument than the proof of proposition 3.4 and in [44], it can be proved that the operator

$$N = \mu J + \eta M, \tag{3.106}$$

where μ and η are arbitrary real constants, is a Hamiltonian operator where J is defined by (3.84) and M is given by

$$M = J\Psi = \Phi J = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \tag{3.107}$$

with

$$M_{11} = \partial + 8\alpha^2 q\partial^{-1}q(p^2+2q^2) + 4\alpha q\partial^{-1}p\partial + 2q\partial^{-1}q - 16\alpha^2 q\partial^{-1}p\partial p\partial^{-1}q - 4\alpha\partial p\partial^{-1}q$$
$$-32\alpha^2 q\partial^{-1}q\partial q\partial^{-1}q + 8\alpha^2 q(p^2+2q^2)\partial^{-1}q + 16\alpha^2 q\partial^{-1}q(p^2+2q^2), \tag{3.108}$$

$$M_{12} = 4\alpha q\partial^{-1}q\partial - \alpha(p^2+2q^2) - 4\alpha^2 q\partial^{-1}p(p^2+2q^2) + 8\alpha^2 q\partial^{-1}p\partial p\partial^{-1}p - q\partial^{-1}p$$
$$+2\alpha\partial p\partial^{-1}p + 16\alpha^2 q\partial^{-1}q\partial q\partial^{-1}p - 4\alpha^2 q(p^2+2q^2)\partial^{-1}p, \tag{3.109}$$

$$M_{21} = \alpha(p^2+2q^2) - p\partial^{-1}q + 16\alpha^2 p\partial^{-1}q\partial q\partial^{-1}q - 4\alpha^2 p\partial^{-1}q(p^2+2q^2) - 2\alpha p\partial^{-1}p\partial$$
$$+8\alpha^2 p\partial^{-1}p\partial p\partial^{-1}q - 4\alpha^2 p(p^2+2q^2)\partial^{-1}q - 4\alpha\partial q\partial^{-1}q, \tag{3.110}$$

$$M_{22} = \frac{1}{2}\partial + \frac{1}{2}p\partial^{-1}p + 2\alpha^2 p\partial^{-1}p(p^2+2q^2) - 4\alpha^2 p\partial^{-1}p\partial p\partial^{-1}p - 8\alpha^2 p\partial^{-1}q\partial q\partial^{-1}p$$
$$-2\alpha p\partial^{-1}q\partial + 2\alpha^2 p(p^2+2q^2)\partial^{-1}p + 2\alpha\partial q\partial^{-1}p. \tag{3.111}$$

Herein Φ is defined by (2.118)-(2.119) and has the following property

$$\Phi = \Psi^{\dagger}. \tag{3.112}$$

Setting $\mu = 0$ and $\eta = 1$ in (3.106) tells that M is a Hamiltonian operator and taking $\mu = \eta = 1$ leads to $J + M$ is a Hamiltonian operator. This implies that J and M compose a Hamiltonian pair (lemma 3.3). Thus, we can say that the generalized Dirac soliton hierarchy (2.93) is bi-Hamiltonian:

$$u_{t_n} = K_n = J\frac{\delta\mathcal{H}_{n+1}}{\delta u} = M\frac{\delta\mathcal{H}_n}{\delta u}, \quad n \geq 1, \tag{3.113}$$

with the second Hamiltonian operator M being given by (3.107) and J , $\mathcal{H}_{n+1}$ being defined in (3.84) and (3.85), respectively.

According to [46], the bi-Hamiltonian hierarchy (3.113) is Liouville integrable, i.e., has infinitely many conserved functions in involution

$$\{\mathcal{H}_m, \mathcal{H}_n\}_J = \int \Big(\frac{\delta\mathcal{H}_m}{\delta u}\Big)^T J\frac{\delta\mathcal{H}_n}{\delta u}dx = 0, \quad m, n \geq 0, \tag{3.114}$$

$$\{\mathcal{H}_m, \mathcal{H}_n\}_M = \int \Big(\frac{\delta\mathcal{H}_m}{\delta u}\Big)^T M\frac{\delta\mathcal{H}_n}{\delta u}dx = 0, \quad m, n \geq 0, \tag{3.115}$$

and commuting symmetries

$$[K_m, K_n] = K'_m(u)[K_n] - K'_n(u)[K_m] = 0, \quad m, n \geq 0. \tag{3.116}$$

Chapter 4

Lump and Interaction Solutions of Two Nonlinear Partial Differential Equations

4.1 Introduction

Scientists are looking for exact solutions to nonlinear evolution equations due to their numerous applications in different areas of sciences. It has been known that finding exact solutions for nonlinear evolution equations including soliton equations can be an intricate and complicated work. In general, a method that solves one nonlinear equation might not be effective for other equations. This led Ryogo Hirota to search for a successful tool to solve many types of nonlinear equations and to establish the direct method which is now called ”Hirota direct method” [22]. In 1971, Hirota found multi-soliton solutions of the KdV equation and derived an explicit expression for its N-soliton solutions by the first application of his direct method [21]. Over the last few decades, exact solutions for many nonlinear evolution equations have been obtained by applying the Hirota direct method [48].

Recently, the idea of finding rational solutions in addition to the rouge wave solutions to nonlinear evolution equations has flourished. Lump solution is a type of rational function solution which localize in all directions in the space [18, 45]. Many studies have been conducted in order to calculate lump solutions for a large number of nonlinear evolution equations. Among these equations are the Kadomtsev petviashvili (KPI) equation [24, 47, 54], the three wave resonant interaction equation [26], and the B-KP equation [18]. In 2015, Ma developed a new direct method, depends on quadratic functions, to generate lump or lump-type solutions [41]. Lump solutions for more nonlinear evolution equations have been found following Ma's method. For instance, the (2+1)-dimensional Boussinesq equation [35], the dimensionally reduced p-gKP and p-gBKP [42] and the BKP equation [72]. In addition to lump solutions, interaction solutions of lump with another kind of solutions including resonance stripe solitons [32] and kink solutions [76,79] have also brought a lot of attention and have been investigated. We begin this chapter by providing some basic notations, definitions, and important results. In Section 4.3, we study lump solutions to a (2+1)-dimensional 5th-order KdV-like equation. In Section 4.4, we seek lump-kind solutions and interaction solutions between lump-kind and kink solutions and between lump-kind and resonance stripe solitons solutions for a Jimbo-Miwa-like equation in (3+1)-dimension.

4.2 Preliminaries

4.2.1 Hirota Derivatives

DEFINITION 4.2.1. *[22] Let $g(t,x)$ and $h(t,x)$ be differentiable functions in t and x. A binary differential operator, called the D-operator, is defined by*

$$\begin{aligned} D_t^m D_x^n g\cdot h &\equiv \Big(\frac{\partial}{\partial t}-\frac{\partial}{\partial s}\Big)^m\Big(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\Big)^n g(t,x)h(s,y)\Big|_{s=t,y=x} \\ &\equiv \frac{\partial^m}{\partial s^m}\frac{\partial^n}{\partial y^n} g(t+s,x+y)h(t-s,x-y)\Big|_{s=0,y=0}, \end{aligned} \tag{4.1}$$

where $m,n=0,1,2,\cdots$. This type of differential operator is called a bilinear operator, due to the obvious linearity in both of its arguments. The D-operators are known as the Hirota derivatives.

EXAMPLE 10. From the definition, it is direct to calculate

$$D_x g\cdot h = g_x h - g h_x, \tag{4.2}$$

$$D_x^2 g\cdot h = g_{xx}h - 2g_x h_x + g h_{xx}, \tag{4.3}$$

$$D_x^3 g\cdot h = g_{xxx}h - 3g_{xx}h_x + 3g_x h_{xx} - g h_{xxx}, \tag{4.4}$$

$$D_x^4 g\cdot h = g_{4x}h - 4g_{xxx}h_x + 6g_{xx}h_{xx} - 4g_x h_{xxx} + g h_{4x}, \tag{4.5}$$

$$D_x D_t g\cdot h = g_{xt}h - g_x h_t - g_t h_x + g h_{xt}, \tag{4.6}$$

$$\begin{aligned} D_x^3 D_t g\cdot h = {}& g_{xxxt}h - 3g_{xxt}h_x + 3g_{xt}h_{xx} - g_t h_{xxx} - g_{xxx}h_t \\ & +3g_{xx}h_{xt} - 3g_x h_{xxt} + g h_{xxxt}, \end{aligned} \tag{4.7}$$

where the subscript g_x denote the partial derivative of g with respect to x and similarly for the others.

DEFINITION 4.2.2. *A nonlinear partial differential equation is said to have a Hirota bilinear form if it is equivalent to*

$$\sum_{i,j=1}^{n} P_{ij}^m(D) g_i\cdot g_j = 0, \quad m=1,\cdots,r, \tag{4.8}$$

where $n,r\geq 1$, $P_{ij}^m(D)$ are linear operators, and g_i's are new dependent variables, under a dependent variable transformation.

Generally, the definition of the D-operators can be extended to high dimensional spaces as follows.

DEFINITION 4.2.3. *Let $M \in \mathbb{N}$, and $x, x' \in \mathbb{R}^M$ such that $x = (x_1, x_2, \cdots, x_M), x' = (x'_1, x'_2, \cdots, x'_M)$. Assume that g and h are infinitely differentiable functions in $\mathbb{R}^M$. Then*

$$D_j^n g \cdot h = D_{x_j}^n g \cdot h \equiv \Big(\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x'_j}\Big)^n g(x)h(x')\Big|_{x'=x} = g_{x_j}(x)h(x) - g(x)h_{x_j}(x), \tag{4.9}$$

and the higher order D-operator is given by

$$\begin{aligned}(D_1^{n_1} D_2^{n_2} \cdots D_M^{n_M}) g \cdot h &= \prod_{j=1}^{M} \left(\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x'_j}\right)^{n_j} g(x)h(x')|_{x'=x} \\ &= \sum_{k_1=0}^{n_1} \cdots \sum_{k_M=0}^{n_M} (-1)^{\sum_{j=1}^{M}(n_j-k_j)} \prod_{j=1}^{M} \binom{n_j}{k_j} \frac{\partial^{k_j}}{\partial x_j^{k_j}} g(x) \frac{\partial^{n_j-k_j}}{\partial x_j^{n_j-k_j}} h(x),\end{aligned} \tag{4.10}$$

where $n_1, n_2, \cdots, n_M \geq 0$, and $\begin{pmatrix} n_j \\ k_j \end{pmatrix} := \dfrac{n_j!}{k_j!(n_j-k_j)!}$ is the binomial coefficient.

EXAMPLE 11. If g is a differentiable function of x, y, z and t, then

$$(D_x D_y D_z D_t) g \cdot g = 2(g_{xyzt}g - g_{xyt}g_z - g_{xyz}g_t - g_{xzt}g_y - g_{yzt}g_x + g_{xy}g_{zt} + g_{xz}g_{yt} + g_{xt}g_{yz}). \tag{4.11}$$

4.2.2 Properties of the D-operators

We list some properties of the operators D_t, D_x [22]. For convenience, we introduce an operator D_z and a differentiation $\frac{\partial}{\partial z}$ by

$$D_z = \mu D_t + \varepsilon D_x, \qquad \frac{\partial}{\partial z} = \mu \frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial x}, \tag{4.12}$$

where μ and ε are constants. The following properties are obtained easily from the definition

1.

$$D_z^m g \cdot 1 = \frac{\partial^m}{\partial z^m} g. \tag{4.13}$$

2. If the functions g and h interchange, then

$$D_z^m g \cdot h = (-1)^m D_z^m h \cdot g, \tag{4.14}$$

from which we have, if m is odd,

$$D_z^m g \cdot g = 0. \tag{4.15}$$

3.

$$D_z^m g \cdot h = D_z^{m-1} D_z g \cdot h = D_z^{m-1}(g_z h - g h_z). \tag{4.16}$$

4. The identity

$$D_z(D_z f \cdot g) \cdot h + D_z(D_z g \cdot h) \cdot f + D_z(D_z h \cdot f) \cdot g = 0, \tag{4.17}$$

holds for any functions f, g and h. If we write $D_z f \cdot g$ as $[f, g]$, then the identity (4.17) can be written as the Jacobi identity

$$[[f, g], h] + [[g, h], f] + [[h, f], g] = 0, \tag{4.18}$$

which indicates one connection between the D-operators and Lie algebras.

4.2.3 Bilinear Forms of Some Nonlinear Evolution Equations

Consider the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0. \tag{4.19}$$

Through the dependent variable transformation

$$u = 2(\log f)_{xx}, \tag{4.20}$$

Eq.(4.19) becomes

$$2f_{xt}f - 2f_x f_t + 2f_{4x}f - 8f_{xxx}f_x + 6f_{xx}^2 = 0, \tag{4.21}$$

with the Hirota bilinear form

$$(D_x D_t + D_x^4) f \cdot f = 0. \tag{4.22}$$

The Kadomtsev-Petviashvili (KP) equation

$$(u_t + 6uu_x + u_{xxx}) - u_{yy} = 0, \tag{4.23}$$

has the bilinear form

$$f f_{xt} - f_x f_t + 3f_{xx}^2 + f f_{4x} - 4f_x f_{xxx} - f_{yy} f + f_y^2 = 0, \tag{4.24}$$

and can be expressed in the Hirota bilinear form

$$(D_tD_x + D_x^4 - D_y^2)f \cdot f = 0, \tag{4.25}$$

under the transformation (4.20).

The Sawada-Kotera equation

$$u_t + u_{5x} + 15uu_{xxx} + 15u_xu_{xx} + 45u^2u_x = 0, \tag{4.26}$$

can be expressed in the bilinear form

$$2f_{xt}f - 2f_xf_t + 2f_{6x}f - 12f_{5x}f + 30f_{4x}f_{xx} - 20f_{xxx}^2 = 0, \tag{4.27}$$

and has the Hirota bilinear form

$$(D_xD_t + D_x^6)f \cdot f = 0, \tag{4.28}$$

through the transformation (4.20).

4.2.4 Hirota Direct Method

Hirota's method is one of the most successful direct techniques for constructing exact solutions to different nonlinear evolution equations. Through this method we can test if a specific equation satisfies the necessary requirements to admit solitary wave solutions and soliton solutions.

The first step in Hirota's method is to transform a nonlinear evolution equation

$$F[u] = F(u, u_x, u_t) = 0, \tag{4.29}$$

into a Hirota bilinear form

$$P(D)f \cdot f = 0, \tag{4.30}$$

under a suitable dependent variable transformation. The next step is to use the perturbation method to find a solution for the bilinear equation which finally leads to N-soliton solutions for the nonlinear evolution equation. The technique applies to any equation that can be written in bilinear form, either as a single bilinear equation or as a system of coupled bilinear equations.

EXAMPLE 12. Consider the KdV equation (4.19). We look for a solution of the Hirota bilinear form (4.22) by expanding f as

$$f = 1 + \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3 + \cdots, \tag{4.31}$$

where ε is a small parameter. Substituting (4.31) into (4.21) and collecting terms with the same power of ε, we obtain

$$\begin{cases} O(\varepsilon) : (D_x D_t + D_x^4)(f_1 \cdot 1 + 1 \cdot f_1) = 0, \\ O(\varepsilon^2) : (D_x D_t + D_x^4)(f_2 \cdot 1 + f_1 \cdot f_1 + 1 \cdot f_2) = 0, \\ O(\varepsilon^3) : (D_x D_t + D_x^4)(f_3 \cdot 1 + f_2 \cdot f_1 + f_1 \cdot f_2 + 1 \cdot f_3) = 0, \\ O(\varepsilon^4) : (D_x D_t + D_x^4)(f_4 \cdot 1 + f_3 \cdot f_1 + f_2 \cdot f_2 + f_1 \cdot f_3 + 1 \cdot f_4) = 0, \\ \dots \end{cases} \tag{4.32}$$

The first equation in (4.32) can be written as a linear differential equation for f_1

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} + \frac{\partial^3}{\partial x^3}\right) f_1 = 0. \tag{4.33}$$

To generate one-soliton solution for the KdV equation we set

$$f_1 = \exp(\eta_1), \qquad \text{with} \qquad \eta_1 = \kappa_1 x + \omega_1 t + \mu_1^0, \tag{4.34}$$

where κ_1, ω_1 and μ_1^0 are constants. Inserting f_1 into Eq.(4.33) we obtain the nonlinear dispersion relation

$$\omega_1 = -\kappa_1^3, \tag{4.35}$$

and the second equation in (4.32) allows to set $f_2 = 0$.

Consequently we can take

$$f_n = 0, \qquad n > 2. \tag{4.36}$$

Finally, letting $\varepsilon = 1$ leads to

$$f = 1 + f_1 = 1 + \exp\bigl(\kappa_1 x - \kappa_1^3 t + \mu_1^0\bigr), \tag{4.37}$$

and by substituting f into (4.20) with (4.35), we get the one-soliton solution of KdV

$$u(x,t) = \tfrac{1}{2}\kappa_1^2 \operatorname{sech}^2\{\tfrac{1}{2}(\kappa_1 x - \kappa_1^3 t + x_{10})\}. \tag{4.38}$$

Similarly, to find two-soliton solution for the KdV equation, we choose

$$f_1 = \exp(\eta_1) + \exp(\eta_2), \qquad \text{with} \qquad \eta_i = \kappa_i x + \omega_i t + \mu_i^0, \quad i = 1, 2, \tag{4.39}$$

to be the solution of (4.33) where κ_i, ω_i and $\mu_i^0 (i = 1, 2)$ are constants.

Note that

$$f_n = 0, \qquad n \geq 3. \tag{4.40}$$

Substituting $f = 1 + \varepsilon f_1 + \varepsilon^2 f_2$ into the relations (4.32) and making the coefficients of ε^k, $k = 0, 1, \cdots, 4$, to vanish gives rise to

$$\omega_i = -\kappa_i^3, \quad i = 1, 2, \tag{4.41}$$

and

$$f_2 = A_{12} \exp(\eta_1 + \eta_2), \tag{4.42}$$

where

$$A_{12} = \left(\frac{\kappa_1 - \kappa_2}{\kappa_1 + \kappa_2}\right)^2. \tag{4.43}$$

At last, we may set $\varepsilon = 1$, hence the two-soliton solution for the KdV equation is obtained from

$$u(x, t) = 2\frac{\partial^2}{\partial x^2}(\log f), \tag{4.44}$$

where

$$f = 1 + \exp(\eta_1) + \exp(\eta_2) + A_{12} \exp(\eta_1 + \eta_2). \tag{4.45}$$

This result implies that two solitons are not destroyed after their interaction. Same calculation as above is applied to generate the three-soliton solution for the KdV equation. The formulation of N-soliton solutions for the KdV equation with $N \geq 3$ is tedious and the analysis becomes more complicated (see [4] for details).

4.2.5 Generalized Bilinear Differential Operators and Bilinear Equations

Though a large class of nonlinear evolution equations possess Hirota bilinear form, there are still some nonlinear equations which can not be written in Hirota bilinear form. In 2011, a kind of generalized bilinear differential operators D_p based on a natural number p has been proposed [38]. The operators D_p allow us to construct more generalized bilinear differential equations, different from Hirota bilinear equations, which still hold nice mathematical properties.

The D_p-operators

DEFINITION 4.2.4. *Suppose that M, p are two natural numbers. The bilinear differential operators D_p are defined as [38]*

$$(D_{p,1}^{n_1} D_{p,2}^{n_2} \cdots D_{p,M}^{n_M}) g \cdot h = (D_{p,x_1}^{n_1} D_{p,x_2}^{n_2} \cdots D_{p,x_M}^{n_M}) g \cdot h$$

$$= \prod_{j=1}^{M} \left(\frac{\partial}{\partial x_j} + \alpha_p \frac{\partial}{\partial x'_j} \right)^{n_j} g(x) h(x')|_{x'=x} \tag{4.46}$$

$$= \sum_{k_1=0}^{n_1} \cdots \sum_{k_M=0}^{n_M} (\alpha_p)^{\sum_{j=1}^{M}(n_j - k_j)} \prod_{j=1}^{M} \binom{n_j}{k_j} \frac{\partial^{k_j}}{\partial x_j^{k_j}} g(x) \frac{\partial^{n_j-k_j}}{\partial x_j^{n_j-k_j}} h(x),$$

where $n_1, n_2, \cdots, n_M \geq 0$, $x, x' \in \mathbb{R}^M$ such that $x = (x_1, x_2, \cdots, x_M), x' = (x'_1, x'_2, \cdots, x'_M)$, and g and h are infinitely differentiable functions in $\mathbb{R}^M$. Here for $s \in \mathbb{Z}$, the s-th power of α_p is defined as follows

$$\alpha_p^s = (-1)^{r_p(s)}, \quad \textit{if } \; s \equiv r_p(s) \mod p, \tag{4.47}$$

with $0 \leq r_p(s) < p$. If g and h are functions of x, t, then (4.46) becomes

$$\begin{aligned}(D_{p,x}^m D_{p,t}^n) g.h &= \left(\frac{\partial}{\partial x} + \alpha_p \frac{\partial}{\partial x'} \right)^m \left(\frac{\partial}{\partial t} + \alpha_p \frac{\partial}{\partial t'} \right)^n g(x,t) h(x',t')|_{x'=x, t'=t} \\ &= \sum_{j=0}^{m} \sum_{k=0}^{n} \binom{m}{j} \binom{n}{k} \alpha_p^j \alpha_p^k \frac{\partial^{m-j}}{\partial x^{m-j}} \frac{\partial^j}{\partial x'^{(j)}} \frac{\partial^{n-k}}{\partial t^{n-k}} \frac{\partial^k}{\partial t'^{(k)}} g(x,t) h(x',t')|_{x'=x, t'=t} \\ &= \sum_{j=0}^{m} \sum_{k=0}^{n} \binom{m}{j} \binom{n}{k} \alpha_p^j \alpha_p^k \frac{\partial^{m+n-j-k}}{\partial x^{m-j} t^{n-k}} g(x,t) \frac{\partial^{j+k}}{\partial x^j t^k} h(x,t),\end{aligned} \tag{4.48}$$

where $m, n \geq 0$.

If p is an even number ($p = 2k, k \in \mathbb{N}$), then $r_p(s) - s$ should be an even number as well. Consequently we have

$$\alpha_p^s = (-1)^{r_p(s)} = (-1)^{r_p(s)-s}(-1)^s = (-1)^s. \tag{4.49}$$

As a result, $D_{2k,x} = D_x$ which means that all the above generalized bilinear operators D_p turn into the Hirota bilinear derivatives.

EXAMPLE 13. Taking $p = 3$, the powers of α^s are

$$\alpha_3 = -1, \quad \alpha_3^2 = \alpha_3^3 = 1, \quad \alpha_3^4 = -1, \quad \alpha_3^5 = \alpha_3^6 = 1, \cdots, \tag{4.50}$$

so the pattern of symbols

$$-,+,+,-,+,+,\cdots \quad (p=3). \tag{4.51}$$

Using the definition with the above pattern of symbols, we can compute

$$D_{3,x}g\cdot h = g_xh - gh_x = D_xg\cdot h, \tag{4.52}$$

$$D_{3,x}^3 g\cdot h = g_{xxx}h - 3g_{xx}h_x + 3g_xh_{xx} + gh_{xxx}, \tag{4.53}$$

$$D_{3,x}^4 g\cdot h = g_{4x}h - 4g_{xxx}h_x + 6g_{xx}h_{xx} + 4g_xh_{xxx} - gh_{4x}, \tag{4.54}$$

$$D_{5,x}^5 g\cdot h = g_{5x}h - 5g_{4x}h_x + 10g_{xxx}h_{xx} + 10g_{xx}h_{xxx} - 5g_xh_{4x} + gh_{5x}, \tag{4.55}$$

$$D_{3,x}D_{3,t}g\cdot h = g_{xt}h - g_xh_t - g_th_x + gh_{xt}, \tag{4.56}$$

$$\begin{aligned} D_{3,x}^3 D_{3,t}g\cdot h &= g_{xxxt}h - 3g_{xxt}h_x + 3g_{xt}h_{xx} + g_th_{xxx} - g_{xxx}h_t \\ &\quad + 3g_{xx}h_{xt} + 3g_xh_{xxt} - gh_{xxxt}. \end{aligned} \tag{4.57}$$

EXAMPLE 14. When $p=5$, we have

$$\alpha_5 = -1, \quad \alpha_5^2 = 1, \quad \alpha_5^3 = -1, \quad \alpha_5^4 = \alpha_5^5 = 1, \quad \alpha_5^6 = -1, \tag{4.58}$$

$$\alpha_5^7 = 1, \quad \alpha_5^8 = -1, \quad \alpha_5^9 = \alpha_5^{10} = 1, \cdots, \tag{4.59}$$

and the pattern of symbols

$$-,+,-,+,+,-,+,-,+,+\cdots \quad (p=5). \tag{4.60}$$

From the definition we find

$$D_{5,x}g\cdot h = g_xh - gh_x = D_xg\cdot h, \tag{4.61}$$

$$D_{5,x}^3 g\cdot h = g_{xxx}h - 3g_{xx}h_x + 3g_xh_{xx} - gh_{xxx}, \tag{4.62}$$

$$D_{3,x}^4 g\cdot h = g_{4x}h - 4g_{xxx}h_x + 6g_{xx}h_{xx} - 4g_xh_{xxx} + gh_{4x}, \tag{4.63}$$

$$D_{5,x}^5 g\cdot h = g_{5x}h - 5g_{4x}h_x + 10g_{xxx}h_{xx} - 10g_{xx}h_{xxx} + 5g_xh_{4x} + gh_{5x}, \tag{4.64}$$

$$D_{5,x}D_{5,t}g\cdot h = g_{xt}h - g_xh_t + g_th_x - gh_{xt}, \tag{4.65}$$

$$\begin{aligned} D_{5,x}^3 D_{5,t}g\cdot h &= g_{xxxt}h - g_{xxx}h_t - 3g_{xxt}h_x + 3g_{xt}h_{xx} - g_th_{xxx} + \\ &\quad + 3g_{xx}h_{xt} - 3g_xh_{xxt} + gh_{xxxt}. \end{aligned} \tag{4.66}$$

Generalized Bilinear Equations

Consider a multivariate polynomial

$$P = P(x_1, x_2, \cdots, x_M). \tag{4.67}$$

A generalized bilinear differential equation is defined by [38]

$$P(D_{p,x_1}, D_{p,x_2}, \cdots, D_{p,x_M})g \cdot g = 0. \tag{4.68}$$

When $p = 2k, k \in \mathbb{N}$, the above relation becomes a Hirota bilinear equation .

EXAMPLE 15. If we choose $p = 3$, then under the logarithmic transformation

$$u = 2\log(f)_{xx}, \tag{4.69}$$

we obtain the generalized bilinear KdV equation [78]

$$(D_{3,x}D_{3,t} + D_{3,x}^4)f \cdot f = 2f_{xt}f - 2f_xf_t + 6f_{xx}^2 = 0, \tag{4.70}$$

and the generalized bilinear Boussinesq equation [59]

$$(D_{3,t}^2 + D_{3,x}^4)f \cdot f = 2f_{tt}f - 2f_t^2 + 6f_{xx}^2 = 0. \tag{4.71}$$

EXAMPLE 16. If we choose $p = 5$, then again under the logarithmic transformation (4.69) we attain the generalized bilinear KdV equation

$$(D_{5,x}D_{5,t} + D_{5,x}^4)f \cdot f = 2f_{xt}f - 2f_xf_t + 2f_{4x}f - 8f_{xxx}f_x + 6f_{xx}^2 = 0, \tag{4.72}$$

and the generalized bilinear Boussinesq equation

$$(D_{5,t}^2 + D_{5,x}^4)f \cdot f = 2f_{tt}f - 2f_t^2 + 2f_{4x}f - 8f_{xxx}f_x + 6f_{xx}^2 = 0. \tag{4.73}$$

Comparing the generalized bilinear equations constructed in the two examples, we conclude that different number p results in different bilinear equation.

In the next two sections, we adopt the generalized bilinear operators D_p given by (4.46) to introduce a (2+1)-dimensional 5th-order KdV-like equation and a (3+1)-dimensional Jimbo-Miwa-like equation with $p = 5$ and $p = 3$, respectively.

4.3 Lump Solutions to a (2+1)-Dimensional 5th-order KdV-like Equation

The (2+1)-dimensional 5th-order KdV equation reads [29]

$$\begin{aligned} KdV_{5th} := {} & 36u_t + u_{5x} + 15u_xu_{xx} + 15uu_{xxx} + 45u^2u_x - 5u_{xxy} - 15uu_y \\ & - 15u_x\int u_y dx - 5\int u_{yy}dx = 0, \end{aligned} \tag{4.74}$$

which is the (2+1)-dimensional analogue of the Caudrey-Dodd-Gibbon-Kotera-Sawada(CDGKS) equation [70]. When $u_y = 0$, Eq.(4.74) reduces to the Sawada-Kotera equation [9, 56]

$$u_t + u_{5x} + 15u_x u_{xx} + 15uu_{xxx} + 45u^2 u_x = 0. \tag{4.75}$$

Eq.(4.74) has a widespread adoption in many physical branches, such as conserved current of Liouville equation, two dimensional quantum gravity gauge field, and conformal field theory [25, 34, 55, 68].

4.3.1 A (2+1)-Dimensional 5th-order KdV-like Equation

Under the dependent variable transformation

$$u = 2(\ln f)_{xx}, \tag{4.76}$$

with $f = f(x, y, t)$, the (2+1)-dimensional fifth-order KdV equation(4.74) becomes the following (2+1)-dimensional Hirota bilinear equation

$$\begin{aligned} B_{5thKdV} &:= (D_x^6 - 5D_x^3 D_y + 36D_x D_t - 5D_y^2) f.f \\ &= 72f_x f_t - 72f_{xt} f + 2f_{6x} f - 12f_{5x} f_x + 30f_{4x} f_{xx} - 20f_{xxx}^2 \\ &\quad +10f_{xxx} f_y - 30f_{xx} f_{xy} + 30f_{xxy} f_x - 10f_{xxxy} f - 10f_{yy} f + 10f_y^2 = 0, \end{aligned} \tag{4.77}$$

where D_x, D_y, and D_t are the Hirota derivatives (4.1).

Utilizing the generalized bilinear operators D_p (4.46) with $p = 5$, we can generalized the Hirota bilinear 5th-order KdV equation(4.77) into

$$\begin{aligned} GB_{5thKdV} &:= (D_{5,x}^6 - 5D_{5,x}^3 D_{5,y} + 36D_{5,x} D_{5,t} - 5D_{5,y}^2) f.f \\ &= 30f_{4x} f_{xx} - 20f_{xxx}^2 - 10f_{xxxy} f + 10f_{xxx} f_y - 30f_{xx} f_{xy} \\ &+ 30f_{xxy} f_x + 72f_{xt} f - 72f_x f_t - 10f_{yy} f + 10f_y^2 = 0. \end{aligned} \tag{4.78}$$

Eq.(4.78) is a generalized bilinear 5th-order KdV equation. Under the transformations

$$u = 6(\ln f)_x, \qquad v = 6(\ln f)_y, \tag{4.79}$$

which was suggested by the Bell polynomial theories [17, 39, 66], Eq.(4.78) is transformed into the following fifth-order KdV-like nonlinear differential equation [5]

$$\begin{aligned} & u_t + \tfrac{11}{279936} u^6 + \tfrac{25}{15552} u^4 u_x + \tfrac{5}{972} u^3 u_{xx} + \tfrac{5}{288} u^2 u_x^2 - \tfrac{5}{2592} u^3 v + \tfrac{5}{54} u u_{xx} u_x + \tfrac{5}{432} u_x^3 - \tfrac{5}{432} u v u_x \\ & - \tfrac{5}{432} u_y u^2 + \tfrac{5}{432} u^2 u_{xxx} + \tfrac{5}{54} u_{xx}^2 - \tfrac{5}{72} u_y u_x + \tfrac{5}{72} u_x u_{xxx} + \tfrac{1}{36} u u_{4x} + \tfrac{1}{36} u_{5x} - \tfrac{5}{36} v_y = 0. \end{aligned} \tag{4.80}$$

Therefore, if f solves the bilinear Eq.(4.77) or (4.78), then $u = 6(\ln f)_{xx}$ or $u = 6(\ln f)_x$ and $v = 6(\ln f)_y$ will solve the nonlinear Eq.(4.74) or (4.80).

4.3.2 Lump Solutions to the 5th-order KdV-like Equation

In this section, we are going to generate lump solutions to Eq.(4.80) by searching for quadratic function solutions to Eq.(4.78) with the assumption

$$\begin{aligned} f &= g^2 + h^2 + a_9, \\ g &= a_1 x + a_2 y + a_3 t + a_4, \\ h &= a_5 x + a_6 y + a_7 t + a_8, \end{aligned} \tag{4.81}$$

where $a_j,\ (1 \leq j \leq 9)$, are real constants to be determined later. Note that using a sum involving one square, in the two dimensional space,will not generate exact solutions which are rationally localized in all directions in the space.

First, we substitute Eq.(4.81) into Eq.(4.78) and then make all coefficients of distinct polynomials of x, y, and t equal to zero by using Maple symbolic computation package. We get a collection of algebraic equations in $a_j,\ (1 \leq j \leq 9)$; solving the collection of algebraic equations, we achieve the following two classes of solutions

Case 1:

$$\begin{cases} a_1 = 0, \qquad a_2 = a_2, \qquad a_3 = -\dfrac{5a_2^3 a_9}{54 a_5^4}, \\[2ex] a_4 = a_4, \qquad a_5 = a_5, \qquad a_6 = -\dfrac{a_2^2 a_9}{3a_5^3}, \\[2ex] a_7 = \dfrac{5a_2^2(-9a_5^6 + a_2^2 a_9^2)}{324 a_5^7}, \qquad a_8 = a_8, \qquad a_9 = a_9, \end{cases} \tag{4.82}$$

where $a_j,\ (j = 2, 4, 5, 8, 9)$, are real free parameters which need to satisfy $a_5 \neq 0$ and $a_9 > 0$ in order to ensure that the corresponding solutions f is well defined and positive, respectively.

The set of parameters in (4.82) construct a class of positive quadratic function solutions to Eq.(4.78):

$$f = \left(-\frac{5a_2^3 a_9 t}{54 a_5^4} + a_2 y + a_4 \right)^2 + \left(\frac{5a_2^2(-9a_5^6 + a_2^2 a_9^2)t}{324 a_5^7} + a_5 x - \frac{a_2^2 a_9 y}{3a_5^3} + a_8 \right)^2 + a_9, \tag{4.83}$$

and the resulting class of quadratic function solutions, in turn, yields a class of lump solutions to the (2+1)-dimensional 5th-order KdV-like equation (4.80) through the dependent variable transformation:

$$
\begin{aligned}
u = 6(\ln f)_{xx} &= \frac{6(f_{xx}f - f_x^2)}{f^2} \\
&= \frac{12(a_1^2 - a_5^2)(-g^2 + h^2) - 48a_1a_5gh + 12(a_1^2 + a_5^2)a_9}{(g^2 + h^2 + a_9)^2},
\end{aligned} \tag{4.84}
$$

where the function f is defined by Eq.(4.81), and the functions g and h are given as follows:

$$
g = -\frac{5a_2^3a_9}{54a_5^4}t + a_2y + a_4, \tag{4.85}
$$

$$
h = \frac{5a_2^2(-9a_5^6 + a_2^2a_9^2)}{324a_5^7}t + a_5x - \frac{a_2^2a_9}{3a_5^3}y + a_8. \tag{4.86}
$$

Case 2:

$$
\begin{cases}
a_1 = a_1, \quad a_2 = a_2, \quad a_3 = \dfrac{5(a_1a_2^2 - a_1a_6^2 + 2a_2a_5a_6)}{36(a_1^2 + a_5^2)}, \\[2ex]
a_4 = a_4, \quad a_5 = a_5, \quad a_6 = a_6, \quad a_7 = \dfrac{5(2a_1a_2a_6 - a_2^2a_5 + a_5a_6^2)}{36(a_1^2 + a_5^2)}, \\[2ex]
a_8 = a_8, \quad a_9 = \dfrac{-3(a_1a_2 + a_5a_6)(a_1^2 + a_5^2)^2}{(a_1a_6 - a_2a_5)^2},
\end{cases} \tag{4.87}
$$

where a_j, $(j = 1, 2, 4, 5, 6, 8)$, are arbitrary constants to be determined with the following restricted conditions

$$
\begin{aligned}
&\Delta_1 : = a_1^2 + a_5^2 = \begin{vmatrix} a_1 & -a_5 \\ a_5 & a_1 \end{vmatrix} \neq 0, \quad \Delta_2 := a_1a_2 + a_5a_6 = \begin{vmatrix} a_1 & -a_5 \\ a_6 & a_2 \end{vmatrix} < 0, \\
&\Delta_3 : = a_1a_6 - a_2a_5 = \begin{vmatrix} a_1 & a_2 \\ a_5 & a_6 \end{vmatrix} \neq 0.
\end{aligned} \tag{4.88}
$$

- Δ_1 makes the corresponding solutions f well-defined,
- Δ_2 assures that the solution f is positive, and
- Δ_3 guarantees the localization of the solutions u in all directions in the (x, y)-plane.

Since these parameters are arbitrary, the corresponding solutions of Eq.(4.80) are more general. The parameters a_1, a_5 indicate that the wave velocity in the x direction is arbitrary and a_2, a_6 illustrate the arbitrariness of the wave velocity in the y direction. The parameters a_4, a_8 represent the invariance of variables and a_3, a_7 show the wave frequency which are represented by other quantities.

This set of parameters, in turn, generates positive quadratic function solutions to Eq.(4.78):

$$f = \left(a_1x + a_2y + \frac{5(a_1a_2^2 - a_1a_6^2 + 2a_2a_5a_6)}{36(a_1^2 + a_5^2)}t + a_4\right)^2 \tag{4.89}$$

$$+\left(a_5x + a_6y + \frac{5(2a_1a_2a_6 - a_2^2a_5 + a_5a_6^2)}{36(a_1^2 + a_5^2)}t + a_8\right)^2 + \frac{-3(a_1a_2 + a_5a_6)(a_1^2 + a_5^2)^2}{(a_1a_6 - a_2a_5)^2}.$$

Consequently, a kind of lump solutions to Eq.(4.80) through the transformation $u = 6(\ln f)_{xx}$ and (4.81) is achieved as follows

$$u = 6(\ln f)_{xx} = \frac{6(f_{xx}f - f_x^2)}{f^2}$$

$$= \frac{12(a_1^2 - a_5^2)(-g^2 + h^2) - 48a_1a_5gh + 12(a_1^2 + a_5^2)a_9}{(g^2 + h^2 + a_9)^2}, \tag{4.90}$$

where the functions g and h are given by:

$$g = a_1x + a_2y + \frac{5(a_1a_2^2 - a_1a_6^2 + 2a_2a_5a_6)}{36(a_1^2 + a_5^2)}t + a_4, \tag{4.91}$$

$$h = a_5x + a_6y + \frac{5(2a_1a_2a_6 - a_2^2a_5 + a_5a_6^2)}{36(a_1^2 + a_5^2)}t + a_8. \tag{4.92}$$

Choosing a special value for the free parameters in case 1 and 2, we can construct specific lump solutions u of Eq.(4.80). One special pair of positive quadratic function solutions and lump solutions with specific values of the parameters in case 1 is given as follows. First, the selection of the parameters

$$a_2 = 4, \quad a_4 = 0, \quad a_5 = 2, \quad a_8 = 0, \quad a_9 = 1, \tag{4.93}$$

leads to

$$f = \frac{34225}{26244}t^2 - \frac{370}{243}ty + \frac{148}{9}y^2 - \frac{350}{81}tx + 4x^2 - \frac{8}{3}xy + 1, \tag{4.94}$$

and the lump solution

$$u = -\frac{u_1}{u_2}, \tag{4.95}$$

with

$$u_1 = 1259712(27025t^2 - 113400tx + 115560ty + 104976x^2 - 69984xy - 408240y^2 - 26244),$$
$$u_2 = (34225t^2 - 113400tx - 39960ty + 104976x^2 - 69984xy + 431568y^2 + 26244)^2. \tag{4.96}$$

If we take a particular choice of the parameters in case 2 as follows

$$a_1 = 1, \quad a_2 = -\frac{1}{2}, \quad a_4 = 0, \quad a_5 = 0, \quad a_6 = 3, \quad a_8 = 0, \tag{4.97}$$

then we have

$$f = \frac{34225}{26244}t^2 - \frac{370}{243}ty + \frac{148}{9}y^2 - \frac{350}{81}tx + 4x^2 - \frac{8}{3}xy + 1, \tag{4.98}$$

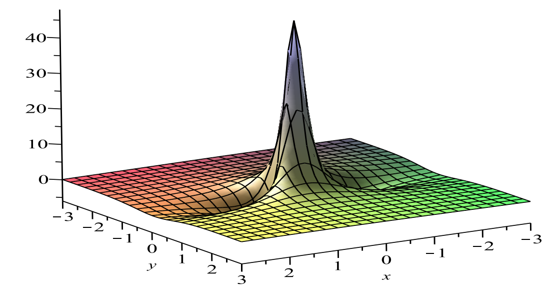
and the lump solution

$$u = -\frac{u_3}{u_4}, \tag{4.99}$$

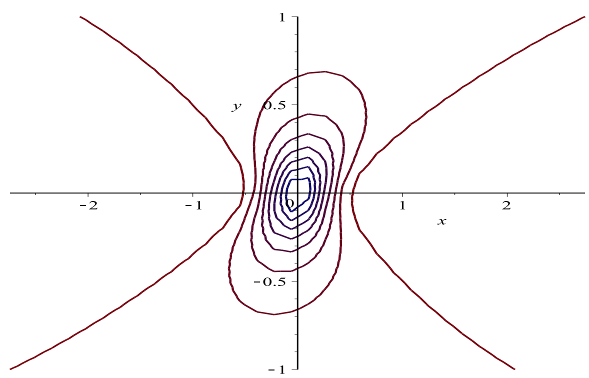
with

$$u_3 = 248832(27025t^2 - 113400tx + 115560ty + 104976x^2 - 69984xy - 408240y^2 - 26244),$$
$$u_4 = (34225t^2 - 113400tx - 39960ty + 104976x^2 - 69984xy + 431568y^2 + 26244)^2. \tag{4.100}$$

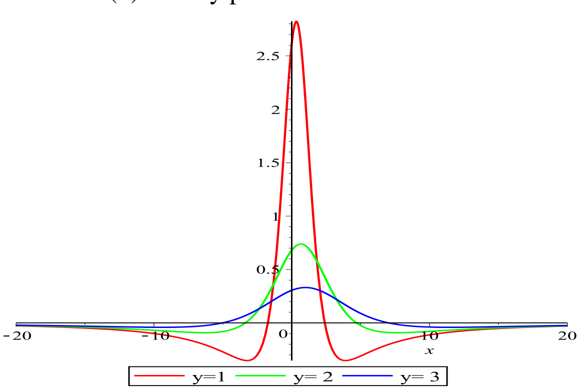
Figure 1 shows the profile of the lump solutions in case 1 with the special choice of the parameters (4.93) at $t = 0$. Figure 2 presents the profile of the lump solutions in case 2 with the special choice of the parameters (4.97) at $t = 0$.



(a) 3D plot

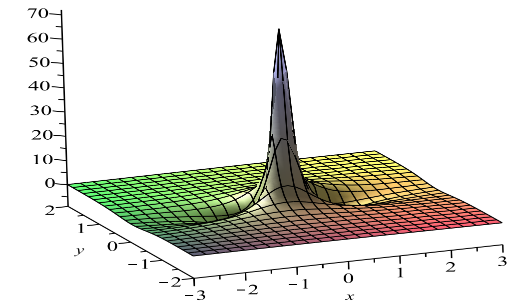


(b) density plot

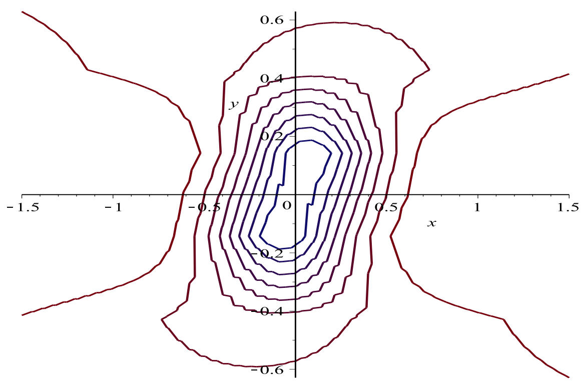


(c) x-curves

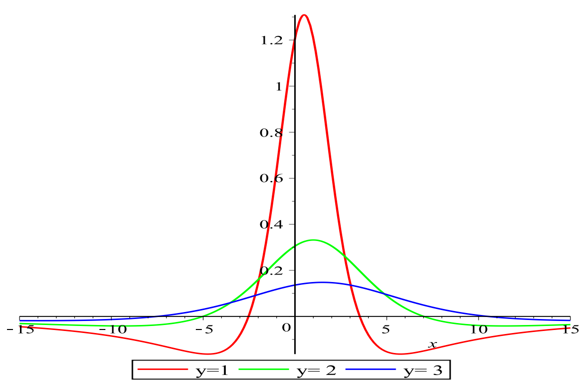
Figure 1.: Plots of the lump solution (4.95)-(4.96) at $t = 0$.



(a) 3D plot



(b) density plot



(c) x-curves

Figure 2.: Plots of the lump solution (4.99)-(4.100) at $t = 0$.

4.4 Interaction Solutions to a Jimbo-Miwa-like Equation in (3+1)-Dimension

In the KP soliton hierarchy [8, 23, 67, 69], the second equation reads

$$u_{xxxy} + 3u_{xx}u_y + 3u_{xy}u_x + 2u_{yt} - 3u_{xz} = 0, \tag{4.101}$$

which is known as the Jimbo-Miwa (JM) equation in (3+1)-dimension [23]. Eq.(4.101) adopted by physicist to explain particular waves in (3+1)-dimension. Exact solutions for the JM Eq.(4.101) are considered especially by using the Hirota direct method [33] even though the equation is not-integrable.

4.4.1 A Jimbo-Miwa-like Equation in (3+1)-Dimension

Using the Cole-Hopf transformation

$$u = 2(\ln f)_x. \tag{4.102}$$

and the Hirota bilinear derivatives (4.1), Eq.(4.101) transforms to the following Hirota bilinear equation

$$\begin{aligned} B_{JM}(f) :&= (D_x^3 D_y + 2D_yD_t - 3D_xD_z)f.f \\ &= 2(ff_{xxxy} - f_yf_{xxx} + 3f_{xy}f_{xx} - 3f_xf_{xxy} + 2f_{ty}f - 2f_tf_y - 3f_{xz}f + 3f_xf_z) = 0, \end{aligned} \tag{4.103}$$

Under the generalized bilinear operators D_p (4.46) with $p = 3$, we can generalize the bilinear JM Eq.(4.103) into

$$\begin{aligned} GB_{JM}(f) :&= (D_{3,x}^3 D_{3,y} + 2D_{3,y}D_{3,t} - 3D_{3,x}D_{3,z})f.f \\ &= 2(3f_{xx}f_{xy} + 2f_{yt}f - 2f_yf_t - 3f_{xz}f + 3f_xf_z) = 0. \end{aligned} \tag{4.104}$$

Employing the relation between f and u defined by (4.102), the generalized bilinear JM Eq.(4.104) becomes [6]

$$\begin{aligned} GP_{JM}(u) :&= \tfrac{9}{8}u^2u_xv + \tfrac{3}{8}u^3u_y + \tfrac{3}{4}uvu_{xx} + \tfrac{3}{4}u_x^2v + \tfrac{3}{4}u^2u_{xy} + \tfrac{9}{4}uu_xu_y \\ &+ \tfrac{3}{2}u_{xx}u_y + \tfrac{3}{2}u_xu_{xy} + 2u_{yt} - 3u_{xz} = 0, \end{aligned} \tag{4.105}$$

where $v_x = u_y$. The transformation (4.102) is a characteristic one in establishing Bell polynomial theories of integral equations [39, 60] and can transform generalized bilinear equations to nonlinear partial differential equations. The definite link between the generalized bilinear Eq.(4.104) and the JM-like Eq.(4.105) is given by

$$GP_{JM}(u) = \left[\frac{GB_{JM}(f)}{f^2}\right]_x. \tag{4.106}$$

The standard JM Eq.(4.101) is different than the JM-like Eq.(4.105) since the later has higher order nonlinearity and more terms and if f is a solution for the generalized bilinear Eq.(4.104), then $u = 2(\ln f)_x$ will be a solution for the JM-like Eq.(4.105).

4.4.2 Lump-type Solutions to the Jimbo-Miwa-like Equation

We begin with searching for positive quadratic function solutions to the bilinear Eq.(4.104) by presenting f as the form

$$\begin{aligned} f &= g^2 + h^2 + a_{11}, \\ g &= a_1x + a_2y + a_3z + a_4t + a_5, \\ h &= a_6x + a_7y + a_8z + a_9t + a_{10}, \end{aligned} \tag{4.107}$$

in order to obtain lump-type solutions for Eq.(4.105) where $a_j,\ (1 \le j \le 11)$, are real constants to be determined. Next, we substitute Eq.(4.107) into Eq.(4.104) and make all the coefficients of distinct polynomials of x, y, z and t equal to zero. A collection of algebraic equations in the parameters $a_j,\ (1 \le j \le 11)$, are obtained and then solved by taking advantage of the computer algebra system Maple. The following sets of solutions are achieved

Case 1:

$$\begin{cases} a_1 = a_1, \qquad a_2 = -\dfrac{a_6a_7}{a_1}, \qquad a_3 = -\dfrac{2}{3}\dfrac{a_4a_6a_7}{a_1^2}, \qquad a_4 = a_4, \\ a_5 = a_5, \qquad a_6 = a_6, \qquad a_7 = a_7, \qquad a_8 = \dfrac{2}{3}\dfrac{a_4a_7}{a_1}, \\ a_9 = \dfrac{a_4a_6}{a_1}, \qquad a_{10} = a_{10}, \qquad a_{11} = a_{11}, \end{cases} \tag{4.108}$$

where $a_j,\ (j = 1, 4, 5, 6, 7, 10, 11)$, are real free parameters that meet the following requirements

- $a_1 \neq 0$ to ensure well-definedness of f, and
- $a_{11} > 0$ to maintain the positiveness of f.

Case 2:

$$\begin{cases} a_1 = a_1, \quad a_2 = -\dfrac{a_6 a_7}{a_1}, \quad a_3 = -\dfrac{a_6 a_8}{a_1}, \quad a_4 = \dfrac{3}{2}\dfrac{a_1 a_8}{a_7}, \\ a_5 = a_5, \quad a_6 = a_6, \quad a_7 = a_7, \quad a_8 = a_8, \\ a_9 = \dfrac{3}{2}\dfrac{a_6 a_8}{a_7}, \quad a_{10} = \dfrac{a_5 a_6}{a_1}, \quad a_{11} = a_{11}, \end{cases} \tag{4.109}$$

where a_j $(j = 1, 5, 6, 7, 8, 11)$ are arbitrary constants which have to fulfill the conditions

- $a_1 a_7 \neq 0$ to make the corresponding solutions f well defined, and
- $a_{11} > 0$ to support the positiveness of f.

Case 3:

$$\begin{cases} a_1 = a_1, \quad a_2 = a_2, \quad a_3 = -\dfrac{2}{3}\dfrac{a_1(a_2 a_4 - a_7 a_9) + a_6(a_2 a_9 + a_4 a_7)}{a_1^2 + a_6^2}, \quad a_4 = a_4, \\ a_5 = a_5, \quad a_6 = a_6, \quad a_7 = a_7, \quad a_8 = \dfrac{2}{3}\dfrac{a_1(a_2 a_9 + a_4 a_7) - a_6(a_2 a_4 + a_7 a_9)}{a_1^2 + a_6^2}, \\ a_9 = a_9, \quad a_{10} = a_{10}, \quad a_{11} = -\dfrac{3}{2}\dfrac{(a_1 a_2 + a_6 a_7)(a_1^2 + a_6^2)^2}{(a_1 a_9 - a_4 a_6)(a_1 a_7 - a_2 a_6)}, \end{cases} \tag{4.110}$$

where a_j, $(j = 1, 2, 4, 5, 6, 7, 9, 10)$, are arbitrary constants to be determined with the following restricted requirements

- $a_1^2 + a_6^2 \neq 0$ to assure the well-definedness of the solutions f ,
- $\dfrac{(a_1 a_2 + a_6 a_7)}{(a_1 a_9 - a_4 a_6)(a_1 a_7 - a_2 a_6)} < 0$ to guarantee that f is positive, and
- $(a_1 a_9 - a_4 a_6)(a_1 a_7 - a_2 a_6) \neq 0$ to make that the corresponding solutions u localize in certain directions in the space.

We get three classes of quadratic function solutions $f_j (j = 1, 2, 3)$, defined by (4.107), from the above cases of solutions, to the bilinear JM Eq.(4.104); and the resulting quadratic function solutions, in turn, yield three classes of lump-type solutions $u_j (j = 1, 2, 3)$, to the JM-like Eq.(4.105) through the transformation (4.102). All the rational function solutions $u_j \to 0, (j = 1, 2, 3)$, when the corresponding sum of squares $g^2 + h^2 \to \infty$. However, they do not approach zero in all directions in the space of x, y, z due to the character of (3+1)-dimensions in the resulting solutions, therefore, they are lump-type solutions but not lump solutions.

Aiming to analyze the dynamic behavior of the solutions $u_j, (j = 1, 2, 3)$, the following values for the parameters are selected:

$$a_1 = 2, \quad a_4 = 1, \quad a_5 = -1, \quad a_6 = 2, \quad a_7 = 3, \quad a_{10} = 4, \quad a_{11} = 1, \tag{4.111}$$

for case 1, then the following lump-type solution is attained:

$$u_1 = \frac{4(4x + 2t + 3)}{4x^2 + 9y^2 + z^2 + t^2 + 6yz + 4tx + 6x + 15y + 5z + 3t + 9}. \tag{4.112}$$

For case 2, we let

$$a_1 = 2, \quad a_5 = -1, \quad a_6 = 2, \quad a_7 = 3, \quad a_8 = 4, \quad a_{11} = 1, \tag{4.113}$$

and we obtain the lump-type solution:

$$u_2 = \frac{16(2x + 4t - 1)}{8x^2 + 18y^2 + 32z^2 + 32t^2 + 48yz + 32tx - 8x - 16t + 3}. \tag{4.114}$$

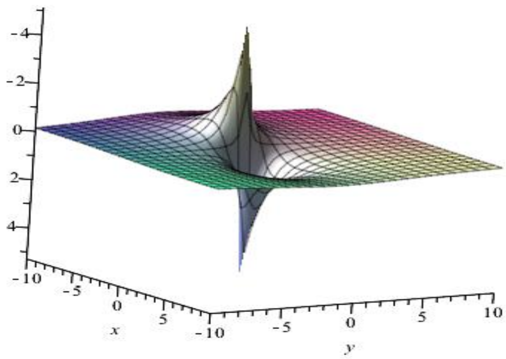
We choose

$$\begin{gathered} a_1 = 1, \quad a_2 = 0, \quad a_4 = 1, \quad a_5 = 2, \quad a_6 = -1, \quad a_7 = 1, \\ a_9 = 0, \quad a_{10} = 0, \end{gathered} \tag{4.115}$$

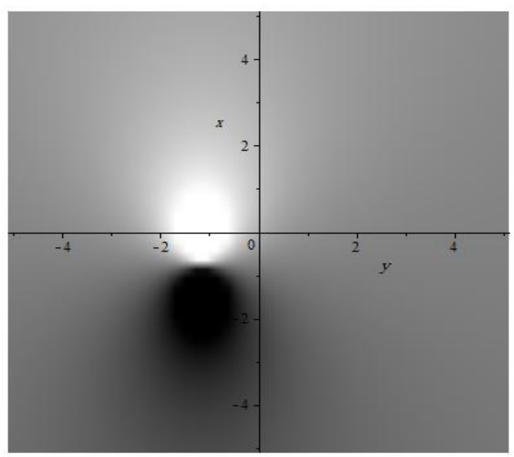
for case 3 and hence the lump-type solution is given by:

$$u_3 = \frac{12(6x - 3y - 2z + 3t + 6)}{18x^2 + 9y^2 + 2z^2 + 9t^2 - 18xy - 12xz + 6yz + 18tx - 6tz + 36x - 12z + 36t + 90}. \tag{4.116}$$

The profiles of the lump-type solutions (4.112)–(4.116) are showed in figures 3–5.

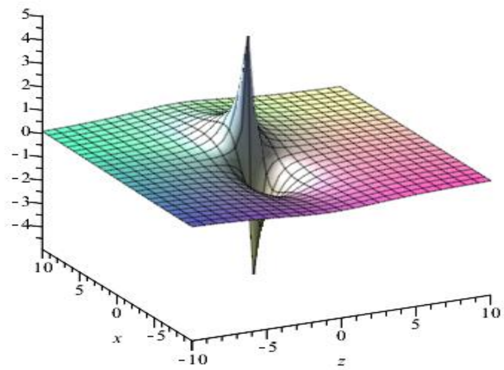


(a) 3D plot

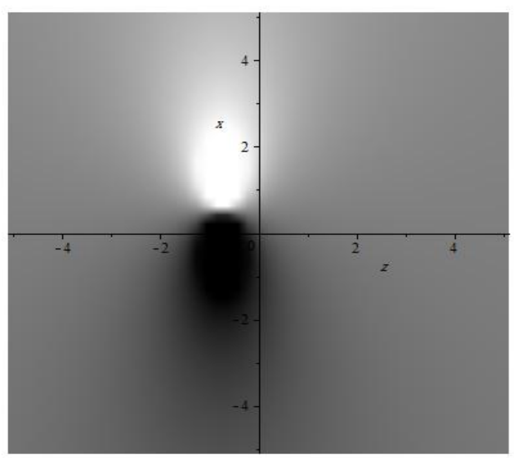


(b) density plot

Figure 3.: Profiles of the lump-type solution (4.112) with $z = 1$ at $t = 0$.

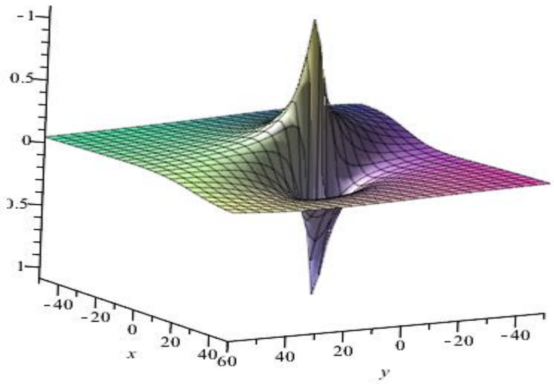


(a) 3D plot

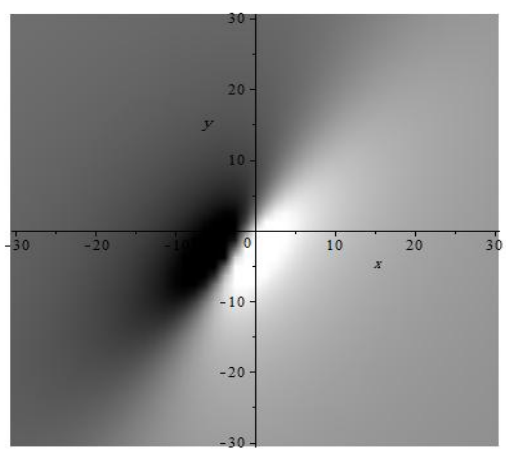


(b) density plot

Figure 4.: Profiles of the lump-type solution (4.114) with $y = 1$ at $t = 0$.



(a) 3D plot



(b) density plot

Figure 5.: Profiles of the lump-type solution (4.116) with $z = 1$ at $t = 1$.

4.4.3 Interaction Solutions of Lump-type Solutions and Kink Solutions

The interaction solutions between a lump-type and a stripe of the JM-like Eq.(4.105) is explored in this section. We add an exponential function to the quadratic function solution (4.107) as

$$
\begin{aligned}
f &= g^2 + h^2 + e^l + a_{16}, \\
g &= a_1 x + a_2 y + a_3 z + a_4 t + a_5, \\
h &= a_6 x + a_7 y + a_8 z + a_9 t + a_{10}, \\
l &= a_{11} x + a_{12} y + a_{13} z + a_{14} t + a_{15},
\end{aligned}
\tag{4.117}
$$

where $a_j,\ (j = 1, ..., 16)$, are real constants to be determined where $a_{16} > 0$. With symbolic computation via Maple on a direct substitution of Eq.(4.117) into Eq.(4.104), and by collecting all the coefficients about $x, y, z, t, e^{a_{11}x + a_{12}y + a_{13}z + a_{14}t + a_{15}}$, we reach the following set of constraining relations among the parameters

$$
\begin{cases}
a_1 = -\dfrac{a_6 a_7}{a_2}, \qquad a_3 = \dfrac{2}{3}\dfrac{a_2 a_9}{a_6}, \qquad a_4 = -\dfrac{a_7 a_9}{a_2}, \\[2ex]
a_8 = -\dfrac{2}{3}\dfrac{a_9 a_7}{a_6}, \qquad a_{12} = a_{13} = 0, \qquad a_{14} = \dfrac{a_9 a_{11}}{a_6}, \\[2ex]
a_j = a_j (j = 2, 5, 6, 7, 9, 10, 11, 15, 16),
\end{cases}
\tag{4.118}
$$

with $a_2 a_6 \neq 0$. Hence, we can express the exact interaction solution of u as follows

$$
u = 2(\ln f)_x = \frac{2 f_x}{f} = \frac{4 a_1 g + 4 a_6 h + 2 a_{11} e^l}{f},
\tag{4.119}
$$

where

$$
\begin{aligned}
f &= g^2 + h^2 + e^l + a_{16}, \\
g &= -\frac{a_6 a_7}{a_2} x + a_2 y + \frac{2}{3}\frac{a_2 a_9}{a_6} z - \frac{a_7 a_9}{a_2} t + a_5, \\
h &= a_6 x + a_7 y - \frac{2}{3}\frac{a_9 a_7}{a_6} z + a_9 t + a_{10}, \\
l &= a_{11} x + \frac{a_9 a_{11}}{a_6} t + a_{15},
\end{aligned}
\tag{4.120}
$$

and $a_2, a_5, a_6, a_7, a_9, a_{10}, a_{11}, a_{15}$ and a_{16} are arbitrary real constants.

The coming after special choices for the parameters is taken to illustrate the interaction phenomena between a lump-type solution and a stripe solution as:

$$\begin{aligned} a_2 = 2, \quad a_5 = 1, \quad a_6 = -1, \quad a_7 = 2, \quad a_9 = 3, \quad a_{10} = 0, \\ a_{11} = 4, \quad a_{15} = -2, \quad a_{16} = 4. \end{aligned} \tag{4.121}$$

Then the mixed lump-type stripe solution to Eq.(4.105) reads:

$$u = -\frac{4(-6t + 2x + 1 + 2e^{-12t+4x-2})}{18t^2 - 12tx + 2x^2 + 8y^2 - 32yz + 32z^2 - 6t + 2x + 4y - 8z + 5 + e^{-12t+4x-2}}. \tag{4.122}$$

The asymptotic behaviors of the solution (4.122) are presented in figures 6 and 7. They exhibit the interaction between the lump-type soliton and the kink wave.

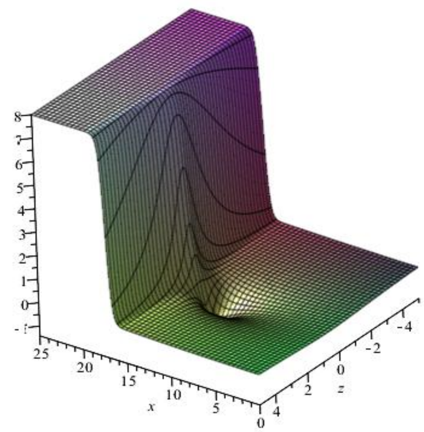
4.4.4 Interaction Solutions of Lump-type Solutions and a Pair of Resonance Stripe Soliton

We investigate the collision between lump-type and a pair of resonance stripe soliton in this section. First, we redefine the quadratic function f as the following form

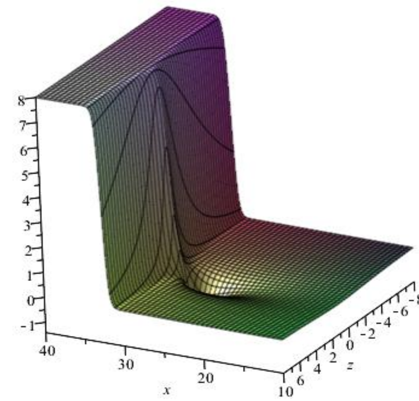
$$\begin{aligned} f &= g^2 + h^2 + \cosh(l) + a_{15}, \\ g &= a_1 x + a_2 y + a_3 z + a_4 t + a_5, \\ h &= a_6 x + a_7 y + a_8 z + a_9 t + a_{10}, \\ l &= a_{11} x + a_{12} y + a_{13} z + a_{14} t, \end{aligned} \tag{4.123}$$

where $a_j, \;\; (j = 1, ..., 15)$, are constants to be determined and $a_{15} > 0$. Next, through substituting Eq.(4.123) into Eq.(4.104) and collecting all the coefficients about $x, y, z, t, \cosh(a_{11}x + a_{12}y + a_{13}z + a_{14}t)$ and $\sinh(a_{11}x + a_{12}y + a_{13}z + a_{14}t)$, the parameters of the solution are determined by

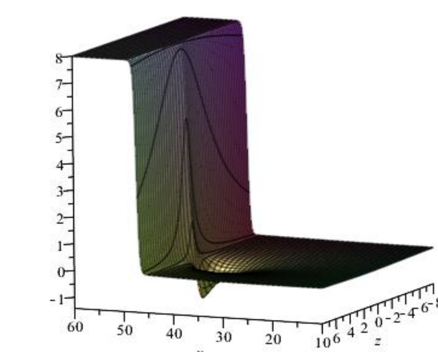
$$\begin{cases} a_1 = -\dfrac{2}{3}\dfrac{a_7 a_9}{a_3}, \qquad a_4 = -\dfrac{a_7 a_9}{a_2}, \qquad a_6 = \dfrac{2}{3}\dfrac{a_2 a_9}{a_3}, \\ a_8 = \dfrac{a_3 a_7}{a_2}, \qquad a_{12} = a_{13} = 0, \qquad a_{14} = \dfrac{3}{2}\dfrac{a_3 a_{11}}{a_2}, \\ a_j = a_j (j = 2, 3, 5, 7, 9, 10, 11, 15), \end{cases} \tag{4.124}$$



(a) $y = 1, t = 5$

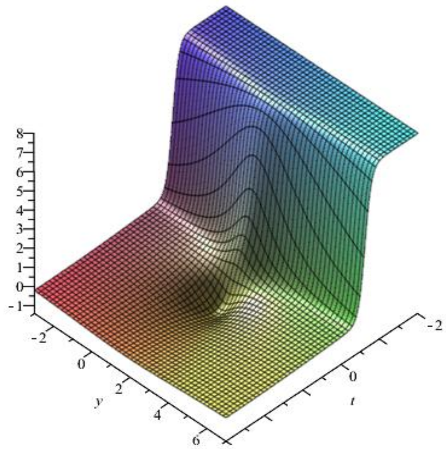


(b) $y = 2, t = 10$

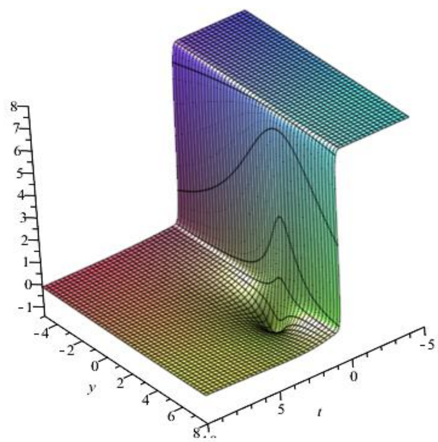


(c) $y = 3, t = 15$

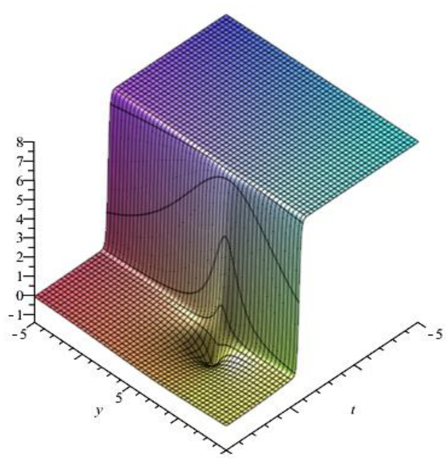
Figure 6.: Profiles of the interaction solution (4.122) with different values of y, t.



(a) $x = 0, z = 1$



(b) $x = 3, z = 2$



(c) $x = 15, z = 4$

Figure 7.: Profiles of the interaction solution (4.122) with different values of x, z.

with $a_2a_3 \neq 0$. Then we can attain a class of explicit solutions of the JM-like equation (4.105) by substituting the above relations into the corresponding equations as we did in Section 4.4.3. The exact expression for u is as follows

$$u = 2(\ln f)_x = \frac{2f_x}{f} = \frac{4a_1 g + 4a_6 h + 2a_{11}\sinh(l)}{f}, \tag{4.125}$$

where

$$f = g^2 + h^2 + \cosh(l) + a_{15}, \tag{4.126}$$

$$g = -\frac{2}{3}\frac{a_7a_9}{a_3}x + a_2y + a_3z + -\frac{a_7a_9}{a_2}t + a_5, \tag{4.127}$$

$$h = \frac{2}{3}\frac{a_2a_9}{a_3}x + a_7y + \frac{a_3a_7}{a_2}z + a_9t + a_{10},$$

$$l = a_{11}x + \frac{3}{2}\frac{a_3a_{11}}{a_2}t,$$

and $a_2, a_3, a_5, a_7, a_9, a_{10}, a_{11}$ and a_{15} are free parameters. Figure 8 illustrates the typical phenomena in the interaction between a lump-type and a resonance soliton with the parameters

$$\begin{aligned} &a_2 = 1, \quad a_3 = 2, \quad a_5 = 4, \quad a_7 = -1, \quad a_9 = 2, \\ &a_{10} = -1, \quad a_{11} = 1, \quad a_{15} = 0. \end{aligned} \tag{4.128}$$

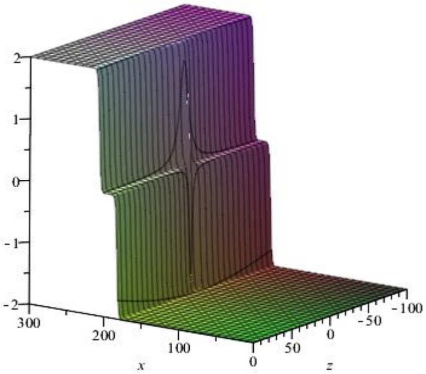
In this case the mixed lump-type resonance soliton solutions to Eq.(4.105) are given by:

$$u = -\frac{u_1}{u_2}, \tag{4.129}$$

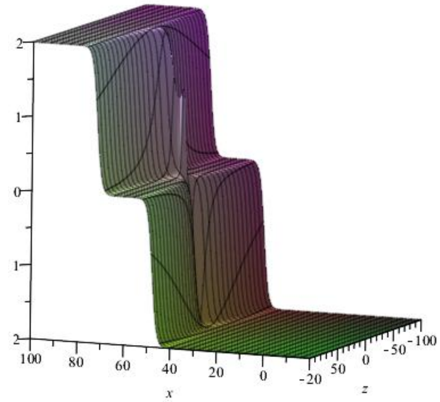
with

$$\begin{aligned} u_1 &= 2(16x + 48t + 9\sinh(3t + x) + 36), \\ u_2 &= 8x^2 + 18y^2 + 72z^2 + 72t^2 + 48tx + 72yz + 36x + 90y + 180z + 108t \\ &\quad + 9\cosh(3t + x) + 153. \end{aligned} \tag{4.130}$$

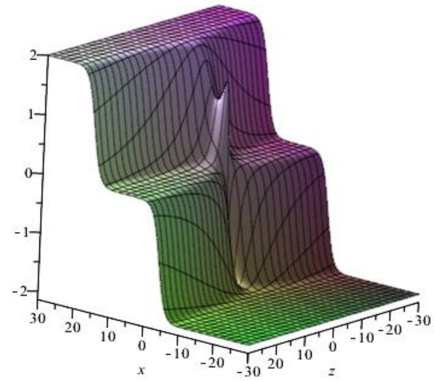
Figure 8 illustrates the interaction of the lump-type and the pair of stripe soliton (4.129)-(4.130).



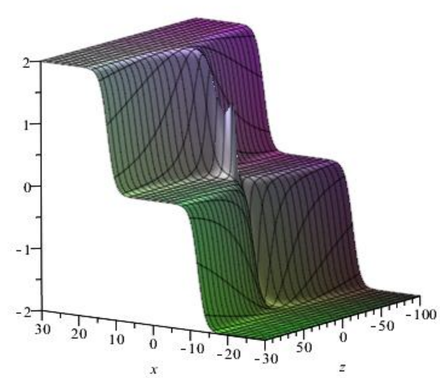
(a) $t=-65$



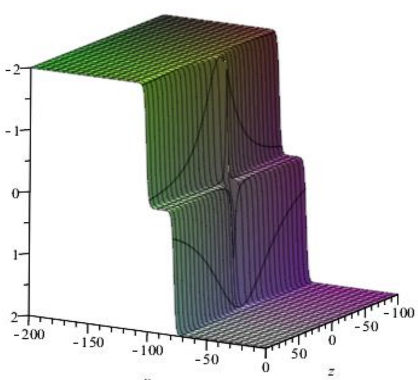
(b) $t=-20$



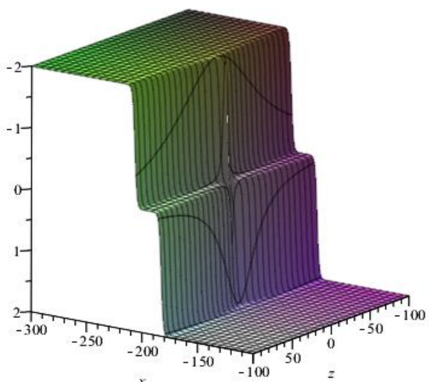
(c) $t=-2$



(d) $t=0$



(e) $t=30$



(f) $t=65$

Figure 8.: Profiles of the interaction solution (4.129) with different values of t and $y=5$.

Chapter 5

N-soliton Solutions of a Coupled Complex Modified-KdV System by the Riemann-Hilbert Approach

5.1 Introduction

The study of nonlinear waves is an interesting research area in Mathematics. Since the discovery of solitons in 1834, nonlinear dispersive wave equations have been used to formulate nonlinear wave phenomenon in many fields such as fluid dynamics, nonlinear plasma and optics among others. The analysis and computation of soliton equations are difficult and challenging because of the nonlinearity. Moreover, finding solitons and explaining their elastic collision needs mathematical methods that are different from those used to solve linear partial differential equations. The Riemann-Hilbert method is one of the powerful tools to construct solutions of integrable equations, specially soliton solutions [16, 20, 28, 49, 65, 71, 75]. The method is introduced as a modern version of the inverse scattering transform method [28, 49, 75], which also can be used to analyze the long time asymptotics of solutions of integrable systems [10]. This chapter is structured as follows. In Section 5.2, we define some terminologies and state some important results. In Section 5.3, we introduce regular and non-regular Riemann-Hilbert problems and present their solutions. In Section 5.4, we re-derive the Ablowitz-Kaup-Newell-Segur (AKNS) integrable hierarchy with two components. In Section 5.5, a Riemann-Hilbert problem of a coupled complex modified Korteweg-de Vries system is formulated and its N-soliton solutions are generated.

5.2 Preliminaries

We start by introducing some basic terminologies and results.

DEFINITION 5.2.1. • *When a complex function f is differentiable at every point in a neighborhood of a point $z = z_0$ then f is called analytic at z_0.*

- *A function f is analytic on a region R of the complex plane $\mathbb{C}$ if it is analytic at every point in R.*
- *We say f is entire if it is analytic in the whole complex plane.*

THEOREM 5.1 (Liouville Theorem). *[2] If a complex-valued function $f(z)$ is entire and bounded in the z plane, then $f(z)$ is constant.*

THEOREM 5.2. *[2] Given a simple closed contour Γ. If a function f is analytic on some simply connected domain containing Γ, then*

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\xi)}{\xi - z} \, d\xi, \qquad \text{for any interior point } z. \tag{5.1}$$

Equation (5.1) is called Cauchy's Integral Formula.

DEFINITION 5.2.2. • *A point $z = z_0$ is said to be a singular point of a complex valued function $f(z)$ if f is not analytic at z_0, i.e., $f'(z_0)$ does not exist.*

- *A point $z = z_0$ is called an isolated singular point of $f(z)$ if f is analytic in some neighborhood of the point z_0 and not at z_0.*

DEFINITION 5.2.3. • *If a function $f(z)$ has the form*

$$f(z) = \frac{\phi(z)}{(z - z_0)^M}, \tag{5.2}$$

then an isolated singularity at the point z_0 of $f(z)$ is called a pole. Here M is a positive integer, $M \geq 1$, and $\phi(z)$ is analytic function in some neighborhood of z_0 with $\phi(z_0) \neq 0$.

- *A function $f(z)$ is said to have a simple pole if $M = 1$, and an M-th order pole if $M \geq 2$.*

EXAMPLE 17. The function

$$f(z) = \frac{z - 1}{(z - 5)(z + 2)^2}, \tag{5.3}$$

has a simple pole at $z = 5$ and a second order pole at $z = -2$.

DEFINITION 5.2.4. • *Assume that $\Phi(\lambda)$ is a matrix function. If $\det \Phi(\lambda_0) = 0$ for some point $\lambda = \lambda_0$, then Φ is said to have a zero at λ_0. Obviously, the inverse matrix $\Phi^{-1}(\lambda)$ has a pole of finite order at this point.*

- *A zero at $\lambda = \lambda_0$ of a matrix function $\Phi(\lambda)$ is called simple if the pole at this point is of the first order.*

DEFINITION 5.2.5. *A complex valued function $f(z)$ which is analytic on a region R except for a set of poles of finite order is called meromorphic on R.*

DEFINITION 5.2.6. *A complex valued function $f(z)$ is said to satisfy a Hölder condition on a contour Σ if for any two points z_1 and z_2 on Σ*

$$|f(z_1) - f(z_2)| \leq \gamma \, |z_1 - z_2|^{\mu}\,, \quad \gamma > 0,\ 0 < \mu \leq 1. \tag{5.4}$$

If $\mu = 1$, then the Hölder condition (5.4) becomes the Lipschitz condition.

DEFINITION 5.2.7. *The principal value of the integral $\int_{\Sigma} \frac{f(\xi)}{\xi - z} d\xi$, denoted by $⨍$, is defined by*

$$⨍_{\Sigma} \frac{f(\xi)}{\xi - z} d\xi = \lim_{\varepsilon \to 0} \int_{\Sigma - \Sigma_{\varepsilon}} \frac{f(\xi)}{\xi - z} d\xi, \quad z \in \Sigma, \tag{5.5}$$

where Σ_{ε} refers to the part of Σ that is centered around z and has length 2ε.

The region on the left side of the positive direction of a contour Σ is labeled by $\oplus$, and the region on right side by $\ominus$.

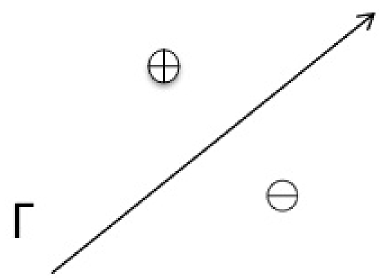


Figure 9.: The $\oplus$ and $\ominus$ regions on the sides of Σ

DEFINITION 5.2.8. *Consider the integral*

$$\Phi(z) = \frac{1}{2\pi i} \int_{\Sigma} \frac{\phi(\xi)}{\xi - z} \, d\xi, \tag{5.6}$$

where Σ is a simple contour and $\phi(\xi)$ is a function satisfying a Hölder condition on Σ. Define the following two limits

$$\Phi^{\pm}(t) = \lim_{z \to t^{\pm}} \frac{1}{2\pi i} \int_{\Sigma} \frac{\phi(\xi)}{\xi - z} d\xi, \tag{5.7}$$

where $t \in \Sigma$, and $\lim\limits_{z \to t^{\pm}}$ means the limits as z approaches t along a curve lying entirely in the $\oplus$, $\ominus$ regions, respectively.

These limits have a crucial role in the theory of Riemann-Hilbert problems and can be found by the following lemma.

LEMMA 5.1 (Plemelj Formulae). *[2] Suppose that Σ is a simple and smooth contour (closed or open) and a function $\phi(\xi)$ satisfy a Hölder condition on Σ. Then the Cauchy type integral $\Phi(z)$, Eq.(5.6), has the limiting values $\Phi^{\pm}(t)$ as z approaches Σ from the left and the right, respectively, and t is not an endpoint of Σ. These limits are given by*

$$\Phi^{\pm}(t) = \pm\frac{1}{2}\phi(t) + \frac{1}{2\pi i}\int_{\Sigma} \frac{\phi(\xi)}{\xi - t}d\xi, \tag{5.8}$$

or equivalently,

$$\Phi^{+}(t) - \Phi^{-}(t) = \phi(t). \tag{5.9}$$

Consider a simple, smooth, closed contour Γ that divides the complex z plane into two regions, namely R^+ and R^-, where R^+ is on the left of the positive direction of Γ.

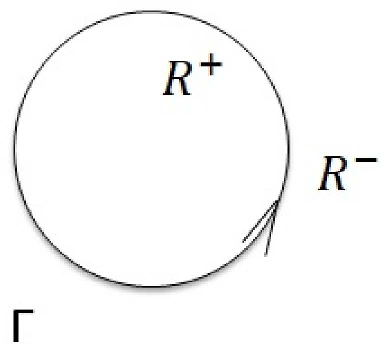


Figure 10.: The contour Γ with the regions R^+ and R^-

DEFINITION 5.2.9. *A sectionally analytic function $\Phi(z)$ is a scalar function that defines in the entire complex plane $\mathbb{C}$ except for points on Γ and satisfies the following:*

1. *$\Phi(z)$ is analytic in each of the regions R^+ and R^- except at $z = \infty$, and*
2. *as z approaches any point t on Γ along any path that lies wholly in either R^+ or R^-, the function $\Phi(z)$ approaches the limiting value, $\Phi^+(t)$ or $\Phi^-(t)$, respectively.*

The values $\Phi^{\pm}(t)$ are called the boundary values of the function $\Phi(z)$.

From above we can say that if a function $\Phi(z)$ has the value $\Phi^+(t)$ on Γ, then it is continuous in the closed region $R^+ \cup \Gamma$. Similarly for the region $R^- \cup \Gamma$.

THEOREM 5.3 (Generalized Liouville's Formula). *[43] Assume that Φ, A and B are square matrices depending on the variable x. If Φ satisfies a linear matrix differential equation*

$$\Phi_x = A\Phi + \Phi B, \tag{5.10}$$

then we have

$$(\det \Phi)_x = [tr(A) + tr(B)] \det \Phi, \tag{5.11}$$

and thus

$$\det \Phi(x) = e^{\int_{x_0}^{x} [tr(A(y)) + tr(B(y))] dy} \det \Phi(x_0), \tag{5.12}$$

where $x_0 \in \mathbb{R}$ is a given initial point.

The following proposition is one of the applications of Theorem 5.3 which is useful in determining solutions to Riemann-Hilbert problems.

PROPOSITION 5.1. *Assume that Φ, A and B are square matrices functions depending on the variable x. If Φ satisfies*

$$\Phi_x = [A, \Phi] + B\Phi, \tag{5.13}$$

then we have

$$(\det \Phi)_x = tr(B) \det \Phi. \tag{5.14}$$

5.3 Riemann-Hilbert Problems

A Riemann-Hilbert problem (RHP) can be stated as follows: Let Γ be a closed contour in the complex λ plane which can be assumed to pass through infinity. Particularly, Γ can be considered to be a real line $-\infty < \lambda < \infty$ treated as a circle in $\mathbb{C} \cup \{\infty\}$ passing through ∞. Assume that $G(\lambda)$ is an $N \times N$ matrix function defined on Γ. We need to assemble matrix functions $P^+(\lambda)$ and $P^-(\lambda)$ that are analytical inside and outside Γ, respectively, in the way

$$P^-(\lambda)P^+(\lambda) = G(\lambda), \tag{5.15}$$

on Γ. When Γ is the real line P^+ has to be analytical in the upper half-plane and P^- has to be analytical in the lower half-plane.

REMARK 1. • The solution of a Riemann Hilbert problem (5.15) is not unique. Indeed, if (P^-, P^+) is a solution and g is any constant non-degenerate matrix, then P^-g, $g^{-1}P^+$ will satisfy

$$(P^-g)(g^{-1}P^+) = P^-P^+ = G. \tag{5.16}$$

Hence, they are a solution of the Riemann-Hilbert problem (5.15) with the same $G(\lambda)$.

- To avoid the non-uniqueness of solutions, the Riemann-Hilbert problem has to be normalized by specifying a value of P^+ or P^- at some point in the domain of the analyticity. Herein, the Riemann-Hilbert problem is normalized by assuming that

$$P^{\pm}(\lambda) \to I, \qquad \lambda \to \infty. \tag{5.17}$$

  This type of normalization is said to be canonical.

5.3.1 Regular Riemann-Hilbert Problems

If P^- and P^+ are degenerate nowhere in their domain of analyticity, i.e., $\det P^{\pm}(\lambda) \neq 0$, then the Riemann-Hilbert problem (5.15) is said to be regular.

PROPOSITION 5.2. *Under the canonical normalization condition (5.17), the solution of a regular Riemann-Hilbert problem is unique.*

Proof. Let (P_1^-, P_1^+) and (P_2^-, P_2^+) be two solutions to the Riemann-Hilbert problem (5.15). Then on the contour Γ

$$P_1^-(\lambda)P_1^+(\lambda) = P_2^-(\lambda)P_2^+(\lambda), \tag{5.18}$$

and thus

$$P_1^+(\lambda)(P_2^+)^{-1}(\lambda) = (P_1^-)^{-1}(\lambda)P_2^-(\lambda), \quad \lambda \in \Gamma. \tag{5.19}$$

Define a function $\chi(\lambda)$ as

$$\chi(\lambda) = P_1^+(\lambda)(P_2^+)^{-1}(\lambda) = (P_1^-)^{-1}(\lambda)P_2^-(\lambda). \tag{5.20}$$

We see that $\det P_1^{\pm} \neq 0$ and $\det P_2^{\pm} \neq 0$ in their analytical domains due to the fact that the Riemann-Hilbert problem is regular. Thus, $\chi(\lambda)$ can be analytically extended from Γ to the whole complex plane $\mathbb{C}$. From the canonical normalization condition (5.17), we have

$$\chi(\lambda) \to I, \qquad \lambda \to \infty, \tag{5.21}$$

Using the Liouville Theorem 5.1, the matrix function $\chi(\lambda)$ is constant, and therefore

$$\chi(\lambda) = P_1^+(\lambda)(P_2^+)^{-1}(\lambda) = (P_1^-)^{-1}(\lambda)P_2^-(\lambda) = I, \qquad \lambda \in \mathbb{C}. \tag{5.22}$$

Hence

$$P_1^{\pm}(\lambda) = P_2^{\pm}(\lambda), \qquad \lambda \in \mathbb{C}. \tag{5.23}$$

□

In what follows, we construct the unique solution for the regular Riemann-Hilbert problem in the λ-complex plane [49]. The variable x is held down in the notation since this construction depends on λ. The solution relies upon solving a system of singular integral equations on Γ.

First, a normalization condition is chosen for the Riemann-Hilbert problem at a point $\lambda_0 \in \mathbb{C}$, say, outside Γ

$$P^{-}(\lambda_0) = I, \tag{5.24}$$

and then set

$$P^{+}(\lambda_0) = g. \tag{5.25}$$

We are looking for a solution of the form

$$(P^{-})^{-1}(\lambda) = h + \int_{\Gamma} \frac{\phi(\xi)}{\xi - \lambda} d\xi, \quad \text{inside the contour } \Gamma, \tag{5.26}$$

and

$$P^{+}(\lambda) = h + \int_{\Gamma} \frac{\phi(\xi)}{\xi - \lambda} d\xi, \quad \text{outside the contour } \Gamma. \tag{5.27}$$

The function h can be determined using the normalization condition as

$$h = g - \int_{\Gamma} \frac{\phi(\xi)}{\xi - \lambda_0} d\xi. \tag{5.28}$$

From the Plemelj formula (5.1), $(P^{-})^{-1}$ and P^{+} on the contour Γ are given by

$$\begin{aligned} (P^{-})^{-1}(\lambda) &= h + \fint_{\Gamma} \frac{\phi(\xi)}{\xi - \lambda} d\xi + \pi i \phi(\lambda), \\ P^{+}(\lambda) &= h + \fint_{\Gamma} \frac{\phi(\xi)}{\xi - \lambda} d\xi - \pi i \phi(\lambda), \end{aligned} \tag{5.29}$$

for $\lambda \in \Gamma$. Substituting Eqs.(5.28) and (5.29) into (5.15), we get the following singular integral equations for ϕ

$$g - \int_{\Gamma} \frac{\phi(\xi)}{\xi - \lambda_0} d\xi + \pi i \phi(\lambda) T(\lambda) + \fint_{\Gamma} \frac{\phi(\xi)}{\xi - \lambda} d\xi = 0, \quad \lambda \in \Gamma, \tag{5.30}$$

where $T = (G+I)(G-I)^{-1}$. The solution of the above integral equation leads to the solution of the regular Riemann-Hilbert problem. If the contour is the real axis and the normalization condition is canonical, so that $h = g = I$, Eq.(5.30) becomes

$$\frac{1}{\pi i} \left[\fint_{-\infty}^{\infty} \frac{\phi(\xi)}{\xi - \lambda} d\xi + I \right] + \phi(\lambda) T(\lambda) = 0. \tag{5.31}$$

Scalar Case

When $N = 1$ and P^-, P^+ are ordinary functions, the solution of the regular Riemann-Hilbert problem can be written in an explicit form. By taking the logarithm of Eq.(5.15), we have

$$\ln\big(P^+\big) - \ln\big(P_-^{-1}\big) = \ln(G). \tag{5.32}$$

Since P^-, and P^+ have no zeros inside their domain of analyticity, their logarithms are also analytical and can be represented in the form of a Cauchy integral. In order to satisfy Eq.(5.32), we take

$$\ln\big(P^+\big) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\ln G(\xi)}{\xi - \lambda} d\xi, \quad \mathrm{Im}(\lambda) < 0, \tag{5.33}$$

$$\ln\big((P^-)^{-1}\big) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\ln G(\xi)}{\xi - \lambda} d\xi, \quad \mathrm{Im}(\lambda) > 0. \tag{5.34}$$

Therefore,

$$\begin{aligned} P^+ &= \exp\Big[\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\ln G(\xi)}{\xi - \lambda} d\xi\Big], \\ P^- &= \exp\Big[-\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\ln G(\xi)}{\xi - \lambda} d\xi\Big]. \end{aligned} \tag{5.35}$$

5.3.2 Non-regular Riemann-Hilbert Problems

In general, Riemann-Hilbert problems are not regular, that is, $\det P^{\pm}(\lambda)$ have zeros at finite points on their domain of analyticity. In this section, we discuss the Riemann-Hilbert problem (5.15) with simple zeros [71] and the contour Γ is taking to be the real line $\mathbb{R}$. The study of the general case where some or all zeros are multiple zeros can be found in [57, 58]. We begin by assuming that the normalization condition is canonical. Suppose that P^+ is an analytical function in the upper half-plane $\mathbb{C}^+ = \{z \in \mathbb{C} |\, \mathrm{Im}(z) > 0\}$ with N simple zeros $\{\lambda_i \in \mathbb{C}^+ : 1 \le i \le N\}$. And suppose that P^- is an analytical function in the lower half-plane $\mathbb{C}^- = \{z \in \mathbb{C} |\, \mathrm{Im}(z) < 0\}$ with N simple zeros $\{\hat{\lambda}_i \in \mathbb{C}^- : 1 \le i \le N\}$. All these zeros are assumed to be off the real line $\mathbb{R}$. Define two functions

$$\tilde{P}^+ = \prod_{i=1}^{n} \frac{\lambda - \hat{\lambda}_i}{\lambda - \lambda_i} P^+, \tag{5.36}$$

$$\tilde{P}^- = \prod_{i=1}^{n} \frac{\lambda - \lambda_i}{\lambda - \hat{\lambda}_i} P^-. \tag{5.37}$$

The functions $\tilde{P}^{\pm}$ are also analytical but have no zeros. It is clear that

$$\tilde{P}^{\pm} \to I, \quad \lambda \to \infty, \tag{5.38}$$

and on the real line $\mathbb{R}$, we have

$$\tilde{P}^{-}\tilde{P}^{+}=P^{-}P^{+}=G. \tag{5.39}$$

Therefore, $\tilde{P}^{\pm}$ constitute a regular Riemann-Hilbert problem and can be determined by (5.35). This shows the if the Riemann-Hilbert problem has zeros then it is not unique. Thus, in addition to the matrix function G, the contour Γ and the normalization condition; the positions of the zeros have to be specified as well. The solution of a non-regular Riemann-Hilbert problem under the canonical normalization condition exists if and only if the number of zeros inside the contour equals to the number of zeros outside the contour. We see that the kernel of $P^{+}(\lambda_i)$ consists of only one column vector v_i and the kernel of $P^{-}(\hat{\lambda}_i)$ has only one row vector $\hat{v}_i$, this is because the zeros $\{\lambda_i \in \mathbb{C}^{+} : 1 \leq i \leq N\}$ and $\{\hat{\lambda}_i \in \mathbb{C}^{-} : 1 \leq i \leq N\}$ are assumed to be simple. Hence,

$$P^{+}(\lambda_i)v_i=0, \quad \hat{v}_iP^{-}(\hat{\lambda}_i)=0, \qquad 1\leq i\leq N. \tag{5.40}$$

The solution of a Riemann-Hilbert problem with zeros is given by the following theorem which reduces a non-regular RHP to a regular one. The theorem was proved first by Kawata [28] and then by Zakharov and Shabat [75] (See also [49, 71]).

THEOREM 5.4. *[28] Under the canonical normalization condition (5.17), the solution to the non-regular Riemann-Hilbert problem*

$$P^{-}(\lambda)P^{+}(\lambda)=G(\lambda), \quad \lambda\in\mathbb{R} \tag{5.41}$$

with zeros(5.40) is given by

$$P^{+}(\lambda)=\widehat{P}^{+}(\lambda)X(\lambda), \tag{5.42a}$$

$$P^{-}(\lambda)=X^{-1}(\lambda)\widehat{P}^{-}(\lambda), \tag{5.42b}$$

with

$$X(\lambda)=I+\sum_{j,k=1}^{N}\frac{v_j(M^{-1})_{jk}\hat{v}_k}{\lambda-\hat{\lambda}_k}, \tag{5.43a}$$

$$X^{-1}(\lambda)=I-\sum_{j,k=1}^{N}\frac{v_j(M^{-1})_{jk}\hat{v}_k}{\lambda-\lambda_j}, \tag{5.43b}$$

M is an $N\times N$ matrix with (j,k)th element given by

$$M_{jk}=\frac{\hat{v}_jv_k}{\hat{\lambda}_j-\lambda_k}, \quad 1\leq j,k\leq N, \tag{5.44}$$

$$\det X(\lambda) = \prod_{k=1}^{N} \frac{\lambda - \lambda_k}{\lambda - \hat{\lambda}_k}, \tag{5.45}$$

and $\widehat{P}^{\pm}(\lambda)$ is the unique solution to the regular Riemann-Hilbert problem:

$$\widehat{P}^{-}(\lambda)\widehat{P}^{+}(\lambda) = X(\lambda)G(\lambda)X^{-1}(\lambda), \quad \lambda \in \mathbb{R} \tag{5.46}$$

with $\widehat{P}^{\pm}(\lambda)$ are analytic in the upper and lower half-plane $\mathbb{C}^{\pm}$, respectively, and $\widehat{P}^{\pm}(\lambda) \to I$ as $\lambda \to \infty$.

Proof. The proof of this theorem depending on the construction method. Let $(\lambda_1, \hat{\lambda}_1)$ be a pair of zeros associated with the vectors $(|v_1\rangle, \langle\hat{v}_1|)$ in Eq.(5.40). We denote column vectors v_j by $|v_j\rangle$ and row vectors $\hat{v}_j$ by $\langle\hat{v}_j|$ to distinguish them from each other. Define a matrix X_1 as

$$X_1(\lambda) = I + \frac{\hat{\lambda}_1 - \lambda_1}{\lambda - \hat{\lambda}_1} \frac{|v_1\rangle\langle\hat{v}_1|}{\langle\hat{v}_1|v_1\rangle}, \tag{5.47}$$

which is meromorphic with a simple pole at $\lambda = \hat{\lambda}_1 \in \mathbb{C}^-$. Simple linear algebra calculation leads us to

$$X_1^{-1}(\lambda) = I - \frac{\hat{\lambda}_1 - \lambda_1}{\lambda - \lambda_1} \frac{|v_1\rangle\langle\hat{v}_1|}{\langle\hat{v}_1|v_1\rangle}, \tag{5.48}$$

$$X_1(\lambda_1)|v_1\rangle = 0, \quad \langle\hat{v}_1| X_1^{-1}(\hat{\lambda}_1) = 0, \tag{5.49}$$

and

$$\det X_1(\lambda) = \frac{\lambda - \lambda_1}{\lambda - \hat{\lambda}_1}. \tag{5.50}$$

Then, let

$$R_1^{+}(\lambda) = P^{+}(\lambda)X_1^{-1}(\lambda), \quad R_1^{-}(\lambda) = X_1(\lambda)P^{-}(\lambda). \tag{5.51}$$

The residue of $R_1^{+}(\lambda)$ at $\lambda = \lambda_1$ is

$$Res(R_1^{+}(\lambda), \lambda_1) = \lim_{\lambda \to \lambda_1} \{(\lambda - \lambda_1)\psi^{+}(\lambda)X_1^{-1}(\lambda)\} \tag{5.52}$$

$$= \lim_{\lambda \to \lambda_1} \left\{(\lambda - \lambda_1)\psi^{+}(\lambda) - (\hat{\lambda}_1 - \lambda_1)\psi^{+}(\lambda)\frac{|v_1\rangle\langle\hat{v}_1|}{\langle\hat{v}_1|v_1\rangle}\right\} = 0, \tag{5.53}$$

where we use Eq.(5.40). Similarly, we can find the residue of $R_1^{-}(\lambda)$ at $\lambda = \hat{\lambda}_1$

$$Res(R_1^{-}(\lambda), \hat{\lambda}_1) = 0. \tag{5.54}$$

Hence, $R^{+}(\lambda)$ and $R^{-}(\lambda)$ are analytic in $\mathbb{C}^{+}$ and $\mathbb{C}^{-}$, respectively. Moreover,

$$\det R_1^{+}(\lambda_1) \neq 0, \quad \det R_1^{-}(\hat{\lambda}_1) \neq 0. \tag{5.55}$$

Next, we construct a set of matrices $\{X_j : 2 \leq j \leq N\}$ as

$$X_j(\lambda) = I + \frac{\hat{\lambda}_j - \lambda_j}{\lambda - \hat{\lambda}_j} \frac{|w_j\rangle \langle \hat{w}_j|}{\langle \hat{w}_j | w_j \rangle}, \quad j = 2, \ldots, N, \tag{5.56}$$

where $(\lambda_j, \hat{\lambda}_j)$ are zeros associated with the vectors $(|v_j\rangle, \langle \hat{v}_j|)$ in Eq.(5.40), and the vectors $(|w_j\rangle, \langle \hat{w}_j|)$ are related to $(|v_j\rangle, \langle \hat{v}_j|)$ as

$$|v_j\rangle = X_1^{-1}(\lambda_j) X_2^{-1}(\lambda_j) \ldots X_{j-1}^{-1}(\lambda_j) |w_j\rangle, \tag{5.57a}$$

and

$$\langle \hat{v}_j| = \langle \hat{w}_j| X_{j-1}(\hat{\lambda} j) X_{j-2}(\hat{\lambda}_j) \ldots X_1(\hat{\lambda}_j). \tag{5.57b}$$

Then we define the matrix functions $R_j^{\pm}(\lambda)$ by

$$R_j^+(\lambda) = R_{j-1}^+(\lambda) X_j^{-1}(\lambda), \quad R_j^-(\lambda) = X_j(\lambda) R_{j-1}^-(\lambda), \quad j = 2, \ldots, N. \tag{5.58}$$

It is easy to verify that

$$\det X_j(\lambda) = \frac{\lambda - \lambda_j}{\lambda - \hat{\lambda}_j}, \quad j = 2, \ldots, N. \tag{5.59}$$

Using the relations (5.40) and (5.57)-(5.58), we observe that

$$R_{j-1}^+(\lambda_j) |w_j\rangle = 0, \qquad \langle \hat{w}_j| R_{j-1}^-(\hat{\lambda}_j) = 0, \quad j = 2, \ldots, N. \tag{5.60}$$

Combining all the above results, we see that the functions $P^{\pm}(\lambda)$ can be written as in (5.42), where

$$X(\lambda) = X_N(\lambda) X_{N-1}(\lambda) \ldots X_1(\lambda). \tag{5.61}$$

Furthermore, the matrix functions $\widehat{P}^{\pm}(\lambda)$ in (5.42) have the following properties

1. analytic in $\mathbb{C}^{\pm}$, respectively,
2. $\det \widehat{P}^{\pm}(\lambda) \neq 0$ in their analyticity domain, and
3. have the asymptotic condition $\widehat{P}^{\pm}(\lambda) \to I$ as $\lambda \to \infty$.

By utilizing formula (5.59) and (5.61), $\det X(\lambda)$ can be obtained by Eq.(5.45). From (5.56) and (5.61), we can find that

$$X^{-1}(\lambda) = X_1^{-1}(\lambda) X_2^{-1}(\lambda) \ldots X_N^{-1}(\lambda). \tag{5.62}$$

and

$$X_j^{-1}(\lambda) = I - \frac{\hat{\lambda}_j - \lambda_j}{\lambda - \lambda_j} \frac{|w_j\rangle \langle \hat{w}_j|}{\langle \hat{w}_j | w_j \rangle}, \quad j = 2, \ \ldots, N, \tag{5.63}$$

From the product formula (5.61) and (5.62), we can see that $X(\lambda)$ and $X^{-1}(\lambda)$ are both meromorphic functions with simple poles at $\{\hat{\lambda}_j, 1 \leq j \leq N\}$ outside Γ and $\{\lambda_j, 1 \leq j \leq N\}$ inside Γ, respectively. To complete the proof of this theorem we need to show that $X(\lambda)$ and $X^{-1}(\lambda)$ have the representation (5.43a)-(5.44), which can be obtained from the following steps.

Step 1 The relations in (5.56) and (5.61) show that $X(\lambda)$ has simple pole singularities at each $\hat{\lambda}_j$. According to Eq.(5.57b), we can find that

$$Res(X(\lambda), \hat{\lambda}_j) = |z_j\rangle \langle \hat{v}_j| , \tag{5.64}$$

for a certain column vector $|z_j\rangle$. Therefore, $X(\lambda)$ can be written as

$$X(\lambda) = I + \sum_{j=1}^{N} \frac{1}{\lambda - \hat{\lambda}_j} |z_j\rangle \langle \hat{v}_j| . \tag{5.65}$$

In the same way, from Eqs(5.57a) and (5.62)-(5.63), we can expand $X^{-1}(\lambda)$ as

$$X^{-1}(\lambda) = I - \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} |v_j\rangle \langle \hat{z}_j| , \tag{5.66}$$

for a certain row vector $\langle \hat{z}_j|$. In the above representations (5.65)-(5.66), vectors $\{|z_j\rangle , \langle \hat{z}_j|\}$ are related to the vectors $\{|v_j\rangle , \langle \hat{v}_j|\}$. To discover this dependence we use the relation

$$X(\lambda)X^{-1}(\lambda) = I. \tag{5.67}$$

Step 2 Substitute Eq.(5.66) into the L.H.S of Eq.(5.67) and equating the residue of X^{-1} at each λ_j to zero, we obtain

$$X(\lambda_j) |v_j\rangle = 0, \quad 1 \leq j \leq N. \tag{5.68}$$

Then substitute Eq.(5.65) into the above equation, we get a linear system of equations as

$$M(|z_1\rangle , |z_2\rangle , \ \ldots, |z_N\rangle) = (|v_1\rangle , |v_2\rangle , \ \ldots, |v_N\rangle), \tag{5.69}$$

where M is the matrix defined by (5.44).

Step 3 By solving the above system for $\{|z_j\rangle\}$ and inserting the result into (5.65), we reach the $X(\lambda)$ representation in (5.43a).

Apply similar calculations as in Steps 2 and 3 to $X(\lambda)$, we can obtain the representation (5.44) for X^{-1}. The proof of the theorem completes. □

5.4 Ablowitz-Kaup-Newell-Segur Hierarchy with Two Components

We begin with the matrix spectral problem

$$\phi_x = U\phi = U(u;\lambda)\phi, \qquad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}, \tag{5.70}$$

where

$$U = \begin{bmatrix} -i\lambda & p_1 & p_2 \\ 5\bar{p}_1 - 3i\bar{p}_2 & i\lambda & 0 \\ 3i\bar{p}_1 + 2\bar{p}_2 & 0 & i\lambda \end{bmatrix}, \tag{5.71}$$

u is the potential column vector

$$u = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}, \tag{5.72}$$

and λ is a spectral parameter. The functions p_1, p_2 are functions of the spatial variable x and the temporal variable t. For convenience we let

$$q_1 = 5\bar{p}_1 - 3i\bar{p}_2, \tag{5.73a}$$

$$q_2 = 3i\bar{p}_1 + 2\bar{p}_2. \tag{5.73b}$$

To formulate the associated Ablowitz-Kaup-Newell-Segur (AKNS) soliton hierarchy [3], we use the Tu-Ma scheme and follow the same procedure we did in Chapter 2. We start by solving the stationary zero curvature equation

$$W_x - [U, W] = 0, \tag{5.74}$$

with a solution W taking to be of the form

$$W = \begin{bmatrix} a & b_1 & b_2 \\ c_1 & d_1 & d_2 \\ c_2 & d_3 & d_4 \end{bmatrix}. \tag{5.75}$$

Then we get

$$\begin{cases} a_x = i(p_1c_1 + p_2c_2 - q_1b_1 - q_2b_2), \\ b_{1,x} = i(-2\lambda b_1 - p_1a + p_1d_1 + p_2d_3), \\ b_{2,x} = i(-2\lambda b_2 - p_2a + p_1d_2 + p_2d_4), \\ c_{1,x} = i(2\lambda c_1 + q_1a - q_1d_1 - q_2d_2), \\ c_{2,x} = i(2\lambda c_2 + q_2a - q_1d_3 - q_2d_4), \\ d_{1,x} = i(q_1b_1 - p_1c_1), \qquad d_{2,x} = i(q_1b_2 - p_2c_1), \\ d_{3,x} = i(q_2b_1 - p_1c_2), \qquad d_{4,x} = i(q_2b_2 - p_2c_2). \end{cases} \tag{5.76}$$

Taking W as a formal series:

$$W = \begin{bmatrix} a & b_1 & b_2 \\ c_1 & d_1 & d_2 \\ c_2 & d_3 & d_4 \end{bmatrix} = \sum_{m=0}^{\infty} W_m \lambda^{-m}, \; W_m = W_m(u) = \begin{bmatrix} a^{[m]} & b_1^{[m]} & b_2^{[m]} \\ c_1^{[m]} & d_1^{[m]} & d_2^{[m]} \\ c_2^{[m]} & d_3^{[m]} & d_4^{[m]} \end{bmatrix}, \quad m \geq 0, \tag{5.77}$$

and comparing the coefficients of the same powers of λ in the system (5.76), we obtain the following recursion relations

$$\begin{cases} b_1^{[m+1]} = -\frac{1}{2}(-ib_{1,x}^{[m]} + p_1a^{[m]} - p_1d_1^{[m]} - p_2d_3^{[m]}), \\ b_2^{[m+1]} = -\frac{1}{2}(-ib_{2,x}^{[m]} + p_2a^{[m]} - p_1d_2^{[m]} - p_2d_4^{[m]}), \\ c_1^{[m+1]} = -\frac{1}{2}(ic_{1,x}^{[m]} + q_1a^{[m]} - q_1d_1^{[m]} - q_2d_2^{[m]}), \\ c_2^{[m+1]} = -\frac{1}{2}(ic_{2,x}^{[m]} + q_2a^{[m]} - q_1d_3^{[m]} - q_2d_4^{[m]}), \qquad m \geq 0. \\ a_x^{[m]} = i(p_1c_1^{[m]} + p_2c_2^{[m]} - q_1b_1^{[m]} - q_2b_2^{[m]}), \\ d_{1,x}^{[m]} = i(q_1b_1^{[m]} - p_1c_1^{[m]}), \qquad d_{2,x}^{[m]} = i(q_1b_2^{[m]} - p_2c_1^{[m]}), \\ d_{3,x}^{[m]} = i(q_2b_1^{[m]} - p_1c_2^{[m]}), \qquad d_{4,x}^{[m]} = i(q_2b_2^{[m]} - p_2c_2^{[m]}), \end{cases} \tag{5.78}$$

with the initial values

$$\begin{cases} b_1^{[0]} = b_2^{[0]} = 0, \qquad c_1^{[0]} = c_2^{[0]} = 0, \\ \\ a_x^{[0]} = 0, \qquad d_{1,x}^{[0]} = d_{2,x}^{[0]} = d_{3,x}^{[0]} = d_{4,x}^{[0]} = 0. \end{cases} \tag{5.79}$$

We choose

$$a^{[0]} = -1, \qquad d_1^{[0]} = d_4^{[0]} = 1, \qquad d_2^{[0]} = d_3^{[0]} = 0, \tag{5.80}$$

and require the following condition on constants of integration in (5.78):

$$W_m|_{u=0} = 0, \qquad m \geq 1, \tag{5.81}$$

that is, we make them equal to zero.

Therefore, the set $\{a^{[m]}, b_1^{[m]}, b_2^{[m]}, c_1^{[m]}, c_2^{[m]}, d_1^{[m]}, d_2^{[m]}, d_3^{[m]}, d_4^{[m]}, m \geq 1\}$ is uniquely determined.

By direct calculations using the recursion relations (5.78), the first four sets of the sequence are given by

$$\begin{cases} b_1^{[1]} = p_1, \qquad b_2^{[1]} = p_2, \\ c_1^{[1]} = q_1, \qquad c_2^{[1]} = q_2, \\ a^{[1]} = 0, \qquad d_1^{[1]} = d_2^{[1]} = d_3^{[1]} = d_4^{[1]} = 0; \end{cases} \tag{5.82}$$

$$\begin{cases} b_1^{[2]} = \frac{1}{2} i p_{1,x}, \qquad b_2^{[2]} = \frac{1}{2} i p_{2,x}, \\ c_1^{[2]} = -\frac{1}{2} i q_{1,x}, \qquad c_2^{[2]} = -\frac{1}{2} i q_{2,x}, \\ a^{[2]} = \frac{1}{2}(p_1 q_1 + p_2 q_2), \\ d_1^{[2]} = -\frac{1}{2} p_1 q_1, \qquad d_2^{[2]} = -\frac{1}{2} p_2 q_1, \\ d_3^{[2]} = -\frac{1}{2} p_1 q_2, \qquad d_4^{[2]} = -\frac{1}{2} p_2 q_2; \end{cases} \tag{5.83}$$

$$\begin{cases} b_1^{[3]} = -\frac{1}{4}\left[p_{1,xx} + 2p_1(p_1 q_1 + p_2 q_2)\right], \qquad b_2^{[3]} = -\frac{1}{4}\left[p_{2,xx} + 2p_2(p_1 q_1 + p_2 q_2)\right], \\ c_1^{[3]} = -\frac{1}{4}\left[q_{1,xx} + 2q_1(p_1 q_1 + p_2 q_2)\right], \qquad c_2^{[3]} = -\frac{1}{4}\left[q_{2,xx} + 2q_2(p_1 q_1 + p_2 q_2)\right], \\ a^{[3]} = -\frac{1}{4} i(p_1 q_{1,x} + p_2 q_{2,x} - p_{1,x} q_1 - p_{2,x} q_2), \\ d_1^{[3]} = -\frac{1}{4} i(p_{1,x} q_1 - p_1 q_{1,x}), \qquad d_2^{[3]} = -\frac{1}{4} i(p_{2,x} q_1 - p_2 q_{1,x}), \\ d_3^{[3]} = -\frac{1}{4} i(p_{1,x} q_2 - p_1 q_{2,x}), \qquad d_4^{[2]} = -\frac{1}{4} i(p_{2,x} q_2 - p_2 q_{2,x}); \end{cases} \tag{5.84}$$

and

$$
\begin{cases}
b_1^{[4]} = -\frac{1}{8} i \left[p_{1,xxx} + 3p_{1,x}(p_1q_1 + p_2q_2) + 3p_1(p_{1,x}q_1 + p_{2,x}q_2) \right], \\
b_2^{[4]} = -\frac{1}{8} i \left[p_{2,xxx} + 3p_{2,x}(p_1q_1 + p_2q_2) + 3p_2(p_{1,x}q_1 + p_{2,x}q_2) \right], \\
c_1^{[4]} = \frac{1}{8} i \left[q_{1,xxx} + 3q_{1,x}(p_1q_1 + p_2q_2) + 3q_1(p_1q_{1,x} + p_2q_{2,x}) \right], \\
c_2^{[4]} = \frac{1}{8} i \left[q_{2,xxx} + 3q_{2,x}(p_1q_1 + p_2q_2) + 3q_2(p_1q_{1,x} + p_2q_{2,x}) \right], \\
a^{[4]} = -\frac{1}{8} \left[3(p_1q_1 + p_2q_2)^2 + p_1q_{1,xx} + p_2q_{2,xx} - p_{1,x}q_{1,x} - p_{2,x}q_{2,x} + p_{1,xx}q_1 + p_{2,xx}q_2 \right], \\
d_1^{[4]} = \frac{1}{8}(3p_1q_1(p_1q_1 + p_2q_2) + p_{1,xx}q_1 - p_{1,x}q_{1,x} + p_1q_{1,xx}), \\
d_2^{[4]} = \frac{1}{8}(3p_2q_1(p_1q_1 + p_2q_2) + p_{2,xx}q_1 - p_{2,x}q_{1,x} + p_2q_{1,xx}), \\
d_3^{[4]} = \frac{1}{8}(3p_1q_2(p_1q_1 + p_2q_2) + p_{1,xx}q_2 - p_{1,x}q_{2,x} + p_1q_{2,xx}) \\
d_4^{[4]} = \frac{1}{8}(3p_2q_2(p_2q_2 + p_2q_2) + p_{2,xx}q_2 - p_{2,x}q_{2,x} + p_2q_{2,xx}).
\end{cases}
\tag{5.85}
$$

Now, we consider the temporal spectral problems

$$
\phi_{t_n} = V^{[n]}\phi = V^{[n]}(u,\lambda)\phi, \tag{5.86}
$$

with a series of Lax matrices

$$
V^{[n]} = (\lambda^n W)_+ = \sum_{m=0}^{n} W_m \lambda^{n-m}, \quad n \geq 0, \tag{5.87}
$$

where P_+ denotes the polynomial part of P in λ.

Substituting (5.71) and (5.87) into the zero curvature equation

$$
U_{t_n} - V_x^{[n]} + [U, V^{[n]}] = 0, \tag{5.88}
$$

we get the AKNS soliton hierarchy with two components

$$
u_{t_n} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}_{t_n} = K_n = i \begin{bmatrix} -2b_1^{[n+1]} \\ -2b_2^{[n+1]} \end{bmatrix}, \qquad n \geq 0. \tag{5.89}
$$

The second nonlinear integrable system in the hierarchy (5.89) is

$$
\begin{cases}
p_{1,t_3} = -\frac{1}{4} \Big[p_{1,xxx} + 6p_{1,x}\big(5|p_1|^2 + |p_2|^2\big) + 9ip_{1,x}\big(\bar{p}_1p_2 - 2p_1\bar{p}_2\big) + 3p_{2,x}\big(2p_1\bar{p}_2 + 3i|p_1|^2\big) \Big], \\
\\
p_{2,t_3} = -\frac{1}{4} \Big[p_{2,xxx} + 3p_{2,x}\big(5|p_1|^2 + 4|p_2|^2\big) + 9ip_{2,x}\big(2\bar{p}_1p_2 - p_1\bar{p}_2\big) + 3p_{1,x}\big(5\bar{p}_1p_2 - 3i|p_2|^2\big) \Big],
\end{cases}
\tag{5.90}
$$

which is a coupled complex modified Korteweg-de Vries (mKdV) system.

5.5 A Riemann-Hilbert Problem for a Coupled Complex Modified-KdV System

The aim of this section is to generate N-soliton solutions for the coupled complex mKdV system (5.90) using the Riemann-Hilbert approach [49], where the spatial variable x is defined on the real line.

5.5.1 Formulating a Riemann-Hilbert Problem

The Lax Pair and Eigenfunctions

One of the important steps in constructing a Riemann-Hilbert problem associated with system (5.90) is to bring in equivalent matrix spectral problems so that we can guarantee the existence of bounded analytical eigenfunctions in the upper or lower half-plane $\mathbb{C}^{\pm}$. The Lax pair of the coupled complex mKdV system (5.90) is

$$\phi_x(x,t;\lambda) = U(x,t)\phi(x,t;\lambda), \tag{5.91a}$$

$$\phi_t(x,t;\lambda) = V(x,t)\phi(x,t;\lambda), \tag{5.91b}$$

where $\phi = (\phi_1, \phi_2, \phi_3)^T$ is a vector or a matrix function, λ is the spectral parameter and

$$U(x,t) = i\lambda\Lambda + iP, \tag{5.92a}$$

$$V(x,t) = i\lambda^3\Lambda + iQ, \tag{5.92b}$$

with

$$\Lambda = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & p_1 & p_2 \\ 5\bar{p}_1 - 3i\bar{p}_2 & 0 & 0 \\ 3i\bar{p}_1 + 2\bar{p}_2 & 0 & 0 \end{bmatrix}, \tag{5.93}$$

and

$$Q = \lambda^2 Q^{(2)} + \lambda Q^{(1)} + Q^{(0)}. \tag{5.94}$$

Herein

$$Q^{(2)} = P, \quad Q^{(1)} = \begin{bmatrix} \frac{1}{2}(p_1q_1 + p_2q_2) & \frac{1}{2}ip_{1,x} & \frac{1}{2}ip_{2,x} \\ -\frac{1}{2}iq_{1,x} & -\frac{1}{2}p_1q_1 & -\frac{1}{2}p_2q_1 \\ -\frac{1}{2}iq_{2,x} & -\frac{1}{2}p_1q_2 & -\frac{1}{2}p_2q_2 \end{bmatrix}, \tag{5.95}$$

and

$$Q^{(0)}=\begin{bmatrix} Q_{11}^{(0)} & Q_{12}^{(0)} & Q_{13}^{(0)} \\ Q_{21}^{(0)} & Q_{22}^{(0)} & Q_{22}^{(0)} \\ Q_{31}^{(0)} & Q_{32}^{(0)} & Q_{33}^{(0)} \end{bmatrix}. \tag{5.96}$$

where

$$\begin{aligned}
&Q_{11}^{(0)}=-\tfrac{1}{4}i(p_1q_{1,x}+p_2q_{2,x}-p_{1,x}q_1-p_{2,x}q_2), \quad Q_{12}^{(0)}=-\tfrac{1}{4}p_{1,xx}-\tfrac{1}{2}(p_1^2q_1+p_1p_2q_2),\\
&Q_{13}^{(0)}=-\tfrac{1}{4}p_{2,xx}-\tfrac{1}{2}(p_1p_2q_1+p_2^2q_2), \quad Q_{21}^{(0)}=-\tfrac{1}{4}q_{1,xx}-\tfrac{1}{2}(p_1q_1^2+p_2q_1q_2),\\
&Q_{22}^{(0)}=-\tfrac{1}{4}i(p_{1,x}q_1-p_1q_{1,x}), \quad Q_{23}^{(0)}=-\tfrac{1}{4}i(p_{2,x}q_1-p_2q_{1,x}),\\
&Q_{31}^{(0)}=-\tfrac{1}{4}q_{2,xx}-\tfrac{1}{2}(p_1q_1q_2+p_2q_2^2), \quad Q_{32}^{(0)}=-\tfrac{1}{4}i(p_{1,x}q_2-p_1q_{2,x}),\\
&Q_{33}^{(0)}=-\tfrac{1}{4}i(p_{2,x}q_2-p_2q_{2,x}).
\end{aligned} \tag{5.97}$$

For convenience we put $q_1=5\bar{p}_1-3i\bar{p}_2$ and $q_2=3i\bar{p}_1+2\bar{p}_2$. The compatibility condition of the Lax pair (5.91) gives the zero curvature equation, $U_t-V_x+[U,V]=0$, which is exactly system (5.90).

To formulate a Riemann-Hilbert problem we assume that the potentials $p_1(x,t)$ and $p_2(x,t)$ are sufficiently smooth functions of (x,t) which decay to zero as $x,t\to\pm\infty$, (to guarantee the existence of ϕ for $x\in(-\infty,\infty)$), and satisfy the integrable conditions

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|x|^{m_1}|t|^{m_2}(|p_1|+|p_2|)dxdt<\infty, \qquad m_1,m_2=0,1. \tag{5.98}$$

The function ϕ defined in Eq.(5.91) is treated as a fundamental matrix of those linear equations. Looking at Eq.(5.91), we observe that when $x,t\to\pm\infty$, we have $\phi\sim e^{i\lambda\Lambda x+i\lambda^3\Lambda t}$. Therefore rather than working with the original form of Lax pair (5.91), it is more convenient to introduce a new eigenfunction $\psi(x,t;\lambda)$ as

$$\phi(x,t;\lambda)=\psi(x,t;\lambda)e^{i\lambda\Lambda x+i\lambda^3\Lambda t}, \tag{5.99}$$

Set $\check{P}=iP$ and $\check{Q}=iQ$. Using (5.99), the new Lax pair equations read

$$\psi_x=\check{P}\psi+i\lambda[\Lambda,\psi], \tag{5.100a}$$

$$\psi_t=\check{Q}\psi+i\lambda^3[\Lambda,\psi]. \tag{5.100b}$$

REMARK 2.

$$tr(\check{P})=tr(\check{Q})=0. \tag{5.101}$$

In the scattering problem, we only consider the spectral analysis of the x-part of (5.100), where the time t is fixed and has been omitted in the notation.

Now let us introduce two matrix solutions $\psi_{\pm}(x,\lambda)$ of Eq.(5.100a) with the following asymptotic conditions

$$\psi_{\pm}(x,\lambda) \to I, \qquad x \to \pm\infty, \tag{5.102}$$

respectively, where I is the 3×3 identity matrix. The subscripts in $\psi_{\pm}$ represent which end of the x-axis the boundary conditions are set.

PROPOSITION 5.3.

$$\det \psi_{\pm}(x,\lambda) = 1, \quad \textit{for all } (x,\lambda). \tag{5.103}$$

Proof. Since $\psi_{\pm}$ satisfies Eq.(5.100a), that is

$$(\psi_{\pm})_x = \check{P}\psi_{\pm} + i\lambda[\Lambda, \psi_{\pm}], \tag{5.104}$$

we can apply the special case of the generalized Liouville's formula using Proposition 5.1 to obtain

$$(\det \psi_{\pm})_x = [tr(\check{P})] \det \psi_{\pm}. \tag{5.105}$$

Since $tr(\check{P}) = 0$, we get

$$\det \psi_{\pm}(x,\lambda) = \text{constant}, \quad \text{for all } x. \tag{5.106}$$

Using the asymptotic condition (5.102), we prove (5.103). □

The Scattering Matrix $S(\lambda)$

Let

$$E(x,\lambda) = e^{i\lambda\Lambda x}, \tag{5.107}$$

and

$$\Phi = \psi_{-}E, \quad \Psi = \psi_{+}E. \tag{5.108}$$

The matrix functions $\Phi(x,\lambda)$ and $\Psi(x,\lambda)$ are linearly dependent since they are both matrix solutions of (5.91a). Hence, they should be related by a matrix, say $S(\lambda)$, as

$$\Phi(x,\lambda) = \Psi(x,\lambda)S(\lambda), \quad \lambda \in \mathbb{R}. \tag{5.109}$$

So we have

$$\psi_- = \psi_+ ESE^{-1}, \quad \lambda \in \mathbb{R}, \tag{5.110}$$

where $S(\lambda) = (s_{jk})_{3\times 3}$ is the scattering matrix. From Eq.(5.103), it is clear that

$$\det S(\lambda) = 1. \tag{5.111}$$

The Definition of $P^{\pm}(\lambda)$

PROPOSITION 5.4. *For $\lambda \in \mathbb{R}$, the relation in (5.100a) can be converted to the Volterra integral equations*

$$\psi_-(x,\lambda) = I + \int_{-\infty}^{x} e^{i\lambda\Lambda(x-y)} \check{P}(y)\psi_-(\lambda,y)e^{i\lambda\Lambda(y-x)}dy, \tag{5.112a}$$
$$\psi_+(x,\lambda) = I - \int_{x}^{\infty} e^{i\lambda\Lambda(x-y)} \check{P}(y)\psi_+(\lambda,y)e^{i\lambda\Lambda(y-x)}dy. \tag{5.112b}$$

Proof. Dealing with Eq.(5.91a) as an inhomogeneous ordinary differential equation where $\check{P}\phi$ is the inhomogeneous term, we see that E is the solution of the homogeneous part

$$\phi_x = i\lambda\Lambda\phi. \tag{5.113}$$

Then through the method of variation of parameters with the asymptotic condition (5.102), we attain Eq.(5.112). □

If the convergence of the integrals on the right hand sides of the Volterra integral equations (5.112) is guaranteed, then the eigenfunctions $\psi_\pm(x,\lambda)$ allow analytical extensions off the real axis $\lambda \in \mathbb{R}$.

LEMMA 5.2. *If we write down $\psi_\pm(x,\lambda)$ as $\psi_\pm = (\psi_\pm^{(1)}, \psi_\pm^{(2)}, \psi_\pm^{(3)})$ where $\psi_\pm^{(i)}$ $(i = 1,2,3)$ are columns, then we have*

1. *$\psi_-^{(1)}$, $\psi_+^{(2)}$, and $\psi_+^{(3)}$ are analytic in $\lambda \in \mathbb{C}^+$,*
2. *$\psi_+^{(1)}$, $\psi_-^{(2)}$, and $\psi_-^{(3)}$ are analytic in $\lambda \in \mathbb{C}^-$.*

Proof. Based on Λ and $\check{P}$, it is clear that the integral equation for the first column of ψ_-, $\psi_-^{(1)}$, consists of only the exponential element $e^{2i\lambda(x-y)}$ that decays while $\lambda \in \mathbb{C}^+$, and the integral equation for the last two columns of ψ_+, $\psi_+^{(2)}$ and $\psi_+^{(3)}$, consists of only the exponential element $e^{2i\lambda(y-x)}$ that also decays while $\lambda \in \mathbb{C}^+$. Therefore, we can analytically extend the three columns $\psi_-^{(1)}, \psi_+^{(2)}$ and $\psi_+^{(3)}$ to the upper half plane $\mathbb{C}^+$. In the same way, the columns $\psi_+^{(1)}, \psi_-^{(2)}$ and $\psi_-^{(3)}$ can be analytically extended to the lower half plane $\mathbb{C}^-$. □

Hence, the matrix solution

$$P^+ = P^+(x,\lambda) = [\psi_-^{(1)}, \psi_+^{(2)}, \psi_+^{(3)}] = \psi_- H_1 + \psi_+ H_2, \tag{5.114}$$

is analytic for $\lambda \in \mathbb{C}^+$, and the matrix solution

$$[\psi_+^{(1)}, \psi_-^{(2)}, \psi_-^{(3)}] = \psi_+ H_1 + \psi_- H_2, \tag{5.115}$$

is analytic for $\lambda \in \mathbb{C}^-$, where

$$H_1 \equiv \text{diag}(1,0,0), \quad H_2 \equiv \text{diag}(0,1,1). \tag{5.116}$$

From the Volterra integral equations (5.112), we get

$$P^+(x,\lambda) \to I, \qquad \lambda \in \mathbb{C}^+ \to \infty, \tag{5.117}$$

and

$$[\psi_+^{(1)}, \psi_-^{(2)}, \psi_-^{(3)}] \to I, \qquad \lambda \in \mathbb{C}^- \to \infty. \tag{5.118}$$

In what follows, the analytic counter part of P^+ in the lower half plane $\mathbb{C}^-$ is constructed. From Eq.(5.100), we find that the adjoint equation of the x-part reads

$$\psi_x^A = -\psi^A \check{P} - i\lambda[\psi^A, \Lambda]. \tag{5.119}$$

PROPOSITION 5.5. *The inverse matrices $\psi_\pm^{-1}$ satisfy the adjoint equations (5.119).*

Proof. Substituting Eq.(5.100a) into the relation

$$0 = (I)_x = (\psi\psi^{-1})_x = \psi_x\psi^{-1} + \psi(\psi^{-1})_x, \tag{5.120}$$

we get

$$(\psi^{-1})_x = -\psi^{-1}\check{P} - i\lambda[\psi^{-1}, \Lambda], \tag{5.121}$$

which means that $\psi_\pm^{-1}$ satisfies the adjoint equation (5.119) □

By expressing $\psi_\pm^A$ as

$$\psi_+^A = \begin{bmatrix} \tilde{\psi}_+^{(1)} \\ \tilde{\psi}_+^{(2)} \\ \tilde{\psi}_+^{(3)} \end{bmatrix}, \quad \psi_-^A = \begin{bmatrix} \tilde{\psi}_-^{(1)} \\ \tilde{\psi}_-^{(2)} \\ \tilde{\psi}_-^{(3)} \end{bmatrix}, \tag{5.122}$$

where $\tilde{\psi}_{\pm}^{(j)}$ denotes the jth row of $\psi_{\pm}^{A}$ $(j = 1, 2, 3)$, and using similar approach as in the proof of Lemma 5.2, we can show that the adjoint matrix solution

$$P^{-} = P^{-}(x, \lambda) = \begin{bmatrix} \tilde{\psi}_{-}^{(1)} \\ \tilde{\psi}_{+}^{(2)} \\ \tilde{\psi}_{+}^{(3)} \end{bmatrix} = H_1 \psi_{-}^{A} + H_2 \psi_{+}^{A}, \tag{5.123}$$

is analytic for λ in the lower half-plane $\mathbb{C}^{-}$, and the adjoint matrix solution

$$\begin{bmatrix} \tilde{\psi}_{+}^{(1)} \\ \tilde{\psi}_{-}^{(2)} \\ \tilde{\psi}_{-}^{(3)} \end{bmatrix} = H_1 \psi_{+}^{A} + H_2 \psi_{-}^{A}, \tag{5.124}$$

is analytic for λ in the upper half-plane $\mathbb{C}^{+}$. Similarly, we find that

$$P^{-}(x, \lambda) \to I, \qquad \lambda \in \mathbb{C}^{-} \to \infty, \tag{5.125}$$

and

$$\begin{bmatrix} \tilde{\psi}_{+}^{(1)} \\ \tilde{\psi}_{-}^{(2)} \\ \tilde{\psi}_{-}^{(3)} \end{bmatrix} \to I, \qquad \lambda \in \mathbb{C}^{+} \to \infty. \tag{5.126}$$

The Riemann-Hilbert Problem and the Time Evolution of the Scattering Data

Now, two matrix functions $P^{\pm}(x, \lambda)$ that are analytic in $\mathbb{C}^{\pm}$, respectively, have been built. On the real line, combining Eqs.(5.110), (5.114), and (5.123), we arrive at

$$P^{-}(x, \lambda) P^{+}(x, \lambda) = G(x, \lambda), \quad \lambda \in \mathbb{R}, \tag{5.127}$$

where

$$G = E(H_1 + H_2 S(\lambda))(H_1 + S^{-1}(\lambda) H_2) E^{-1},$$

$$= E \begin{bmatrix} 1 & \hat{s}_{12} & \hat{s}_{13} \\ s_{21} & 1 & 0 \\ s_{31} & 0 & 1 \end{bmatrix} E^{-1}. \tag{5.128}$$

$S^{-1}(\lambda) = (S(\lambda))^{-1} = (\hat{s}_{jk})_{3\times 3}$ is the inverse of the scattering matrix $S(\lambda)$. Hence, the associated matrix Riemann-Hilbert problem we desire to formulate for the coupled complex mKdV system (5.90) is defined by the relations (5.127)-(5.128). The canonical normalization condition for this RHP is determined from (5.117) and (5.125) as

$$P^{\pm}(x,\lambda) \to I, \qquad \lambda \in \mathbb{C}^{\pm} \to \infty. \tag{5.129}$$

To complete the direct scattering transform, the time evolution of the scattering data

$$\{\hat{s}_{12}(\lambda), \hat{s}_{13}(\lambda), s_{21}(\lambda), s_{31}(\lambda), \lambda \in \mathbb{R}; \quad \lambda_k, \hat{\lambda}_k, v_k, \hat{v_k}, 1 \leq k \leq N\}, \tag{5.130}$$

has to be determined.

PROPOSITION 5.6. *The scattering coefficients* $s_{12}(\lambda), s_{13}(\lambda), s_{21}(\lambda), s_{31}(\lambda)$ *are time dependent while all other scattering coefficients are time independent.*

Proof. Recall Eq.(5.110)

$$\psi_{-}E = \psi_{+}ES, \quad \lambda \in \mathbb{R}. \tag{5.131}$$

Since $\psi_{\pm}$ satisfies Eq.(5.100b), then multiplying Eq.(5.100b) by E we find that $\psi_{-}E$, that is, $\psi_{+}ES$ also complies with the temporal equation (5.100b).

Substituting $\psi_{+}ES$ into Eq.(5.100b) leads to

$$(\psi_{+}ES)_t = \check{Q}(\psi_{+}ES) + i\lambda^3[\Lambda, \psi_{+}ES]. \tag{5.132}$$

By Taking the limit of the above equation as $x \to \infty$, and utilizing the asymptotic condition (5.102) for ψ_{+} in addition to the vanishing condition $\check{Q} \to 0$ as $x \to \pm\infty$, we obtain

$$S_t = i\lambda^3[\Lambda, S]. \tag{5.133}$$

This equation leads to

$$\begin{cases} s_{12}(\lambda,t) = s_{12}(\lambda,0)e^{2i\lambda^3 t}, \qquad s_{13}(\lambda,t) = s_{13}(\lambda,0)e^{2i\lambda^3 t}, \\ s_{21}(\lambda,t) = s_{21}(\lambda,0)e^{-2i\lambda^3 t}, \qquad s_{31}(\lambda,t) = s_{31}(\lambda,0)e^{-2i\lambda^3 t}, \end{cases} \tag{5.134}$$

and all other scattering coefficients are time independent. □

5.5.2 N-Soliton Solutions of the Coupled Complex Modified-KdV System

In this section, we analyze the solution for the Riemann-Hilbert problem with zeros. As mentioned in Section 5.3, the solution to the non-regular Riemann-Hilbert problem (5.127) is not unique except if we specify the zeros of $\det P^+$ in the upper half of the λ-plane and $\det P^-$ in the lower half of the λ-plane and find the structures of $\ker P^\pm$ at these zeros.

Based on the definition of $P^\pm$, Eq.(5.114), Eq.(5.123) and the scattering relation(5.110), we have

$$\det P^+(\lambda) = s_{11}(\lambda), \quad \det P^-(\lambda) = s_{22}(\lambda)s_{33}(\lambda) - s_{23}(\lambda)s_{32}(\lambda), \tag{5.135}$$

and since $\det S = 1$, we get

$$\det P^+(\lambda) = s_{11}(\lambda), \quad \det P^-(\lambda) = \hat{s}_{11}(\lambda). \tag{5.136}$$

The number of zeros for $\det P^+$ and $\det P^-$ should be the same, or otherwise the associated Riemann-Hilbert problem is not solvable. Hence, let N be a natural number and assume that s_{11} has N simple zeros $\{\lambda_k \in \mathbb{C}^+, 1 \leq k \leq N\}$, and $\hat{s}_{11}$ has N simple zeros $\{\hat{\lambda}_k \in \mathbb{C}^-, 1 \leq k \leq N\}$. Then each of $\ker P^+(\lambda_k)$ ($\ker P^-(\hat{\lambda}_k)$) contains only a single column vector v_k (row vector $\hat{v}_k$), respectively,

$$P^+(\lambda_k)v_k = 0, \quad \hat{v}_k P^-(\hat{\lambda}_k) = 0, \qquad 1 \leq k \leq N. \tag{5.137}$$

The Riemann-Hilbert problem (5.127) with the canonical normalization condition (5.129) and the zero structure (5.137) can be solved using Theorem 5.4 and hence the potential P can be reconstructed as follows. By expanding P^+ in the way that

$$P^+(x,\lambda) = I + \frac{1}{\lambda}P_1^+(x) + O(\frac{1}{\lambda^2}), \qquad \lambda \to \infty, \tag{5.138}$$

and substituting this expansion into Eq.(5.100a) (noting that P^+ is a solution to this equation), and comparing terms of the same order in λ^{-1}, we observe that

$$\check{P} = -i[\Lambda, P_1^+]. \tag{5.139}$$

Consequently, the potential matrix P can be presented by

$$P = -[\Lambda, P_1^+] = \begin{bmatrix} 0 & 2(P_1^+)_{12} & 2(P_1^+)_{13} \\ -2(P_1^+)_{21} & 0 & 0 \\ -2(P_1^+)_{31} & 0 & 0 \end{bmatrix}, \tag{5.140}$$

where $P_1^+ = ((P_1^+)_{jl})_{3\times 3}$. This implies that the potentials p_1 and p_2 can be obtained by

$$p_1 = 2(P_1^+)_{12}, \quad p_2 = 2(P_1^+)_{13}. \tag{5.141}$$

The Symmetry Properties

Notice that the potential matrix

$$P = \begin{bmatrix} 0 & p_1 & p_2 \\ 5\bar{p}_1 - 3i\bar{p}_2 & 0 & 0 \\ 3i\bar{p}_1 + 2\bar{p}_2 & 0 & 0 \end{bmatrix}, \tag{5.142}$$

has the following symmetry property

$$P^{\dagger} = -BPB^{-1}, \tag{5.143}$$

where the superscript "†" refers to the Hermitian conjugate, and the anti-Hermitian matrix B is given by

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 3i \\ 0 & -3i & 5 \end{bmatrix}. \tag{5.144}$$

The following proposition shows the symmetry properties of the scattering matrix $S(\lambda)$ and the matrix solution $\psi_{\pm}$.

PROPOSITION 5.7. *Zeros of* $\det P^{+}$ *and* $\det P^{-}$ *satisfy the symmetry property*

$$\hat{\lambda}_k = \bar{\lambda}_k, \qquad 1 \le k \le N, \tag{5.145}$$

and vectors in $\ker P^{+}(\lambda_k)$ *and* $\ker P^{-}(\hat{\lambda}_k)$ *have the involution relation*

$$\hat{v}_k = v_k^{\dagger} B, \qquad 1 \le k \le N, \tag{5.146}$$

Proof. First we find the involution property of the matrix function $\psi_{\pm}$.
Taking the Hermitian conjugate of Eq.(5.100a), we have

$$(\psi_{\pm}^{\dagger}(\bar{\lambda}))_x = -i\lambda[\psi_{\pm}^{\dagger}(\bar{\lambda}), \Lambda] - \psi_{\pm}^{\dagger}(\bar{\lambda})\check{P}^{\dagger}. \tag{5.147}$$

Then multiplying Eq.(5.147) by B from the right and utilizing the symmetry property (5.143), gives

$$(\psi_{\pm}^{\dagger}(\bar{\lambda})B)_x = -i\lambda[\psi_{\pm}^{\dagger}(\bar{\lambda})B, \Lambda] - \psi_{\pm}^{\dagger}(\bar{\lambda})B\check{P}, \tag{5.148}$$

this means that $\psi_{\pm}^{\dagger}(\bar{\lambda})B$ is a matrix solution for the adjoint equation (5.119).

Noting that $\psi_{\pm}^{-1}(\lambda)$ is also a matrix solution of Eq.(5.119), this implies that $\psi_{\pm}^{\dagger}(\bar{\lambda})B$ and $\psi_{\pm}^{-1}(\lambda)$ must be linearly related, say, $\psi_{\pm}^{\dagger}(\bar{\lambda})B = C\psi_{\pm}^{-1}(\lambda)$, where the matrix C does not depend on x. Through the boundary conditions (5.102) of $\psi_\pm(\lambda)$, we conclude that $C = B$. Thus, the involution property illustrated as follows

$$\psi_\pm^\dagger(\bar{\lambda}) = B\psi_\pm^{-1}(\lambda)B^{-1}, \tag{5.149}$$

is satisfied by the matrix solutions $\psi_\pm$, where $\bar{\lambda}$ denotes the complex conjugate of λ.

Second, it follows from the scattering relation (5.110) that S satisfies

$$S^\dagger(\bar{\lambda}) = BS^{-1}(\lambda)B^{-1}, \tag{5.150}$$

which gives rise to the following relation

$$s_{11}(\bar{\lambda}) = \hat{s}_{11}(\lambda). \tag{5.151}$$

Hence, we have

$$\hat{\lambda}_k = \bar{\lambda}_k, \quad 1 \leq k \leq N. \tag{5.152}$$

Finally, the involution property of $P^\pm$ comes from the above property in addition to Eqs(5.114) and (5.123) as

$$(P^+)^\dagger(\bar{\lambda}) = BP^-(\lambda)B^{-1}. \tag{5.153}$$

The symmetry properties (5.146) for the vectors v_k and $\hat{v}_k$ is obtained by taking the Hermitian conjugate of the left equation in (5.40) and using the properties (5.145) and (5.153) to attain

$$v_k^\dagger BP^-(\hat{\lambda}_k) = 0, \quad 1 \leq k \leq N. \tag{5.154}$$

As a result of comparing the above equation with the second equation in (5.137), one obtain

$$\hat{v}_k = v_k^\dagger B, \quad 1 \leq k \leq N. \tag{5.155}$$

□

The Spatial and Temporal Evolutions

Now we determine the spatial and temporal evolutions for the kernel vectors $(v_k, \hat{v}_k)$, $(1 \leq k \leq N)$ as follows. Differentiate the first equation in (5.137) with respect to x and note that P^+ satisfies Eq.(5.100a), we have

$$P^+(\lambda_k, x)\Big(\frac{dv_k}{dx} - i\lambda_k\Lambda v_k\Big) = 0, \quad 1 \leq k \leq N. \tag{5.156}$$

This means that for each $k = 1 \dots N$, $\dfrac{dv_k}{dx} - i\lambda_k \Lambda v_k$ is in the kernel of $P^+(\lambda_k)$ and so is a constant multiple of v_k. With out loss of generality, we may assume that

$$\frac{dv_k}{dx} = i\lambda_k \Lambda v_k, \quad 1 \le k \le N. \tag{5.157}$$

On the other hand, we take the t-derivative of $P^+(\lambda_k)v_k = 0$ and utilize Eq.(5.100b) to get

$$\frac{dv_k}{dt} = i\lambda_k^3 \Lambda v_k, \quad 1 \le k \le N. \tag{5.158}$$

By solving the two equations above, we reach

$$v_k(x,t) = e^{i\lambda_k \Lambda x + i\lambda_k^3 \Lambda t} v_{k,0}, \quad 1 \le k \le N, \tag{5.159}$$

where each $v_{k,0} = v_k(x=0,t)$, $1 \le k \le N$, is an arbitrary constant column vector. Adopting similar arguments for $\hat{v}_k$, one attains

$$\hat{v}_k(x,t) = \hat{v}_{k,0} e^{-i\hat{\lambda}_k \Lambda x - i\hat{\lambda}_k^3 \Lambda t}, \quad 1 \le k \le N, \tag{5.160}$$

where each $\hat{v}_{k,0} = \hat{v}_k(x=0,t)$, $1 \le k \le N$, is an arbitrary constant row. From the involution property of the vector $\hat{v}_k(x,t)$ (5.146), it follows that

$$\hat{v}_k(x,t) = v_{k,0}^\dagger e^{-i\bar{\lambda}_k \Lambda x - i\bar{\lambda}_k^3 \Lambda t}, \quad 1 \le k \le N. \tag{5.161}$$

N-Soliton Solutions

In order to generate N-soliton solutions we take the jump matrix G to be the identity matrix I in the Riemann-Hilbert problem (5.127). In this case, the scattering data $(s_{21}, s_{31}, \hat{s}_{12}, \hat{s}_{13})$ are all zeros, and the corresponding scattering equation (5.100) is called reflectionless. This special Riemann-Hilbert problem has the unique solution given by Theorem 5.4 as

$$P^+(\lambda) = I - \sum_{j,k=1}^{N} \frac{v_j (M^{-1})_{jk} \hat{v}_k}{\lambda - \hat{\lambda}_k}, \tag{5.162}$$

and

$$P^-(\lambda) = I + \sum_{j,k=1}^{N} \frac{v_j (M^{-1})_{jk} \hat{v}_k}{\lambda - \lambda_j}. \tag{5.163}$$

where the matrix $M = (M_{jk})_{N \times N}$ and its entries M_{jk} are given by Eq.(5.44).

From the expansion of P^+ (5.138), we see that

$$P_1^+(x,t) = -\sum_{j,k=1}^{N} v_j (M^{-1})_{jk} \hat{v}_k. \tag{5.164}$$

Using the symmetry property (5.145) and the involution property (5.146), we see that $P_1^+(x,t)$ satisfies

$$(P_1^+)^\dagger = -BP_1^+B^{-1}. \tag{5.165}$$

Therefore, through the reconstruction formula (5.141) we obtain the N-soliton solutions to the coupled complex mKdV system (5.90):

$$p_1 = 2(P_1^+)_{12} = -2\sum_{j,k=1}^{N} v_{j,1}(M^{-1})_{jk}\hat{v}_{k,2}, \tag{5.166}$$

$$p_2 = 2(P_1^+)_{13} = -2\sum_{j,k=1}^{N} v_{j,1}(M^{-1})_{jk}\hat{v}_{k,3}, \tag{5.167}$$

where the vectors $v_k = (v_{k,1}, v_{k,2}, v_{k,3})^T$ and $\hat{v}_k = (\hat{v}_{k,1}, \hat{v}_{k,2}, \hat{v}_{k,3})$, $1 \leq k \leq N$, are given by (5.159) and (5.161), respectively.

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