February 2020

Non-Associative Algebraic Structures in Knot Theory

Emanuele Zappala
University of South Florida

Non-Associative Algebraic Structures in Knot Theory

by

Emanuele Zappala

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
Department of Mathematics & Statistics
College of Arts and Sciences
University of South Florida

Co-major Professor: Mohamed Elhamdadi, Ph.D.
Co-major Professor: Masahico Saito, Ph.D.
Dmytro Savchuk, Ph.D.
Boris Shekhtman, Ph.D.

Date of Approval:
November 1, 2019

Keywords: Self-Distributivity, Quandle, Rack, Cohomology, Yang-Baxter Equation,
Symmetric Monoidal Category.

DEDICATION

To Pippo,

the epitome of fatherhood

ACKNOWLEDGEMENTS

This dissertation work is the product of three different projects I have worked on, during my doctoral studies, in collaboration with M. Elhamdadi and M. Saito. Although this work solely reflects my point of view and my interpretation of our results, I have greatly benefited from their advice and wise perspective. They have also offered good guidance and, therefore, I feel thankful for that. I would also like to thank W.E. Clark for sharing his ideas, deep insight and some unpublished results that have played a pivotal role in proving one of the theorems present in this work.

TABLE OF CONTENTS

LIST OF FIGURES

ABSTRACT

In this dissertation we investigate self-distributive algebraic structures and their cohomologies, and study their relation to topological problems in knot theory. Self-distributivity is known to be a set-theoretic version of the Yang-Baxter equation (corresponding to Reidemeister move III) and is therefore suitable for producing invariants of knots and knotted surfaces. We explore three different instances of this situation. The main results of this dissertation can be, very concisely, described as follows. We introduce a cohomology theory of topological quandles and determine a class of topological quandles for which the cohomology can be computed, at least in principle, by means of the cohomology groups of smaller and discrete quandles. We utilize a diagrammatic description of higher self-distributive structures in terms of framed links via a functorial procedure called *doubling*, and generalize previously known (co)homology theories to introduce a cocycle invariant of framed links. Finally, we study a class of ternary self-distributive structures called *heaps*, and introduce two cohomology theories that classify their extensions. We show that heap cohomology is related to both group cohomology (via a long exact sequence) and ternary self-distributive cohomology (the heap second cohomology group canonically injects into the ternary self-distributive one with modified coefficients). We also develop the theory in the context of symmetric monoidal categories.

CHAPTER 1 : INTRODUCTION

Knot theory is the study and classifications of the embeddings of $\mathbb{S}^1$, the unit circle in the plane $\mathbb{R}^2$, into the three dimensional euclidean space $\mathbb{R}^3$ or its compactification $\mathbb{S}^3$. Such an embedding is what we refer to as a *knot*. Two knots are considered to be equivalent if there exists an ambient isotopy transforming one into the other. In other words, given two knots K_1 and K_2, we say that they are equivalent, and write $K_1 \cong K_2$, if there exists a continuous map $F : [0,1] \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that $F(0, K_1) = K_1$, $F(1, K_1) = K_2$ and $F(t, \bullet)$ is required to be a homeomorphism for all t, see [PS97] for instance. It is often required in the literature a "smoothness" assumption, meaning that the embeddings are differentiable maps with nonsingular differential, the isotopies depend smoothly on the first variable $t \in [0,1]$ and for a fixed value of the parameter t, they induce a diffeomorphism of $\mathbb{R}^3$ onto itself. Equivalently, we can require embeddings to be piecewise linear. The two types of requirements are essentially the same, in the sense that the two theories can be seen to correspond "bijectively". Smooth (and pieciewise linear) knots admitting an immersion in $\mathbb{R}^2$, obtained as a projection, with finitely many double points (no tangent points allowed) are called *tame*, in contrast with *wild* knots, to which no extra requirement is applied. More generally, a *link* is an embedding of finitely many copies of $\mathbb{S}^1$. The definition of isotopy in the case of links is essentially the same as in the case of knots.

1.1 Knot diagrams

Of fundamental importance to the classification of knots and links, is the concept of *diagram* of a knot or a link. This is a generically immersed closed plane curve together with over/under crossing information corresponding to each double point. See the description in Chapter 1 of [CS98], for a detailed account of knot diagrams and their generalizations to knotted surfaces. See also [CJK$^+$, PS97]. The relevance of the concept of diagram relies in the fact that it allows to translate topological problems into combinatorial ones.

Given two diagrams D_1 and D_2, it is natural to ask whether they represent the same knot/link or not. Consider for instance the "O" shaped and the "8" shaped diagrams. It is intuitively clear that they both correspond to the same embedding of the circle, since it is possible to untwist the 8. The answer to this question has been given by K. Reidemeister in the 1930's. To this purpose, he has introduced three kinds of diagram manipulations that now are referred to as Reidemeister moves of type I, II and III. We refer the reader to [PS97, CS98] for a diagrammatic depiction of Reidemeister moves.

Theorem 1.1.1. *Two link diagrams correspond to isotopic links if and only if one can be obtained from the other via plane isotopies and finitely many applications of Reidemeister moves.*

This fundamental result provides the correspondence between topology of knots/links and combinatorics. To determine isotopic classes of links is the same as to determine classes of diagrams up to plane isotopy and Reidmeister moves.

1.2 Knot Invariants

In order to classify knots/links (up to ambient isotopy), it is of central importance the notion of *knot invariant*, i.e. a quantity that does not depend on the representatiave of the equivalence class of the knot/link.

Famous examples of knot invariants include polynomial invariants like Alexander polynomial, Jones polynomial, HOMFLY-PT polynomial; homological invariants such as Khovanov homology (a categorification of Jones polynomial); categorical invariants such as the Reshetikin-Turaev invariant for ribbon graphs.

Of specific interest to us, will be the notion of cocycle invariant introduced by Carter, Jelsovsky, Kamada, Langfor and Saito in [CJK$^+$], and generalized by Carter, Elhamdadi, Grana and Saito in [CEGnS] to the case of quandle homology with non abelian coefficients. See also [CES, CENS]. We give a brief overview of this invariant, along with the notion of quandle and its homology in Section 1.3 below.

1.3 Quandles and Cocycle Invariants

Quandles are algebraic objects that encode the essence of Reidmeister moves. They have been introduced in the 1920's in [BM29], under the name of Distributive Groups. See also [Tak43].

Matveev, in [Mat], and Joyce, in [Joy82], later showed that for a given a knot, using a procedure similar to the Wirtinger presentation of the fundamental group (i.e. the first homotopy group), it is possible to construct what is called *fundamental quandle* of the knot. They also showed that the fundamental quandle is a complete invariant of a knot, up to mirror symmetry and orientation reversal. Unfortunaly, this invariant is usually computationally extremely difficult to determine. The books [Nos17, EN] are good references regarding the theory of quandles.

Definition 1.3.1. A quandle is a set X togehter with a binary operation $* : X \times X \longrightarrow X$ satisfying the following three axioms

- $x * x = x$, for all $x \in X$,
- the right multiplicaiton map $- * x : X \longrightarrow X$ is a bijection for all $x \in X$, where $-$ is a placeholder,
- $(x * y) * z = (x * z) * (y * z)$, for all $x, y, z \in X$.

Remark 1.3.1. The three axioms in the definition of quandle correspond to Reidmeister moves of type I, II and III.

A binary operation satisfying only the third axiom (self-distributivity) is called *shelf*, while a binary operation satisfying second and third axioms is called *rack*. Therefore a quandle is an idempotent rack.

Given two quandles (resp. racks or shelfs) $(X, *_X)$ and $(Y, *_Y)$, we define a homomorphism of quandles (resp. racks or shelfs) $f : X \longrightarrow Y$, to be a map satisfying $f(x *_X y) = f(x) *_Y f(y)$ for all $x, y \in X$. Quandles (resp. racks or shelfs) together with their homomorphisms give rise therefore to a category. The isomorphisms in this categories are the bijective homomorphisms.

Example 1.3.2. Every group G, endowed with the operation of conjugation $x * y := yxy^{-1}$ defines a quandle structure. This quandle is called *conjugation quandle.*

Example 1.3.3. A group G with the operation $a * b := ba^{-1}b$ is a quandle called the *core quandle* of G.

Example 1.3.4. Any $\Lambda(= \mathbb{Z}[t, t^{-1}])$-module M is a quandle with $a * b := ta + (1-t)b$, for $a, b \in M$, and is called an *Alexander quandle.*

Example 1.3.5. Given a group G and an automorpism $f \in \mathrm{Aut}(G)$, it is easy to show that $x*y := f(xy^{-1})y$ defines a quandle structure. This is called a *generalized Alexander quandle.*

Remark 1.3.6. If X is a rack, then by the second axiom of Definition 1.3.1, it follows that the right multiplication map R_x is a bijection for all $x \in X$. Moreover, using the third axiom of Definition 1.3.1, it follows that R_x is a rack automorphism. Consider now the subgroup of $\mathrm{Aut}(X)$ generated by the right multiplication maps R_x, indicated by $\mathrm{Int}(X)$, and called the interior automorphisms group. A rack is said to be *indecomposable* if $\mathrm{Int}(X)$ acts transitively on X. The word *connected* is also commonly found in the literature. We prefer to use "indecomposable" mostly because of the possible ambiguity arising in the topological context of Chapter 2.

In [CJK$^+$], a cohomology theory of quandles has been introduced, and utilized to construct invariants of knots, called *cocycle invariants.* We briefly recall the definition of quandle (co)homology.

Let $(X, *)$ be a quandle. Define the chain group of order n, written $C_n(X)$, to be the free group generated by n-tuples $(x_1, \ldots, x_n) \in X^{\times n}$. We define differentials $\partial_n : C_n(X) \longrightarrow C_{n-1}(X)$ on generators by the assignment

$$\partial_n(x_1, \ldots, x_n) := \sum_{i=2}^{n} [(x_1, \ldots, \widehat{x}_i, \ldots, x_n) - (x_1 * x_i, x_2 * x_i, \ldots, \widehat{x}_i, \ldots, x_n)],$$

where, as usual, the symbol $\widehat{}$ indicates omission of the underlying element. It is easy to see that the differentials ∂_n satisfy $\partial_{n-1}\partial_n = 0$, therefore defining a chain complex. As usual, $Z_n(X)$ indicates the group of n-cycles, and $B_n(X)$ indicates the subgroup of n-boundaries. Given an abelian group A, we obtain a cohomology theory by dualization. We refer the reader to [Moc, Nos] for examples of computations of quandle cohomology groups and constructions that relate the quandle cohomology to invariant theory, respectively.

Definition 1.3.2 ([CJK$^+$])**.** A coloring of a link diagram D by a quandle X, is a function $\mathcal{C} : R \longrightarrow X$, where R is the set of arcs of the diagram D, with the following property. Suppose we have a crossing as in Figure 1.1 left, where we assume the orientation of the over arc to be downward, with the over arc r given the color $C(r) = y$, and the under arcs r_1 and r_2, reading from top to bottom. Then it is required that if $C(r_1) = x$, r_2 is given color: $C(r_2) = x * y$.

Remark 1.3.7. A crossing as in the left diagram of Figure 1.1, with arrows oriented downwards, is called a positive crossing. In the same situation, with inverted over-passing/underpassing arrows, we say that the crossing is negative. A good mnemonic rule to remember how to determine if a crossing is positive or negative is given by the "right hand rule", as for the cross product. See Figure 4 in [CJK+].

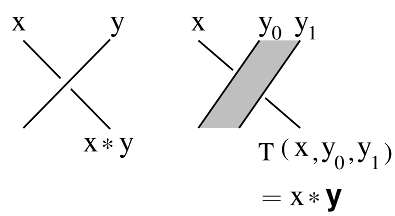

Figure 1.1: Diagrammatic representations of a binary (left) and ternary (right) operations

In order to define the cocycle invariant, we need one more preliminary definition.

Definition 1.3.3 ([CJK+]). Let $\phi \in Z^2(X, A)$ be a quandle 2-cocycle. A Boltzmann weight $B(\tau, \mathcal{C})$, at the crossing τ is defined in the following way. Let y be the color of the over arc and x and $x * y$ be the colors of the under arcs accoding to the rules in Definition 1.3.2. Then we set $B(\tau, \mathcal{C}) := \phi(x, y)^{\epsilon(\tau)}$, where $\epsilon(\tau) = \pm 1$ for a positive (resp. negative) crossing τ.

Finally, we are able to introduce the cocycle invariant.

Definition 1.3.4 ([CJK+]). Given $\phi \in Z^2(X, A)$, the Botlzmann state sum is given by the expression

$$\sum_{\mathcal{C}} \prod_{\tau} B(\tau, \mathcal{C}),$$

where the sum is taken over all the possible colorings of the link diagrams, and for a given coloring, τ varies among all the crossings.

Remark 1.3.8. Observe that in Definition 1.3.3 and Definition 1.3.4, the group A is assumed to be in multiplicative notation. The Boltzmann state sum is an element of the group ring of A.

In [CJK$^+$] is then proved that the Boltzmann state sum is indeed a link invariant. Specifically, we have the following result.

Theorem 1.3.9. *The Boltzmann state sum in Definition 1.3.4 is invariant under Reidmeister moves. It therefore defines an invariant of links denoted by* $\Phi(K)$.

As we will see in Chapter 3, it is possible to generalize this construction to the case of framed links and their diagrams using ternary quandle cohomology and a functorial procedure that we call doubling.

1.4 Ternary and Higher Arity Self-Distributivity

The notion of quandle has been recently generalized to ternary and higher arity operations, see for instance [CEGM, Gre]. As will be described in Chapter 3, a diagrammatic interpretation of these operations requires now more strings at once and is particularly suitable to describe framed links. We hereby recall the definition of ternary self-distributive operation and ternary cohomology. The natural generalization to higher arities is obtained by introducing the appropriate number of variables and does not present a particular hindrance.

We begin with the following definitions.

Definition 1.4.1. Let (X,T) be a set equipped with a ternary operation $T : X \times X \times X \to X$. The operation T is said to be *ternary self-distributive* if it satisfies the following condition for all $x, y, z, u, v \in X$,

$$T(T(x,y,z),u,v) = T(T(x,u,v),T(y,u,v),T(z,u,v)).$$

Definition 1.4.2. Let $T : X \times X \times X \to X$ be a ternary distributive operation on a set X. If for all $a, b \in X$, the map $R_{a,b} : X \to X$ given by $R_{a,b}(x) = T(x,a,b)$ is invertible, then (X,T) is said to be a *ternary rack*. If further T satisfies

$$T(x,x,x) = x$$

, for all $x \in X$. Then (X,T) is called a *ternary quandle*.

Example 1.4.1. The following constructions are found in [EGM].

- Let $(X,*)$ be a rack and define a ternary operation on X by $T(x,y,z) = (x*y)*z$, for all $x, y, z \in X$. It is straightforward to see that (X,T) is a ternary rack. Note that in this case $R_{a,b} = R_b \circ R_a$. We will say that this ternary rack is induced by a (binary) rack.

  In particular, if $(X,*)$ is an Alexander quandle with $x*y = tx + (1-t)y$, then the ternary rack coming from X has the operation

$$T(x,y,z) = t^2x + t(1-t)y + (1-t)z.$$

- Let M be any Λ-module where $\Lambda = \mathbb{Z}[t^{\pm 1}, s]$. The operation $T(x, y, z) = tx + sy + (1 - t - s)z$ defines a ternary rack structure on M. We call this an *affine* ternary rack.

  In particular, consider $\mathbb{Z}_8$ with the ternary operation $T(x, y, z) = 3x + 2y + 4z$. This affine ternary rack given in [EGM] is not induced by an Alexander quandle structure as described in the preceding item since 3 is not a square in $\mathbb{Z}_8$.

- Any group G with the ternary operation $T(x, y, z) = xy^{-1}z$ gives a ternary rack. This operation is well known and called *heap* (sometimes also called groud) of the group G.

For a ternary distributive operation T on X, we also use the notation

$$x * \mathbf{y} := T(x, y_0, y_1),$$

where $\mathbf{y} = (y_0, y_1)$. Although strictly speaking $T(x, y_0, y_1)$ is not equal to $T(x, (y_0, y_1))$, no confusion is likely to happen by this convention. Furthermore, for $\mathbf{x} = (x_0, x_1)$, we use the notation $\mathbf{x} * \mathbf{y}$ to represent

$$(x_0 * \mathbf{y}, x_1 * \mathbf{y}) = (T(x_0, y_0, y_1), T(x_1, y_0, y_1)).$$

In this notation the ternary distributivity can be written as

$$(x * \mathbf{y}) * \mathbf{z} = (x * \mathbf{z}) * (\mathbf{y} * \mathbf{z})$$

in analogy to the binary case.

We also recall the definition of homology of ternary racks [EGM]. Define first $C_n(X)$ to be the free abelian group generated by $(2n+1)$-tuples $(x_0, x_1, \ldots, x_{2n})$ of elements of a ternary rack (X, T). Define the differentials $\partial_n : C_n(X) \longrightarrow C_{n-1}(X)$ as:

$$\begin{aligned}
&\partial_n(x_0, x_1, \ldots, x_{2n}) \\
&= \sum_{i=1}^{n}(-1)^i[(x_0, \ldots, \widehat{x}_{2i-1}, \widehat{x}_{2i}, \ldots, x_{2n}) \\
&\quad -(T(x_0, x_{2i-1}, x_{2i}), \ldots, T(x_{2i-2}, x_{2i-1}, x_{2i},), \widehat{x}_{2i-1}, \widehat{x}_{2i}, \ldots, x_{2n})].
\end{aligned}$$

Definition 1.4.3. The n^{th} homology group of the ternary rack X is defined to be:

$$H_n(X) = \ker\partial_n / \mathrm{im}\partial_{n+1}.$$

By dualizing the chain complex given above, we get a cohomology theory for ternary racks.

CHAPTER 2 : CONTINUOUS COHOMOLOGY OF TOPOLOGICAL QUANDLES

Topological quandles were introduced by R.L. Rubinsztein in [Rub07] to construct an invariant of links, indicated with the symbol $J_Q(L)$, for a link L and a fixed topological quandle Q. Roughly speaking, this invariant is a topological space consisting of the fixed points of the action of an element of the braid group B_n on a topological quandle Q. Two other possible interpretations of the invariant are as follows. Given a link L, we can contruct the fundamental quandle $Q(L)$ associated to it. It is possible to show that the space of quandle homomorphism $\mathrm{Hom}_q(Q(L), Q)$ endowed with the compact-open topology is homeomorphic to $J_Q(L)$. Lastly, one can interpret $J_Q(L)$ as the space of coloring of a fixed diagram D of the link L, with colors belonging to Q. This point of view is due to Oleg Viro (see [Rub07]).

The main purpose of this chapter is to introduce a cohomology theory for topological quandles and techniques to compute continuous cohomology groups. One natural question that arises is whether or not the continuous cohomology differs from the standard (discrete) one. We will show that it is indeed the case that the two theories differ, providing an explicit example in which the continuous (i.e. topological chomology is zero while the discrete cohomology is not.

The chapter is organized as follows. In the first section we recall the definition and basic examples of topological quandles. In the second section we introduce a cohomology theory for topological quandles and study general properties of the first and second cohomology groups. In particular we will see that, as usual, the second cohomology group classifies extensions. In the third section we introduce the notion of inverse and direct limit of quandles and utilize it to provide a computation of cohomology groups. Furthermore we will see that the cohomology groups of inverse limits of quandles are isomorphic, under certain hypothesis to the direct limit of the cohomology of the components, a result analogous to one which is quite known in the group theoretic context.

The present chapter is based on the article [ESZ19].

2.1 Basics of Topological Quandles

Definition 2.1.1. Let X be a topological space together with a continuous operation $* : X \times X \longrightarrow X$, usually indicated as $*(x, y) = x * y$. We say that X is a *topological quandle* if $*$ satisfies the properties

- for all $x \in X$ we have $x * x = x$, i.e. the operation is idempotent;
- for all $y \in X$ the right multiplication map $X \longrightarrow X$, given by $x \mapsto x * y$ is a homeomorphism;
- for all $x, y, z \in X$ $(x * y) * z = (x * z) * (y * z)$.

Here we have a list of typical examples of topological quandles encountered in the literature.

Example 2.1.1. Any topological group G becomes a topological quandle with operation $*$ given by conjugation: $x * y = y^{-1}xy$. This quandle is denoted $\mathrm{Conj}(G)$, the conjugation quandle associated to G.

For a topological group G and a continuous automorphism $f : G \to G$, $x * y = f(xy^{-1})y$ for $x, y \in G$ defines a topological quandle structure on G. This is called a *generalized Alexander quandle* and is denoted by (G, f). If G is abelian, then the conjugation quandle is called an Alexander quandle.

In particular, for any $T \in \mathrm{GL}(n, \mathbb{R})$, $\mathbb{R}^n$ can be given a topological quandle structure by defining $x * y = Tx + (I - T)y$, for all $x, y \in \mathbb{R}^n$, where I denotes the identity matrix.

The following two examples can be found in [Rub07].

Example 2.1.2. Consider the n-dimensional sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$. The operation $x * y = 2\langle x|y\rangle\, y - x$, for all $x, y \in \mathbb{S}^n$ endows the sphere with a topological quandle structure, where $\langle x|y\rangle$ denotes the standard inner product in $\mathbb{R}^{n+1}$. Also, this operation induces a topological quandle structure on the real projective space $\mathbb{RP}^n$ giving a topological quandle homomorphism $S^n \to \mathbb{RP}^n$.

Example 2.1.3. Let V be a finite dimensional complex vector space and let $q \in \mathbb{C}$ be a modulus one complex number. For each $1 \leq k \leq dim(V)$, consider the grassmannian $Gr_k(V)$. Given an element $U \in Gr_k(V)$, choose an orthonormal basis $\{u_1, \ldots, u_k\}$ of U and define the map $\iota_U^q : V \longrightarrow V$ by the assignment $\iota_U^q(v) = qv + (1-q)\sum_i \langle v|u_i\rangle\, v$, where $\langle -|-\rangle$ stands for the standard inner product. We can define an operation on $Gr_k(V)$ by

$U * V = \iota_V^q(U)$. This operation turns the grassmannian into a topological quandle.

We can define the category of topological quandles, denoted $\mathcal{TQ}$, as follows. The objects are topological quandles and, given two objects X and Y, a morphism $f : X \longrightarrow Y$ is a morphism of quandles that is continuous with respect to the topologies of X and Y. It is clear that $\mathcal{TQ}$ is a subcategory of $\mathcal{Q}$, the category of quandles.

2.2 Continuous Cohomology

In this section we define a cohomology theory for topological quandles. We would also like to point out that a similar construction for smooth quandles has been introduced by Nosaka in [Nos18].

Let X be a topological quandle. Let A be a topological abelian group, $T : A \to A$ be a continuous automorphism, and A is also considered with the generalized Alexander quandle structure (A, T). We define the n-cochain group to be the set of continuous maps from n-tuples $(x_1, \dots, x_n) \in X^n$ to A, endowed with the abelian group structure induced by pointwise addition in A, where X^n is given the product topology. We indicate the n-cochain group by the symbol $\Gamma^n(X, A)$. Define the maps $\Gamma^n(X, A) \longrightarrow \Gamma^{n+1}(X, A)$, $n \in \mathbb{N}$ as in the discrete case, namely

$$\delta_0^i f(x_1, \dots, x_{n+1}) = f(x_1, \dots, \widehat{x}_i, \dots, x_{n+1});$$

$$\delta_1^i f(x_1, \dots, x_{n+1}) = f(x_1 * x_i, \dots, x_{i-1} * x_i, x_{i+1}, \dots, x_{n+1}).$$

We now set the differentials to be

$$\delta^n = \sum_{i=1}^{n+1} (-1)^i [T\delta_0^i - \delta_1^i].$$

It is easy to show that the differentials satisfy $\delta^{n+1}\delta^n = 0$. We therefore define the n^{th}-cohomology group as usual and indicate them by $H^n_{TC}(X, A)$. We assume further that the map δ^0 is defined to be the canonical inclusion of the trivial group into $\Gamma^1(X, A)$, i.e. $H^1_{TC}(X, A) = \Gamma^1(X, A)$.

When $T = 1$, the groups $H^n_{TC}(X, A)$ are called (untwisted) continuous quandle cohomology groups and will be denoted $H^n_C(X, A)$.

Continuous cohomology groups in low dimensions take the following form.

Example 2.2.1. Let X be a topological quandle and (A, T) be a topological Alexander quandle. Then a continuous map $\eta : X \to A$ is a continuous 1-cocycle if it satisfies $T[\eta(y) - \eta(x)] - [\eta(y) - \eta(x * y)] = 0$, that is, η is a continuous quandle homomorphism, $\eta(x * y) = T\eta(x) + (1 - T)\eta(y)$. If, in particular, $T = 1$ and A is considered as a topological abelian group with trivial quandle structure, then the 1-cocycle condition is $\eta(x * y) = \eta(x)$.

A continuous map $\phi : X^2 \to A$ is a 2-cocycle if and only if it satisfies the condition:

$$T\phi(x_1, x_2) + \phi(x_1 * x_2, x_3) = T\phi(x_1, x_3) + (1 - T)\phi(x_2, x_3) + \phi(x_1 * x_3, x_2 * x_3)$$

and $\phi(x, x) = 0$. These considerations appear in [CES] except the requirement of continuity.

The notation that we will use throughout the rest of this chapter, to indicate the various type of cohomologies, is summarized as follows.

H_{Q}	:	Original (untwisted)	H_{T}	:	Original twisted
H_{C}	:	Continuous (untwisted)	H_{TC}	:	Continuous twisted
H_{GC}	:	Continuous generalized (quandle module)			

The cohomology H_{GC} is a continuous version of Andruskiewitsch-Grana's generalized quandle cohomology, see [AGn], and will be treated in Section 2.5.

2.3 First Continuous Cohomology Groups

We determine next, the first continuous cohomology group of certain topological quandles satisfying some suitable hypothesis. First, we have the following result for untwisted continuous cohomology.

Proposition 2.3.1. *Let X be a topological quandle and A be an Alexander topological qunadle. If X is indecomposable, then the first cohomology group $H^1_C(X, A)$ is isomorphic to A.*

Remark 2.3.2. We note the similarity between this result and the more traditional case of the first cohomology group of a path-connected topological space.

Proof. Let $x, x' \in X$ be arbitrary elements of X and let $f : X \to A$ be a 1-cocycle. By indecomposability of X there exist $y_1, \dots, y_n \in X$ such that $(\cdots(x *^{\epsilon_1} y_1) *^{\epsilon_2} \cdots *^{\epsilon_n} y_n) = x'$, such that $\epsilon_i = \pm 1$, where $*^{-1}$ is defined by $x *^{-1} y = z$ if $z * y = x$. Recall that in this case the 1-cocycle condition is $f(x * y) = f(x)$, which also implies $f(x *^{-1} y) = f(x)$. Therefore

$$f(x') = f(\cdots (x *^{\epsilon_1} y_1) *^{\epsilon_2} \cdots *^{\epsilon_n} y_n) = f(\cdots (x *^{\epsilon_1} y_1) *^{\epsilon_2} \cdots *^{\epsilon_{n-1}} y_{n-1})$$

where the second equality follows from the 1-cocycle condition for f. Inductively it follows that f is a constant map. On the other hand, any constant map satisfies the cocycle condition and is continuous, hence it is in $H^1_C(X, A) = Z^1_C(X, A)$. As a consequence there is a bijective correspondence between $H^1_C(X, A)$ and A that respects the group structures as the group operation of cocycles is pointwise. ■

We also have the following result regarding the first twisted continuous cohomology groups.

Proposition 2.3.3. *Let $X = (\mathbb{R}^n, S)$ and $A = (\mathbb{R}^m, T)$ be indecomposable Alexander quandles, where S, T are continuous additive automorphisms. Then $H^1_{TC}(X, A)$ is isomorphic to*

$$\{\ F + a : \mathbb{R}^n \to \mathbb{R}^m \mid a \in A,\ F \textit{ is linear, } FS = TF\ \}.$$

Proof. Since X and A are indecomposable, we have $I - S$ and $I - T$ invertible. Let $G \in H^1_{TC}(X, A)$. Then G is a continuous quandle homomorphism $G : X \to A$. Then for all $a \in A$, $G + a \in H^1_{TC}(X, A)$. For any $G \in H^1_{TC}(X, A)$, there is $a \in A$ such that $(G + a)(0) = 0$. By certain results due to E.W. Clark, see Appendix, we have that $F = G + a$ is linear, and $FS = TF$. Hence the result follows. ■

2.4 Second Continuous Cohomology Groups

Our next objective is to estabilish a bijective correspondence between extensions of topological quandles and second continuous cohomology groups, and utilize this result to determine families of topological quandles having non trivial second continuous cohomology groups. We start with the following

Definition 2.4.1. Assume we are given a topological quandle X and a topological Alexander quandle (A, T). For a continuous 2-cocycle $\psi \in Z^2_{TC}(X, A)$ (so that $\psi(x, x) = 0$ for all $x \in X$), a quandle structure is defined on $X \times A$ by

$$(x, a) * (y, b) = (x * y, a * b + \psi(x, y))$$

for all $x, y \in X$ and $a, b \in A$, as in [CENS]. The resulting quandle is denoted by $X \times_\psi A$ and called a *topological extension* of X by A.

Remark 2.4.1. The projection $\pi : X \times_\psi A \to X$ is a topological quandle homomorphism.

We define morphisms in the class of extensions of X by the abelian group A and, consequently, define an equivalence relation corresponding to the isomorphism classes. The class of extensions of X by A can be viewed therefore as a category. Consider two topological extensions $X \times_\psi A$ and $X \times_\phi A$ where ψ and ϕ are two 2-cocycles. A morphism $X \times_\psi A \to X \times_\phi A$ of extensions of X by A is a morphism of topological quandles $f : X \times_\psi A \to X \times_\phi A$ making the following diagram commute

$$\begin{array}{ccc} X \times_\psi A & \xrightarrow{f} & X \times_\phi A \\ \downarrow & & \downarrow \\ X & = & X \end{array}$$

In particular, if f is an isomorphism of topological quandles with the property of making the above diagram commute, it will be called an *isomorphism of topological extensions*. Two extensions are *equivalent* if there is an isomorphism f as above. We now prove the following result, analogous to the classification of the second cohomology group for group cohomology and the corresponding result for discrete quandles, as in [CENS, CES].

Proposition 2.4.2. *There is a bijective correspondence between equivalence classes of topological abelian extensions of X by A and the second cohomology group $H^2_{TC}(X, A)$ of X with coefficients in A.*

Proof. Although computations below are similar to those in [CES], we examine topological aspects of the argument. Assume $X \times_\psi A$ and $X \times_\phi A$ are two topological extension of X with ψ and ϕ cohomologus 2-cocycles (i.e. they differ by a coboundary). Consider the map $f : X \times A \to X \times A$, $(x, a) \mapsto (x, a + g(x))$, where $g : X \to A$ is such that $\delta g = \psi - \phi$. Since $g \in Z^1_{TC}(X, A)$, g is continuous, and so is f. We have

$$f((x, a) * (y, b)) = f(x * y, a * b + \phi(x, y)) = (x * y, a * b + \phi(x, y) + g(x * y)).$$

On the other hand we have

$$f(x,a) * f(y,b) = (x, a+g(x)) * (y, b+g(y)) = (x*y, a*b+g(x)*g(y)+\psi(x,y)).$$

These two terms are equal since $\phi = \psi + \delta g$, hence f is an isomorphism of quandles. Since it is also a homeomorphism and clearly makes the required diagram commute, we get that $X \times_\psi A$ and $X \times_\phi A$ are equivalent.

Conversely, assume $X \times_\psi A$ and $X \times_\phi A$ are equivalent. Say $f : X \times A \to X \times A$ is an isomorphism of topological extensions. Since, by definition, both $\pi(x,a) = x$ and $\pi(f(x,a)) = x$, the map f is determined by its second component. Using the group structure of A we can also write f as $f(x,a) = (x, a+g(x))$ for some map $g : X \to A$. Now the continuity of f implies the continuity of g. Since f is a morphism of quandles we get, for all $x, y \in X$ and all $a, b \in A$,

$$\begin{aligned}(x*y, a*b+g(x*y)+\psi(x,y)) &= f((x,a)*(y,b)) \\ &= f(x,a)*f(y,b) = (x*y, a*b+g(x)*g(y)+\phi(x,y)).\end{aligned}$$

Equating the second component, we find that ψ and ϕ differ by δg, i.e. they are representatives of the same cohomology class, since g is continuous. ■

Let $(G,+)$ be a topological abelian group. Consider G^m with the binary operation given by the rule

$$(a_1, \dots, a_m) * (b_1, \dots, b_m) = (a_1, a_2 + b_1 - a_1, \dots, a_m + b_{m-1} - a_{m-1}).$$

By direct computation we see that the operation just defined respects the defining axioms for a quandle structure and it is continuous, hence define a topological quandle structure on G^m.

Proposition 2.4.3. *Let $(G, +)$ be a topological abelian group, $x \neq 0$, and $(G^m, *)$ be the topological quandle defined as above. Then $H^2_{\mathrm{C}}(G^m, G) \neq 0$.*

Proof. Consider the following 2-cycle (in the usual sense of discrete homology):

$$\alpha = (0, \dots, 0) \times (x, 0, \dots, 0) + (0, x, 0, \dots, 0) \times (-x, x, 0, \dots, 1) \ ,$$

where $\times$ has been used to better indicate that α is an element of $G^m \times G^m$. By direct computation using the boundary map, it follows that α is indeed a 2-cycle. Consider also the 2-cocycle:

$$\phi : G^m \times G^m \longrightarrow G$$

defined by

$$\phi(\,(a_1, \dots, a_m) \times (b_1, \dots, b_m)\,) = b_m - a_m \ .$$

Again by direct computation using the coboundary map it can be shown that ϕ is a cocycle. Applying ϕ to α we get $\phi(\alpha) = x \neq 0$. The lemma below shows therefore that ϕ is not null-cohomologus, and we obtain $H^2_{\mathrm{C}}(G^m, G) \neq 0$. ■

Lemma 2.4.4. *Let X be a topological quandle, and (A, T) be a topological Alexander quandle. Let $\alpha \in Z^{\mathrm{T}}_n(X, A)$ be an n-cycle (in the usual sense of discrete homology), and $\phi \in Z^n_{\mathrm{TC}}(X, A)$ be a continuous n-cocycle. If $\phi(\alpha) \neq 0$, then $[\phi] \neq 0 \in H^n_{\mathrm{TC}}(X, A)$.*

Theorem 2.4.5. *Let $X = (\mathbb{R}^n, S)$ and $A = (\mathbb{R}^m, T)$ be Alexander quandles, where $S \in \mathrm{GL}_n(\mathbb{R})$ and $T \in \mathrm{GL}_m(\mathbb{R})$, respectively. Then $H^2_{\mathrm{TC}}(X, A) \neq 0$ if the following conditions hold for $k > 1$: $\sum_{i=0}^{k+1}(-S)^i = 0 = \sum_{i=0}^{k+1}(-T)^i$, and there exists an $n \times m$ matrix C such that $\sum_{\ell=0}^{k}(-T)^\ell C(\sum_{j=1}^{k-\ell+1}(-S)^j) \neq 0$.*

Proof. Let $w = \sum_{i=0}^{k} T^i(u_i, v_i)$. One computes

$$\begin{aligned}
\partial w &= \sum_{i=0}^{k} T^i[T(u_i) + (1-T)(v_i) - (Su_i + (1-S)v_i)] \\
&= [(v_0) - (Su_0 + (1-S)v_0)] \\
&\quad + \sum_{i=1}^{k-1} T^i[(u_{i-1}) - (v_{i-1}) + (v_i) - (Su_i + (1-S)v_i)] \\
&\quad + T^{k+1}[(u_k) - (v_k)].
\end{aligned}$$

By setting

$$v_1 = Su_0 + (1-S)v_0, \quad u_{i-1} = Su_i + (1-S)v_i, \quad v_j = v_{j-2}, \quad \text{and} \quad u_k = v_{k-1} \tag{2.1}$$

for $i = 1, \ldots, k$ and $j = 2, \ldots, k$, we obtain

$$\partial w = (\sum_{\ell=0}^{k+1} (-T)^\ell)[(v_0) - (v_1)].$$

Hence the condition (2.1) and the assumption $\sum_{\ell=0}^{k+1}(-T)^\ell = 0$ implies $\partial w = 0$.

For k odd, set

$$\begin{aligned}
u_{k-2i} &= (\sum_{j=2}^{2i+1} (-S)^j) u_0 + (1 - \sum_{j=2}^{2i+1} (-S)^j) v_0 \\
u_{k-(2i+1)} &= (-\sum_{j=1}^{2i+2} (-S)^j) u_0 + (\sum_{j=0}^{2i+2} (-S)^j) v_0
\end{aligned}$$

and for even k, set

$$\begin{aligned}
u_{k-2i} &= (-\sum_{j=1}^{2i+1} (-S)^j) u_0 + (\sum_{j=0}^{2i+1} (-S)^j) v_0 \\
u_{k-(2i+1)} &= (\sum_{j=2}^{2i+2} (-S)^j) u_0 + (1 - \sum_{j=2}^{2i+2} (-S)^j) v_0.
\end{aligned}$$

Then it is checked by induction that these satisfy Equations (2.1).

For $\phi(x_1, x_2) = C(x_1 - x_2)$ one computes

$$\phi(w) = \sum_{\ell=0}^{k} \phi(T^\ell(u_\ell, v_\ell)) = \sum_{\ell=0}^{k} T^\ell C(u_\ell - v_\ell) = \sum_{\ell=0}^{k} (-T)^\ell C(\sum_{j=1}^{k-\ell+1} (-S)^j)(u_0 - v_0)$$

as desired. The last equality is obtained by substituting the formulas for u_{k-2i} and $u_{k-(2i+1)}$

for each case of k odd and even. ■

Example 2.4.6. Let $n = 4$, $m = 2$, $S = T \oplus T$, and $T = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$. Then $S^2 - S + 1 = 0 = T^2 - T + 1$. Let $C = (I, I)$ where I is 2×2 identity matrix. Then $C - TCS = (I - S^2, I - S^2)$ is not the zero matrix, and applying Theorem 2.4.5 we obtain $H^2_{\mathrm{TC}}(X, A) \neq 0$.

2.5 Continuous Cohomology with Quandle Modules

The goal of this section is to introduce a topological version of the cohomology theory generalized in [AGn] and exhibit explicit examples with non-trivial continuous cohomology. We adapt the definition of quandle module, see the original paper [AGn]), to the topological case by requiring the triple (A, η, τ) to consist of a topological abelian group and continuous morphisms. In this setting, consider the abelian groups $\Gamma^n(X, A)$, δ_0^i and δ_1^i as defined above in Section 2.2. Define the differentials by the following formula

$$\begin{aligned} \delta^n \ := \ & \sum_{i=2}^{n+1} (-1)^i \left(\eta_{[x_1, \cdots, \widehat{x}_i, \cdots, x_{n+1}],[x_i, \cdots, x_{n+1}]} \delta_0^i - \delta_1^i \right) \\ & + \tau_{[x_2, x_3, \ldots, x_{n+1}],[x_1, x_3, \ldots, x_{n+1}]} \delta_0^1 \end{aligned}$$

where

$$[x_1, x_2, x_3, \ldots, x_n] = ((\cdots (x_1 * x_2) * x_3) \cdots) * x_n.$$

As in the discrete case, it is easy to see that we obtain a cochain complex

$$\cdots \to \Gamma^n(X, A) \xrightarrow{\delta} \Gamma^{n+1}(X, A) \to \cdots$$

The resulting cohomology groups are denoted by $H^n_{GC}(X, A)$.

Following [AGn], if X is a topological quandle and (A, η, τ) is a topological quandle module, we can define a topological quandle structure on $X \times A$ by

$$(x, a) * (y, b) = (x * y, \eta_{x,y}(a) + \tau_{x,y}(b) + \kappa_{x,y}),$$

for all $x, y \in X$ and $a, b \in A$, where $X \times A$ is given the product topology. This formula defines a topological quandle structure if and only if $\kappa_{x,y}$ is a 2-cocycle of this cohomology theory.

Similar definition and arguments as in Section 2.4 show that there is a bijective correspondence between the second generalized continuous cohomology group of X, with coefficients in A, and equivalence classes of extensions of X by A. We will leave the details to the reader.

Example 2.5.1. Let G be the subgroup of $\mathrm{GL}(2n, \mathbb{R})$ for a positive integer n, consisting of block matrices of the form

$$G = \left\{ E = \begin{pmatrix} S & O \\ C & T \end{pmatrix} \middle| \; S, T \in \mathrm{GL}(n, \mathbb{R}), C \in \mathrm{M}(n, \mathbb{R}) \right\},$$

where O denotes the zero matrix, and consider $X = G \times \mathbb{R}^n$ with quandle operation

$$(E_0, x_0) * (E_1, x_1) = (E_1 E_0 E_1^{-1}, S_1 x_0 + (I - S_1) x_1),$$

where $E_i = \begin{pmatrix} S_i & O \\ C_i & T_i \end{pmatrix}$ for $i = 0, 1$. Let $A = \mathbb{R}^n$ and consider endomorphisms of A defined by $\eta_{(E_0,x_0),(E_1,x_1)}(a) = T_1 a$ and $\tau_{(E_0,x_0),(E_1,x_1)}(a) = (I - T_1)a$. It is checked by computation that these define an X-module structure on A.

Theorem 2.5.2. *Let X and A be as in Example 2.5.1. Then we have*

$$H^2_{GC}(X, A) \neq 0$$

Proof. The quandle operation on $X \times A = G \times \mathbb{R}^{2n}$ defined by

$$(E_0, u_0) * (E_1, u_1) = (E_1 E_0 E_1^{-1}, E_1 u_0 + (I - E_1) u_1),$$

where $u_i = (x_i, a_i)$ and $a_i \in A$ for $i = 0, 1$ are computed as operation on $X \times A$ as

$$
\begin{aligned}
& [\ (E_0, x_0), a_0\ ] * [\ (E_1, x_1), a_1\ ] \\
&= \ [\ E_1 E_0 E_1^{-1}, S_1 x_0 + (I - S_1) x_1, T_1 a_0 + (I - T_1) a_1 + C_1(x_0 - x_1))].
\end{aligned}
$$

Let $p : X \times A \to X$ by $p(\ [(E, x), a)]\ ) = (E, x)$. Then we find that p defines the extension of X by the X-module A, with the 2-cocycle $\kappa_{(E_0,x_0),(E_1,x_1)} = C_1(x_0 - x_1)$.

We show that κ is not a coboundary. Let $E = \begin{pmatrix} -I & O \\ C & -I \end{pmatrix}$, and $w = [(E, x), (E, 0)] - [(E, -x), (E, 0)]$ be a 2-chain. Since $\partial(\ (x, y)\ ) = \eta_{x,y}(x) + \tau_{x,y}(y) - (x * y)$, one computes

$$
\begin{aligned}
\partial(w) &= (-I)(E, x) + (2I)(E, 0) - (E, (-I)x + (2I)0) \\
&- \ [(-I)(E, -x) + (2I)(E, 0) - (E, (-I)(-x) + (2I)0) \quad = \quad 0.
\end{aligned}
$$

Hence w is a 2-cycle. One also computes

$$
\begin{aligned}
\kappa_{(E,x),(E,0)} - \kappa_{(E,-x),(E,0)} &= C_1(x - 0) - C_1(-x - 0) \\
&= C_1(2x)
\end{aligned}
$$

and by choosing x, C_1 such that $C_1 x \neq 0$, we obtain that κ is not a coboundary by the argument similar to Lemma 2.4.4. ■

2.6 Continuous Cohomology vs Discrete Cohomology

Given a topological quandle X, there is an obvious forgetful functor $\mathcal{F} : \mathcal{TQ} \longrightarrow \mathcal{Q}$, from the category of topological quandles to the category of (discrete) quandles. Therefore, for a given topological abelian group A, associated to any topological quandles there are two type of cohmology. The classical discrete cohomology, $H^n_T(\mathcal{F}(X); A)$, where we also use the discrete topology on the group A, and the topological one, $H^n_{TC}(X; A)$. The main purpose of this section is to show that the two theories are different. In other words, there exists topological quandles having $H^n_T(\mathcal{F}(X); A) \neq H^n_{TC}(X; A)$.

Remark 2.6.1. We also observe that there is another obvious "discretization" functor $\mathcal{D} : \mathcal{Q} \longrightarrow \mathcal{TQ}$, that turns a discrete quandle into a topological one by endowing it with the discrete topology, and works on morphisms as the identity. It is clear that in this case, we have $H^n_{TC}(\mathcal{D}(X); A) = H^n_T(X; A)$, for all $n \in \mathbb{N}$.

Remark 2.6.2. The pair of functors $(\mathcal{F}, \mathcal{D})$ is an adjoint pair.

Let A be a topological abelian group and $p : E \to X$ be a principal A-bundle; a fiber bundle with a fiber preserving right action of A on E that acts freely and transitively.

Definition 2.6.1 (cf. [Eis])**.** Let E, X be connected topological quandles and A be a topological abelian group. A *principal (abelian) quandle extension by A* is a continuous surjective quandle homomorphism $p : E \to X$ that is a principal A-bundle such that for all $x, y \in X$ and $a \in A$, the following conditions hold:

(i) $(x * y) \cdot a = (x \cdot a) * (y \cdot a)$, (equivariance),

(ii) $(x \cdot a) * y = (x * y) \cdot a$. (commutativity of right actions).

In particular, the quandle homomorphism in Example 2.1.2 is a principal abelian quandle extension by $\mathbb{Z}_2$.

Lemma 2.6.3. *Let A be a topological abelian group and $p : E \to X$ be a principal abelian quandle extension by A. Let $s : X \to E$ be a set-theoretic section; $p \circ s = \mathrm{id}_X$. Then for all $x, y \in X$, there exists a unique element $a \in A$ such that $s(x) * s(y) = s(x * y) \cdot a$.*

Proof. Since p is a quandle homomorphism, we have

$$p(s(x) * s(y)) = (ps)(x) * (ps)(y) = x * y = (ps)(x * y).$$

Since A acts freely and transitively, there is a unique a such that $s(x) * s(y) = s(x * y) \cdot a$. ■

Remark 2.6.4. In the preceding lemma, the unique element a is determined by $x, y \in X$, so that we denote it by $a = \phi(x, y)$. Then we obtain a function $\phi : X \times X \to A$.

Lemma 2.6.5. *Let $\phi : X \times X \to A$ be defined as above. Then ϕ is a quandle (abelian) 2-cocycle.*

Proof. We perform the following computations analogous to those in [CENS] and [Eis]:

$$\begin{aligned}
(s(x) * s(y)) * s(z) &= [\, s(x * y) \cdot \phi(x, y) \,] * s(z) \\
&= [\, s(x * y) * s(z) \,] \cdot \phi(x, y)
\end{aligned}$$

$$
\begin{aligned}
&= s((x*y)*z)\cdot[\phi(x*y,z)\phi(x,y)],\\
(s(x)*s(z))*(s(y)*s(z)) &= [\,s(x*z)\cdot\phi(x,z)\,]*[\,s(y*z)\cdot\phi(y,z)\,]\\
&= ([\,s(x*z)\cdot\phi(x,z)\phi(y,z)^{-1}\,]*s(y*z))\cdot\phi(y,z)\\
&= (s(x*z)*s(y*z))\cdot[(\phi(x,z)\phi(y,z)^{-1})\phi(y,z)]\\
&= s((x*z)*(y*z))\cdot[(\phi(x*z,y*z)\phi(x,z)],
\end{aligned}
$$

and $s(x)*s(x)=s(x*x)\cdot\phi(x,x)$ gives $\phi(x,x)=0$. Hence ϕ satisfies the 2-cocycle condition.

$\blacksquare$

Remark 2.6.6 (Nosaka)**.** The argument works also for non-abelian groups A. See [AGn] for non-abelian 2-cocycles.

Example 2.6.7. Consider $p: S^2 \to \mathbb{RP}^2$ as in Example 2.1.2. Let

$$P_+ := \{(x,y,z)\in S^2 : z>0 \text{ or } z=0, y>0 \text{ or } y=z=0, x>0\}$$

and $P_- := S^2 \setminus P_+$. Let $s:\mathbb{RP}^2 \to S^2$ be a set-theoretic section defined by $s([x]) = x$ where $x \in P_+$. Then the map ϕ of the preceding lemma provides a non-zero 2-cocycle. For example, $\phi([1,0,0],[0,1,0]) = 1 \in \mathbb{Z}_2$. In this case, as a set S^2 is regarded as $\mathbb{RP}^2 \times \mathbb{Z}_2$.

Remark 2.6.8. Let $p: E \to X$ be a principal abelian quandle extension by A, and fix a set-theoretic section $s: X \to E$. For any given $u \in E$, let $x = p(u)$, then there is a unique $a = a_s(u)$ such that $u = s(x)\cdot a$. Similarly for $v \in E$ let $y = p(v)$ and $v = s(y)\cdot b$. Then one computes

$$u * v = (s(x) \cdot a) * (s(y) \cdot b) = [(s(x) \cdot (ab^{-1})) * s(y)] \cdot b = [s(x) * s(y)] \cdot a = s(x * y) \cdot (a\phi(x, y)).$$

Note that this equality $(s(x) \cdot a) * (s(y) \cdot b) = s(x * y) \cdot (a\,\phi(x, y))$ compares to the equality $(x, a) * (y, b) = (x * y, a + \phi(x, y))$ for $E = X \times_\phi A$ in the case $T = 1$.

Proposition 2.6.9. *$H^2_{\mathrm{Q}}(\mathbb{RP}^2, \mathbb{Z}_2) \neq 0$, yet $H^2_C(\mathbb{RP}^2, \mathbb{Z}_2) = 0$.*

Proof. Let ϕ be the quandle 2-cocycle constructed in Example 2.6.7. By Lemma 2.6.5 and Example 2.6.7, ϕ is a (discrete) quandle 2-cocycle, that yields a non-trivial extension, and therefore, ϕ is non-trivial in $H^2_{\mathrm{Q}}(\mathbb{RP}^2, \mathbb{Z}_2)$. Any continuous 2-cocycle gives rise to the trivial extension. Indeed, let ϕ be such a 2-cocycle. By continuity assumption, it is the constant map. But since $\phi(x, x) = 0$ for all x, it follows that ϕ is the zero map. Therefore $H^2_C(\mathbb{RP}^2, \mathbb{Z}_2) = 0$. ∎

Remark 2.6.10. The difference in Proposition 2.6.9 intercurring between the discrete and continuous cases can be interpreted in the following way. Since $\mathbb{RP}^2$ is not (homeomorphic to) a cartesian product, we can construct set theoretic sections, but these ought not to be continuous. Since a 2-cocycle with coefficients in $\mathbb{Z}_2$ gives rise to a section, a discrete 2-cocycle can be nonzero, but a topological 2-cocycle cannot.

2.7 Inverse Limits of Quandles and their Cohomology

In this section we introduce the notion of inverse and direct limits of quandles and show that, under suitable hypothesis, the continuous cohomology of an inverse limit is isomorphic to the direct limit of the cohomology groups of the components.

Suppose we are given a projective system of quandles $(X_n, \psi_n)_{n\in\mathbb{N}}$:

$$X_1 \xleftarrow{\psi_1} X_2 \xleftarrow{\psi_2} \cdots \longleftarrow X_n \xleftarrow{\psi_n} \cdots$$

where each ψ_n is a quandle morphism. We define the inverse limit of the projective system, $\varprojlim X_n$, as the subset of $\prod_{n\in\mathbb{N}} X_n$ of sequences $(x_0, x_1, \ldots, x_n, \ldots)$ satisfying $\psi_n(x_{n+1}) = x_n$ for all $n \geq 1$. We give $\varprojlim X_n$ the $*$ operation induced componentwise by the operations of the X_n. This construction together with the canonical projection maps $\varprojlim X_n \to X_i$, for each $i \in \mathbb{N}$, satisfies the usual universal property for an inverse limit of a projective system indexed by the natural numbers, see Remark 2.7.2 below.

The same construction can be defined for a projective system of topological quandles, where each morphism is now required to be continuous, and $\varprojlim X_n$ is endowed with the initial topology with respect to the projection maps. The initial topology is the coarsest topology that makes projections $p_i : \varprojlim X_n \to X_i$ continuous, and in our case, it is the same topology as the subspace topology of the product space.

Example 2.7.1. Fix a prime $p \in \mathbb{N}$. Put $X_n = \mathbb{Z}/p^n\mathbb{Z}$ together with the standard dihedral quandle operation $x * y = 2y - x$. There are canonical projections $\psi_n : X_{n+1} \to X_n$

obtained by reducing *mod* p^n a representative of a class modulo p^{n+1}. These maps are ring homomorphisms and, as a consequence, quandle morphisms, since the quandle structure on X_n is obtained from the ring operations. By definition, the inverse limit of this projective system is the ring of p-adic integers $\mathbb{Z}_p$ and it inherits a topological quandle structure from the dihedral quandles X_n. To be precise, the same quandle operation would be obtained on $\mathbb{Z}_p$ defining the Alexander quandle structure with $T = -1$, via the ring operations on $\mathbb{Z}_p$. That is, $(\varprojlim X_n, *)$ is isomorphic to $(\mathbb{Z}_p, -1)$.

Remark 2.7.2. More generally, if we start with a directed set I and a projective system of (topological) quandles $(X_i, \psi_{i,j})$, where the morphisms $\psi_{i,j}$ satisfy the usual compatibility relations, we can define

$$\varprojlim X_n = \{x \in \prod_{n\in I} X_n \mid \psi_{i,j}\pi_j(x) = \pi_i(x) \text{ for all } i, j \text{ with } j \geq i\}$$

where π_i is the canonical projection onto the i^{th} factor. They satisfy the universal property depicted below.

$$\begin{tikzcd}
& Y \arrow[ddl] \arrow[ddr] \arrow[d, dashed, "\exists!"] & \\
& \varprojlim X_n \arrow[dl] \arrow[dr] & \\
X_j \arrow[rr, "\psi_{i,j}"] & & X_i
\end{tikzcd}$$

Then we can endow it again with the $*$ operation induced pointwise by the quandle operations in each X_i and get an inverse limit for the projective system we started with. If we start with topological quandles, the topology of $\varprojlim X_i$ will be again the initial topology with respect to the projections. By definition it follows that the inverse limit of (topological) quandles, is the usual inverse limit in the category of topological spaces, equipped with a continuous binary operation that turns it into a topological quandle.

We have now the main result of the section.

Theorem 2.7.3. *Let (X_n, ψ_n) be a projective system of discrete quandles and (A_m, ϕ_m) a direct system of discrete abelian groups. Then ther cohomology groups $H_T^\bullet(X_n, A_n)$ can be arranged in an inductive system such that the following isomorphisms hold*

$$H^\bullet_{TC}(\varprojlim X_k, \varinjlim A_l) \cong \varinjlim H^\bullet_T(X_n, A_n).$$

Proof. We first define an inductive system of cohomology groups, whose direct limit is $\varinjlim H^\bullet_T(X_n, A_n)$. Below we suppress subscripts of cochain groups for simplicity. Define cochain maps $C^\bullet(X_n, A_n) \longrightarrow C^\bullet(X_{n+1}, A_{n+1})$ by the following diagram

$$\begin{tikzcd}
C^\bullet(X_n, A_n) \arrow[r, "\psi_n^*"] \arrow[dr, dashed, "\tau_n"] & C^\bullet(X_{n+1}, A_n) \arrow[d, "\phi_n\circ -"] \\
& C^\bullet(X_{n+1}, A_{n+1})
\end{tikzcd}$$

where the vertical map is the change of coefficients induced by ϕ_n and the horizontal map is the dual map of the projection ψ_n. We obtain consequently an inductive system in the category of groups. Consider now the following diagram

$$\begin{tikzcd}
 & \vdots \arrow[d, "\delta^{\bullet-1}"] & \vdots \arrow[d, "\delta^{\bullet-1}"] & \\
\cdots \arrow[r] & C^{\bullet}(X_n, A_n) \arrow[r, "\tau_n"] \arrow[d, "\delta^{\bullet}"] & C^{\bullet}(X_{n+1}, A_{n+1}) \arrow[r] \arrow[d, "\delta^{\bullet}"] & \cdots \\
\cdots \arrow[r] & C^{\bullet+1}(X_n, A_n) \arrow[r, "\tau_n"] \arrow[d, "\delta^{\bullet+1}"] & C^{\bullet+1}(X_{n+1}, A_{n+1}) \arrow[r] \arrow[d, "\delta^{\bullet+1}"] & \cdots \\
 & \vdots & \vdots &
\end{tikzcd}$$

where δ indicates the cohomology differentials and the maps τ are defined above. Since ψ_n^* and $\phi_n \circ -$ commute with differentials, the diagram is commutative, so that each τ_n induces a map on cohomology (which we will still denote by τ_n). Thus we obtain the inductive systems of cohomology groups:

$$H_T^{\bullet}(X_0, A_0) \xrightarrow{\tau_0} H_T^{\bullet}(X_1, A_1) \xrightarrow{\tau_1} \cdots \xrightarrow{\tau_{n-1}} H_T^{\bullet}(X_n, A_n) \xrightarrow{\tau_n} \cdots$$

from which we derive their inductive limits: $\varinjlim H_T^{\bullet}(X_n, A_n)$. Next, we construct a homomorphism of groups

$$\Psi : \varinjlim H_T^{\bullet}(X_n, A_n) \longrightarrow H_{TC}^{\bullet}(\varprojlim X_k, \varinjlim A_l)$$

Consider the following diagram, corresponding to any representative f of a class $[f] \in H_T^{\bullet}(X_n, A_n)$

$$\begin{tikzcd}
(\varprojlim X_k)^\bullet \arrow[r, dashed] \arrow[d, "\pi_n^\bullet"] & \varinjlim A_l \\
X_n^\bullet \arrow[r, "f"] & A_n \arrow[u, "\iota_n"']
\end{tikzcd}$$

where $\pi_n^\bullet$ is the canonical projection $(\varprojlim X_k)^\bullet \to X_n^\bullet$ and $\iota_n : A_n \to \varinjlim A_l$ is the natural morphism of A_n into the direct limit. Observe that if $\iota_n f \pi_n^\bullet$ is as above, then it factors through $X_n^\bullet$ by definition and, in particular, any preimage of a subset in $\varinjlim A_l$ is a basis element of the topology of $\varprojlim X_k^\bullet$ since each X_k is a discrete topological space and $\varprojlim X_k$ is endowed with the projective limit topology. Since $\varinjlim A_l$ is discrete, being a direct limit of discrete spaces; it follows then that $\iota_n f \pi_n^\bullet$ is continuous. Also, the correspondence $\{f : X_n^\bullet \to A_n\} \xrightarrow{\iota_n f \pi_n^\bullet} \{(\varprojlim X_k)^\bullet \to \varinjlim A_l\}$ respects equivalence classes, so it induces a well defined map $\sigma_n : H_T^\bullet(X_n, A_n) \to H_{TC}^\bullet(\varprojlim X_k, \varinjlim A_l)$. We obtain therefore the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H_T^\bullet(X_{n-1}, A_{n-1}) \arrow[r] \arrow[dr, "\sigma_{n-1}"] & H_T^\bullet(X_n, A_n) \arrow[r] \arrow[d, "\sigma_n"] & H_T^\bullet(X_{n+1}, A_{n+1}) \arrow[r] \arrow[dl, "\sigma_{n+1}"] & \cdots \\
& & H_{TC}^\bullet(\varprojlim X_k, \varinjlim A_l) & &
\end{tikzcd}$$

which can be seen to be commutative by a direct inspection. By the universal property of colimits we obtain a unique morphism $\varinjlim H_T^\bullet(X_n, A_n) \xrightarrow{\Psi} H_{TC}^\bullet(\varprojlim X_k, \varinjlim A_l)$. Our, last step is to prove that Ψ is indeed an isomorphism.

Suppose $\alpha \in \varinjlim H_T^\bullet(X_n, A_n)$ is mapped to zero by Ψ. By construction of Ψ, it means that there exists some $i \in \mathbb{N}$ such that $[\iota_i f \pi_i^\bullet]$ is the zero class in $H_{TC}^\bullet(\varprojlim X_k, \varinjlim A_l)$, where f is a representative of a class $[f] \in H_T^\bullet(X_i, A_i)$, $\pi_i^\bullet$ is the projection on the i^{th} factor and ι_i is the natural map $A_i \to \varinjlim A_l$. So $\iota_i f \pi_i^\bullet$ is a coboundary of some continuous $g : (\varprojlim X_k)^{\bullet -1} \to \varinjlim A_l$ which factors through some $X_j^{\bullet -1}$, $j \in \mathbb{N}$. Choosing $t \in \mathbb{N}$ large enough it follows that $[\iota_t f \pi_t^\bullet] = 0$ in $H_T^\bullet(X_t, A_t)$. Since $[\iota_i f \pi_i^\bullet] \sim [\iota_t f \pi_t^\bullet]$ in $\varinjlim H_T^\bullet(X_n, A_n)$ it follows that α is the zero class.

Suppose now we are given a class $[\beta] \in H_{TC}^\bullet(\varprojlim X_k, \varinjlim A_l)$. Since β is continuous and $\varinjlim A_l$ is discrete, it factors through some $X_i^\bullet$. Therefore its image in $\varinjlim A_l$ is finite and it will be contained in some A_j. Choosing $t \in \mathbb{N}$ large enough, we get that $[\beta]$ is the image $T\overline{[h]}$, for some $[h] \in H_T^\bullet(X_t, A_t)$, where the bar symbol indicates that we are considering a representative class in $\varinjlim H_T^\bullet(X_n, A_n)$. This concludes the proof. ■

Remark 2.7.4. Observe that the construction of the morphism Ψ and the proof above are still valid if we consider topological compact Hausdorff quandles X_n and replace each $H_T^\bullet(X_n, A_n)$ by their continuous counterparts. The proof depends indeed on categorical properties of direct and inverse limits, and a key factorization property of conitnuous maps between compact Hausdorff spaces and direct limits of discrete spaces. See [ESZ19], for a brief account pertaining this property.

As an application of Theorem 2.7.3, we are able to establish more computations of continuous cohomology groups. We first have the following

Example 2.7.5. Fix an odd prime $p \in \mathbb{Z}$ and choose $u \in \mathbb{Z}$ such that $(u, p) = 1$. Then multiplications by u and by $1 - u$ define automorphisms of $\mathbb{Z}/p^n\mathbb{Z}$, for any $n \in \mathbb{N}$. Thus we obtain an Alexander quandle structure on $\mathbb{Z}/p^n\mathbb{Z}$, which will be denoted by X_n^u. Also recall that there are natural maps $\mathbb{Z}/p^n\mathbb{Z} \longrightarrow \mathbb{Z}/p^{n+1}\mathbb{Z}$ given by multiplication by p, which commute with the action by $\mathbb{Z}$ and consequently a direct system, whose direct limit is the Prüfer group $\mathbb{Z}(p^\infty)$. Thus we have an Alexander quandle $(\mathbb{Z}(p^\infty), u)$, denoted also by $\mathbb{Z}(p^\infty)_u$.

Corollary 2.7.6. $H^1_{TC}(\varprojlim X^u_n, \mathbb{Z}(p^\infty)_u) \cong \mathbb{Z}(p^\infty)_u \times \mathbb{Z}(p^\infty)_u$ *for any* $u \in \mathbb{Z}$ *such that* $(u, p) = 1$.

Proof. As in Section 2.3 it is possible to show that the first twisted (discrete) cohomology group $H^1_T(X^u_n, \mathbb{Z}/p^n\mathbb{Z})$ is the abelian group of affine maps $\{f_{\alpha,\beta} : \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z} \mid f(x) = \alpha x + \beta,\ \alpha, \beta \in \mathbb{Z}/p^n\mathbb{Z}\}$ which can be seen to be isomorphic to $\mathbb{Z}/p^n\mathbb{Z} \times \mathbb{Z}/p^n\mathbb{Z}$ for all $n \in \mathbb{N}$, via the isomorphism $f_{\alpha,\beta} \mapsto \alpha \times \beta$. Using the definition of $\sigma_n : H^1_T(X^u_n, \mathbb{Z}/p^n\mathbb{Z}) \to H^1_T(X^u_{n+1}, \mathbb{Z}/p^{n+1}\mathbb{Z})$ we have that $(\sigma_n f)(x) = (pf\pi_n)[x] = p\alpha[x] + p\beta$, from which we obtain the direct limit

$$\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \to \cdots \to \mathbb{Z}/p^n\mathbb{Z} \times \mathbb{Z}/p^n\mathbb{Z} \to \cdots$$

where each map is just multiplication by p on each coordinate.

It follows that $\varinjlim H^1_T(X^u_n, \mathbb{Z}/p^n\mathbb{Z}) = \mathbb{Z}(p^\infty)_u \times \mathbb{Z}(p^\infty)_u$ and the result follows. ■

Corollary 2.7.7. $H^3_C(\varprojlim R_{p^n}, \mathbb{Z}/p\mathbb{Z}) = 0$, *where* R_{p^n} *denotes the dihedral quandle on* p^n *elements and the cohomology group is meant to be untwisted.*

Proof. For a given odd prime, it has been computed by Mochizuki [Moc], that $H^3(R_{p^n}, \mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$, where R_{p^n}. Directly from the proof in [Moc] it also follows that the map $H^3(R_{p^n}, \mathbb{Z}/p\mathbb{Z}) \to H^3(R_{p^{n+1}}, \mathbb{Z}/p\mathbb{Z})$ induced by the canonical projection $R_{p^{n+1}} \to R_{p^n}$ is the trivial map. We obtain the inductive system:

$$\mathbb{Z}/p\mathbb{Z} \xrightarrow{0} \mathbb{Z}/p\mathbb{Z} \xrightarrow{0} \cdots \xrightarrow{0} \mathbb{Z}/p\mathbb{Z} \xrightarrow{0} \cdots$$

whose direct limit is the trivial group. ■

CHAPTER 3 : HIGHER ARITY SELF-DISRIBUTIVE OPERATIONS

In this chapter we introduce new constructions regarding self-distributive operations of arbitrary arity and study the relations intercurring between their cohomology theories. We introduce a diagrammatic interpretation of higher arity self-distributivity and relate our constructions to diagrams of ribbon tangles. Our ultimate, though somehow elusive, goal is to introduce new (quantum) invariants of ribbon graphs and tangles via state-sums of Boltzmann weights of higher arity 2-cocycles. See [CJK$^+$, CEGnS] for the binary (quandle) analogue of this construction. It is therefore of crucial importance to find new ways to define higher order self-distributive operations and compute their (co)homology groups. We will introduce, partially after [Prz] and in parallel to [IIJ], certain classes of well behaved ternary self-distributive oeprations via the doubling procedure and relate their cohomology groups to those of the binary "building blocks" used to construct them. The procedure is general and it is seen to extend to higher arities. We also introduce a categorical version of self-dsitributivity making use of comonoidal objects in symmetric monoidal categories. This categorical generalization is expected to be particularly fruitful in the construction of interesting examples.

The present chapter is based on [ESZb].

3.1 Doubling Functor

In this section we introduce a functor that produces a "doubled" binary operation from two binary operations under certain assumptions. We can interpret this construction as the algebraic counterpart of doubling a string into a ribbon. Recall the following definition, due to J. Przytycki ([Prz]).

Definition 3.1.1. Let X be a set and $*_0$ and $*_1$ be two binary rack operations on X. We call the pair $(*_0, *_1)$ *mutually distributive* if $*_\epsilon$ is self-distributive for $\epsilon = 0, 1$, and the equalities $(x *_0 y) *_1 z = (x *_1 z) *_0 (y *_1 z)$ and $(x *_1 y) *_0 z = (x *_0 z) *_1 (y *_0 z)$ hold for all $x, y, z \in X$. We call $(X, *_0, *_1)$ a *mutually distributive rack.*

Przytycki calls $(X, *_0, *_1)$ a distributive set and defines this object for more then two operations. We provide a few examples of mutually distributive racks.

Example 3.1.1. Let $(X, *_X)$, $(Y, *_Y)$ be racks. Define $*_0, *_1$ on $X \times Y$, respectively, by $(x_0, y_0) *_0 (x_1, y_1) = (x_0 *_X x_1, y_0)$ and $(x_0, y_0) *_1 (x_1, y_1) = (x_0, y_0 *_Y y_1)$. Then computation shows that $(*_0, *_1)$ are mutually distributive.

Example 3.1.2. The following example appears in [II] and provides a way of constructing mutually distributive rack operations from a given rack. Denote by $*^n$ the rack operation on X defined by n-fold leftmost product $x *^n y = (\cdots (x * y) * y) * \cdots * y$. Then $*_0 = *^m$ and $*_1 = *^n$ are mutually distributive for positive integers m and n.

More generally, the following appears in [IIJ, Prz]. Let X be a group, and let $f_0, f_1 \in \mathrm{Aut}(X)$ be mutually commuting automorphisms. Let $*_\epsilon$ be the generalized Alexander quandles with respect to f_ϵ for $\epsilon = 0, 1$. Thus $x *_\epsilon y = (xy^{-1})^{f_\epsilon} y$, where the action is denoted in exponential notation. Then computations show that $*_0$ and $*_1$ are mutually distributive.

Lemma 3.1.3. *Let $(X, *_0, *_1)$ be mutually distributive racks. Define the operation for $(x_0, x_1), (y_0, y_1) \in X \times X$ by*

$$(x_0, x_1) * (y_0, y_1) := ((x_0 *_0 y_0) *_1 y_1, (x_1 *_0 y_0) *_1 y_1).$$

Then $(X \times X, *)$ is a rack.

Proof. We have

$$\begin{aligned}
&[(x_0, x_1) * (y_0, y_1)] * (z_0, z_1) \\
&= ((x_0 *_0 y_0) *_1 y_1, (x_1 *_0 y_0) *_1 y_1) * (z_0, z_1) \\
&= (([(x_0 *_0 y_0) *_1 y_1] *_0 z_0) *_1 z_1, ([(x_1 *_0 y_0) *_1 y_1] *_0 z_0) *_1 z_1) \\
&= ([((x_0 *_0 y_0) *_0 z_0) *_1 (y_1 *_0 z_0)] *_1 z_1, \\
&\qquad [((x_1 *_0 y_0) *_0 z_0) *_1 (y_1 *_0 z_0)] *_1 z_1) \\
&= ([(x_0 *_0 z_0) *_0 (y_0 *_0 z_0)) *_1 (y_1 *_0 z_0)] *_1 z_1], \\
&\qquad [((x_1 *_0 z_0) *_0 (y_0 *_0 z_0)) *_1 (y_1 *_0 z_0)] *_1 z_1) \\
&= ([\{(x_0 *_0 z_0) *_1 z_1\} *_0 \{(y_0 *_0 z_0) *_1 z_1\}] *_1 [(y_1 *_0 z_0) *_1 z_1], \\
&\qquad [\{(x_1 *_0 z_0) *_1 z_1\} *_0 \{(y_0 *_0 z_0) *_1 z_1\}] *_1 [(y_1 *_0 z_0) *_1 z_1])
\end{aligned}$$

$$= \quad ((x_0 *_0 z_0) *_1 z_1, (y_1 *_0 z_0) *_1 z_1) * ((y_0 *_0 z_0) *_1 z_1, (x_1 *_0 z_0) *_1 z_1)$$

$$= \quad [(x_0, x_1) * (z_0, z_1)] * [(y_0, y_1) * (z_0, z_1)].$$

The fact that the right multiplication is bijective is straightforward. ■

Remark 3.1.4. We observe that in Lemma 3.1.3 we have used both the equalities in Definition 3.1.1.

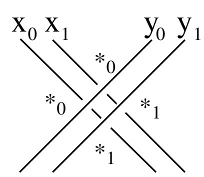

Figure 3.1: Diagrammatic representation of doubling

A diagrammatic representation of the preceding lemma is depicted in Figure 3.1, and the computations in its proof are facilitated by the corresponding Reidemeister type III move with doubled strings (i.e. ribbons).

Definition 3.1.2. Let $\mathcal{R}_M$ be the category defined as follows. The objects consist of $(X, *_0, *_1)$, where X is a set and $(*_0, *_1)$ is mutually distributive. For objects $(X, *_0, *_1)$ and $(X', *_0', *_1')$, a morphism f is a map $f : X \to X'$ that is a rack morphism for both $(*_0, *_0')$ and $(*_1, *_1')$.

We observe that if $f : X \to X'$ is a morphism in the sense of this definition, then f will automatically respect the mutual distributivity. Specifically, simple computations imply the following.

Lemma 3.1.5. *If $f:(X,*_0,*_1)\to(X',*'_0,*'_1)$ is a morphism in $\mathcal{R}_M$, then it holds that*

$$f((x *_0 y) *_1 z) = (f(x) *'_1 f(z)) *'_0 (f(y) *'_1 f(z)).$$

We also have the following result.

Lemma 3.1.6. *Let $(X,*_0,*_1)$ and $(X',*'_0,*'_1)$ be two mutually distributive racks, and $(X\times X,*)$ and $(X'\times X',*')$ be racks as in Lemma 3.1.3. If $f:(X,*_0,*_1)\to(X',*'_0,*'_1)$ is a morphism in $\mathcal{R}_M$, then the map $F:(X\times X,*)\to(X'\times X',*')$ defined by $F(x,y)=(f(x),f(y))$ is a rack morphism.*

Definition 3.1.3. The functor $\mathcal{D}_R$ from $\mathcal{R}_M$ to the category $\mathcal{R}$ of binary racks defined on objects by $\mathcal{D}_R(X,*_0,*_1)=(X\times X,*)$ through Lemma 3.1.3 and on morphisms by $\mathcal{D}_R(f)=f\times f$ through Lemma 3.1.6, is called the *doubling* functor.

Remark 3.1.7. The functor $\mathcal{D}_R$ is injective on objects and morphisms, but not surjective on either.

It is natural to ask if there is a relation between the cohomology groups of the mutually distributive racks $(X,*_0,*_1)$ and the cohomology of the double racks obtained via $\mathcal{D}_R$. Toward this direction, we introduce the following

Definition 3.1.4. Let $(X,*_0)$ and $(X,*_1)$ be two binary racks and let ϕ_0 be a 2-cocycle for $(X,*_0)$ and ϕ_1 be a 2-cocycle for $(X,*_1)$ both with coefficients in an abelian group A. We say that (ϕ_0,ϕ_1) is a pair of mutually distributive rack 2-cocycles, if the following two conditions are satisfied

$$\begin{aligned} \phi_0(x, y) + \phi_1(x *_0 y, z) &= \phi_1(x, z) + \phi_0(x *_1 z, y *_1 z), \\ \phi_1(x, y) + \phi_0(x *_1 y, z) &= \phi_0(x, z) + \phi_1(x *_0 z, y *_0 z). \end{aligned}$$

Example 3.1.8. Let $(X, *_X)$, $(Y, *_Y)$ be racks, and $(*_0, *_1)$ be mutually distributive operations defined on $X \times Y$ in Example 3.1.1. Let ϕ_X and ϕ_Y be 2-cocycles of $(X, *_X)$ and $(Y, *_Y)$, respectively. Define 2-cocycles of $X \times Y$ corresponding to $*_0$, $*_1$, respectively, by $\phi_0((x_0, y_0), (x_1, y_1)) = \phi_X(x_0, x_1)$ and $\phi_1((x_0, y_0), (x_1, y_1)) = \phi_Y(y_0, y_1)$. Then computations show that (ϕ_0, ϕ_1) are mutually distributive.

Example 3.1.9. The following construction is found in [II]. Let $(X, *)$ be a rack, $\phi : X \times X \to A$ be a 2-cocycle, and $(E = X \times A, \tilde{*})$ be the corresponding extension. Recall that $*^n$ denotes the n-fold leftmost product $x *^n y = (\cdots (x * y) * y) * \cdots * y$. Then the function ϕ_n defined by

$$\phi_n(x, y) = \phi(x, y) + \phi(x * y, y) + \cdots + \phi(x *^{n-1} y, y)$$

is a 2-cocycle.

Let $(X, *_0 = *^m, *_1 = *^n)$ be the mutually distributive rack defined in Example 3.1.2, and let ϕ_m, ϕ_n be 2-cocycles defined above. Then ϕ_m and ϕ_n are mutually distributive. This is seen by a diagrammatic interpretation of parallel strings.

We show next that given a pair of mutually distributive 2-cocycles, we can produce a 2-cocycle for the doubled rack. Specifically, we have the following result.

Theorem 3.1.10. *Let $(X, *_0, *_1)$ and $(X \times X, *)$ be as described in Lemma 3.1.3. Let ϕ_0, ϕ_1 be rack 2-cocycles of $(X, *_0)$ and $(X, *_1)$, respectively, that satisfy the mutually distributive rack 2-cocycle condition. Then*

$$\phi((x_0, x_1), (y_0, y_1)) = \phi_0(x_0, y_0) + \phi_1(x_0 *_0 y_0, y_1) + \phi_0(x_1, y_0) + \phi_1(x_1 *_0 y_0, y_1)$$

is a rack 2-cocycle of $(X \times X, *)$.

We defer the proof of Theorem 3.1.10 and prove instead a preliminary result, relating extensions of racks defined from mutually distributive 2-cocycles.

Lemma 3.1.11. *Let $(X, *_0, *_1)$ be a mutually distributive rack, and (ϕ_0, ϕ_1) be mutually distributive rack 2-cocycles. Let $(E, \tilde{*}_\epsilon)$ be abelian extensions of $(X, *_\epsilon)$ with respect to ϕ_ϵ,*

$$(x, a)\ \tilde{*}_\epsilon\ (y, b) = (x *_\epsilon y, a + \phi_\epsilon(x, y))$$

for $\epsilon = 0, 1$. Then $(E, \tilde{*}_0, \tilde{*}_1)$ is a mutually distributive rack.

Proof. We have

$$\begin{aligned}
&((x, a)\tilde{*}_0(y, b))\tilde{*}_1(z, c) \\
&= (x *_0 y, a + \phi_0(x, y))\tilde{*}_1(z, c) \\
&= ((x *_0 y) *_1 z, a + \phi_0(x, y + \phi_1(x *_0 y, z)))
\end{aligned}$$

$$
\begin{aligned}
&= ((x *_1 z) *_0 (y *_1 z), a + \phi_1(x, z) + \phi_0(x *_1 z, y *_1 z)) \\
&= ((x *_1 z), a + \phi(x, z)) *_0 (y *_1 z, b + \phi_1(y, z)) \\
&= ((x, a) *_1 (z, c)) *_0 ((y, b) *_1 (z, c)).
\end{aligned}
$$

A similar computation shows the second equality in Definition 3.1.4. ■

We construct then, a ternary version of the binary doubling functor. We start with the following definition, which can be considered as a ternary counterpart to Definition 3.1.1.

Definition 3.1.5. Let T_0 and T_1 be two ternary operations on a set X. We say that T_0 and T_1 are *compatible* if they satisfy

$$
\begin{aligned}
&T_0(T_0(x_0, y_0, y_1), z_0, z_1) \\
&\quad = T_0(T_0(x_0, z_0, z_1), T_0(y_0, z_0, z_1), T_1(y_1, z_0, z_1)), \\
&T_1(T_1(x_1, y_0, y_1), z_0, z_1) \\
&\quad = T_1(T_1(x_1, z_0, z_1), T_0(y_0, z_0, z_1), T_1(y_1, z_0, z_1)).
\end{aligned}
$$

Doubling of ternary operations can be interpreted diagrammatically as a crossing of ribbons and the compatibility conditions correspond to a ribbon Redemeister move III. We depict this ideas in Figure 3.2.

Example 3.1.12. Consider a Λ-module M where $\Lambda = \mathbb{Z}[t^{\pm 1}, t'^{\pm 1}, s, s']$. The following two ternary operation $T_0(x, y, z) = tx + sy + (1 - t - s)z$ and $T_1(x, y, z) = t'x + s'y + (1 - t' - s')z$

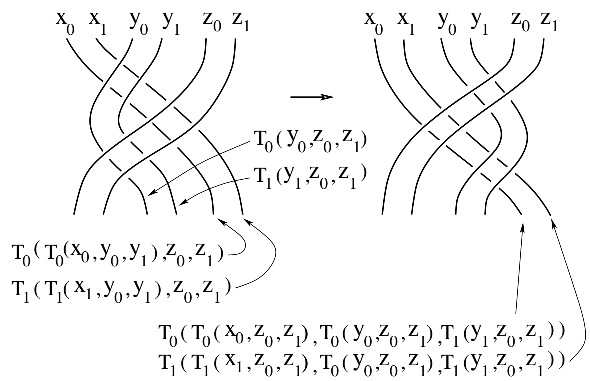

Figure 3.2: Diagrammatic representation of compatible ternary rack operations

are compatible if and only if the following conditions hold

$$\begin{cases} (1-t-s)(t'-t)=0 \\ (1-t-s)(s'-s)=0 \end{cases} \quad \text{and} \quad \begin{cases} (1-t'-s')(t-t')=0 \\ (1-t'-s')(s-s')=0\ . \end{cases}$$

For example, one can choose $M = \mathbb{Z}_8$ with $T_0(x,y,z) = 3x+2y+4z$ and $T_1(x,y,z) = -x+2y$.

Theorem 3.1.13. *Let (T_0, T_1) be compatible ternary distributive operations on X. Then $T : X^2 \times X^2 \times X^2 \to X^2$ defined by*

$$\begin{aligned} &T((x_0,x_1),(y_0,y_1),(z_0,z_1)) \\ &= \ (T_0(T_0(x_0,y_0,y_1),z_0,z_1), T_1(T_1(x_1,y_0,y_1),z_0,z_1)) \end{aligned}$$

is a ternary distributive operation on X^2.

Proof. We need to establish

$$
\begin{aligned}
&T(T((x_0,x_1),(y_0,y_1),(z_0,z_1)),(u_0,u_1),(v_0,v_1))\\
&= \; T(\; T((x_0,x_1),(u_0,u_1),(v_0,v_1)),\\
&\qquad\qquad T((y_0,y_1),(u_0,u_1),(v_0,v_1)),T((z_0,z_1),(u_0,u_1),(v_0,v_1))\;).
\end{aligned}
$$

A diagrammatic representation of this equality is depicted in Figure 3.3. This diagrammatic equality follows from a sequence of moves as in Figure 3.2. Thus calculations are obtained by applications of defining relations of compatibility accordingly. ■

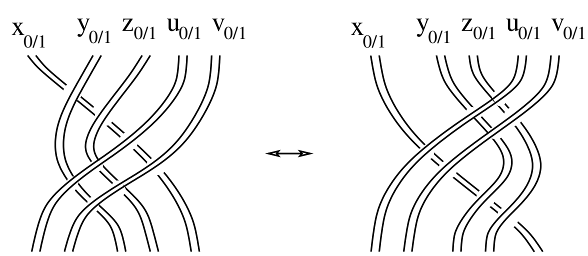

Figure 3.3: Diagrammatic representation of Theorem 3.1.13

Definition 3.1.6. The category $\mathcal{T}_C$ of compatible ternary distributive racks is defined as follows. The objects consist of triples (X,T_0,T_1) where X is a set and (T_0,T_1) are compatible ternary operations on X. A morphism between two objects (X,T_0,T_1) and (Y,T_0',T_1') is a map $f:X\to Y$ which is morphism in the ternary category for both (T_0,T_0') and (T_1,T_1').

Observe that if (X,T_0,T_1) and (X',T_0',T_1') are mutually distributive racks and $f: X\longrightarrow X'$ is a morphism according to Definition 3.1.6, then f automatically respects the mutual distributivity conditions. This is the content of the next Lemma.

Lemma 3.1.14. *If $f : (X, T_0, T_1) \to (X', T_0', T_1')$ is a morphism in $\mathcal{T}_M$, then it holds that*

$$f(T_0(T_0(x_0, y_0, y_1), z_0, z_1)) =$$

$$T_0'(T_0'(f(x_0), f(z_0), f(z_1)), T_0'(f(y_0), f(z_0), f(z_1)), T_1'(f(y_1), f(z_0), f(z_1))),$$

$$f(T_1(T_1(x_0, y_0, y_1), z_0, z_1)) =$$

$$T_1'(T_1'(f(x_0), f(z_0), f(z_1)), T_0'(f(y_0), f(z_0), f(z_1)), T_1'(f(y_1), f(z_0), f(z_1))).$$

Proof. A direct computation. ■

The following is analogous to Lemma 3.1.6 and is shown by direct computations.

Lemma 3.1.15. *Let (X, T_0, T_1) and (X', T_0', T_1') be sets with mutually distributive ternary operations, and $(X \times X, T)$ and $(X' \times X', T')$ be ternary distributive racks constructed in Theorem 3.1.13. If $f : (X, T_0, T_1) \to (X', T_0', T_1')$ is a morphism in $\mathcal{T}_C$, then F defined from f by $f \times f$ is a morphism of $\mathcal{T}_C$.*

Definition 3.1.7. We denote the functor from $\mathcal{T}_M$ to the category of ternary racks defined on objects by $\mathcal{D}_T(X, T_0, T_1) = (X \times X, T)$ and on morphisms by $\mathcal{D}_T(f) = f \times f$, and call it *doubling*.

Remark 3.1.16. The functor $\mathcal{D}_T$ is injective on both objects and morphisms, but is not surjective on either.

Definition 3.1.8. Let (T_0, T_1) be compatible ternary distributive operations on X. Let ψ_0, ψ_1 be 2-cocycles with respect to T_0 and T_1, respectively. Then the following are called the compatibility conditions for ψ_0 and ψ_1:

$$\begin{aligned}
&\psi_0(x_0, y_0, y_1) + \psi_1(T_1(x_1, y_0, y_1), z_0, z_1) \\
&\quad = \quad \psi_1(x_1, z_0, z_1) + \psi_0(T_0(x_0, z_0, z_1), T_0(y_0, z_0, z_1), T_1(y_1, z_0, z_1)), \\
&\psi_1(x_1, y_0, y_1) + \psi_0(T_0(x_0, y_0, y_1), z_0, z_1) \\
&\quad = \quad \psi_0(x_0, z_0, z_1) + \psi_1(T_1(x_0, z_0, z_1), T_0(y_0, z_0, z_1), T_1(y_1, z_0, z_1)).
\end{aligned}$$

We assert now that given a pair of compatible ternary 2-cocycles, as in Definition 3.1.8 above, we can construct a 2-cocycle of the doubled tenrary rack. Specifically, we have the following result.

Theorem 3.1.17. *Let (T_0, T_1) be compatible ternary distributive operations on X. Let T be the doubled ternary operation defined in Theorem 3.1.13. Let ψ_0, ψ_1 be 2-cocycles with respect to T_0 and T_1, respectively, that satisfy the compatibility condition defined in Definition 3.1.8. Then*

$$\begin{aligned}
&\psi((x_0, x_1), (y_0, y_1), (z_0, z_1)) \\
&\quad = \quad \psi_0(x_0, y_0, y_1) + \psi_1(x_1, y_0, y_1) \\
&\qquad\quad + \psi_0(T_0(x_0, y_0, y_1), z_0, z_1) + \psi_1(T_1(x_1, y_0, y_1), z_0, z_1)
\end{aligned}$$

is a ternary rack 2-cocycle of $(X \times X, T)$.

We call ψ the *doubled ternary rack 2-cocycle.* We will prove Theorem 3.1.17 along with Theorem 3.1.10 in Section 3.4.

3.2 From Binary Racks to Ternary Racks

In this section, we construct a functor from the category of mutually distributive racks $\mathcal{R}_M$, as in Definition 3.1.2, to the category of ternary racks. We start with the assignments on objects.

Lemma 3.2.1. *Let $(X, *_0, *_1)$ be a mutually distributive rack. Then the operation T given by*

$$T(x, y_0, y_1) := (x *_0 y_0) *_1 y_1$$

is a ternary distributive operation, that is,

$$T(T(x, y_0, y_1), z_0, z_1) = T(T(x, z_0, z_1), T(y_0, z_0, z_1), T(y_1, z_0, z_1)).$$

Proof. We have

$$\begin{aligned} & T(T(x, y_0, y_1), z_0, z_1) \\ = \quad & [[(x *_0 y_0) *_1 y_1] *_0 z_0] *_1 z_1 \\ = \quad & [[(x *_0 y_0) *_0 z_0] *_1 (y_1 *_0 z_0)] *_1 z_1 \\ = \quad & [[(x *_0 z_0) *_0 (y_0 *_0 z_0)] *_1 (y_1 *_0 z_0)] *_1 z_1 \\ = \quad & ([(x *_0 z_0) *_0 (y_0 *_0 z_0)] *_1 z_1) *_1 [(y_1 *_0 z_0) *_1 z_1] \end{aligned}$$

$$= \; ([(x *_0 z_0) *_1 z_1] *_0 [(y_0 *_0 z_0) *_1 z_1]) *_1 [(y_1 *_0 z_0) *_1 z_1]$$

$$= \; T(T(x, z_0, z_1), T(y_0, z_0, z_1), T(y_1, z_0, z_1))$$

where the second and the fifth equalities follow from the mutual distributivity of $*_0$ and $*_1$. ■

Remark 3.2.2. Note that the order in which the two operations $*_0$ and $*_1$ in Lemma 3.2.1 appear is important. In other words, the two ternary structures $T(x, y, z) = (x *_0 y) *_1 z$ and $T'(x, y, z) = (x *_1 y) *_0 z$ may not be isomorphic in general, as the following example shows.

Consider the set $\mathbb{Z}_3$ with the two binary operations $x *_0 y = x$ and $x *_1 y = 2y - x$. The induced ternary structures $T(x, y, z) = (x *_0 y) *_1 z$ and $T'(x, y, z) = (x *_1 y) *_0 z$ are not isomorphic. In fact, if $f : (\mathbb{Z}_3, T) \to (\mathbb{Z}_3, T')$ is an isomorphism then for all x, y, z in $\mathbb{Z}_3$, we have $f(T(x, y, z)) = T'(f(x), f(y), f(z))$. Then $f(2z - x) = 2f(y) - f(x)$. One obtains then a contradition, for example, by setting $x = z = 0$.

Definition 3.2.1. The assignment on objects defined by Lemma 3.2.1 is denoted by $\mathcal{F} : \mathcal{R}_M \to \mathcal{T}$, where $\mathcal{F}(X, *_0, *_1) = (X, T)$.

Let $f : (X, *_0, *_1) \to (X', *'_0, *'_1)$ be a morphism of mutually distributive sets. Define an assignment on morphisms by $\mathcal{F}(f) = f$. Lemma 3.1.5 implies that $\mathcal{F}$ is a functor.

By definition $\mathcal{F}$ is injective and surjective on morphisms.

Proposition 3.2.3. *Let $\mathcal{Q}_M$ denote the subcategory of $\mathcal{R}_M$ restricted on quandles. Then the functor $\mathcal{F}|_{\mathcal{Q}_M}$ is not surjective on objects.*

Proof. Let $(X,T) = \mathcal{F}(X, *_0, *_1)$, where $(X, *_0, *_1)$ is a set with mutually distributive quandle operations. Then for all $x, y \in X$, it holds that

$$T(x,x,y) = (x *_0 x) *_1 y = x *_1 y = (x *_0 y) *_1 y = T(x,y,y).$$

On the other hand, a heap (X, T'), where X is a group and $T'(x,y,z) = xy^{-1}z$, satisfies $T'(x,x,y) = y$ and $T'(x,y,y) = x$, so that it is not in the image of $\mathcal{F}$ for a non-trivial group X. ■

The functor $\mathcal{F}$ and extensions commute in the following sense.

Proposition 3.2.4. *Let $(X, *_0, *_1)$ be a mutually distributive rack, and (ϕ_0, ϕ_1) be mutually distributive rack 2-cocycles. Let $(E, \tilde{*}_0, \tilde{*}_1)$ be a mutually distributive rack that is an abelian extension with respect to (ϕ_0, ϕ_1) obtained by Lemma 3.1.11. Let $\mathcal{F}(X, *_0, *_1) = (X, T)$, and ψ be the ternary 2-cocycle obtained in Theorem 3.2.5. Let $(E, \tilde{T})$ be the abelian extension with respect to ψ obtained in Lemma* **??**. *Then we have $\mathcal{F}(E, \tilde{*}_0, \tilde{*}_1) = (E, \tilde{T})$.*

Proof. Suppose ϕ_0 and ϕ_1 are 2-cocycles with coefficients in an abelian group A. Let $\tilde{*}_0$ and $\tilde{*}_1$ be the binary distributive structures obtained on $X \times A$ from ϕ_0 and ϕ_1, respectively. Then, by Lemma 3.1.11, $(X \times A, \tilde{*}_0, \tilde{*}_1)$ is a mutually distributive rack. We want to show that the ternary structure obtained on $X \times A$ by applying $\mathcal{F}$, is the same as the ternary structure obtained on $X \times A$ as a ternary extension, with cocycle ψ as in Theorem 3.2.5 and T obtained from $*_0$ and $*_1$. By direct computation, $\mathcal{F}(E, \tilde{*}_0, \tilde{*}_1)$ has ternary operation given by:

$$
\begin{aligned}
&\tilde{T}((x,a),(y_0,b_0),(y_1,b_1)) \\
&\quad = \quad ((x,a)\ \tilde{*}_0\ (y_0,b_0))\ \tilde{*}_1\ (y_1,b_1) \\
&\quad = \quad ((x *_0 y_0) *_1 y_1, a + \phi_0(x,y_0) + \phi_1(x *_0 y_0, y_1)) \\
&\quad = \quad (T(x,y_0,y_1), a + \psi(x,y_0,y_1)),
\end{aligned}
$$

which is the ternary abelian extension on $\mathcal{F}(X,*_0,*_1)$ corresponding to the cocycle ψ. ■

The situation of the proposition above is represented by the following commutative diagram.

$$
\begin{tikzcd}
(E,\tilde{*}_0,\tilde{*}_1) \arrow[r, "\mathcal{F}"] & (E,\tilde{T}) \\
(X,*_0,*_1) \arrow[u, "{(\phi_0,\phi_1)}"] \arrow[r, "\mathcal{F}"] & (X,T) \arrow[u, "\psi"']
\end{tikzcd}
$$

It is natural to demand a way to define a correspondence between binary 2-cocycles of mutually distributive racks, and ternary 2-cocycles of the ternary self-distributive structure they produce via the functor $\mathcal{F}$. This is the content of the main result of this section, Theorem 3.2.5 below.

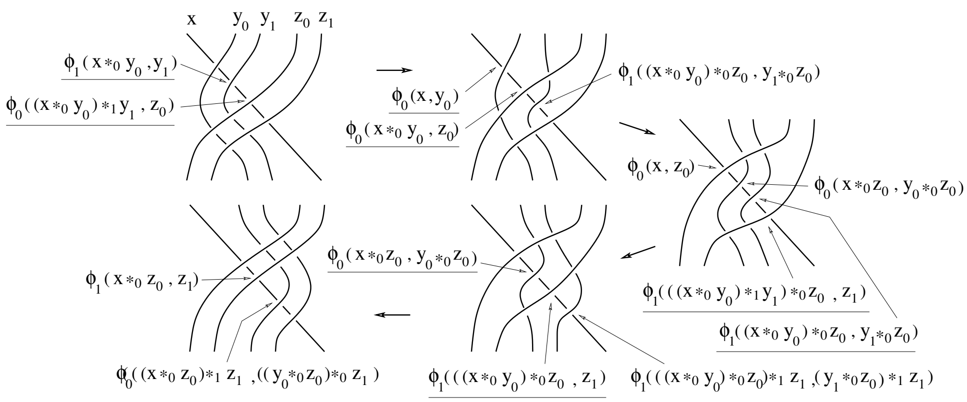

Figure 3.4: Diagrammatic proof of 2-cocycle conditions

Theorem 3.2.5. *Let* $(X, *_0, *_1)$ *be a mutually distributive rack, and* (ϕ_0, ϕ_1) *be mutually distributive rack 2-cocycles. Then* $\psi(x, y, z)$ *given by*

$$\psi(x, y, z) := \phi_0(x, y) + \phi_1(x *_0 y, z)$$

is a ternary 2*-cocycle for the ternary distributive set* $(X, T) = \mathcal{F}(X, *_0, *_1)$.

Proof. It is sufficient to check that the map ψ satisfies the following equation

$$\psi(x, y_0, y_1) + \psi(T(x, y_0, y_1), z_0, z_1) =$$
$$\psi(x, z_0, z_1) + \psi(T(x, z_0, z_1), T(y_0, z_0, z_1), T(y_1, z_0, z_1)).$$

The computations below are aided by diagrams shown in Figure 3.4, where each equality is represented by a type III Reidemeister move. In the figure and the computations below, underlines highlight those terms to which the cocycle condition is applied. We compute

$$\begin{aligned}
\text{LHS} &= \phi_0(x, y_0) + \underline{\phi_1(x *_0 y_0, y_1) + \phi_0((x *_0 y_0) *_1 y_1), z_0)} \\
&\quad +\phi_1((x *_0 y_0) *_1 y_1) *_0 z_0, z_1) \\
&= \underline{\phi_0(x, y_0) + \phi_0(x *_0 y_0, z_0)} + \phi_1((x *_0 y_0) *_1 y_1) *_0 z_0, z_1) \\
&\quad +\phi_1((x *_0 y_0) *_0 z_0, y_1 *_0 z_0) \\
&= \phi_0(x, z_0) + \phi_0(x *_0 z_0, y_0 *_0 z_0) + \underline{\phi_1((x *_0 y_0) *_0 z_0) *_0 z_0, y_1 *_0 z_0)} \\
&\quad +\underline{\phi_1(((x *_0 y_0) *_1 y_1) *_0 z_0, z_1)} \\
&= \phi_0(x, z_0) + \underline{\phi_0(x *_0 z_0, y_0 *_0 z_0) + \phi_1((x *_0 y_0) *_0 z_0), z_1)} \\
&\quad +\phi_1((x *_0 y_0) *_0 z_0) *_1 z_1, (y_1 *_0 z_0) *_1 z_1)
\end{aligned}$$

$$\begin{aligned}
&= \phi_0(x, z_0) + \phi_1(x *_0 z_0, z_1) \\
&\quad +\phi_1((x *_0 y_0) *_0 z_0) *_1 z_1, (y_1 *_0 z_0) *_1 z_1) \\
&\quad +\phi_0((x *_0 z_0) *_0 z_0) *_1 z_1, (y_1 *_0 z_0) *_1 z_1) \quad = \quad \text{RHS}
\end{aligned}$$

as desired. ■

The result given by Theorem 3.2.5. We would like to construct a map from the binary rack second cohomology groups of $(X, *_0)$ and $(X, *_1)$ to the tarnary second cohomology group of $\mathcal{F}(X, *_0, *_1)$. A first hindrance to such a construction is that in general, Theorem 3.2.5 defines a map from a subset of $C^2((X, *_0); A) \times C^2((X, *_1); A)$ to $C^2_T(\mathcal{F}(X, *_0, *_1); A)$, domain of which is not part of a (co)chain complex in a natural way. We now construct a new chain complex, from the binary rack complexes, that encodes the compatibility conditions between pairs of rack 2-cocycles and enables us to obtain Theorem 3.2.5 as part of a more general construction.

We construct a complex by taking the direct sum of n copies of $C^n(X;A) = \{f : X^n \to A\}$ for all n, and combining the rack differentials corresponding to $*_0$ and $*_1$ according to certain rules. The situation is better described by means of the diagram below

$$\begin{tikzcd}
\vdots & & & & \\
C^4 \arrow[u, "\delta_1"] \arrow[r, "\delta_0"] & C^5 & & & \\
C^3 \arrow[u, "\delta_1"] \arrow[r, "\delta_0"] & C^4 \arrow[u, "\delta_1"] \arrow[r, "\delta_0"] & C^5 & & \\
C^2 \arrow[u, "\delta_1"] \arrow[r, "\delta_0"] & C^3 \arrow[u, "\delta_1"] \arrow[r, "\delta_0"] & C^4 \arrow[u, "\delta_1"] \arrow[r, "\delta_0"] & C^5 & \\
C^1 \arrow[u, "\delta_1"] \arrow[r, "\delta_0"] & C^2 \arrow[u, "\delta_1"] \arrow[r, "\delta_0"] & C^3 \arrow[u, "\delta_1"] \arrow[r, "\delta_0"] & {} \arrow[u, "\delta_1"] \arrow[r, "\delta_0"] & \cdots
\end{tikzcd}$$

where we take direct sum along diagonals at 135 degrees, δ_i stands for the rack differential of the rack $(X, *_i)$ and the signs corresponding to each differential will be described in detail below.

Remark 3.2.6. Observe that starting from C^1 and proceeding vertically, we have just compositions of δ_1 we obtain the cohomology of $(X, *_1)$. Similarly horizontally we obtain the cohomology of $(X, *_0)$.

Recall the following Definition.

Definition 3.2.2 ([Prz]). Let $*_j$, $j = 1, \dots, k$, be distributive binary operations on X that are pairwise mutually distributive. Then we call $(X, \{*_j\}_{j=1}^k)$ a mutually distributive set.

We first introduce a chain complex and its associated homology, whose dualized cochain complex corresponds to the cohomology sought for. This chain complex generalizes those found in [EGM, IIJ, Prz].

Definition 3.2.3. Let $(X, \{*_j\}_{j=1}^k)$ be a mutually distributive set. Let $\vec{\epsilon} = (\epsilon_1, \dots, \epsilon_{n-1})$ be a vector such that $\epsilon_i \in \{j\}_{j=1}^k$ for $i = 1, \dots, n-1$. Let chain groups $C_n^{\vec{\epsilon}}(X)$ be defined by the free abelian group generated by tuples $\mathbf{x} = (x_0, (x_1, \epsilon_1), \dots, (x_{n-1}, \epsilon_{n-1}))$. Define $C_n(X) = \oplus_{\vec{\epsilon}}\, C_n^{\vec{\epsilon}}(X)$ where the direct sum ranges over all possible vectors $\vec{\epsilon}$. Define the differential $\partial_n^{\vec{\epsilon}} : C_n^{\vec{\epsilon}}(X) \to C_{n-1}(X)$ by

$$\begin{aligned}
\partial_n^{\vec{\epsilon}}(\mathbf{x}) &= \sum_{i=2}^{n-1} (-1)^i [(x_0 *_{\epsilon_i} x_i, (x_1 *_{\epsilon_i} x_i, \epsilon_1), \dots, (x_{i-1} *_{\epsilon_i} x_i, \epsilon_{i-1}), \widehat{(x_i, \epsilon_i)}, \\
&\qquad\qquad (x_{i+1}, \epsilon_{i+1}), \dots, (x_{n-1}, \epsilon_{n-1})) \\
&\qquad - (x_0, (x_1, \epsilon_1), \dots, \widehat{(x_i, \epsilon_i)}, \dots, (x_{n-1}, \epsilon_{n-1}))],
\end{aligned}$$

and let

$$\partial_n = \sum_{\vec{\epsilon}} \partial_n^{\vec{\epsilon}} : C_n(X) \to C_{n-1}(X).$$

In Section 3.5 we show that the differentials given in Definition 3.5.3 indeed satisfy the condition $\partial_{n-1}\partial_n = 0$.

Lemma 3.2.7. *Let $(X, \{*_j\}_{j=1}^k)$ be a mutually distributive set. Then the sequence $(C_n(X), \partial_n)$ defines a chain complex.*

Proof. The proof is the binary case of the higher arity (vector) version given in Section 3.5. ■

Definition 3.2.4. The chain complex defined by Definition 3.5.3 and the homology that it induces will be called labeled chain complex and labeled homology and will be denoted $C^L_\bullet(X)$ and $H^L_\bullet(X)$, respectively.

Remark 3.2.8. The chain complex in Definition 3.5.3 has a diagrammatic interpretation as in Figure 3.5. In particular, the mutual distributivity condition, takes the same form as in the curtain homology of [PW].

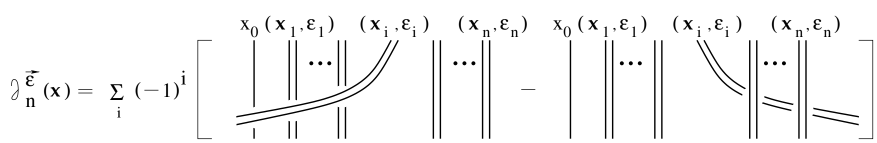

Figure 3.5: Curtain diagram representing chain maps

Remark 3.2.9. For a given abelian group A, we obtain a labeled cochain complex with coefficients in A, upon dualizing the chain complex in Definition 3.5.3. We will write $C^n_L(X; A)$ and $H^n_L(X; A)$ to indicate the labeled n^{th} cochain and cohomology groups with coefficients in A, respectively. We observe that the cochain complex encodes the compatibility conditions, as described in the next proposition.

Proposition 3.2.10. *Let $(X, *_0, *_1)$ be a mutually distributive rack and let $C^2_L(X; A)$ be the second labeled cochain group with coefficients in A, as in Remark 3.2.9. Then the labeled 2-cocycle conditions corresponding to $\delta^{(01)}\psi = 0$ and $\delta^{(10)}\psi = 0$ are equivalent to the mutual distributive rack 2-cocycle condition in Definition 3.1.4.*

Proof. By definition, $C^L_2(X)$ splits in the direct sum of labeled cycles. Dualizing, a 2-cocycle $\psi \in C^2_L(X; A)$ is a pair (ϕ_0, ϕ_1), where $\phi_i \in \mathrm{Hom}(C^{(i)}_2, A)$, $i = 0, 1$. Similarly, $C^L_3(X)$ consists of a direct sum of four terms labeled by vectors (00), (01), (10) and (11). It follows therefore that $\delta\psi$ consists of four summands, obtained by dualizing the labeled differential and precomposing with each of the two components of ψ. Specifically, the component corresponding to the differential $\partial^{(01)}$ reads

$$\begin{aligned}\delta^{(01)}\psi(x, (y, *_0), (z, *_1)) &= \phi_1(x *_0 y, (z, *_1)) - \phi_1(x, (z, *_1)) \\ &\quad - \phi_0(x *_1 z, (y *_1 z, *_0)) + \phi_0(x, (y, *_0)).\end{aligned}$$

This gives us the first condition in Definition 3.1.4. Similarly, from $\delta^{(ii)}$ we obtain the 2-cocycle condition for ϕ_i with respect to the rack $(X, *_i)$, and $\delta^{(10)}$ gives the second equation in Definition 3.1.4. ■

Definition 3.2.5. We define maps $\mathcal{F}_{\sharp,n} : C^T_n(X) \longrightarrow C^L_n(X)$, from the tenrary cochain complex, to the chain complex defined by Lemma 3.5.5 for $n = 1, 2, 3$. Explicitly:

$$\begin{aligned}
\mathcal{F}_{\sharp,1} &= \mathbf{1} \\
\mathcal{F}_{\sharp,2}(x, y_0, y_1) &= (x, y_0)_0 + (x *_0 y_0, y_1)_1 \\
\mathcal{F}_{\sharp,3}(x, y_0, y_1, z_0, z_1) &= (x, y_0, z_0)_{00} + (x *_0 z_0, y_0 *_0 z_0, z_1)_{01} + \\
&\quad (x *_0 y_0, y_1, z_0)_{10} + ((x *_0 y_0) *_0 z_0, y_1 *_0 z_0, z_1)_{11},
\end{aligned}$$

where we put the labels as a subscript.

Definition 3.2.6. Let $(X, *_0, *_1)$ be a mutually distributive racks. Let $\mathcal{F}^{\sharp,n} : C^n_L(X) \to C^n_T(X)$ for $n = 2, 3$ be the maps obtained from $\mathcal{F}_{\sharp,n}$ by dualization.

Lemma 3.2.11. *For $n = 2, 3$ the maps $\mathcal{F}_{\sharp,n}$ define chain maps. Therefore they define induced homomorphisms $\mathcal{F}_{\sharp,n} : H^T_n(X) \to H^L_n(X)$ in homology and $\mathcal{F}^{*,n} : H^n_L(X) \to H^n_T(X)$ in cohomology.*

Proof. For a ternary 2-chain (x, y_0, y_1) we have:

$$\partial\mathcal{F}_{\sharp,2}(x, y_0, y_1) =$$

$$-(x *_0 y_0) + (x) - ((x *_0 y_0) *_1 y_1) + (x *_0 y_0) = \partial_T(x, y_0, y_1).$$

By direct computation, we also have:

$$
\begin{aligned}
&\mathcal{F}_{\sharp,2}\partial_T(x, y_0, y_1, z_0, z_1)\\
&= (x, z_0)_0 + (x *_0 z_0, z_1)_1 - (T(x, y_0, y_1), z_0)_0\\
&\quad -(T(x, y_0, y_1) *_0 z_0, z_1)_1 - (x, y_0)_0 - (x *_0 y_0, y_1)_1\\
&\quad +(T(x, z_0, z_1), T(y_0, z_0, z_1)_0\\
&\quad +(T(x, z_0, z_1) *_0 T(y_0, z_0, z_1), T(y_1, z_0, z_1))_1.
\end{aligned}
$$

On the other hand, the following holds:

$$
\begin{aligned}
&\partial\mathcal{F}_{\sharp,3}(x, y_0, y_1, z_0, z_1)\\
&= (x, z_0)_0 - (x *_0 y_0, z_0)_0 - (x, y_0)_0\\
&\quad +(x *_0 z_0, y_0 *_0 z_0)_0 + (x *_0 z_0, z_1)_1 - ((x *_0 z_0) *_0 (y_0 *_0 z_0), z_1)_1\\
&\quad -(x *_0 z_0, y_0 *_0 z_0)_0 + (T(x, z_0, z_1), T(y_0, z_0, z_1))_0\\
&\quad +(x *_0 y_0, z_0)_0 - (T(x, y_0, y_1), z_0)_0\\
&\quad -(x *_0 y_0, y_1)_1 + ((x *_0 y_0) *_0 z_0, y_1 *_0 z_0)_1\\
&\quad +((x *_0 y_0) *_0 z_0, z_1)_1 - (((x *_0 y_0) *_0 z_0) *_1 (y_1 *_0 z_0), z_1)_1\\
&\quad -((x *_0 y_0) *_0 z_0, y_1 *_0 z_0)_1 + (((x *_0 y_0) *_0 z_0) *_1 z_1, T(y_1, z_0, z_1))_1.
\end{aligned}
$$

The two quantities can be seen to be equal, making use of the identity:

$$T(x, z_0, z_1) *_0 T(y_0, z_0, z_1) = ((x *_0 y_0) *_0 z_0) *_1 z_1.$$

Therefore we obtain $\mathcal{F}_{\sharp,2}\partial_T = \partial\mathcal{F}_{\sharp,3}$, which concludes the proof of the first statement. The second statement follows easily from the first one by standard arguments in homological algebra. ■

We can conclude the section with a refined version of Theorem 3.2.5.

Theorem 3.2.12. *The construction given in Theorem 3.2.5 induces a well defined map between second cohomology groups $H^2_L(X; A)$ and $H^2_T(X; A)$.*

Proof. In virtue of Lemma 3.2.11, it is enough to show that the map $\mathcal{F}^{\sharp,2} : C^2_L(X) \to C^2_T(X)$ in Definition 3.2.6 coincide with the map ψ defined in Theorem 3.2.5. This follows from a direct inspection. ■

Remark 3.2.13. We observe that Lemma 3.2.11 actually serves two purposes. On the one hand it formalizes the construction in Theorem 3.2.5 providing a map of second cohomology groups. On the other hand it also provides a version of the construction for third cohomology groups as well.

3.3 From Ternary Racks to Binary Racks

In this section we present a construction of a rack structure on the product $X \times X$ from ternary distributive operations (T_0, T_1) on X, and describe the functor that arises from it.

Lemma 3.3.1. *Let T_0 and T_1 be two compatible ternary rack operations. Then the binary operation on the cartesian product $X \times X$ defined by*

$$(x_0, x_1) * (y_0, y_1) := (T_0(x_0, y_0, y_1), T_1(x_1, y_0, y_1)) = (x_0 *_0 \mathbf{y}, x_1 *_1 \mathbf{y})$$

gives a rack structure $(X \times X, *)$.

Proof. We have the distributivity as follows

$$\begin{aligned}
&[(x_0, x_1) * (y_0, y_1)] * (z_0, z_1) \\
&= (T_0(x_0, y_0, y_1), T_1(x_1, y_0, y_1)) * (z_0, z_1) \\
&= (T_0(T_0(x_0, y_0, y_1), z_0, z_1), T_1(T_1(x_1, y_0, y_1), z_0, z_1)) \\
&= (T_0[T_0(x_0, z_0, z_1), T_0(y_0, z_0, z_1), \\
&\qquad T_1(y_1, z_0, z_1)], T_1[T_1(x_1, z_0, z_1), \\
&\qquad\quad T_0(y_0, z_0, z_1), T_1(y_1, z_0, z_1)]) \\
&= (T_0(x_0, z_0, z_1), T_1(x_1, z_0, z_1)) * (T_0(y_0, z_0, z_1), T_1(y_1, z_0, z_1)) \\
&= [(x_0, x_1) * (z_0, z_1)] * [(y_0, y_1) * (z_0, z_1)].
\end{aligned}$$

The invertibility of the map $R_{(y_0,y_1)} : X^2 \to X^2$ comes from the fact that, for given $y_0, y_1 \in X$, the maps $T_i(-, y_0, y_1)$ from X to X sending u to $T_i(u, y_0, y_1)$ are invertible for $i = 0, 1$. ■

Definition 3.3.1. The functor defined by Lemma 3.3.1 is denoted by $\mathcal{G} : \mathcal{T}_C \to \mathcal{R}$, where $\mathcal{G}(X, T_0, T_1) = (X \times X, *)$ on objects, and $\mathcal{G}(f) = f \times f$ on morphisms.

Observe that $\mathcal{G}$ is injective on objects and on morphisms.

Proposition 3.3.2. *the functor $\mathcal{G}$ is not surjective on objects.*

Proof. Consider the binary rack structure on $\mathbb{Z} \times \mathbb{Z}$ defined by

$$(x_0, x_1) * (y_0, y_1) = (x_0 + x_1, x_1).$$

This rack is not in the image of $\mathcal{G}$ since the first entry depends on both x_0 and x_1. ■

Theorem 3.3.3. *Let (X, T_0, T_1) be an object in $\mathcal{T}_C$, and $(X \times X, *) = \mathcal{G}(X, T_0, T_1)$ be as in Lemma 3.3.1. Suppose ψ_0 and ψ_1 are compatible ternary 2-cocycles of respectively (X, T_0) and (X, T_1). Then*

$$\phi((x_0, x_1), (y_0, y_1)) := \psi_0(x_0, y_0, y_1) + \psi_1(x_1, y_0, y_1)$$

defines a 2-cocycle ϕ of $(X \times X, *)$.

Proof. We check that ϕ satisfies the following equation

$$\begin{aligned}
&\phi((x_0,x_1),(y_0,y_1)) + \phi((x_0,x_1)*(y_0,y_1),(z_0,z_1)) \\
= \;& \phi((x_0,x_1),(z_0,z_1)) + \phi((x_0,x_1)*(z_0,z_1),(y_0,y_1)*(z_0,z_1)).
\end{aligned}$$

We have

$$\begin{aligned}
\text{LHS} &= \psi_0(x_0,y_0,y_1) + \psi_1(x_1,y_0,y_1) + \\
&\quad \psi_0(T_0(x_0,y_0,y_1),z_0,z_1) + \psi_1(T_1(x_1,y_0,y_1),z_0,z_1), \\
\text{RHS} &= \psi_0(x_0,z_0,z_1) + \psi_1(x_1,z_0,z_1) + \\
&\quad \psi_0(T_0(x_0,z_0,z_1),T_0(y_0,z_0,z_1),T_1(y_1,z_0,z_1)) + \\
&\quad \psi_1(T_1(x_1,z_0,z_1),T_0(y_0,z_0,z_1),T_1(y_1,z_0,z_1)).
\end{aligned}$$

The compatibility conditions of ψ_0 and ψ_1 show that LHS and RHS coincide. ■

The functors $\mathcal{F}, \mathcal{G}, \mathcal{D}_R$ and $\mathcal{D}_T$ are not indipendent, as the proposition below shows.

Proposition 3.3.4. *It holds that $\mathcal{G} \circ \mathcal{F} = \mathcal{D}_R$ and $\mathcal{F} \circ \mathcal{G} = \mathcal{D}_T$.*

Proof. Let $(X, *_0, *_1)$ be a set with mutually distributive rack operations. Let $(X,T) = \mathcal{F}(X, *_0, *_1)$. Then by definition $T(x, y_0, y_1) = (x *_0 y_0) *_1 y_1$. Lemma 3.3.1 implies that

$(X \times X, *) = \mathcal{G}(X, T, T)$ is a rack, since T is mutually distributive over itself. One computes

$$\begin{aligned} \mathcal{G}(X,T,T) &= (x_0, x_1) * (y_0, y_1) \\ &= (T(x_0, y_0, y_1), T(x_1, y_0, y_1)) \\ &= ((x_0 *_0 y_0) *_1 y_1, (x_1 *_0 y_0) *_1 y_1) \\ &= \mathcal{D}_R(X, *_0, *_1) \end{aligned}$$

as desired.

Let (X, T_0, T_1) be a set with mutually distributive ternary rack operations. Let $(X \times X, *) = \mathcal{G}(X, T_0, T_1)$. Then by definition $(x_0, x_1) * (y_0, y_1) = (T_0(x_0, y_0, y_1), T_1(x_1, y_0, y_1))$. Since $*$ is mutually distributive over itself, Lemma 3.2.1 implies that $(X \times X, T) = \mathcal{F}(X \times X, *, *)$ is a rack. One computes

$$\begin{aligned} \mathcal{F}(X \times X, *, *) &= T((x_0, x_1), (y_0, y_1), (z_0, z_1)) \\ &= [(x_0, x_1) * (y_0, y_1)] * (z_0, z_1) \\ &= (T_0(x_0, y_0, y_1), T_1(x_1, y_0, y_1)) * (z_0, z_1) \\ &= (T_0(T_0(x_0, y_0, y_1), z_0, z_1), T_1(T_1(x_1, y_0, y_1), z_0, z_1) \\ &= \mathcal{D}_T(X, T_0, T_1) \end{aligned}$$

as desired. ■

3.4 Proofs of Theorem 3.1.10 and Theorem 3.1.17

In this section we provide the long delayed proofs of Theorem 3.1.10 and Theorem 3.1.17. The relations intercurring between the funcotrs $\mathcal{F}, \mathcal{G}, \mathcal{D}_R$ and $\mathcal{D}_T$ as in Proposition 3.3.4, play a central role in the proofs.

Proof of Theorem 3.1.10. Let $(*_0, *_1)$ be mutually distributive rack operations on X. Let $(X, T) = \mathcal{F}(X, *_0, *_1)$. By Lemma 3.2.1, (X, T) is a ternary rack. Let ϕ_0, ϕ_1 be mutualy distributive rack 2-cocycles of $(X, *_0)$ and $(X, *_1)$, respectively. Then by Theorem 3.2.5,

$$\psi(x, y_0, y_1) := \phi_0(x, y_0) + \phi_1(x *_0 y_0, y_1)$$

is a ternary rack 2-cocycle of (X, T). Since T is compatible over itself,

$$\begin{aligned}
&(\mathcal{G} \circ \mathcal{F})(X, *_0, *_1)((x_0, x_1), (y_0, y_1), (z_0, z_1)) \\
&= \mathcal{G}(X \times X, T, T)((x_0, x_1), (y_0, y_1), (z_0, z_1)) \\
&= (T(T(x_0, y_0, y_1), z_0, z_1), T(T(x_1, y_0, y_1), z_0, z_1)
\end{aligned}$$

is a rack operation by Theorem 3.1.13. Then Theorem 3.3.3 applied to $(X \times X, T, T)$ with mutually distributive cocycles (ψ, ψ) implies that

$$
\begin{aligned}
&\phi((x_0, x_1), (y_0, y_1)) \\
&\quad = \quad \psi(x_0, y_0, y_1) + \psi(x_1, y_0, y_1) \\
&\quad = \quad \phi_0(x, y_0) + \phi_1(x *_0 y_0, y_1) + \phi_0(x_1, y_0) + \phi_1(x_1 *_0 y_0, y_1)
\end{aligned}
$$

as desired. ■

Proof of Theorem 3.1.17. Let (T_0, T_1) be compatible ternary distributive operations on X, and $(X \times X, *) = \mathcal{G}(X, T_0, T_1)$. By Lemma 3.3.1, $(X \times X, *)$ is a rack. Let ψ_0, ψ_1 be compatible ternary 2-cocycles of (X, T_0) and (X, T_1), respectively. Then by Theorem 3.3.3,

$$\phi((x_0, x_1), (y_0, y_1)) := \psi_0(x_0, y_0, y_1) + \psi_1(x_1, y_0, y_1)$$

is a rack 2-cocycle of $(X \times X, *)$. Since $*$ is mutually distributive over itself,

$$
\begin{aligned}
&(\mathcal{F} \circ \mathcal{G})(X, T_0, T_1)((x_0, x_1), (y_0, y_1), (z_0, z_1)) \\
&\quad = \quad T((x_0, x_1), (y_0, y_1), (z_0, z_1)) \\
&\quad = \quad [(x_0, x_1) * (y_0, y_1)] * (z_0, z_1)
\end{aligned}
$$

is a ternary rack operation by Lemma 3.2.1. Then Theorem 3.2.5 applied to $(X \times X, *, *)$

with mutually distributive cocycles (ϕ, ϕ) implies that

$$
\begin{aligned}
& \psi((x_0, x_1), (y_0, y_1), (z_0, z_1)) \\
= \quad & \phi((x_0, x_1), (y_0, y_1)) + \phi((x_0, x_1) * (y_0, y_1), (z_0, z_1)) \\
= \quad & \phi((x_0, x_1), (y_0, y_1)) + \phi((T_0(x_0, y_0, y_1), T_1(x_1, y_0, y_1)), (z_0, z_1)) \\
= \quad & \psi_0(x_0, y_0, y_1) + \psi_1(x_1, y_0, y_1) \\
& + \psi_0((T_0(x_0, y_0, y_1), z_0, z_1) + \psi_1(T_1(x_1, y_0, y_1)), z_0, z_1)
\end{aligned}
$$

as desired. ■

3.5 General n-Ary Compositions

In this section we generalize compositions of mutual distributive operations to n-ary cases. We recall the vector notation for n-ary operations given in the Introduction. Let (X, W) be an n-ary distributive set. Let $\mathbf{y} = (y_1, \ldots, y_{n-1}) \in X^{n-1}$. Then the operation $W : X^n \to X$ is denoted by $W(x, y_1, \ldots, y_{n-1}) = W(x, \mathbf{y})$. An n-ary operation is also denoted by $x * \mathbf{y} := W(x, \mathbf{y})$. Here the extra parentheses caused by the vector notation is ignored, i.e., for $\mathbf{y} = (y_1, \ldots, y_{n-1})$ and $\mathbf{z} = (z_1, \ldots, z_{n-1})$, the concatenation $(\mathbf{y}, \mathbf{z})$ or simply $\mathbf{y}, \mathbf{z}$ denotes $(y_1, \ldots, y_{n-1}, z_1, \ldots, z_{n-1})$. Furthermore, for $\mathbf{x} = (x_1, \ldots, x_m) \in X^m$ and $\mathbf{y} \in X^{n-1}$, denote $(W(x_1, \mathbf{y}), \ldots, W(x_m, \mathbf{y}))$ by $W(\mathbf{x}, \mathbf{y})$ or $\mathbf{x} * \mathbf{y}$.

Definition 3.5.1. Let W_m and W_n be m-ary and n-ary distributive operations on X, respectively. The two operations W_m and W_n are called *mutually distributive* if they satisfy

$$\begin{aligned} W_n(W_m(x,\mathbf{y}),\mathbf{z}) &= W_m(W_n(x,\mathbf{z}),W_n(\mathbf{y},\mathbf{z})) \\ W_m(W_n(x,\mathbf{u}),\mathbf{v}) &= W_n(W_m(x,\mathbf{v}),W_m(\mathbf{u},\mathbf{v})) \end{aligned}$$

for all $x \in X$, $\mathbf{y},\mathbf{v} \in X^{m-1}$ and $\mathbf{z},\mathbf{u} \in X^{n-1}$.

Example 3.5.1. Let X be a module over $\mathbb{Z}[u^{\pm 1}, t^{\pm 1}, s]$ and $*$, T be affine binary and ternary rack operations, respectively, defined by

$$\begin{aligned} x * y &= ux + (1-u)y, \\ T(x,y,z) &= tx + sy + (1-t-s)z. \end{aligned}$$

Then computations show that $*$ and T are mutually distributive.

Proposition 3.5.2. *Let W_m and W_n be mutually distributive m-ary and n-ary distributive operations on X. Then $W : X^{m+n-1} \to X$ defined by*

$$W(x,\mathbf{y},\mathbf{z}) = W_n(W_m(x,\mathbf{y}),\mathbf{z})$$

is an $(m+n-1)$-ary distributive operation.

Proof. We establish the equality

$$W(W(x,\mathbf{y},\mathbf{z}),\mathbf{u},\mathbf{v}) = W(W(x,\mathbf{u},\mathbf{v}),W(\mathbf{y},\mathbf{u},\mathbf{v}),W(\mathbf{z},\mathbf{u},\mathbf{v})).$$

We replace $W_n(x,\mathbf{y})$ by the notation $x *_n \mathbf{y}$. Thus we have

$$W(x,\mathbf{y},\mathbf{z}) := (x *_m \mathbf{y}) *_n \mathbf{z}.$$

Then we compute

$$\begin{aligned}
& W(W(x,\mathbf{y},\mathbf{z}),\mathbf{u},\mathbf{v}) \\
& = [[(x *_m \mathbf{y}) *_n \mathbf{z}] *_m \mathbf{u}] *_n \mathbf{v} \\
& = ([(x *_m \mathbf{y}) *_m \mathbf{u}] *_n [\mathbf{z} *_m \mathbf{u}]) *_n \mathbf{v} \\
& = [[(x *_m \mathbf{u}) *_m (\mathbf{y} *_m \mathbf{u})] *_n (\mathbf{z} *_m \mathbf{u})] *_n \mathbf{v} \\
& = [[(x *_m \mathbf{u}) *_m (\mathbf{y} *_m \mathbf{u})] *_n \mathbf{v}] *_n [(\mathbf{z} *_m \mathbf{u}) *_n \mathbf{v}] \\
& = [[(x *_m \mathbf{u}) *_n \mathbf{v}] *_m [(\mathbf{y} *_m \mathbf{u}) *_n \mathbf{v}]] *_n [(\mathbf{z} *_m \mathbf{u}) *_n \mathbf{v}]
\end{aligned}$$

$$= W(W(x,\mathbf{u},\mathbf{v}), W(\mathbf{y},\mathbf{u},\mathbf{v}), W(\mathbf{z},\mathbf{u},\mathbf{v})),$$

where the second and the fifth equalities follow from the mutual distributivity of $*_m$ and $*_n$. This concludes the proof. ■

Remark 3.5.3. We note that for a group G, the core binary operation $(x * y = yx^{-1}y)$ and the ternary operation heap $(x \hat{*} (y_0, y_1) = xy_0^{-1}y_1)$ satisfy $(x * y) \hat{*} \mathbf{z} = (x \hat{*} \mathbf{z}) * (y \hat{*} \mathbf{z})$ but not $(x \hat{*} \mathbf{y}) * z = (x * z) \hat{*} (\mathbf{y} * z)$.

Definition 3.5.2. Let $*_{n_j}$, $j = 1, \ldots, k$, be distributive n_j-ary operations on X that are pairwise mutually distributive. Then we call $(X, \{*_{n_j}\}_{j=1}^k)$ a mutually distributive set.

Computations give the following.

Lemma 3.5.4. *Let $\{*, *_0, *_1\}$ be a mutually distributive binary set. Let $(X, T) = \mathcal{F}(X, *_0, *_1)$. Then $\{*, T\}$ are mutually distributive.*

Recall the following definition.

Definition 3.5.3. Let $(X, \{*_{n_j}\}_{j=1}^k)$ be a mutually distributive set. Let $\vec{\epsilon} = (\epsilon_1, \ldots, \epsilon_{n-1})$ be a vector such that $\epsilon_i \in \{n_j\}_{j=1}^k$ for $i = 1, \ldots, n-1$. Let chain groups $C_n^{\vec{\epsilon}}(X)$ be defined by the free abelian group generated by tuples $\mathbf{x} = (x_0, (\mathbf{x}_1, \epsilon_1), \ldots, (\mathbf{x}_{n-1}, \epsilon_{n-1}))$. Define $C_n(X) = \oplus_{\vec{\epsilon}}\, C_n^{\vec{\epsilon}}(X)$ where the direct sum ranges over all possible vectors $\vec{\epsilon}$. Define the differential $\partial_n^{\vec{\epsilon}} : C_n^{\vec{\epsilon}}(X) \to C_{n-1}(X)$ by

$$\begin{aligned} \partial_n^{\vec{\epsilon}}(\mathbf{x}) \;=\; & \sum_{i=2}^{n-1} (-1)^i [(x_0 *_{\epsilon_i} \mathbf{x}_i, (\mathbf{x}_1 *_{\epsilon_i} \mathbf{x}_i, \epsilon_1), \ldots, (\mathbf{x}_{i-1} *_{\epsilon_i} \mathbf{x}_i, \epsilon_{i-1}), \widehat{(\mathbf{x}_i, \epsilon_i)}, \\ & \qquad (\mathbf{x}_{i+1}, \epsilon_{i+1}), \ldots, (\mathbf{x}_{n-1}, \epsilon_{n-1})) \\ & \qquad - (x_0, (\mathbf{x}_1, \epsilon_1), \ldots, \widehat{(\mathbf{x}_i, \epsilon_i)}, \ldots, (\mathbf{x}_{n-1}, \epsilon_{n-1}))], \end{aligned}$$

and let

$$\partial_n = \sum_{\vec{\epsilon}} \partial_n^{\vec{\epsilon}} : C_n(X) \to C_{n-1}(X).$$

We now prove that the construction in Definition 3.5.3 gives a chain complex.

Lemma 3.5.5. *Let $(X, \{*_{n_j}\}_{j=1}^k)$ be a mutually distributive set. Then the sequence $(C_n(X), \partial_n)$ defines a chain complex.*

Proof. We define linear maps

$$\begin{aligned}\partial_n^{i\vec{\epsilon}} \quad = \quad & \sum_{i=1}^{n}(-1)^i[(x_0 *_{\epsilon_i} \mathbf{x}_i, (\mathbf{x}_1 *_{\epsilon_i} \mathbf{x}_i, \epsilon_1), \ldots, (\mathbf{x}_{i-1} *_{\epsilon_i} \mathbf{x}_i, \epsilon_{i-1}), \widehat{(\mathbf{x}_i, \epsilon_i)}, \\ & (\mathbf{x}_{i+1}, \epsilon_{i+1}), \ldots, (\mathbf{x}_{n-1}, \epsilon_{n-1})) \\ & -(x_0, (\mathbf{x}_1, \epsilon_1), \ldots, \widehat{(\mathbf{x}_i, \epsilon_i)}, \ldots, (\mathbf{x}_{n-1}, \epsilon_{n-1}))].\end{aligned}$$

Therefore by definition, $\partial_n^{\vec{\epsilon}} = \sum_i (-1)^i \partial_n^{i\vec{\epsilon}}$. It is enough to show now that the maps $\partial_n^{i\vec{\epsilon}}$ satisfy the pre-simplicial complex relation: $\partial_n^{i\vec{\epsilon}}\partial_n^{j\vec{\epsilon}} = \partial_n^{j\vec{\epsilon}}\partial_n^{i+1\vec{\epsilon}}$ for each $n \in \mathbb{N}$ whenever $j < i$.

Fix a vector $\vec{\epsilon} = (\epsilon_1, \ldots, \epsilon_{n-1})$ and consider an element $(x_0, (x_1, \epsilon_1), \ldots, (x_{n-1}, \epsilon_{n-1})) \in C_n^{\vec{\epsilon}}(X)$. Then we have:

$$\begin{aligned}& \partial_n^i \partial_n^j (x_0, (\mathbf{x}_1, \epsilon_1), \ldots, (\mathbf{x}_{n-1}, \epsilon_{n-1})) = \\ & ((x_0 *_{\epsilon_j} \mathbf{x}_j) *_{\epsilon_{i+1}} \mathbf{x}_{i+1}, ((\mathbf{x}_1 *_{\epsilon_j} \mathbf{x}_j) *_{\epsilon_{i+1}} \mathbf{x}_{i+1}, \epsilon_1), \ldots, \widehat{(\mathbf{x}_j, \epsilon_j)}, \\ & (\mathbf{x}_{j+1} * \mathbf{x}_{i+1}, \epsilon_{j+1}), \ldots, \widehat{(\mathbf{x}_i, \epsilon_i)}, \ldots, (\mathbf{x}_{n-1}, \epsilon_{n-1})) \\ & -(x_0, (\mathbf{x}_1, \epsilon_1), \ldots, \widehat{(\mathbf{x}_j, \epsilon_j)}, \\ & (\mathbf{x}_{j+1}, \epsilon_{j+1}), \ldots, \widehat{(\mathbf{x}_{i+1}, \epsilon_{i+1})}, \ldots, (\mathbf{x}_{n-1}, \epsilon_{n-1})).\end{aligned}$$

On the other hand we have:

$$\partial_n^{j\vec{\epsilon}}\partial_n^{(i+1)\vec{\epsilon}}(x_0,(\mathbf{x}_1,\epsilon_1),\dots,(\mathbf{x}_{n-1},\epsilon_{n-1}))=$$

$$((x_0 *_{\epsilon_{i+1}} \mathbf{x}_{i+1}) *_{\epsilon_j} (\mathbf{x}_j *_{\epsilon_{i+1}} \mathbf{x}_{i+1}),((\mathbf{x}_1 *_{\epsilon_{i+1}} \mathbf{x}_{i+1}) *_{\epsilon_j} (\mathbf{x}_j *_{\epsilon_{i+1}} \mathbf{x}_{i+1}),\epsilon_1),$$

$$\dots\widehat{(\mathbf{x}_j,\epsilon_j)},(\mathbf{x}_{j+1} *_{\epsilon_{i+1}} \mathbf{x}_{i+1},\epsilon_{j+1}),\dots,\widehat{(\mathbf{x}_{i+1},\epsilon_{i+1})},\dots,(\mathbf{x}_{n-1},\epsilon_{n-1}))$$

$$-(x_0,(\mathbf{x}_1,\epsilon_1),\dots,\widehat{(\mathbf{x}_j,\epsilon_j)},$$

$$(\mathbf{x}_{j+1},\epsilon_{j+1}),\dots,\widehat{(\mathbf{x}_{i+1},\epsilon_{i+1})},\dots,(\mathbf{x}_{n-1},\epsilon_{n-1})),$$

where we have used the vector notation introduced in Section 3.5. The two quantities are equal, in virtue of the mutual distributivity property of the set $\{*_{n_j}\}_{j=1}^k$. ∎

Similarly to the binary case, we have the following definition.

Remark 3.5.6. The multiplication on binary operations condidered in [Prz] can be directly generalized to n-ary operations as follows. Given a nonempty set X, let $\mathrm{Dist}_M(X)$ denotes the set of all n-ary mutually distributive operations on X. Define the following multiplication on $\mathrm{Dist}_M(X)$:

$$(W \cdot W')(x,\mathbf{y}) := W(W'(x,\mathbf{y}),\mathbf{y})$$

for all $x \in X$ and $\mathbf{y} \in X^{n-1}$. Then it is straightforward to see that the multiplication defined above makes $\mathrm{Dist}_M(X)$ into a monoid with identity W_0 given by $W_0(x,\mathbf{y}) = x$, for all $x \in X$ and $\mathbf{y} \in X^{n-1}$.

For example, let (X, T) be a ternary rack. Define, inductively,

$$T^n(x, y_0, y_1) = T(T^{n-1}(x, y_0, y_1), y_0, y_1).$$

Then (X, T^n) is a ternary distributive set for all positive integer n.

We digress now from the main topic, to introduce a braid group action on n-ary operations. Let $(X, *)$ be a rack. Let B_m denote the m-string braid group, and let $\beta \in B_m$. As in [Bri], B_m acts on X^m via $*$ by

$$(x_1, \ldots, x_m)^{\sigma_i} = (x_1, \ldots, x_{i-1}, x_{i+1}, x_i * x_{i+1}, x_{i+2}, \ldots, x_m),$$

where the right action is denoted by the exponent, and σ_i, $i = 1, \ldots, m-1$ denotes the standard generator of B_m.

Lemma 3.5.7. *Let $*$ and $\hat{*}$ be mutually distributive binary and n-ary operations on X. Let $\beta \in B_m$ and $\mathbf{x} \in X^m$, $\mathbf{y} \in X^{n-1}$. Then we have*

$$\mathbf{x}^\beta \,\hat{*}\, \mathbf{y} = (\mathbf{x} \,\hat{*}\, \mathbf{y})^\beta,$$

where the action of B_m on X^m is defined by $*$.

Proof. It suffices to prove it for standard generators, and hence, the case $\beta = \sigma_1 \in B_2$. Then the mutual distributivity implies

$$
\begin{aligned}
\mathbf{x}^{\sigma_1} \,\hat{*}\, \mathbf{y} &= (x_1, x_2)^{\sigma_1} \,\hat{*}\, \mathbf{y} \\
&= (x_2, x_1 * x_2) \,\hat{*}\, \mathbf{y} \quad = \quad (x_2 \,\hat{*}\, \mathbf{y}, (x_1 * x_2) \,\hat{*}\, \mathbf{y}) \\
&= (x_2 \,\hat{*}\, \mathbf{y}, (x_1 \,\hat{*}\, \mathbf{y}) * (x_2 \,\hat{*}\, \mathbf{y})) \quad = \quad (\mathbf{x} \,\hat{*}\, \mathbf{y})^{\beta}
\end{aligned}
$$

as desired. ■

The following establishes braid group actions on the n-ary operations.

Theorem 3.5.8. *Let $*$ and $\hat{*}$ be mutually distributive binary and n-ary operations on X. Let $\beta \in B_{n-1}$ and $x \in X$, $\mathbf{y} \in X^{n-1}$. Define the action of B_{n-1} on X^{n-1} by $*$. Then the operation defined by $x \,\hat{*}^{\beta}\, \mathbf{y} := x \,\hat{*}\, (\mathbf{y}^{\beta})$ is an n-ary distributive operation.*

Furthermore, for any $\beta_0, \beta_1 \in B_{n-1}$, the operations $\hat{*}^{\beta_0}$ and $\hat{*}^{\beta_1}$ are mutually distributive.

Proof. It suffices to show the second statement with possibility of $\beta_0 = \beta_1$. One computes, for $x \in X$ and $\mathbf{y}, \mathbf{z} \in X^{n-1}$,

$$
\begin{aligned}
&(x \,\hat{*}^{\beta_0}\, \mathbf{y}) \,\hat{*}^{\beta_1}\, (\mathbf{z}) \\
&= (x \,\hat{*}\, \mathbf{y}^{\beta_0}) \,\hat{*}\, \mathbf{z}^{\beta_1} \\
&= (x \,\hat{*}\, \mathbf{z}^{\beta_1}) \,\hat{*}\, (\mathbf{y}^{\beta_0} \,\hat{*}\, \mathbf{z}^{\beta_1}) \\
&= (x \,\hat{*}\, \mathbf{z}^{\beta_1}) \,\hat{*}\, (\mathbf{y} \,\hat{*}\, \mathbf{z}^{\beta_1})^{\beta_0}
\end{aligned}
$$

$$= \quad (x \mathbin{\hat{*}}^{\beta_1} \mathbf{z}) \mathbin{\hat{*}}^{\beta_0} (\mathbf{y} \mathbin{\hat{*}}^{\beta_1} \mathbf{z}),$$

where the third equality follows from Lemma 3.5.7. ■

Example 3.5.9. Let $\{*, *_0, *_1\}$ be mutually distributive binary rack operations on X. Let $(X, T) = \mathcal{F}(X, *_0, *_1)$. Then $\{*, T\}$ are mutually distributive by Lemma 3.5.4. The preceding theorem provides a new ternary operations $\hat{*}^{\sigma_1^m}$ from $\hat{*} = T$ for all integers m, where the braid action is defined by $*$. In particular, Alexander quandles can be used for $\{*, *_0, *_1\}$.

3.6 Internalization of Higher Order Self-Distributivity

We begin this section with the definition of n-ary self-distributive object in a symmetric monoidal category, providing therefore a higher arity version of the work in [CCES]. We will use the symbol $\boxtimes$ to indicate the tensor product in the symmetric monoidal category $\mathcal{C}$, not to confuse the general setting with the standard tensor product in vector spaces, to be found in the examples. We remind the reader first, that a symmetric monoidal category is a monoidal category $\mathcal{C}$ together with a family of isomorphisms $\tau_{X,Y} : X \boxtimes Y \longrightarrow Y \boxtimes X$, natural in X and Y, satisfying the following conditions (Section 11 in [ML71]). The hexagon

$$
\begin{tikzcd}[column sep=small]
& X \boxtimes (Y \boxtimes Z) \arrow[dr, "\tau_{X,Y\boxtimes Z}"] & \\
(X \boxtimes Y) \boxtimes Z \arrow[ur, "\alpha_{X,Y,Z}"] \arrow[d, "\tau_{X,Y}\boxtimes \mathbb{1}"'] & & (Y \boxtimes Z) \boxtimes X \arrow[d, "\alpha_{Y,Z,X}"] \\
(Y \boxtimes X) \boxtimes Z \arrow[dr, "\alpha_{Y,X,Z}"'] & & Y \boxtimes (Z \boxtimes X) \\
& Y \boxtimes (X \boxtimes Z) \arrow[ur, "\mathbb{1}\boxtimes\tau_{X,Z}"'] &
\end{tikzcd}
$$

is commutative for all objects X,Y and Z in $\mathcal{C}$, where $\alpha_{X,Y,Z}$ indicates the associator of the monoidal category. We further have the following identity for all objects X and Y

$$\tau_{Y,X}\tau_{X,Y} = \mathbb{1}_{X\boxtimes Y}.$$

For the sake of simplicity, we work on a strict symmetric monoidal category for the rest of the paper and forget therefore to keep track of the bracketing. We recall also that a comonoid in a symmetric monoidal category is an object $X \in \mathcal{C}$ endowed with morphisms $\Delta : X \longrightarrow X \boxtimes X$, called comultiplication or diagonal, and $\epsilon : X \longrightarrow I$, called counit, where I is the unit object of the monoidal category. The comultiplication and the counit satisfy the usual coherence diagrams analogous to the coalgebra axioms. In virtue of the coassociative axiom we can inductively define an n-diagonal $\Delta_n : X \longrightarrow X^{\boxtimes n}$ by the assignment: $\Delta_n = (\Delta \boxtimes \mathbb{1})\Delta_{n-1}$, for all $n \in \mathbb{N}$. Let us define the isomorphism $\tau_{i,i+1} : X^{\boxtimes n} \longrightarrow X^{\boxtimes n}$ as $\tau_{i,i+1} = \mathbb{1}^{\boxtimes(i-1)} \boxtimes \tau_{X,X} \boxtimes \mathbb{1}^{\boxtimes(n-i-1)}$. It is easy to verify that the morphisms $\tau_{i,i+1}$ satisfy the relations of the transposition $(i, i+1)$ in $\mathbb{S}_n$, the symmetric group on n letters. We therefore obtain, for every object X, an action of $\mathbb{S}_n$ on $X^{\boxtimes n}$, by mapping $(i, i+1)$ to $\tau_{i,i+1}$, and extending to a homomorphsim of groups between $\mathbb{S}_n$ and $\mathrm{Aut}(X^{\boxtimes n})$, the automorphism group of $X^{\boxtimes n}$. In particular we will make use of the automorphism of $X^{\boxtimes n^2}$, corresponding to the permutation

$$
\begin{aligned}
\sqcup\!\sqcup_n \quad = \quad & (2, n+1)(3, 2n+1)\cdots(n, (n-1)n+1) \\
& (n+3, 2n+2)(n+4, 3n+2)\cdots(2n, (n-1)n+2) \\
& \cdots((n-2)n+n, (n-1)n+n-1).
\end{aligned}
$$

We are ready now to define n-ary self-distributive objects in a symmetric monoidal category $\mathcal{C}$.

Definition 3.6.1. An n-ary self-distributive object in a symmetric monoidal category $\mathcal{C}$ is a pair (X, W), where X is a comonoid object in $\mathcal{C}$ and $W : X^{\boxtimes n} \longrightarrow X$ is a morphism making the following diagram commute

$$
\begin{tikzcd}
& X^{\boxtimes n^2} \arrow[dl, "\sqcup\!\sqcup_n"'] & & X^{\boxtimes(2n-1)} \arrow[ll, "\mathbb{1}^{\boxtimes n}\boxtimes\Delta_n^{\boxtimes(n-1)}"'] \arrow[dr, "W\boxtimes\mathbb{1}^{\boxtimes(n-1)}"] & \\
X^{\boxtimes n^2} \arrow[d, "W\boxtimes\cdots\boxtimes W"'] & & & & X^{\boxtimes n} \arrow[d, "W"] \\
X^{\boxtimes n} \arrow[rrrr, "W"'] & & & & X
\end{tikzcd}
$$

Remark 3.6.1. The need of defining a self-distributive object by means of a diagonal map

Δ seems to be intrinsic to self-distributivity itself. In other words it appears that self-distributivity is a properadic, (see [Mar06] for a definition of properad), rather than operadic, property.

Example 3.6.2. Clearly, any n-ary rack is an n-ary self-distributive object in the symmetric monoidal category of sets, with τ and Δ defined in the obvious way.

In the rest of this section we will make use of Sweedler notation in the following form: $\Delta(x) = x^{(1)} \otimes x^{(2)}$.

Example 3.6.3. Let H be an involutive Hopf algebra. Define a ternary operation T : $H \otimes H \otimes H \longrightarrow H$ by the assignment $T(x \otimes y \otimes z) = xS(y)z$, extended by linearity, where we use juxtaposition as a shorthand to indicate the multiplication μ of H and S is the antipode. By direct computation on tensor monomials we obtain, for the left hand side of ternary self-distributivity:

$$\begin{aligned} & T(T(x \otimes y \otimes z) \otimes u \otimes z) \\ &= T(xS(y)z \otimes u \otimes v) \\ &= xS(y)zS(u)v. \end{aligned}$$

The right hand side is:

$$\begin{aligned} & TT^{\otimes 3} \sqcup_3 (\mathbb{1}^{\otimes 3} \otimes (\Delta \otimes \mathbb{1})\Delta \otimes (\Delta \otimes \mathbb{1})\Delta)(x \otimes y \otimes z \otimes u \otimes v) \\ &= TT^{\otimes 3}((x \otimes u^{(11)} \otimes v^{(11)} \otimes (y \otimes u^{(12)} \otimes v^{(12)}) \otimes (z \otimes u^{(2)} \otimes v^{(2)})) \end{aligned}$$

$$
\begin{aligned}
&= T(xS(u^{(11)})v^{(11)}) \otimes yS(u^{(12)})v^{(12)} \otimes zS(u^{(2)})v^{(2)}) \\
&= xS(u^{(11)})v^{(11)}S(yS(u^{(12)})v^{(12)})zS(u^{(2)})v^{(2)} \\
&= xS(u^{(11)})v^{(11)}S(v^{(12)})S^2(u^{(12)})S(y)zS(u^{(2)})v^{(2)} \\
&= xS(u^{(11)})\epsilon(v^{(1)} \cdot 1)S^2(u^{(12)})S(y)zS(u^{(2)})v^{(2)} \\
&= xS(\epsilon(u^{(1)}) \cdot 1)S(y)zS(u^{(2)})\epsilon(v^{(1)})v^{(2)} \\
&= xS(y)zS(\epsilon(u^{(1)})u^{(2)})v \\
&= xS(y)zS(u)v.
\end{aligned}
$$

This ternary strucutre is the Hopf algebra analogue of the heap operation in group theory, which is known to be ternary self-distributive.

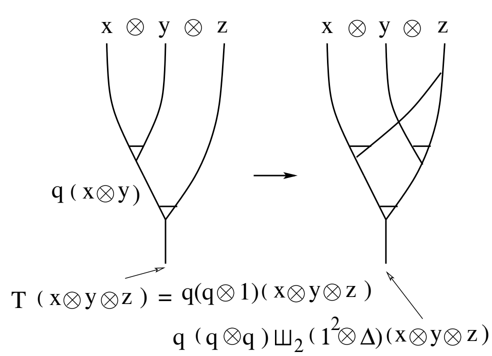

Figure 3.6: Diagrammatic representation of categorical distributivity

n Figure 3.6, a diagrammatic representation of categorical distributivity is depicted. It is read from top to bottom, where the top 3 end points of both sides represent $x \otimes y \otimes z$, a trivalent vertex with a small triangle represents a self-distributive morphism $q : X \otimes X \to X$, and the left-hand side represents $T = q(q \otimes \mathbb{1})$.

Given a symmetric monoidal category $\mathcal{C}$, we define categories $n\mathcal{SD}$, for each $n \in \mathbb{N}$, as follows. The objects are n-ary self-distrifbutive objects in $\mathcal{C}$, as in Definition 3.6.1. Given two objects (X, q) and (X', q'), we define the morphism class between them to be the class of morphism $f : X \longrightarrow X'$ in $\mathcal{C}$, such that $f \circ q = q' \circ f^{\boxtimes n}$. In particular we define $\mathcal{BSD} = 2\mathcal{SD}$ and $\mathcal{TSD} = 3\mathcal{SD}$, $\mathcal{B}$ and $\mathcal{T}$ standing for binary and ternary, respectively.

We will make use of the following results in Theorem 3.6.6.

Lemma 3.6.4. *Let $\mathcal{C}$ be a strict symmetric monoidal category. Suppose (X, Δ, ϵ) is a comonoid in $\mathcal{C}$. Then the swithcing morphism and the comultiplication commute. More specifically, we have: $\Delta \boxtimes \mathbb{1} \circ \tau_{X,Y} = \tau_{X,Y^{\boxtimes 2}} \circ \mathbb{1} \boxtimes \Delta$.*

This lemma is represented in Figure 3.7 (A) below.

Proof. We consider the following diagram

$$\begin{tikzcd}
X \boxtimes Y \arrow[r, "\tau_{X,Y}"] \arrow[dr, "\mathbb{1}\boxtimes\Delta"'] & Y \boxtimes X \arrow[r, "\Delta\boxtimes\mathbb{1}"] & Y^{\boxtimes 2} \boxtimes X \\
& X \boxtimes Y^{\boxtimes 2} \arrow[ur, "\tau_{X,Y^{\boxtimes 2}}"'] \arrow[r, "\tau_{X,Y}\boxtimes\mathbb{1}"'] & Y \boxtimes X \boxtimes Y \arrow[u, "\mathbb{1}\boxtimes\tau_{X,Y}"']
\end{tikzcd}$$

The outmost diagram commutes by naturality of switching map $\tau_{X,Y}$ with respect to X and Y. The lower right triangle commutes by the hexagon axiom.

$$
\begin{tikzcd}
 & X \boxtimes (Y \boxtimes Y) \arrow[dl, "\tau_{X,Y^{\boxtimes 2}}"'] \arrow[dr, equal] & \\
(Y \boxtimes Y) \boxtimes X \arrow[d, equal] & & (X \boxtimes Y) \boxtimes Y \arrow[d, "\tau_{X,Y} \boxtimes \mathbb{1}"] \\
Y \boxtimes (Y \boxtimes X) & & (Y \boxtimes X) \boxtimes Y \arrow[dl, equal] \\
 & Y \boxtimes (X \boxtimes Y) \arrow[ul, "\mathbb{1} \boxtimes \tau_{X,Y}"] &
\end{tikzcd}
$$

The assertion now follows. ■

Lemma 3.6.5. *Let (X, q) be a binary self-distributive object in a strict symmetric monoidal category $\mathcal{C}$. Then the switching morphism and the self-distributive operation commute. More specifically, we have: $\tau_{X,Y} \circ q \boxtimes \mathbb{1} = \mathbb{1} \boxtimes q \circ \tau_{X^{\boxtimes 2},Y}$.*

This lemma is represented in Figure 3.7 (B) below.

Proof. Similar to Lemma 3.6.4 and left to the reader. ■

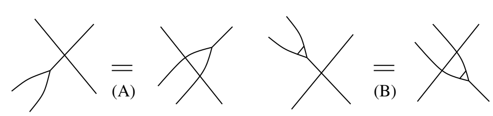

Figure 3.7: The switching morphism commutes with comultiplication and binary self-distributive operation

In general, the following result is useful to produce ternary self-distributive objects in the category of vector spaces, starting from binary self-distributive objects (see also [CCES]). Compare it to the construction of Section 3.2.

Theorem 3.6.6. *Let (X,Δ) be a comonoid in a (strict) symmetric monoidal $\mathcal{C}$ (e.g. a coalgebra in the category of vector spaces). Let $q : X \boxtimes X \longrightarrow X$ be a morphism such that (X,q) is a binary self-distributive object in $\mathcal{C}$. Then the pair (X,T), where $T = q(q \boxtimes \mathbb{1})$, defines a ternary self-distributive object in $\mathcal{C}$. The construction defines a functor $\mathcal{F} : \mathcal{BSD} \to \mathcal{TSD}$.*

Proof. We define $\mathcal{F}$ on objects as $\mathcal{F}(X,q) = (X,T)$ and as the identity on morphisms. To show that the map $T = q(q \boxtimes \mathbb{1})$ is ternary self-distributive, we can proceed as in Figure 3.8. In the left column of the figure, the part of the diagram representing each $T = q(q \boxtimes \mathbb{1})$ are indicated by dotted circles. At each step we are using the definition of T, the binary self-distributivity of q and Lemmas 3.6.4 and 3.6.5. If $f : (X,q) \longrightarrow (Y,q')$ is a morphism in $\mathcal{BSD}$, we can show that f is also a morphism in $\mathcal{TSD}$ between $(X, T = q(q \boxtimes \mathbb{1}))$ and $(Y, T' = q'(q' \boxtimes \mathbb{1}))$ via the following diagram

$$\begin{tikzcd}
X \boxtimes X \boxtimes X \arrow[r, "q \boxtimes \mathbb{1}"] \arrow[d, "f \boxtimes f \boxtimes f"] & X \boxtimes X \arrow[r, "q"] \arrow[d, "f \boxtimes f"] & X \arrow[d, "f"] \\
Y \boxtimes Y \boxtimes Y \arrow[r, "q' \boxtimes \mathbb{1}"] & Y \boxtimes Y \arrow[r, "q'"] & Y
\end{tikzcd}$$

where the commutativity of the left and right squares is just a restatement of the fact that q is a morphism in $\mathcal{BSD}$. The consequent commutativity of the outer rectangle means that f is a morphism in $\mathcal{TSD}$ as well. It is also clear that $\mathcal{F}$ preserves composition of morphisms. ■

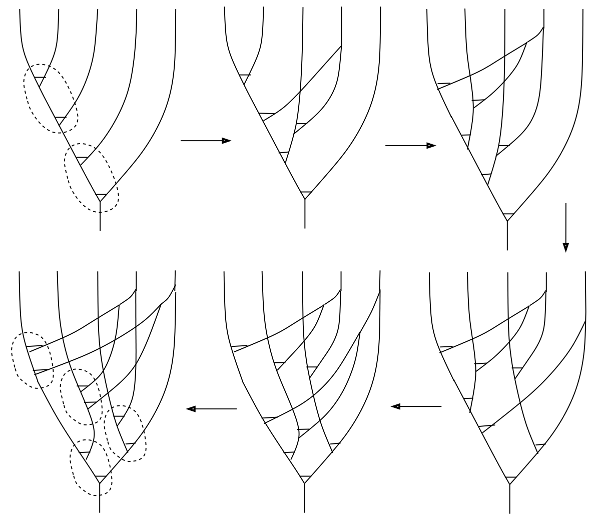

Figure 3.8: Diagrammatic proof of doubling procedure

The following is a rephrased version of Lemma 3.3 in [CCES], adapted to our languange in the present article.

Lemma 3.6.7. *Let L be a Lie algebra over a ground field $\Bbbk$. Define $X = \Bbbk \oplus L$ and endow it with a comultiplication Δ, defined by $(a, x) \mapsto (a, x) \otimes (1, 0) + (1, 0) \otimes (0, x)$, and a counit ϵ, defined by $(a, x) \mapsto a$. Then (X, Δ, ϵ) is a comonoid in the symmetric monoidal category of vector spaces. The morphsim $q : X \otimes X \longrightarrow X$ defined by $(a, x) \otimes (b, y) \mapsto (ab, bx + [x, y])$ turns X into a binary self-distributive object.*

Proof. By direct computation making use of the Jacobi identity. This is done explicitly in Lemma 3.3 in [CCES]. ■

Example 3.6.8. Let L be a Lie algebra and let $X = \Bbbk \oplus L$ be as in Lemma 3.6.7. The map

$T : X \otimes X \otimes X \longrightarrow X$ defined by

$$(a, x) \otimes (b, y) \otimes (c, z) \longmapsto (abc, bcx + c[x, y] + b[x, z] + [[x, y], z]),$$

and extended by linearity, is such that (X, T) is a tenrary self-distributive object in the category of vector spaces by an easy application of Theorem 3.6.6. An explicit, and tedious, computation that shows the self-distributivity of T directly, is postponed to Appendix 3.6.8.

If H is a Hopf algebra, we can use the adjoint map to produce a ternary self distributive map, as the following example shows:

Example 3.6.9. The map defined by $T(x \otimes y \otimes z) = S(z^{(1)})S(y^{(1)})xy^{(2)}z^{(2)}$ is ternary self-distributive, as an easy direct computation shows. This is the Hopf algebra analogue of the iterated conjugation quandle.

The following definition can be considered a ternary analogue of an augmented rack [FR].

Definition 3.6.2. Let X be a set with a right G-action denoted by $X \times G \ni (x, g) \mapsto x \cdot g \in X$. Let G act on the right of $X \times X$ diagonally, $(y_0, y_1) \cdot g = (y_0 \cdot g, y_1 \cdot g)$ for $y_0, y_1 \in X$ and $g \in G$. An *(double) augmentation* of X is a map $p : X \times X \to G$ satisfying the condition

$$p((y_0, y_1) \cdot g) = g^{-1} p((y_0, y_1))g$$

for all $y_0, y_1 \in X$ and $g \in G$.

The following is a direct analogue of binary augmented rack and, therefore, the proof is omitted.

Lemma 3.6.10. *Let X be a set with an augmentation $p : X \times X \to G$. Then the ternary operation $T : X^3 \to X$ defined by*

$$T(x, y_0, y_1) := x \cdot p((y_0, y_1))$$

is ternary self-distributive.

Definition 3.6.3. Let X be a set with an augmentation $p : X^2 \to G$ and T be a ternary operation defined in Lemma 3.6.10. Then (X, T) is called an *augmented ternary shelf.*

The following is a Hopf algebra version of ternary augmented rack.

Definition 3.6.4. Let X be a coalgebra, and let H be a Hopf algebra such that X is a right H-module, therefore $X^{\otimes 2}$ is also a right H-module via the comultiplication in H. The map of coalgebras $p : X^{\otimes 2} \longrightarrow H$ is a ternary augmented shelf if, for all $z \in X^{\otimes 2}$ and $g \in H$, we have:

$$p(z \cdot \Delta(g)) = S(g^{(1)})p(z)g^{(2)}.$$

This axiom is depicted diagrammatically in Figure 3.9, where solid lines refer to X, and dashed lines refer to H. We have used Δ, m and S to indicate comultiplication, multiplication and antipode in the Hopf algebra H, while μ stands for the action of H on X.

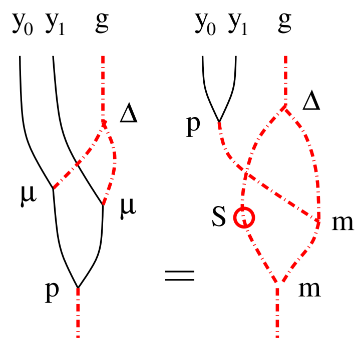

Figure 3.9: Augmented ternary shelf axiom

We have the following result.

Theorem 3.6.11. *Let $p : X^{\otimes 2} \longrightarrow H$ be a ternary augmented shelf. Then the ternary operation defined on monomials via $x \otimes y \otimes z \longmapsto x \cdot p(y \otimes z)$, and extended by linearity, is self-distributive.*

Proof. By direct computation we have, for the right hand side of self-distributivity axiom:

$$
\begin{aligned}
& TT^{\otimes 3} \sqcup_3 (\mathbb{1}^{\otimes 3} \otimes (\Delta \otimes \mathbb{1})\Delta \otimes (\Delta \otimes \mathbb{1})\Delta)(x \otimes y_0 \otimes y_1 \otimes z_0 \otimes z_1) \\
&= T(x \cdot p(z_0^{(1)} \otimes z_1^{(1)}) \otimes y_0 \cdot p(z_0^{(2)} \otimes z_1^{(2)}) \otimes y_1 \cdot p(z_0^{(3)} \otimes z_1^{(3)})) \\
&= x \cdot (p(z_0^{(1)} \otimes z_1^{(1)}) p(y_0 \cdot p(z_0^{(2)} \otimes z_1^{(2)}) \otimes y_1 \cdot p(z_0^{(3)} \otimes z_1^{(3)}))) \\
&= x \cdot (p(z_0^{(1)} \otimes z_1^{(1)})(y_0 \otimes y_1 \cdot \Delta p((z_0 \otimes z_1)^{(2)}))) \\
&= x \cdot (p(z_0 \otimes z_1)^{(1)} S(p((z_0 \otimes z_1)^{(2)})) p(y_0 \otimes y_1) p((z_0 \otimes z_1))^{(3)}) \\
&= x \cdot (\epsilon(p(z_0 \otimes z_1)^{(1)}) \cdot 1 p(y_0 \otimes y_1) p((z_0 \otimes z_1))^{(2)}) \\
&= x \cdot (p(y_0 \otimes y_1) p(z_0 \otimes z_1)),
\end{aligned}
$$

where we have used the fact that p is a coalgebra morphism in the third equality, the defining axiom for augmented ternary shelf in the fourth equality, the antipode and the counit axioms to obtain the fifth and sixth equations respectively. It is easy to see that it coincide with the left hand side of self-distributivity. ■

Example 3.6.12. Let H be a Hopf algebra and let $X = H$. Then, H acts on X via the multiplication. Define p to be the map given by $x \otimes y \longmapsto S(x)y$ and extended by linearity. The ternary rack structure obtained is the one in Example 3.6.3. A diagrammatic proof that the given p satisfies the augmented ternary rack axiom is shown in Figure 3.10.

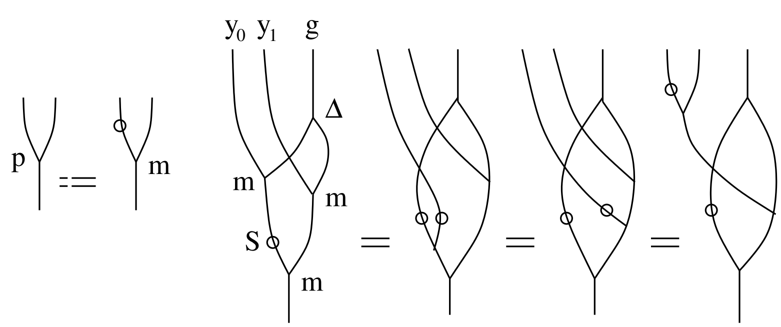

Figure 3.10: Hopf algebra heap as an augmented ternary shelf

Remark 3.6.13. It is possible, a priori, to develop the theory of higher self-distributivity in braided monoidal categories, where the switching morphism satisfies the hexagon axiom but we do not require $\tau_{Y,X}\tau_{X,Y} = \mathbb{1}_{X\boxtimes Y}$. Similarly as above we have an action of the braid group on n strings on every object $X^{\boxtimes n}$ and the shuffle map $\sqcup_n$ takes now into account over passing and under passing of the strings.

3.7 Ternary Cocycle Framed Link Invariants

We introduce, following [CJK$^+$, CEGnS], an invariant of framed links, using colorings of ribbon tangles by ternrary quandles, and ternary quandle 2-cocycles.

Let X be a tarnary quandle and let $\mathcal{D}$ be a diagram of a ribbon tangle. Suppose for the moment that the tangle has a single component, in the sense that its closure is a diagram of a framed knot. To each ribbon arc in $\mathcal{D}$, we associate a color by a pair of elements $(x_1, x_2) \in X \times X$. We visualize a crossing of framed links as in Figure 3.1, where we assume all the orientations to be downwards. We define the notion of positive/negative crossing in a similar fashion to Section 1.3. At a crossing τ of $\mathcal{D}$, where the arcs colored by (x_1^τ, x_2^τ) and (y_1^τ, y_2^τ) meet, we let the overpassing ribbon mantain the same color, while we change the color of the underpassing ribbon to $(T(x_1, y_1, y_2), T(x_2, y_1, y_2))$.

Definition 3.7.1. A ternary quandle coloring, $\mathcal{C}$, of a ribbon tangle diagram $\mathcal{D}$, is a map

$$\{\text{arcs of } \mathcal{D}\} \longrightarrow X \times X,$$

that is consistent with the coloring rule above.

Suppose ϕ is a ternary quandle 2-cocycle of X, with coefficients in A. For a given crossing τ, we define the Boltzmann weight at τ, depending on the coloring $\mathcal{C}$ and the 2-cocycle ϕ by $(\phi(x_1^\tau, y_1^\tau, y_2^\tau), \phi(x_2^\tau, y_1^\tau, y_2^\tau))^{\epsilon(\tau)} \in A \times A$, where $\epsilon(\tau)$ is the sign of the crossing, similarly defined as in the case of knots/links. We give an ordering to the crossings of the diagram in the following way. We consider the closure of the diagram and obtain a framed knot. We arbitrarily choose a base point and consider the crossings according to order we meet them going from the base point, along the direction of the framed knot.

Definition 3.7.2. Let $\mathcal{D}$ be a ribbon tangle diagram having k crossings $\tau_1, \ldots, \tau_k$, $\mathcal{C}$ a coloring of $\mathcal{D}$ and $\phi \in Z^2(X, A)$ a ternary 2-cocycle. Define the Boltzmann weight at the crossing τ, by the 2-cocycle ϕ associated to the coloring $\mathcal{C}$ as

$$\mathcal{B}(\phi, \tau, \mathcal{C}) = (\phi(x_1^{\tau_i}, y_1^{\tau_i}, y_2^{\tau_i})^{\epsilon(\tau_i)}, \phi(x_2^{\tau_i}, y_1^{\tau_i}, y_2^{\tau_i})^{\epsilon(\tau_i)}).$$

The ternary cocycle invariant is defined by the assignment

$$\Theta(\mathcal{D}) = \sum_{\mathcal{C}} \prod_{\tau} \mathcal{B}(\phi, \tau, \mathcal{C}).$$

Theorem 3.7.1. *The cocycle invariant does not depend on the equivalence class of the ribbon tangle diagram $\mathcal{D}$. Therefore, it is well defined and it is an invariant of framed knots.*

Proof. There is a bijective correspondence between colorings of a framed link, before and after performing one of the moves T1-T6 in [FY89], or rel_1-rel_{10} in [RT]. Let ϕ be a ternary 2-cocycle of X, with coefficients in the abelian group A, $\mathcal{D}$ a ribbon tangle diagram and $\mathcal{C}$ a coloring of $\mathcal{D}$ by X. It is enough to establish that a Boltzmann sum does not change under the 6 relations T1-T5 and T6_f given in [FY89] or, equivalently, [RT] rel_1-rel_8. Relations T1 and T2 (resp. rel_5 to rel_6) hold by construction of the invariant, and the definition of ternary self-distributive cohomology as in Figure 3.2. Relation T3 (resp. rel_1 to rel_4) is trivially satisfied.

We need to verify relations T4, T5 and T6_f (resp. rel_8 to $\mathrm{rel}_1 0$). We will consider just T6_f (resp. rel_8), and leave to the reader the remaining cases. Suppose (x, y) is the coloring of the incoming arc in T6_f. Since the first crossing is positive, the second crossing is negative and $\mathcal{C}$ is a coloring of $\mathcal{D}$ by X, the contribution of the left hand side of T6_f to the total Boltzmann sum is $(\phi(T(x,x,y),x,y), \phi(T(y,x,y),x,y))(\phi(T(x,x,y),x,y)^{-1}, \phi(T(y,x,y),x,y)^{-1})$. The Boltzmann sum is therefore invariant under this relation. The remaining cases can be seen in a similar fashion. ■

Remark 3.7.2. The invariant takes values in the group ring of the product $A \times A$, similarly to the case of the original cocycle link invariant. It is therefore useful to observe that there is a natural isomorphism of rings $\mathbb{Z}[A \times A] \cong \mathbb{Z}[A] \otimes \mathbb{Z}[A]$.

To generalize the previous construction to the case in which the closure of the ribbon tangle diagram is a framed link with more than one component, we can just proceed analogously, once per component. The vector whose entries are Boltzmann sums relative to each component is seen to be invariant by an iterated version of the previous proof.

CHAPTER 4 : HEAP COHOMOLOGY AND TSD COHOMOLOGY

As we have seen in Chapter 3, ternary self-distibutive (TSD for short) cohomology can be used to produce invariants of framed links via Boltzmann state-sums of 2-cocycles as in [CJK$^+$, CEGnS]. It becomes crucial then, to introduce new methods to construct TSD 2-cocycles. This issue has been treated in Chapter 3, for certain kind of TSD operations obtained as a combination of binary self-distributive operations. In particular, it has been proved therein that knowledge of the second cohomology groups of the binary operations, can be translated into knowledge of the second cohomology group of the TSD operation they give rise to. In this chapter, we will give another useful construction, whose inspiration comes from Example 3.6.3 of Chapter 3, in which it has been shown that a Hopf algebra version of the heap operation satisfies the TSD property. We introduce a cohomology theory for heaps and show that heap 2-cocycles give rise to TSD 2-cocycles in a natural way. Our main definitions in the present chapter are justified by a deformation theoretical approach.

The content of this chapter is a revised version of the paper [ESZa].

4.1 Generalities on Heaps

We begin by recalling the definition of heap and some notations that will be used in the rest of the chapter. A heap is an abstraction of the ternary operation $a \times b \times c \mapsto ab^{-1}c$

in a group, that allows to "forget" which element of the group is the unit. In fact the operation just described extends to a functor that determines an equivalence between the category of pointed (i.e. an element specified) heaps and the category of groups. It is also interesting that, for a given category $\mathcal{C}$ and a fixed object $X \in \mathcal{C}$, there is an obvious way to define a group structure on the automorphism set of X, $\mathrm{Aut}(X)$. If we cosider, instead, two isomorphic objects X and Y in $\mathcal{C}$, the set of isomorphisms Iso(X, Y) cannot be endowed in general with a "natural" notion of group structure. In this more general case we can define a heap strucutre on $\mathrm{Iso}(X, Y)$ with a rule formally analogous to $a \times b \times c \mapsto ab^{-1}c$. This point of view can be found in [Kon99]. We introduce first some notation that will be used throughout the whole chapter.

Given a set with a ternary operation, we call the equalities

$$
\begin{aligned}
[[x_1, x_2, x_3], x_4, x_5] &= [x_1, x_2, [x_3, x_4, x_5]] \\
[[x_1, x_2, x_3], x_4, x_5] &= [x_1, [x_4, x_3, x_2], x_5] \\
[x_1, x_2, [x_3, x_4, x_5]] &= [x_1, [x_4, x_3, x_2], x_5]
\end{aligned}
$$

the type 0, 1, 2 para-associativity (or simply equality), respectively. Observe that any pair of equalities of type 0, 1 or 2 implies that the remaining one also holds. If a ternary operation satisfies all types of para-associativity, then it is called para-associative. We call the condition $[x, x, y] = y$ and $[x, y, y] = x$ the degeneracy (conditions).

Definition 4.1.1. A heap is a non-empty set with a ternary operation satisfying para-associativity and degeneracy conditions.

We mention that algebraic structures satisfying the para-associativity conditions, without the degeneracy conditions, are also called semiheaps in [HL17]. A typical example of a heap, as mentioned above, is a group G where the ternary operation is given by $[x, y, z] = xy^{-1}z$, which we call a *group heap*. If G is abelian, we call it an *abelian (group) heap*. Conversely, given a heap X with a fixed element e, then one defines a binary operation on X by $x * y = [x, e, y]$ which makes $(X, *)$ into a group with e as the identity and the inverse of x is $[e, x, e]$ for any $x \in X$.

We refer the reader to the classical reference [BH], chapter IV, where it can also be found a short historical background and a description in terms of universal algebra. Heaps are also known under different names such as *torsor* and *groud*. In [Sko07] a quantum version of heap was introduced and it has been shown, in analogy to the "classical" case, that the category of quantum heaps is equivalent to the category of pointed Hopf algebras. Further developments of the thematics introduced in [Sko07] can also be found in [Gru02, Sch]. Other sources include [Kon99, BS11]. We observe that the definition of quantum heap given in [Sko07] is in some sense dual to the notion of heap object in a symmetric monoidal category, that we introduce in Section 4.6. Our heap objects in symmetric monoidal categories are much in the same spirit as in the definition of non-commutative torsor treated in [BS11].

4.2 Heap Cohomology

In this section we introduce the heap cohomology. Let X be a set with a ternary operation $[-]$ and A be an abelian group. We focus on para-associative (and in particular heap) operations. Define the n-dimensional *cochain group* $C^n(X, A)$ as the abelian group of functions $\{f : X^{2n-1} \to A\}$ for positive integers n, where the abelian group structure is given by pointwise summation in A.

Definition 4.2.1. Let X be a set with a para-associative ternary operation $[-]$, A be an abelian group and define $C^1_{PA}(X, A) = C^1(X, A)$, $C^2_{PA}(X, A) = C^2(X, A)$. Then the 1-dimensional coboundary map $\delta^1 : C^1_{\mathrm{PA}}(X, A) \to C^2_{\mathrm{PA}}(X, A)$ is defined for $f \in C^1_{\mathrm{PA}}(X)$ by

$$\delta^1 f(x, y, z) = f([x, y, z]) - f(x) + f(y) - f(z).$$

The kernel $Z^1_{\mathrm{PA}}(X, A)$ of δ^1 is called the 1-dimensional cocycle group. In this case we define 1-dimensional cohomology group $H^1_{\mathrm{PA}}(X, A)$ to be $Z^1_{\mathrm{PA}}(X, A)$.

We observe that $f \in Z^1_{\mathrm{PA}}(X)$ if and only if f is a para-associative homomorphism from X to A regarded as an abelian heap.

We determine $Z^1_{\mathrm{PA}}(X, A)$ for two examples.

Example 4.2.1. Consider $\mathbb{Z}_2$ endowed with the para-associative ternary operation $[x, y, z] = x + y + z$. We want to compute $Z^1_{\mathrm{PA}}(\mathbb{Z}_2; \mathbb{Z}_2)$. Given three variables, x, y and z, at least two of them need to coincide. Consider the case when $x = y$, the 1-cocycle condition becomes $f([x, x, z]) = f(z)$ which is trivially satisfied since $[x, x, z] = 2x + z = z$. The other cases are analogous. It follows that $Z^1_{\mathrm{PA}}(\mathbb{Z}_2;\mathbb{Z}_2) = C^1_{\mathrm{PA}}(\mathbb{Z}_2;\mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Observe that $\mathbb{Z}_2$ just defined is actually an abelian heap (and therefore a ternary quandle, see Lemma 4.5.1).

Example 4.2.2. We proceed to compute $Z^1_{\mathrm{PA}}(\mathbb{Z}_3,\mathbb{Z}_n)$, where $\mathbb{Z}_3$ is given the same ternary operation as before: $[x,y,z] = x+y+z$. Observe that this operation does not define a heap, but it is para-associative. Take $x = y = 1$ in the 1-cocycle condition. We obtain $f(z+2) = f(z)$, which implies that f is the constant map. It follows that $Z^1_{\mathrm{PA}}(\mathbb{Z}_3,\mathbb{Z}_n) \cong \mathbb{Z}_n$ for all odd integers n.

Definition 4.2.2. Define $C^3_{\mathrm{PA(i)}}(X,A)$ for $i = 0,1,2$ to be an isomorphic copy of $C^3(X,A)$. The 2-dimensional coboundary map $\delta^2_{(i)} : C^2_{\mathrm{PA}}(X,A) \to C^3_{\mathrm{PA(i)}}(X,A)$ of type $i = 0,1,2$, respectively, are defined by

$$\begin{aligned}\delta^2_{(0)}\eta(x_1,x_2,x_3,x_4,x_5) &= \eta(x_1,x_2,x_3) + \eta([x_1,x_2,x_3],x_4,x_5)\\ &\quad -\eta(x_3,x_4,x_5) - \eta(x_1,x_2,[x_3,x_4,x_5])),\end{aligned}$$

$$\begin{aligned}\delta^2_{(1)}\eta(x_1,x_2,x_3,x_4,x_5) &= \eta(x_1,x_2,x_3) + \eta([x_1,x_2,x_3],x_4,x_5)\\ &\quad +\eta(x_4,x_3,x_2) - \eta(x_1,[x_4,x_3,x_2],x_5)),\end{aligned}$$

$$\begin{aligned}\delta^2_{(2)}\eta(x_1,x_2,x_3,x_4,x_5) &= \eta(x_3,x_4,x_5) + \eta(x_1,x_2,[x_3,x_4,x_5])\\ &\quad +\eta(x_4,x_3,x_2) - \eta(x_1,[x_4,x_3,x_2],x_5)).\end{aligned}$$

Direct calculations give the following.

Lemma 4.2.3. *If X has a para-associative operation $[-]$, then $\delta^2_{(i)}\delta^1 = 0$ for $i = 0, 1, 2$.*

Definition 4.2.3. Let X be a para-associative operation $[-]$ and A be an abelian group. Define $C^3_{\mathrm{PA}}(X, A) = C^3_{\mathrm{PA}(1)}(X, A) \oplus C^3_{\mathrm{PA}(2)}(X, A)$. Then $\delta^2 = \delta^2_{(1)} \oplus \delta^2_{(2)}$ defines a homomorphism $C^2_{\mathrm{PA}}(X, A) \to C^3_{\mathrm{PA}}(X, A)$. Define the 2$^{\text{nd}}$ cocycle group $Z^2_{\mathrm{PA}}(X, A)$ by $\ker(\delta^2)$. Define the 2$^{\text{nd}}$ coboundary group $B^2_{\mathrm{PA}}(X, A)$ by $\mathrm{im}(\delta^1)$. Then define the 2$^{\text{nd}}$ cohomology group is defined as usual: $H^2_{\mathrm{PA}}(X, A) = Z^2_{\mathrm{PA}}(X, A)/B^2_{\mathrm{PA}}(X, A)$.

Definition 4.2.4. Let X be a para-associative operation $[-]$ and A be an abelian group. A 2-cocycle $\eta \in Z^2_{\mathrm{PA}}(X, A)$ is said to satisfy the *degeneracy condition* if the following holds for all $x, y \in X$: $\eta(x, x, y) = 0 = \eta(x, y, y)$.

We observe that 2-coboundaries $\delta^1 f$ satisfy the degeneracy condition.

Definition 4.2.5. Let X be a heap and A be an abelian group. The 2$^{\text{nd}}$ heap cocycle group $Z^2_{\mathrm{H}}(X, A)$ is defined as the subgroup of $Z^2_{\mathrm{PA}}(X, A)$ generated by the 2-cocycles that satisfy the degeneracy conditions. The 2$^{\text{nd}}$ heap cohomology group $H^2_{\mathrm{H}}(X, A)$ is defined as the quotient $Z^2_{\mathrm{H}}(X, A)/B^2_{\mathrm{PA}}(X, A)$.

Example 4.2.4. Let $X = \mathbb{Z}_2$ with group heap operation and $A = \mathbb{Z}_2$. Computations show that $\eta \in Z^2_{\mathrm{PA}}(X, A)$ if and only if η satisfies the following set of equations:

$$\eta(0,0,0) = \eta(0,0,1) = \eta(1,0,0),$$

$$\eta(1,1,1) = \eta(1,1,0) = \eta(0,1,1),$$

$$\eta(0,0,0) + \eta(1,1,1) + \eta(0,1,0) + \eta(1,0,1) = 0.$$

Express $\eta = \sum \eta(x,y,z)\chi_{(x,y,z)}$ by characteristic functions $\chi_{(x,y,z)}$. By setting $\eta(0,0,0) = a$, $\eta(1,1,1) = b$ and $\eta(0,1,0) = c$, the last equation above implies $\eta(1,0,1) = -(a+b+c)$. Then η is expressed as

$$\begin{aligned} \eta &= a(\chi_{(0,0,0)} + \chi_{(0,0,1)} + \chi_{(1,0,0)} - \chi_{(1,0,1)}) \\ &+ b(\chi_{(1,1,1)} + \chi_{(1,1,0)} + \chi_{(0,1,1)} - \chi_{(1,0,1)}) \\ &+ c(\chi_{(0,1,0)} - \chi_{(1,0,1)}). \end{aligned}$$

Since the group of coboundaries is zero from Example 4.2.1, it follows that $H^2_{\mathrm{PA}}(\mathbb{Z}_2, \mathbb{Z}_2) = Z^2_{\mathrm{PA}}(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Since the degeneracy condition implies $a = b = 0$, we have $H^2_{\mathrm{H}}(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2$.

Definition 4.2.6. Let X be a heap, A an abelian group and $\eta : X \times X \times X \longrightarrow A$ a 2-cochain. We define the heap extension of X by the 2-cochain η with coefficients in A, denoted $X \times_\eta A$, as the cartesian product $X \times A$ with ternary operation given by:

$$[(x,a),(y,b),(z,c)] = ([x,y,z], a - b + c + \eta(x,y,z)).$$

Lemma 4.2.5. *The abelian extension by a 2-cochain η satisfies the equality of type 1, 2, and degeneracy if and only if η is a heap 2-cocycles of type 1, 2, and with degeneracy condition, respectively. In particular, a 2-cochain η defines an extension heap if and only if it satisfies all conditions.*

Proof. We prove the Lemma for type 1 equality and type 1 cocycle condition, the remaining cases being completely analogous.

Let $\eta : X \times X \times X \longrightarrow A$ be a 2-chain. We have:

$$\begin{aligned}
&[[(x,a),(y,b),(z,c)],(u,d),(v,e)] \\
&\quad = \quad ([[x,y,z],u,v], a-b+c+\eta(x,y,z)-d+e+\eta([x,y,z],u,v)).
\end{aligned}$$

It also holds:

$$\begin{aligned}
&[(x,a),[(u,d),(z,c),(y,b)],(v,e)] \\
&\quad = \quad ([x,[u,z,y],v], a-d+c-b-\eta(u,z,y)+e+\eta(x,[u,z,y],v).
\end{aligned}$$

The two terms coincide (and hence equality 1 holds) if and only if η satisfies the 2-cocycle condition of type 1. ■

Example 4.2.6. The following is a common construction applied to the heap. Let $0 \to A \xrightarrow{\iota} E \xrightarrow{\pi} G \to 0$ be a short exact sequence of abelian groups, and $s : G \to E$ be a set-theoretic section ($\pi s = \mathrm{id}$). Since s is a section, we have that $s(x) - s(y) + s(z) - s([x,y,z])$ is in the kernel of π for all $x, y, z \in G$, so that there is $\eta : G \times G \times G \to A$ such that

$$\iota\eta(x, y, z) = s(x) - s(y) + s(z) - s([x, y, z]).$$

Then computations show the following.

Lemma 4.2.7. *$\eta \in Z^2_{\mathrm{H}}(G, A)$.*

Example 4.2.8. For a positibve integer $n > 0$, let $0 \to \mathbb{Z}_n \xrightarrow{\iota} \mathbb{Z}_{n^2} \xrightarrow{\pi} \mathbb{Z}_n \to 0$ be as above, where $s(x) \bmod(n^2) = x$, representing elements of $\mathbb{Z}_m$ by $\{0, \ldots, m-1\}$. Then for all $x, y, z \in G = \mathbb{Z}_n$, $\iota\eta(x, y, z)$ is divisible by n in $E = \mathbb{Z}_{n^2}$, so that the value of η is computed by $\eta(x, y, z) = \iota\eta(x, y, z)/n$. For example, for $n = 3$, $\eta(2, 0, 2) = [s(2) - s(0) + s(2) - s([2, 0, 2])]/3 = 1 \in \mathbb{Z}_3$. We will show in Example 4.2.13, that $[\eta] \neq 0$ and therefore $H^2_H(\mathbb{Z}_3, \mathbb{Z}_3)$ is nontrivial.

Definition 4.2.7. Let $X \times_\eta A$ and $X \times_{\eta'} A$ be two heap extensions with coefficients in the abelian group A, by two 2-cocycles η and η' of type 1,2 and with degeneracy condition. We define a morphism of extensions, indicated by $\phi : X \times_\eta A \longrightarrow X \times_{\eta'} A$, to be a morphism of heaps making the following diagram (of sets) commutes.

$$\begin{tikzcd} X \times A \arrow[rr, "f"] \arrow[dr, "\pi"'] & & X \times A \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

An invertible morphism of extensions is also called isomorphism of extensions.

Remark 4.2.9. We have a natural equivalence classes decomposition of heap extensions by heap 2-cocycles.

We prove next, that the second heap cohomology group classifies heap extensions.

Proposition 4.2.10. *There is a bijective correspondence between isomorphism classes of heap extensions by A, and the second heap cohomology group $H^2_H(X; A)$.*

Proof. Let us first assume that $\phi : X \times_\eta A \longrightarrow X \times_{\eta'} A$ is an isomorphism of extensions. Since both (x, a) and $\phi(x, a)$ project onto the same element x, we can write the map ϕ, as $\phi(x, a) = (x, a + f(x))$, for some set-theoretic function $f : X \longrightarrow A$. We therefore have

$$\phi([(x, a), (y, b), (z, c)]) = ([x, y, z], a - b + c + \eta(x, y, z) + f([x, y, z])).$$

Similarly, we have

$$[\phi(x, a), \phi(y, b), \phi(z, c)] = ([x, y, z], a + f(x) - b - f(y) + c + f(z) + \eta'(x, y, z)).$$

Since ϕ is a morphism of heaps, we can equate the two expressions. It follows that $\eta = \eta' + \delta^1 f$ and therefore η and η' are in the same cohomology class.

Viceversa, if η and η' represent the same cohomology class, they differ by $\delta^1 f$, for some 1-cochain f. Define ϕ to be $\phi(x, a) = (x, a + f(x))$. Using the same equations as above we see that ϕ is a morphism of heaps. The fact that it is an isomorphism of extensions comes from the fact that it is easily seen to be bijective and obviously $\pi_1(x, a) = x = \pi_1\phi(x, a)$. ∎

Lemma 4.2.11. *Let η_0 be a heap 2-cocycle of type 0 that satisfies the degeneracy condition. Then the following equality holds:*

$$\eta_0(x_1, x_2, x_3) + \eta_0([x_1, x_2, x_3], x_3, x_2) = 0.$$

Proof. Observe that the 2-cocycle condition of type 0 applied to $\eta_0(x_1, x_2, x_3, x_3, x_2)$ reads

$$\begin{aligned}
&\delta^1_{(0)}\eta_0(x_1, x_2, x_3, x_3, x_2) \\
&= \eta_0(x_1, x_2, x_3) + \eta_0([x_1, x_2, x_3], x_3, x_2) \\
&\quad -\eta_0(x_3, x_3, x_2) - \eta_0(x_1, x_2, [x_3, x_3, x_2]).
\end{aligned}$$

Applying the degeneracy condition and the degeneracy heap axiom, we obtain the result. ■

Definition 4.2.8. Let (ζ_1, ζ_2) be a pair of heap 3-cochains. Define a heap 3-cochain of type 0 corresponding to (ζ_1, ζ_2) by

$$\zeta_0(x_1, x_2, x_3, x_4, x_5) = \zeta_1(x_1, x_2, x_3, x_4, x_5) - \zeta_2(x_1, x_4, x_3, x_2, x_5).$$

Definition 4.2.9. Let X be a set with para-associative operation $[\,-\,]$ and A be an abelian group. Let $C^4_{\mathrm{PA(i)}}(X, A)$ be three isomorphic copies of the free abelian group generated by functions $\{f : X^7 \to A\}$ for $i = 1, 2, 3$. Let $C^4_{\mathrm{PA}}(X, A) = \oplus_{i=1,2,3} C^4_{\mathrm{PA(i)}}(X, A)$. For $(\zeta_1, \zeta_2) \in C^3_{\mathrm{PA}}(X, A) = C^3_{\mathrm{PA(1)}}(X, A) \oplus C^3_{\mathrm{PA(2)}}(X, A)$, define $\delta^3_{(i)} : C^3_{\mathrm{PA}}(X, A) \to C^4_{\mathrm{PA(i)}}(X, A)$, for $i = 1, 2, 3$, as follows.

$$\begin{aligned}
&\delta^3_{(1)}(\zeta_1, \zeta_2)(x_1, x_2, x_3, x_4, x_5, x_6, x_7) \\
&= \zeta_1([x_1, x_2, x_3], x_4, x_5, x_6, x_7) + \zeta_1(x_1, x_2, x_3, [x_6, x_5, x_4], x_7) \\
&\quad -\zeta_1(x_6, x_5, x_4, x_3, x_2) - \zeta_1(x_1, x_2, x_3, x_4, x_5) \\
&\quad +\zeta_2(x_1, x_2, x_3, x_4, x_5) - \zeta_1(x_1, x_2, [x_3, x_4, x_5], x_6, x_7),
\end{aligned}$$

$$\begin{aligned}
&\delta^3_{(2)}(\zeta_1, \zeta_2)(x_1, x_2, x_3, x_4, x_5, x_6, x_7) \\
&= \zeta_2(x_1, x_2, x_3, x_4, [x_5, x_6, x_7]) + \zeta_2(x_1, [x_4, x_3, x_2], x_5, x_6, x_7) \\
&\quad -\zeta_2(x_6, x_5, x_4, x_3, x_2) - \zeta_2(x_3, x_4, x_5, x_6, x_7) \\
&\quad +\zeta_1(x_3, x_4, x_5, x_6, x_7) - \zeta_2(x_1, x_2, [x_3, x_4, x_5], x_6, x_7),
\end{aligned}$$

$$\begin{aligned}
&\delta^3_{(3)}(\zeta_1, \zeta_2)(x_1, x_2, x_3, x_4, x_5, x_6, x_7) \\
&= \zeta_2([x_1, x_2, x_3], x_4, x_5, x_6, x_7) + \zeta_1(x_1, x_2, x_3, [x_6, x_5, x_4], x_7) \\
&\quad +\zeta_1(x_6, x_5, x_4, x_3, x_2) - \zeta_1(x_1, x_2, x_3, x_4, [x_5, x_6, x_7]) \\
&\quad -\zeta_2(x_1, [x_4, x_3, x_2], x_5, x_6, x_7) - \zeta_2(x_6, x_5, x_4, x_3, x_2).
\end{aligned}$$

Then define $\delta^3 := \oplus_{i=1,2,3} \delta^3_{(i)} : C^3_{\mathrm{PA}}(X, A) \to C^4_{\mathrm{PA}}(X, A)$.

Let X be a heap, and let $x_i \in X$ for $i = 1, \ldots, 5$. We utilize the following dia-

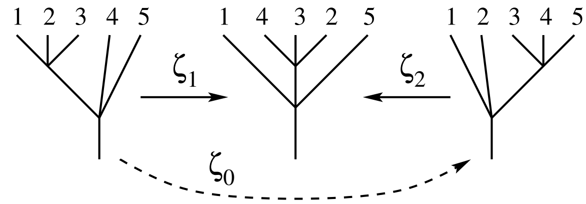

Figure 4.1: Heap 3-cocycle notations

grammatic representations of heap 3-cocycles in Theorem 4.2.12. In Figure 4.1, 3-cocycles are associated to changes of diagrams. The three tree diagrams with top vertices labeled represent the elements in the equality

$$[[x_1, x_2, x_3], x_4, x_5] = [x_1, [x_4, x_3, x_2], x_5] = [x_1, x_2, [x_3, x_4, x_5]],$$

from left to right, respectively. The 3-cocycle $\zeta_1(x_1, x_2, x_3, x_4, x_5)$ (resp. $\zeta_2(x_1, x_2, x_3, x_4, x_5)$) is associated to the change from the left to the middle (resp. the right to the middle) tree diagrams as depicted by the solid arrows. The 3-cocycle $\zeta_0(x_1, x_2, x_3, x_4, x_5)$ is associated to the left to the right, and depicted by the dotted arrow.

In Figure 4.2, the 3-cocycle conditions are represented by diagrams with 7 elements. In the figure, labeled arrows represent 3-cocycles as described above. In the middle, there is a hexagon formed by labeled arrows, and has double arrow labeled by (3). This hexagon represents the differential $\delta^3_{(3)}$. The definition of the differentials, as well as the proof of Theorem 4.2.12, are aided by this figure. This procedure is analogous to the one relating Hochschild cohomology and the Stasheff pentagon.

Theorem 4.2.12. *The composition $\delta^3\delta^2$ vanishes.*

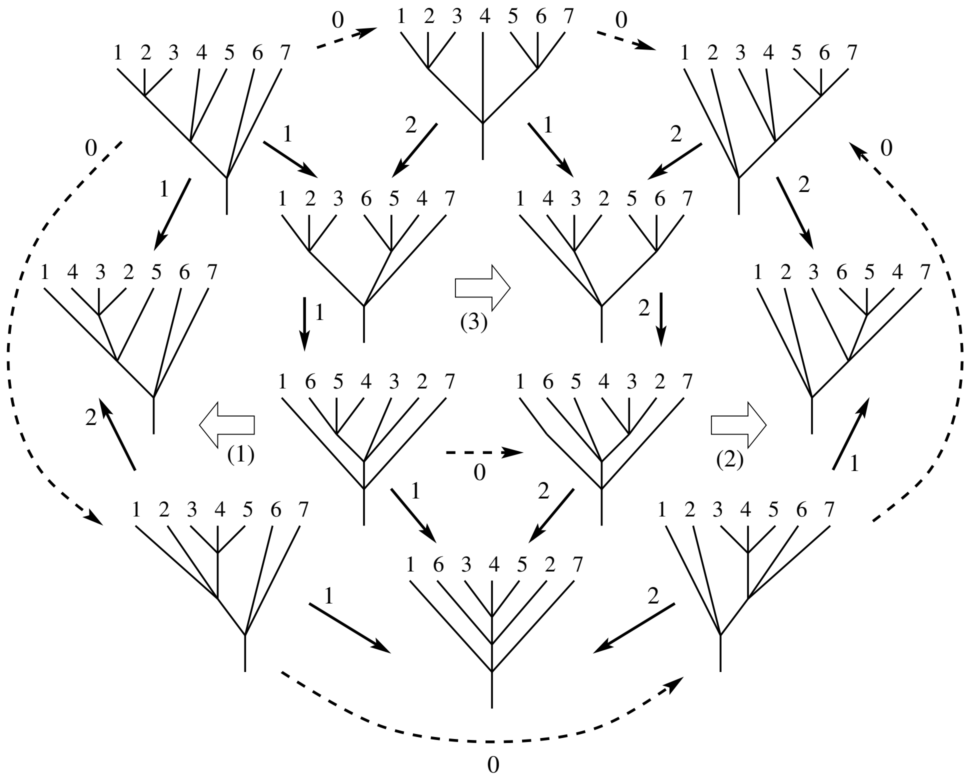

Figure 4.2: Heap 3-cocycle conditions

Proof. This follows by proving, for $\eta \in C^2_{\mathrm{PA}}(X, A)$ and $\zeta_i = \delta^2_{(i)}\eta$ for $i = 1, 2$, that $\delta^3_{(j)}(\zeta_1, \zeta_2) = 0$ for $j = 1, 2, 3$. For $\delta^3_{(3)}(\zeta_1, \zeta_2) = 0$, first we compute positive terms:

$$
\begin{aligned}
&\zeta_2([x_1, x_2, x_3], x_4, x_5, x_6, x_7) \\
&+\zeta_1(x_1, x_2, x_3, [x_6, x_5, x_4], x_7) + \zeta_1(x_6, x_5, x_4, x_3, x_2) \\
&\quad = \quad \{\eta(x_5, x_6, x_7) + \eta([x_1, x_2, x_3], x_4, [x_5, x_6, x_7]) \\
&\qquad\quad -\underline{\eta(x_6, x_5, x_4)} - \underline{\eta([x_1, x_2, x_3], [x_6, x_5, x_4], x_7)}\} \\
&\qquad\quad +\{\eta(x_1, x_2, x_3) + \underline{\eta([x_1, x_2, x_3], [x_6, x_5, x_4], x_7)} \\
&\qquad\quad -\underline{\eta([x_6, x_5, x_4], x_3, x_2)} - \eta(x_1, [[x_6, x_5, x_4], x_3, x_2], x_7)\}
\end{aligned}
$$

$$+\{\underline{\eta(x_6,x_5,x_4)}+\underline{\eta([x_6,x_5,x_4],x_3,x_2)}$$

$$-\eta(x_3,x_4,x_5)-\eta(x_6,[x_3,x_4,x_5],x_2)\}$$

where canceling terms are underlined. For the remaining terms, one computes

$$\begin{aligned}
&\zeta_1(x_1,x_2,x_3,x_4,[x_5,x_6,x_7])\\
&+\zeta_2(x_1,[x_4,x_3,x_2],x_5,x_6,x_7)+\zeta_2(x_6,x_5,x_4,x_3,x_2)\\
&= \quad \{\eta(x_1,x_2,x_3)+\eta([x_1,x_2,x_3],x_4,[x_5,x_6,x_7])\\
&\quad -\underline{\eta(x_4,x_3,x_2)}-\underline{\eta(x_1,[x_4,x_3,x_2],[x_5,x_6,x_7])}\}\\
&\quad +\{\eta(x_5,x_6,x_7)+\underline{\eta(x_1,[x_4,x_3,x_2],[x_5,x_6,x_7])}\\
&\quad -\underline{\eta(x_6,x_5,[x_4,x_3,x_2])}-\eta(x_1,[x_6,x_5,[x_4,x_3,x_2]],x_7)\}\\
&\quad +\{\underline{\eta(x_4,x_3,x_2)}+\underline{\eta(x_6,x_5,[x_4,x_3,x_2])}\\
&\quad -\eta(x_3,x_4,x_5)-\eta(x_6,[x_3,x_4,x_5],x_2)\}
\end{aligned}$$

and all terms cancel.

The conditions $\delta^3_{(1)}(\zeta_1,\zeta_2)=0$ and $\delta^3_{(2)}(\zeta_1,\zeta_2)=0$ follow similarly from direct computations. ■

For type 0 condition, a chain complex is defined in a manner similar to the group cohomology as follows.

Definition 4.2.10. Let X be a heap. The n-th heap chain group with coefficients in the abelian group A, denoted by the symbol $C_H^{2n-1}(X;A)$, is defined to be the dual of the free abelian group on tuples $(x_1, \ldots, x_{2n-1})$, $x_i \in X$, and the boundary map $\delta^n_{(0)} : C_H^{2n-1}(X) \to C_H^{2n+1}(X)$ is defined by

$$\begin{aligned}
&\delta^n_{(0)} f(x_1, \ldots, x_{2n+1}) \\
&\quad = \; -f(x_3, \ldots, x_{2n-1}) \\
&\qquad + \sum_{i=1}^{n} (-1)^{i+1} f(x_1, \ldots, x_{2i-2}, [x_{2i-1}, x_{2i}, x_{2i+1}], x_{2i+2}, \ldots, x_{2n+1}) \\
&\qquad + f(x_1, \ldots, x_{2n-1}),
\end{aligned}$$

for $n \geq 2$ and $\delta^1_{(0)}$ is set to be δ^1 as in Definition 4.2.1.

It is straightforward to verify that the boundary maps defined above do indeed satisfy the differential condition and define therefore a chain complex. The dual cochain groups with coefficient group A and their dual differential maps coincide with those in Definitions 4.2.1 and 4.2.9 for (type 0) cochain maps.

Definition 4.2.11. The homology of the chain complex introduced in Definition 4.2.10 is called *type 0* para-associative (PA) homology, and written $H_n^{(0)}(X)$.

We note that $\partial_2^{(0)}$ defined above is dual to δ^1 in Definition 4.2.1. Therefore if ϕ a 2-coboundary and α is a 2-cycle, then

$$\phi(\alpha) = \delta^1 f(\alpha) = f(\partial_2^{(0)} \alpha) = f(0) = 0.$$

Hence the standard argument applies that if $\phi(\alpha) \neq 0$ for a 2-cycle α then ϕ is not nullcohomologous.

We conclude this section with an example of a non-trivial heap 2-cocycle.

Example 4.2.13. Cosnider $0 \to \mathbb{Z}_3 \xrightarrow{\iota} \mathbb{Z}_9 \xrightarrow{\pi} \mathbb{Z}_3 \to 0$ as in Example 4.2.8 and the corresponding η. The 2-chain $\alpha := (1,0,2) + (0,1,0) + (1,2,0)$ is easily seen to be a heap 2-cycle and $\eta(\alpha) = 1 \neq 0$. Therefore, the discussion above implies that η is not the cohomologous to the trivial cochain. Therefore, $H^2_H(\mathbb{Z}_3, \mathbb{Z}_3) \neq 0$.

4.3 Relation Between Heap and Group (Co)Homology

4.3.1 Group homology and Type 0 Heap Homology

Proposition 4.3.1. *Let X be a heap, $e \in X$, and G be the associated group, so that $x \cdot y = [x, e, y]$ for all $x, y \in X$. Let $\Psi_n : C^G_n \to C^{(0)}_n(X)$ be the map on chain groups defined by*

$$\Psi_n(x_1, \ldots, x_n) = (x_1, e, x_2, e, \ldots, e, x_n).$$

Then $\Psi_\bullet$ is a chain map and therefore induces a well defined map

$$\bar{\Psi}_n : H^{(0)}_n(X) \to H^G_n(X)$$

.

Proof. This is a direct computation, using the fact that $[x_{2i}, e, x_{2i+1}] = x_{2i} \cdot x_{2i+1}$ by definition.

$\blacksquare$

Remark 4.3.2. By dualizing Proposition 4.3.1, the previous result, we obtain a cochain map between type zero heap cohomology and group cohomology. In the specific case of the second cohomology group, we observe that Proposition 4.3.1 corresponds to the construction of a group from a heap through extensions as follows. Let X be a heap, A an abelian group, $E = X \times A$ the heap extension defined in Lemma 4.2.6 with a heap 2-cocycle η. Let $(e, c) \in E$ be a fixed element. Then the group structure on E defined from the heap structure on E is computed as

$$\begin{aligned}
&(x, a) \cdot (y, b) \\
&\quad = \quad [(x, a), (e, c), (y, b)] \\
&\quad = \quad (\ [x, e, y], a - c + b + \eta(x, e, y)\ ) \\
&\quad = \quad (xy, a + b + \theta(x, y))
\end{aligned}$$

giving rise to the relation $\theta(x, y) = \eta(x, e, y) - c$, a difference of a constant comparing to Proposition 4.3.1.

Let X be a heap, and $e \in X$. We define chain subgroups $\widehat{C}_n^{(0)}(X)$ by the free abelian group generated by

$$\{((x_1, \ldots, x_{2n-1}) \in C_n^{(0)}(X) \mid x_{2i} = e,\ i = 1, 2, \ldots, n-1\},$$

and $\widehat{C}_1^{(0)}(X) = C_1^{(0)}(X)$. It is checked by direct computation that $\partial_n^{(0)}(\widehat{C}_n^{(0)}(X)) \subset \widehat{C}_{n-1}^{(0)}(X)$, so that $\{\widehat{C}_n^{(0)}(X), \partial_n^{(0)}\}$ forms a chain subcomplex. Let $\widehat{H}_n^{(0)}(X)$ denote the homology groups of this subcomplex, and let $\widetilde{H}_n^{(0)}(X)$ denote the relative homology groups for the quotient $C_n^{(0)}(X)/\widehat{C}_n^{(0)}(X)$. We now have the following result.

Theorem 4.3.3. *In the same setting as in Proposition 4.3.1, the map $\bar{\Psi}_n$ is an injection for all n. Furthermore, there is a long exact sequence of homology groups*

$$\cdots \to H_n^G(X) \to H_n^{(0)}(X) \to \widetilde{H}_n^{(0)}(X) \xrightarrow{\partial} H_{n-1}^G(X) \to \cdots .$$

Proof. The chain map Ψ gives an isomorphism between chain groups $C_n^G(X)$ and $\widehat{C}_n^{(0)}(X)$ and commute with differentials, giving rise to an isomorphism of chain complexes. Through the map $\bar{\Psi}_n$, $H_n^G(X)$ is identified with $\widehat{H}_n^{(0)}(X)$.

The second statement follows from the short exact sequence of chain complexes

$$0 \to \widehat{C}_\bullet^{(0)}(X) \to C_\bullet^{(0)}(X) \to C_\bullet^{(0)}(X)/\widehat{C}_\bullet^{(0)}(X) \to 0$$

using the isomorphism $\Psi_\bullet$ and defining ∂ as the usual connecting homomorphism, via the Snake Lemma. ∎

Definition 4.3.1. The homology $\widetilde{H}_\bullet^{(0)}(X)$ is called the type zero *essential* heap homology.

Remark 4.3.4. The essential homology of a group heap X is regarded as a measure of how far is group homology from being isomorphic to the type zero heap homology.

Example 4.3.5. We show that $\widetilde{H}^{(0)}_{\bullet}(X)$ can be nontrivial. Consider the group heap correspoding to $\mathbb{Z}_2$. The 2-chain $(0,1,1)$ is easily seen to be a type zero 2-cycle. We show that the class $[(0,1,1)] \in \widetilde{H}^{(0)}_2(X)$ is nontrivial. The 2-cochain $\eta(1,1,1) = \eta(1,1,0) = \eta(0,1,1) = 1$, and zero otherwise is a heap 2-cocycle, as seen in Example 4.2.4. As previously observed, a heap 2-cocycle is also a type zero 2-cocycle. Furthermore, $\partial^{(0)}_1$ is dual to $\delta^1_{\rm PA}$, so that η is nontrivial as a type zero heap cocycle. Suppose that $[(0,1,1)] = 0$ in $\widetilde{H}^{(0)}_2(X)$. Then there is a 3-chain α such that $\partial^{(0)}_3\alpha - (0,1,1) \in \widehat{C}^{(0)}_2(X)$. Therefore $\eta(\partial^{(0)}\alpha - (0,1,1)) = 0$, since by definition η vanishes on $\widehat{C}^{(0)}_2(X)$. Since $\eta(\partial^{(0)}) = \delta^{(0)}\eta$ and η is a type zero 2-cocycle, we have obtained that $\eta(0,1,1) = 0$, in contradiction with the choice of η. Therefore $[(0,1,1)]$ is nontrivial in $\widetilde{H}^{(0)}_2(X)$.

4.3.2 From Group Cocycles to PA Cocycles

In this section we present a construction of PA 2-cocycles from group 2-cocycles. The following gives an answer to a natural question on how the relation between groups and heaps descends to relations in their homology theories. It also provides a construction of ternary self-distributive 2-cocycles from group 2-cocycles through heap 2-cocycles (Section 4.5). We recall that the group 2-cocycle condition with trivial action on the coefficient group is written as

$$\theta(x,y) + \theta(xy,z) = \theta(y,z) + \theta(x,yz)$$

for all $x,y,z \in G$ of a group G. The *normalized* 2-cocycle satisfies $\theta(x,1) = 0 = \theta(1,x)$, and it follows that normalized 2-cocycles satisfy $\theta(x,x^{-1}) = \theta(x^{-1},x)$. The notion of normalized cocycles easily generalizes to n-cocycles and they form a sub-complex of the group cochain complex. We indicate the normalized cohomology by the symbol $\widehat{H}^n_G(X)$. We refer the reader to the classical reference [Bro82], for more details regarding group cohomology.

Theorem 4.3.6. *Let G be a group, and X be the associated heap defined by $[x,y,z]=xy^{-1}z$ for $x,y,z\in G$. Let θ be a normalized group 2-cocycle with trivial action on the coefficient group A. Then*

$$\eta(x,y,z) := \theta(x,y^{-1}) + \theta(xy^{-1},z) - \theta(y,y^{-1})$$

is a PA 2-cocycle. This construction defines a cohomology map $\bar{\Phi}:\widehat{H}^2_G(X)\to H^2_{PA}(X)$.

Proof. First we note that for an extension group 2-cocycle θ, the condition $y^{-1}(zu^{-1}) = ((uz^{-1})y)^{-1}$ implies following identity

$$\begin{aligned}
&\theta(z,u^{-1}) + \theta(y^{-1},zu^{-1}) - \theta(y,y^{-1}) - \theta(u,u^{-1}) \\
&= \theta(u,z^{-1}) + \theta(z,z^{-1}) + \theta(u^{-1},uz^{-1}) - \theta(y,y^{-1}) \\
&= \theta(u,z^{-1}) + \theta(uz^{-1},y) - \theta(z,z^{-1}) - \theta(uz^{-1}y, y^{-1}zu^{-1}),
\end{aligned}$$

which we call product-inversion relation. For $\delta^2_{(1)}(\eta)=0$, one computes

$$\begin{aligned}
&\eta(x,y,z) + \eta([x,y,z],u,v) \\
&= \underline{\theta(x,y^{-1})} + \underline{\theta(xy^{-1},z)} - \theta(y,y^{-1}) \\
&\quad +\theta(xy^{-1}z,u^{-1}) + \theta(xy^{-1}zu^{-1},v) - \theta(u,u^{-1})
\end{aligned}$$

$$
\begin{aligned}
&= \theta(y^{-1},z)+\underline{\theta(x,y^{-1}z)}-\theta(y,y^{-}1)\\
&\quad +\underline{\theta(xy^{-1}z,u^{-1})}-\theta(u,u^{-1})+\theta(xy^{-1}zu^{-1},v)\\
&= \underline{\theta(y^{-1},z)}+\underline{\theta(y^{-1}z,u^{-1})}+\theta(x,y^{-1}zu^{-1})\\
&\quad -\theta(y,y^{-1})-\theta(u,u^{-1})+\theta(xy^{-1}zu^{-1},v)\\
&= \theta(z,u^{-1})+\theta(y^{-1},zu^{-1})+\theta(x,y^{-1}zu^{-1})\\
&\quad +\theta(xy^{-1}zu^{-1},v)-\theta(y,y^{-1})-\theta(u,u^{-1})\\
&= \theta(u,z^{-1})+\theta(uz^{-1},y)-\theta(z,z^{-1})\\
&\quad -\theta(uz^{-1}y,y^{-1}zu^{-1})+\theta(x,y^{-1}zu^{-1})+\theta(xy^{-1}zu^{-1},v)\\
&= \eta(u,z,y)+\eta(x,[u,z,y],v),
\end{aligned}
$$

where we have underlined the terms undergoing the group 2-cocycle relation at each step, and used the product-inverse relation in the penultimate equality. Similar computations prove the equality $\delta^2_{(2)}(\eta)=0$.

To complete the proof, consider the maps $\Phi_1 := -\mathbb{1} : \widehat{C}^1_G(X) \to C^1_{PA}(X)$, and $\Phi_2 : \widehat{C}^2_G(X) \longrightarrow C^2_{PA}(X)$, $\theta \mapsto \eta$, as in the previous part of the proof. It is easy to see that $\delta^1_G\Phi_2 = \Phi_1\delta^1_{PA}$, therefore showing that $\bar{\Phi}$ is well defined on cohomology groups. ■

Remark 4.3.7. Extensions of groups and heaps, in this case, are related as in Remark 4.3.2. The group extension is defined, for a group G and the coefficient abelian group A, by

$$(x,a)\cdot(y,b)=(xy,a+b+\theta(x,y))$$

for $x, y \in G$ and $a, b \in A$. For the heap $E = G \times A$ constructed from the group $E = G \times A$ defined above, one computes

$$\begin{aligned}
&[\ (x,a),(y,b),(z,c)\ ] \\
&= (x,a)(y,b)^{-1}(z,c) \\
&= (x,a)(y^{-1}, -b-\theta(y,y^{-1}))(z,c) \\
&= (xy^{-1}z, a-b+c+\theta(x,y^{-1})+\theta(xy^{-1},z)-\theta(y,y^{-1}))
\end{aligned}$$

so that we obtain the correspondence

$$\eta(x,y,z) = \theta(x,y^{-1}) + \theta(xy^{-1},z) - \theta(y,y^{-1}).$$

4.4 Ternary Self-Distributive Cohomology with Heap Coefficients

In this section we introduce a cohomology theory of ternary self-distributive (TSD) operations with abelian heap coefficients, and investigate extension theory by 2-cocycles.

Definition 4.4.1. Let (X, T) be a ternary shelf. We define the n^{th} chain group of X with heap coefficients in A, denoted by $C_n^{\text{SD}}(X)$, to be the free abelian group on $(2n-1)$-tuples

X^{2n-1}. We introduce differentials $\partial_n : C_n^{\mathrm{SD}}(X) \longrightarrow C_{n-1}^{\mathrm{SD}}(X)$ by the formula

$$\begin{aligned}
&\partial_n(x_1, x_2, \ldots, x_{2n-2}, x_{2n-1}) \\
&= \sum_{i=2}^{n} (-1)^{n+i} [(x_1, \ldots, \widehat{x_{2i}}, \widehat{x_{2i+1}}, \ldots, x_{2n-1}) \\
&\quad -(T(x_1, x_{2i}, x_{2i+1}), \ldots, T(x_{2i-1}, x_{2i}, x_{2i+1}), \widehat{x_{2i}}, \widehat{x_{2i+1}}, \ldots, x_{2n-1})] \\
&\quad +(-1)^{n+1}[(x_1, x_4, \ldots, x_{2n-1}) - (x_2, x_4, \ldots, x_{2n-1}) \\
&\quad +(x_3, x_4, \ldots, x_{2n-1}) - (T(x_1, x_2, x_3), x_4, \ldots, x_{2n-1})],
\end{aligned}$$

where $\widehat{\ }$ denotes the deletion of that factor.

Proposition 4.4.1. *The differential maps in Definition 4.4.1 satisfy the condition $\partial^2 = 0$.*

Proof. We can write the differential in the following form:

$$\partial_n = (-1)^n \sum_{i=1}^{n} (-1)^i \partial_n^i,$$

where ∂_n^i is defined, for $i \geq 2$, by the formula

$$\begin{aligned}
&\partial_n^i(x_1, \ldots, x_{2n-1}) \\
&= (x_1, \ldots, \widehat{x_{2i}}, \widehat{x_{2i+1}}, \ldots, x_{2n-1}) \\
&\quad -f(T(x_1, x_{2i}, x_{2i+1}), \ldots, T(x_{2i-1}, x_{2i}, x_{2i+1}), \widehat{x_{2i}}, \widehat{x_{2i+1}}, \ldots, x_{2n-1}),
\end{aligned}$$

and, for $i = 1$, is defined by the formula

$$\begin{aligned}
&\partial_n^i(x_1, \ldots, x_{2n-1}) \\
&= (x_1, x_4, \ldots, x_{2n-1}) - (x_2, x_4, \ldots, x_{2n-1}) \\
&\quad +(x_3, x_4, \ldots, x_{2n-1}) - (T(x_1, x_2, x_3), x_4, \ldots, x_{2n-1}).
\end{aligned}$$

Now it remains to prove the relations $\partial_{n-1}^{j-1}\partial_n^i = \partial_{n-1}^i\partial_n^j$ for $i < j$. The cases with $i \geq 2$ are standard, while the remaining cases $1 = i < j$ can be checked directly. ■

The cycle, boundary and homology groups and their duals with respect to an abelian group A are defined as usual. Our focus is on significance and constructions of 2-cocycles in relation to heaps for this theory, so that we provide explicit cocycle conditions in low dimensions below.

Example 4.4.2. Let (X, T) be a ternary shelf, and A be an abelian group heap. Then cochain groups and differentials dual to Definition 4.4.1 in low dimensions are formulated as follows. The cochain groups $C^n_{\mathrm{SD}}(X, A)$ are defined to be the abelian groups of functions $\{f : X^{2n-1} \to A\}$. The differentials $\delta^n = \delta^n_{\mathrm{SD}} : C^n_{\mathrm{SD}}(X, A) \to C^{n+1}_{\mathrm{SD}}(X, A)$ are formulated for

$n = 1, 2, 3$ as follows.

$$\begin{aligned}
&\delta^1\xi(x_1, x_2, x_3)\\
&\quad= \xi(x_1) - \xi(x_2) + \xi(x_3) - \xi(T(x_1, x_2, x_3)),\\
&\delta^2\eta(x_1, x_2, x_3, x_4, x_5)\\
&\quad= \eta(x_1, x_2, x_3) + \eta(T(x_1, x_2, x_3), x_4, x_5)\\
&\qquad -\eta(x_1, x_4, x_5) + \eta(x_2, x_4, x_5) - \eta(x_3, x_4, x_5)\\
&\qquad -\eta(T(x_1, x_4, x_5), T(x_2, x_4, x_5), T(x_3, x_4, x_5)),\\
&\delta^3\psi(x_1, x_2, x_3, x_4, x_5, x_6, x_7)\\
&\quad= \psi(x_1, x_2, x_3, x_4, x_5) + \psi(T(x_1, x_4, x_5), T(x_2, x_4, x_5), T(x_3, x_4, x_5), x_6, x_7)\\
&\qquad +\psi(x_1, x_4, x_5, x_6, x_7) - \psi(x_2, x_4, x_5, x_6, x_7) + \psi(x_3, x_4, x_5, x_6, x_7)\\
&\qquad -\psi(T(x_1, x_2, x_3), x_4, x_5, x_6, x_7) - \psi(x_1, x_2, x_3, x_6, x_7)\\
&\qquad -\psi(T(x_1, x_6, x_7), T(x_2, x_6, x_7), T(x_3, x_6, x_7), T(x_4, x_6, x_7), T(x_5, x_6, x_7)).
\end{aligned}$$

The case $n = 0$ is defined by convention that $C^0_{\mathrm{SD}}(X, A) = 0$.

Definition 4.4.2. Let (X, T) be a ternary self-distributive set, A an abelian group heap and $\eta : X \times X \times X \longrightarrow A$ a 2-cochain of X with values in A. We define the self-distributive cocycle extension of X with heap coefficients in A, by the cocycle η to be the cartesian product $X \times A$, endowed with the ternary operation T' given by

$$(x, a) \times (y, b) \times (z, c) \mapsto (T(x, y, z), a - b + c + \eta(x, y, z)).$$

In this situation we denote the extension by $X \times_\eta A$.

Lemma 4.4.3. *The TSD 2-cocycle condition gives extension cocycles of TSDs with abelian group heap coefficients. Specifically, the ternary operation in Definition 4.4.2, corresponding to a 2-cocycle η satisfying the second condition $\delta^2\eta = 0$ in 4.4.2, is ternary self-distributive.*

Definition 4.4.3. Given two extensions $X \times_\eta A$ and $X \times_{\eta'} A$, we define a morpshim of extensions to be a morphism of ternary self-distributive sets making commutative a diagram identical to the one in Definition 4.2.7. An invertible morphism of extensions is called isomorphism.

Similarly to Definition 4.2.7, we have a subdivision in equivalence classes of extensions. We have the following result.

Proposition 4.4.4. *There is a bijective correspondence between $H^2_{\mathrm{SD}}(X, A)$ and equivalence classes of extensions.*

Proof. Similar to the group-theoretic case and Proposition 4.2.10. ∎

Example 4.4.5. Let $X = \mathbb{Z}_2$ with the TSD operation $T(x, y, z) = x + y + z \in \mathbb{Z}_2$. This is in fact the abelian heap $\mathbb{Z}_2$ and by Lemma 4.5.1 below, the same operation is self-distributive. In this example we compute the first cohomology group $H^1_{\mathrm{SD}}(X, \mathbb{Z}_2)$ and the second cohomology group $H^2_{\mathrm{SD}}(X, \mathbb{Z}_2)$ with coefficients in the abelian heap $\mathbb{Z}_2$. For a function $f : X \to \mathbb{Z}_3$, a straightforward computation gives that $\delta^1(f) = 0$. This gives $H^1_{\mathrm{SD}}(X, \mathbb{Z}_2) \cong C^1_{\mathrm{SD}}(X, \mathbb{Z}_2)$. To compute the kernel of δ^2, let us write an element $\phi : X^3 \to \mathbb{Z}_2$ in term of characteristic functions as $\phi = \sum_{x,y,z} \phi(x,y,z)\chi_{(x,y,z)}$. Then $\delta^2(\phi) = 0$ gives the following system of equations in $\mathbb{Z}_2$:

$$\begin{cases} \phi(1,1,1) + \phi(0,0,0) = 0 \\ \phi(1,1,0) + \phi(0,0,1) = 0 \\ \phi(1,0,1) + \phi(0,1,0) = 0 \\ \phi(1,0,0) + \phi(0,1,1) = 0 \end{cases}$$

implying that $\ker(\delta^2)$ is 4-dimensional with a basis $\chi_{(1,1,1)} + \chi_{(0,0,0)}$, $\chi_{(1,1,0)} + \chi_{(0,0,1)}$, $\chi_{(1,0,1)} + \chi_{(0,1,0)}$, and $\chi_{(1,0,0)} + \chi_{(0,1,1)}$. Since $\mathrm{im}(\delta^1) = 0$ we then obtain that $H^2_{\mathrm{SD}}(X, \mathbb{Z}_2) \cong {\mathbb{Z}_2}^{\oplus 4}$.

Example 4.4.6. In this example we compute the first cohomology group $H^1_{\mathrm{SD}}(X, \mathbb{Z}_3)$ and the second cohomology group $H^2_{\mathrm{SD}}(X, \mathbb{Z}_3)$ for the same $X = \mathbb{Z}_2$ as above, with coefficients in the abelian heap $\mathbb{Z}_3$. For a function $f : X \to \mathbb{Z}_3$, a direct computation gives that $\delta^1(f)(1,0,1) = f(0) - f(1)$, $\delta^1(f)(0,1,0) = f(1) - f(0)$ and all other unspecified values of $\delta^1(f)(x,y,z)$ are zeros. This gives $H^1_{\mathrm{SD}}(X, \mathbb{Z}_3) \cong \mathbb{Z}_3$. To compute the kernel of δ^2, let us write an element $\phi : X^3 \to \mathbb{Z}_3$ in term of characteristic functions as $\phi = \sum_{x,y,z} \phi(x,y,z)\chi_{(x,y,z)}$. Then hand computations give that $\ker(\delta^2)$ is 3-dimensional with a basis $\chi_{(1,1,1)} + \chi_{(0,0,0)} + \chi_{(1,0,0)} + \chi_{(0,1,1)}$, $\chi_{(1,1,0)} + \chi_{(0,0,1)} - \chi_{(0,1,0)}$ and $\chi_{(1,0,1)} - \chi_{(0,1,0)}$. Since $\mathrm{im}(\delta^1)$ is generated by $\chi_{(1,0,1)} - \chi_{(0,1,0)}$, we then obtain that $H^2_{\mathrm{SD}}(X, \mathbb{Z}_3) \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3$.

Example 4.4.7. If the TSD set X is trivial, that is $T(x,y,z) = x$ for all $x,y,z \in X$, then the differentials δ^1 and δ^2 take the following simpler forms:

$$\begin{aligned}
\delta^1\xi(x,y,z) &= \xi(z) - \xi(y), \\
\delta^2\eta(x,y,z,u,v) &= \eta(y,u,v) - \eta(z,u,v).
\end{aligned}$$

This gives, for an abelian group A,

$$\operatorname{im}(\delta^1) = \{\eta : X^3 \to A, \eta(x,y,z) = \xi(z) - \xi(y), \text{for some map } \xi : X \to A\}.$$

Thus $H^1_{\mathrm{SD}}(X,A) = Z^1_{\mathrm{SD}}(X,A)$ is the group of constant functions, which is isomorphic to A. The kernel of δ^2 is given by

$$\ker(\delta^2) = \{\eta : X^3 \to A, \eta(x,y,z) = \eta(x',y,z), \forall x,x',y,z \in X\}$$

that are functions constant on the first variable. Hence $Z^2_{\mathrm{SD}}(X,A)$ is isomorphic to $A^{X\times X}$, the group of functions $A^{X\times X}$ from $X \times X$ to A. This group has the subgroup $B^1_{\mathrm{SD}}(X,A) = \operatorname{im}(\delta^1) = \{\eta(x,y,z) = \xi(z) - \xi(y) \mid \xi \in A^X\}$.

For example, if X is an n element set and $A = \mathbb{Z}_p$ for a prime p, then $Z^2_{\mathrm{SD}}(X,A) \cong \mathbb{Z}_p^{\oplus n^2}$, $B^1_{\mathrm{SD}}(X,A) \cong \mathbb{Z}_p^{\oplus n}$ and $H^1_{\mathrm{SD}}(X,A) \cong \mathbb{Z}_p^{\oplus n}$.

The following provides an algebraic meaning of the TSD 3-cocycle condition.

Proposition 4.4.8. *The TSD 3-cocycle condition gives obstruction cocycles of TSDs for short exact sequences of coefficients. Specifically, let X be a TSD set and consider a short* *exact sequence of abelian groups,*

$$0 \longrightarrow H \xrightarrow{\iota} E \xrightarrow{\pi} A \longrightarrow 0,$$

where E is the extension heap corresponding to the 2-cocycle $\phi \in Z^2(X, A)$, and a section $s : A \longrightarrow E$, such that $s(0) = 0$, the obstruction for $s\phi$ to satisfy the 2-cocycle condition is a 3-cocycle with heap coefficients in H.

Proof. We construct the mapping $\alpha : X^5 \longrightarrow H$ by the equality

$$\begin{aligned}
&\iota\alpha(x_1, \ldots, x_5) \\
&= \quad s\phi(x_1, x_2, x_3) - s\phi(T(x_1, x_4, x_5), T(x_2, x_4, x_5), T(x_3, x_4, x_5)) \\
&\quad\quad + s\phi(T(x_1, x_2, x_3), x_4, x_5) - s\phi(x_1, x_4, x_5) \\
&\quad\quad + s\phi(x_2, x_4, x_5) - s\phi(x_3, x_4, x_5).
\end{aligned}$$

Since ϕ satisfies the 2-cocycle condition, we see that $\pi\alpha$ is the zero map, where $\pi : E \to A$ is the projection. It follows that there is $\alpha : X^5 \longrightarrow H$ satisfying the above equality. We claim that $\alpha : X^5 \longrightarrow H$ so defined satisfies the 3-cocycle condition with heap coefficients in H. To shorten the computation we introduce the following notation. The ternary operation will be indicated in exponential form, $T(x, y, z) = x^{yz}$, we will identify the element x_i with its index i, will not write $s\phi$ in front of the parenthesis and, finally, we will use bars to separate the entries in the tuples. For example we will write $(1^{23}|4|5)$ for $s\phi(T(x_1, x_2, x_3), x_4, x_5)$. The

3-cocycle condition applied on α now reads

$$
\begin{aligned}
&\delta^3\alpha(x_1,\ldots,x_7)\\
&= (1|2|3)-(1^{45}|2^{45}|3^{45})+(1^{23}|4|5)-(1|4|5)\\
&\quad +(2|4|5)-(3|4|5)+(1^{45}|2^{45}|3^{45})-(1^{(45)(67)}|2^{(45)(67)}|3^{(45)(67)})\\
&\quad +(1^{(45)(2^{45}3^{45})}|6|7)-(1^{45}|6|7)+(2^{45}|6|7)-(3^{45}|6|7)\\
&\quad +(1|4|5)-(1^{67}|4^{67}|5^{67})+(1^{45}|6|7)-(1|6|7)\\
&\quad +(4|6|7)-(5|6|7)-(2|4|5)+(2^{67}|4^{67}|5^{67})\\
&\quad -(2^{45}|6|7)+(2|6|7)-(4|6|7)+(5|6|7)\\
&\quad +(3|4|5)-(3^{67}|4^{67}|5^{67})+(3^{45}|6|7)-(3|6|7)\\
&\quad +(4|6|7)-(5|6|7)-(1^{23}|4|5)+(1^{(23)(67)}|4^{67}|5^{67})\\
&\quad -(1^{(23)(45)}|6|7)+(1^{23}|6|7)-(4|6|7)+(5|6|7)\\
&\quad -(1|2|3)+(1^{67}|2^{67}|3^{67})-(1^{23}|6|7)+(1|6|7)\\
&\quad -(2|6|7)+(3|6|7)-(1^{67}|2^{67}|3^{67})\\
&\qquad\qquad\qquad +((1^{(67)(4^{67}5^{67})}|2^{(67)(4^{67}5^{67})}|3^{(67)(4^{67}5^{67})}))\\
&\quad -(1^{(67)(2^{67}3^{67})}|4^{67}|5^{67})+(1^{67}|4^{67}|5^{67})-(2^{67}|4^{67}|5^{67})+(3^{67}|4^{67}|5^{67}).
\end{aligned}
$$

Most of the terms cancel right away, while the remaining terms can be seen to cancel after one application of the TSD property. This concludes the proof. ∎

4.5 From Heap Cocycles to TSD Cocycles

In this section, we show that heaps and their 2-cocycles give rise to those for TSDs. In particular, combining this result with Lemma 4.3.1, we obtain a way to construct TSD 2-cocycles with heap coefficients from group 2-cocycles. We start with a preliminary result, implicitly present in [ESZb] and Chapter 3, showing that heaps are a particular instance of TSD structures.

Lemma 4.5.1. *If* $[-]$ *is a heap operation on* X*, then the same operation is ternary self-distributive.*

Proof. First we note that for a heap operation it holds that

$$[[x, y, z], z, y] = [x, y, [z, z, y]] = [x, y, y] = x.$$

Then one computes

$$\begin{aligned}
&T(T(x_1, x_4, x_5), T(x_2, x_4, x_5), T(x_3, x_4, x_5)) \\
&= \ [\ [x_1, x_4, x_5], [x_2, x_4, x_5], [x_3, x_4, x_5]\ ] \\
&= \ [\ x_1, [\ [x_2, x_4, x_5], x_5, x_4], [x_3, x_4, x_5]\ ] \\
&= \ [\ x_1, x_2, [x_3, x_4, x_5]\ ] \\
&= \ [\ [x_1, x_2, x_3], x_4, x_5] \\
&= \ T(T(x_1, x_2, x_3), x_4, x_5))
\end{aligned}$$

as desired. The notation $T(x,y,z) = [x,y,z]$ was used for clarification. ■

We now state and prove the main result of the section.

Theorem 4.5.2. *Let X be a heap, with the operation regarded as a TSD operation by Lemma 4.5.1, and let A be an abelian group. Suppose that $\eta \in Z^2_{\mathrm{H}}(X,A)$, that is, η satisfies $\delta^2_{(1)}\eta = 0 = \delta^2_{(2)}\eta$ and the degeneracy condition. Then η is a TSD 2-cocycle, $\eta \in Z^2_{\mathrm{SD}}(X,A)$. This assignment induces an injection of $H^2_H(X,A)$ into $H^2_{SD}(X,A)$.*

Proof. We note that $\delta^2_{(1)}\eta = 0 = \delta^2_{(2)}\eta$ also implies $\delta^2_{(0)}\eta = 0$, and the equality $[[x,y,z],z,y] = x$ from the proof of Lemma 4.5.1. One computes

$$
\begin{aligned}
& \underline{\eta(x_1,x_4,x_5)} - \eta(x_2,x_4,x_5) + \eta(x_3,x_4,x_5) \\
& \qquad + \underline{\eta(T(x_1,x_4,x_5), T(x_2,x_4,x_5), T(x_3,x_4,x_5))} \\
= {} & -\eta([x_2,x_4,x_5],x_5,x_4) + \underline{\eta(x_1,[[x_2,x_4,x_5],x_5,x_4],[x_3,x_4,x_5])} \\
& \qquad\qquad\qquad (= \eta(x_1,x_2,[x_3,x_4,x_5])\ ) \\
& -\eta(x_2,x_4,x_5) + \underline{\eta(x_3,x_4,x_5)} \\
= {} & \eta(x_1,x_2,x_3) + \eta([x_1,x_2,x_3],x_4,,x_5) \\
& -\underline{\eta([x_2,x_4,x_5],x_5,x_4)} - \underline{\eta(x_2,x_4,x_5)} \\
= {} & \eta(x_1,x_2,x_3) + \eta(T(x_1,x_2,x_3),x_4,,x_5)
\end{aligned}
$$

as desired. The equalities follow from $\delta^2_{(1)}\eta = 0$, $\delta^2_{(0)}\eta = 0$ and Lemma 4.2.11, respectively, and the underlined terms indicate where they are applied. This proves that we have an inclusion $h : Z^2_{\mathrm{H}}(X,A) \hookrightarrow Z^2_{\mathrm{SD}}(X,A)$. Since we have the equality $C^1_{\mathrm{H}}(X,A) = C^1_{\mathrm{SD}}(X,A)$ and the first cochain differentials for heap and TSD cohomologies coincide up to sign, $\delta^1_{\mathrm{H}} = -\delta^1_{\mathrm{SD}}$, we have $h(\delta^1_{\mathrm{H}}(f)) = -\delta^1_{\mathrm{SD}}(h(f))$ and $h(B^2_{\mathrm{H}}(X,A)) \subset B^2_{\mathrm{SD}}(X,A)$, so that h induces a homomorphism $\bar{h} : H^2_{\mathrm{H}}(X,A) \to H^2_{\mathrm{SD}}(X,A)$. Lastly, the map $\bar{h}$ is injective. Indeed, for $\eta \in Z^2_{\mathrm{H}}(X,A)$, assume that $h(\eta) \in Z^2_{\mathrm{SD}}(X,A)$ is null-cohomologous. Then $h(\eta) = \delta^1_{\mathrm{SD}}(\xi')$ for some $\xi' \in C^1_{\mathrm{SD}}(X,A)$. For $\xi = -\xi' \in C^1_{\mathrm{H}}$, we have $\eta = \delta^1_{\mathrm{H}}(\xi)$, so that η is null-cohomologous in $Z^2_{\mathrm{H}}(X,A)$. ■

Example 4.5.3. In Example 4.2.13, a nontrivial heap 2-cocycle η was given for $X = \mathbb{Z}_3 = A$. By Theorem 4.5.2, η is a non-trivial TSD 2-cocycle. Hence we obtain $H^2_{\mathrm{TSD}}(X,A) \neq 0$.

Remark 4.5.4. The construction in Theorem 4.5.2 and taking extensions commute (c.f. Remarks 4.3.2 and 4.3.7). Indeed, for a heap X and an abelian heap A, the heap extension $X \times A$ by a heap 2-cocycle η is defined by

$$[(x,a),(y,b),(z,c)] = (\,[x,y,z], a - b + c + \eta(x,y,z)\,),$$

and Lemma 4.5.1 states that this heap operation gives a ternary shelf. On the other hand, this is the extension of a ternary shelf by a TSD 2-cocycle η with the heap coefficient A by Definition 4.4.2.

4.6 Categorical Heaps

In this section we proceed to introduce a more general version of heap, in symmetric monoidal categories. This construction is particularly suitable to produce objects that behave in a similar fashion with respect to the set-theoretic version studied thus far, and therefore interesting to from a point of view of self-distributive property and low-dimensional topology related to it. The definition introduced in this section is similar to the one previously sudied by Booker and Street in [BS11]. The reasons that lead us to it, are quite different from the scopes set in [BS11], though.

Throughout the section all symmetric monoidal categories are strict (the associator $(A \boxtimes B) \boxtimes C \to A \boxtimes (B \boxtimes C)$, the right and left unitors $I \boxtimes X \to X$ and $X \boxtimes I \to X$ are all identity maps, where I is the unit object).

Let $(\mathcal{C}, \boxtimes)$ be a symmetric monoidal category, (X, Δ, ϵ) be a comonoid object in $\mathcal{C}$ and consider a morphism $\mu : X \boxtimes X \boxtimes X \to X$. We translate the heap axioms of Section 4.1 into commutative diagrams in the category $\mathcal{C}$. The equalities of type 1 and 2 para-associativity are defined by the commutative diagram

$$\begin{tikzcd}
X^{\boxtimes 3} \arrow[dr, "\mu"'] & X^{\boxtimes 5} \arrow[l, "\mu \boxtimes \mathbb{1}^2"'] \arrow[r, "\mathbb{1}^2 \boxtimes \mu"] \arrow[d] & X^{\boxtimes 3} \arrow[dl, "\mu"] \\
 & X &
\end{tikzcd}$$

where the central arrow corresponds to the morphism

$$\mu(\mathbb{1} \boxtimes \mu \boxtimes \mathbb{1})(\mathbb{1} \boxtimes \tau \boxtimes \mathbb{1}^2)(\mathbb{1}^2 \boxtimes \tau \boxtimes \mathbb{1}).$$

The type 0 para-associativity is defined by

$$\begin{tikzcd}
X^{\boxtimes 5} \arrow[r, "\mu\boxtimes\mathbb{1}^2"] \arrow[d, "\mathbb{1}^2\boxtimes\mu"'] & X^{\boxtimes 3} \arrow[d, "\mu"] \\
X^{\boxtimes 3} \arrow[r, "\mu"'] & X
\end{tikzcd}$$

and follows from those of types 1 and 2. The degeneracy conditions are formulated as commutativity of the following diagrams.

$$\begin{tikzcd}
X \boxtimes X \arrow[r, "\mathbb{1}\boxtimes\Delta"] \arrow[d, "\mathbb{1}\boxtimes\epsilon"'] & X \boxtimes X \boxtimes X \arrow[dl, "\mu"] \\
X &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
X \boxtimes X \arrow[r, "\Delta\boxtimes\mathbb{1}"] \arrow[d, "\epsilon\boxtimes\mathbb{1}"'] & X \boxtimes X \boxtimes X \arrow[dl, "\mu"] \\
X &
\end{tikzcd}$$

Definition 4.6.1. A heap object in a symmetric monoidal category is a comonoid object (X, Δ, ϵ), where $\epsilon : X \to I$ is a counital morphism to the unit object I, endowed with a morphism of comonoids $\mu : X^{\boxtimes 3} \to X$ making all the diagrams above commute.

Example 4.6.1. A (set-theoretic) heap in the sense of Section 4.1 is a heap object in the category of sets.

The following appeared implicitly in [ESZb].

Example 4.6.2. Let H be an involutory Hopf algebra (i.e. $S^2 = \mathbb{1}$) over a field $\Bbbk$. Then H is a heap object in the monoidal category of vector spaces and tensor products, with the ternary operation μ induced by the assignment

$$x \otimes y \otimes z \mapsto xS(y)z$$

for single tensors. Indeed, we have

$$\begin{aligned}
&\mu(\mu(x \otimes y \otimes z) \otimes u \otimes v) \\
&\quad = \quad xS(y)zS(u)v \\
&\quad = \quad xS(y)S^2(z)S(u)v \\
&\quad = \quad xS(uS(z)y)v \\
&\quad = \quad \mu(x \otimes \mu(u \otimes z \otimes y) \otimes v)
\end{aligned}$$

corresponding to the commutativity of the diagram representing equality of type 1. Observe that we have used the involutory hypothesis to obtain the second equality. We also have

$$\begin{aligned}
&(\mathbb{1} \otimes \Delta)(x \otimes y) \\
&\quad = \quad \mu(x \otimes y^{(1)} \otimes y^{(2)}) \\
&\quad = \quad xS(y^{(1)})y^{(2)} \\
&\quad = \quad \epsilon(y)x
\end{aligned}$$

which shows the left degeneracy constraint. The rest of the axioms can be checked in a similar manner.

The opposite direction in the group-theoretic case is the assertion that a pointed heap generates a group by means of the operation $xy = [x, e, y]$. The following is a Hopf algebra version and can be obtained by calculations. More general statement of this can be found in [BS11] and below.

Proposition 4.6.3. *Let $(X, [-])$ be a heap object in a coalgebra category, and let $e \in X$ be a group-like element (i.e., $\Delta(e) = E \otimes e$ and $\epsilon(e) = 1$). Then X is an involutory Hopf algebra with multiplication $m(x \otimes y) := \mu_e(x \otimes y) := [x \otimes e \otimes y]$, unit e, and antipode $S(x) := [e \otimes x \otimes e]$.*

Proof. We use Sweedler's notation $\Delta(x) = x^{(1)} \otimes x^{(2)}$. The associativity of m follows from the type 0 para-associativity of μ. A unit condition is computed by

$$m(e \otimes x) = \mu(e \otimes e \otimes x) = \mu(\Delta(e) \otimes x) = x$$

by the degeneracy condition and the assumption that e is group-like. The other condition

$m(x \otimes e) = x$ is similar. The compatibility between m and Δ is computed as

$$\begin{aligned}
\Delta m(x \otimes y) &= \Delta\mu(x \otimes e \otimes y) \\
&= \mu\tau(\Delta(x) \otimes \Delta(e) \otimes \Delta(y)) \\
&= \mu\tau(x^{(1)} \otimes x^{(2)} \otimes e \otimes e \otimes y^{(1)} \otimes y^{(2)}) \\
&= \mu(x^{(1)} \otimes e \otimes y^{(1)}) \otimes \mu(x^{(2)} \otimes e \otimes y^{(2)}) \\
&= m(x^{(1)} \otimes y^{(1)}) \otimes m(x^{(2)} \otimes y^{(2)})
\end{aligned}$$

as desired, where τ is an appropriate permutation that give the third equality, and the group-like assumption is used in the second equality. An antipode condition is computed as

$$\begin{aligned}
&m(S \otimes \mathbb{1})\Delta(x) \\
&= \mu(\mu(e \otimes x^{(1)} \otimes e) \otimes e \otimes x^{(2)}) \\
&= \mu(e \otimes x^{(1)} \otimes \mu(e \otimes e \otimes x^{(2)})) \\
&= \mu(e \otimes x^{(1)} \otimes \mu(\Delta(e) \otimes x^{(2)})) \\
&= \mu(e \otimes x^{(1)} \otimes x^{(2)})\epsilon(e) \\
&= e\epsilon(x)
\end{aligned}$$

as desired, where the group-like condition $\epsilon(e)$ and the degeneracy condition for μ were used. The other case $m(\mathbb{1} \otimes S)\Delta(x) = e\epsilon(x)$ is similar. This completes the proof. ■

Remark 4.6.4. Observe that S so defined, is involutory. This observation corroborates the necessity of including the involutory hypothesis in Example 4.6.2.

Remark 4.6.5. We observe a relation between a choice of a group-like element e in Proposition 4.6.3 and a *coaugmentation map* of a coalgebra. Let (X, Δ, ϵ) be a coalgebra. A coaugmentation is a coalgebra morphism $\eta : \Bbbk \to X$ (i.e., $\Delta\eta = (\eta \otimes \eta)j$, where $j : \Bbbk \to \Bbbk \otimes \Bbbk$ is the canonical isomorphism, $j(1) = 1 \otimes 1$) such that $\epsilon\eta = \mathbb{1}|_{\Bbbk}$. Let $e = \eta(1)$. We show that e is group-like. One computes $\Delta(e) = \Delta\eta(1) = \eta(1) \otimes \eta(1) = e \otimes e$, and $\epsilon(e) = \epsilon(\eta(1)) = 1$ as desired. We do not know, however, in general the converse holds, i.e., whether for any group-like element e, there exists a coaugmentation map η such that $\eta(1) = e$. We observe that an advantage of using coaugmentation map is the desired condition can be stated by a map, without mention of particular elements, which becomes fruitful in categorical definitions as we see below.

We generalize Example 4.6.2 and Proposition 4.6.3 to symmetric monoidal categories as follows, using Remark 4.6.5. For this purpose first we define a *coaugmentation* of a comonoid object (X, Δ, ϵ) in a symmetric monoidal category with a unit object I as a comonoidal morphism $\eta : I \to X$ such that $\epsilon\eta = \mathbb{1}$.

Definition 4.6.2. Let $\mathcal{C}$ be a symmetric monoidal category. We define the *category of heap objects* in $\mathcal{C}$, $\mathcal{H}_{\mathcal{C}}$, as follows. The objects of $\mathcal{H}_{\mathcal{C}}$ are the heap objects as in Definition 4.6.1. The morphisms are defined to be the morpshisms of $\mathcal{C}$ commuting with the heap maps and the comonoidal structures. A heap object X, is called *pointed*, if it is endowed with a coaugmentation $\eta : I \to X$. The *category of pointed heap objects* in $\mathcal{C}$, $\mathcal{H}^*_{\mathcal{C}}$, is the category consisting of pointed heap objects over $\mathcal{C}$, and morphisms of heap objects commuting with the coaugmentations.

An *involutory* Hopf monoid (object) in a symmetric monoidal category is equipped with a monoidal product m, a unit object I, a comonoidal product Δ, an antipodal morphism S with $S^2 = \mathbb{1}$, a unit morphism $\eta : I \to X$ that satisfies the left and right unital conditions $m(\eta \boxtimes \mathbb{1}) = \mathbb{1}$ and $m(\mathbb{1} \boxtimes \eta) = \mathbb{1}$, and a counit morphism $\epsilon : X \to I$ that satisfies the left and right counital conditions $(\epsilon \boxtimes \mathbb{1})\Delta = \mathbb{1}$ and $(\mathbb{1} \boxtimes \epsilon)\Delta = \mathbb{1}$.

Theorem 4.6.6. *Let $\mathcal{C}$ be a symmetric monoidal category. There is an equivalence of categories between the category $\mathcal{H}^*_{\mathcal{C}}$ and the category of involutory Hopf monoids in $\mathcal{C}$, $sH_{\mathcal{C}}$.*

Proof. We define a functor $\mathcal{F} : \mathcal{H}^*_{\mathcal{C}} \longrightarrow sH_{\mathcal{C}}$ as follows. Let $(X, \eta, \mu, \epsilon, \Delta)$ be a pointed heap object in $\mathcal{C}$, define $\mathcal{F}(X) := (X, I, \eta, \lambda_\eta, \rho_\eta, m, \epsilon, \Delta, S)$, where multiplication $m := \mu \circ \mathbb{1} \boxtimes \eta \boxtimes \mathbb{1}$, antipode $S := \mu \circ \eta \boxtimes \mathbb{1} \boxtimes \eta$, the unit object I, the left and right unitors λ_η, ρ_η by $\lambda_\eta := m(\eta \boxtimes \mathbb{1}) : I \boxtimes X \to X$ and $\rho_\eta := m(\mathbb{1} \boxtimes \mathbb{1}) : X \boxtimes I \to X$, and comonoidal structure unchanged. The functor $\mathcal{F}$ is defined to be the identity on morphisms. The fact that $\mathcal{F}(X)$ is a Hopf monoid in $\mathcal{C}$ is a translation of the computations in Example 4.6.2 in commutative diagrams or series of composite morphisms.

Specifically, defining conditions are verified as follows. The associativity of m follows from the para-associativity as before. The unit η and the counit ϵ are unchanged and a left unital condition is check by

$$m(\eta \boxtimes \mathbb{1}) = \mu(\eta \boxtimes \eta \boxtimes \mathbb{1})$$
$$= \mu(\Delta\eta \boxtimes \mathbb{1}) = (\epsilon \boxtimes \mathbb{1})(\eta \boxtimes \mathbb{1}) = \epsilon\eta \boxtimes \mathbb{1} = \mathbb{1} \boxtimes \mathbb{1}$$

as desired. The right unital condition is computed similarly. The compatibility between m and Δ is computed as

$$(m \boxtimes m)\tau(\Delta \boxtimes \Delta) = [(\mu(\mathbb{1} \boxtimes \eta \boxtimes \mathbb{1}) \boxtimes (\mu(\mathbb{1} \boxtimes \eta \boxtimes \mathbb{1})]\tau(\Delta \boxtimes \Delta)$$
$$= (\mu \boxtimes \mu)\tau(\Delta \boxtimes \Delta\eta \boxtimes \Delta) = \Delta\mu(\mathbb{1} \boxtimes \eta \boxtimes \mathbb{1}) = \Delta m.$$

We note that we have implicitly used the naturality of the switching morphisms multiple times. The antipode condition is computes as

$$\begin{aligned}
&m(S \boxtimes \mathbb{1})\Delta \\
&= \mu(\mathbb{1} \boxtimes \eta\mathbb{1})(\mu(\eta \boxtimes \eta) \boxtimes \mathbb{1})\Delta \\
&= \mu(\eta \boxtimes \mathbb{1} \boxtimes \mu(\eta \boxtimes \eta \boxtimes \mathbb{1}))\Delta \\
&= \mu(\eta \boxtimes \mathbb{1} \boxtimes (\epsilon \boxtimes \mathbb{1}))(\mathbb{1} \boxtimes \eta)\Delta.
\end{aligned}$$

The other antipode condition is similar.

Similarly, we define a functor $\mathcal{G} : iH_{\mathcal{C}} \longrightarrow \mathcal{H}^*_{\mathcal{C}}$ by the assignment on objects

$$\mathcal{G}(X, \eta, m, \epsilon, \Delta, S) := (X, \eta, \mu, \epsilon, \Delta)$$

, with $\mu := m(m \boxtimes \mathbb{1})(\mathbb{1} \boxtimes S \boxtimes \mathbb{1})$. Also, $\mathcal{G}$ is the identity on morphisms. We leave the details of the proof to the reader. ■

Next we show that Lemma 4.5.1 holds for a coalgebra (i.e. a comonoid in the category of vector spaces). Although this is a special case of Theorem 4.6.9, we include its statement and proof here to illustrate and further motivate Theorem 4.6.9. For this goal, we slightly modify the definition of TSD maps in a symmetric monoidal categories, given in [ESZb].

Definition 4.6.3. Let (X, Δ, ϵ) be a comoidal object in a symmetric monoidal category $\mathcal{C}$. A *ternary self-distributive object* $(X, \Delta, \epsilon, \mu)$ in $\mathcal{C}$ is a comonoidal object that satisfies the following condition:

$$\mu(\mu^{\boxtimes 3}) \sqcup\!\sqcup_3 (\Delta' \boxtimes \mathbb{1})\Delta \boxtimes (\Delta' \boxtimes \mathbb{1})\Delta) = \mu(\mu \boxtimes \mathbb{1}^{\boxtimes 2})$$

where $\sqcup\!\sqcup_3$ denotes the composition of switching maps corresponding to transpositions $(2,\ 4)(3,\ 7)(6,\ 8)$ and $\Delta' = \tau\Delta$.

This differs from the definition found in [ESZb] only in the use of Δ' instead of Δ. The main examples, set theoretical ones and Hopf algebras, satisfy both definitions.

Proposition 4.6.7. *Let H and μ be as in the Example 4.6.2. Then μ defines a ternary self-distributive object in the category of vector spaces.*

Proof. One proceeds as in the proof of Lemma 4.5.1 as follows. We use the Sweedler notation $\Delta(x) = x^{(1)} \otimes x^{(2)}$ and $(\Delta \otimes \mathbb{1})\Delta(x) = x^{(11)} \otimes x^{(12)} \otimes x^{(2)}$. Then one computes

$$\begin{aligned}
&\mu(\mu(x \otimes y^{(1)} \otimes z^{(1)}) \otimes z^{(2)} \otimes y^{(2)}) \\
&= \mu(x \otimes y^{(1)} \otimes \mu(z^{(2)} \otimes z^{(1)} \otimes y^{(2)}) \\
&= \mu(x \otimes y^{(1)} \otimes y^{(2)})\epsilon(z) \quad = \quad x\ \epsilon(y)\epsilon(z).
\end{aligned}$$

We note that the last equality $S(y^{(2)})y^{(1)} = \epsilon(y)$ follows from the assumption that H is involutory. Then we obtain

$$\begin{aligned}
&\mu(\mu(x_1 \otimes x_4^{(12)} \otimes x_5^{(12)}) \otimes \mu(x_2 \otimes x_4^{(11)} \otimes x_5^{(11)}) \otimes \mu(x_3 \otimes x_4^{(2)} \otimes x_5^{(2)})) \\
&= \mu(x_1 \otimes \mu(\mu(x_2 \otimes x_4^{(11)} \otimes x_5^{(11)}) \otimes x_5^{(12)} \otimes x_4^{(12)}) \otimes \mu(x_3 \otimes x_4^{(2)} \otimes x_5^{(2)})\ ) \\
&= \mu(x_1 \otimes x_2\ \epsilon(x_4^{(1)})\epsilon(x_5^{(1)}) \otimes \mu(x_3 \otimes x_4^{(2)} \otimes x_5^{(2)})\ ) \\
&= \mu(\mu(x_1 \otimes x_2 \otimes x_3) \otimes \epsilon(x_4^{(1)})\epsilon(x_5^{(1)})(x_4^{(2)} \otimes x_5^{(2)})) \\
&= \mu(\mu(x_1 \otimes x_2 \otimes x_3) \otimes x_4 \otimes x_5))
\end{aligned}$$

as desired. ■

Our goal, next, is to show that a more general version of Lemma 4.5.1 and Proposition 4.6.7 holds in an arbitrary symmetric monoidal category. We first have the following preliminary result.

Lemma 4.6.8. *Let $(X, \Delta, \epsilon, \mu)$ be a heap object in a symmetric monoidal category with tensor product $\boxtimes$ and switching morphism τ. Then the following identity of morphisms holds*

$$\mu(\mu \boxtimes \mathbb{1}^{\boxtimes 2})\tau_{4,5}\tau_{3,4}(\mathbb{1}\boxtimes\Delta \boxtimes \Delta) = \mathbb{1} \boxtimes \epsilon \boxtimes \epsilon.$$

Proof. We observe that the following commutative diagram implies our statement.

$$\begin{tikzcd}
X^{\boxtimes 5} \arrow[rr, "\tau_{4,5}\tau_{3,4}"] & & X^{\boxtimes 5} \arrow[ddr, bend left, "\mathbb{1}^2\boxtimes\mu"] \arrow[ddrr, bend left, "\mu\boxtimes\mathbb{1}^2"] & & \\
X^{\boxtimes 4} \arrow[u, "\mathbb{1}^3\boxtimes\Delta"'] \arrow[rr, "\tau_{3,4}"] \arrow[drrr, "\mathbb{1}^3\boxtimes\epsilon"'] & & X^{\boxtimes 4} \arrow[u, "\mathbb{1}^2\boxtimes\Delta\boxtimes\mathbb{1}"] \arrow[dr, "\mathbb{1}^2\boxtimes\epsilon\boxtimes\mathbb{1}"] & & \\
& & & X^{\boxtimes 3} \arrow[ddd, "\mu"] & X^{\boxtimes 3} \arrow[dddl, "\mu"] \\
& X^{\boxtimes 2} \arrow[urr, "\mathbb{1}\boxtimes\Delta"] \arrow[ddrr, "\mathbb{1}\boxtimes\epsilon"] & & & \\
X^{\boxtimes 3} \arrow[uuu, "\mathbb{1}\boxtimes\Delta\boxtimes\mathbb{1}"'] \arrow[uuuu, bend left, "\mathbb{1}\boxtimes\Delta\boxtimes\Delta"] \arrow[ur, "\mathbb{1}^2\boxtimes\epsilon"] \arrow[drrr, "\mathbb{1}\boxtimes\epsilon\boxtimes\epsilon"'] & & & & \\
& & & X &
\end{tikzcd}$$

where the rectangle on top, and the two triangles below commute because of naturality of the braiding, while the other parts of the diagram commute by heap and comonoid axioms.

■

Theorem 4.6.9. *Let $(X, \Delta, \epsilon, \mu)$ be a heap object in a symmetric monoidal category $\mathcal{C}$. Then $(X, \Delta, \epsilon, \mu)$ is also a ternary self-distributive object in $\mathcal{C}$.*

Proof. Since (X, Δ, ϵ) is a comonoid in $\mathcal{C}$ by hypothesis, we just need to prove that ternary self-distributivity of μ. We use the following commutative diagram

$$\begin{tikzcd}
 & X^9 \arrow[r, "\tau"] \arrow[drr, "\mathbb{1}^6\mu"] & X^9 \arrow[rr, "\mathbb{1}^3\mu\mu"] & & X^5 \arrow[d, "\mu\mathbb{1}^2"] \arrow[ddl, "\tau\circ(\mathbb{1}\mu\mathbb{1})"'] \\
X^7 \arrow[ur, "\mathbb{1}^2\Delta'\Delta'\mathbb{1}^3"] \arrow[r, "\mathbb{1}^4\mu"] & X^5 \arrow[r, "\mathbb{1}^2\Delta'\Delta'\mathbb{1}"] \arrow[drr, bend right=15, "\mathbb{1}\epsilon\mathbb{1}\epsilon\mathbb{1}"'] & X^7 \arrow[r, "\tau"] & X^7 \arrow[d, "(\mathbb{1}\mu\mathbb{1})\circ(\mathbb{1}\mu\mathbb{1}^3)"'] & X^3 \arrow[dd, "\mu"] \\
X^7 \arrow[u, "\tau"] \arrow[r, "\mu\mathbb{1}^4"] & X^5 \arrow[rr, "\mathbb{1}\epsilon\mathbb{1}\epsilon\mathbb{1}"'] & & X^3 \arrow[dr, "\mu"] & \\
X^5 \arrow[u, "\mathbb{1}^3\Delta\Delta"] \arrow[urrr, "\mathbb{1}^2\mu"] \arrow[rr, "\mu\mathbb{1}^2"] & & X^3 \arrow[rr, "\mu"] & & X
\end{tikzcd}$$

where we have omitted the symbol $\boxtimes$ in the product of morphisms, omitted the subscripts corresponding to the switching morphisms τ, to slightly shorten the notation and, finally, we have used the notation $\circ$ to indicate the composition of morphisms. The leftmost $\tau : X^7 \to X^7$ is the composition of symmetry constraints corresponding to the transposition $(5, 6)(4, 5)(5, 6)(4, 5)(3, 4)$, proceeding clockwise, $\tau : X^9 \to X^9$ corresponds to $(4, 5)(3, 4)(4, 5)(3, 4)(2, 3)$. The reader can easily find the correct compositions corresponding to the remaining τ's by a diagrammatic approach. The triangles on the right and on the bottom are instances of type 1 and type 0 axioms, respectively. The middle triangle commutes as a consequence of Lemma 4.6.8. The other diagrams can be seen to be commutative either by applying the comonoid axioms or naturality of the braiding. Finally, by direct inspection we can see that the upper perimeter of the diagram corresponds to the LHS of TSD, as stated in Definition 4.6.3. ∎

CHAPTER 5 : FUTURE WORK

In this final chapter, we provide a brief description of future projects we intend to embark upon. This list includes projects that are currently at a germinal stage (i.e. some proofs have been done but they lack an overall structure), or at a hypothetical stage (i.e. based on preliminary considerations we believe it is possible to obtain certain results).

5.1 Non-Associative/Quantum Algebra and Knot Theory

The cocycle invariant introduced in Section 3.7 has not been computed in any practical examples. It is therefore of fundamental importance to determine the invariant in some specific cases. In particular, we ask whether or not the invariant distinguishes certain framed links. Is it possible to find certain families of framed links for which it is possible to compute the cocycle invariant?

Graña proved in [Gn02] that cocycle invariants are "quantum" invariants, in the sense that it is possible to obtain them as the trace of a certain endomorphism in an appropirate category. Does an analogous result hold in the case of framed link cocycle invariant? Is it possible to utilize the categorical doubling procedure described in Chapter 3 to generalize Graña's proof to the framed cocycle link invariant case? In this perspective, we expect that the internalization prcoedure of categorical self-distributivity will play a crucial role.

Lastly, we intend to complete a currently ongoing project (with M. Elhamdadi and M. Saito) regarding Yang-Baxter (YB) operators and their homology theory. It has been conjectured, based on strong computational evidence, that the homology groups of a certain YB operator have rank 2 and a torsion described by means of the Fibonacci sequence [PW]. In [ESZc], we are making progress towards the understanding of this conjecture [ESZc] using skein theoretic procedures. Our current results are in concordance with the conjecture and solve part of it. Homology and cohomology theories of Yang-Baxter operators are highly promising tools to develop new invariants of knots but it is still an open problem to determine these invariants and relate them to well known invariants. Explicit computations of homology groups for specific YB operators are exiguous and the computational methods applied so far are rather rudimentary. The skein techniques introduced in [ESZc] seem to be applicable to a vast range of YB operators and to be suitable to systematically (algorithmically) compute homology.

5.2 Koszul Duality for Operads and its Ramifications

More recently I have been increasingly interested in the theory of operads and, more specifically, operadic Koszulity and (co)homology. I am currently studying Koszul duality theory of the Jordan operad (in [Zap]), whose algebras are the well known Jordan algebras. For years, the problem of Koszul duality of Jordan operad has been considered not well posed, due to the fact that the Jordan operad is cubic. In this paper it is estabilished that, with a certain suitable presentation, the Jordan operad is quadratic-linear Koszul, whose main impact lies in the fact that the cobar construction of its dual cooperad is a resolution of it. An explicit description of the notion of Homotopy Jordan Algebra is given, as a corollary of the aforementioned Koszul duality, by means of Maurer-Cartan elements in the enveloping differential-graded Lie algebra. My next goal is to proceed to study infinity morphisms, homotopy transfer theorem and the deformation complex in this context. We expect that these results might be applied to the study of Jordan super-algebras.

The following project is similar, in spirit, to the previous one. Hartwig, Larsson and Silvestrov have introduced in [HLS06], a generalized version of the Jacobi identity and studied in subsequent works, what are now known as "Hom" versions of famous algebraic structures (i.e. Hom-Lie algebras, Hom-associative algebras, Hom-Nambu brackets, Hom-Jordan algebras etc.). Whereas the main motivation to study these kind of structures comes from Theoretical Physics (Conformal Field Theories and Quantum Gravity among others), it is inherently interesting to understand these "deformed" algebras. Based on our considerations and current understanding, we pose the following:

Conjecture 5.2.1. The operad controlling the Hom-Jacobi identity is Koszul, and the Homotopy Hom-Lie algebras obtained via the standard cobar resolution produce the n-ary Hom-Nambu Lie brackets introduced in [AMS09].

The implications of such a result, if true, in Theoretical Physics provide quite an interesting perspective for future work.

5.3 Applied Mathematics

In this section, we give a brief explanation of a project we have started to work on during Fall 2019, in an Internship at the Biomedical & Clinical Informatics Lab at University of Michigan, Ann Arbor.

It has recently proposed in [ZNL18], a method to construct Neural Networks by means of Tropical Algebra and Tropical Geoemtry. In [ZNL18], the authors show that there is a bijective correspondence between Feedforward Neural Networks and Tropical Rational Functions (i.e. he tropical version of rational functions in Algebraic Geoemtry). It is possible therefore, at least in principle, to produce decision making algorithms based on Tropical Hypersurfaces. We intend to develop a method of diagnosis of heart failure based on "tropical networks".

REFERENCES

[AGn] Nicolás Andruskiewitsch and Matias Graña. From racks to pointed hopf algebras. *Adv. Math.*, 178(2):177–243.

[AMS09] H. Ataguema, A. Makhlouf, and S. Silvestrov. Generalization of n-ary nambu algebras and beyond. *Journal of Mathematical Physics*, 50(8):083501, 2009.

[Bar] Stephen Barnett. *Matrices. Methods and applications.* Oxford Applied Mathematics and Computing Science Series. The Clarendon Press, Oxford University Press, New York.

[BH] George M. Bergman and Adam O. Hausknecht. *Cogroups and co-rings in categories of associative rings*, volume 45 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI.

[BM29] C. Burstin and W. Mayer. Distributive gruppen von endlicher ordnung. *Journal fur die Reine und Angewandte Mathematik*, 1929(160):111–130, 1929.

[Bri] E. Brieskorn. Automorphic sets and braids and singularities. pages 45–115.

[Bro82] Kenneth S. Brown. *Cohomology of Groups*, volume 87 of *Graduate Texts in Mathematics.* Springer-Verlag New York, 1982.

[BS11] Thomas Booker and Ross Street. Torsors, herds and flocks. *Journal of Algebra*, 330(1):346 – 374, 2011.

[CCES] J. Scott Carter, Alissa S. Crans, Mohamed Elhamdadi, and Masahico Saito. Cohomology of categorical self-distributivity. *J. Homotopy Relat. Struct.*, 3(1):13–63.

[CEGM] Rasika Churchill, Mohamed Elhamdadi, Matthew Green, and Abdenacer Makhlouf. Ternary and n-ary f-distributive structures. *Open Math.*, 16:32–45.

[CEGnS] J. Scott Carter, Mohamed Elhamdadi, Matias Graña, and Masahico Saito. Cocycle knot invariants from quandle modules and generalized quandle homology. *Osaka J. Math.*, 42(3):499–541.

[CENS] J. Scott Carter, Mohamed Elhamdadi, Marina Appiou Nikiforou, and Masahico Saito. Extensions of quandles and cocycle knot invariants. *J. Knot Theory Ramifications*, 12(6):725–738.

[CES] J. Scott Carter, Mohamed Elhamdadi, and Masahico Saito. Twisted quandle homology theory and cocycle knot invariants. *Algebr. Geom. Topol.*, 2:95–135.

[CJK+] J. Scott Carter, Daniel Jelsovsky, Seiichi Kamada, Laurel Langford, and Masahico Saito. Quandle cohomology and state-sum invariants of knotted curves and surfaces. *Trans. Amer. Math. Soc.*, 355(10):3947–3989.

[CS98] J. Scott Carter and Masahico Saito. *Knotted Surfaces and Their Diagrams*, volume 55 of *Mathematical Surveys and Monographs*. American Mathematical Society, 1998.

[EGM] Mohamed Elhamdadi, Matthew Green, and Abdenacer Makhlouf. Ternary distributive structures and quandles. *Kyungpook Math. J.*, 56(1):1–27.

[Eis] Michael Eisermann. Quandle coverings and their galois correspondence. *Fund. Math.*, 225:103–167.

[EN] Mohamed Elhamdadi and Sam Nelson. *Quandles—an introduction to the algebra of knots*, volume 74 of *Student Mathematical Library*. American Mathematical Society, Providence, RI.

[ESZa] Mohamed Elhamdadi, Masahico Saito, and Emanuele Zappala. Heap cohomology and self-distributive cohomology. *arXiv:1910.02877*.

[ESZb] Mohamed Elhamdadi, Masahico Saito, and Emanuele Zappala. Higher arity self-distributive operations in cascades and their cohomology. *arXiv:1905.00440*.

[ESZc] Mohamed Elhamdadi, Masahico Saito, and Emanuele Zappala. Homology of yang-baxter operators and wang's conjecture. *in progress*.

[ESZ19] Mohamed Elhamdadi, Masahico Saito, and Emanuele Zappala. Continuous cohomology of topological quandles. *J. Knot Theory Ramifications*, 28(06):1950036, 2019.

[FR] Roger Fenn and Colin Rourke. Racks and links in codimension two. *J. Knot Theory Ramifications*, 01(04).

[FY89] Peter J Freyd and David N Yetter. Braided compact closed categories with applications to low dimensional topology. *Advances in Mathematics*, 77(2):156 – 182, 1989.

[Gn02] Matias Graña. Quandle knot invariants are quantum knot invariants. *Journal of Knot Theory and its Ramifications*, 11, 08 2002.

[Gre] Matthew J. Green. Generalizations of quandles and their cohomologies. *PhD dissertation, University of South Florida.*

[Gru02] Cyril Grunspan. Quantum torsors. *Journal of Pure and Applied Algebra*, 184, 05 2002.

[HL17] Christopher Hollings and Mark Lawson. *Wagner's Theory of Generalised Heaps*. Springer International Publishing, 2017.

[HLS06] Jonas T. Hartwig, Daniel Larsson, and Sergei D. Silvestrov. Deformations of lie algebras using ÃŔÂČ-derivations. *Journal of Algebra*, 295(2):314 – 361, 2006.

[II] Atsushi Ishii and Masahide Iwakiri. Quandle cocycle invariants for spatial graphs and knotted handlebodies. *Canad. J. Math.*, 64(1):102–122.

[IIJ] Atsushi Ishii, Masahide Iwakiri, and Kanako Jang, Yeonhee andOshiro. A g-family of quandles and handlebody-knots. *Illinois J. Math.*, 57:817–838.

[Joy82] David Joyce. A classifying invariant of knots, the knot quandle. *Journal of Pure and Applied Algebra*, 23(1):37 –65, 1982.

[Kon99] Maxim Kontsevich. Operads and motives in deformation quantization. *Letters in Mathematical Physics*, 48(1):35–72, Apr 1999.

[Mar06] Martin Markl. Operads and props. *Handbook of Algebra*, 5:87–140, 01 2006.

[Mat] S. V. Matveev. Distributive groupoids in knot theory. *Mat. Sb. (N.S.)*, 119(161)(1):78–88, 160.

[ML71] Saunders Mac Lane. *Categories for the Working Mathematician*. Graduate Texts in Mathematics. Springer-Verlag New York, 1971.

[Moc] Takuro Mochizuki. Some calculations of cohomology groups of finite alexander quandles. *J. Pure Appl. Algebra*, 179(3):287–330.

[Nel] Sam Nelson. Classification of finite alexander quandles. *Topology Proc.*, 27(1):245–258.

[Nos] Takefumi Nosaka. Quandle cocycles from invariant theory. *Adv. Math.*, 245:423–438.

[Nos17] Takefumi Nosaka. *Quandles and Topological Pairs*, volume 5 of *SpringerBriefs in Mathematics*. Springer Singapore, 2017.

[Nos18] Takefumi Nosaka. de rham theory and cocycles of cubical sets from smooth quandles. *Kodai Mathematical Journal*, 42, 04 2018.

[Prz] Józef H. Przytycki. Distributivity versus associativity in the homology theory of algebraic structures. *Demonstratio Math.*, 44(4):823–869.

[PS97] V. Prasolov and A. Sossinsky. *Knots, Links, Braids, and 3–Manifolds*, volume 154 of *Translations of Mathematical Monographs*. American Mathematical Society, 1997.

[PW] Józef H. Przytycki and Xiao Wang. Equivalence of two definitions of set-theoretic yang-baxter homology and general yang-baxter homology. *J. Knot Theory Ramifications*, 27(7):1841013, 15.

[Ree] Daniel Reem. Remarks on the cauchy functional equation and variations of it. *Aequationes Math.*, 91(2):237–264.

[RT] N. Yu. Reshetikhin and V. G. Turaev. Ribbon graphs and their invariants derived from quantum groups. *Comm. Math. Phys.*, 127(1):1–26.

[Rub07] Ryszard L. Rubinsztein. Topological quandles and invariants of links. *Journal of Knot Theory and Its Ramifications*, 16(06):789–808, 2007.

[Sch] Peter Schauenburg. Quantum torsors and hopf-galois objects. *arXiv:math/0208047*.

[Sko07] Zoran Skoda. Quantum heaps, cops and heapy categories. *Math. Commun.*, 12:1–9, 2007.

[Tak43] M. Takasaki. Abstractions of symmetric functions. *Tohoku Math. J.*, 49:143–207, 1943.

[Zap] Emanuele Zappala. Koszulity of jordan operad.

[ZNL18] Liwen Zhang, Gregory Naitzat, and Lek-Heng Lim. Tropical geometry of deep neural networks. *CoRR*, abs/1805.07091, 2018.

APPENDIX A : CONTINUOUS ISOMORPHISMS OF TOPOLOGICAL QUANDLES

In this appendix we show that continuous isomorphism classes differ from algebraic isomorphism classes for topological quandles. The results of this appendix were obtained by W. Edwin Clark.

Lemma .0.1. *[Ree] If $T : \mathbb{R}^n \to \mathbb{R}^m$ is additive and continuous, then T is $\mathbb{R}$-linear.*

Remark .0.2. We recall [Nel] that an Alexander quandle (X, T) is indecomposable if and only if $I - T$ is surjective. If $(\mathbb{R}^n, T)$ is a topological generalized Alexander quandle that is indecomposable such that T is additive, then $I - T$ is surjective, and Lemma .0.1 implies that T is linear. Hence $I - T$ is invertible.

Lemma .0.3. *Let $(\mathbb{R}^n, S)$ and $(\mathbb{R}^m, T)$ be topological Alexander quandles, such that $I - S$ and $I - T$ are invertible. Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a continuous quandle homomorphism such that $F(0) = 0$. Then S, T, F are $\mathbb{R}$-linear and the condition $FS = TF$ holds.*

Proof. First from Lemma .0.1, S and T are linear. Since F is a quandle homomorphism,

$$F(Sx + (I - S)y) = TF(x) + (I - T)F(y) \tag{1}$$

holds for all $x, y \in \mathbb{R}^n$. By setting $x = 0$ and $y = 0$ respectively in Equation(1) and using the assumption $F(0) = 0$, we obtain

$$F((I - S)y) = (I - T)F(y), \tag{2}$$

and

$$F(Sx) = TF(x), \tag{3}$$

which is the condition $FS = TF$. These Equations (2) and (3) also imply

$$F(Sx + (I - TS)y) = F(Sx) + F((I - S)y). \tag{4}$$

By the invertibility assumptions, we have $\{Sx \mid x \in \mathbb{R}^n\} = \mathbb{R}^n$ and $\{(I-S)y \mid y \in \mathbb{R}^n\} = \mathbb{R}^n$. Hence Equation (4) implies that $F(a+b) = F(a)+F(b)$ for all $a, b \in \mathbb{R}^n$. Since F is additive and continuous, Lemma .0.1 implies that F is linear. ■

Solving the matrix equation $FS = TF$ can be found, for example, in [Bar]. A direct calculation gives the following lemma.

Lemma .0.4. *Let* $(\mathbb{R}^n, S)$ *and* $(\mathbb{R}^m, T)$ *be Alexander quandles. Let* $F : \mathbb{R}^n \to \mathbb{R}^m$ *be a quandle homomorphism. Let* $a \in \mathbb{R}^m$*. Then* $F+a : \mathbb{R}^n \to \mathbb{R}^m$ *defined by* $(F+a)(x) = F(x)+a$ *for* $x \in \mathbb{R}^n$ *is a quandle homomorphism.*

Proposition .0.5. *Let* $(\mathbb{R}^n, S)$ *and* $(\mathbb{R}^n, T)$ *be indecomposable topological Alexander quandles. If* $F : \mathbb{R}^n \to \mathbb{R}^n$ *is a continuous quandle isomorphism such that* $F(0) = 0$*, then* S, T, F *are* $\mathbb{R}$*-linear and* S *and* T *are similar:* $T = FSF^{-1}$*.*

Proof. By Lemma .0.1, S, T, F are linear. By Lemma .0.3, S and T are similar via F. ■

Proposition .0.6. *There is a family with continuum cardinality of topological quandle structures on* $\mathbb{R}^n$ *for all* $n > 0$*, such that its elements are pairwise non-isomorphic as topological quandles but are isomorphic as algebraic quandles.*

Proof. Let $Q(u)$ be the field of rational functions over $\mathbb{R}$ with variable u. Let $s \in \mathbb{R}$ be a transcendental number. Let $Q(u)$ act on $\mathbb{R}^n$ by the scalar multiple $f(u) \cdot x = f(s)x$. Let s, t be distinct transcendentals. Then there are two vector space structures on $\mathbb{R}^n$ over $Q(u)$ by multiples by s and t. They have the same dimension as vector spaces, and therefore, there is a vector space isomorphism $F : \mathbb{R}^n \to \mathbb{R}^n$ over $Q(u)$, and it satisfies $F(sx) = tF(x)$. Hence F is a quandle isomorphism. If F is continuous, then F is linear over $\mathbb{R}$ by Lemma .0.3, and $Fs = sF = tF$ and $s = t$, a contradiction. Hence F is a quandle isomorphism that is not continuous. ■

APPENDIX B : EXAMPLE 3.6.8 REVISITED

In this appendix we explicitly show that the map in Example 3.6.8 is indeed self-distributive. Each equality is obtained by applying the Jacobi identity as in the proof of Lemma 3.3 in [CCES]. In fact, each step corresponds to one of the diagrams in the proof of Theorem 3.6.6 (cf. figure 3.8). Recall also the definition of the diagonal Δ, from Lemma 3.6.7, and the inductive definition for Δ_3 at the beginning of Section 3.6. Explicitly, we have for Δ_3:

$$\Delta_3(a,x) = (a,x)\otimes(1,0)\otimes(1,0) + (1,0)\otimes(0,x)\otimes(1,0) + (1,0)\otimes(1,0)\otimes(0,x).$$

To make the steps easier for the reader, we declare the terms that are going to be replaced according to the Jacobi identity, and underline the replacing terms in the subsequent equality. We obtain therefore:

$$\begin{aligned}
&T(T((a,x)\otimes(b_0,y_0)\otimes(b_1,y_1))\otimes(c_0,z_0)\otimes(c_1,z_1))\\
&\quad = \quad (ab_0b_1c_0c_1, b_0b_1c_0c_1x + b_1c_0c_1[x,y_0] + b_0c_0c_1[x,y_1]\\
&\qquad\qquad +c_0c_1[[x,y_0],y_1] + b_0b_1c_1[x,z_0] + b_1c_1[[x,y_0],z_0]\\
&\qquad\qquad +b_0c_1[[x,y_1],z_0] + c_1[[[x,y_0],y_1],z_0] + b_0b_1c_0[x,z_1]\\
&\qquad\qquad +b_1c_0[[x,y_0],z_1] + b_0c_0[[x,y_1],z_1] + c_0[[[x,y_0],y_1],z_1]\\
&\qquad\qquad +b_0b_1[[x,z_0],z_1] + b_1[[[x,y_0],z_0],z_1] + b_0[[[x,y_1],z_0],z_1]\\
&\qquad\qquad +[[[[x,y_0],y_1],z_0],z_1]).
\end{aligned}$$

Applying the Jacobi identity to the terms $b_0c_1[[x,y_1],z_0]$, $c_1[[[x,y_0],y_1],z_0]$, $b_0[[[x,y_1],z_0],z_1]$ and $[[[[x,y_0],y_1],z_0],z_1]$ we obtain:

$$\begin{aligned}
= \quad &(ab_0b_1c_0c_1, b_0b_1c_0c_1x + b_1c_0c_1[x,y_0] + b_0b_1c_1[x,z_0]\\
&+b_1c_1[[x,y_0],z_0] + b_0c_0c_1[x,y_1] + c_0c_1[[x,y_0],y_1]\\
&+\underline{b_0c_1[[x,z_0],y_1]} + \underline{c_1[[[x,y_0],z_0],y_1]} + \underline{b_0c_1[x,[y_1,z_0]]}\\
&+\underline{c_1[[x,y_0],[y_1,z_0]]} + b_0b_1c_0[x,z_1] + b_1c_0[[x,y_0],z_1]\\
&+b_0b_1[[x,z_0],z_1] + b_1[[[x,y_0],z_0],z_1] + b_0c_0[[x,y_1],z_1]\\
&+c_0[[[x,y_0],y_1],z_1] + \underline{b_0[[[x,z_0],y_1],z_1]} + \underline{[[[[x,y_0],z_0],y_1],z_1]}\\
&+\underline{b_0[[x,[y_1,z_0]],z_1]} + \underline{[[[x,y_0],[y_1,z_0]],z_1]}).
\end{aligned}$$

We now apply the Jacoby identity to the term $b_1c_1[[x,y_0],z_0]$, $b_1[[[x,y_0],z_0],z_1]$, $c_1[[[x,y_0],z_0],y_1]$

and $[[[[x,y_0],z_0],y_1],z_1]$ to obtain:

$$
\begin{aligned}
= \quad & (ab_0b_1c_0c_1, b_0b_1c_0c_1x + b_0b_1c_1[x,z_0] + b_1c_0c_1[x,y_0] \\
& + \underline{b_1c_1[[x,z_0],y_0]} + b_0c_0c_1[x,y_1] + b_0c_1[[x,z_0],y_1] \\
& + c_0c_1[[x,y_0],y_1]] + \underline{c_1[[[x,z_0],y_0],y_1]} + \underline{b_1c_1[x,[y_0,z_0]]} \\
& + \underline{c_1[[x,[y_0,z_0]],y_1]} + b_0c_1[x,[y_1,z_0]] + c_1[[x,y_0],[y_1,z_0]] \\
& + b_0b_1c_0[x,z_1] + b_0b_1[[x,z_0],z_1] + b_1c_0[[x,y_0],z_1] \\
& + \underline{b_1[[[x,z_0],y_0],z_1]} + b_0c_0[[x,y_1],z_1] + b_0[[[x,z_0],y_1],z_1] \\
& + c_0[[[x,y_0],y_1],z_1] + \underline{[[[[x,z_0],y_0],y_1],z_1]} + \underline{b_1[[x,[y_0,z_0]],z_1]} \\
& + \underline{[[[x,[y_0,z_0]],y_1],z_1]} + b_0[[x,[y_1,z_0]],z_1] + [[[x,y_0],[y_1,z_0]],z_1]).
\end{aligned}
$$

Next, we use the Jacoby identity on the terms $b_0c_0[[x,y_1],z_1]$, $b_0[[[x,z_0],y_1],z_1]$, $b_0[[x,[y_1,z_0]],z_1]$, $c_0[[[x,y_0],y_1],z_1]$, $[[[[x,z_0],y_0],y_1],z_1]$, $[[[x,[y_0,z_0]],y_1],z_1]$ and $[[[x,y_0],[y_1,z_0]],z_1]$.

$$
\begin{aligned}
= \quad & (ab_0b_1c_0c_1, b_0b_1c_0c_1x + b_0b_1c_1[x,z_0] + b_1c_0c_1[x,y_0] \\
& + b_1c_1[[x,z_0],y_0] + b_0b_1c_0[x,z_1] + b_0b_1[[x,z_0],z_1] \\
& + b_1c_0[[x,y_0],z_1] + b_1[[[x,z_0],y_0],z_1] + b_0c_0c_1[x,y_1] \\
& + b_0c_1[[x,z_0],y_1] + c_0c_1[[x,y_0],y_1] + c_1[[x,[y_0,z_0]],y_1] \\
& + b_1c_1[x,[y_0,z_0]] + \underline{b_0c_0[[x,z_1],y_1]} + \underline{b_0[[[x,z_0],z_1],y_1]} \\
& + \underline{c_0[[[x,y_0],z_1],y_1]} + b_1[[x,[y_0,z_0]],z_1] + \underline{[[[[x,z_0],y_0],z_1],y_1]} \\
& + \underline{[[[x,[y_0,z_0]],z_1],y_1]} + b_0c_1[x,[y_1,z_0]] + c_1[[x,y_0],[y_1,z_0]] \\
& + \underline{b_0[[x,z_1],[y_1,z_0]]} + \underline{[[[x,y_0],z_1],[y_1,z_0]]} + \underline{b_0c_0[x,[y_1,z_1]]} \\
& + \underline{b_0[[x,z_0],[y_1,z_1]]} + c_1[[[x,z_0],y_0],y_1] + \underline{c_0[[x,y_0],[y_1,z_1]]} \\
& + \underline{[[[x,z_0],y_0],[y_1,z_1]]} + \underline{[[x,[y_0,z_0]],[y_1,z_1]]} + \underline{b_0[x,[[y_1,z_0],z_1]]} \\
& + \underline{[[x,y_0],[[y_1,z_0],z_1]]}),
\end{aligned}
$$

Lastly, making use of the Jacobi identity on the terms $b_1c_0[[x,y_0],z_1]$, $b_1[[[x,z_0],y_0],z_1]$, $c_0[[[x,y_0],z_1],y_1]$, $b_1[[x,[y_0,z_0]],z_1]$, $[[[[x,z_0],y_0],z_1],y_1]$, $[[[x,[y_0,z_0]],z_1],y_1]$ and $[[[x,y_0],z_1],[y_1,z_0]]$ we obtain:

$$
\begin{aligned}
= \quad & (ab_0b_1c_0c_1, b_0b_1c_0c_1x + b_0b_1c_1[x,z_0] + b_0b_1c_0[x,z_1] \\
& +b_0b_1[[x,z_0],z_1] + b_1c_0c_1[x,y_0] + b_1c_1[[x,z_0],y_0] \\
& +\underline{b_1c_0[[x,z_1],y_0]} + \underline{b_1[[[x,z_0],z_1],y_0]} + b_0c_0c_1[x,y_1] \\
& +b_0c_1[[x,z_0],y_1] + b_0c_0[[x,z_1],y_1] + b_0[[[x,z_0],z_1],y_1] \\
& +c_0c_1[[x,y_0],y_1] + c_1[[[x,z_0],y_0],y_1] + \underline{c_0[[[x,z_1],y_0],y_1]} \\
& +\underline{[[[[x,z_0],z_1],y_0],y_1]} + b_0c_1[x,[y_1,z_0]] + b_0[[x,z_1],[y_1,z_0]] \\
& +c_1[[x,y_0],[y_1,z_0]] + \underline{[[[x,z_1],y_0],[y_1,z_0]]} + b_1c_1[x,[y_0,z_0]] \\
& +\underline{b_1[[x,z_1],[y_0,z_0]]} + c_1[[x,[y_0,z_0]],y_1] + \underline{[[[x,z_1],[y_0,z_0]],y_1]} \\
& +\underline{b_1c_0[x,[y_0,z_1]]} + \underline{b_1[[x,z_0],[y_0,z_1]]} + \underline{c_0[[x,[y_0,z_1]],y_1]} \\
& +\underline{[[[x,z_0],[y_0,z_1]],y_1]} + \underline{b_1[x,[[y_0,z_0],z_1]]} + \underline{[[x,[[y_0,z_0],z_1]],y_1]} \\
& +\underline{[[x,[y_0,z_1]],[y_1,z_0]]} + b_0c_0[x,[y_1,z_1]] + b_0[[x,z_0],[y_1,z_1]] \\
& +c_0[[x,y_0],[y_1,z_1]] + [[[x,z_0],y_0],[y_1,z_1]] + [[x,[y_0,z_0]],[y_1,z_1]] \\
& +b_0[x,[[y_1,z_0],z_1]] + [[x,y_0],[[y_1,z_0],z_1]]).
\end{aligned}
$$

This last term can be seen to coincide with the right-hand side of the self-distributivity equation:

$$T(T^{\otimes 3}) \sqcup_3 (\mathbb{1}^3 \otimes \Delta_3^{\otimes 2})((a,x) \otimes (b_0,y_0) \otimes (b_1,y_1)) \otimes (c_0,z_0) \otimes (c_1,z_1)).$$

It follows therefore, that the map T turns X into a ternary self-distributive object in the category of vector spaces.

APPENDIX C : COPYRIGHT CLEARANCE FORM

Below is the permission for the material included in Chapter 2.

2/3/2020 University of South Florida Mail - Authorization JKTR

USF

Emanuele Zappala <zae@mail.usf.edu>

Authorization JKTR

Jozef Przytycki <przytyck@gmail.com> Wed, Sep 18, 2019 at 11:13 PM
Reply-To: przytyck@gwu.edu
To: Emanuele Zappala <zae@mail.usf.edu>

Dear Emanuele Zappala,
You can use for sure the material from your published paper; on the other hand
I think you cannot thus put scan of the paper in JKTR.
People often put arXiv version and that is ok (in references you mention JKTR).
If you have more technical questions I can pass it to JKTR.
Yours Jozef

Jozef H. Przytycki
(George Washington University)
http://home.gwu.edu/~przytyck/
https://en.wikipedia.org/wiki/J%C3%B3zef_H._Przytycki

[Quoted text hidden]

https://mail.google.com/mail/u/1?ik=4eb4f6546d&view=pt&search=all&permmsgid=msg-f%3A1645071871086785810&simpl=msg-f%3A1645071871086785810 1/1

APPENDIX D : ARXIV LICENSE

According to arXiv license information, I have the right to freely utilize material from the papers I have submitted to arXiv, without any specific permission.

We gratefully acknowledge support from
the Simons Foundation and member institutions.

Search... All fields Search

Help | Advanced Search

Login

Search Help Search

arXiv / Help / arXiv License Information

arXiv License Information

arXiv is a repository for scholarly material, and perpetual access is necessary to maintain the scholarly record. As such, arXiv keeps a permanent record of every submission and replacement announced.

arXiv does not ask that copyright be transferred. However, we require sufficient rights to allow us to distribute submitted articles in perpetuity. In order to submit an article to arXiv, the submitter must either:

- grant arXiv.org a non-exclusive and irrevocable license to distribute the article, and certify that the submitter has the right to grant this license;
- certify that the work is available under one of the following Creative Commons licenses and that the submitter has the right to assign this license:
  - Creative Commons Attribution license (CC BY 4.0)
  - Creative Commons Attribution-ShareAlike license (CC BY-SA 4.0)
  - Creative Commons Attribution-Noncommercial-ShareAlike license (CC BY-NC-SA 4.0);
- or dedicate the work to the public domain by associating the Creative Commons Public Domain Dedication (CC0 1.0) with the submission.

In the most common case, authors have the right to grant these licenses because they hold copyright in their own work.

We currently support three of the Creative Commons licenses. If you wish to use a different CC license, then select arXiv's non-exclusive license to distribute in the arXiv submission process and indicate the desired Creative Commons license in the actual article.

Note: if you intend to submit, or have submitted, your article to a journal then you should verify that the license you select during arXiv submission does not conflict with the journal's license or copyright transfer agreement. Many journal agreements permit submission to arXiv using the non-exclusive license to distribute, which arXiv has used since 2004. Yet the CC BY and CC BY-SA licenses permit commercial reuse and may therefore conflict with some journal agreements.

Metadata license

To the extent that the depositor or arxiv has a copyright interest in metadata accompanying the submission, a Creative Commons CC0 1.0 Universal Public Domain Dedication will apply.