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The Systems of Post and Post Algebras: A Demonstration of an Obvious Fact

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by

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Dedication

For those who have been pushed to the margins for shining a light where darkness was with the sole intent of making the universe a more hospitable place for creation.

> I have declared a spiritual war upon all coercion that restricts man's creative activity. There are two kinds of coercion. One of them is *physical*... the other... is *logical*. We must accept self–evident principles and the theorems resulting therefrom.... That coercion originated with the rise of Aristotelian logic and Euclidean geometry.
>
> —Jan Łukasiewicz, *Selected Works*

> I have proclaimed the glory of thy works to the people who will read these demonstrations, to the extent that the limitations of my spirit would allow.
>
> —Johannes Kepler, *Harmonia Mundi*

> The idea does not belong to the soul; it is the soul that belongs to the idea.
>
> —Charles Sanders Peirce, *Collected Papers, Vol. 1*

> Civilization does not consist in progress as such and in mindless destruction of old values, but in developing and refining the good that has been won.
>
> —Carl Gustav Jung, *Collected Works, Vol. 2*

> I wish we could open our eyes to see in all directions at the same time.
>
> —Death Cab for Cutie, "Marching Bands of Manhattan"

> At least we have tomorrow if we have tomorrow. Tomorrow's just a day beyond today if tomorrow comes tomorrow.
>
> —Built to Spill, "Tomorrow"

> I know that I don't know everything there is to know, but I know that if you plant a seed you can live to see it grow.
>
> —Brian Ernst, "Change"

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Abstract

In 1942, Paul C. Rosenbloom put out a definition of a *Post algebra* after Emil L. Post published a collection of systems of many–valued logic. Post algebras became easier to handle following George Epstein's alternative definition. As conceived by Rosenbloom, Post algebras were meant to capture the algebraic properties of Post's systems; this fact was not verified by Rosenbloom nor Epstein and has been assumed by others in the field. In this thesis, the long–awaited demonstration of this oft–asserted assertion is given.

After an elemental history of many–valued logic and a review of basic Classical Propositional Logic, the systems given by Post are introduced. The definition of a Post algebra according to Rosenbloom together with an examination of the meaning of its notation in the context of Post's systems are given. Epstein's definition of a Post algebra follows the necessary concepts from lattice theory, making it possible to prove that Post's systems of many–valued logic do in fact form a Post algebra.

Chapter 1

Introduction

In the first two years of the 1920s, systems of many–valued logic were introduced in the respective articles [5] and [6] of Jan Łukasiewicz and Emil L. Post, making many–valued logic an accepted area of research. Although it is a convention that in the West many–valued logic has grown from a seed that sprouted in Ancient Greece, it is worth noting that outside of the West, particularly in Asia, many–valued logic has been essential to the view of the world fostered by Buddhism. Today, many–valued logic is a dynamic branch of mathematics, but this dynamism did not come about through Jan Łukasiewicz and Emil L. Post alone. In 1909, Charles Sanders Peirce wrote down, in the manuscript we call the *Logic Notebook* (MS 339), a formal system of many–valued logic; the author is not aware of any earlier formal system. Unfortunately, it does not seem that Peirce pursued this subject sufficiently far. Łukasiewicz and Post developed their systems of many–valued logic independently of Peirce and of each other.

Two decades after Post put forward his systems of many–valued logic, Paul C. Rosenbloom gave a definition of an algebraic structure that served as an interpretation of Post's systems; these structures are called Post algebras. Rosenbloom's definition consists of plenty of notation and axioms, making it quite heavy to build a theory on. Almost two more decades passed before George Epstein proposed an equivalent definition that is comparatively easy to understand; since then, much research has been done regarding Post algebras. However, the author's investigation has uncovered a question—does the definition of a Post algebra actually capture Post's systems of many–valued logic?—that has been repeatedly answered in the affirmative, but without any extant justification or proof found; in this thesis, a demonstration of this assertion shall be given, making this the focal point of the thesis. In Chapter 2, after introducing Post's propositional systems of many–valued logic (Section 2.1) and Post algebras according to Rosenbloom's definition (Section 2.2), we shall construct the equivalent definition given by Epstein (Subsection 2.2.2). Finally, we shall show the connection between Post algebras and the set of formulas of the propositional systems of Post, when partitioned by a suitable equivalence relation (Subsection 2.2.3). For the moment, let us begin with a formalization of Classical Propositional Logic and a prelude to many–valued logic.

1.1 A Review of Classical Propositional Logic

The precursor to the system known today as Classical Propositional Logic (CPL) is found in the texts *Prior Analytics* and *On Interpretation* by Aristotle[1]. In these books, Aristotle attempted to systematize, while creating a model of, what he considered to be proper reasoning. In CPL, each member of a set of *propositional variables* is assigned an element from the set of *truth values* $\{\top, \bot\}$, where $\top$ and $\bot$ are to be interpreted as *true* and *false*, respectively. The set of formulas of CPL is defined as the smallest set containing the set of propositional variables, and closed with respect to a unary operator $\neg$ and a binary operator $\Vee$; these operators are called *negation* and *disjunction*, respectively. Propositional variables are called *atomic formulas*. Operators on sets of formulas are also known as *connectives*.

For two formulas φ and ψ of CPL, $(\neg\varphi)$ and $(\varphi \Vee \psi)$ are defined through the following *truth tables*:

φ	$(\neg\varphi)$
$\top$	$\bot$
$\bot$	$\top$

	$\Vee$	ψ: $\top$	ψ: $\bot$
φ	$\top$	$\top$	$\top$
φ	$\bot$	$\top$	$\bot$

Observe that the truth value of the negation of a formula is the «opposite» of its truth value and the truth value of the disjunction of a pair of formulas is the truth value of the «truest» disjunct. For readability, we shall not use outermost parentheses in formulas unless the meaning is ambiguous.

Using $\neg$ and $\Vee$, other connectives of CPL can be defined. For example, *conjunction*, a binary connective denoted by $\Wedge$, is defined as $\neg\,((\neg\varphi) \Vee (\neg\psi))$, where φ and ψ are formulas of CPL. A simple computation reveals the truth table for $\varphi \Wedge \psi$ as follows:

	$\Wedge$	ψ: $\top$	ψ: $\bot$
φ	$\top$	$\top$	$\bot$
φ	$\bot$	$\bot$	$\bot$

The connection between CPL and the English language is very straighforward. In *On Interpretation*, Aristotle tells us that a *sentence* is considerable speech, various parts of which may have meaning. Certain sentences—those that are either true or false—Aristotle called *propositions*. The class of propositions is the intended realm of CPL. A proposition not containing in it another proposition is known as a *simple proposition*. Simple propositions correspond to propositional variables in CPL; all other propositions are called

[1] The author is not claiming that the historical figure known by the same name is the actual writer of these works; unless it is specified, anytime Aristotle is referred to, it is meant the person (or collection of persons) who wrote these works.

compound propositions. In CPL, the formulas containing connectives can be thought of as representing compound propositions.

The set of propositions in the English language can be partitioned into two *polarity classes*, called *affirmative* and *negative*. If the formulas of CPL are interpreted as propositions in the English language, it should not be a surprise that the connective $\neg$ corresponds to the operation which changes the polarity class of a proposition. Certain words in the English language are reserved as *coordinators*; these words are the mechanism through which two or more propositions are joined together to form another proposition. A moment of thinking can show that the binary connectives $\vee$ and $\wedge$ act as the coordinators *or* and *and*, respectively.

In CPL, the so–called *Principle of Bivalence*, telling us that every formula is (or has truth value) either true or false, is upheld. A *tautology* of CPL is a formula for which every entry in its truth table is $\top$; that is, a formula is a tautology of CPL if it is true no matter what the truth values of its component formulas are. Similarly, a formula whose truth table consists only of $\bot$ is called a *contradiction*.

If CPL is to be a model of proper reasoning, then two facts, among others, must hold. Firstly, the *Law of Excluded Middle* tells us that for every formula, the formula or its negation is true. In the language of CPL, the Law of Excluded Middle is equivalent to $\varphi \vee (\neg\varphi)$ being a tautology for any formula φ; this can be observed in the following truth table:

φ	$\neg\varphi$	$\varphi \vee (\neg\varphi)$
$\top$	$\bot$	$\top$
$\bot$	$\top$	$\top$

The *Law of Non–Contradiction* asserts that it is not the case that a formula and its negation are true at the same time; in CPL this means that $\varphi \wedge (\neg\varphi)$ is a contradiction or, equivalently, that $\neg(\varphi \wedge (\neg\varphi))$ is a tautology, for any formula φ. The following truth table confirms the Law of Non–Contradiction in CPL:

φ	$\neg\varphi$	$\varphi \wedge (\neg\varphi)$	$\neg(\varphi \wedge (\neg\varphi))$
$\top$	$\bot$	$\bot$	$\top$
$\bot$	$\top$	$\bot$	$\top$

1.2 An Excavation of Many–Valued Logic

Through *Prior Analytics* and *On Interpretation*, Aristotle's ideas on logic came to be the prominent and most prevailing theory to this day, influencing every form of human thought and organization; we may never know how convinced Aristotle, the historical figure, was that his work was adequate enough to deserve having such an impact on what was to be the future of our species. A century before Aristotle is supposed to have roamed a corner of our planet, a collection of ideas that—like Aristotle's—has been capable of surviving in the conceptual sphere of human consciousness was in the process of being realized. Today, that collection of ideas is called Buddhism.

Buddhism is filled with propositions that do not obey the Law of Excluded Middle nor the Law of Non–Contradiction when viewed from the standpoint of Classical Propositional Logic; in [8], Graham Priest provides the following example courtesy of the Buddhist philosopher Nagarjuna: "The nature of things is to have no nature." A way to capture the notions set forth in Buddhism involves the rejection of the Principle of Bivalence in favor of allowing some formulas to have truth values distinct from $\top$ and $\bot$. Systems of logic in which more than two truth values are involved are called *systems of many–valued logic*. See [7] and [8] for a detailed description of how Buddhism can be approached through a system of many–valued logic.

Despite the fact that the system passed down in Aristotle's works has served as the primary model of reasoning, many–valued logic made an appearance in these writings, notably regarding propositions about the future known as *future contingents*. Future contingents would come back in the twentieth century to motivate the work of Jan Łukasiewicz and help solidify many–valued logic as an accepted area of research. The author invites the reader to examine Chapter 1 of [9] and Chapter 4 of [4] for a general history of many–valued logic dating back to Aristotle.

As an example of a system of many–valued logic, let us look at Charles Sanders Peirce's system, which was exposed by Max Fisch and Atwell Turquette in the 1966 article [3]. In the *Logic Notebook*, Peirce tells us that certain propositions declaring properties about objects are not conclusively true nor false, but at the limit between true and false. In Peirce's system, formulas are allowed to have any of the three truth values V, L, and F, which are to represent *verum* (true), *limit*, and *falsum* (false), respectively. Peirce's system consists of four unary connectives and six binary connectives shown in the following tables, where x and y are formulas:

x	$\overline{x}$	$\mathring{\overline{x}}$	$\grave{x}$	$\acute{x}$
V	F	L	F	L
L	L	L	V	F
F	V	L	L	V

	Φ	y: V	L	F
	V	V	V	V
x	L	V	L	F
	F	V	F	F

	Θ	y: V	L	F
	V	V	V	V
x	L	V	L	L
	F	V	L	F

			y	
	$\boldsymbol{\Psi}$	V	L	F
	V	V	V	F
x	L	V	L	F
	F	F	F	F

			y	
	$\mathbf{Z}$	V	L	F
	V	V	L	F
x	L	L	L	F
	F	F	F	F

			y	
	$\boldsymbol{\Omega}$	V	L	F
	V	V	L	F
x	L	L	L	L
	F	F	L	F

			y	
	$\boldsymbol{\Upsilon}$	V	L	F
	V	V	L	V
x	L	L	L	L
	F	V	L	F

As to what moved Peirce to settle for the connectives given above is a matter of speculation. Atwell Turquette published subsequent articles exploring Peirce's system of many–valued logic and the interested reader is encouraged to inspect [11, 12, 13].

Chapter 2

An Algebraic Perspective of the Propositional Systems of Emil L. Post

2.1 The Propositional Systems of Post: Syntax and Semantics

For an integer $n \geq 2$, the formulas of *Post's propositional n–valued logic* are strings of symbols containing the *rotational* (unary) *connective* $'$, and the *disjunctive* (binary) *connective* $\vee$, with *parentheses* as auxiliary symbols. The *formulas* are obtained after finitely many applications of the following rules:

- Any element from a countable set of *propositional variables* Λ is a formula.
- If φ is a formula, so is (φ'), called the *rotation of* φ.
- If φ and ψ are formulas, so is $(\varphi \vee \psi)$, which is called the *disjunction of* φ *and* ψ.

The set of formulas of Post's propositional n–valued logic shall be denoted by $\mathbf{P}_n$. If a formula φ is in Λ, then φ is called an *atomic formula*; otherwise, φ is said to be a *compound formula*. Provided the meaning is clear, when we find it convenient, we shall suppress the use of parentheses.

To each atomic formula φ, we assign an element from the set

$$\Delta_n := \left\{ \frac{i}{n-1} \,;\, i = 0, \ldots, n-1 \right\},$$

called its *truth value* and denoted by $\hat{\mathbf{v}}(\varphi)$; hence, $\hat{\mathbf{v}} \colon \Lambda \to \Delta_n$ and we shall call $\hat{\mathbf{v}}$ a *truth assignment of* Λ. Once a truth assignment $\hat{\mathbf{v}}$ of Λ has been agreed upon, we can extend $\hat{\mathbf{v}}$ to all formulas in $\mathbf{P}_n$ and obtain a map $\mathbf{v} \colon \mathbf{P}_n \to \Delta_n$, called a *valuation of* $\mathbf{P}_n$, which agrees with $\hat{\mathbf{v}}$ on Λ and gives the truth values of all compound formulas according to the rules below:

- If $\theta = \varphi'$, then $\mathbf{v}(\theta) := \begin{cases} 1 & \text{if } \mathbf{v}(\varphi) = 0, \\ \mathbf{v}(\varphi) - \frac{1}{n-1} & \text{otherwise.} \end{cases}$
- If $\theta = \varphi \vee \psi$, then $\mathbf{v}(\theta) := \max\{\mathbf{v}(\varphi), \mathbf{v}(\psi)\}$.

We can organize the information above into a *truth table* in which, given an integer $n \geq 2$, for a formula or a pair of formulas in $\mathbf{P}_n$, all truth values or combinations of truth values, respectively, along with their respective rotation and disjunction are displayed. For example, with $n = 2$, the set of truth values is given by $\Delta_2 = \{0, 1\}$ and we have the following truth tables:

φ	φ'
1	0
0	1

	$\vee$	ψ: 1	0
φ	1	1	1
	0	1	0

Note that the truth values of the formulas in $\mathbf{P}_2$ agree with the truth values of the formulas of Classical Propositional Logic when $\top$ is replaced by 1, $\bot$ is replaced by 0, and the rotational and disjunctive connectives are seen as ⫬ and ⩔, respectively.

When $n = 3$, we have that $\Delta_3 = \{0, 1/2, 1\}$ and the truth tables are given by:

φ	φ'
1	$1/2$
$1/2$	0
0	1

	$\vee$	ψ: 1	$1/2$	0
	1	1	1	1
φ	$1/2$	1	$1/2$	$1/2$
	0	1	$1/2$	0

The truth tables show that the formulas in $\mathbf{P}_3$ have truth values coinciding with the three–valued logic of Charles Sanders Peirce when replacing 1, $1/2$, and 0 by V, L, and F, respectively, and rotation and disjunction are seen as Peirce's ´ and **Θ**, respectively.

To see these systems under a better light, let us see what happens for a larger value of n. In the case $n = 5$, $\Delta_5 = \{0, 1/4, 1/2, 3/4, 1\}$ with the following truth tables:

φ	φ'
1	$3/4$
$3/4$	$1/2$
$1/2$	$1/4$
$1/4$	0
0	1

	$\vee$	ψ: 1	$3/4$	$1/2$	$1/4$	0
	1	1	1	1	1	1
	$3/4$	1	$3/4$	$3/4$	$3/4$	$3/4$
φ	$1/2$	1	$3/4$	$1/2$	$1/2$	$1/2$
	$1/4$	1	$3/4$	$1/2$	$1/4$	$1/4$
	0	1	$3/4$	$1/2$	$1/4$	0

For an arbitrary integer $n \geq 2$, we have that

$$\Delta_n = \left\{0, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, \frac{n-3}{n-1}, \frac{n-2}{n-1}, 1\right\}$$

and the truth tables are given by:

φ	φ'
1	$\frac{n-2}{n-1}$
$\frac{n-2}{n-1}$	$\frac{n-3}{n-1}$
$\vdots$	$\vdots$
$\frac{2}{n-1}$	$\frac{1}{n-1}$
$\frac{1}{n-1}$	0
0	1

		ψ					
	$\vee$	1	$\frac{n-2}{n-1}$	$\cdots$	$\frac{2}{n-1}$	$\frac{1}{n-1}$	0
	1	1	1	$\cdots$	1	1	1
	$\frac{n-2}{n-1}$	1	$\frac{n-2}{n-1}$	$\cdots$	$\frac{n-2}{n-1}$	$\frac{n-2}{n-1}$	$\frac{n-2}{n-1}$
	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
φ	$\frac{2}{n-1}$	1	$\frac{n-2}{n-1}$	$\cdots$	$\frac{2}{n-1}$	$\frac{2}{n-1}$	$\frac{2}{n-1}$
	$\frac{1}{n-1}$	1	$\frac{n-2}{n-1}$	$\cdots$	$\frac{2}{n-1}$	$\frac{1}{n-1}$	$\frac{1}{n-1}$
	0	1	$\frac{n-2}{n-1}$	$\cdots$	$\frac{2}{n-1}$	$\frac{1}{n-1}$	0

Thus, we can see that rotation permutes the truth values of a formula in $\mathbf{P}_n$ cyclically and the truth value of the disjunction of a pair of formulas is the maximum of the truth values of its disjuncts.

DEFINITION 1 A formula $\varphi \in \mathbf{P}_n$ is a *tautology of* $\mathbf{P}_n$ if $\mathbf{v}(\varphi) = 1$ for every valuation $\mathbf{v}$ of $\mathbf{P}_n$; φ is a *contradiction of* $\mathbf{P}_n$ if $\mathbf{v}(\varphi) = 0$ for every $\mathbf{v}$. The sets consisting of tautologies and contradictions of $\mathbf{P}_n$ shall be denoted by **t** and **c**, respectively; that is,

$$\mathbf{t} := \{\varphi\,;\ \varphi \in \mathbf{P}_n,\ \mathbf{v}(\varphi) = 1 \text{ for every } \mathbf{v}\colon \mathbf{P}_n \to \Delta_n\}$$

and

$$\mathbf{c} := \{\varphi\,;\ \varphi \in \mathbf{P}_n,\ \mathbf{v}(\varphi) = 0 \text{ for every } \mathbf{v}\colon \mathbf{P}_n \to \Delta_n\}\,.$$

That the sets **t** and **c** are not empty has been shown by Emil L. Post in [6]. Post also showed that the set of connectives $\{\,'\,,\vee\}$ is *truth–functionally complete* in the sense that any truth table is the truth table of some formula in $\mathbf{P}_n$.

DEFINITION 2 A binary relation $\equiv$ on a set Ξ is an *equivalence relation* if it is reflexive, symmetric, and transitive, that is, if for every $\theta, \varphi, \psi \in \Xi$:

- $\theta \equiv \theta$.
- If $\varphi \equiv \psi$, then $\psi \equiv \varphi$.
- If $\theta \equiv \varphi$ and $\varphi \equiv \psi$, then $\theta \equiv \psi$.

DEFINITION 3 Two formulas $\varphi, \psi \in \mathbf{P}_n$ are (*logically*) *equivalent*, denoted by $\varphi \risingdotseq \psi$, if $\mathbf{v}(\varphi) = \mathbf{v}(\psi)$ for every valuation $\mathbf{v}$ of $\mathbf{P}_n$.

Observe that $\risingdotseq$ in Definition 3 is an equivalence relation on $\mathbf{P}_n$.

2.1.1 Post's Interpretation of the Propositional Systems

Emil L. Post suggested a semantical interpretation for the systems of propositional logic he defined. Post did not elaborate in depth regarding the implications of his interpretation. In this section, we shall give an overview of Post's intepretation of his many–valued propositional logic as is found in [6] accounting for notational changes.

Let $n \geq 2$ be an integer. The set Λ of atomic formulas of $\mathbf{P}_n$ consists of sequences of $n-1$ atomic formulas of Classical Propositional Logic with increasing truth values; for simplicity, we shall use 1 for $\top$ and 0 for $\bot$. Hence, $\Lambda \subseteq \mathrm{K}^{n-1}$, where K is the set of atomic formulas of Classical Propositional Logic. Let $\hat{\mathbf{u}} \colon \mathrm{K} \to \{0,1\}$ be a truth assignment of K. Then, φ is in Λ if

$$\varphi = \langle p_1, p_2, \ldots, p_{n-1} \rangle,$$

where $p_i \in \mathrm{K}$, $i = 1, \ldots, n-1$, with $\hat{\mathbf{u}}(p_{j-1}) \leq \hat{\mathbf{u}}(p_j)$, $j = 2, \ldots, n-1$. From $\hat{\mathbf{u}}$, we obtain a truth assignment $\hat{\mathbf{v}} \colon \Lambda \to \Delta_n$ of Λ as follows:

$$\hat{\mathbf{v}}(\varphi) = \begin{cases} 1 & \text{if } \hat{\mathbf{u}}(p_1) = 1, \\ \frac{n-i}{n-1} & \text{if } \hat{\mathbf{u}}(p_i) = 1 \text{ and } \hat{\mathbf{u}}(p_{i-1}) = 0, \text{ where } i = 2, \ldots, n-1, \\ 0 & \text{if } \hat{\mathbf{u}}(p_{n-1}) = 0. \end{cases}$$

With Λ as given above, the compound formulas $\varphi \vee \psi$ and φ' in $\mathbf{P}_n$, where $\varphi = \langle p_1, p_2, \ldots, p_{n-1} \rangle$ and $\psi = \langle q_1, q_2, \ldots, q_{n-1} \rangle$, are defined as

$$\varphi \vee \psi = \langle p_1 \Cup q_1, p_2 \Cup q_2, \ldots, p_{n-1} \Cup q_{n-1} \rangle$$

and

$$\varphi' = \langle \mathrel{\ulcorner} (p_1 \Cup \cdots \Cup p_{n-1}), \mathrel{\ulcorner} (p_1 \Cup \cdots \Cup p_{n-1}) \Cup (p_1 \Cap p_2), \ldots, \\ \mathrel{\ulcorner} (p_1 \Cup \cdots \Cup p_{n-1}) \Cup (p_{n-2} \Cap p_{n-1}) \rangle,$$

respectively. Now, $\hat{\mathbf{v}}$ can be extended to all formulas in $\mathbf{P}_n$, giving us a valuation $\mathbf{v}$ of $\mathbf{P}_n$ according to the following rule with $\varphi = \langle p_1, p_2, \ldots, p_{n-1} \rangle$:

$$\mathbf{v}(\varphi) = \begin{cases} 1 & \text{if } \mathbf{u}(p_1) = 1, \\ \frac{n-i}{n-1} & \text{if } \mathbf{u}(p_i) = 1 \text{ and } \mathbf{u}(p_{i-1}) = 0, \text{ where } i = 2, \ldots, n-1, \\ 0 & \text{if } \mathbf{u}(p_{n-1}) = 0, \end{cases}$$

where $\mathbf{u}$ is a valuation of the formulas of Classical Propositional Logic agreeing with $\hat{\mathbf{u}}$ on K and with truth values conforming to the truth tables for Classical Propositional Logic as given in Chapter 1, with the exception that $\top$ and $\bot$ are replaced by 1 and 0, respectively.

As an example, let $\hat{\mathbf{u}}$ be a truth assignment of K and consider $\varphi \in \Lambda$, where

$$\varphi = \langle p_1, p_2, p_3 \rangle$$

with $p_1, p_2, p_3 \in$ K such that $\hat{\mathbf{u}}(p_1) = \hat{\mathbf{u}}(p_2) = 0$ and $\hat{\mathbf{u}}(p_3) = 1$. Observe that φ is in $\mathbf{P}_4$. Then, using $\hat{\mathbf{u}}$, we have a truth assignment $\hat{\mathbf{v}} \colon \Lambda \to \Delta_4$ of Λ with

$$\hat{\mathbf{v}}(\varphi) = \frac{4-3}{4-1} = \frac{1}{3}.$$

We can now compute the truth value of the φ'. To that end, let $\varphi' = \langle \varphi_1, \varphi_2, \varphi_3 \rangle$, where, by definition,

$$\begin{aligned}
&\varphi_1 = \neg\!\!\!\neg \left((p_1 \mathbin{\Cup} p_2) \mathbin{\Cup} p_3 \right), \\
&\varphi_2 = \neg\!\!\!\neg \left((p_1 \mathbin{\Cup} p_2) \mathbin{\Cup} p_3 \right) \mathbin{\Cup} (p_1 \mathbin{\Cap} p_2), \text{ and} \\
&\varphi_3 = \neg\!\!\!\neg \left((p_1 \mathbin{\Cup} p_2) \mathbin{\Cup} p_3 \right) \mathbin{\Cup} (p_2 \mathbin{\Cap} p_3).
\end{aligned}$$

Letting $\mathbf{u}$ be the extension of $\hat{\mathbf{u}}$ to all formulas of Classical Propositional Logic, a straightforward computation tells us that $\mathbf{u}(\varphi_1) = 0$, $\mathbf{u}(\varphi_2) = 0$, and $\mathbf{u}(\varphi_3) = 0$. Thus, since $\mathbf{u}(\varphi_3) = 0$, we have that $\mathbf{v}(\varphi') = 0$, where $\mathbf{v}$ is a valuation of $\mathbf{P}_4$; note that this result agrees with the definition of the rotational connective.

Similarly, for $\psi = \langle q_1, q_2, q_3 \rangle \in \mathbf{P}_4$, where q_1, q_2, and q_3 are formulas of Classical Propositional Logic with $\mathbf{u}(q_1) = 0$ and $\mathbf{u}(q_2) = \mathbf{u}(q_3) = 1$, we can compute the truth value of

$$\varphi \vee \psi = \langle p_1 \mathbin{\Cup} q_1, p_2 \mathbin{\Cup} q_2, p_3 \mathbin{\Cup} q_3 \rangle.$$

Observe that

$$\mathbf{v}(\psi) = \frac{4-2}{4-1} = \frac{2}{3}.$$

Now, we have that

$$\begin{aligned}
&\mathbf{u}(p_1 \mathbin{\Cup} q_1) = 0, \\
&\mathbf{u}(p_2 \mathbin{\Cup} q_2) = 1, \text{ and} \\
&\mathbf{u}(p_3 \mathbin{\Cup} q_3) = 1.
\end{aligned}$$

Thus,

$$\mathbf{v}(\varphi \vee \psi) = \frac{4-2}{4-1} = \frac{2}{3} = \max\left\{\mathbf{v}(\varphi), \mathbf{v}(\psi)\right\},$$

as the definition of disjunction prescribes.

2.2 Post Algebras

In his 1942 article [10], Paul C. Rosenbloom defined an algebra to capture the many–valued logic put forth by Post, which Rosenbloom called a Post algebra.

DEFINITION 4 Given an integer $n \geq 2$, the triple $\langle \Xi, \curlyvee, \sim \rangle$, where Ξ is a set, $\curlyvee$ is a binary operation on Ξ, and $\sim$ is a unary operation on Ξ, is a *Post algebra* (*of type* n) if the axioms below are satisfied subject to the following notational conventions:

Notations:

1. $\theta \curlyvee \varphi \curlyvee \psi = (\theta \curlyvee \varphi) \curlyvee \psi$.
2. $\displaystyle\bigcurlyvee_{i=a}^{b} f(i) = f(a) \curlyvee \cdots \curlyvee f(b)$.
3. $\overset{0}{\sim} \varphi = \varphi, \qquad \overset{m+1}{\sim} \varphi = \sim \overset{m}{\sim} \varphi$.
4. ${}^{1}\varphi = \displaystyle\bigcurlyvee_{m=0}^{n-1} \overset{m}{\sim} \varphi$.
5. ${}^{2}\varphi = \sim\left({}^{1}\varphi\right), \qquad {}^{m}\varphi = \overset{m-1}{\sim}\left({}^{1}\varphi\right). \qquad (2 \leq m \leq n-1)$.
6. ${}^{0}\varphi = \sim\left({}^{n-1}\varphi\right) = \overset{n-1}{\sim}\left({}^{1}\varphi\right)$.
7. $\varphi^{1} = \overset{n-1}{\sim}\left(\displaystyle\bigcurlyvee_{m=1}^{n-1} \overset{m}{\sim} \varphi\right)$.
8. $\varphi^{m} = \overset{n-1}{\sim}\left(\overset{n-1}{\sim}\left(\varphi \curlyvee {}^{2}\varphi\right) \curlyvee \overset{m}{\sim} \varphi\right). \qquad (2 \leq m \leq n-1)$.
9. $\varphi_{i}^{j} = \left(\overset{i}{\sim} \varphi\right)^{j}$.
10. $\varphi^{-i} = \varphi_{n-i+1}^{1}$.
11. $-\varphi = \displaystyle\bigcurlyvee_{m=1}^{n-1} \varphi_{m}^{m}$.
12. $\varphi \curlywedge \psi = -\left((-\varphi) \curlyvee (-\psi)\right)$.
13. $\theta \curlywedge \varphi \curlywedge \psi = (\theta \curlywedge \varphi) \curlyvee \psi$.

Axioms:

1. Ξ contains at least two distinct elements.

2. $\curlyvee$ is commutative and associative.

3. For every $\varphi \in \Xi$, $\varphi \curlyvee \varphi = \varphi$.

4. For every $\varphi \in \Xi$, ${}^1\varphi = \overset{n}{\sim}\left({}^1\varphi\right) = \sim\left({}^0\varphi\right)$.

5. For every $\theta, \varphi, \psi \in \Xi$, $\theta \curlyvee (\varphi \curlywedge \psi) = (\theta \curlyvee \varphi) \curlywedge (\theta \curlyvee \psi)$.

6. For every $\varphi, \psi \in \Xi$, $\displaystyle\bigcurlyvee_{m=0}^{n-1} \left(\varphi \curlywedge \overset{m}{\sim}\psi\right) = \varphi$.

7. For every $\varphi \in \Xi$,

$$\varphi = \varphi^1 \curlyvee \left({}^2\varphi \curlywedge \varphi^1_{n-1}\right) \curlyvee \bigcurlyvee_{i=3}^{n-3} \left(\left({}^i\varphi \curlyvee {}^{i+1}\varphi\right) \curlywedge \varphi^{-i}\right) \curlyvee$$
$$\left({}^{n-2}\varphi \curlywedge \varphi^1_3\right) \curlyvee \left({}^{n-1}\varphi \curlywedge \varphi^1_2\right) \curlyvee \left({}^0\varphi \curlywedge \varphi^1_1\right).$$

8. For every $\varphi, \psi_0, \ldots, \psi_{n-1} \in \Xi$, $\sim\left(\displaystyle\bigcurlyvee_{i=0}^{n-1} \left(\psi_i \curlywedge \varphi^1_i\right)\right) = \displaystyle\bigcurlyvee_{i=0}^{n-1} \left(\sim\psi_i \curlywedge \varphi^1_i\right)$.

2.2.1 A Closer Look at the Definition in the Context of Post's Propositional Systems

In what follows, we shall attempt at making better sense of what the notations given in Definition 4 mean in the context of the systems of Post. Firstly, we shall adapt the notational conventions given in Definition 4.

DEFINITION 5 Given formulas $\theta, \varphi, \psi \in \mathbf{P}_n$, we have the following notational definitions:

1. $\theta \vee \varphi \vee \psi := (\theta \vee \varphi) \vee \psi$.

2. $\displaystyle\bigvee_{i=a}^{b} f(i) := f(a) \vee \cdots \vee f(b)$.

3. $\mathsf{R}^0(\varphi) := \varphi, \qquad \mathsf{R}^{m+1}(\varphi) := (\mathsf{R}^m(\varphi))'$.

4. ${}^1\varphi := \displaystyle\bigvee_{m=0}^{n-1} \mathsf{R}^m(\varphi)$.

5. ${}^2\varphi := \left({}^1\varphi\right)', \qquad {}^m\varphi := \mathsf{R}^{m-1}\left({}^1\varphi\right). \qquad (2 \leq m \leq n-1).$

6. ${}^0\varphi := \left({}^{n-1}\varphi\right)' = \mathsf{R}^{n-1}\left({}^1\varphi\right)$.

7. $\varphi^1 := \mathsf{R}^{n-1}\left(\displaystyle\bigvee_{m=1}^{n-1} \mathsf{R}^m(\varphi)\right)$.

8. $\varphi^m := \mathsf{R}^{n-1}\left(\mathsf{R}^{n-1}\left(\varphi \vee {}^2\varphi\right) \vee \mathsf{R}^m(\varphi)\right). \qquad (2 \leq m \leq n-1).$

9. $\varphi_i^j := \left(\mathsf{R}^i(\varphi)\right)^j$.

10. $\overline{\varphi} := \bigvee_{m=1}^{n-1} \varphi_m^m$.

11. $\varphi \wedge \psi := \overline{\overline{\varphi} \vee \overline{\psi}}$.

For a clearer understanding of the notations given in Definition 5, let us construct truth tables for a formula $\varphi \in \mathbf{P}_4$. Note that $\mathsf{R}^1(\varphi) = \varphi'$, $\mathsf{R}^2(\varphi) = \varphi''$, and $\mathsf{R}^3(\varphi) = \varphi'''$. Notations 3–6 in Definition 5 are captured in the following truth tables:

φ	φ'	φ''	φ'''	${}^1\varphi$	${}^2\varphi$	${}^3\varphi$	${}^0\varphi$
1	$2/3$	$1/3$	0	1	$2/3$	$1/3$	0
$2/3$	$1/3$	0	1	1	$2/3$	$1/3$	0
$1/3$	0	1	$2/3$	1	$2/3$	$1/3$	0
0	1	$2/3$	$1/3$	1	$2/3$	$1/3$	0

Continuing in Definition 5, the truth tables corresponding to Notations 7–10 are given by:

φ	φ^1	φ^2	φ^3	φ_1^1	φ_2^2	φ_3^3	$\overline{\varphi}$
1	1	$2/3$	$1/3$	0	0	0	0
$2/3$	0	0	0	0	0	$1/3$	$1/3$
$1/3$	0	0	0	0	$2/3$	0	$2/3$
0	0	0	0	1	0	0	1

Finally, for formulas $\varphi, \psi \in \mathbf{P}_4$, Notation 11 in Definition 5 can be portrayed as follows:

		ψ			
	$\wedge$	1	$2/3$	$1/3$	0
	1	1	$2/3$	$1/3$	0
	$2/3$	$2/3$	$2/3$	$1/3$	0
φ	$1/3$	$1/3$	$1/3$	$1/3$	0
	0	0	0	0	0

As we can see from the truth tables above, the formulas ${}^m\varphi$, where $0 \leq m \leq 3$, take one truth value uniformly for every valuation. Note that the truth tables for φ^m consist of the truth value 0 for every valuation $\mathbf{v}$ of $\mathbf{P}_4$, unless $\mathbf{v}(\varphi) = 1$, in which case, $\mathbf{v}(\varphi^m) = \mathbf{v}({}^m\varphi)$. In the truth tables for φ_m^m, the truth values of φ^m are permuted in such a way that $\overline{\varphi}$ coincides with negation as was defined by Łukasiewicz. Lastly, the entries in the truth table for $\varphi \wedge \psi$ are the minimum of corresponding truth values of φ and ψ. In the remainder of this section, we shall show the properties just mentioned.

PROPOSITION 2.1 For any formula $\varphi \in \mathbf{P}_n$ and any valuation $\mathbf{v}$ of $\mathbf{P}_n$,

$$\mathbf{v}\big((\mathsf{R}^m(\varphi))'\big) = \mathbf{v}\big(\mathsf{R}^m(\varphi')\big)\,,$$

where $m \geq 0$ is any integer.

Proof. We shall proceed by induction on m.

In the base case, $m = 0$, we have that

$$\mathbf{v}\Big(\big(\mathsf{R}^0(\varphi)\big)'\Big) = \mathbf{v}(\varphi') = \mathbf{v}\big(\mathsf{R}^0(\varphi')\big)\,.$$

Assume the statement holds for some integer m. Then

$$\begin{aligned}\mathbf{v}\Big(\big(\mathsf{R}^{m+1}(\varphi)\big)'\Big) &= \mathbf{v}\Big(\big((\mathsf{R}^m(\varphi))'\big)'\Big)\\ &= \mathbf{v}\Big(\big(\mathsf{R}^m(\varphi')\big)'\Big)\\ &= \mathbf{v}\big(\mathsf{R}^{m+1}(\varphi')\big)\,.\end{aligned}$$

■

COROLLARY 2.1.1 Let $\mathbf{v}$ be a valuation of $\mathbf{P}_n$. For any formula $\varphi \in \mathbf{P}_n$,

$$\mathbf{v}(\mathsf{R}^{m_1}(\mathsf{R}^{m_2}(\varphi))) = \mathbf{v}\big(\mathsf{R}^{m_1+m_2}(\varphi)\big)\,,$$

where $m_1, m_2 \geq 0$ are integers.

LEMMA 2.1 Let $\mathbf{v}$ be a valuation of $\mathbf{P}_n$ and $\varphi \in \mathbf{P}_n$ be a formula. Then,

$$\mathbf{v}(\mathsf{R}^m(\varphi)) = \begin{cases}\mathbf{v}(\varphi) + \frac{n-m}{n-1} & \text{if } \ \mathbf{v}(\varphi) \leq \frac{m-1}{n-1},\\ \mathbf{v}(\varphi) - \frac{m}{n-1} & \text{if } \ \mathbf{v}(\varphi) \geq \frac{m}{n-1},\end{cases}$$

where m, $1 \leq m \leq n-1$, is an integer.

Proof. We shall use induction on m for both parts of this proof.

For the first part, in the base case, $m = 1$, $\mathbf{v}(\varphi) \leq 0$, which implies that $\mathbf{v}(\varphi) = 0$. Thus,

$$\mathbf{v}\big(\mathsf{R}^1(\varphi)\big) = 1 = 0 + \frac{n-1}{n-1}.$$

Suppose the first part of the statement holds for some integer $m < n-1$. Observe that $\mathbf{v}(\mathsf{R}^m(\varphi)) \neq 0$; if it were the case that $\mathbf{v}(\mathsf{R}^m(\varphi)) = 0$, then

$$\mathbf{v}(\varphi) = \frac{m-n}{n-1} < 0,$$

which is a contradiction since $\mathbf{v}(\varphi) \geq 0$. Suppose $\mathbf{v}(\varphi) \leq \frac{m}{n-1}$. Then

$$\begin{aligned}\mathbf{v}\big(\mathsf{R}^{m+1}(\varphi)\big) &= \mathbf{v}\big((\mathsf{R}^m(\varphi))'\big) \\ &= \mathbf{v}(\mathsf{R}^m(\varphi)) - \frac{1}{n-1} \\ &= \mathbf{v}(\varphi) + \frac{n-m}{n-1} - \frac{1}{n-1} \\ &= \mathbf{v}(\varphi) + \frac{n-(m+1)}{n-1},\end{aligned}$$

as was desired.

For the second part, in the base case, $m = 1$, we have that $\mathbf{v}(\varphi) \geq \frac{1}{n-1}$, so that $\mathbf{v}\big(\mathsf{R}^1(\varphi)\big) = \mathbf{v}(\varphi) - \frac{1}{n-1}$.

For the inductive step, suppose the second part of the statement holds for some integer $m < n-1$. Assume $\mathbf{v}(\varphi) \geq \frac{m+1}{n-1}$. Then $\mathbf{v}(\varphi) \neq 0$, which implies that $\mathbf{v}(\varphi') = \mathbf{v}(\varphi) - \frac{1}{n-1}$. Thus,

$$\begin{aligned}\mathbf{v}\big(\mathsf{R}^{m+1}(\varphi)\big) &= \mathbf{v}\big(\mathsf{R}^m\left(\varphi'\right)\big) \\ &= \mathbf{v}\big(\varphi'\big) - \frac{m}{n-1} \\ &= \mathbf{v}(\varphi) - \frac{1}{n-1} - \frac{m}{n-1} \\ &= \mathbf{v}(\varphi) - \frac{m+1}{n-1},\end{aligned}$$

which concludes the proof. ■

OBSERVATION 2.1.1 Let $\mathbf{v}$ be a valuation of $\mathbf{P}_n$ and $\varphi \in \mathbf{P}_n$ be a formula. If $\mathbf{v}(\varphi) = \frac{m}{n-1}$, where m, $1 \leq m \leq n-1$, is an integer, then

$$\mathbf{v}(\mathsf{R}^m(\varphi)) = \mathbf{v}(\varphi) - \frac{m}{n-1} = 0.$$

OBSERVATION 2.1.2 For any valuation $\mathbf{v}$ of $\mathbf{P}_n$ and formula $\varphi \in \mathbf{P}_n$, if $\mathbf{v}(\varphi) = 1$, then, for any integer m, $1 \leq m \leq n-1$,

$$\mathbf{v}(\mathsf{R}^m(\varphi)) = 1 - \frac{m}{n-1}.$$

OBSERVATION 2.1.3 Let $\varphi \in \mathbf{P}_n$ be a formula. For any valuation $\mathbf{v}$ of $\mathbf{P}_n$, if $\mathbf{v}(\varphi) = 1$, then

$$\mathbf{v}\big(\mathsf{R}^{m+1}(\varphi)\big) < \mathbf{v}(\mathsf{R}^m(\varphi))\,,$$

where m, $1 \leq m \leq n-2$, is an integer.

OBSERVATION 2.1.4 Let $\mathbf{v}$ be a valuation of $\mathbf{P}_n$ and $\varphi \in \mathbf{P}_n$ be a formula. If $\mathbf{v}(\varphi) \neq 1$, then $\mathbf{v}(\varphi) = \frac{m-1}{n-1}$ for some integer m such that $1 \leq m \leq n-1$, which implies that

$$\mathbf{v}(\mathsf{R}^m(\varphi)) = \mathbf{v}(\varphi) + \frac{n-m}{n-1} = \frac{m-1}{n-1} + \frac{n-m}{n-1} = 1.$$

OBSERVATION 2.1.5 Let $\mathbf{v}$ be a valuation of $\mathbf{P}_n$ and $\varphi \in \mathbf{P}_n$ be a formula. If $\mathbf{v}(\varphi) \neq \frac{m-1}{n-1}$, where m, $1 \leq m \leq n-1$, is an integer, then $\mathbf{v}(\mathsf{R}^m(\varphi)) \neq 1$; for, if it were the case that $\mathbf{v}(\mathsf{R}^m(\varphi)) = 1$, then either $\mathbf{v}(\varphi) + \frac{n-m}{n-1} = 1$ or $\mathbf{v}(\varphi) - \frac{m}{n-1} = 1$. In the first case, $\mathbf{v}(\varphi) = \frac{m-1}{n-1}$, which is a contradiction; in the second case, $\mathbf{v}(\varphi) = 1 + \frac{m}{n-1} > 1$, which cannot happen since $\mathbf{v}(\varphi) \leq 1$.

LEMMA 2.2 Let $\mathbf{v}$ be a valuation of $\mathbf{P}_n$ and $\varphi \in \mathbf{P}_n$ be a formula. Then

$$\mathbf{v}\left(\bigvee_{m=0}^{n-1} \mathsf{R}^m(\varphi)\right) = 1.$$

Proof. Observe that if $\mathbf{v}(\varphi) = 1$, then, since $\mathbf{v}\big(\mathsf{R}^0(\varphi)\big) = \mathbf{v}(\varphi)$, we have that

$$\max\{\mathbf{v}(\mathsf{R}^m(\varphi))\ ;\ m = 0, \ldots, n-1\} = 1.$$

Assume $\mathbf{v}(\varphi) \neq 1$. Then $\mathbf{v}(\varphi) = \frac{i-1}{n-1}$ for some integer i, $1 \leq i \leq n-1$. It follows, by Observation 2.1.4, that $\mathbf{v}\big(\mathsf{R}^i(\varphi)\big) = 1$, implying that

$$\max\{\mathbf{v}(\mathsf{R}^m(\varphi))\ ;\ m = 0, \ldots, n-1\} = 1.$$

The proof is complete since

$$\mathbf{v}\left(\bigvee_{m=0}^{n-1} \mathsf{R}^m(\varphi)\right) = \max\{\mathbf{v}(\mathsf{R}^m(\varphi))\ ;\ m = 0, \ldots, n-1\}.$$

■

PROPOSITION 2.2 Given a valuation $\mathbf{v}$ of $\mathbf{P}_n$ and a formula $\varphi \in \mathbf{P}_n$:

(a) $\mathbf{v}({}^m\varphi) = \frac{n-m}{n-1}$, where m, $1 \leq m \leq n-1$, is an integer;

(b) $\mathbf{v}\big({}^0\varphi\big) = 0$.

Proof. (a) Recall that

$${}^1\varphi = \bigvee_{i=0}^{n-1} \mathsf{R}^i(\varphi)$$

and

$${}^m\varphi = \mathsf{R}^{m-1}\big({}^1\varphi\big),$$

where m, $2 \leq m \leq n-1$, is an integer.

We shall proceed by induction on m. The base case, $m = 1$, is obvious by Lemma 2.2.

Now, suppose the statement holds for some integer $m < n-1$. Then $\mathbf{v}({}^{m}\varphi) \neq 0$. Thus,

$$\begin{aligned}\mathbf{v}\big({}^{m+1}\varphi\big) &= \mathbf{v}\big(\mathsf{R}^{m}\big({}^{1}\varphi\big)\big)\\ &= \mathbf{v}\Big(\big(\mathsf{R}^{m-1}\big({}^{1}\varphi\big)\big)'\Big)\\ &= \mathbf{v}\big(({}^{m}\varphi)'\big)\\ &= \frac{n-m}{n-1} - \frac{1}{n-1}\\ &= \frac{n-(m+1)}{n-1}.\end{aligned}$$

(b) Recall that ${}^{0}\varphi = \big({}^{n-1}\varphi\big)'$. By part (a),

$$\mathbf{v}\big({}^{n-1}\varphi\big) = \frac{n-(n-1)}{n-1} = \frac{1}{n-1} \neq 0.$$

It follows that

$$\mathbf{v}\big({}^{0}\varphi\big) = \mathbf{v}\big({}^{n-1}\varphi\big) - \frac{1}{n-1} = 0.$$

■

PROPOSITION 2.3 For any formula $\varphi \in \mathbf{P}_n$ and any valuation $\mathbf{v}$ of $\mathbf{P}_n$, we have that

$$\mathbf{v}\left(\bigvee_{m=1}^{n-1} \mathsf{R}^{m}(\varphi)\right) = \begin{cases} \mathbf{v}\big({}^{2}\varphi\big) & \text{if } \mathbf{v}(\varphi) = 1,\\ 1 & \text{otherwise.}\end{cases}$$

Proof. Recall that

$$\mathbf{v}\left(\bigvee_{m=1}^{n-1} \mathsf{R}^{m}(\varphi)\right) = \max\left\{\mathbf{v}(\mathsf{R}^{m}(\varphi))\ ;\ m = 1, \ldots, n-1\right\}.$$

If $\mathbf{v}(\varphi) = 1$, then, by Observation 2.1.2,

$$\mathbf{v}\big(\mathsf{R}^{1}(\varphi)\big) = 1 - \frac{1}{n-1} = \frac{n-2}{n-1}.$$

Thus, by Observation 2.1.3 and Proposition 2.2(a),

$$\max\left\{\mathbf{v}(\mathsf{R}^{m}(\varphi))\ ;\ m = 1, \ldots, n-1\right\} = \mathbf{v}\big(\mathsf{R}^{1}(\varphi)\big) = \mathbf{v}\big({}^{2}\varphi\big).$$

Assume $\mathbf{v}(\varphi) \neq 1$. Then $\mathbf{v}(\varphi) = \frac{i-1}{n-1}$ for some integer i, $1 \leq i \leq n-1$, which, by Observation 2.1.4, implies that $\mathbf{v}\big(\mathsf{R}^{i}(\varphi)\big) = 1$ and we conclude that

$$\max\left\{\mathbf{v}(\mathsf{R}^{m}(\varphi))\ ;\ m = 1, \ldots, n-1\right\} = \mathbf{v}\big(\mathsf{R}^{i}(\varphi)\big) = 1.$$

■

PROPOSITION 2.4 For any formula $\varphi \in \mathbf{P}_n$ and any valuation $\mathbf{v}$ of $\mathbf{P}_n$,

$$\mathbf{v}(\varphi^1) = \begin{cases} 1 & \text{if } \mathbf{v}(\varphi) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Recall that

$$\varphi^1 = \mathsf{R}^{n-1}\left(\bigvee_{m=1}^{n-1} \mathsf{R}^m(\varphi)\right).$$

If $\mathbf{v}(\varphi) = 1$, then, by Proposition 2.3,

$$\mathbf{v}\left(\bigvee_{m=1}^{n-1} \mathsf{R}^m(\varphi)\right) = \frac{n-2}{n-1},$$

from which it follows, by Observation 2.1.4, that

$$\mathbf{v}(\varphi^1) = \mathbf{v}\left(\mathsf{R}^{n-1}\left(\bigvee_{m=1}^{n-1} \mathsf{R}^m(\varphi)\right)\right) = 1.$$

If $\mathbf{v}(\varphi) \neq 1$, then Proposition 2.3 tells us that

$$\mathbf{v}\left(\bigvee_{m=1}^{n-1} \mathsf{R}^m(\varphi)\right) = 1.$$

Hence, by Observation 2.1.1,

$$\mathbf{v}(\varphi^1) = \mathbf{v}\left(\mathsf{R}^{n-1}\left(\bigvee_{m=1}^{n-1} \mathsf{R}^m(\varphi)\right)\right) = 0.$$

■

PROPOSITION 2.5 Let $\varphi \in \mathbf{P}_n$ be a formula and $\mathbf{v}$ be a valuation of $\mathbf{P}_n$. Then,

$$\mathbf{v}(\varphi^m) = \begin{cases} \mathbf{v}({}^m\varphi) & \text{if } \mathbf{v}(\varphi) = 1, \\ 0 & \text{otherwise,} \end{cases}$$

where m, $1 \leq m \leq n-1$, is an integer.

Proof. That the statement holds when $m = 1$ follows immediately from Propositions 2.4 and 2.2(a).

Recall that for an integer m, $2 \leq m \leq n-1$,

$$\varphi^m = \mathsf{R}^{n-1}\big(\mathsf{R}^{n-1}\big(\varphi \vee {}^2\varphi\big) \vee \mathsf{R}^m(\varphi)\big).$$

To continue the demonstration for the case when $2 \leq m \leq n-1$, let us simplify the problem. Observe that

$$\mathbf{v}\big(\varphi \vee {}^2\varphi\big) = \max\left\{\mathbf{v}(\varphi)\,, \mathbf{v}\big({}^2\varphi\big)\right\} = \begin{cases} \mathbf{v}(\varphi) & \text{if } \mathbf{v}(\varphi) = 1, \\ \mathbf{v}\big({}^2\varphi\big) & \text{otherwise.} \end{cases}$$

If $\mathbf{v}(\varphi) = 1$, then $\mathbf{v}\big(\varphi \vee {}^2\varphi\big) = 1$. Thus, by Observation 2.1.1,

$$\mathbf{v}\big(\mathsf{R}^{n-1}\big(\varphi \vee {}^2\varphi\big)\big) = 0.$$

If $\mathbf{v}(\varphi) \neq 1$, then

$$\mathbf{v}\big(\varphi \vee {}^2\varphi\big) = \frac{n-2}{n-1},$$

which implies, by Observation 2.1.4, that

$$\mathbf{v}\big(\mathsf{R}^{n-1}\big(\varphi \vee {}^2\varphi\big)\big) = 1.$$

We conclude that

$$\mathbf{v}(\psi) = \begin{cases} 0 & \text{if } \mathbf{v}(\varphi) = 1, \\ 1 & \text{otherwise,} \end{cases}$$

where $\psi = \mathsf{R}^{n-1}\big(\varphi \vee {}^2\varphi\big)$.

Next, note that when $2 \leq m \leq n-1$, if $\mathbf{v}(\varphi) = 1$, then Observation 2.1.2 tells us that

$$\mathbf{v}(\mathsf{R}^m(\varphi)) = 1 - \frac{m}{n-1},$$

from which it follows that

$$\mathbf{v}(\psi \vee \mathsf{R}^m(\varphi)) = \max\left\{0, \mathbf{v}(\mathsf{R}^m(\varphi))\right\} = 1 - \frac{m}{n-1}.$$

We also have that if $\mathbf{v}(\varphi) \neq 1$, then

$$\mathbf{v}(\psi \vee \mathsf{R}^m(\varphi)) = \max\left\{1, \mathbf{v}(\mathsf{R}^m(\varphi))\right\} = 1.$$

Hence, we can write

$$\mathbf{v}(\psi \vee \mathsf{R}^m(\varphi)) = \begin{cases} 1 - \frac{m}{n-1} & \text{if } \mathbf{v}(\varphi) = 1, \\ 1 & \text{otherwise.} \end{cases}$$

If $\mathbf{v}(\varphi) = 1$, then

$$\mathbf{v}(\psi \vee \mathsf{R}^m(\varphi)) = \frac{n-m-1}{n-1},$$

implying, by Observation 2.1.4, that

$$\mathbf{v}\big(\mathsf{R}^{n-m}(\psi \vee \mathsf{R}^m(\varphi))\big) = 1.$$

It follows, by Corollary 2.1.1, Observation 2.1.2, and Proposition 2.2(a), that

$$\begin{aligned}\mathbf{v}\big(\mathsf{R}^{n-1}(\psi \vee \mathsf{R}^m(\varphi))\big) &= \mathbf{v}\big(\mathsf{R}^{m-1}\big(\mathsf{R}^{n-m}(\psi \vee \mathsf{R}^m(\varphi))\big)\big)\\ &= 1 - \frac{m-1}{n-1}\\ &= \frac{n-m}{n-1}\\ &= \mathbf{v}({}^m\varphi)\,.\end{aligned}$$

If $\mathbf{v}(\varphi) \neq 1$, then $\mathbf{v}(\psi \vee \mathsf{R}^m(\varphi)) = 1$. We have, by Observation 2.1.1, that

$$\mathbf{v}\big(\mathsf{R}^{n-1}(\psi \vee \mathsf{R}^m(\varphi))\big) = 0.$$

The statement follows since $\varphi^m = \mathsf{R}^{n-1}(\psi \vee \mathsf{R}^m(\varphi))$. ■

PROPOSITION 2.6 For any formula $\varphi \in \mathbf{P}_n$ and any valuation $\mathbf{v}$ of $\mathbf{P}_n$,

$$\mathbf{v}(\varphi_m^m) = \begin{cases}\mathbf{v}({}^m\varphi) & \text{if } \mathbf{v}(\varphi) = \frac{m-1}{n-1},\\ 0 & \text{otherwise,}\end{cases}$$

where m, $1 \leq m \leq n-1$, is an integer.

Proof. Recall that $\varphi_m^m = (\mathsf{R}^m(\varphi))^m$.

Assume $\mathbf{v}(\varphi) = \frac{m-1}{n-1}$. Then Observation 2.1.4 tells us that $\mathbf{v}(\mathsf{R}^m(\varphi)) = 1$. Hence, by Proposition 2.5,

$$\mathbf{v}((\mathsf{R}^m(\varphi))^m) = \mathbf{v}({}^m\varphi)\,.$$

Suppose $\mathbf{v}(\varphi) \neq \frac{m-1}{n-1}$, then Observation 2.1.5 tells us that $\mathbf{v}(\mathsf{R}^m(\varphi)) \neq 1$. Therefore, by Proposition 2.5,

$$\mathbf{v}((\mathsf{R}^m(\varphi))^m) = 0.$$

■

PROPOSITION 2.7 Let $\varphi \in \mathbf{P}_n$ be a formula and $\mathbf{v}$ be a valuation of $\mathbf{P}_n$. Then

$$\mathbf{v}(\varphi) + \mathbf{v}(\overline{\varphi}) = 1.$$

Proof. Recall that

$$\overline{\varphi} = \bigvee_{m=1}^{n-1} \varphi_m^m.$$

If $\mathbf{v}(\varphi) = 1$, then Proposition 2.6 tells us that $\mathbf{v}(\varphi_m^m) = 0$ for every integer m, $1 \leq m \leq n-1$. Thus,

$$\mathbf{v}(\overline{\varphi}) = \max\{\mathbf{v}(\varphi_m^m)\ ;\ m = 1, \ldots, n-1\} = 0 = 1 - \mathbf{v}(\varphi).$$

Let $\mathbf{v}(\varphi) = \frac{i-1}{n-1}$, where i, $1 \leq i \leq n-1$, is an integer. Then, by Propositions 2.6 and 2.2(a),

$$\begin{aligned}
\mathbf{v}(\overline{\varphi}) &= \max\{\mathbf{v}(\varphi_m^m)\ ;\ m = 1, \ldots, n-1\} \\
&= \mathbf{v}\left({}^i\varphi\right) \\
&= \frac{n-i}{n-1} \\
&= 1 - \frac{i}{n-1} \\
&= 1 - \mathbf{v}(\varphi).
\end{aligned}$$

■

LEMMA 2.3 Let $a, b \in \Delta_n$. Then

$$\min\{1-a, 1-b\} = 1 - \max\{a, b\}.$$

Proof. Without loss of generality, assume $1 - a = \min\{1-a, 1-b\}$. Then $1 - a \leq 1 - b$, from which it follows that $b \leq a$, implying that $a = \max\{a, b\}$. We conclude that

$$\min\{1-a, 1-b\} = 1 - a = 1 - \max\{a, b\}.$$

■

PROPOSITION 2.8 Let $\varphi, \psi \in \mathbf{P}_n$ be two formulas and $\mathbf{v}$ be a valuation of $\mathbf{P}_n$. Then

$$\mathbf{v}(\varphi \wedge \psi) = \min \{\mathbf{v}(\varphi), \mathbf{v}(\psi)\}.$$

Proof. Recall that $\varphi \wedge \psi = \overline{\overline{\varphi} \vee \overline{\psi}}$. By Proposition 2.7 and Lemma 2.3, we have that

$$\begin{aligned}
\mathbf{v}(\varphi \wedge \psi) &= 1 - \mathbf{v}\big(\overline{\varphi} \vee \overline{\psi}\big) \\
&= 1 - \max\big\{\mathbf{v}(\overline{\varphi}), \mathbf{v}\big(\overline{\psi}\big)\big\} \\
&= \min\big\{1 - \mathbf{v}(\overline{\varphi}), 1 - \mathbf{v}\big(\overline{\psi}\big)\big\} \\
&= \min\{\mathbf{v}(\varphi), \mathbf{v}(\psi)\}.
\end{aligned}$$

■

DEFINITION 6 Let φ and ψ be formulas in $\mathbf{P}_n$ with truth values $\mathbf{v}(\varphi)$ and $\mathbf{v}(\psi)$, respectively.

- The formula $(\overline{\varphi})$ is the *negation of* φ and, in accordance with Proposition 2.7, its truth value is given by $\mathbf{v}(\overline{\varphi}) = 1 - \mathbf{v}(\varphi)$.
- The formula $(\varphi \wedge \psi)$ is the *conjunction of* φ *and* ψ; according to Proposition 2.8, its truth value is given by $\mathbf{v}(\varphi \wedge \psi) = \min\{\mathbf{v}(\varphi), \mathbf{v}(\psi)\}$.

2.2.2 Lattices and Post Algebras

In 1959, George Epstein's article [2] reduced the number of axioms required in Definition 4 and, by using a larger number of operations, gave an alternative definition of a Post algebra. Epstein's definition relied on concepts from lattice theory, which we review presently.

DEFINITION 7 A triple $\langle \Xi, \curlyvee, \curlywedge \rangle$, where Ξ is a set, and $\curlyvee$ and $\curlywedge$ are binary operations on Ξ, is a *lattice* if $\curlyvee$ and $\curlywedge$ are commutative, associative, and mutually absorptive, that is, if for every $\theta, \varphi, \psi \in \Xi$:

- $\varphi \curlyvee \psi = \psi \curlyvee \varphi$ and $\varphi \curlywedge \psi = \psi \curlywedge \varphi$.
- $(\theta \curlyvee \varphi) \curlyvee \psi = \theta \curlyvee (\varphi \curlyvee \psi)$ and $(\theta \curlywedge \varphi) \curlywedge \psi = \theta \curlywedge (\varphi \curlywedge \psi)$.
- $\varphi \curlyvee (\varphi \curlywedge \psi) = \varphi$ and $\varphi \curlywedge (\varphi \curlyvee \psi) = \varphi$.

PROPOSITION 2.9 Let $\langle \Xi, \curlyvee, \curlywedge \rangle$ be a lattice. Then, for every $\varphi, \psi \in \Xi$, $\varphi \curlyvee \psi = \psi$ if and only if $\varphi \curlywedge \psi = \varphi$.

Proof. If $\varphi \curlyvee \psi = \psi$, then

$$\varphi \curlywedge \psi = \varphi \curlywedge (\varphi \curlyvee \psi) = \varphi.$$

Conversely, suppose $\varphi \curlywedge \psi = \varphi$. Then

$$\varphi \curlyvee \psi = (\varphi \curlywedge \psi) \curlyvee \psi = \psi \curlyvee (\varphi \curlywedge \psi) = \psi \curlyvee (\psi \curlywedge \varphi) = \psi.$$

■

DEFINITION 8 For a lattice $\langle \Xi, \curlyvee, \curlywedge \rangle$, let $\preccurlyeq$ be the binary relation on Ξ, where $\varphi \preccurlyeq \psi$ if $\varphi \curlyvee \psi = \psi$ or $\varphi \curlywedge \psi = \varphi$.

DEFINITION 9 A binary relation $\sqsubseteq$ on a set Ξ is a *partial order* if $\sqsubseteq$ is reflexive, antisymmetric, and transitive, that is, if for every $\theta, \varphi, \psi \in \Xi$:

- $\theta \sqsubseteq \theta$.
- If $\varphi \sqsubseteq \psi$ and $\psi \sqsubseteq \varphi$, then $\varphi = \psi$.
- If $\theta \sqsubseteq \varphi$ and $\varphi \sqsubseteq \psi$, then $\theta \sqsubseteq \psi$.

DEFINITION 10 Given a set Ξ and a partial order $\sqsubseteq$ on Ξ, the couple $\langle \Xi, \sqsubseteq \rangle$ is a *partially ordered set*.

PROPOSITION 2.10 Let $\langle \Xi, \curlyvee, \curlywedge \rangle$ be a lattice. The relation $\preccurlyeq$ in Definition 8 is a partial order on Ξ.

Proof. Let $\theta, \varphi, \psi \in \Xi$. Then $\theta \curlyvee (\theta \curlywedge \varphi) = \theta$, from which it follows that

$$\theta \curlywedge \theta = \theta \curlywedge (\theta \curlyvee (\theta \curlywedge \varphi)) = \theta$$

since $\curlyvee$ and $\curlywedge$ are mutually absorptive, implying that $\theta \preccurlyeq \theta$. Thus, $\preccurlyeq$ is reflexive.

Suppose $\varphi \preccurlyeq \psi$ and $\psi \preccurlyeq \varphi$. Then

$$\varphi = \varphi \curlywedge \psi = \psi \curlywedge \varphi = \psi.$$

Hence, $\preccurlyeq$ is antisymmetric.

Assume $\theta \preccurlyeq \varphi$ and $\varphi \preccurlyeq \psi$. Then $\theta \curlywedge \varphi = \theta$ and $\varphi \curlywedge \psi = \varphi$. Thus,

$$\theta \curlywedge \psi = (\theta \curlywedge \varphi) \curlywedge \psi = \theta \curlywedge (\varphi \curlywedge \psi) = \theta \curlywedge \varphi = \theta.$$

It follows that $\theta \preccurlyeq \psi$. We conclude that $\preccurlyeq$ is transitive. ■

DEFINITION 11 A partially ordered set $\langle \Xi, \sqsubseteq \rangle$ is a *chain* if for every $\varphi, \psi \in \Xi$, $\varphi \sqsubseteq \psi$ or $\psi \sqsubseteq \varphi$.

DEFINITION 12 Let $\langle \Xi, \sqsubseteq \rangle$ be a partially ordered set and $\Gamma \subseteq \Xi$. Then $\varphi \in \Gamma$ is a *least element* (or *minimum*) *of* Γ if $\varphi \sqsubseteq \psi$ for every $\psi \in \Gamma$.

In a similar fashion, $\varphi \in \Gamma$ is a *greatest element* (or *maximum*) *of* Γ if $\psi \sqsubseteq \varphi$ for every $\psi \in \Gamma$.

If they exist, a least and a greatest element of Ξ shall be called a *zero* and a *unit of* Ξ, respectively.

Observe that given a partially ordered set $\langle \Xi, \sqsubseteq \rangle$, if least and greatest elements of a subset Γ of Ξ exist, then they are unique since a partial order is antisymmetric. Hence, provided they exist, we shall denote the zero and unit elements of Ξ by o and ι, respectively. A partially ordered set $\langle \Xi, \sqsubseteq \rangle$ with zero o and unit ι shall be denoted by $\langle \Xi, \sqsubseteq, o, \iota \rangle$.

DEFINITION 13 Let $\langle \Xi, \sqsubseteq \rangle$ be a partially ordered set and $\Gamma \subseteq \Xi$. An element $\varphi \in \Xi$ is a *lower bound of* Γ if $\varphi \sqsubseteq \psi$ for every $\psi \in \Gamma$. If φ is a lower bound of Γ such that $\theta \sqsubseteq \varphi$ for every lower bound θ of Γ, then φ is a *greatest lower bound* (or *infimum*) *of* Γ.

Similarly, $\varphi \in \Xi$ is an *upper bound of* Γ if $\psi \sqsubseteq \varphi$ for every $\psi \in \Gamma$; φ is a *least upper bound* (or *supremum*) *of* Γ if φ is an upper bound of Γ such that $\varphi \sqsubseteq \theta$ for every upper bound θ of Γ.

Note that for a subset Γ of Ξ, where $\langle \Xi, \sqsubseteq \rangle$ is a partially ordered set, provided they exist, the infimum and supremum of Γ are unique since they are greatest and least elements, respectively, of subsets of Ξ. The infimum and supremum of Γ, if they exist, shall be denoted by $\bigsqcap \Gamma$ and $\bigsqcup \Gamma$, respectively. If $\Gamma = \{\varphi, \psi\}$, where $\varphi, \psi \in \Xi$, we shall denote its infimum and supremum by $\varphi \sqcap \psi$ and $\varphi \sqcup \psi$, respectively.

PROPOSITION 2.11 Let $\langle \Xi, \curlyvee, \curlywedge \rangle$ be a lattice. Then, in the partially ordered set $\langle \Xi, \preccurlyeq \rangle$, for any $\varphi, \psi \in \Xi$:

(a) $\varphi \sqcap \psi = \varphi \curlywedge \psi$ and

(b) $\varphi \sqcup \psi = \varphi \curlyvee \psi$.

Proof. (a) Note that

$$(\varphi \curlywedge \psi) \curlyvee \varphi = \varphi \curlyvee (\varphi \curlywedge \psi) = \varphi,$$

which implies that $\varphi \curlywedge \psi \preccurlyeq \varphi$. Moreover, since $\psi \curlywedge \varphi \preccurlyeq \psi$ and $\psi \curlywedge \varphi = \varphi \curlywedge \psi$, we have that $\varphi \curlywedge \psi \preccurlyeq \psi$. Hence, $\varphi \curlywedge \psi$ is a lower bound of $\{\varphi, \psi\}$.

Let $\theta \in \Xi$ be such that $\theta \preccurlyeq \varphi$ and $\theta \preccurlyeq \psi$. Then

$$\theta = \theta \curlywedge \varphi = (\theta \curlywedge \psi) \curlywedge \varphi = \theta \curlywedge (\psi \curlywedge \varphi) = \theta \curlywedge (\varphi \curlywedge \psi).$$

Thus, $\theta \preccurlyeq \varphi \curlywedge \psi$, from which it follows $\varphi \curlywedge \psi$ is the infimum of $\{\varphi, \psi\}$.

(b) A similar method to the one used in part (a) is used to show that $\varphi \curlyvee \psi$ is the supremum of $\{\varphi, \psi\}$. ■

From Propositions 2.10 and 2.11, we conclude that, given a lattice $\langle \Xi, \curlyvee, \curlywedge \rangle$, the partially ordered set $\langle \Xi, \preccurlyeq \rangle$ is such that the infimum and the supremum of every subset $\{\varphi, \psi\}$ of Ξ exist, and are given by $\varphi \curlywedge \psi$ and $\varphi \curlyvee \psi$, respectively. It has been shown in [1] that if we begin with a partially ordered set for which every pair of elements has a greatest lower bound and a least upper bound, then two commutative, associative, and mutually absorptive binary operations on the underlying set are obtained; that is, a partially ordered set with the property that every pair of elements has an infimum and a supremum gives rise to a lattice. Thus, there is a correspondence between lattices and partially ordered sets in which every pair of elements has an infimum and a supremum. This correspondence shall be taken for granted in the rest of this section and there should be no confusion when we talk about a lattice or its corresponding partially ordered set. The lattice corresponding to the partially ordered set $\langle \Xi, \preccurlyeq, \mathrm{o}, \iota \rangle$ with zero o and unit ι shall be denoted by $\langle \Xi, \curlyvee, \curlywedge, \mathrm{o}, \iota \rangle$. Before we give Epstein's definition of a Post algebra, let us introduce another term which shall be encountered.

DEFINITION 14 A lattice $\langle \Xi, \curlyvee, \curlywedge \rangle$ is said to be *distributive* if for every $\theta, \varphi, \psi \in \Xi$,

$$\theta \curlywedge (\varphi \curlyvee \psi) = (\theta \curlywedge \varphi) \curlyvee (\theta \curlywedge \psi) .$$

Now, we are ready for the definition of a Post algebra given by Epstein, which he showed to be equivalent to Definition 4.

DEFINITION 15 For an integer $n \geq 2$, a distributive lattice $\langle \Xi, \curlyvee, \curlywedge, \mathrm{o}, \iota \rangle$ is a *Post algebra* (*of type* n) if:

1. Ξ contains n fixed elements, denoted by $\mathrm{o} = \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{n-2}, \varepsilon_{n-1} = \iota$, such that:

   a. The set $\{\varepsilon_i \, ; \; i = 0, \ldots, n-1\}$ is a chain with $\varepsilon_{i-1} \preccurlyeq \varepsilon_i$.

   b. If $\theta \in \Xi$ and $\theta \curlywedge \varepsilon_1 = \mathrm{o}$, then $\theta = \mathrm{o}$.

   c. If $\theta \in \Xi$ and $\theta \curlyvee \varepsilon_{i-1} = \varepsilon_i$ for some i, then $\theta = \varepsilon_i$.

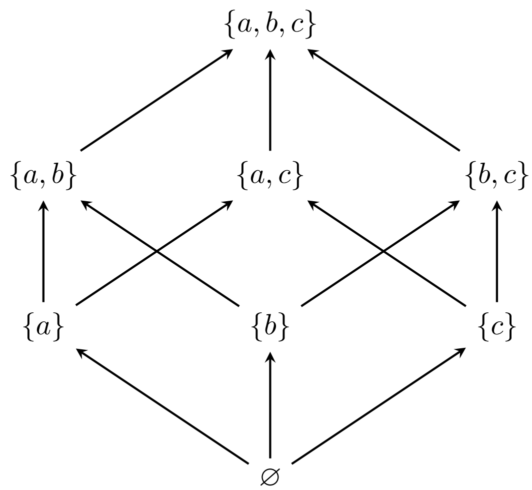
2. For every $\theta \in \Xi$, there exist $\mathsf{C}_0^\theta, \mathsf{C}_1^\theta, \ldots, \mathsf{C}_{n-1}^\theta \in \Xi$ such that:

   a. $\mathsf{C}_i^\theta \curlywedge \mathsf{C}_j^\theta = \mathrm{o}$ for $i \neq j$.

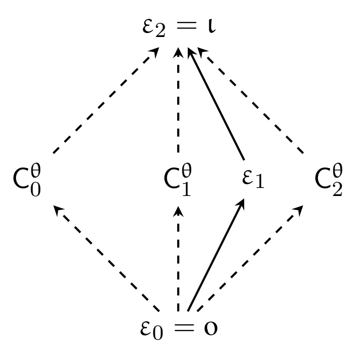
   b. $\bigsqcup \left\{ \mathsf{C}_i^\theta \, ; \; i = 0, \ldots, n-1 \right\} = \iota$.

   c. $\theta = \left(\varepsilon_0 \curlywedge \mathsf{C}_0^\theta\right) \curlyvee \left(\varepsilon_1 \curlywedge \mathsf{C}_1^\theta\right) \curlyvee \cdots \curlyvee \left(\varepsilon_{n-2} \curlywedge \mathsf{C}_{n-2}^\theta\right) \curlyvee \left(\varepsilon_{n-1} \curlywedge \mathsf{C}_{n-1}^\theta\right)$.

A lattice and its corresponding partially ordered set can be depicted using a *Hasse diagram*. The elements of the lattice are displayed with arrows between them according to the partial order induced by the lattice operations; the number of arrows should be minimal and exhibit the relevant structure unambiguously. For example, it is a well–known fact that for a finite set S, the triple $\langle 2^S, \cup, \cap\rangle$, where 2^S is the power set of S, is a lattice and the partially ordered set corresponding to this lattice is given by $\langle 2^S, \subseteq\rangle$; with $S = \{a, b, c\}$, the Hasse diagram of the aforementioned lattice and partially ordered set is given below:



Let us try to better understand the definition of a Post algebra through a Hasse diagram. As may be expected, Hasse diagrams for Post algebras are very complicated so we shall give a very rudimentary example. To that end, let $\langle \Xi, \curlyvee, \curlywedge, \mathrm{o}, \iota\rangle$ be a Post algebra of type 3. Then, for an arbitrary $\theta \in \Xi$, a «slice» of the Hasse diagram is given by:



Note that, in the Hasse diagram above, the requirements that C_0^θ, C_1^θ, and C_2^θ be mutually disjoint and $\bigsqcup\left\{\mathsf{C}_0^\theta, \mathsf{C}_1^\theta, \mathsf{C}_2^\theta\right\} = \iota$ are satisfied. The chain $\{\varepsilon_0, \varepsilon_1, \varepsilon_2\}$ consisting of fixed elements of Ξ is connected by solid arrows while the set of elements of Ξ that depend on θ by dashed arrows.

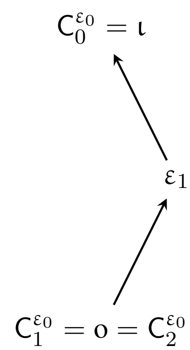
Now, for $\theta = \varepsilon_0$, we must have that

$$\varepsilon_0 = (\varepsilon_0 \curlywedge \mathsf{C}_0^{\varepsilon_0}) \curlyvee (\varepsilon_1 \curlywedge \mathsf{C}_1^{\varepsilon_0}) \curlyvee (\varepsilon_2 \curlywedge \mathsf{C}_2^{\varepsilon_2}),$$

which implies that

$$\begin{aligned} \mathrm{o} &= \mathrm{o} \curlyvee (\varepsilon_1 \curlywedge \mathsf{C}_1^{\varepsilon_0}) \curlyvee \mathsf{C}_2^{\varepsilon_0} \\ &= (\varepsilon_1 \curlywedge \mathsf{C}_1^{\varepsilon_0}) \curlyvee \mathsf{C}_2^{\varepsilon_0}. \end{aligned}$$

Hence, $\varepsilon_1 \curlywedge \mathsf{C}_1^{\varepsilon_0} = \mathrm{o}$ and $\mathsf{C}_2^{\varepsilon_0} = \mathrm{o}$. Since $\varepsilon_1 \curlywedge \mathsf{C}_1^{\varepsilon_0} = \mathsf{C}_1^{\varepsilon_0} \curlywedge \varepsilon_1$, Axiom 1b of Definition 15 tells us that $\mathsf{C}_1^{\varepsilon_0} = \mathrm{o}$. Finally, since $\bigsqcup \{\mathrm{o}, \mathsf{C}_0^{\varepsilon_0}\} = \iota$, we must have that $\mathsf{C}_0^{\varepsilon_0} = \iota$. Thus, for ε_0, the «slice» of the Post algebra $\langle \Xi, \curlyvee, \curlywedge, \mathrm{o}, \iota \rangle$ of type 3 can be represented in the Hasse diagram below:



2.2.3 Post Algebras and the Propositional Systems of Post

In this section, we shall connect Post algebras with the systems of propositional logic given by Post. Recall that, in Definition 3, $\varphi \Bumpeq \psi$ if $\mathbf{v}(\varphi) = \mathbf{v}(\psi)$ for every valuation $\mathbf{v}$ of $\mathbf{P}_n$, where $\varphi, \psi \in \mathbf{P}_n$ are formulas, in which case φ and ψ are said to be equivalent.

To achieve the intended goal, for any formula $\varphi \in \mathbf{P}_n$, the set of formulas equivalent to φ shall be denoted by $\boldsymbol{\varphi}$; that is,

$$\boldsymbol{\varphi} := \{\psi\, ;\ \psi \in \mathbf{P}_n,\ \varphi \Bumpeq \psi\}\,.$$

The set of equivalence classes of $\mathbf{P}_n$ by $\Bumpeq$ shall be denoted by $\mathcal{P}_n$; hence,

$$\mathcal{P}_n := \mathbf{P}_n / \Bumpeq\,.$$

DEFINITION 16 For two equivalence classes $\boldsymbol{\varphi}, \boldsymbol{\psi} \in \mathcal{P}_n$, let $\Vee$ and $\Wedge$ be binary operations on $\mathcal{P}_n$, where

- $\boldsymbol{\varphi} \Vee \boldsymbol{\psi} := \{\theta\, ;\ \theta \in \mathbf{P}_n,\ \theta \Bumpeq \varphi \vee \psi,\ \varphi \in \boldsymbol{\varphi},\ \psi \in \boldsymbol{\psi}\}$ and
- $\boldsymbol{\varphi} \Wedge \boldsymbol{\psi} := \{\theta\, ;\ \theta \in \mathbf{P}_n,\ \theta \Bumpeq \varphi \wedge \psi,\ \varphi \in \boldsymbol{\varphi},\ \psi \in \boldsymbol{\psi}\}$.

PROPOSITION 2.12 The operations on $\mathcal{P}_n$ in Definition 16 are well–defined.

Proof. Let $\boldsymbol{\varphi}_i, \boldsymbol{\psi}_i \in \mathcal{P}_n$, $i = 1, 2$, be equivalence classes. Suppose $\boldsymbol{\varphi}_1 = \boldsymbol{\varphi}_2$ and $\boldsymbol{\psi}_1 = \boldsymbol{\psi}_2$, and let $\varphi_i \in \boldsymbol{\varphi}_i$ and $\psi_i \in \boldsymbol{\psi}_i$, $i = 1, 2$, be formulas. So that for any valuation $\mathbf{v}$ of $\mathbf{P}_n$, $\mathbf{v}(\varphi_1) = \mathbf{v}(\varphi_2)$ and $\mathbf{v}(\psi_1) = \mathbf{v}(\psi_2)$.

Then $\theta \in \boldsymbol{\varphi}_1 \curlyvee \boldsymbol{\psi}_1$ if and only if $\theta \backsimeq \varphi_1 \vee \psi_1$, that is, if and only if

$$\mathbf{v}(\theta) = \mathbf{v}(\varphi_1 \vee \psi_1) = \max\{\mathbf{v}(\varphi_1), \mathbf{v}(\psi_1)\} = \max\{\mathbf{v}(\varphi_2), \mathbf{v}(\psi_2)\} = \mathbf{v}(\varphi_2 \vee \psi_2).$$

Thus, $\theta \backsimeq \varphi_2 \vee \psi_2$, which means that $\theta \in \boldsymbol{\varphi}_2 \curlyvee \boldsymbol{\psi}_2$. We conclude that $\boldsymbol{\varphi}_1 \curlyvee \boldsymbol{\psi}_1 = \boldsymbol{\varphi}_2 \curlyvee \boldsymbol{\psi}_2$.

That $\boldsymbol{\varphi}_1 \curlywedge \boldsymbol{\psi}_1 = \boldsymbol{\varphi}_2 \curlywedge \boldsymbol{\psi}_2$ follows in an analogous way. ■

PROPOSITION 2.13 The triple $\langle \mathcal{P}_n, \curlyvee, \curlywedge \rangle$ is a lattice.

Proof. Let $\boldsymbol{\theta}, \boldsymbol{\varphi}, \boldsymbol{\psi} \in \mathcal{P}_n$ be equivalence classes. That $\curlyvee$ and $\curlywedge$ are commutative and associative operations on $\mathcal{P}_n$ follows since, for $\circ \in \{\vee, \wedge\}$, $\varphi \circ \psi \backsimeq \psi \circ \varphi$ and $(\theta \circ \varphi) \circ \psi \backsimeq \theta \circ (\varphi \circ \psi)$, where $\theta \in \boldsymbol{\theta}$, $\varphi \in \boldsymbol{\varphi}$, and $\psi \in \boldsymbol{\psi}$ are formulas.

For mutual absorptivity, given a valuation $\mathbf{v}$ of $\mathbf{P}_n$, let $\varphi \in \boldsymbol{\varphi}$ and $\psi \in \boldsymbol{\psi}$ be formulas and suppose $\mathbf{v}(\varphi) \geq \mathbf{v}(\psi)$. Then

$$\mathbf{v}(\varphi \wedge \psi) = \min\{\mathbf{v}(\varphi), \mathbf{v}(\psi)\} = \mathbf{v}(\psi),$$

which implies that

$$\mathbf{v}(\varphi \vee (\varphi \wedge \psi)) = \max\{\mathbf{v}(\varphi), \mathbf{v}(\varphi \wedge \psi)\} = \mathbf{v}(\varphi).$$

Now, assume $\mathbf{v}(\varphi) < \mathbf{v}(\psi)$. It follows that $\mathbf{v}(\varphi \wedge \psi) = \mathbf{v}(\varphi)$. Hence,

$$\mathbf{v}(\varphi \vee (\varphi \wedge \psi)) = \mathbf{v}(\varphi).$$

We conclude that $\varphi \vee (\varphi \wedge \psi) \backsimeq \varphi$. Thus, $\boldsymbol{\varphi} \curlyvee (\boldsymbol{\varphi} \curlywedge \boldsymbol{\psi}) = \boldsymbol{\varphi}$.

The remaining absorption is treated in a similar way. Hence, $\boldsymbol{\varphi} \curlywedge (\boldsymbol{\varphi} \curlyvee \boldsymbol{\psi}) = \boldsymbol{\varphi}$. ■

PROPOSITION 2.14 The lattice $\langle \mathcal{P}_n, \curlyvee, \curlywedge \rangle$ is distributive.

Proof. Let $\boldsymbol{\theta}, \boldsymbol{\varphi}, \boldsymbol{\psi} \in \mathcal{P}_n$ be equivalence classes. Given a valuation $\mathbf{v}$ of $\mathbf{P}_n$, let $\theta \in \boldsymbol{\theta}$, $\varphi \in \boldsymbol{\varphi}$, and $\psi \in \boldsymbol{\psi}$ be formulas.

Suppose $\mathbf{v}(\theta) \leq \mathbf{v}(\varphi)$ and $\mathbf{v}(\theta) \leq \mathbf{v}(\psi)$. Then

$$\mathbf{v}(\theta) \leq \max\{\mathbf{v}(\varphi), \mathbf{v}(\psi)\} = \mathbf{v}(\varphi \vee \psi).$$

Observe that $\mathbf{v}(\theta \wedge \varphi) = \mathbf{v}(\theta)$ and $\mathbf{v}(\theta \wedge \psi) = \mathbf{v}(\theta)$. Thus,

$$\begin{aligned} \mathbf{v}((\theta \wedge \varphi) \vee (\theta \wedge \psi)) &= \max \{\mathbf{v}(\theta \wedge \varphi), \mathbf{v}(\theta \wedge \psi)\} \\ &= \mathbf{v}(\theta) \\ &= \min \{\mathbf{v}(\theta), \mathbf{v}(\varphi \vee \psi)\} \\ &= \mathbf{v}(\theta \wedge (\varphi \vee \psi)). \end{aligned}$$

Now, assume $\mathbf{v}(\theta) \geq \mathbf{v}(\varphi)$ and $\mathbf{v}(\theta) \geq \mathbf{v}(\psi)$. Note that $\mathbf{v}(\theta \wedge \varphi) = \mathbf{v}(\varphi)$ and $\mathbf{v}(\theta \wedge \psi) = \mathbf{v}(\psi)$. Then

$$\mathbf{v}(\theta) \geq \max \{\mathbf{v}(\varphi), \mathbf{v}(\psi)\} = \mathbf{v}(\varphi \vee \psi).$$

It follows that

$$\begin{aligned} \mathbf{v}((\theta \wedge \varphi) \vee (\theta \wedge \psi)) &= \max \{\mathbf{v}(\theta \wedge \varphi), \mathbf{v}(\theta \wedge \psi)\} \\ &= \max \{\mathbf{v}(\varphi), \mathbf{v}(\psi)\} \\ &= \mathbf{v}(\varphi \vee \psi) \\ &= \min \{\mathbf{v}(\theta), \mathbf{v}(\varphi \vee \psi)\} \\ &= \mathbf{v}(\theta \wedge (\varphi \vee \psi)). \end{aligned}$$

Lastly, suppose $\mathbf{v}(\varphi) \leq \mathbf{v}(\theta) \leq \mathbf{v}(\psi)$. Then $\mathbf{v}(\theta \wedge \varphi) = \mathbf{v}(\varphi)$ and $\mathbf{v}(\theta \wedge \psi) = \mathbf{v}(\theta)$. Since $\mathbf{v}(\varphi \vee \psi) = \mathbf{v}(\psi)$, it follows that

$$\begin{aligned} \mathbf{v}((\theta \wedge \varphi) \vee (\theta \wedge \psi)) &= \max \{\mathbf{v}(\varphi), \mathbf{v}(\theta)\} \\ &= \mathbf{v}(\theta) \\ &= \min \{\mathbf{v}(\theta), \mathbf{v}(\psi)\} \\ &= \min \{\mathbf{v}(\theta), \mathbf{v}(\varphi \vee \psi)\} \\ &= \mathbf{v}(\theta \wedge (\varphi \vee \psi)). \end{aligned}$$

The case when $\mathbf{v}(\psi) \leq \mathbf{v}(\theta) \leq \mathbf{v}(\varphi)$ is treated in an analogous way.

Hence, $\theta \wedge (\varphi \vee \psi) \Bumpeq (\theta \wedge \varphi) \vee (\theta \wedge \psi)$. It follows that $\boldsymbol{\theta} \Wedge (\boldsymbol{\varphi} \Vee \boldsymbol{\psi}) = (\boldsymbol{\theta} \Wedge \boldsymbol{\varphi}) \Vee (\boldsymbol{\theta} \Wedge \boldsymbol{\psi})$. ■

The partial order associated with the lattice $\langle \mathcal{P}_n, \Vee, \Wedge \rangle$ shall be denoted by $\lll$. Thus, $\boldsymbol{\varphi} \lll \boldsymbol{\psi}$ if $\boldsymbol{\varphi} \Vee \boldsymbol{\psi} = \boldsymbol{\psi}$ or $\boldsymbol{\varphi} \Wedge \boldsymbol{\psi} = \boldsymbol{\varphi}$, where $\boldsymbol{\varphi}, \boldsymbol{\psi} \in \mathcal{P}_n$.

Recall, from Definition 1, that the sets of contradictions and tautologies of $\mathbf{P}_n$ are denoted by $\mathfrak{c}$ and $\mathfrak{t}$, respectively.

PROPOSITION 2.15 In the partially ordered set $\langle \mathcal{P}_n, \lll \rangle$, the elements $\mathfrak{c} \in \mathcal{P}_n$ and $\mathfrak{t} \in \mathcal{P}_n$ are the zero and unit, respectively, of $\mathcal{P}_n$.

Proof. Let $o \in \mathfrak{c}$ and $\iota \in \mathfrak{t}$. Observe that given a valuation $\mathbf{v}$ of $\mathbf{P}_n$, $\mathbf{v}(o) = 0$ and $\mathbf{v}(\iota) = 1$.

Let $\boldsymbol{\varphi} \in \mathcal{P}_n$ be an equivalence class and $\varphi \in \boldsymbol{\varphi}$ be a formula. Then

$$\mathbf{v}(o \wedge \varphi) = \min\{\mathbf{v}(o), \mathbf{v}(\varphi)\} = \mathbf{v}(o)$$

and

$$\mathbf{v}(\varphi \vee \iota) = \max\{\mathbf{v}(\varphi), \mathbf{v}(\iota)\} = \mathbf{v}(\iota).$$

Thus, $o \wedge \varphi \Leftrightarrow o$ and $\varphi \vee \iota \Leftrightarrow \iota$, implying that $\mathfrak{c} \wedge\!\!\!\wedge \boldsymbol{\varphi} = \mathfrak{c}$ and $\boldsymbol{\varphi} \vee\!\!\!\vee \mathfrak{t} = \mathfrak{t}$. We conclude that, for every equivalence class $\boldsymbol{\varphi} \in \mathcal{P}_n$, $\mathfrak{c} \lll \boldsymbol{\varphi}$ and $\boldsymbol{\varphi} \lll \mathfrak{t}$. ■

LEMMA 2.4 Let $\varphi \in \mathbf{P}_n$ be a formula. Then, for any valuation $\mathbf{v}$ of $\mathbf{P}_n$,

$$\mathbf{v}\big(\varphi^1_{m+1}\big) = \begin{cases} 1 & \text{if } \mathbf{v}(\varphi) = \frac{m}{n-1}, \\ 0 & \text{otherwise,} \end{cases}$$

where m, $0 \leq m \leq n-1$, is an integer.

Proof. From Notations 7 and 9 of Definition 5, observe that

$$\varphi^1_{m+1} = \big(\mathsf{R}^{m+1}(\varphi)\big)^1 = \mathsf{R}^{n-1}\left(\bigvee_{k=1}^{n-1} \mathsf{R}^k\big(\mathsf{R}^{m+1}(\varphi)\big)\right).$$

If $\mathbf{v}(\varphi) = \frac{m}{n-1}$, then, by Lemma 2.1,

$$\mathbf{v}\big(\mathsf{R}^{m+1}(\varphi)\big) = \mathbf{v}(\varphi) + \frac{n-(m+1)}{n-1} = \frac{m}{n-1} + \frac{n-m-1}{n-1} = 1$$

It follows, by Proposition 2.4, that

$$\mathbf{v}\Big(\big(\mathsf{R}^{m+1}(\varphi)\big)^1\Big) = 1.$$

Suppose $\mathbf{v}(\varphi) \neq \frac{m}{n-1}$. Then, for $0 \leq m \leq n-2$, Observation 2.1.5 tells us that $\mathbf{v}\big(\mathsf{R}^{m+1}(\varphi)\big) \neq 1$. Thus, by Proposition 2.4,

$$\mathbf{v}\Big(\big(\mathsf{R}^{m+1}(\varphi)\big)^1\Big) = 0.$$

When $m = n-1$, we have that $\mathbf{v}(\varphi) \neq 1$, which implies that $\mathbf{v}(\varphi) \leq \frac{n-2}{n-1}$. By Lemma 2.1,

$$\mathbf{v}\big(\mathsf{R}^{n-1}(\varphi)\big) = \mathbf{v}(\varphi) + \frac{n-m}{n-1}.$$

Observe that $\mathbf{v}\big(\mathsf{R}^{n-1}(\varphi)\big) \neq 0$; if it were the case that $\mathbf{v}\big(\mathsf{R}^{n-1}(\varphi)\big) = 0$, then

$$\mathbf{v}(\varphi) = \frac{m-n}{n-1} < 0,$$

giving us a contradiction. It follows that

$$\mathbf{v}(\mathsf{R}^{n}(\varphi)) = \mathbf{v}\big(\mathsf{R}^{n-1}(\varphi)\big) - \frac{1}{n-1}.$$

Note that $\mathbf{v}(\mathsf{R}^{n}(\varphi)) \neq 1$ since $\mathbf{v}\big(\mathsf{R}^{n-1}(\varphi)\big) \neq 0$. By Proposition 2.4,

$$\mathbf{v}\Big(\Big(\mathsf{R}^{(n-1)+1}(\varphi)\Big)^{1}\Big) = \mathbf{v}\Big((\mathsf{R}^{n}(\varphi))^{1}\Big) = 0,$$

as was desired. ■

DEFINITION 17 For an integer m, $0 \leq m \leq n-1$, the set of formulas $\varphi \in \mathbf{P}_n$ such that $\mathbf{v}(\varphi) = \frac{m}{n-1}$ for every valuation $\mathbf{v}$ of $\mathbf{P}_n$ shall be denoted by $\mathfrak{f}_m$; that is,

$$\mathfrak{f}_m := \left\{\varphi\,;\ \varphi \in \mathbf{P}_n,\ \mathbf{v}(\varphi) = \frac{m}{n-1} \text{ for every } \mathbf{v}\colon \mathbf{P}_n \to \Delta_n\right\}.$$

LEMMA 2.5 Let $\theta \in \mathbf{P}_n$ and $\varphi_m \in \mathfrak{f}_m$ be formulas, where m, $0 \leq m \leq n-1$, is an integer. For any valuation $\mathbf{v}$ of $\mathbf{P}_n$,

$$\mathbf{v}\big(\varphi_m \wedge \theta^{1}_{m+1}\big) = \begin{cases} \mathbf{v}(\varphi_m) & \text{if } \mathbf{v}(\theta) = \frac{m}{n-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If $\mathbf{v}(\theta) = \frac{m}{n-1}$, then Lemma 2.4 tells us that

$$\mathbf{v}\big(\varphi_m \wedge \theta^{1}_{m+1}\big) = \min\left\{\frac{m}{n-1}, 1\right\} = \frac{m}{n-1} = \mathbf{v}(\varphi_m)\,.$$

Similarly, if $\mathbf{v}(\theta) \neq \frac{m}{n-1}$, by another application of Lemma 2.4,

$$\mathbf{v}\big(\varphi_m \wedge \theta^{1}_{m+1}\big) = \min\left\{\frac{m}{n-1}, 0\right\} = 0.$$

■

THEOREM For an integer $n \geq 2$, the distributive lattice $\langle \mathcal{P}_n, \Cup, \Cap, \mathfrak{c}, \mathfrak{t}\rangle$ is a Post algebra of type n.

Proof. Let $\mathbf{v}$ be a valuation of $\mathbf{P}_n$, $\mathcal{F} = \{\mathfrak{f}_i\,;\ i = 0, \ldots, n-1\}$, and $\varphi_i \in \mathfrak{f}_i$ be formulas. Observe that $\mathfrak{f}_0 = \mathfrak{c}$ and $\mathfrak{f}_{n-1} = \mathfrak{t}$. That $\mathcal{F}$ is a chain with $\mathfrak{f}_{m-1} \lll \mathfrak{f}_m$ follows immediately since, for any integer m, $1 \leq m \leq n-1$,

$$\mathbf{v}(\varphi_{m-1} \vee \varphi_m) = \max\{\mathbf{v}(\varphi_{m-1})\,, \mathbf{v}(\varphi_m)\} = \mathbf{v}(\varphi_m)\,,$$

which implies that $\varphi_{m-1} \vee \varphi_m \approx \varphi_m$, so that $\mathfrak{f}_{m-1} \curlyvee \mathfrak{f}_m = \mathfrak{f}_m$. Hence, $\mathfrak{f}_{m-1} \lll \mathfrak{f}_m$.

Let $\boldsymbol{\theta} \in \mathcal{P}_n$ be an equivalence class and $\theta \in \boldsymbol{\theta}$ be a formula. Suppose $\boldsymbol{\theta} \curlywedge \mathfrak{f}_1 = \mathfrak{c}$. Then

$$\mathbf{v}(\theta \wedge \varphi_1) = \min\{\mathbf{v}(\theta)\,, \mathbf{v}(\varphi_1)\} = \mathbf{v}(\varphi_0)\,.$$

Since $\mathbf{v}(\varphi_1) = \frac{1}{n-1}$ and $\mathbf{v}(\varphi_0) = 0$, it follows that $\mathbf{v}(\theta) = \mathbf{v}(\varphi_0)$, implying that $\theta \approx \varphi_0$. Thus, $\boldsymbol{\theta} = \mathfrak{c}$.

Assume $\boldsymbol{\theta} \curlyvee \mathfrak{f}_{i-1} = \mathfrak{f}_i$ for some i. Then

$$\mathbf{v}(\theta \vee \varphi_{i-1}) = \max\{\mathbf{v}(\theta)\,, \mathbf{v}(\varphi_{i-1})\} = \mathbf{v}(\varphi_i)\,.$$

Note that $\mathbf{v}(\varphi_{i-1}) = \frac{i-1}{n-1}$ and $\mathbf{v}(\varphi_i) = \frac{i}{n-1}$. Hence, $\mathbf{v}(\theta) = \mathbf{v}(\varphi_i)$, which implies that $\theta \approx \varphi_i$. Therefore, $\boldsymbol{\theta} = \mathfrak{f}_i$.

For each $\boldsymbol{\theta} \in \mathcal{P}_n$, let $\mathsf{C}_0^{\boldsymbol{\theta}}, \mathsf{C}_1^{\boldsymbol{\theta}}, \ldots, \mathsf{C}_{n-1}^{\boldsymbol{\theta}} \in \mathcal{P}_n$ be such that

$$\theta_{m+1}^1 \in \mathsf{C}_m^{\boldsymbol{\theta}},$$

where $\theta \in \boldsymbol{\theta}$. Let $\mathfrak{C}^{\boldsymbol{\theta}} = \left\{\mathsf{C}_m^{\boldsymbol{\theta}}\,;\ \boldsymbol{\theta} \in \mathcal{P}_n,\ m = 0, \ldots, n-1\right\}$ and $\mathsf{C}_i^{\boldsymbol{\theta}}, \mathsf{C}_j^{\boldsymbol{\theta}} \in \mathfrak{C}^{\boldsymbol{\theta}}$ with $i \neq j$.

If $\mathbf{v}\big(\theta_{i+1}^1\big) = \mathbf{v}\Big(\theta_{j+1}^1\Big) = 0$, then

$$\mathbf{v}\big(\theta_{i+1}^1 \wedge \theta_{j+1}^1\big) = 0 = \mathbf{v}(\varphi_0)\,.$$

If $\mathbf{v}\big(\theta_{i+1}^1\big) \neq 0$, then, by Lemma 2.4, $\mathbf{v}\big(\theta_{i+1}^1\big) = 1$, which is the case only when $\mathbf{v}(\theta) = \frac{i}{n-1}$. Thus, since $i \neq j$, it follows that $\mathbf{v}\Big(\theta_{j+1}^1\Big) = 0$ by Lemma 2.4. Hence,

$$\mathbf{v}\big(\theta_{i+1}^1 \wedge \theta_{j+1}^1\big) = 0 = \mathbf{v}(\varphi_0)\,;$$

the case when $\mathbf{v}\Big(\theta_{j+1}^1\Big) \neq 0$ is analogous. We conclude that $\theta_{i+1}^1 \wedge \theta_{j+1}^1 \approx \varphi_0$, which implies that

$$\mathsf{C}_i^{\boldsymbol{\theta}} \curlywedge \mathsf{C}_j^{\boldsymbol{\theta}} = \mathfrak{c}$$

for $i \neq j$.

To show that $\bigsqcup \mathfrak{C}^{\boldsymbol{\theta}} = \mathfrak{t}$, first observe that $\mathfrak{t}$ is an upper bound of $\mathfrak{C}^{\boldsymbol{\theta}}$ since $\mathfrak{t}$ is the unit of $\mathcal{P}_n$. We show that $\mathfrak{t}$ is the least upper bound of $\mathfrak{C}^{\boldsymbol{\theta}}$ by contradiction. To that end, suppose there exists an upper bound $\boldsymbol{\psi} \in \mathcal{P}_n$ of $\mathfrak{C}^{\boldsymbol{\theta}}$ such that $\boldsymbol{\psi} \lll \mathfrak{t}$ but $\boldsymbol{\psi} \neq \mathfrak{t}$ and let $\psi \in \boldsymbol{\psi}$ be a formula. Since $\boldsymbol{\psi}$ is an upper bound of $\mathfrak{C}^{\boldsymbol{\theta}}$, we have that $\mathsf{C}_k^{\boldsymbol{\theta}} \curlyvee \boldsymbol{\psi} = \boldsymbol{\psi}$ for every integer k, $0 \leq k \leq n-1$, which implies that $\theta_{k+1}^1 \vee \psi \approx \psi$ for every k. Thus, for every valuation $\mathbf{u}$ of $\mathbf{P}_n$,

$$\mathbf{u}\big(\theta_{k+1}^1 \vee \psi\big) = \max\left\{\mathbf{u}\big(\theta_{k+1}^1\big)\,, \mathbf{u}(\psi)\right\} = \mathbf{u}(\psi)\,.$$

It follows that $\mathbf{u}\big(\theta^1_{k+1}\big) \leq \mathbf{u}(\psi)$ for every $\mathbf{u}$. Note that, since $\boldsymbol{\psi} \neq \mathbf{t}$, there exists a valuation $\mathbf{w}$ of $\mathbf{P}_n$ such that $\mathbf{w}(\psi) \neq 1$, which implies that $\mathbf{w}(\psi) < 1$. Now, by Lemma 2.4, $\mathbf{w}\big(\theta^1_{k+1}\big) = 1$ if $\mathbf{w}(\theta) = \frac{k}{n-1}$. Thus, when $\mathbf{w}(\theta) = \frac{k}{n-1}$, we have that $\mathbf{w}\big(\theta^1_{k+1}\big) > \mathbf{w}(\psi)$, which is a contradiction, from which we conclude that $\boldsymbol{\psi} = \mathbf{t}$. Therefore,

$$\bigsqcup \left\{ \mathsf{C}^{\boldsymbol{\theta}}_m \, ; \; m = 0, \ldots, n-1 \right\} = \mathbf{t}.$$

Finally, let $\mathbf{v}(\theta) = \frac{i}{n-1}$. Then, by Lemma 2.5,

$$\mathbf{v}\big(\varphi_i \wedge \theta^1_{i+1}\big) = \mathbf{v}(\varphi_i) = \frac{i}{n-1} = \mathbf{v}(\theta)$$

and $\mathbf{v}\left(\varphi_j \wedge \theta^1_{j+1}\right) = 0$ for every $j \neq i$. It follows that

$$\mathbf{v}\left(\bigvee_{m=0}^{n-1} \left(\varphi_m \wedge \theta^1_{m+1}\right) \right) = \max\left\{0, \mathbf{v}(\varphi_i)\right\} = \mathbf{v}(\varphi_i) = \mathbf{v}(\theta)\,.$$

Hence,

$$\bigvee_{m=0}^{n-1} \left(\varphi_m \wedge \theta^1_{m+1}\right) \Leftrightarrow \theta.$$

Since $\varphi_m \in \mathfrak{f}_m$ and $\theta^1_{m+1} \in \mathsf{C}^{\boldsymbol{\theta}}_m$, for every integer m, $0 \leq m \leq n-1$, we have that

$$\boldsymbol{\theta} = \left(\mathfrak{f}_0 \Cap \mathsf{C}^{\boldsymbol{\theta}}_0\right) \Cup \left(\mathfrak{f}_1 \Cap \mathsf{C}^{\boldsymbol{\theta}}_1\right) \Cup \cdots \Cup \left(\mathfrak{f}_{n-2} \Cap \mathsf{C}^{\boldsymbol{\theta}}_{n-2}\right) \Cup \left(\mathfrak{f}_{n-1} \Cap \mathsf{C}^{\boldsymbol{\theta}}_{n-1}\right),$$

as was desired. This concludes the demonstration. ■

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