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Hamiltonian structures and Riemann-Hilbert problems of integrable systems

by

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A dissertation submitted in partial fulfillment
of the requirements for the degree of
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Dedication

This dissertation is dedicated to my dear parents and sister, as well as all the people who have persistently trusted me and supported me with their mentorship or friendship.

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First of all, I must express my sincerest gratitude to my Ph.D. advisor, Prof. Dr. Wen-Xiu Ma. Prof. Ma is a distinguished scholar and a Thomson Reuters' highly-cited researcher in mathematics, as well as a kind and wise academic mentor. He works conscientiously and efficiently; he has burning passion in mathematics; he shares experience and innovative ideas with us; he is quickly available to help and encourage us whenever we have difficulties . . . His wisdom, methodology and enthusiasm of enjoying mathematics & life had impacted me a lot, and definitely will also illuminate some of the future paths ahead of me.

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Abstract

We begin this dissertation by presenting a brief introduction to the theory of solitons and integrability (plus some classical methods applied in this field) in Chapter 1, mainly using the Korteweg-de Vries equation as a typical model. At the end of this Chapter a mathematical framework of notations and terminologies is established for the whole dissertation.

In Chapter 2, we first introduce two specific matrix spectral problems (with 3 potentials) associated with matrix Lie algebras $\mathrm{sl}(2,\mathbb{R})$ and $\mathrm{so}(3,\mathbb{R})$, respectively; then we engender two soliton hierarchies. Analysis of their Hamiltonian structures based on the trace identity affirms that the obtained hierarchies are Liouville integrable. This chapter shows the entire process of how a soliton hierarchy is engendered by starting from a proper matrix spectral problem.

In Chapter 3, at first we elucidate the Gauge equivalence among three types u-linear Hamiltonian operators, and construct then the corresponding Bäcklund transformations among them explicitly. Next we derive the if-and-only-if conditions under which the linear coupling of the discussed u-linear operators and matrix differential operators with constant coefficients is still Hamiltonian. Very amazingly, the derived conditions show that the resulting Hamiltonian operators is truncated only up to the 3rd differential order. Finally, a couple of relevant examples of integrable hierarchies are illustrated.

In Chapter, 4 we first present a generalized modified Korteweg-de Vries hierarchy. Then for one of the equations in this hierarchy, we build the associated Riemann-Hilbert problems with some equivalent spectral problems. Next, computation of soliton solutions is performed by reducing the Riemann-Hilbert problems to those with identity jump matrix, i.e., those correspond to reflectionless inverse scattering problems. Finally a special reduction of the original matrix spectral problem will be briefly discussed.

Chapter 1
Introduction

1.1 Solitons

Solitons are meant in mathematics and physics to be nonlinear self-reinforcing wave packets that preserve their shapes while traveling at constant speeds. The story about the discovery of solitary waves by Scott Russell on the Union Canal at Hermiston in 1834, is well known to soliton scientists just like Genesis to the Christians [1]. Russell even spent time making his own water tanks to investigate these waves, but in the middle of 19th century his experimental results cannot be explained by the existing Newton's or Bernoulli's theories of hydrodynamics or water waves.

In 1870s, as an early attempt of theoretical exploration to the above phenomenon, J. Boussinesq and L. Rayleigh published an effective approximation treatment and presented the corresponding solutions [2, 3, 4]. Then in 1895, Diederik Korteweg and Gustav de Vries formally proposed a partial differential equation (PDE) for the description of wave propagation on shallow water surfaces (like what Russell had observed), which is known to all of us today as the **Korteweg-de Vries equation** (KdV equation) [5].

But the KdV equation then immediately slept in silence for almost seven decades till it was woken up in 1965—by using a finite difference method, N. Zabusky from Bell Labs and M. Kruskal in Princeton University first introduced computer simulations to gain analytical insights upon soliton behavior in media governed by the KdV equation [6]. Then by associating this result with that of an earlier numerical simulation performed by Fermi, Pasta, Ulam and Tsingou [7] (a 1-dimensional lattice of equal point masses connected by nonlinear anharmonic strings), they also showed that the KdV equation can be viewed as the continuum limit of this FPUT system, whose behavior remained unexplained till then (Indeed the modern discoveries associated with the KdV equation later on was also more or less stimulated by the unexpected numerical results yielded from this FPUT model).

In 1967, Gardner, Greene, Kruskal and Miura proposed the inverse scattering transform method that propelled the study of exact solutions of the KdV equation on its right way [8]. Meanwhile, the mathematical tools of Lax pairs and Lax's equation developed by Peter Lax [9] have extended this to computing solutions of various soliton-related models.

Mathematically, *solitons* are solutions of a class of nonlinear PDEs which describe those nonlinear travelling waves that are characterized by:

- The waves do not disperse, so they preserve stabilized profiles (see eqn. (1.22) below) and can travel rather long distances if not much disturbed;
- They are localized (on every dimension), also see eqn. (1.22);
- Due to the nonlinearity, the traveling speed of such a wave is dependent on its amplitude (usually the speed increases with the growth of amplitude, though not necessarily in proportion, see eqn. (1.22) again), and multiple waves do not interact according to the superposition principle;
- The waves interact elastically.

The very last item above, i.e., the elastic interaction between soliton waves, can be basically pictured in one spatial dimension as: suppose two soliton waves are initially well separated, the one with larger speed is located to the left of the other, and both are travelling from left to right; then the taller, faster wave will catch up with the smaller, slower one, and overlap it through a nonlinear interaction; finally both retrieve their profiles unchanged from the collision, and still keep traveling with their own speeds independently, leaving but only a phase shift as the evidence of undergoing a collision.

Physically, the origin of solitons lies in the mechanism of a balanced competition between nonlinear and dispersive effects in the medium—or in another word, in the cases when the effect that causes the wave dispersion is properly compensated by the involved nonlinear effect, such an subtle interplay between the two may give rise to a stable, localized and uniformly traveling wave profile.

So far solitons have been observed not only in shallow water surfaces, but also through experiments in fiber optics [10], in magnets described by Landau-Lifshitz equation or continuum Heisenberg model [11], in the low-frequency collective motion in proteins and DNA [12] and in plasmas, etc.

1.2 The KdV equation as an integrable model

The KdV equation reads

$$u_t + 6uu_x + u_{xxx} = 0, \tag{1.1}$$

where $u = u(x,t)$ is the to-be-determined function depending spatially on x and temporally on t, was first introduced by Joseph Boussinesq in 1877, and particularly revisited by Diederik Korteweg and Gustav de Vries in 1895 as a mathematical model to depict the waves propagating on shallow water surfaces. The remarkable significance of the KdV equation lies in the fact that it is the prototypical example of an

integrable model, i.e., a nonlinear partial differential (or difference) equation whose solutions to well-posed initial or boundary valued problems can be presented [13] in terms of elementary functions, through a finite number of algebraic operations, composition of functions, and computing limits (including differentiations, integrals, etc.).

The KdV equation can be solved in various ways depending on how the initial or boundary values are posed. The most fundamental and amazing property of the KdV equation (indeed of most integrable models) is that it can be expressed as the compatibility condition of a pair of linear differential equations. Suppose that L and M are two operators. Let φ be an eigenfunction of L, with $\lambda \in \mathbb{C}$ (called a **spectral parameter**) being the corresponding eigenvalue; and let the action of M upon ϕ give the associated temporal evolution of the eigenfunction, i.e.:

$$L\varphi = \lambda\varphi, \quad \varphi_t = M\varphi. \tag{1.2}$$

By taking the partial derivative of (1.2) with respect to t, one obtains:

$$\begin{aligned} &\frac{\partial}{\partial t}L\varphi = L_t\varphi + L\varphi_t = \lambda_t\varphi + \lambda\varphi_t, \\ &\quad \Rightarrow L_t\varphi + LM\varphi = \lambda_t\varphi + \lambda M\varphi = \lambda_t\varphi + M(\lambda\varphi) = \lambda_t\varphi + ML\varphi, \\ &\quad \Rightarrow (L_t + [L, M])\varphi = \lambda_t\varphi, \end{aligned} \tag{1.3}$$

where $[L, M] = LM - ML$ is the operator commutator of L and M. Hence it follows from the nontriviality of the eigenfunction $\varphi(x, t)$ in the isospectral case of $\lambda_t = 0$ that the so-called **Lax's equation** holds:

$$L_t + [L, M] = 0. \tag{1.4}$$

Peter Lax discovered in 1968 that given $L = \frac{\partial^2}{\partial x^2} + u$ which is the Sturm-Liouville operator, so that $L\varphi = \lambda\varphi$ reproduces the one-dimensional linear Schrödinger equation, and if M is a third-order differential operator

$$M = -4\frac{\partial^3}{\partial x^3} - 6u\frac{\partial}{\partial x} - 3u_x, \tag{1.5}$$

then the Lax's equation is equivalent to the KdV equation, i.e.,

$$L_t + [L, M] = u_t + 6uu_x + u_{xxx} = 0. \tag{1.6}$$

More generally, if a nonlinear PDE, in a way similar to that for the KdV equation presented above, arises as the compatibility condition of two operators L and M, (1.4) for the corresponding pair of L, M is then called the *Lax representation* of that PDE, and L and M are said to make a *Lax pair*.

The Lax's equation in one-dimensional space actually can be presented in a more general form. Consider two matrix spectral problems, with a potential function u and a spectral parameter λ, i.e., a pair of first-order linear matrix differential equations, with respect to the one-dimensional coordinate x and time t, respectively:

$$\begin{cases} \phi_x = U(x,t,u,\lambda)\phi, \\ \phi_t = V(x,t,u,\lambda)\phi, \end{cases} \tag{1.7}$$

where ϕ is an n-dimensional column vector, and U and V are $n \times n$ matrices. By imposing the compatibility condition $(\phi_x)_t = (\phi_t)_x$, one obtains the so-called *zero curvature equation*:

$$U_t - V_x + [U, V] = 0. \tag{1.8}$$

The zero curvature equation admits spatial eigenvalue dependence other than $L\phi = \lambda\phi$, and is hence more general than the Lax's equation.

From the view of Lagrangian mechanics, if one sets $u(x,t) = \psi_x(x,t)$, the KdV equation can also be derived as the Euler-Lagrange equation of motion associated with the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_x\psi\partial_t\psi + (\partial_x\psi)^3 - \frac{1}{2}(\partial_x^2\psi)^2. \tag{1.9}$$

A "milestone" discovery in the development of general methods for solving the KdV equation is that the KdV may possess infinitely many independent conservation laws. A conservation law of a system is mathematically characterized by $\frac{\partial}{\partial t}T + \frac{\partial}{\partial x}X = 0$ (see Definition 1.4.7 below), from which one can derive $\frac{d}{dt}\int_\Omega T dx = 0$ by applying the Green's formula upon a domain $\Omega \subseteq \mathbb{R}^N$, and so $\int_\Omega T dx$ comes out to be a conserved quantity.

The first two conservation laws of the KdV equation were found with ease due to their mathematical simplicity and physical significance:

$$u_t = -(3u^2 + u_{xx})_x, \tag{1.10a}$$

$$(u^2)_t = -(4u^3 + 2uu_{xx} - u_x^2)_x, \tag{1.10b}$$

which correspond obviously to the conservation laws of momentum and energy in the system, respectively. The third conservation of the KdV was then found by Whitham in 1965 to be:

$$(u^3 - \frac{1}{2}u_x^2)_t = -\left(\frac{9}{2}u^4 + 3u^2u_xx - 6uu_x^2 - u_xu_{xxx} + \frac{1}{2}u_{xx}^2\right)_x. \tag{1.11}$$

Kruskal and Zabusky later discovered the fourth and fifth conservation laws of the KdV. This number kept growing till Miura found the tenth. The thus-arisen conjecture that the KdV equation may have an infinite

number of independent conservation laws was finally proved by Miura, Gardner and Kruskal in 1968 by using the Miura transformation which bridges the KdV equation and the so-called modified KdV equation (we shall omit the very much detail here)

$$v_t - 6v^2 v_x + v_{xxx} = 0 \tag{1.12}$$

in terms of $u = -(v^2 + v_x)$ (Note that through this transformation, if v solves (1.12), then u solves the KdV, but not necessarily the converse).

Another amazing property of the KdV equation is that, as an evolutionary equation in the form of $u_t = K[u] = K(x, u^{(k)})$, $K[u]$ on the r.h.s can be written in the form of a Hamiltonian operator acting on a Hamiltonian functional in two alternative ways. Firstly,

$$u_t = \frac{\partial}{\partial x}(-u_{xx} - 3u^2) = \mathcal{J}_1 \delta \mathcal{H}_1, \tag{1.13}$$

where δ denotes the variational derivative with respect to u (See Definition 1.4.5 below) and

$$\mathcal{J}_1 = \frac{\partial}{\partial x}, \quad \mathcal{H}_1 = \int \Big(\frac{1}{2}u_x^2 - u^3\Big) dx. \tag{1.14}$$

Secondly, which is less obvious, $K[u]$ can be alternatively written as

$$u_t = (\partial_x^3 + 4u\partial_x + 2u_x)u = \mathcal{J}_2 \delta \mathcal{H}_2, \tag{1.15}$$

where

$$\mathcal{J}_2 = \partial_x^3 + 4u\partial_x + 2u_x, \quad \mathcal{H}_2 = \int \Big(-\frac{1}{2}u^2\Big) dx. \tag{1.16}$$

Both $\mathcal{J}_1$ and $\mathcal{J}_2$ can be shown to be Hamiltonian because they satisfy the skew-symmetric properties and the Jacobi identity; both $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hamiltonian functionals (these concepts will also be briefly introduced below). This fantastic fact indicates that not only the KdV equation is a Hamiltonian system, but also it possesses the so-called bi-Hamiltonian structure. This bi-Hamiltonian structure is in fact intimately associated with the infinitely many conservation laws mentioned just above, and the Liouville integrability of the KdV equation as a Hamiltonian system. We shall elucidate a bit more about the connections among them below.

1.3 Methodologies for solving the KdV equation

We shall begin with using a traditional method to obtain a solution of the KdV equation with a stable localized wave shape that travels to the right at a constant speed c (i.e., a simplest one-soliton solution). Such a solution can intuitively be conjectured to take the form $u(x,t) = f(x - ct) = f(w)$, where $w = x - ct$.

By substituting this into the KdV equation $u_t + 6uu_x + u_{xxx} = 0$, we are given an ordinary differential equation (ODE)

$$-c\frac{df}{dw} + 6f\frac{df}{dw} + \frac{d^3f}{dw^3} = 0, \tag{1.17}$$

which can be integrated directly to give

$$-cf + 3f^2 + \frac{d^2f}{dw^2} = C_1, \tag{1.18}$$

where C_1 is a constant of integration. Next we multiply (1.18) with $\frac{df}{dw}$ and integrate again to yield a first order ODE

$$-cf\frac{df}{dw} + 3f^2\frac{df}{dw} + \frac{d^2f}{dw^2}\frac{df}{dw} = C_1\frac{df}{dw} \quad \Rightarrow -\frac{c}{2}f^2 + f^3 + \frac{1}{2}\left(\frac{df}{dw}\right)^2 = C_1f + C_2.$$

where C_2 involves as the second constant of integration. The condition that $u(x,t) = f(w)$ is localized requires $f, \frac{df}{dw}, \frac{d^2f}{dw^2} \to 0$ as $x \to \pm\infty$, from which it follows apparently that $C_1 = C_2 = 0$.

It remains therefore to solve

$$-\frac{c}{2}f^2 + f^3 + \frac{1}{2}\left(\frac{df}{dw}\right)^2 = 0 \quad \Rightarrow \quad \left(\frac{df}{dw}\right)^2 = f^2(c - 2f), \tag{1.19}$$

or equivalently, to integrate

$$\int_{f(0)}^{f}(w)\frac{dz}{z\sqrt{c-2z}} = \int_0^w d\eta. \tag{1.20}$$

For this purpose we introduce the change of variable $2z = c\text{sech}^2 y$, such that one derives easily

$$c - 2z = c(1 - \text{sech}^2 y) = c\text{tanh}^2 y, \quad dz = -c\frac{\text{sinh}y}{\text{cosh}^3 y}dy \quad \Rightarrow \quad \frac{dz}{z\sqrt{c-2z}} = -\frac{2}{\sqrt{c}}dy.$$

By this change of variable, the upper and lower limits of the integral on l.h.s of (1.20) must also be transformed to $\text{sech}^{-1}\sqrt{2f/c}$ and $\text{sech}^{-1}\sqrt{2f(0)/c}$, respectively. The integration therefore gives

$$-\frac{2}{\sqrt{c}}\left(\text{sech}^{-1}\sqrt{\frac{2f}{c}} - \text{sech}^{-1}\sqrt{\frac{2f(0)}{c}}\right) = w = x - ct, \tag{1.21}$$

from which it follows immediately that

$$f = \frac{c}{2}\text{sech}^2\left(\frac{\sqrt{c}}{2}(x - ct) + a\right). \tag{1.22}$$

where the phase shift a can be easily determined by the value of f at $w = 0$, i.e., $a = \text{sech}^{-1}\sqrt{\frac{2f(0)}{c}}$.

Below we shall introduce a few systematical methodologies which are pretty well-developed in solving the KdV equation as an integrable model.

1.3.1 Bäcklund transform

The study of Bäcklund transformations can be traced back to originate from the work of Albert Victor Bäcklund, a swedish geometrist. When he was investigating surfaces with negative constant curvatures, he discovered that one can build a transformation between the two existing solutions u_1 and u_2 to the sine-Gordon equation (to his knowledge at that time) $u_{xy} = \sin u$, which reads

$$u_{2,x} = u_{1,x} + 2a\sin\big(\frac{u_1+u_2}{2}\big), \quad u_{2,y} = -u_{1,y} - \frac{2}{a}\sin\big(\frac{u_1-u_2}{2}\big),$$

where a is an arbitrary constant. This is the well-known (auto) Bäcklund transformation for the sine-Gordon equation. Just like the KdV equation, Bäcklund transformations did not find quick applications and slept in a even longer silence, before it resurrected in 1960s.

The fundamental idea of Bäcklund transform in solving a nonlinear PDE is that, by starting from a "trivial" solution (i.e., a solution pretty obvious and easy to guess), the Bäcklund transform can map it to a new solution which is less trivial [15] (if the new solution solves the same PDE, this Bäcklund transform is said to be *Auto-Bäcklund*; but in general, Bäcklund transform can also map the original solution to something which solves another related PDE).

To show how Bäcklund transform works for the KdV equation, we begin by letting $u = v_x$ for some function v, and this leads to

$$u_t + 6uu_x + u_{xxx} = (v_x)_t + 6v_xv_{xx} + (v_x)_{xxx} = \frac{\partial}{\partial x}(v_t + 3v_x^2 + v_{xxx}) = 0. \tag{1.23}$$

It follows then that $v_t + 3v_x^2 + v_{xxx}$ must be dependent of t only, i.e.,

$$v_t + 3v_x^2 + v_{xxx} = f(t). \tag{1.24}$$

We now introduce

$$w = v - \int^t f(s)ds, \tag{1.25}$$

such that w clearly satisfy

$$w_t + 3w_x^2 + w_{xxx} = 0. \tag{1.26}$$

Next we introduce an Auto-Bäcklund transform given by

$$\tilde{v}_x = \beta - w_x - \frac{1}{2}(\tilde{v}-w)^2, \tag{1.27a}$$

$$\tilde{v}_t = -w_t + (\tilde{v}-w)(\tilde{v}_{xx} - w_{xx}) - 2(\tilde{v}_x^2 + \tilde{v}_x w_x + w_x^2), \tag{1.27b}$$

where the involved β is called the Bäcklund parameter. Through a routine computation one can derive that $\tilde{v}$ satisfies also $\tilde{v}_t + 3\tilde{v}_x^2 + \tilde{v}_{xxx} = 0$, as w does.

At this point we'd like to trivially pick $w = 0$, such that (1.27a), (1.27b) now reduce to

$$\tilde{v}_x = \beta - \frac{1}{2}\tilde{v}^2, \tag{1.28a}$$

$$\tilde{v}_t = \tilde{v}\tilde{v}_{xx} - 2\tilde{v}_x^2. \tag{1.28b}$$

We can easily solve (1.28a) by integration and substitute the result into (1.28b) to find a solution $\tilde{v}$. One of the solutions we can have is

$$\tilde{v}(x,t) = \sqrt{2\beta}\text{tanh}\Big[\sqrt{\frac{\beta}{2}}(x - 2\beta t) + a\Big], \tag{1.29}$$

(where, just like in (1.22), the phase shift a arises as a constant of integration) from which it follows finally the one-soliton solution

$$u(x,t) = \tilde{v}_x = \beta\text{sech}^2\Big[\sqrt{\frac{\beta}{2}}(x - 2\beta t) + a\Big]. \tag{1.30}$$

(1.30) is clearly equivalent to (1.22) since the latter is reproduced by setting $\beta = \frac{c}{2}$ in the former.

Bäcklund transform can also be used to compute multiple-soliton solutions of the KdV equation.

1.3.2 Inverse scattering transform

The inverse scattering transform (IST) is in all senses a very typical example of the so-called "inverse problems" in mathematics. In contrast to direct problems, in which we straightforwardly derive consequences (that we expect to observe) based on given causal factors, in an inverse problem we try to figure out from a set of observed consequences the causal factors that lead to these consequences (which in general, is more difficult). I.S.T. was developed originally because of its importance in exploring atomic and particle structures in physics, i.e., via an scattering experiment, the physicists expect to reconstruct the potential that causes plenty of atomic and subatomic phenomena from the phase shifts of the various scattered waves. Very famous, for example, is the Rutherford scattering experiment carried out in 1911, through which the existence of an atomic nucleus in an atom (a milestone in exploration of atomic structure) was demonstrated.

But mathematically, the inverse scattering transform method is usually called the "nonlinear version of Fourier analysis" in the following sense: For example, when we solve a PDE about a dependent variable u depending on a spatial independent variable x and a temporal independent variable t, i.e., $u = u(x,t)$, with initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$, we first perform a Fourier transform w.r.t x—this converts the original PDE into an ODE of a Fourier coefficient $U(k,t) = \mathcal{F}(u)(k)$ depending on time t (with k transformed from x now being the Fourier parameter); then by solving this ODE, we find the temporal evolution of the Fourier coefficient; finally through the inverse Fourier transform, we obtain the solution $u(x,t)$ that describes the system for any later time $t > 0$.

Based on this understanding, the steps of running an inverse scattering transform can be sketched as follows (of course, solving specified Cauchy problems of the KdV equation also follow these steps; but since I.S.T. is in this thesis only an auxiliary tool for understanding part of the materials in Chapter 4, and since presentation for an specific model costs a bit lengthy text, we are going to present only its spirit instead of detailed computation, and the interested readers may refer to [8, 32, 14]):

- **Step 1.** Proceed with forward scattering (which is the "direct problem" counterpart of inverse scattering) to find the Lax pair of the nonlinear PDE that we are studying. The routine procedure is that of solving a matrix spectral problem, the readers will see how this works in Chapter 4;

- **Step 2.** Collect the so-called **scattering data** and compute its time evolution. Scattering data refers to a set of information that encloses the eigenfunctions associated with each eigenvalue λ, the normalizing constants and the reflection coefficient. In most cases, the information of scattering data is carried by a matrix (said to be the **scattering matrix**). Generally the time evolution of the scattering data is described by a system of linear ODEs that is solvable.

- **Step 3.** Finally, one performs the inverse scattering procedure through solving the so-called Gel'fand-Levitan-Marchenko equation, which is a linear integral equation, to obtain the final solution of the original nonlinear PDE. Generally all the scattering data will be called in this step, though the process could be simplified a lot if the reflection coefficient happens to be zero.

So to summarize, Fig. 1 gives a schematic diagram of the inverse scattering procedure. If one replaces in this figure "scattering" by "Fourier transform", as well as "Scattering data S" by $\mathcal{F}(k)(u)$, respectively, the figure reduces to a chart flow diagram for solving a linear PDE via the Fourier transform. Therefore one sees that scattering data for a nonlinear PDE plays the role similar to that of the Fourier transform of a solution in a linear PDE.

The inverse scattering procedure, in the case of a nonlinear PDE with only one space dimension, can always be reformulated as a Riemann-Hilbert problem. In higher space dimensions, one obtains instead a "nonlocal" Riemann-Hilbert problem or a $\bar{\partial}-$problem [16, 17] (i.e., the problem of solving any differential equation involving the $\bar{\partial}$ derivative, e.g., $\bar{\partial} f(z, \bar{z}) = g(z, \bar{z})$, $z \in D$, where $\bar{\partial} = \frac{\partial}{\partial \bar{z}}$, g is a given function and D is a simply-connected domain of the complex plane).

1.3.3 Hirota bilinear method

There are a bunch of ways through which one can compute multi-soliton solutions of the KdV equation, such that the elastic interaction between distinct solitons can be investigated and interpreted in mathematical

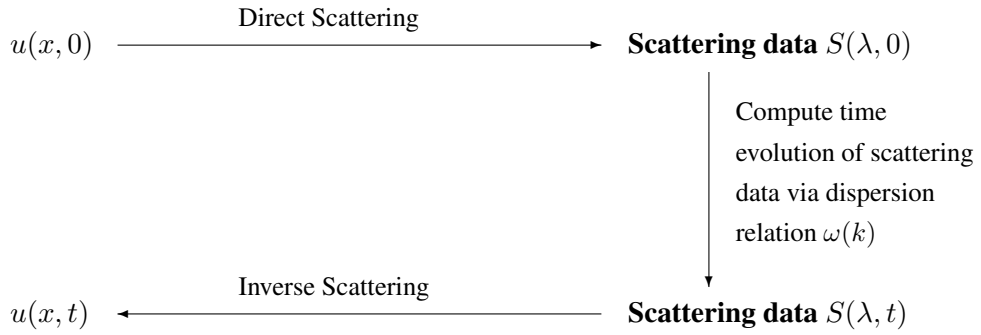

Figure 1.: Schematic Diagram of the Inverse Scattering Transform

detail. The most effective one of them is probably the Hirota bilinear method [18].

Hirota first realized that a PDE's most popular "face" may not be the best one for its mathematical analysis, and for the KdV equation he insightfully saw the bilinear properties hidden behind, and proposed the bilinear differential operator based on this understanding:

Let $f(x,t)$ and $g(x,t)$ be sufficiently smooth functions of x and t. Let $m,n \geq 0$ be integers. The action of the **Hirota differential operators** D_x and D_t upon the product $f \cdot g$ is given by

$$\begin{aligned} D_x^n D_t^m f \cdot g &= \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m f(x,t)g(x',t')|_{x'=x,t'=t} \\ &= \frac{\partial^n}{\partial \tilde{x}^n}\frac{\partial^m}{\partial \tilde{t}^m} f(x+\tilde{x},t+\tilde{t})g(x-\tilde{x},t-\tilde{t})|_{\tilde{x}=\tilde{t}=0}, \end{aligned} \tag{1.31}$$

in which the involved two ways of definition are obviously equivalent.

It would be straightforward to derive the following properties of the bilinear differential operators: (i) $D_x^n D_t^m f \cdot g = (-1)^{m+n} D_x^n D_t^m g \cdot f$ (from which $D_x^n D_t^m f \cdot f = 0$, when $m+n$ is odd, is an immediate consequence); (ii) $D_x^n D_t^m f \cdot 1 = \partial_x^n \partial_t^m f$; (iii) Given that $\xi_1 = \omega_1 t + k_1 x + \varphi_1$ and $\xi_2 = \omega_2 t + k_2 x + \varphi_2$, where $\omega_1, \omega_2, k_1, k_2$ are constants, $D_x^n D_t^m \mathrm{e}^{\xi_1} \cdot \mathrm{e}^{\xi_2} = (\omega_1 - \omega_2)^m (k_1 - k_2)^n \mathrm{e}^{\xi_1 + \xi_2}$—in particular, $D_x^n D_t^m \mathrm{e}^{\xi_1} \cdot \mathrm{e}^{\xi_1} = 0$.

In order to apply this to the KdV equation, Hirota also introduced the transformation

$$u = 2\frac{\partial^2}{\partial x^2} \ln f = 2(\ln f)_{xx} = \frac{2(f_{xx}f - f_x^2)}{f^2}, \tag{1.32}$$

through which the KdV equation is converted into

$$u_t + 6uu_x + u_{xxx} = \left[\frac{(D_x^4 + D_x D_t) f \cdot f}{f^2}\right]_x,$$

where by definition of the bilinear differential operator,

$$(D_x^4 + D_x D_t) f \cdot f = 2(f_{xt} f - f_x f_t + f_{xxxx} f - 4 f_{xxx} f_x + 3 f_{xx}^2). \tag{1.33}$$

Therefore f solves the above bilinear version of Kdv equation $(D_x^4 + D_x D_t) f \cdot f = 0$ implies that $u = 2(\ln f)_{xx}$ will solve the original KdV equation (but not necessarily the converse). The bilinear property carried by (1.33) is indeed quite advantageous for constructing explicit exact solutions.

We shall work with this tool to compute multi-soliton solutions of the KdV equation. We suppose that $f = f(x,t)$ can be expanded as a power series of ε, which is introduced here as a parameter:

$$f(x,t) = 1 + \varepsilon f^{(1)} + \varepsilon^2 f^{(2)} + \cdots = 1 + \sum_{k=1}^{\infty} \varepsilon^k f^{(k)}, \tag{1.34}$$

By substituting this expansion of f into $(D_x^4 + D_x D_t) f \cdot f = 0$, and equating the coefficients of all powers of ε yields

$$f_{tx}^{(1)} + f_{xxxx}^{(1)} = 0, \tag{1.35a}$$

$$2(f_{tx}^{(2)} + f_{xxxx}^{(2)}) = -(D_x D_t + D_x^4) f^{(1)} \cdot f^{(1)}, \tag{1.35b}$$

$$f_{tx}^{(3)} + f_{xxxx}^{(3)} = -(D_x D_t + D_x^4) f^{(1)} \cdot f^{(2)}, \tag{1.35c}$$

$$2(f_{tx}^{(4)} + f_{xxxx}^{(4)}) = -(D_x D_t + D_x^4)(2 f^{(1)} \cdot f^{(3)} + f^{(2)} \cdot f^{(2)}), \tag{1.35d}$$

$\cdots\cdots$

One can easily derive that a linear exponential function in the form of

$$f^{(1)} = \mathrm{e}^{\eta_1}, \quad \eta_1 = \omega_1 t + k_1 x + \phi_1, \quad \omega_1 = -k_1^3 \tag{1.36}$$

nontrivially solves (1.35a). By substituting now $f^{(1)}$ in (1.35b) by (1.36), one obtains from the properties of bilinear differential operators that

$$f_{tx}^{(2)} + f_{xxxx}^{(2)} = 0. \tag{1.37}$$

If one takes at this point trivially that $f^{(2)} = 0$, it follows from (1.35c) that

$$f_{tx}^{(3)} + f_{xxxx}^{(3)} = 0. \tag{1.38}$$

Again one can take $f^{(3)} = 0$. By always applying this logic trivially, it follows naturally that $f^{(4)} = f^{(5)} = \cdots = 0$. The power series is thus truncated to be finite, and when $\varepsilon = 1$ it gives

$$f(x,t) = 1 + \mathrm{e}^{\eta_1} \quad \Rightarrow \quad u = 2(\ln f)_{xx} = \frac{k_1^2}{2} \mathrm{sech}^2 \frac{\omega_1 t + k_1 x + \phi_1}{2}. \tag{1.39}$$

So far we obtain the simplest solution, one-soliton solution.

Since the equation (1.35a) is linear (i.e., principle of superposition is here applicable), one can naturally generalize the above one-soliton solution to a two-soliton solution by assuming

$$f^{(1)} = 1 + \mathrm{e}^{\eta_1} + \mathrm{e}^{\eta_2}, \tag{1.40}$$

where

$$\eta_1 = \omega_1 t + k_1 x + \phi_1, \quad \eta_1 = \omega_2 t + k_2 x + \varphi_2, \quad \omega_1 = -k_1^3, \quad \omega_2 = -k_2^3.$$

By substituting (1.40) into (1.35b), one obtains (nontrivially, $\omega_1 k_2 - \omega_2 k_1 \neq 0$ is assumed)

$$f_{tx}^{(2)} + f_{xxxx}^{(2)} = 3k_1 k_2 (k_1 - k_2)^2 \mathrm{e}^{\eta_1 + \eta_2}, \tag{1.41}$$

from which one solves

$$f^{(2)} = C\mathrm{e}^{\eta_1 + \eta_2}, \quad \text{with} \quad C = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2. \tag{1.42}$$

Now $f^{(1)}$, $f^{(2)}$ and (1.35c) together will imply

$$f_{tx}^{(3)} + f_{xxxx}^{(3)} = 0, \tag{1.43}$$

for which we again can trivially take $f^{(3)} = 0$ (similar to what we did for the one-soliton solution). Likewise it follows then that $f^{(4)} = f^{(5)} = \cdots = 0$. Therefore by admitting $\varepsilon = 1$ we obtain that

$$f(x,t) = 1 + \mathrm{e}^{\eta_1} + \mathrm{e}^{\eta_2} + C\mathrm{e}^{\eta_1 + \eta_2} = 1 + \mathrm{e}^{\eta_1} + \mathrm{e}^{\eta_2} + \mathrm{e}^{\eta_1 + \eta_2 + A_{12}}, \tag{1.44}$$

where we have assumed $C = \mathrm{e}^{A_{12}}$. Therefore the two-solitons' solution of the KdV equation reads

$$u = 2[\ln(1 + \mathrm{e}^{\eta_1} + \mathrm{e}^{\eta_2} + \mathrm{e}^{\eta_1 + \eta_2 + A_{12}})]_{xx}. \tag{1.45}$$

Likewise one can compute the N-soliton solution with $N > 2$. Obviously the complexity of the involved computation as well as of the results increases with the growth of N. For example, the 3-soliton solution of the KdV equation is found to possess the general form

$$\begin{aligned} u &= 2(\ln f)_{xx}, \\ f &= 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3} + e^{\eta_1 + \eta_2 + A_{12}} + e^{\eta_1 + \eta_3 + A_{13}} + e^{\eta_2 + \eta_3 + A_{23}} \\ &\quad + e^{\eta_1 + \eta_2 + \eta_3 + A_{12} + A_{13} + A_{23}}, \end{aligned} \tag{1.46}$$

where

$$\eta_j = \omega_j t + k_j x + \varphi_j, \quad \mathrm{e}^{A_{jl}} = \left(\frac{k_j - k_l}{k_j + k_l}\right)^2, \quad \omega_j k_l - \omega_l k_j \neq 0, \quad 1 \leq j < l \leq 3. \tag{1.47}$$

Satsuma, Freeman and Nimmo ingeniously showed that the multi-soliton solutions to the KdV equation can be expressed in terms of the Wronskian determinant [19, 20]; Matveev then generalized the Wronskian determinant to verify the existence of another important class of exact solutions, the so-called positons, to the KdV equation [21, 22] . Ma and You also greatly enriched these work in 2005 [23].

1.3.4 Riemann-Hilbert problems

Riemann-Hilbert problems (RHP), named after the giant German mathematicians Bernhard Riemann and David Hilbert, are a class of models which can be used to solve certain differential equations with the assistance of complex analysis techniques such as analytical continuation. RHP can be presented in slightly different ways when treating different problems [24]. Here we introduce only the most typical representation that is widely used in integrable systems.

Let $\gamma \subset \mathbb{C}$ be an oriented contour in the complex λ-plane. The orientation defines traditionally the $\mathbb{D}_+$ and $\mathbb{D}_-$ sides of γ as being on the left and right sides of the direction arrow, respectively. Let G be a map from γ into the set of $N \times N$ invertible matrices (with complex entries), which we shall denote by $GL_N(\mathbb{C})$. An RHP associated with the pair $(\gamma; G)$ consists in finding an $N \times N$ matrix-valued function $\Phi(\lambda)$ ($\lambda \in \mathbb{C}$) characterized by [26]:

- $\Phi(\lambda)$ is holomorphic in $\mathbb{C}\backslash\gamma$;
- $\Phi^+(\lambda) = \Phi^-(\lambda)G(\lambda)$ for all $\lambda \in \gamma$, where

$$\Phi^+(\lambda) = \lim_{\lambda' \to \lambda, \lambda' \in \mathbb{D}_+} \Phi(\lambda'), \quad \text{and} \quad \Phi^-(\lambda) = \lim_{\lambda' \to \lambda, \lambda' \in \mathbb{D}_-} \Phi(\lambda'); \tag{1.48}$$

  $G(\lambda)$ involved here is often called the **jump matrix** in this model; more generally, $\Phi^+(\lambda)$ and $\Phi^-(\lambda)$ can be defined as $\Phi(\lambda)$ restricted to $\lambda \in \mathbb{D}^+$ and $\lambda \in \mathbb{D}^-$, respectively (hence $\Phi^+(\lambda)$ and $\Phi^-(\lambda)$ are holomorphic in $\mathbb{D}^+$ and $\mathbb{D}^-$, respectively);

- Both $\Phi^+(\lambda)$ and $\Phi^-(\lambda)$ approach the identity matrix as $\lambda \to \infty$ (canonical normalization condition).

To solve the simplest scalar case $N = 1$, one can rewrite the original multiplicative jump condition into an additive form with the help of the logarithmic function

$$\log \Phi_+(\lambda) = \log \Phi_-(\lambda) + \log G(\lambda), \tag{1.49}$$

which can always be solved by using the Cauchy-Plemelj-Sokhotskii formula [27, 28]

$$\log \Phi(\lambda) = \frac{1}{2\pi i} \int_\gamma \frac{\log G(z)}{z - \lambda} dz. \tag{1.50}$$

RHPs with $N \geq 1$ can be solved also explicitly by (1.50), whenever the involved matrix multiplication for G is abelian, i.e., when $[G(\lambda_1), G(\lambda_2)] := G(\lambda_1)G(\lambda_2) - G(\lambda_2)G(\lambda_1) = 0$ for all $\lambda_1, \lambda_2 \in \gamma$; that is

$$\Phi(\lambda) = \exp\left(\frac{1}{2\pi i}\int_\gamma \frac{\log G(z)}{z-\lambda}dz\right). \tag{1.51}$$

For a more general non-abelian matrix RHP, formula (1.50) or (1.51) unfortunately ceases to work–it is so far believed that in such cases the RHP cannot be solved in analytical form by means of contour integrals. However, reformulating the original problem into an RHP still makes much sense, since it can always be reduced to the study of a linear singular-integral equation. Indeed, nonabelian RHPs usually arise when the original problem is nonlinear, so the value of the Riemann-Hilbert reformulation lies in the fact that it linearizes a nonlinear system effectively.

The Riemann-Hilbert approach has acquired wide applications in integrable systems, orthogonal polynomials, random matrices, and asymptotic analysis. In particular for many integrable systems, the inverse spectral or inverse scattering problems associated particularly with the Cauchy problems for 1+1 dimensional (typically, space x and time t) PDEs, or the construction of soliton solutions for these systems, can be formulated as RHPs on the real line $\mathbb{R}$.

In many applications, the jump matrices $G(\lambda)$ in play often exhibit an oscillatory dependence on x (or t) as the latter approaches $\pm\infty$. The asymptotic estimate of the solution $\Phi(x, t; \lambda)$ of the RHP as $x, t \to \pm\infty$ will then involve evaluating the asymptotics of oscillatory contour integrals by using the classical method of steepest descent or stationary phase. Starting from the pioneering works in 1973 by Shabat, Manakov, Ablowitz and Newell [29, 30], the asymptotic analysis of integrable systems reached in around 20 years a new peak marked by the nonlinear steepest descent method for oscillatory RHPs, which was introduced in 1993 by Deift and Zhou [31].

In Chapter 4 we will use the Riemann-Hilbert approach to compute multi-soliton solutions of a generalized mKdV system, hence we'd like to postpone some further discussions till that chapter (with a target-oriented model, this more effectively helps the readers to see how a RHP works).

What have been so far elucidated or discussed above are based mainly on the KdV equation, since as an integrable system it is so perfect as well as simple enough for an introduction chapter. Certainly this does not mean the KdV equation is the only integrable system. Indeed, soliton-generating, possessing Hirota bilinear forms, infinitely many conservation laws or bi-Hamiltonian structures, as well as the mathematical tools preliminarily mentioned above, are (to more or less extents) the common characteristics of many other physically important systems in the frame of integrable systems; for example, the Ablowitz-Kaup-Newell-Segur equation [32], the Kadomtsev-Petviashvili equation [33] and the Davey-Stewartson equation [34], etc.

Today mathematicians are still working hard trying to find more such systems and study their behavior.

1.4 Mathematical framework

From now on we shall try to present the work in the language of mathematics. By this chance we set up also the major notations, concepts and definitions for all chapters in this dissertation.

Let n and m be positive integers. Let $t \in \mathbb{R}$, $x = (x^1, \dots, x^n) \in X = \mathbb{R}^n$, and $\partial_i = \frac{\partial}{\partial x^i}$. Let $u = (u^1, \dots, u^m)^T \in U \subseteq \mathbb{R}^m$, where $u^i = u^i(x,t)$ $(1 \leq i \leq m)$ are functions of x and t. Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be a n-tuple multi-index that has $\alpha_i \geq 0$ for all $1 \leq i \leq n$. Define accordingly

$$u^i_\alpha = D^\alpha u^i, \qquad 1 \leq i \leq m, \quad \text{with} \quad D^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}. \tag{1.52}$$

Let $M \subseteq X \times U$ be a connected open subset (can be sufficiently large for the description of the whole system). Assume that $\mathcal{A}$ denotes the space of all smooth local functions $f(x, u^{(k)})$ ($k \geq 0$ is a finite integer), where $u^{(k)}$ stands for the k-th *prolongation* of u (w.r.t. to x), i.e. to be brief, the set of u and all its possible partial derivatives with respect to x up to order $k \geq 0$ evaluated at the point x (for example, if $m = 1$ and $n = 2$, then $u^{(2)} = (u; u_{x_1}, u_{x_2}; u_{x_1x_1}, u_{x_1x_2}, u_{x_2x_2})$). The locality here implies that the dependence of f on u, i.e. $(f(u))(x)$, is completely determined by the values of u in an arbitrarily small neighborhood of x. In this dissertation, to make things concise as well as to remind the readers that $f(x, u^{(k)})$ is a function of x, u and the derivatives of u, very often we use also the abbreviated symbol $f[u] := f(x, u^{(k)})$.

1.4.1 Evolution equations

DEFINITION 1.4.1 A system of **evolution equations** is generally meant to be a system of PDEs of $u(x,t) = (u^1(x,t), \dots, u^m(x,t))^T \in \mathbb{R}^m$ ($t \in \mathbb{R}$, $x \in \mathbb{R}^n$) given specifically in the form

$$\frac{\partial u}{\partial t} = K[u] = K(x, u^{(k)}), \tag{1.53}$$

where $K(x, u^{(k)}) \in \mathcal{A}$ for some $k \geq 0$ (Note that K does not depend explicitly on t).

Certainly an evolution equation can also be an ODE. Even some second order equation, $u_{tt} = f(u)$ for instance, can be reformulated into a first order system by introducing an auxiliary variable v, such that the equivalent system

$$\begin{cases} u_t = v, \\ v_t = f(u) \end{cases}$$

can be viewed as evolutionary.

DEFINITION 1.4.2 A **Lax pair** consists of a spectral operator L and a Lax operator M that may depend on x, u and $u_x, u_{xx}, \ldots$, etc. (but not explicitly on t), such that the Lax's equation

$$L_t + [L, M] = 0,$$

where $[L, M] = LM - ML$ is the commutator of L and M (being operators, obviously they could be non-commutative), represents a a nonlinear system of evolution equations of $u = u(x, t)$.

The Sturm-Liouville operator $L = \partial_x^2 + u$ and $M = -4\partial_x^3 - 6u\partial_x - 3u_x$ constitute the Lax pair of the KdV equation.

As a somewhat informal definition, a (matrix) **spectral problem** is usually meant in the domain of integrable systems to be a first-order differential eigenvalue problem in the form of

$$\phi_x = M(x, u^{(k)}, \lambda)\phi,$$

where M is a square matrix whose entries are functions of x,u and derivatives of u w.r.t. x (up to some order k), as well as of λ (λ serves as the spectral parameter).

Ablowitz, Kaup, Newell and Segur published in 1974 a matrix formalism for the Lax pair [25, 32]. They introduced a system that consists of a pair of matrix spectral problems with a potential function u and a spectral parameter λ:

$$\phi_x = U(t, x, u^{(k)}, \lambda)\phi, \tag{1.54a}$$

$$\phi_t = V(t, x, u^{(k)}, \lambda)\phi, \tag{1.54b}$$

where ϕ is an n-dimensional vector, and U and V are $n \times n$ matrices. Through applying the isospectral ($\lambda_t = 0$) compatibility condition $\phi_{xt} = \phi_{tx}$, one obtains $U_t - V_x + [U, V] = 0$.

DEFINITION 1.4.3 The **zero curvature equation** $U_t - V_x + [U, V] = 0$, also known as the matrix's Lax equation, is given as the compatibility condition $\phi_{xt} = \phi_{tx}$ of the matrix spectral problems (1.54a) and (1.54b). It also represents a nonlinear system of evolution equations.

The matrices U, V are usually also called a Lax pair (of matrix form).

1.4.2 Variational calculus

DEFINITION 1.4.4 [38] Let $X = \mathbb{R}^n$. Let Ω be an open, connected subset of X with smooth boundary $\partial\Omega$. The goal of a **variational problem** is to address the extrema of a functional

$$\mathcal{L}[u] = \int_\Omega L(x, u^{(n)})dx \tag{1.55}$$

for a certain class of functions $u = u(x)$ (i.e., they commonly possess some specified properties) defined on Ω. The integrand $L(x, u^{(n)})$ in (1.55) is said to be the **Lagrangian** of the variational problem.

DEFINITION 1.4.5 [38] Let $u = u(x), v = v(x)$ both be m-tuple smooth functions on Ω, and v has a compact support $K \subseteq \Omega$. Let $\mathcal{L}[u]$ be the functional of a variational problem. The **variational derivative** is meant to be the m-tuple function $\delta\mathcal{L}[u] = (\delta_1\mathcal{L}[u], \ldots, \delta_m\mathcal{L}[u])$, such that

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\mathcal{L}[u + \varepsilon v] = \int_\Omega \delta\mathcal{L}[u(x)] \cdot v(x)dx = \sum_{k=1}^{m}\int_\Omega \delta_k\mathcal{L}[u]v^k(x)dx. \tag{1.56}$$

where $v = (v^1, \ldots, v^m)^T$. $\delta_k\mathcal{L}[u]$ is often written also as $\frac{\delta\mathcal{L}[u]}{\delta u^k}$ $(1 \leq k \leq m)$. (1.56) can be easily proved by using integration by parts and the fact that v has a compact support $K \subseteq \Omega$.

1.4.3 Conservation laws and symmetries

DEFINITION 1.4.6 Let $K, S \in \mathcal{A}^m$. The **Gateaux derivative** of K in the direction of S at the point u is defined by

$$K'[S] = K'(u)[S] = \frac{\partial}{\partial\epsilon}K(u + \epsilon S)|_{\epsilon=0}. \tag{1.57}$$

If the Gateaux derivative $K'(u)[S]$ exists for every $S \in \mathcal{A}^m$, then one says K is Gateaux-differentiable at u.

DEFINITION 1.4.7 For a PDE

$$\triangle(x, t, u(x, t)) = 0, \tag{1.58}$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}^n$ are the spatial and temporal independent variables, respectively, u is the dependent variable, and $\triangle$ is a smooth function of x, t, u and the partial derivatives of u with respect to x and t up to some order, a **conservation law** is meant to be an equation characterized as

$$\frac{\partial}{\partial t}T + \text{Div}X = 0, \tag{1.59}$$

(where Div stands for the spatial divergence) which holds for all u that solve (1.58); $T = T(x, t, u)$ and $X = X(x, t, u)$ are here functions of x, t and u, as well as the partial derivatives of u up to some order. Usually, T is said to be the **conserved density**, and X, the **conserved flow** [35].

The meaning of conservation law lies basically in the fact that if $u, X(x, t, u) \to 0$ sufficiently rapidly as $|x| \to \infty$, one derives in the case of one spatial dimension

$$\frac{d}{dt}\int_{-\infty}^{\infty} Tdx = 0, \tag{1.60}$$

or in the case of n spatial dimensions

$$\frac{d}{dt}\int_{\Omega} T dx = 0, \tag{1.61}$$

where $\Omega \subseteq \mathbb{R}^N$ is a domain in $\mathbb{R}^n$ (it could be $\Omega = \mathbb{R}^n$), which implies that

$$\int_{-\infty}^{\infty} T dx = \text{const}, \quad \text{or} \quad \int_{\Omega} T dx = \text{const}, \tag{1.62}$$

where the inetgral is therefore called a **conserved functional**. Hence a conserved functional yields a conservation law.

Below in this subsection we present the basic concepts related to symmetries according to the manner of [36].

DEFINITION 1.4.8 A function $S = S(x, u^{(k)}) \in \mathcal{A}^m$ is said to be a **symmetry** of the (1.53), provided S solves the linearized equation of (1.53), i.e.,

$$\frac{\partial}{\partial \varepsilon}(u + \varepsilon S)_t \Big|_{\varepsilon=0} = \frac{\partial}{\partial \varepsilon} K(u + \varepsilon S)\Big|_{\varepsilon=0} \quad \Rightarrow \quad \frac{dS}{dt} = K'(u)[S], \tag{1.63}$$

where d/dt stands for the total derivative with respect to t, and $K'(u)[S]$ is the Gateaux derivative defined above.

Obviously, the definition of symmetry can also be equivalently written as

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial u} u_t = \frac{\partial S}{\partial t} + S'(u)[K] = K'(u)[S] \quad \Rightarrow \quad \frac{\partial S}{\partial t} == [K, S]. \tag{1.64}$$

where $[K, S] := K'(u)[S] - S'(u)[K]$. It is advantageous to use this commutator $[K, S]$ ($K, S \in \mathcal{A}^m$) in some cases, because $\mathcal{A}^m$ is known to form a Lie algebra [38] with respect to this commutator.

Let $L(\mathcal{A}^m)$ be the set of linear operators that map $\mathcal{A}^m$ into itself. Let $\mathcal{U}$ be the set of differentiable operators $\Phi = \Phi(x, u)$ that map $\mathbb{R}^n \times \mathcal{A}^m$ into $L(\mathcal{A}^m)$. Thus $\Phi K = \Phi(x, u)K$ for $\Phi \in \mathcal{U}$ and $K \in \mathcal{A}^m$ is meant to be the action of the operator Φ upon K.

DEFINITION 1.4.9 Let $\Phi \in \mathcal{U}$ and $K \in \mathcal{A}^m$. One defines the **Lie derivative** $L_K \Phi \in \mathcal{U}$ of Φ with respect to K as

$$(L_K \Phi)S = \Phi[K, S] - [K, \Phi S], \quad \forall S \in \mathcal{A}^m. \tag{1.65}$$

DEFINITION 1.4.10 An operator $\Phi \in \mathcal{U}$ is said to be a **hereditary symmetry**, if

$$\Phi^2[K, S] + [\Phi K, \Phi S] - \Phi[K, \Phi S] - \Phi[\Phi K, S] = 0, \quad \forall K, S \in \mathcal{A}^m. \tag{1.66}$$

Evidently, an operator $\Phi \in \mathcal{U}$ is a hereditary symmetry if and only if

$$L_{\Phi K}\Phi = \Phi L_K \Phi, \quad \forall K \in \mathcal{A}^m. \tag{1.67}$$

Proof. Since $\forall S \in \mathcal{A}^m$, $(L_{\Phi K}\Phi)S = (\Phi L_K \Phi)S \Rightarrow \Phi[\Phi K, S] - [\Phi K, \Phi S] = \Phi(\Phi[K, S] - [K, \Phi S])$, which is simply a rewriting of (1.66). □

DEFINITION 1.4.11 If the action of an operator $\Phi \in \mathcal{U}$ upon one symmetry of (1.53) gives another symmetry of (1.53), then Φ is said to be a **strong symmetry** (or a recursion operator) of (1.53).

It is also easy to verify that an operator $\Phi \in \mathcal{U}$ is a recursion operator of (1.53), if and only if

$$\frac{\partial \Phi}{\partial t} + L_K \Phi = 0. \tag{1.68}$$

Moreover, one can also define

$$\Phi'[K]S = \frac{\partial}{\partial \varepsilon}\Phi(u + \varepsilon K)S\Big|_{\varepsilon=0}, \quad \forall S \in \mathcal{A}^m, \tag{1.69}$$

based on which one can derive

$$L_K \Phi = \Phi'[K] - [K', \Phi] = \Phi'[K] - K'\Phi + \Phi K'. \tag{1.70}$$

The condition for Φ to be a hereditary symmetry, eqn. (1.66), can thus also be equivalently formulated as

$$\Phi'[\Phi K]S - \Phi'[\Phi S]K - \Phi(\Phi'[K]S - \Phi'[S]K) = 0, \quad \forall K, S \in \mathcal{A}^m. \tag{1.71}$$

1.4.4 Hamiltonian structures

DEFINITION 1.4.12 Suppose $P, Q \in \mathcal{A}$. Define P to be **equivalent to** Q, denoted by $P \sim Q$, if there exists $R = (R_1, \ldots, R_n) \in \mathcal{A}^n$, such that $Q - P = \mathrm{div}R = \sum_{k=1}^{n} \partial_k R_k$. Obviously $\sim$ defines an **equivalence relation** on $\mathcal{A}$. Denote $\tilde{P} = \int P dx$ as the equivalence class that $P \in \mathcal{A}$ belongs to. One then defines the quotient space of $\mathcal{A}$ with respect to $\sim$, and denote it by $\mathcal{F}$, i.e. $\mathcal{F} := \mathcal{A}/\sim$.

In particular, provided that the functions in $\mathcal{A}$ that we are interested in are also all from the Schwartz space $\mathcal{S}$ [37], and P is a conserved density, i.e., $\int P dx$ is a conserved functional, then if $Q \sim P$, it follows directly that $\int (P - Q)dx = \int \mathrm{div} R dx = 0$ (since $R \in \mathcal{S}^n$); that is, Q is also a conserved density, and $\int Q dx$ $(= \int P dx)$ is a conserved functional as well since it possesses the same constant of motion as $\int P dx$ does.

Thus we can define also the **inner product**

$$(P,Q)=\int P^TQdx=\int\sum_{i=1}^{r}P_iQ_idx,$$

for $P=(P_1,\ldots,P_r)^T$, $Q=(Q_1,\ldots,Q_r)^T\in\mathcal{A}^r$.

DEFINITION 1.4.13 [38] Let $\mathcal{J}:\mathcal{A}^m\to\mathcal{A}^m$ be a linear operator. Let $P,Q\in\mathcal{F}$. The **Poisson bracket** of P and Q with respect to $\mathcal{J}$ is defined formally as

$$\{P,Q\}_{\mathcal{J}}=\int\delta P\cdot\mathcal{J}\delta Qdx. \tag{1.72}$$

The Poisson bracket defined above is indeed the infinite-dimensional generalization of the Poisson bracket applied in N-dimensional classical Hamiltonioan mechanics [38]:

$$\{f,g\}=\sum_{k=1}^{N}\left(\frac{\partial f}{\partial q_k}\frac{\partial g}{\partial p_k}-\frac{\partial f}{\partial p_k}\frac{\partial g}{\partial q_k}\right). \tag{1.73}$$

for two given functions $f(q_1,\ldots,q_N;p_1,\ldots,p_N;t)$ and $g(q_1,\ldots,q_N;p_1,\ldots,p_N;t)$. Obviously in both cases, the Poisson bracket is a binary operation; and its particular importance lies in the fact that mathematically, the time evolution of a Hamiltonian dynamical system is intimately related to the Poisson bracket.

DEFINITION 1.4.14 The **adjoint** $\mathcal{D}^*$ of a linear operator $\mathcal{D}:\mathcal{A}^m\to\mathcal{A}^m$ is given by

$$\int P\cdot\mathcal{D}Qdx=\int(\mathcal{D}^*P)\cdot Qdx,\quad\forall P,Q\in\mathcal{A}^m. \tag{1.74}$$

DEFINITION 1.4.15 [38] Let $\mathcal{J}:\mathcal{A}^m\to\mathcal{A}^m$ be a linear operator. If for all $\mathcal{P},\mathcal{Q},\mathcal{R}\in\mathcal{F}$, the Poisson bracket with respect to $\mathcal{J}$ satisfies always both
(i) the condition of skew-symmetry, i.e.,

$$\{\mathcal{P},\mathcal{Q}\}_{\mathcal{J}}=-\{\mathcal{Q},\mathcal{P}\}_{\mathcal{J}}; \tag{1.75}$$

and (ii) the Jacobi identity, i.e.,

$$\{\{\mathcal{P},\mathcal{Q}\}_{\mathcal{J}},\mathcal{R}\}_{\mathcal{J}}+\{\{\mathcal{R},\mathcal{P}\}_{\mathcal{J}},\mathcal{Q}\}_{\mathcal{J}}+\{\{\mathcal{Q},\mathcal{R}\}_{\mathcal{J}},\mathcal{P}\}_{\mathcal{J}}=\{\{\mathcal{P},\mathcal{Q}\}_{\mathcal{J}},\mathcal{R}\}_{\mathcal{J}}+\text{cycle}(\mathcal{P},\mathcal{Q},\mathcal{R})=0, \tag{1.76}$$

then $\mathcal{J}$ is said to be **Hamiltonian**.

It follows immediately from the above definitions that a Hamiltonian operator $\mathcal{J}$ must be skew-adjoint, i.e., $\mathcal{J}^*=-J$. Moreover, it is easy to derive that a skew-adjoint operator $\mathcal{J}$ is Hamiltonian if and only if for all $P,Q,R\in\mathcal{A}^m$,

$$\int P^T\mathcal{J}'(u)[\mathcal{J}Q]Rdx+\text{cycle}(P,Q,R)=0, \tag{1.77}$$

where $'(u)[.]$ denotes the Gateaux derivative defined above.

DEFINITION 1.4.16 A system of evolution equations is said to be **Hamiltonian**, provided it can be formulated in the form of

$$\frac{\partial u}{\partial t} = K[u] = \mathcal{J} \cdot \delta \mathcal{H}, \tag{1.78}$$

where $\mathcal{J}$ is a Hamiltonian operator and $\mathcal{H}$ is a Hamiltonian functional, i.e., $\mathcal{H} = \int H dx \in \mathcal{F}$.

DEFINITION 1.4.17 [44] Let G be a differential function. Define its **one-form** to be

$$dG := \sum_{k=1}^{n} \frac{\partial G}{\partial x^k} dx^k + \frac{\partial G}{\partial t} dt + \sum_{\alpha=1}^{m} \sum_{\#I \geq 0} \frac{\partial G}{\partial u_I^\alpha} du_I^\alpha, \tag{1.79}$$

where $u_I^\alpha = u^\alpha$ in case $\#I = 0$ (i.e., cardinality of I is 0); if the cardinality $\#I = l \geq 1$, it is understood that $u_I^\alpha = \frac{\partial^l u^\alpha}{\partial x^{i_1} \ldots \partial x^{i_l}}$ for $I = (i_1, \ldots, i_l)$, $1 \leq i_k \leq m$, $1 \leq k \leq l$.

Let J_∞ be the so-called **infinite jet space**, which is the conceptual extension of $(x, u^{(k)})$ by admitting $k \to \infty$ in the k-th prolongation $u^{(k)}$.

DEFINITION 1.4.18 [44] Let K be a set consisting of integer elements and $m > 0$ be an integer. We define a set of m-tuples of differential functions

$$\{G_k = (G_k^1, \ldots, G_k^m)^T | k \in K\}$$

to be **independent**, if all the m-tuples of one forms

$$\{dG_k = (dG_k^1, \ldots, dG_k^m)^T | k \in K\},$$

are linearly independent everywhere in the infinite jet space J_∞. A class of conserved functionals $\{\mathcal{H}_k\}_{k \in K}$ of a Hamiltonian system (1.78) is called independent, if all the characteristics $\{\mathcal{J} \frac{\delta \mathcal{H}_k}{\delta u}\}_{k \in K}$ of the Hamiltonian vector fields associated with $\mathcal{J}$ are independent.

Based on Definitions 1.4.17 and 1.4.18, one can now come up with:

DEFINITION 1.4.19 [38, 44] One claims that a Hamiltonian system (1.78) is **Liouville integrable**, if there is a sequence of conserved functionals $\{\mathcal{H}_k\}_{k=0}^{\infty}$ that are in involution pairwise, i.e., the Poisson bracket of any pair of conserved functionals from $\{\mathcal{H}_k\}_{k=0}^{\infty}$ associated with the Hamiltonian operator $\mathcal{J}$ is identical to 0:

$$\{\mathcal{H}_{k_1}, \mathcal{H}_{k_2}\}_{\mathcal{J}} = \int \Big(\frac{\delta \mathcal{H}_{k_1}}{\delta u}\Big)^T \mathcal{J} \frac{\delta \mathcal{H}_{k_2}}{\delta u} = 0, \quad k_1, k_2 \geq 0; \tag{1.80}$$

and the characteristics of the Hamiltonian vector fields associated with $\mathcal{J}$

$$K_k := \mathcal{J} \frac{\delta \mathcal{H}_k}{\delta u}, \quad k \geq 0, \tag{1.81}$$

are independent.

DEFINITION 1.4.20 [38] A pair of $m \times m$ matrix differential operators $\mathcal{J}$ and $\mathcal{M}$ is said to construct a **Hamiltonian pair** if $c_1\mathcal{J} + c_2\mathcal{M}$ is always a Hamiltonian for all $c_1, c_2 \in \mathbb{R}$.

If a system of evolution equations can be formulated as

$$u_t = K[u] = \mathcal{J}\delta\mathcal{H}_1 = \mathcal{M}\delta\mathcal{H}_0, \tag{1.82}$$

where $\mathcal{J}, \mathcal{M}$ forms a Hamiltonian pair, and both $\mathcal{H}_1$ and $\mathcal{H}_0$ are proper Hamiltonian functionals, then (1.82) is said to be a **bi-Hamiltonian system**.

F. Magri had first introduced the concepts of bi-Hamiltonian structures and systems [86], major conclusions about bi-Hamiltonian systems can be briefly summarized in the following theorem [38, 86]:

THEOREM 1.4.1 *[38] Let $\mathcal{J}$ and $\mathcal{M}$ be a Hamiltonian pair, and $\mathcal{J}$ is non-degenerate (i.e., if $\mathcal{J}$ is $\mathcal{A}^r \to \mathcal{A}^s$, and $\mathcal{D}$ is $\mathcal{A}^s \to \mathcal{A}$, then $\mathcal{D} \cdot \mathcal{J} = 0$ always implies $\mathcal{D} = 0$; the non-degeneracy guarantees the existence of $\mathcal{J}^{-1}$). Let*

$$u_t = K_1[u] = \mathcal{J}\delta\mathcal{H}_1 = \mathcal{M}\delta\mathcal{H}_2 \tag{1.83}$$

be the corresponding bi-Hamiltonian system of evolution equations, which generates the corresponding recursion operator $\mathcal{R} = \mathcal{M} \cdot \mathcal{J}^{-1}$. Let $K_0 = \mathcal{J}\delta\mathcal{H}_0$. With the aid of $\mathcal{R}$ we can recursively generate the sequence

$$K_n = \mathcal{R}K_{n-1}, \quad n \geq 1, \tag{1.84}$$

where K_{n-1} lies in the image of $\mathcal{R}$ for every $n \geq 1$ is assumed. Then there exists accordingly a sequence of functionals $\{\mathcal{H}_k\}_{k=1}^{\infty}$, such that

(i) the sequence

$$u_{t_n} = K_n[u] = \mathcal{J}\delta\mathcal{H}_n = \mathcal{M}\delta\mathcal{H}_{n-1} \tag{1.85}$$

($n \geq 1$) engenders a hierarchy of bi-Hamiltonian systems of evolution equations;

(ii) the corresponding evolutionary vector fields $\boldsymbol{v}_n = \boldsymbol{v}_{K_n}$ are all pairwise commutative, i.e.,

$$[\boldsymbol{v}_m, \boldsymbol{v}_n] = 0, \quad m, n \geq 0; \tag{1.86}$$

(iii) the Hamiltonian functionals in the sequence $\{\mathcal{H}_k\}_{k=1}^{\infty}$ are all pairwise in involution w.r.t. the Poisson brackets associated with both $\mathcal{J}$ and $\mathcal{M}$:

$$\{\mathcal{H}_m, \mathcal{H}_n\}_{\mathcal{J}} = \{\mathcal{H}_m, \mathcal{H}_n\}_{\mathcal{M}} = 0, \quad m, n \geq 0. \tag{1.87}$$

The sequence of Hamiltonian functionals $\{\mathcal{H}_k\}_{k=1}^{\infty}$ thus provides an infinite collection of conservation laws for each bi-Hamiltonian system $u_{t_k} = K_k (k \geq 1)$.

Chapter 2

Integrable Hamiltonian hierarchies with three potentials based on matrix spectral problems associated with sl$(2,\mathbb{R})$ and so$(3,\mathbb{R})$

2.1 Matrix spectral problems

A pretty matured and powerful mathematical tool for generating integrable soliton hierarchies is to start from a well-selected matrix spectral problem, construct the corresponding zero curvature formulation and solve it. The Korteweg-de Vries hierarchy [9], the Ablowitz-Kaup-Newell-Segur hierarchy [32], the Kaup-Newell hierarchy [50], the Dirac hierarchy [49] and the Heisenberg hierarchy [51], etc., are known as celebrated examples of soliton hierarchies that are generated by using this technique. This technique is also equipped with a well-developed Analysis toolkit for Hamiltonian structures in terms of the trace identity[52, 74], or in more general sense the variational identity[47, 48], through which it has been shown that all the above-mentioned soliton hierarchies are characterized by Hamiltonian or bi-Hamiltonian (or even multiple-Hamiltonian) structures, and are Liouville integrable[14, 39, 40].

We recall that in Chapter 1, a matrix spectral problem is informally defined as a first-order differential eigenvalue problem in the form of

$$\varphi_x = M(x, u^{(k)}, \lambda)\varphi,$$

where M is a square matrix whose entries are functions of x,u and derivatives of u w.r.t. x (up to a certain order k), as well as of a spectral parameter λ that is also the eigenvalue of this problem.

In the frame of integrable systems, the major purpose of solving matrix spectral problems is to find the key element—Lax pair which has been mentioned in Chapter 1. To put it more exactly, once given an appropriate spectral matrix $U(u,\lambda)$, in order to engender a soliton hierarchy, one has to correspondingly find a sequence of Lax matrices $\{V^{[k]}(u,\lambda)\}$, each of which together with U, constitute a Lax pair $(U, V^{[k]})$ that satisfies the zero curvature equation

$$U_{t_k} - V_x^{[k]} + [U, V^{[k]}] = 0.$$

This sequence of zero curvature equations represents in turn a sequence of nonlinear evolutationary equations, which, by running through some subsequent analysis (e.g. computing Hamiltonian structures), very

often can finally be demonstrated to be Liouville integrable or possess bi- or multi-Hamiltonian structures.

First of all, we shall review briefly the procedures for constructing soliton hierarchies by means of zero curvature equations [52, 74]. To begin with, we select an appropriate matrix loop algebra $\tilde{\mathfrak{g}}$ associated usually with a semisimple matrix Lie algebra $\mathfrak{g}$, with $[P, Q] = PQ - QP$ ($\forall P, Q \in \mathfrak{g}$) being the commutator, and consider a spatial matrix spectral problem based on $\tilde{\mathfrak{g}}$:

$$\varphi_x = U\varphi, \quad U = U(u, \lambda) \in \tilde{\mathfrak{g}}, \tag{2.1}$$

where λ is the spectral parameter, and U represents a matrix whose entries involve the potential functions of x and t. Our goal is to look for a solution expressed in terms of a Laurent expansion in λ

$$W = W(u, \lambda) = \sum_{k \geq 0} W_{0,k} \lambda^{-k}, \quad W_{0,k} \in \mathfrak{g}, \quad i \geq 0, \tag{2.2}$$

to the stationary zero curvature equation (i.e., by assuming $U_t = 0$ in the zero curvature equation)

$$W_x = [U, W]. \tag{2.3}$$

The solution W is important not only in computing the soliton hierarchy, but also plays key roles in the subsequent Hamiltonian analysis by means of trace or variational identity. As long as W is determined, we shall attempt to construct a sequence of matrix spectral problems with respect to t, that is, we formulate for all $m \geq 0$ the Lax matrices

$$V^{[m]} = V^{[m]}(u, \lambda) = (\lambda^m W)_+ + \delta_m \in \tilde{\mathfrak{g}}, \tag{2.4}$$

with $(\lambda^m W)_+$ denoting the polynomial part of $\lambda^m W$ in λ, to construct

$$\varphi_{t_m} = V^{[m]}\varphi = V^{[m]}(u, \lambda)\varphi, \quad m \geq 0, \tag{2.5}$$

in terms of the Lax matrices

$$V^{[m]} = V^{[m]}(u, \lambda) = (\lambda^m W)_+ + \delta_m \in \tilde{\mathfrak{g}}, \quad m \geq 0. \tag{2.6}$$

The δ_m introduced above represents a modification term which is used to correct the Lax matrices in case of necessity, to make sure that the zero curvature equations (i.e., the compatibility conditions of (2.1) and (2.5))

$$U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0, \quad m \geq 0, \tag{2.7}$$

will generate a soliton hierarchy

$$u_{t_m} = K_m(u), \quad m \geq 0. \tag{2.8}$$

Such a soliton hierarchy, in general, is characterized by the following Hamiltonian structures

$$u_{t_m} = K_m(u) = J\frac{\delta \mathcal{H}_m}{\delta u}, \quad m \geq 0, \tag{2.9}$$

where J is a Hamiltonian operator, and $\{\mathcal{H}_m\}_{m=0}^{\infty}$ is a sequence of Hamiltonian functionals that can in principle be determined using the trace identity [52, 74] if the associated matrix Lie algebras are semi-simple:

$$\frac{\delta}{\delta u}\int \mathrm{tr}\Big(W\frac{\partial U}{\partial \lambda}\Big)dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\mathrm{tr}\Big(W\frac{\partial U}{\partial u}\Big), \quad \gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\mathrm{tr}(W^2)|, \tag{2.10}$$

or through the variational identity [47, 48] provided the involved matrix Lie algebras are non-semi-simple:

$$\frac{\delta}{\delta u}\int \langle\frac{\partial U}{\partial \lambda}, W\rangle dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\langle\frac{\partial U}{\partial u}, W\rangle, \quad \gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\langle W, W\rangle|. \tag{2.11}$$

Here, $\langle ., .\rangle$ is an ad-invariant, nondegenerate and symmetric bilinear form defined on the underlying matrix loop algebra $\tilde{\mathfrak{g}}$. The Hamiltonian structures of the whole hierarchy are regarded as well established, as long as J and $\{\mathcal{H}_m\}_{m=0}^{\infty}$ can be worked out in a recursive manner. In most cases, the recursion structure revealed by such an analysis will lead to Liouville integrability, rather often to bi-Hamiltonian structures as well (could be even multi-Hamiltonian structures).

When restricted to the real field, there are only two 3-dimensional Lie algebras, i.e., the real special linear Lie algebra $\mathrm{sl}(2,\mathbb{R})$ and the real special orthogonal Lie algebra $\mathrm{so}(3,\mathbb{R})$. They have derived Lie algebras exactly identical to themselves (hence are also three-dimensional). In particular, $\mathrm{so}(3,\mathbb{R})$ offers us a different basis as well as a new starting point for constructing soliton hierarchies, and such works were therefore frequently reported in recent years [41, 42, 43, 45, 46]. This Lie algebra consists simply of all 3×3 antisymmetric real matrices, and is realized by simply using the following 3×3 matrices as the basis:

$$e_1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{2.12}$$

This basis has a set of beautiful cyclic commutators (so many people start liking to use this Lie algebra):

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2. \tag{2.13}$$

In addition, the basis of $\mathrm{sl}(2,\mathbb{R})$ is (i.e. $\mathrm{sl}(2,\mathbb{R})$ Lie algebra consists of all 2×2 real matrices with a vanishing trace)

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \tag{2.14}$$

with pairwise commutators being

$$[e_1, e_2] = 2e_2, \quad [e_1, e_3] = -2e_3, \quad [e_2, e_3] = e_1. \tag{2.15}$$

The Lie algebra $\mathrm{sl}(2, \mathbb{R})$ and $\mathrm{so}(3, \mathbb{R})$, when extended to be on the complex field $\mathbb{C}$, become $\mathrm{sl}(2, \mathbb{C})$ and $\mathrm{so}(3, \mathbb{C})$, respectively. The latter two can be proved to be isomorphic to one another, and are therefore the same Lie algebra. However, when restricted to the real field $\mathbb{R}$, $\mathrm{sl}(2, \mathbb{R})$ and $\mathrm{so}(3, \mathbb{R})$ are still pretty different things.

It has already been many years since $\mathrm{sl}(2, \mathbb{R})$ was applied to construction of soliton hierarchies by starting from matrix spectral problems [9, 32, 49, 50, 51, 52].

We introduce now the corresponding matrix loop algebras of $\mathrm{sl}(2, \mathbb{R})$ and $\mathrm{so}(3, \mathbb{R})$ as below, which will be called frequently in this chapter, i.e.:

$$\widetilde{\mathrm{sl}}(2, \mathbb{R}) = \left\{ \sum_{k \geq 0} L_k \lambda^{n-k} | L_k \in \mathrm{sl}(2, \mathbb{R}), k \geq 0, n \in \mathbb{Z} \right\} \tag{2.16}$$

and

$$\widetilde{\mathrm{so}}(3, \mathbb{R}) = \left\{ \sum_{k \geq 0} L_k \lambda^{n-k} | L_k \in \mathrm{so}(3, \mathbb{R}), k \geq 0, n \in \mathbb{Z} \right\}, \tag{2.17}$$

It is obvious that these two matrix loop algebras contain the following linear combinations:

$$c_1 \lambda^{k_1} e_1 + c_2 \lambda^{k_2} e_2 + c_3 \lambda^{k_3} e_3, \tag{2.18}$$

where k_1, k_2, $k_3 \in \mathbb{Z}$, and c_1, c_2, c_3 are real constants, and e_1, e_2, e_3 can be chosen as either (2.12) or (2.14).

We shall organize this chapter in the following way: in Section 2.2, somewhat enlightened by the Dirac hierarchy, we shall introduce a new matrix spectral problem associated with $\widetilde{\mathrm{sl}}(2, \mathbb{R})$, and proceed with computing the corresponding soliton hierarchy by means of the zero curvature equation; in Section 2.3, then, computations are performed for a formally identical matrix spectral problem whose loop algebra is but substituted by $\widetilde{\mathrm{so}}(3, \mathbb{R})$. Hamiltonian structures are computed by using the trace identity in these two sections to indicate that the two engendered soliton hierarchies are both Liouville integrable. Finally we make a few concluding remarks in Section 2.4.

2.2 A new soliton hierarchy based on the $\mathrm{sl}(2, \mathbb{R})$ Lie algebra

2.2.1 From spectral problem to soliton hierarchy

In this subsection we shall begin with a matrix spectral problem based on the $\mathrm{sl}(2, \mathbb{R})$ Lie algebra. That is, we bring forth a 2×2 matrix spectral problem

$$\varphi_x = U\varphi = U(u,\lambda)\varphi, \quad \text{where} \quad u = (p,q,r)^T \quad \text{and} \quad \varphi = (\varphi_1,\varphi_2)^T, \tag{2.19}$$

for which the spectral matrix $U \in \widetilde{\mathrm{sl}}(2,\mathbb{R})$ reads

$$U = U(u,\lambda) = pe_1 + (q+\lambda)e_2 + (r+\lambda)e_3 = \begin{bmatrix} p & q+\lambda \\ r+\lambda & -p \end{bmatrix}, \tag{2.20}$$

with the basis (e_1, e_2 and e_3) being given by (2.14). The mathematical profile U looks similar to the matrix spectral problem

$$U = U(u,\lambda) = pe_1 + (q+\lambda)e_2 + (q-\lambda)e_3 = \begin{bmatrix} p & q+\lambda \\ q-\lambda & -p \end{bmatrix} \tag{2.21}$$

of the Dirac hierarchy [49], of which a generalization for $\mathrm{so}(3,\mathbb{R})$ has been reported [43]. There are yet two fundamental distinctions between (2.20) and (2.21). Firstly, in (2.20) we get same signs "+" leading λ, but in (2.21) these signs are different; Secondly, in (2.20) there are 3 dependent variables p, q and r, however (2.21) has merely 2 variables p and q.

We shall follow the procedures given in Section 2.1 by beginning with solving the stationary zero curvature equation

$$W_x - [U,W] = 0, \quad W \in \widetilde{\mathrm{sl}}(2,\mathbb{R}). \tag{2.22}$$

We assume that W has the form

$$W = ae_1 + (b+c)e_2 + (b-c)e_3 = \begin{bmatrix} a & b+c \\ b-c & -a \end{bmatrix}, \tag{2.23}$$

and so the stationary zero curvature equation (2.22) gives

$$\begin{cases} a_x = -2\lambda c - (r-q)b - (r+q)c, \\ b_x = (r-q)a + 2pc, \\ c_x = -2a\lambda + 2pb - a(r+q). \end{cases} \tag{2.24}$$

Next we assume that a, b and c can be expressed in terms of Laurent expansions

$$a = \sum_{i\geq 0} a_i\lambda^{-i}, \quad b = \sum_{i\geq 0} b_i\lambda^{-i}, \quad c = \sum_{i\geq 0} c_i\lambda^{-i}, \tag{2.25}$$

and apply the initial values

$$a_0 = c_0 = 0, \quad b_0 = 1, \tag{2.26}$$

through a comparison of the coefficients of all powers of λ in (2.24), we arrive at:

$$\begin{cases} a_{i+1} = -\frac{1}{2}c_{i,x} + pb_i - \frac{1}{2}(r+q)a_i, \\ c_{i+1} = -\frac{1}{2}a_{i,x} - \frac{1}{2}(r-q)b_i - \frac{1}{2}(r+q)c_i, \qquad i \geq 0. \\ b_{i+1,x} = (r-q)a_{i+1} + 2pc_{i+1}, \end{cases} \tag{2.27}$$

We impose the following condition while proceeding with the above recursive calculation:

$$a_i|_{u=0} = b_i|_{u=0} = c_i|_{u=0} = 0, \quad i \geq 1, \tag{2.28}$$

such that the whole sequence of functions $\{a_i, b_i, c_i | i \geq 1\}$ can be uniquely determined. By running (2.27), we list here the three members that are leading this sequence as follows:

$$\begin{aligned} &a_1 = p, \quad b_1 = 0, \quad c_1 = -\frac{1}{2}(r-q); \\ &a_2 = \frac{1}{4}(r_x - q_x) - \frac{1}{2}p(r+q), \quad b_2 = \frac{1}{8}(r-q)^2 - \frac{1}{2}p^2, \quad c_2 = -\frac{1}{2}p_x + \frac{1}{4}(r^2 - q^2); \\ &a_3 = \frac{1}{4}p_{xx} + \frac{1}{8}p(3r^2 + 3q^2 + 2rq) - \frac{1}{2}p^3 - \frac{3}{8}(rr_x - qq_x) + \frac{1}{8}(rq_x - qr_x), \\ &b_3 = \frac{1}{2}p^2(r+q) - \frac{1}{4}p(r_x - q_x) + \frac{1}{4}p_x(r-q) - \frac{1}{8}(r+q)(r-q)^2, \\ &c_3 = -\frac{1}{8}(r_{xx} - q_{xx}) + \frac{1}{4}p(r_x + q_x) + \frac{1}{2}p_x(r+q) + \frac{1}{4}p^2(r-q) \\ &\qquad - \frac{1}{16}(r-q)(3r^2 + 3q^2 + 2rq). \end{aligned}$$

These data indicate that $\{a_1, b_1, c_1; a_2, b_2, c_2; a_3, b_3, c_3\}$ are all local (since they are all differential polynomials whose values at a specified point x_0 are completely determined by $u = (p, q)^T$ and its derivatives with respect to x in an arbitrarily small neighborhood of x_0), and may naturally suggest that the entire sequence of $\{a_i, c_i, b_i\}_{i=1}^{\infty}$ could be all local. This is actually the truth that can be justified by applying the method of mathematical induction as follows: First by following (2.22), and observing that both U and W are from $\widetilde{\mathrm{sl}}(2, \mathbb{R})$, we get (recalling that trace has the remarkable properties of $\mathrm{tr}(X+Y) = \mathrm{tr}X + \mathrm{tr}Y$, and $\mathrm{tr}(XY) = \mathrm{tr}(YX)$)

$$\begin{aligned} [\mathrm{tr}(W^2)]_x &= \mathrm{tr}[(W^2)_x] = \mathrm{tr}(WW_x + W_xW) \\ &= 2\mathrm{tr}(WW_x) = 2\mathrm{tr}(W[U, W]) \equiv 0. \end{aligned}$$

Hence $\mathrm{tr}(W^2)$ must be a constant. Noticing also that $\mathrm{tr}(W^2) = 2(a^2 + b^2 - c^2)$, and recalling (2.26), it follows

$$a^2 + b^2 - c^2 \equiv (a^2 + b^2 - c^2)|_{u=0} = 1. \tag{2.29}$$

Then we substitute a, b, c by their Laurent expressions, apply again the initial values (2.26), and through setting the coefficients of λ^{-i} in (2.29) to 0, we arrive at

$$b_i = -\frac{1}{2} \sum_{m+n=i, m,n\geq 1} (a_m a_n + b_m b_n - c_m c_n), \quad i \geq 1. \tag{2.30}$$

By keeping (2.30) in mind and recalling the first two recursion relations in (2.27), and directly applying the mathematical induction, it follows that for all $i \geq 1$, a_i, b_i, c_i appear in the face of differential polynomials of u, and thus, they must be all local. The mathematical induction indicates also that the differential order of a_i reads $i-1$, the differential order of b_i reads $i-2$, and the differential order of c_i reads $i-1$ ($i \geq 2$).

With the knowledge of $\{a_i, b_i, c_i | i \geq 1\}$ in hand, let $(\lambda^m W)_+$ denote the polynomial part of $\lambda^m W$ in λ, we thus arrive at

$$\begin{aligned}
&\frac{d}{dx}(\lambda^m W)_+ - [U, (\lambda^m W)_+] \\
&= \sum_{k=0}^{m} [a_{k,x} e_1 + (b_{k,x} + c_{k,x}) e_2 + (b_{k,x} - c_{k,x}) e_3] \lambda^{m-k} \\
&\quad - \left[p e_1 + (\lambda + q) e_2 + (\lambda + r) e_3, \sum_{k=0}^{m} \big(a_k e_1 + (b_k + c_k) e_2 + (b_k - c_k) e_3\big) \lambda^{m-k} \right] \\
&= \sum_{k=0}^{m} [a_{k,x} e_1 + (b_{k,x} + c_{k,x}) e_2 + (b_{k,x} - c_{k,x}) e_3] \lambda^{m-k} \\
&\quad - \sum_{k=0}^{m} 2p(b_k + c_k) \lambda^{m-k} e_2 + \sum_{k=0}^{m} 2p(b_k - c_k) \lambda^{m-k} e_3 + \sum_{k=0}^{m} 2(\lambda + q) a_k \lambda^{m-k} e_2 \\
&\quad - \sum_{k=0}^{m} (\lambda + q)(b_k - c_k) \lambda^{m-k} e_1 - \sum_{k=0}^{m} 2(\lambda + r) a_k \lambda^{m-k} e_3 + \sum_{k=0}^{m} (\lambda + r)(b_k + c_k) \lambda^{m-k} e_1 \\
&= \sum_{k=0}^{m} \big(a_{k,x} + 2\lambda c_k - (q - r) b_k + (q + r) c_k\big) \lambda^{m-k} e_1 \\
&\quad + \sum_{k=0}^{m} (b_{k,x} + c_{k,x} + 2\lambda a_k + 2q a_k - 2p b_k - 2p c_k) \lambda^{m-k} e_2 \\
&\quad + \sum_{k=0}^{m} (b_{k,x} - c_{k,x} - 2\lambda a_k - 2r a_k + 2p b_k - 2p c_k) \lambda^{m-k} e_3 \\
&= \sum_{k=0}^{m} \big(a_{k,x} + 2c_{k+1} - (q - r) b_k + (q + r) c_k\big) \lambda^{m-k} e_1 - 2c_{m+1} e_1 + 2c_0 \lambda^{m+1} e_1 \\
&\quad + \sum_{k=0}^{m} (b_{k,x} + c_{k,x} + 2a_{k+1} + 2q a_k - 2p b_k - 2p c_k) \lambda^{m-k} e_2 - 2a_{m+1} e_2 + 2a_0 \lambda^{m+1} e_2 \\
&\quad + \sum_{k=0}^{m} (b_{k,x} - c_{k,x} - 2a_{k+1} - 2r a_k + 2p b_k - 2p c_k) \lambda^{m-k} e_3 - 2a_{m+1} e_3 + 2a_0 \lambda^{m+1} e_3
\end{aligned}$$

$$= -2c_{m+1}e_1 - 2a_{m+1}e_2 + 2a_{m+1}e_3$$
$$= \begin{bmatrix} -2c_{m+1} & -2a_{m+1} \\ 2a_{m+1} & 2c_{m+1} \end{bmatrix}, \quad m \geq 0, \tag{2.31}$$

where

$$a_{k,x} + 2c_{k+1} - (q-r)b_k + (q+r)c_k = 0,$$
$$b_{k,x} + c_{k,x} + 2a_{k+1} + 2qa_k - 2pb_k - 2pc_k = 0,$$
$$b_{k,x} - c_{k,x} - 2a_{k+1} - 2ra_k + 2pb_k - 2pc_k = 0,$$

for all $k \geq 0$, which can be concluded from (2.27), are applied in the last several steps of derivation. This computation also convinces us why the initial data (2.26) were chosen this way.

This expression, $-2c_{m+1}e_1 - 2a_{m+1}e_2 + 2a_{m+1}e_3$, does not generally agree with $U_{t_m} = p_{t_m}e_1 + q_{t_m}e_2 + r_{t_m}e_3$, hence one must amend modification terms to fix the imbalance. We observe that

$$[U, e_2] = -(r+\lambda)e_1 + 2pe_2, \qquad [U, e_3] = (q+\lambda)e_1 - 2pe_3,$$

we thus employ a sequence of Lax operators that are amended by modification terms

$$V^{[m]} = (\lambda^m W)_+ + \delta_m, \qquad \text{with} \quad \delta_m = \beta b_{m+1}e_3 + \beta b_{m+1}e_2, \quad m \geq 0,$$

with β being an arbitrary constant. To this stage we obtain

$$V_x^{[m]} - [U, V^{[m]}]$$
$$= \begin{bmatrix} -2c_{m+1} + \beta(r-q)b_{m+1} & -2a_{m+1} + \beta b_{m+1,x} - 2\beta p b_{m+1} \\ 2a_{m+1} + \beta b_{m+1,x} + 2\beta p b_{m+1} & 2c_{m+1} - \beta(r-q)b_{m+1} \end{bmatrix},$$

and then, the associated zero curvature equations

$$U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0, \quad m \geq 0,$$

will generate a soliton hierarchy

$$u_{t_m} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}_{t_m} = K_m = \begin{bmatrix} -2c_{m+1} - \beta(q-r)b_{m+1} \\ -2a_{m+1} + \beta b_{m+1,x} - 2\beta p b_{m+1} \\ 2a_{m+1} + \beta b_{m+1,x} + 2\beta p b_{m+1} \end{bmatrix}, \quad m \geq 0. \tag{2.32}$$

Since the sequence $\{a_i, b_i, c_i | i \geq 1\}$ is known to be all local, as a consequence, it naturally follows that the entire hierarchy of soliton equations are all local. We list here only the first two nonlinear members:

$$u_{t_1} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}_{t_1} = K_1 = \begin{bmatrix} K_{1,a} \\ K_{1,b} \\ K_{1,c} \end{bmatrix},$$

where

$$\begin{aligned}
K_{1,a} &= p_x - \frac{1}{2}(r^2 - q^2) + \frac{1}{8}\beta(r-q)^3 - \frac{1}{2}\beta(r-q)p^2, \\
K_{1,b} &= -\frac{1}{2}(r_x - q_x) + p(r+q) + \frac{1}{4}\beta(r-q)(r_x - q_x) - \beta p p_x - \frac{1}{4}\beta p(r-q)^2 + \beta p^3, \\
K_{1,c} &= \frac{1}{2}(r_x - q_x) - p(r+q) + \frac{1}{4}\beta(r-q)(r_x - q_x) - \beta p p_x + \frac{1}{4}\beta p(r-q)^2 - \beta p^3,
\end{aligned}$$

and

$$u_{t_2} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}_{t_2} = K_2 = \begin{bmatrix} K_{2,a} \\ K_{2,b} \\ K_{2,c} \end{bmatrix},$$

where

$$\begin{aligned}
K_{2,a} = {} & \frac{1}{4}(r_{xx} - q_{xx}) - \frac{1}{2}p(r_x + q_x) - p_x(r+q) - \frac{1}{2}p^2(r-q) \\
& + \frac{1}{8}(r-q)(3r^2 + 3q^2 + 2rq) + \frac{1}{2}\beta p^2(r^2 - q^2) - \frac{1}{4}\beta p(r-q)(r_x - q_x) \\
& + \frac{1}{4}\beta p_x(r-q)^2 - \frac{1}{8}\beta(r+q)(r-q)^3,
\end{aligned}$$

$$\begin{aligned}
K_{2,b} = {} & -\frac{1}{2}p_{xx} - \frac{1}{4}p(3r^2 + 3q^2 + 2rq) + p^3 + \frac{3}{4}(rr_x - qq_x) - \frac{1}{4}(rq_x - qr_x) \\
& - \beta p^3(r+q) + \frac{1}{2}\beta p^2(r_x - q_x) - \frac{1}{2}\beta p p_x(r-q) + \frac{1}{4}\beta p(r+q)(r-q)^2 \\
& + \frac{1}{2}\beta(r_x + q_x)p^2 + (r+q)\beta p p_x - \frac{1}{4}\beta p(r_{xx} - q_{xx}) + \frac{1}{4}\beta p_{xx}(r-q) \\
& - \frac{1}{8}\beta(r-q)(3rr_x - q_x r + qr_x - 3qq_x),
\end{aligned}$$

$$\begin{aligned}
K_{2,c} = {} & \frac{1}{2}p_{xx} + \frac{1}{4}p(3r^2 + 3q^2 + 2rq) - p^3 - \frac{3}{4}(rr_x - qq_x) + \frac{1}{4}(rq_x - qr_x) \\
& + \beta p^3(r+q) - \frac{1}{2}\beta p^2(r_x - q_x) + \frac{1}{2}\beta p p_x(r-q) - \frac{1}{4}\beta p(r+q)(r-q)^2 \\
& + \frac{1}{2}\beta(r_x + q_x)p^2 + (r+q)\beta p p_x - \frac{1}{4}\beta p(r_{xx} - q_{xx}) + \frac{1}{4}\beta p_{xx}(r-q) \\
& - \frac{1}{8}\beta(r-q)(3rr_x + qr_x - q_x r - 3qq_x).
\end{aligned}$$

2.2.2 Analysis of Hamiltonian structures

The Hamiltonian structures of a soliton hierarchy typically has the following mathematical expression:

$$u_{t_k} = K_k(u) = J\frac{\delta \mathcal{H}_k}{\delta u}, \quad k \geq 0, \tag{2.33}$$

where J is a Hamiltonian operator, and $\{\mathcal{H}_k\}_{k=0}^{\infty}$ is a sequence of Hamiltonian functionals that, in principle, can be determined through the trace identity (2.10), or in general by using the variational identity (2.11).

To this stage, because the involved Lie algebra $\mathrm{sl}(2,\mathbb{R})$ is semisimple, for the soliton hierarchy (2.32) we shall employ the trace identity to produce Hamiltonian structures. As the first step, a quick differentiation directly gives

$$\frac{\partial U}{\partial \lambda} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \frac{\partial U}{\partial p} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \frac{\partial U}{\partial q} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \frac{\partial U}{\partial r} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

and

$$\begin{aligned} &\mathrm{tr}\Big(W\frac{\partial U}{\partial p}\Big) = 2a, \quad \mathrm{tr}\Big(W\frac{\partial U}{\partial q}\Big) = -(c-b), \\ &\mathrm{tr}\Big(W\frac{\partial U}{\partial r}\Big) = c+b, \quad \mathrm{tr}\Big(W\frac{\partial U}{\partial \lambda}\Big) = 2b. \end{aligned}$$

Insert all the above into the trace identity, and thus the identity is indeed equivalent to

$$\frac{\delta}{\delta u}\int 2b\,dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\begin{bmatrix} 2a \\ b-c \\ b+c \end{bmatrix}.$$

Balancing the coefficients that are leading all powers of λ, we have that for all $m \geq 0$,

$$\frac{\delta}{\delta u}\int 2b_{m+1}dx = (\gamma - m)\begin{bmatrix} 2a_m \\ b_m - c_m \\ b_m + c_m \end{bmatrix}.$$

Because γ is a constant, it can be determined by looking at a special case, for example, one picks $m = 1$ and concludes that it must be $\gamma = 0$; the variational identity thus become

$$\frac{\delta}{\delta u}\int \Big(-\frac{2b_{m+1}}{m}\Big)dx = \begin{bmatrix} 2a_m \\ b_m - c_m \\ b_m + c_m \end{bmatrix}, \quad m \geq 1.$$

Naturally it follows that one can take the Hamiltonian functionals to be

$$\mathcal{H}_m = \int \Big(-\frac{2b_{m+1}}{m}\Big)dx, \quad m \geq 1. \tag{2.34}$$

Recalling (2.33) and particularly that $b_{m+1,x} = 2pc_{m+1} + (r-q)a_{m+1}$, we arrive at

$$K_m = J \begin{bmatrix} 2a_{m+1} \\ b_{m+1} - c_{m+1} \\ b_{m+1} + c_{m+1} \end{bmatrix} = \begin{bmatrix} -2c_{m+1} + (r-q)\beta b_{m+1} \\ -2a_{m+1} - 2p\beta b_{m+1} + \beta b_{m+1,x} \\ 2a_{m+1} + 2p\beta b_{m+1} + \beta b_{m+1,x} \end{bmatrix}, \tag{2.35}$$

where

$$J = \begin{bmatrix} 0 & 1 + \frac{1}{2}(r-q)\beta & -1 + \frac{1}{2}(r-q)\beta \\ -1 - \frac{1}{2}(r-q)\beta & \beta\partial & \beta\partial - 2\beta p \\ 1 - \frac{1}{2}(r-q)\beta & \beta\partial + 2\beta p & \beta\partial \end{bmatrix}. \tag{2.36}$$

A rather routine computation (could be performed with Maple) reveals that the Poisson bracket of J is anti-symmetric and satisfies the Jacobi identity, therefore the operator J is Hamiltonian. Therefore the Hamiltonian structures of the hierarchy (2.32) can thus be written down as:

$$u_{t_m} = K_m = J\frac{\delta \mathcal{H}_{m+1}}{\delta u}, \quad m \geq 0, \tag{2.37}$$

with the Hamiltonian operator J and the Hamiltonian functionals $\mathcal{H}_m$'s being explicitly given by (2.36) and (2.34), respectively.

According to Definitions 1.4.17–1.4.19 (whose statements effectively take advantages of the characteristics of the associated Hamiltonian vector fields), we must show two properties before we can claim that a soliton hierarchy is Liouville integrable: (i) the infinitely many conserved functionals are pairwise commutative; and (ii) these conserved functionals have characteristics that are functionally independent. For our soliton hierarchy (2.32), the fact that the conserved functionals are commutative is ensured by their zero curvature representations that actually form a Virasoro algebra which is commutative (refer to related literatures [53, 54, 55, 56]). We should emphasize that all these Hamiltonian functionals actually correspond to common conservation laws shared by every soliton equation in the soliton hierarchy (2.32).

Keeping in mind also the Hamiltonian structures from (2.37) and the distinct differential orders of the sequence $\{a_i, b_i, c_i | i \geq 1\}$, the differential recursion structure of the hierarchy thus tells us (because clearly, a group of differential functions is said to be independent, provided that the differential orders of its members are all pairwise distinct), the corresponding characteristics of the conserved functionals are functionally independent. To summarize, the soliton hierarchy (2.32) is Liouville integrable in accordance with the Definition 1.4.19. To put it more accurately, all equations in the soliton hierarchy (2.32) have an infinite

number of independent conserved functionals that pairwise commute in the sense of (1.80), and an infinite number of generalized symmetries that pairwise commute in the sense of:

$$[K_n, K_l] := K_n'(u)[K_l] - K_l'(u)[K_n] = J\frac{\delta\{\mathcal{H}_n, \mathcal{H}_l\}_J}{\delta u} = 0, \qquad n, l \geq 0, \tag{2.38}$$

with K' standing for the Gateaux derivative of K.

2.3 An so$(3, \mathbb{R})$-counterpart

2.3.1 From spectral problem to soliton hierarchy

We continue in this section to present an so$(3, \mathbb{R})$-counterpart of the matrix spectral problem in Section 2.2. We again begin with a matrix spectral problem:

$$\varphi_x = U\varphi = U(u, \lambda)\varphi, \qquad \text{where} \quad u = (p, q, r)^T \quad \text{and} \quad \varphi = (\varphi_1, \varphi_2, \varphi_3)^T, \tag{2.39}$$

(thus the wave function φ accordingly has 3 components) for which the spectral matrix U belonging to $\widetilde{\text{so}}(3, \mathbb{R})$ reads

$$U = U(u, \lambda) = pe_1 + (q + \lambda)e_2 + (r + \lambda)e_3 = \begin{bmatrix} 0 & -r - \lambda & -p \\ r + \lambda & 0 & -q - \lambda \\ p & q + \lambda & 0 \end{bmatrix}, \tag{2.40}$$

it differs however from that in (2.20) because the basis is that of so$(3, \mathbb{R})$ given in (2.12). This matrix spectral problem has 3 dependent variables, too.

Just like what we have done in Section 2.2, we first work on the stationary zero curvature equation

$$W_x - [U, W] = 0, \qquad W \in \widetilde{\text{so}}(3, \mathbb{R}). \tag{2.41}$$

Similarly, by assuming W to be

$$W = ae_1 + (b + c)e_2 + (b - c)e_3 = \begin{bmatrix} 0 & -b + c & -a \\ b - c & 0 & -b - c \\ a & b + c & 0 \end{bmatrix}, \tag{2.42}$$

the stationary zero curvature equation $[U, W] - W_x = 0$ yields

$$\begin{cases} a_x = -(r - q)b - (r + q + 2\lambda)c, \\ b_x = \frac{1}{2}(r - q)a + pc, \\ c_x = \frac{1}{2}(r + q + 2\lambda)a - pb. \end{cases} \tag{2.43}$$

Again by applying the assumption that a, b and c are Laurent expansions in λ

$$a = \sum_{i\geq 0} a_i \lambda^{-i}, \quad b = \sum_{i\geq 0} b_i \lambda^{-i}, \quad c = \sum_{i\geq 0} c_i \lambda^{-i}, \tag{2.44}$$

and employing the initial conditions

$$a_0 = c_0 = 0, \quad b_0 = 1, \tag{2.45}$$

we obtain through balancing the coefficients leading all powers of λ

$$\begin{cases} a_{i+1} = pb_i - \frac{1}{2}(r+q)a_i + c_{i,x}, \\ c_{i+1} = -\frac{1}{2}(r-q)b_i - \frac{1}{2}(r+q)c_i - \frac{1}{2}a_{i,x}, \qquad i \geq 0. \\ b_{i+1,x} = +\frac{1}{2}(r-q)a_{i+1} + pc_{i+1}, \end{cases} \tag{2.46}$$

Note that we still use the following condition while running the above recursive steps:

$$a_i|_{u=0} = b_i|_{u=0} = c_i|_{u=0} = 0, \quad i \geq 1, \tag{2.47}$$

such that the whole sequence $\{a_i, b_i, c_i | i \geq 1\}$ is worked out in a unique manner. Upon running (2.46), the first three members of $\{a_i, b_i, c_i | i \geq 1\}$ are computed and listed below:

$$\begin{aligned} &a_1 = p, \quad b_1 = 0, \quad c_1 = -\frac{1}{2}(r-q); \\ &a_2 = -\frac{1}{2}(r_x - q_x) - \frac{1}{2}p(r+q), \quad b_2 = -\frac{1}{8}(r-q)^2 - \frac{1}{4}p^2, \quad c_2 = -\frac{1}{2}p_x + \frac{1}{4}(r^2 - q^2); \\ &a_3 = -\frac{1}{2}p_{xx} + \frac{1}{8}p(r^2+q^2+6rq) - \frac{1}{4}p^3 + \frac{3}{4}(rr_x - qq_x) - \frac{1}{4}(rq_x - qr_x), \\ &b_3 = \frac{1}{4}p^2(r+q) - \frac{1}{4}p_x(r-q) + \frac{1}{4}p(r_x - q_x) - \frac{1}{8}(r+q)(r-q)^2, \\ &c_3 = \frac{1}{4}(r_{xx} - q_{xx}) + \frac{1}{4}p(r_x + q_x) + \frac{1}{2}p_x(r+q) + \frac{1}{8}p^2(r-q) \\ &\qquad - \frac{1}{16}(r-q)(r^2+q^2+6rq). \end{aligned}$$

Likewise, by running mathematical induction as what we have done in Section 2.2, we can again prove the localness of the sequence $\{a_i, b_i, c_i | i \geq 1\}$, and that the differential order of a_i reads $i-1$, the differential order of b_i reads $i-2$, and the differential order of c_i reads $i-1$ ($i \geq 2$).

Upon having the results of $\{a_i, b_i, c_i | i \geq 1\}$ that are computed recursively, next we calculate (we omit here the lengthy intermediate steps that we have presented for sake of clarity in Sec 2.2)

$$\begin{aligned} &\frac{d}{dx}(\lambda^m W)_+ - [U, (\lambda^m W)_+] \\ &\qquad = -2c_{m+1}e_1 + a_{m+1}e_2 - a_{m+1}e_3 \\ &\qquad = \begin{bmatrix} 0 & a_{m+1} & 2c_{m+1} \\ -a_{m+1} & 0 & -a_{m+1} \\ -2c_{m+1} & a_{m+1} & 0 \end{bmatrix}, \quad m \geq 0. \end{aligned} \tag{2.48}$$

This however does not agree generally with the shape of $U_{t_m} = p_{t_m}e_1 + q_{t_m}e_2 + r_{t_m}e_3$, and hence modification terms again have be introduced to correct the Lax operators. Likewise by noticing

$$[U, e_2] = -(r+\lambda)e_1 + pe_3, \qquad [U, e_3] = (q+\lambda)e_1 - pe_2,$$

we choose a sequence of Lax operators corrected by modification terms

$$V^{[m]} = (\lambda^m W)_+ + \delta_m, \qquad \text{with } \delta_m = \beta b_{m+1}e_3 + \beta b_{m+1}e_2, \quad m \geq 0,$$

where β is an arbitrary constant. To this stage we obtain

$$V_x^{[m]} - [U, V^{[m]}] = \begin{bmatrix} 0 & -Z_3 & -Z_1 \\ Z_3 & 0 & -Z_2 \\ Z_1 & Z_2 & 0 \end{bmatrix},$$

where

$$\begin{aligned} Z_1 &= -2c_{m+1} + \beta(r-q)b_{m+1}, \\ Z_2 &= a_{m+1} + \beta p b_{m+1} + \beta b_{m+1,x}, \\ Z_3 &= -a_{m+1} - \beta p b_{m+1} + \beta b_{m+1,x}, \end{aligned}$$

and then, the associated zero curvature equations

$$U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0, \quad m \geq 0,$$

will generate a hierarchy of solition equations

$$u_{t_m} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}_{t_m} = K_m = \begin{bmatrix} -2c_{m+1} + \beta(r-q)b_{m+1} \\ a_{m+1} + \beta p b_{m+1} + \beta b_{m+1,x} \\ -a_{m+1} - \beta p b_{m+1} + \beta b_{m+1,x} \end{bmatrix}, \quad m \geq 0. \tag{2.49}$$

Since the sequence of functions $\{a_i, b_i, c_i | i \geq 1\}$ is known to be all local, as a consequence, we can therefore claim with confidence that the whole hierarchy of soliton equations are local. Again we write down here the very first two nonlinear members from this hierarchy:

$$u_{t_1} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}_{t_1} = K_1 = \begin{bmatrix} K_{1,a} \\ K_{1,b} \\ K_{1,c} \end{bmatrix},$$

where

$$
\begin{aligned}
K_{1,a} &= p_x - \frac{1}{2}(r^2 - q^2) - \frac{1}{8}\beta(r-q)^3 - \frac{1}{4}\beta(r-q)p^2, \\
K_{1,b} &= -\frac{1}{2}(r_x - q_x) - \frac{1}{2}p(r+q) - \frac{1}{4}\beta(r-q)(r_x - q_x) - \frac{1}{2}\beta p p_x \\
&\quad - \frac{1}{8}\beta p(r-q)^2 - \frac{1}{4}\beta p^3, \\
K_{1,c} &= \frac{1}{2}(r_x - q_x) + \frac{1}{2}p(r+q) - \frac{1}{4}\beta(r-q)(r_x - q_x) - \frac{1}{2}\beta p p_x \\
&\quad + \frac{1}{8}\beta p(r-q)^2 + \frac{1}{4}\beta p^3,
\end{aligned}
$$

and

$$
u_{t_2} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}_{t_2} = K_2 = \begin{bmatrix} K_{2,a} \\ K_{2,b} \\ K_{2,c} \end{bmatrix},
$$

where

$$
\begin{aligned}
K_{2,a} &= -\frac{1}{2}(r_{xx} - q_{xx}) - \frac{1}{2}p(r_x + q_x) - p_x(r+q) - \frac{1}{4}p^2(r-q) \\
&\quad + \frac{1}{8}(r-q)(r^2 + q^2 + 6rq) + \frac{1}{4}\beta p^2(r^2 - q^2) + \frac{1}{4}\beta p(r-q)(r_x - q_x) \\
&\quad - \frac{1}{4}\beta p_x(r-q)^2 + \frac{1}{8}\beta(r-q)^3(r+q),
\end{aligned}
$$

$$
\begin{aligned}
K_{2,b} &= -\frac{1}{2}p_{xx} + \frac{1}{8}p(r^2 + q^2 + 6rq) - \frac{1}{4}p^3 + \frac{3}{4}(rr_x - qq_x) - \frac{1}{4}(rq_x - qr_x) \\
&\quad + \frac{1}{4}\beta p^3(r+q) - \frac{1}{4}\beta p p_x(r-q) + \frac{1}{4}\beta p^2(r_x - q_x) + \frac{1}{8}\beta p(r+q)(r-q)^2 \\
&\quad + \frac{1}{4}\beta(r_x + q_x)p^2 + \frac{1}{2}\beta p p_x(r+q) + \frac{1}{4}\beta p(r_{xx} - q_{xx}) - \frac{1}{4}\beta p_{xx}(r-q) \\
&\quad + \frac{1}{8}\beta(r-q)(3rr_x - rq_x + r_x q - 3qq_x),
\end{aligned}
$$

$$
\begin{aligned}
K_{2,c} &= \frac{1}{2}p_{xx} - \frac{1}{8}p(r^2 + q^2 + 6rq) + \frac{1}{4}p^3 - \frac{3}{4}(rr_x - qq_x) + \frac{1}{4}(rq_x - qr_x) \\
&\quad - \frac{1}{4}\beta p^3(r+q) + \frac{1}{4}\beta p p_x(r-q) - \frac{1}{4}\beta p^2(r_x - q_x) - \frac{1}{8}\beta p(r+q)(r-q)^2 \\
&\quad + \frac{1}{4}\beta(r_x + q_x)p^2 + \frac{1}{2}\beta p p_x(r+q) + \frac{1}{4}\beta p(r_{xx} - q_{xx}) - \frac{1}{4}\beta p_{xx}(r-q) \\
&\quad + \frac{1}{8}\beta(r-q)(3rr_x - rq_x + r_x q - 3qq_x).
\end{aligned}
$$

2.3.2 Analysis of Hamiltonian structures

We shall follow the routine steps to perform Hamiltonian analysis as we did in subsection 2.2.2. First, it comes up straightforwardly with

$$\frac{\partial U}{\partial \lambda} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \frac{\partial U}{\partial p} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$
$$\frac{\partial U}{\partial q} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \frac{\partial U}{\partial r} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and

$$\mathrm{tr}\Big(W\frac{\partial U}{\partial p}\Big) = -2a, \quad \mathrm{tr}\Big(W\frac{\partial U}{\partial q}\Big) = -2(c+b),$$
$$\mathrm{tr}\Big(W\frac{\partial U}{\partial r}\Big) = 2(c-b), \quad \mathrm{tr}\Big(W\frac{\partial U}{\partial \lambda}\Big) = -4b.$$

The trace identity then connects these results to give

$$\frac{\delta}{\delta u}\int 2b\,dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\begin{bmatrix} a \\ b+c \\ b-c \end{bmatrix}.$$

Balancing now the coefficients leading all powers of λ, we arrive at

$$\frac{\delta}{\delta u}\int 2b_{m+1}dx = -(m-\gamma)\begin{bmatrix} a_m \\ b_m + c_m \\ b_m - c_m \end{bmatrix}, \quad m \geq 0.$$

Just like in section 2.2.2, γ can be determined by working on a special case, for example, one picks $m = 1$ and works out $\gamma = 0$. The variational identity thus gives rise to

$$\frac{\delta}{\delta u}\int\Big(-\frac{2b_{m+1}}{m}\Big)dx = \begin{bmatrix} a_m \\ b_m + c_m \\ b_m - c_m \end{bmatrix}, \quad m \geq 1.$$

This indicates that we can choose the Hamiltonian functionals to be

$$\mathcal{H}_m = \int -\frac{2b_{m+1}}{m}dx, \quad m \geq 1. \tag{2.50}$$

Recalling that $b_{m+1,x} = pc_{m+1} + \frac{r-q}{2}a_{m+1}$, we can now present the Hamiltonian structures for (2.49)

$$K_m = J \begin{bmatrix} a_{m+1} \\ b_{m+1} + c_{m+1} \\ b_{m+1} - c_{m+1} \end{bmatrix} = \begin{bmatrix} -2c_{m+1} + (r-q)\beta b_{m+1} \\ a_{m+1} + p\beta b_{m+1} + \beta b_{m+1,x} \\ -a_{m+1} - p\beta b_{m+1} + \beta b_{m+1,x} \end{bmatrix}, \tag{2.51}$$

where

$$J = \begin{bmatrix} 0 & -1 + \frac{1}{2}\beta(r-q) & 1 + \frac{1}{2}\beta(r-q) \\ 1 - \frac{1}{2}\beta(r-q) & \beta\partial & \beta\partial + p\beta \\ -1 - \frac{1}{2}\beta(r-q) & \beta\partial - \beta p & \beta\partial \end{bmatrix} \tag{2.52}$$

is also a Hamiltonian operator since both the anti-symmetry and the Jacobi identity are satisfied. The Hamiltonian structures of the hierarchy (2.49) can therefore be formulated as:

$$u_{t_m} = K_m = J\frac{\delta \mathcal{H}_{m+1}}{\delta u}, \quad m \geq 0, \tag{2.53}$$

with J and the Hamiltonian functionals $\mathcal{H}_m$'s being given by (2.50) and (2.52), respectively. By following what we have argued at the end of Section 2.2.2, once again we arrive at the conclusion that the soliton hierarchy (2.49) is Liouville integrable.

2.4 Concluding remarks

In this chapter, by starting from two spectral problems associated with the matrix loop algebras $\widetilde{\mathrm{sl}}(2,\mathbb{R})$ and $\widetilde{\mathrm{so}}(3,\mathbb{R})$, we have engendered two soliton hierarchies that are known to be integrable. The analysis of Hamiltonian structures via the trace identity reveals that, all the members created recursively in the two hierarchies of soliton equations are Liouville integrable.

The systems reported in this chapter provide two new examples of soliton hierarchies carrying three potential variables. We expect to find more interesting and convincing examples of such soliton hierarchies which have three, or even four to five dependent variables. That certainly requires insightful understandings and smart applications of, for instance, computer algebras, trace identities as well as variational identities.

Using computer algebras (Maple, Mathematica, etc.) to automate the whole procedures of engendering a soliton hierachy by starting from a matrix spectral problem (hopefully including the computation of Hamiltonian structures as well) may greatly increase the efficiency of the mathematicians' work in this field. There are certainly many subtle details that can challenge the software's intelligence, such as judging whether a modification term is needed or not and how to determine it, or drawing the conclusion about the Liouville integrability, etc. Therefore it would be more wise to make the software into something that mathematicians can intervene (if absolutely necessary) at some critical steps.

Plenty of matrix spectral problems in higher-order were also reported to be able to yield soliton hierarchies [61, 62, 63, 64, 65, 66, 67]. At the end of this chapter, we'd also like to mention that integrable couplings associated with enlarged matrix loop algebras [68, 69, 70, 71] is a potential powerful tool that can help a lot in enriching specific examples of soliton hierarchies.

Chapter 3

A type of Hamiltonian operators with two potentials and their integrable hierarchies

3.1 Introduction

DEFINITION 3.1.1 Let $u = (u^1, \ldots, u^m)^T$ ($m > 0$ is an integer) be a vector potential. A u-dependent operator $A[u] = A[(u^1, \ldots, u^m)^T]$ is said to be u-**linear**, provided $A[cu] = cA[u]$, where c is an arbitrary real constant, always holds.

Many matrix spectral problems can yield Hamiltonian operators that have one or several components which are u-linear and are very often themselves Hamiltonian, too, in their Hamiltonian structures. In particular when $m = 2$ and a simplified notation $u = (p, q)^T$ is frequently applied in this case, such u-linear Hamiltonian components can be encountered in many soliton hierarchies explored during recent decades, for example, the generalized WKI hierarchy [72], the coupled Burgers hierarchy [73], etc. Experienced mathematicians working in this field are always very careful about such a fact: that is, the original matrix spectral problems of some soliton hierarchies may have very different faces, but can be eventually verified to be mutually convertible in terms of Bäcklund type transformations [75, 76, 77, 78, 79]. As a part of this chapter, we shall discuss several u-linear Hamiltonian operators and argue via mathematical proofs that there exist Bäcklund type transformation among these operators, hence they can be viewed as Gauge equivalent. These Hamiltonian operators are frequently encountered in some integrable systems, but it seems the implications of the Gauge equivalence therein are still not sufficiently understood and realized by people.

Mathematicians of integrable theory are always also interested in constructing new Hamiltonian operators using those existing ones. A rather well-known and effective method is to couple some u-linear Hamiltonian operators with matrix differential operators with constant coefficients (we shall for brevity call the latter constant matrix differential operators in this chapter) to create new Hamiltonian operators [81]. People thought that such kind of couplings might be working at very high (perhaps infinite) differential orders, and in those cases the underlying mathematics is expected to extremely complicated. We shall explore this phenomenon in this chapter for a specific class of u-linear Hamiltonian operators, and will argue that these couplings could be effective only up to a finite k-th differential order, addressed to be $k = 3$ for this study.

Recall now most of the notations given at the beginning of Sec. 1.4. Recall also that $\mathcal{A}$ is defined to be the space of all smooth local functions $f(x, u^{(k)})$ (k is a nonnegative integer). We'd like to define furthermore $\mathcal{B}$ as the space of smooth functions $f(x, u^{(k)})$, with the only distinction that we allow the functions in $\mathcal{B}$ to be nonlocal. For example $\int_{-2}^{x^4} x'^4 \partial_3 u^1 dx'^4$ is acceptable in $\mathcal{B}$, however not in $\mathcal{A}$–notice but that the superscript 4 of x' and x here refers to the fourth component of x, not the power. The r-th power direct products of $\mathcal{A}$ and $\mathcal{B}$ can thus be defined as

$$\mathcal{A}^r = \{(f_1, \ldots, f_r)^T | f_i \in \mathcal{A}, 1 \leq i \leq r\}, \quad \mathcal{B}^r = \{(f_1, \ldots, f_r)^T | f_i \in \mathcal{B}, 1 \leq i \leq r\}, \tag{3.1}$$

respectively.

Now the definitions of Gateaux derivative, equivalence and equivalence class of functions in $\mathcal{A}$, the action of linear operators and their adjoints upon $\mathcal{A}$, Hamiltonian operators and Hamiltonian pair can all be generalized to $\mathcal{B}$ or $\mathcal{B}^r$ without too much difficulties.

Though we have already some vague ideas about Bäcklund transformation in Chapter 1 (Sec. 1.3.1), we'd like to review it in a more strict mathematical language, by using (as we think more compatible for this chapter) the definition from Fuchssteiner and Fokas [77]:

DEFINITION 3.1.2 Let $B(u, \tilde{u})$ be a function of two sets of arguments $u \in M$ and $\tilde{u} \in \tilde{M}$ with function values in a vector space S', where M and $\tilde{M}$ are two manifolds with their fibers of tangent bundles being S and $\tilde{S}$, respectively (generally, S and $\tilde{S}$ are different from S'). Let B_u and $B_{\tilde{u}}$ denote the partial derivatives of B w.r.t. u and $\tilde{u}$, respectively; B is said to be **admissible** provided the implicit function defined by $B(u, \tilde{u}) = 0$ establishes a one-to-one map between the corresponding tangent spaces; i.e., it is required that for $B = 0$ the linear maps B_u and $B_{\tilde{u}}$ from S to S' and $\tilde{S}$ to S' are both invertible.

Let

$$u_t = K[u], \quad u \in M, \tag{3.2a}$$

$$\tilde{u}_t = G[\tilde{u}], \quad \tilde{u} \in \tilde{M} \tag{3.2b}$$

be two systems of evolution equations. We call an admissible function $B(u, \tilde{u})$ (or the implicit function relating u and $\tilde{u}$ given by $B(u, \tilde{u}) = 0$) a **Bäcklund transformation** between the equations (3.2a) and (3.2b), if $B(u(t), \tilde{u}(t)) \equiv 0$ for all t, whenever $B(u(0), \tilde{u}(0)) = 0$.

The above definition implies that, Bäcklund transformations are connections between partial differential equations and their solutions. The two functions u and $\tilde{u}$ solve (3.2a) and (3.2b), respectively, thus when $B(u, \tilde{u})$ is admissible, either u or $\tilde{u}$ is said to be a Bäcklund transformation of the other.

A Bäcklund transformation may often relate a solution of a PDE (in which we are interested) to a solution of another PDE which we know better or which is easier to solve. In particular, a Bäcklund transformation that maps between solutions of the same PDE is said to be an **auto-Bäcklund transformation**. If an auto-Bäcklund transformation of a PDE can be found, then it could be effectively applied in mapping existing solutions of this PDE to some new solutions which are probably unknown. Therefore Bäcklund transformation is a powerful tool in searching for solutions to PDEs, especially those PDEs in soliton theory and integrable systems.

In general, Bäcklund transformations are viewed as a type of Gauge transformations [80]. Therefore two systems linked by a Bäcklund transformation are usually also considered in this sense to be Gauge equivalent to each other. So throughout this chapter, the terminology "Gauge equivalence" is used to indicate that two systems can be mapped into each other via a Gauge transformation in Bäcklund type.

To be as simple as possible, in this chapter unless otherwise stated, below we always set $n = 1$ and $m = 2$; i.e., we shall apply $u = u(x,t) = (p,q)^T$ (where p, q are functions of x and t). Also, we define $\partial = \frac{d}{dx}$, $\partial^{-1} = \int dx$, where in ∂^{-1} the involved constant of integration is properly selected to make sure that $\partial\partial^{-1} = \partial^{-1}\partial = 1$ always holds. Moreover, below in this chapter, we let $\partial^m g$ denote the m-th order derivative of g w.r.t. x, except sometimes when $m = 1$ or 2, for convenience we use g_x or g_{xx} instead.

Let g be a function of x, t and u which is smooth enough. Let $k, l \geq 0$ be integers. Based on a quick and direct computation via integration by parts one can conclude that,

$$x^k\partial^l g \sim \begin{cases} 0, & \text{if } k < l, \\ (-1)^l k(k-1)\dots(k-l+1)x^{k-l}g, & \text{if } k \geq l; \end{cases}$$

where $\sim$ means the equivalent relation in the sense of Definition 1.4.12. In particular, $x^k\partial^k g \sim (-1)^k k!g$. We shall call this conclusion from time to time in this chapter.

This chapter runs as follows: We investigate in Sec.3.2 a class of matrix differential Hamiltonian operators H and the Gauge equivalence among them. Then we study the coupling of H to constant matrix differential operators in Sec.3.3, where it got proved that this kind of coupling no longer takes effect at differential orders $k \geq 4$. Finally in Sec.3.4 several illustrative examples are presented so that the readers might catch the ideas how the Gauge equivalence elucidated in Sec.3.2 actually works. We end this chapter with a couple of concluding remarks.

3.2 The Gauge equivalence of a class of Hamiltonian matrix differential operators

Ma [81] proved that matrix u-linear differential operators in the profile of

$$H_1 = \begin{bmatrix} p\partial + \partial p & q\partial \\ \partial q & 0 \end{bmatrix}, \tag{3.3}$$

where $u = (p, q)^T$ is a column vector that contains two potentials, are Hamiltonian. Generalizing this result a bit, matrix differential operators carrying the form

$$H_2 = \begin{bmatrix} 0 & \partial p \\ p\partial & q\partial + \partial q \end{bmatrix} \tag{3.4}$$

can also be proved to be Hamiltonian in a similar way.

What's more, people can claim that matrix differential operators that can be expressed as linear combinations of H_1 and H_2 via real constants c_1, c_2, that is,

$$H = c_1 H_1 + c_2 H_2 = \begin{bmatrix} c_1(p\partial + \partial p) & c_2\partial p + c_1 q\partial \\ c_2 p\partial + c_1\partial q & c_2(q\partial + \partial q) \end{bmatrix} \tag{3.5}$$

are Hamiltonian, too. To prove this, certainly people can apply the conventional method reported in the original paper. There is however another approach that takes advantage of the of the Gauge transformation of Bäcklund type among Hamiltonian operators, which could be more concise as well as more enlightening, since it reveals the fact that the three types of matrix differential operators H_1, H_2 and H are Gauge equivalent.

THEOREM 3.2.1 *Suppose H_1, H_2 and H are given by Eqs. (3.3), (3.4) and (3.5), respectively. Suppose that $c_1^2 + c_2^2 \neq 0$. Then H_1, H_2 and H are pairwise Gauge equivalent. Furthermore, H is both u-linear and Hamiltonian.*

Proof. Let H be given by (3.5). It is apparently u-linear. First let

$$D = \begin{bmatrix} 1 & 0 \\ -c_2 & c_1 \end{bmatrix} \quad \text{which suggests} \quad \begin{cases} \tilde{p} = p, \\ \tilde{q} = (-c_2 p + c_1 q). \end{cases} \tag{3.6}$$

If $c_1 \neq 0$, the following computations immediately guide us to

$$\begin{aligned} \tilde{H} &= DHD^* \\ &= \begin{bmatrix} c_1 p\partial + c_1\partial p & c_1(-c_2 p + c_1 q)\partial \\ c_1\partial(-c_2 p + c_1 q) & 0 \end{bmatrix} = c_1 \begin{bmatrix} \tilde{p}\partial + \partial\tilde{p} & \tilde{q}\partial \\ \partial\tilde{q} & 0 \end{bmatrix}, \end{aligned} \tag{3.7}$$

where D^* stands for the Hermitian conjugate of D (D^* reduces to D's transpose whenever c_1, $c_2 \in \mathbb{R}$), is a Hamiltonian operator of H_1-type with a scaler c_1 (therefore still of H_1-type). According to Fuchssteiner and Fokas [76], H is Gauge equivalent to H_1. On the other hand, in case of $c_2 \neq 0$, one again finds that H is Gauge equivalent to the H_2-type Hamiltonian operator with a scaler c_2

$$\tilde{H}' = c_2 \begin{bmatrix} 0 & \partial\tilde{p}' \\ \tilde{p}'\partial & \tilde{q}'\partial + \partial\tilde{q}' \end{bmatrix} \tag{3.8}$$

via

$$D' = \begin{bmatrix} c_2 & -c_1 \\ 0 & 1 \end{bmatrix} \quad \text{that suggests} \quad \begin{cases} \tilde{p}' = c_2 p - c_1 q, \\ \tilde{q}' = q. \end{cases} \tag{3.9}$$

The transitivity of an equivalence relation implies that H_1 and H_2 are Gauge equivalent as well. This Gauge-equivalence among H_1, H_2 and H is therefore pairwise, and thus guarantees that H is Hamiltonian. □

The above theorem gives us insights that: the three classes of Hamiltonian operators H, H_1 and H_2 defined by Eqs. (3.5), (3.3) and (3.4) respectively, are actually Gauge equivalent in such a sense that any two of them can be mapped into each other via Gauge transformations in Bäcklund type. Not many people however looked into this fact so far, though proving it just comes out to be rather brief and concise.

3.3 The coupling between H and constant matrix differential operators

Mathematicians already knew the way of generating new Hamiltonian operators through coupling matrix differential operators of H_1, H_2 or H types to constant matrix differential operators with low orders [81, 82, 83, 84, 85]. Let

$$K = \sum_{k=0}^{m} B_k \partial^k, \quad B_k = (b_{ijk})_{2\times 2} \tag{3.10}$$

be a matrix differential operator summed up to the m-th order which has constant parameters b_{ijk} (where $1 \leq i, j \leq 2$ and $0 \leq k \leq m$). It can be easily verified that when and only when

$$b_{ijk} = (-1)^{k+1} b_{jik} \tag{3.11}$$

holds, K comes out to be skew-symmetric. This condition also automatically guarantees K Hamiltonian. To generalize the work from [81], We now state and prove the following theorem:

THEOREM 3.3.1 *Suppose that H and K are defined by Eqs. (3.5) and (3.10), respectively, where the parameters b_{ijk} in K satisfy Eq. (3.11). Suppose that $c_1^2 + c_2^2 \neq 0$. Then the if-and-only-if conditions for the*

matrix differential operator

$$\mathcal{J} = K + H \tag{3.12}$$

to be a Hamiltonian operator read as follows:

$$K = \begin{bmatrix} b_{111} & b_{121} \\ b_{121} & b_{221} \end{bmatrix} \partial + \begin{bmatrix} 0 & b_{122} \\ -b_{122} & 0 \end{bmatrix} \partial^2 + \begin{bmatrix} b_{113} & b_{123} \\ b_{123} & b_{223} \end{bmatrix} \partial^3, \tag{3.13}$$

where b_{111}, b_{121}, b_{221} *and* b_{122} *are arbitrary real constants; and* b_{113}, b_{123} *and* b_{223} *are real constants satisfying*

$$b_{113}c_2 - b_{123}c_1 = 0 \quad \textit{and} \quad b_{223}c_1 - b_{123}c_2 = 0. \tag{3.14}$$

Proof. It is rather straightforward to show the skew-symmetry required by being Hamiltonian. Hence in this proof we concentrate on verifying the Jacobi Identity.

First of all, we already know from Sec. 3.2 that the H-part of $\mathcal{J}$ is Hamiltonian. The computation thus reduces directly to (for example, for $\{X, Y, Z\}$ in the Jacobi identity)

$$\{X, Y, Z\} = \sum_{k=0}^{m} \int [f_k(X, Y, Z) + g_k(X, Y, Z) + h_k(X, Y, Z) + j_k(X, Y, Z)]dx,$$

where (since all the matrix operators involved are 2×2, it is obvious that X, Y, Z are arbitrarily picked from $\mathcal{A}^2$ (or $\mathcal{B}^2$), i.e., $X = (X_1, X_2)^T$, $Y = (Y_1, Y_2)^T$, $Z = (Z_1, Z_2)^T$)

$$f_k(X, Y, Z) = c_1(X_1 Z_{1,x} - X_{1,x} Z_1) b_{11k} \partial^k Y_1, \tag{3.15a}$$

$$\begin{aligned} g_k(X, Y, Z) = {} & c_1(X_1 Z_{1,x} - X_{1,x} Z_1) b_{12k} \partial^k Y_2 + c_2(X_2 Z_{1,x} - X_{1,x} Z_2) b_{11k} \partial^k Y_1 \\ & + c_1(X_1 Z_{2,x} - X_{2,x} Z_1) b_{21k} \partial^k Y_1, \end{aligned} \tag{3.15b}$$

$$\begin{aligned} h_k(X, Y, Z) = {} & c_2(X_2 Z_{1,x} - X_{1,x} Z_2) b_{12k} \partial^k Y_2 + c_1(X_1 Z_{2,x} - X_{2,x} Z_1) b_{22k} \partial^k Y_2 \\ & + c_2(X_2 Z_{2,x} - X_{2,x} Z_2) b_{21k} \partial^k Y_1, \end{aligned} \tag{3.15c}$$

$$j_k(X, Y, Z) = c_2(X_2 Z_{2,x} - X_{2,x} Z_2) b_{22k} \partial^k Y_2, \tag{3.15d}$$

for all $0 \leq k \leq m$. Considering the subscripts and the differential orders of P_i, Q_i, R_i, one can tell that for $H + K$, the Jacobi identity is satisfied if and only if

$$\bar{f}_k(X, Y, Z) = f_k(X, Y, Z) + \text{cycle}(X, Y, Z) \sim 0, \tag{3.16a}$$

$$\bar{g}_k(P, Q, R) = g_k(X, Y, Z) + \text{cycle}(X, Y, Z) \sim 0, \tag{3.16b}$$

$$\bar{h}_k(X, Y, Z) = h_k(X, Y, Z) + \text{cycle}(X, Y, Z) \sim 0, \tag{3.16c}$$

$$\bar{j}_k(X, Y, Z) = j_k(X, Y, Z) + \text{cycle}(X, Y, Z) \sim 0, \tag{3.16d}$$

where $0 \leq k \leq m$.

First we look into the case of $k = 0$. Considering that $b_{110} = b_{220} = 0$ and $b_{210} = -b_{120}$, a particular choice of (3.16b) with $X_1 = r$, $Y_1 = 0$, $Z_1 = x$ and $Y_2 = 1$, and a special choice of (3.16c) with $X_1 = 1$, $X_2 = 0$, $Y_2 = s$ and $Z_2 = x$ (one can choose s and r to be any smooth functions of x), give rise to

$$\bar{g}_0(X, Y, Z) = c_1 b_{120}(r - x r_x) \sim 2 c_1 b_{120} r,$$
$$\bar{h}_0(X, Y, Z) = c_2 b_{120}(s - x s_x) \sim 2 c_2 b_{120} s,$$

respectively. One concludes quickly from the "arbitrariness" of r and s that $b_{120} = 0$. Hence when $k = 0$, (3.16a)-(3.16d) automatically holds if and only if $B_0 = 0$.

As the second step, we check when happens when $k = 1$. Automatically we already have $\bar{f}_1(X, Y, Z) = \bar{j}_1(X, Y, Z) = 0$. Furthermore the condition $b_{211} = b_{121}$ ensures that $\bar{g}(X, Y, Z) = \bar{h}_1(X, Y, Z) = 0$. So in case of $k = 1$ nothing more than $B_1^T = B_1$ is required.

Next we look into the case of $k = 2$. The prerequisites $b_{112} = b_{222} = 0$ and $b_{212} = -b_{122}$ ensure that $\bar{f}_2(X, Y, Z) = \bar{j}_2(X, Y, Z) = 0$ and

$$\bar{g}_2(X, Y, Z) = \frac{d}{dx}[c_1 b_{122}(X_1 Z_{1,x} - X_{1,x} Z_1) Y_{2,x} + \text{cycle}(X, Y, Z)] \sim 0,$$
$$\bar{h}_2(X, Y, Z) = \frac{d}{dx}[c_2 b_{122}(X_2 Y_{2,x} - X_{2,x} Y_2) Z_{1,x} + \text{cycle}(X, Y, Z)] \sim 0.$$

Hence for $k = 2$ it is only required that $B_2^T = -B_2$.

For the case of $k = 3$ we need a slightly longer discussion. On the one hand, we have

$$\bar{f}_3(X, Y, Z) = \frac{d}{dx}[c_1 b_{113} X_1 (Z_{1,x} Y_{1,xx} - Y_{1,x} Z_{1,xx}) + \text{cycle}(X, Y, Z)] \sim 0,$$
$$\bar{j}_3(X, Y, Z) = \frac{d}{dx}[c_2 b_{223} X_2 (Z_{2,x} Y_{2,xx} - Y_{2,x} Z_{2,xx}) + \text{cycle}(X, Y, Z)] \sim 0.$$

By again making use of the "arbitrariness" of r and s, a particular choice of (3.16b) with $X_1 = r$, $Z_1 = x$, $Y_2 = \frac{x^3}{3!}$ and $Y_1 = X_2 = Z_2 = 0$ gives

$$\bar{g}_3(X, Y, Z) = c_1 b_{123}(r - x r_x + \frac{1}{2} x^3 \partial^3 r) - \frac{1}{6} c_2 b_{113} x^3 \partial^3 r \sim (-c_1 b_{123} + c_2 b_{113}) r,$$

based on which, (3.16b) would thus require

$$c_1 b_{123} - c_2 b_{113} = 0. \tag{3.17}$$

Likewise, a particular choice of (3.16c) with $X_2 = s$, $Y_1 = \frac{x^3}{3!}$, $Z_2 = x$ and $X_1 = Z_1 = Y_2 = 0$ leads to

$$c_1 b_{223} - c_2 b_{123} = 0. \tag{3.18}$$

Whereas on the other hand, if (3.17) and (3.18) are satisfied, we arrive at

$$\bar{g}_3(X,Y,Z) = \gamma_1 \frac{d}{dx}[(X_1Z_{1,x} - X_{1,x}Z_1)Y_{2,xx} + (Y_{2,x}Z_1 - Y_2Z_{1,x})X_{1,xx} + (X_{1,x}Y_2 - X_1Y_{2,x})Z_{1,xx} + \text{cycle}(X,Y,Z)] \sim 0,$$

where $\gamma_1 = c_2 b_{113} = c_1 b_{123}$, and

$$\bar{h}_3(X,Y,Z) = \gamma_2 \frac{d}{dx}[(X_1Z_{2,x} - X_{1,x}Z_2)Y_{2,xx} + (Y_{2,x}Z_2 - Y_2Z_{2,x})X_{1,xx} + (X_{1,x}Y_2 - X_1Y_{2,x})Z_{2,xx} + \text{cycle}(X,Y,Z)] \sim 0,$$

where $\gamma_2 = c_2 b_{123} = c_1 b_{223}$. Thus, (3.17) and (3.18) combine to guarantee that (3.16a)-(3.16d) hold in the case of $k = 3$.

We discuss now the very last case of $m \geq k \geq 4$. A particular choice of (3.16a) with $X_1 = r$, $Y_1 = \frac{x^k}{k!}$ and $Z_1 = x$, and a particular choice of (3.16d) with $X_2 = s$, $Y_2 = \frac{x^k}{k!}$ and $Z_2 = x$, will produce

$$\begin{aligned} \bar{f}_k &= c_1 b_{11k}[r - r_x x - (1-k)\frac{x^k}{k!}\partial^k r] \sim c_1 b_{11k}[2 - (-1)^k(1-k)]r, \\ \bar{j}_k &= c_2 b_{22k}[s - x s_x - (1-k)\frac{x^k}{k!}\partial^k s] \sim c_2 b_{22k}[2 - (-1)^k(1-k)]s, \end{aligned} \tag{3.19}$$

respectively. The condition $k \geq 4$ indicates that both

$$c_1 b_{11k} = 0, \quad \text{and} \quad c_2 b_{22k} = 0 \tag{3.20}$$

are needed to make sure that $\bar{f}_k \sim 0$ and $\bar{j}_k \sim 0$ will hold. Then, by using (3.20), two special choices of (3.16b) with $X_1 = r$, $Y_1 = 0$, $Z_1 = x$ and $Y_2 = \frac{x^k}{k!}$, and of (3.16c) with $X_1 = 0$, $Y_1 = \frac{x^k}{k!}$, $Z_2 = x$ and $X_2 = s$, will yield

$$\begin{aligned} \bar{g}_k(X,Y,Z) &= c_1 b_{12k}(r - x r_x) - \frac{1}{k!} c_2 b_{11k} x^k \partial^k r + \frac{1}{(k-1)!} c_1 b_{21k} x^k \partial^k r \\ &\sim c_1 b_{12k}(2-k)r - (-1)^k c_2 b_{11k} r, \end{aligned} \tag{3.21}$$

$$\begin{aligned} \bar{h}_k(X,Y,Z) &= c_2 b_{21k}(s - x s_x) - \frac{1}{k!} c_1 b_{22k} x^k \partial^k s + \frac{1}{(k-1)!} c_2 b_{12k} x^k \partial^k s \\ &\sim c_2 b_{21k}(2-k)s - (-1)^k c_1 b_{22k} s. \end{aligned} \tag{3.22}$$

Since $c_1^2 + c_2^2 \neq 0$, first we assume $c_1 \neq 0$ without loss of generality. Then, (3.20) implies $b_{11k} = 0$, from which (3.21) will lead to $b_{12k} = 0$. It follows $b_{21k} = 0$ as a consequence of (3.11), from which $b_{22k} = 0$ is finally again concluded from (3.22). We will similarly arrive at $b_{11k} = b_{12k} = b_{21k} = b_{22k} = 0$ if $c_2 \neq 0$ is assumed at first. Thus in brief, the conditions in (3.20), (3.21) and (3.22) together give rise to the only possibility: that is, $B_k = 0$ for $m \geq k \geq 4$.

Hence to summarize, that (3.13) and (3.14) being satisfied by the coefficients in K is the sufficient and necessary condition for $H + K$ to be Hamiltonian. The proof is completed. □

Based on early Hamiltonian theory [86], Fokas and Fuchssteiner proved in [87] that if two differential operators J and M constitute a Hamiltonian pair, where J is invertible, then one can construct a hereditary symmetry $\Phi = MJ^{-1}$. Following this remarkable result, for $M = H + K$ (which is Hamiltonian) with H, K being defined by (3.5), (3.13), respectively (and of course all the coefficients in K satisfy the conditions given by (3.14)), we let

$$J = \begin{bmatrix} d_1 & d_2 \\ d_2 & d_3 \end{bmatrix} \partial + \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} \partial^2 + \begin{bmatrix} d_4 & d_5 \\ d_5 & d_6 \end{bmatrix} \partial^3, \tag{3.23}$$

where $\frac{d_4}{b_{113}} = \frac{d_5}{b_{123}} = \frac{d_6}{b_{223}}$ if they are all nonzero, or $b_{113} = 0$ if and only if $d_4 = 0$, and likewise for b_{123} and d_5, b_{223} and d_6 (this condition shows up here to guarantee that an arbitrary linear combination of J and M will always yield a Hamiltonian operator, which is the if-and-only-if condition that J and M will make a Hamiltonian pair). What's more, a and d_1, d_2, d_3 must be properly chosen to make sure that J is invertible. Thus the linear combination of J and M always gives Hamiltonian operators, and (J, M) therefore gives rise to a Hamiltonian pair. Two typical examples of such J given by (3.23) could be, say, presented by

$$J = \begin{bmatrix} d_1 & d_2 \\ d_2 & d_3 \end{bmatrix} \partial \quad \Rightarrow \quad J^{-1} = \frac{1}{\triangle} \begin{bmatrix} d_3 & -d_2 \\ -d_2 & d_1 \end{bmatrix} \partial^{-1}, \tag{3.24}$$

with $\triangle = d_1 d_3 - d_2^2 \neq 0$; or another one which is less frequently used,

$$J = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} \partial^2 + \begin{bmatrix} d_4 & d_5 \\ d_5 & d_6 \end{bmatrix} \partial^3 \quad \Rightarrow$$

$$J^{-1} = \frac{1}{a^2} \begin{bmatrix} d_6 \partial^{-1} & -d_5 \partial^{-1} - a\partial^{-2} \\ -d_5 \partial^{-1} + a\partial^{-2} & d_4 \partial^{-1} \end{bmatrix},$$

with $a \neq 0$ and $d_4 d_6 - d_5^2 = 0$. One can illustrate more examples of such Hamiltonian operators. Therefore it comes up immediately with

THEOREM 3.3.2 *Let H, K and J be given by (3.5), (3.13), and (3.23), respectively. Let $M = H + K$. Then $\Phi = MJ^{-1}$ gives rise to a hereditary symmetry.*

The hereditary symmetries Φ given by Theorem 3.3.2depends only implicitly on x, and is thus translational invariant w.r.t. x. Therefore $\forall S \in \mathcal{B}^2$,we always have a Lie derivative which is identical to 0 [88], i.e.,

$$(L_{u_x}\Phi)S = \Phi[u_x, S] - [u_x, \Phi S] = 0. \tag{3.25}$$

Therefore by Corollary 1 in [89], we can conclude

$$[\Phi^m u_x, \Phi^n u_x] = 0, \quad \forall m, n \geq 0,$$

and it follows naturally also that

THEOREM 3.3.3 *Suppose that H, K and J are defined by (3.5), (3.13) and (3.23), respectively. Let $M = H + K$ and $\Phi = MJ^{-1}$. Then the hierarchy of evolution equations*

$$u_t = \Phi^m u_x, \quad m \geq 0 \tag{3.26}$$

constitutes a sequence of infinitely many common symmetries $\{K_m = \Phi^m u_x\}$ ($m \geq 0$), with Φ being a common hereditary strong symmetry.

The hierarchy (3.26) generally is integrable in the sense that each member of this hierarchy has infinitely many K-symmetries [87, 89].

3.4 Examples

We present in this section a couple of illustrative examples, aiming at indicating the implications behind the discussed Gauge equivalence.

3.4.1 Example #.1

As the first example, it was reported in [74] the following matrix spectral problem:

$$\phi_x = U\phi = U(u, \lambda)\phi, \quad \text{where} \quad u = [p, q]^T \quad \text{and} \quad \phi = [\phi_1, \phi_2]^T,$$

for which the spectral matrix $U \in \widetilde{\text{sl}}(2, \mathbb{R})$ possesses the form

$$U = (-q - \lambda)e_1 + pe_2 + \gamma e_3 = \begin{bmatrix} -q - \lambda & p \\ \gamma & q + \lambda \end{bmatrix}, \tag{3.27}$$

with $\gamma \neq 0$ being a constant, and

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \tag{3.28}$$

are the three elements that form the basis of $\text{sl}(2, \mathbb{R})$. The soliton hierarchy engendered by this matrix spectral problem is Liouville integrable and possesses the bi-Hamiltonian structure

$$u_{t_k} = J\frac{\delta \mathcal{H}_k}{\delta u} = M\frac{\delta \mathcal{H}_{k-1}}{\delta u}, \quad \text{for all } k \geq 1, \tag{3.29}$$

where $\mathcal{H}_k$ ($k \geq 0$) are the involved Hamiltonian functionals, and J, M together form the Hamiltonian pair:

$$J = -\frac{1}{\gamma}\begin{bmatrix} 0 & \partial \\ \partial & 0 \end{bmatrix}, \qquad M = \begin{bmatrix} \frac{1}{\gamma}(p\partial + \partial p) & \frac{1}{2\gamma}\partial^2 + \frac{1}{\gamma}q\partial \\ -\frac{1}{2\gamma}\partial^2 + \frac{1}{\gamma}\partial q & -\frac{1}{2}\partial \end{bmatrix}. \tag{3.30}$$

J is obviously invertible, and M clearly has the shape of the $\mathcal{J}$ operator presented in Theorem 3.3.1 with $\alpha = 0$ (i.e., the H-component of $\mathcal{J}$ reduces to H_1 given in (3.3)). If one specifically sets $\gamma = 1/\beta$, and modifies the matrix spectral problem (3.27) into

$$\tilde{\phi}_x = \tilde{U}\tilde{\phi} = \tilde{U}(u,\lambda)\tilde{\phi}, \qquad \tilde{U} = \begin{bmatrix} -\lambda - \alpha\tilde{p} - \beta\tilde{q} & \tilde{p} \\ \frac{1}{\beta} & \lambda + \alpha\tilde{p} + \beta\tilde{q} \end{bmatrix} \tag{3.31}$$

via the transformation through the matrix D

$$\begin{bmatrix} p \\ q \end{bmatrix} = D\begin{bmatrix} \tilde{p} \\ \tilde{q} \end{bmatrix}, \quad \text{with} \quad D = \begin{bmatrix} 1 & 0 \\ \alpha & \beta \end{bmatrix}, \tag{3.32}$$

this will give rise to the Hamiltonian operator

$$\tilde{M} = \begin{bmatrix} \beta(\tilde{p}\partial + \partial\tilde{p}) & \frac{1}{2}\partial^2 - \alpha\partial\tilde{p} + \beta\tilde{q}\partial \\ -\frac{1}{2}\partial^2 - \alpha\tilde{p}\partial + \beta\partial\tilde{q} & -\frac{1}{2\beta^2}\partial - \alpha(\tilde{q}\partial + \partial\tilde{q}) \end{bmatrix} \tag{3.33}$$

which apparently has again the shape of the $\mathcal{J}$ operator presented in Theorem 3.3.1. Also by Theorem 3.3.1, $\tilde{M}$ is connected with M in (3.30) through

$$\tilde{M} = D^{-1}MD^{-1^*}. \tag{3.34}$$

Especially, the K-component in M, while undergoing the same Bäcklund type transformation, is observed to remain as a (Hamiltonian) K-component in $\tilde{M}$. The above facts justify our statements in Theorem 3.3.1.

3.4.2 Example #.2

The Wadati-Konno-Ichikawa (WKI) hierarchy was generalized in 2002 by Xu [72, 90] through computing the following matrix spectral problem (for simplicity of computation, the matrix problem is presented here in a slightly different form from the original work; but they are in no doubt exactly equivalent):

$$\phi_x = U\phi = U(u,\lambda)\phi, \quad \text{with} \quad u = [p,q]^T \quad \text{and} \quad \phi = [\phi_1, \phi_2]^T,$$

where $U \in \widetilde{\mathrm{sl}}(2,\mathbb{R})$ is given by

$$U = (\lambda - \frac{\alpha}{2}p)e_1 + \lambda p e_2 + \lambda q e_3 = \begin{bmatrix} \lambda - \frac{\alpha}{2}p & \lambda p \\ \lambda q & -\lambda + \frac{\alpha}{2}p \end{bmatrix}. \tag{3.35}$$

By routinely solving the stationary zero-curvature equation $W_x = [U, W]$, also with $W \in \widetilde{\mathrm{sl}}(2, \mathbb{R})$ chosen to be

$$W = \begin{bmatrix} (\lambda - \frac{\alpha}{2}p)a - \frac{\alpha b_x}{2\lambda} & \lambda p a + b_x \\ \lambda q a + c_x & -(\lambda - \frac{\alpha}{2}p)a + \frac{\alpha b_x}{2\lambda} \end{bmatrix}, \tag{3.36}$$

where a, b and c are functions of λ, x and t expressed in terms of Laurent expansions of λ:

$$a = \sum_{i\geq 0} a_i \lambda^{-i}, \quad b = \sum_{i\geq 0} b_i \lambda^{-i}, \quad c = \sum_{i\geq 0} c_i \lambda^{-i}, \tag{3.37}$$

it was demonstrated that this generalized WKI soliton hierarchy possesses a bi-Hamiltonian structure $u_{t_m} = M\frac{\delta \mathcal{H}_m}{\delta u} = J\frac{\delta \mathcal{H}_{m-1}}{\delta u}$ $(m \geq 1)$, where $\mathcal{H}_m$, $\mathcal{H}_{m-1}$ are the corresponding Hamiltonian functionals, with J and M being the Hamiltonian pair. In particular here M reads

$$M = \begin{bmatrix} 0 & \partial^2 + \alpha\partial p \\ -\partial^2 + \alpha p\partial & \alpha(q\partial + \partial q) \end{bmatrix}, \tag{3.38}$$

which has the characteristics of the $\mathcal{J}$ operator presented in Theorem 3.3.1 when $\beta = 0$ (or in another word, the H-component of $\mathcal{J}$ reduces to H_2 given in (3.4)). Interestingly, this soliton hierarchy is also Liouville integrable, though J in the Hamiltonian pair does not have the form as that of (3.23). This matrix spectral problem (3.35) of the generalized WKI hierarchy, if modified to

$$\tilde{\phi}_x = \tilde{U}\tilde{\phi} = \tilde{U}(u, \lambda)\tilde{\phi}, \qquad \tilde{U} = \begin{bmatrix} \lambda - \frac{\alpha}{2}(\alpha\tilde{p} - \beta\tilde{q}) & \alpha\lambda\tilde{p} - \beta\lambda\tilde{q} \\ \lambda\tilde{q} & -\lambda + \frac{\alpha}{2}(\alpha\tilde{p} - \beta\tilde{q}) \end{bmatrix} \tag{3.39}$$

by using the transformation

$$\begin{bmatrix} p \\ q \end{bmatrix} = T \begin{bmatrix} \tilde{p} \\ \tilde{q} \end{bmatrix}, \quad \text{with} \quad T = \begin{bmatrix} \alpha & -\beta \\ 0 & 1 \end{bmatrix}, \tag{3.40}$$

will convert the the Hamiltonian operator M into

$$\tilde{M} = \begin{bmatrix} \beta(\tilde{p}\partial + \partial\tilde{p}) & \frac{1}{\alpha}\partial^2 + \alpha\partial\tilde{p} + \beta\tilde{q}\partial \\ -\frac{1}{\alpha}\partial^2 + \alpha\tilde{p}\partial + \beta\partial\tilde{q} & \alpha(\tilde{q}\partial + \partial\tilde{q}) \end{bmatrix}. \tag{3.41}$$

Here $\tilde{M}$ apparently carries also the characteristics of the $\mathcal{J}$ operator presented in Theorem 3.3.1, and is associated with M in (3.38) through

$$\tilde{M} = T^{-1}MT^{-1^*}. \tag{3.42}$$

Once again, the K-component in M, while undergoing the same Bäcklund type transformation, is observed to remain as a (Hamiltonian) K-component in $\tilde{M}$. The above facts again justify our statements in Theorem 3.3.1.

It was also reported in [91] an alternative form of the generalized WKI hierarchy, for which rather similar arguments about a Gauge transformation of Bäcklund type can be made.

3.4.3 Example #.3

In 2015 Zhang et al. published the coupled Burgers hierarchy based on the matrix spectral problem [73]

$$\phi_x = U\phi = U(u,\lambda)\phi, \quad \text{where} \quad u = (p,q)^T \quad \text{and} \quad \phi = (\phi_1,\phi_2)^T,$$

in which the spectral matrix U belonging to $\widetilde{\mathrm{sl}}(2,\mathbb{R})$ possesses the mathematical profile

$$U = (-\lambda+\alpha p+\beta q)e_1 + pe_2 + qe_3 = \begin{bmatrix} -\lambda+\alpha p+\beta q & p \\ q & \lambda-\alpha p-\beta q \end{bmatrix}. \tag{3.43}$$

One thing that needs to be emphasized here is, in this model the constants α, β are subjected to the constraint $\alpha\beta = -1/4$. The soliton hierarchy engendered by this matrix spectral problem is also Liouville integrable and has a bi-Hamiltonian structure $u_{t_m} = J\frac{\delta \mathcal{H}_m}{\delta u} = M\frac{\delta \mathcal{H}_{m-1}}{\delta u}$ $(m \geq 1)$, for which the Hamiltonian pair is

$$J = \begin{bmatrix} 0 & \partial \\ \partial & 0 \end{bmatrix}, \qquad M = \begin{bmatrix} \beta(p\partial+\partial p) & -\frac{1}{2}\partial^2+\alpha\partial p+\beta q\partial \\ \frac{1}{2}\partial^2+\alpha p\partial+\beta\partial q & \alpha(q\partial+\partial q) \end{bmatrix}, \tag{3.44}$$

where J is definitely invertible, and M has the characteristics of the $\mathcal{J}$ operator presented in Theorem 3.3.1 (note that $\alpha\beta = -1/4$). This matrix spectral problem (3.43), if modified into

$$\tilde{\phi}_x = \tilde{U}\tilde{\phi} = \tilde{U}(u,\lambda)\tilde{\phi}, \qquad \tilde{U} = \begin{bmatrix} -\lambda+2\alpha\tilde{p}+\tilde{q} & \tilde{p} \\ \frac{\alpha}{\beta}\tilde{p}+\frac{1}{\beta}\tilde{q} & \lambda-2\alpha\tilde{p}-\tilde{q} \end{bmatrix}, \tag{3.45}$$

by using the Gauge transformation

$$\begin{bmatrix} p \\ q \end{bmatrix} = D\begin{bmatrix} \tilde{p} \\ \tilde{q} \end{bmatrix}, \quad \text{with} \quad D = \begin{bmatrix} 1 & 0 \\ \frac{\alpha}{\beta} & \frac{1}{\beta} \end{bmatrix}, \tag{3.46}$$

will convert M into

$$\tilde{M} = \begin{bmatrix} \beta(\tilde{p}\partial+\partial\tilde{p}) & -\frac{1}{2}\beta\partial^2+\beta\tilde{q}\partial \\ \frac{1}{2}\beta\partial^2+\beta\partial\tilde{q} & 0 \end{bmatrix}. \tag{3.47}$$

$\tilde{M}$ again has the characteristics of the $\mathcal{J}$ operator, with its H-component reducing to that of H_1's form in (3.3), presented in Theorem 3.3.1. The matrix spectral problem can also be modified into

$$\hat{\phi}_x = \hat{U}\hat{\phi} = \hat{U}(u,\lambda)\hat{\phi}, \qquad \hat{U} = \begin{bmatrix} -\lambda-\hat{p}+2\beta\hat{q} & -\frac{1}{\alpha}\hat{p}+\frac{\alpha}{\beta}\hat{q} \\ \hat{q} & \lambda+\hat{p}-2\beta\hat{q} \end{bmatrix} \tag{3.48}$$

via the transformation

$$\begin{bmatrix} p \\ q \end{bmatrix} = D\begin{bmatrix} \hat{p} \\ \hat{q} \end{bmatrix}, \quad \text{with} \quad D = \begin{bmatrix} -\frac{1}{\alpha} & \frac{\beta}{\alpha} \\ 0 & 1 \end{bmatrix}, \tag{3.49}$$

to yield the Hamiltonian operator

$$\hat{M} = \begin{bmatrix} 0 & \frac{1}{2}\alpha\partial^2 + \alpha\partial\hat{p} \\ -\frac{1}{2}\alpha\partial^2 + \alpha\hat{p}\partial & \alpha(\hat{q}\partial + \partial\hat{q}) \end{bmatrix}, \tag{3.50}$$

which has also the characteristics of the $\mathcal{J}$ operator, with its H-component reducing to the shape of H_2 in (3.4), presented in Theorem 3.3.1. Both $\tilde{M}$ given by (3.47) and $\hat{M}$ given by (3.50) in this example can be obtained from M in (3.44) by using $\tilde{M}(\text{or } \hat{M}) = D^{-1}MD^{-1^*}$.

3.4.4 Example #.4

As a somehow different final example, we consider now the Hamiltonian operator that carries such a shape:

$$J = B_1\partial + B_2\partial^2 + B_3\partial^3 = \begin{bmatrix} d_1 & d_2 \\ d_2 & d_3 \end{bmatrix}\partial + \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}\partial^2 + \begin{bmatrix} d_4 & d_5 \\ d_5 & d_6 \end{bmatrix}\partial^3, \tag{3.51}$$

where a is nonzero, and $\det B_3 = d_4d_6 - d_5^2 = 0$. J of (3.51) has a formal inverse which be presented as

$$J^{-1} = \begin{bmatrix} d_3 + d_6\partial^2 & d_2 + a\partial + d_5\partial^2 \\ d_2 - a\partial + d_5\partial^2 & d_1 + d_4\partial^2 \end{bmatrix} \triangle^{-1}\,\partial^{-1}, \tag{3.52}$$

where

$$\triangle = d_1d_3 - d_2^2 + (d_1d_6 + d_3d_4 + a^2 - 2d_2d_5)\partial^2. \tag{3.53}$$

Let $M = H$ (i.e., M has merely the H-component of $\mathcal{J} = H + K$ in Theorem 3.3.1) satisfy the conditions in Theorems 3.3.2 and 3.3.3, one arrives thus at the conclusion that $\Phi = MJ^{-1}$ (the recursion operator) gives rise to a hereditary strong symmetry, and the hierarchy of evolution equations $u_t = \Phi^m u_x$ ($m \geq 0$) therefore possesses a sequence of infinitely many common symmetries $\{K_m = \Phi^m u_x\}_{m=0}^{\infty}$. The Hamiltonian operator J given by (3.51) involved in this example has a third order differential component, and therefore carries some characteristics of a Camassa-Holm type hierarchy [92], due to the presence of $\triangle$ in (3.53).

By introducing another two implicit variables r, s (when dealing with Camassa-Holm type systems, this is a typical recipe) that are related to p, q in terms of

$$\begin{cases} p = C_1r + C_2r_{xx}, \\ q = C_1s + C_2s_{xx}, \end{cases} \tag{3.54}$$

where $C_1 = d_1d_3 - d_2^2$, and $C_2 = d_1d_6 + d_5d_2 + a^2 - 2d_3d_4$. The first nonlinear member of this hierarchy, $u_t = (p_t, q_t)^T = \Phi u_x$ ($\Phi = MJ^{-1}$) can be presented in a bit complicated form:

$$\begin{cases} p_t = 2\beta p(d_3r_x - d_2s_x) + \beta p_x(d_3r - d_2s) - \beta q(d_2r_x - d_1s_x) \\ \quad -\beta q(d_5r_{xxx} - d_4s_{xxx}) + 2\beta p(d_6r_{xxx} - d_5s_{xxx}) \\ \quad +\beta p_x(d_6r_{xx} - d_5s_{xx}) - \beta a(p_xs_x + 2ps_{xx} - qr_{xx}) \\ \quad +\alpha[p(d_1s - d_2r)]_x + \alpha[p(d_4s_{xx} - d_5r_{xx})]_x + \alpha a(pr_x)_x, \\ \\ q_t = \beta[q(d_3r - d_2s)]_x + \beta[q(d_6r_{xx} - d_5s_{xx})]_x - \beta a(qs_x)_x \\ \quad -2\alpha q(d_2r_x - d_1s_x) - \alpha q_x(d_2r - d_1s) + \alpha p(d_3r_x - d_2s_x) \\ \quad +\alpha p(d_6r_{xxx} - d_5s_{xxx}) - 2\alpha q(d_5r_{xxx} - d_4s_{xxx}) \\ \quad -\alpha q_x(d_5r_{xx} - d_4s_{xx}) + \alpha a(q_xr_x + 2qr_{xx} - ps_xx). \end{cases} \tag{3.55}$$

If alternatively, one imposes in (3.51) $B_2 = 0$ (i.e. $a = 0$), and admits but $\det B_1 < 0$, or to put it more precisely,

$$\det B_1 = d_1d_3 - d_2^2 = -\frac{d_1^2}{d_4^2}\left(d_5 - \frac{d_2d_4}{d_1}\right)^2, \tag{3.56}$$

which is an alternative equivalent version of $d_1d_6 + d_4d_3 - 2d_2d_5 = 0$, and J has an explicit inverse guaranteed by these assumptions:

$$J^{-1} = \frac{1}{\det B_1}\begin{bmatrix} d_6\partial + d_3\partial^{-1} & -d_5\partial - d_2\partial^{-1} \\ -d_5\partial - d_2\partial^{-1} & d_4\partial + d_1\partial^{-1} \end{bmatrix}. \tag{3.57}$$

Rather similarly, the Hamiltonian operator J in (3.51) satisfying (3.56), and a second Hamiltonian operator $M = H + K$, will together produce a hereditary strong symmetry $\Phi = MJ^{-1}$. The hierarchy of evolution equations $u_t = \Phi^m u_x$ ($m \geq 0$) indeed corresponds to a sequence of infinitely many common symmetries $\{K_m = \Phi^m u_x\}_{m=0}^{\infty}$.

3.5 Concluding remarks

We have shown in this chapter that matrix differential operators carrying the profiles of H_1, H_2 or H defined by (3.3), or (3.4) and (3.5), respectively, are u-linear Hamiltonian operators that are also pairwise gauge equivalent. The local transformations among them can be performed by means of constant 2×2 matrices. Hopefully, such gauge equivalence among H_1, H_2 and H can be accordingly generalized to more Hamiltonian operators, and may help to shed light on the classification of Hamiltonian operators.

Further it is proved that new Hamiltonian operators can be generated by coupling these matrix differential operators to constant differential operators. The corresponding sufficient and necessary conditions (3.13)

and (3.14) were derived as well. The shining point in Theorem 3.3.1 is that this kind of coupling takes effect up to a finite order only (for the operators studied in this chapter, it has been determined to be the 3rd order) of the constant differential operators. This piece of remarkably succinct result (conciser and simpler than we used to expect) would encourage us to explore further what might be the interesting mathematics behind those Hamiltonian structures.

Chapter 4

Computation of soliton solutions of a generalized eight-component mKdV system based on Riemann-Hilbert problems

The inverse scattering transform (IST) is a powerful tool for addressing the Cauchy problems of many nonlinear evolution PDEs associated with integrable models, as mentioned in Chapter 1, in modern soliton theory [39, 93]. Such problems of I.S.T., preferably for those nonlinear PDEs involving only one spatial dimension, can always be formulated into a Riemann-Hilbert problem (RHP) [39] characterized by a jump on the real line $\mathbb{R}$ (subjected typically to canonical normalization condition), with bounded eigenfunctions that are analytically continuable to

$$\mathbb{D}^+ := \{\lambda \in \mathbb{C} | \text{Im}\lambda > 0\} \quad \text{or} \quad \mathbb{D}^- := \{\lambda \in \mathbb{C} | \text{Im}\lambda < 0\}, \tag{4.1}$$

and continuous in their closures $\bar{\mathbb{D}}^+$ or $\bar{\mathbb{D}}^-$, through which the involved nonlinear elements can be effectively linearized.

In particular, soliton solutions can be obtained by reducing such an RHP to reflectionless scattering problem (mathematically performed by admitting an identity jumping matrix in the RHP) [39, 104]. By taking special limits of these soliton solutions, one may obtain periodic solutions, lump solutions [94], or even complexiton solutions [95].

The standard procedure of formulating an RHP associated with a scattering problem on the real axis $\mathbb{R}$ runs as follows:

Step 1. One starts from a pair of matrix spectral problems that can be written in the shape of

$$-i\phi_x = U(u,\lambda)\phi, \quad -i\phi_t = V(u,\lambda)\phi, \tag{4.2}$$

where λ is the spectral parameter, ϕ is an $m \times m$ matrix eigenfunction, u is a (column) vector of potentials, and

$$U(u,\lambda) = U_0(\lambda) + P(u,\lambda), \quad V(u,\lambda) = V_0(\lambda) + P(u,\lambda),$$

with U_0, V_0 being constant commuting (usually diagonal) $m \times m$ matrices, and P, Q being traceless (i.e., $\text{tr}P = \text{tr}Q = 0$). The complex unit -$i$ here is introduced to greet the appetite of the direct scattering

method—since this method prefers the solution to the matrix spectral problem has "planar wave" behavior at infinity (on the real line), which is guaranteed by the presence of this $-i$ (could be i, too) as well as the fast decreasing of the entries in p and q described below. The zero curvature equation (if admitted by the system) for this pair of matrix spectral problems (4.2) apparently is

$$U_t - V_x + i[U, V] = 0. \tag{4.3}$$

Step 2. By introducing

$$\phi = \psi E_g, \quad \text{with} \quad E_g = \mathrm{e}^{i[U_0(\lambda)x + V_0(\lambda)t]}, \tag{4.4}$$

we can formulate the equivalent matrix spectral problems with respect to another $m \times m$ matrix eigenfunction ψ:

$$\psi_x = i[U_0(\lambda), \psi] + \tilde{P}\psi, \quad \psi_t = i[V_0(\lambda), \psi] + \tilde{Q}\psi, \tag{4.5}$$

where $\tilde{P} = iP$, $\tilde{Q} = iQ$. Such an equivalence is guaranteed by the commutativity of U_0 and V_0. One of the major advantages of having this equivalent form of a zero curvature equation is that: since $\phi \to E_g$ as $x \to \pm\infty$ (generally, the potential vector u, and so $P(u, \lambda)$ and $Q(u, \lambda)$, are assumed to be rapidly decreasing to 0 as $x \to \pm\infty$; see (4.38) below), the eigenfunction ψ possesses now the more preferable constant asymptotic behavior

$$\psi \to I_m, \quad \text{as} \quad x, t \to \pm\infty, \tag{4.6}$$

where I_m is the $m \times m$ identity matrix.

Step 3. From the $\psi^{\pm}$ obtained in Step 2, one then tries to construct two matrix function $Y^{\pm}(x, t, \lambda)$ that are analytical in $\mathbb{D}^+$ and $\mathbb{D}^-$, as well as continuous in $\bar{\mathbb{D}}^+$ and $\bar{\mathbb{D}}^-$ (as defined in (4.1)), respectively, to formulate an RHP on the real axis:

$$G^+(x, t, \lambda) = G^-(x, t, \lambda)G(x, t, \lambda), \quad \lambda \in \mathbb{R}, \tag{4.7}$$

where

$$G^+(x, t, \lambda) = \lim_{\mu \in \mathbb{D}^+, \mu \to \lambda} Y^+(x, t, \mu), \quad (G^-)^{-1}(x, t, \lambda) = \lim_{\mu \in \mathbb{D}^-, \mu \to \lambda} Y^-(x, t, \mu).$$

Step 4. The RHP, when reduced to reflectionless scattering problem (by taking $J = I_m$), can usually be solved to yield soliton solutions by computing the asymptotics of the matrix functions $Y^{\pm}$ (subjected to canonical normalization condition) at $|\lambda| \to \infty$.

Study of solutions to nonlinear evolution equations using the Riemann-Hilbert approach have been reported on a few integrable equations, for instance the multiple wave interaction equations [39], the Harry-Dym equations [96], the general coupled nonlinear Schrödinger equations [97, 98], the generalized Sasa-

Satsuma equation [99], and the coupled multiple nonlinear Schrödinger equations and the coupled mKdV equations [100, 101, 102], etc.

We shall present in this chapter a new class of generalized modified Korteweg-de Vries (mKdV) system and compute N-soliton solutions for one special case using the Riemann-Hilbert approach formulated from the inverse scattering transform, by following the steps in Chapter 2 (when solving the matrix spectral problem in zero curvature formulation) and the Steps 1-4 mentioned just above (when formulating and solving the related RHP).

4.1 Matrix spectral problems of a class of generalized modified KdV systems

Let $m, n > 0$ be arbitrary integers. We shall investigate the following matrix spectral problem

$$-i\phi_x = U(u,\lambda)\phi, \quad U = [U_{ij}]_{(m+n)\times(m+n)} = \begin{bmatrix} \alpha_1\lambda I_m & p \\ q & \alpha_2\lambda I_n \end{bmatrix}, \tag{4.8}$$

where λ stands for the spectral parameter, α_1, α_2 are real constants, $p = (p_{ij})_{m\times n}, q = (q_{ij})_{n\times m}$ are submatrices whose entries denote scalar potential components, and u is a column potential vector constructed by arranging all the $2mn$ entries of p, q in the shape of

$$\begin{aligned} u = (&p_{11}, \ldots, p_{1n}, p_{21}, \ldots, p_{2n}, \ldots, p_{m1}, \ldots, p_{mn}; \\ &q_{11}, \ldots, q_{n1}, q_{12}, \ldots, q_{n2}, \ldots, q_{1m}, \ldots, q_{nm})^T. \end{aligned} \tag{4.9}$$

By following the routine procedures of solving matrix spectral problems to engender the associated integrable hierarchy, we accordingly look for a solution W in the form

$$W = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{4.10}$$

to the stationary zero curvature equation $W_x = i[U, W]$, with the submatrices

$$a = (a_{ij})_{m\times m}, \quad b = (b_{ij})_{m\times n}, \quad c = (c_{ij})_{n\times m}, \quad d = (d_{ij})_{n\times n} \tag{4.11}$$

being arranged correspondingly. Upon letting $\alpha = \alpha_1 - \alpha_2$, the stationary zero curvature equation is thus equivalently formulated as

$$\begin{cases} a_x = i(pc - qb), \\ b_x = i(\alpha\lambda b + pd - ap), \\ c_x = i(-\alpha\lambda c + qa - dq), \\ d_x = i(qb - cp). \end{cases} \tag{4.12}$$

Then, upon writing W formally as a Laurent series of λ:

$$W = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \sum_{k=0}^{\infty} W_k \lambda^{-k}, \quad W_k = W_k(u) = \begin{bmatrix} a^{(k)} & b^{(k)} \\ c^{(k)} & d^{(k)} \end{bmatrix}, \quad k \geq 0, \tag{4.13}$$

one derives immediately the equivalent recursive form of the stationary zero curvature equation:

$$\begin{cases} b^{(k+1)} = \frac{1}{\alpha}[-ib_x^{(k)} - (pd^{(k)} - a^{(k)}p)], \\ c^{(k+1)} = \frac{1}{\alpha}[ic_x^{(k)} + (qa^{(k)} - d^{(k)}q)], \\ a_x^{(k+1)} = i(pc^{(k+1)} - b^{(k+1)}q), \\ d_x^{(k+1)} = i(qb^{(k+1)} - c^{(k+1)}p), \end{cases} \quad k \geq 0, \tag{4.14}$$

Upon running Eqn. (4.14), one derives recursively the sequences of submatrices $\{a^{(k)}, b^{(k)}, c^{(k)}, d^{(k)}\}_{k=0}^{\infty}$. Thus by admitting

$$a^{(0)} = \beta_1 I_m, \quad d^{(0)} = \beta_2 I_n, \quad b^{(0)} = {c^{(0)}}^T = (0)_{m\times n}, \tag{4.15}$$

and introducing Lax matrices in size of $(m+n)\times(m+n)$ as follows:

$$V^{(r)} = V^{(r)}(u,\lambda) = (\lambda^r W)_+ = \sum_{k=0}^{r} W_k \lambda^{r-k}, \qquad \forall r \geq 1 \tag{4.16}$$

the compatibility conditions, i.e., the zero curvature equations, will straightforwardly yield:

$$U_{t_k} = \begin{bmatrix} 0 & p_{t_k} \\ q_{t_k} & 0 \end{bmatrix} = \begin{bmatrix} 0 & i\alpha b^{(k+1)} \\ -i\alpha c^{(k+1)} & 0 \end{bmatrix}, \quad k \geq 0. \tag{4.17}$$

Due to the redundancy of U_{t_k}, as well as the a bit complicated recursion relations (4.14) since b and c are here matrices, we shall use u_{t_k} with u given by (4.9). Through a subtle computation we obtain

$$\begin{pmatrix} \bar{c}^{(k+1)} \\ \bar{b}^{(k+1)} \end{pmatrix} = \Psi \begin{pmatrix} \bar{c}^{(k)} \\ \bar{b}^{(k)} \end{pmatrix}, \quad k \geq 0, \tag{4.18}$$

where

$$\bar{b} = (b_{11}, \ldots, b_{1n}, b_{21}, \ldots, b_{2n}, \ldots, b_{m1}, \ldots, b_{mn})^T,$$
$$\bar{c} = (c_{11}, \ldots, c_{n1}, c_{12}, \ldots, c_{n2}, \ldots, c_{1m}, \ldots, c_{nm})^T,$$

and

$$\Psi = \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{bmatrix}, \tag{4.19}$$

with Ψ_{11}, Ψ_{12}, Ψ_{21} and Ψ_{22} being $mn \times mn$ submatrices that read

$$\begin{aligned}
\Psi_{11} &= \frac{i}{\alpha}\left((\partial + p^T\partial^{-1}q^T)I_n\delta_{jl} + (p_j\partial^{-1}q_l)_{n\times n}\right)_{m\times m},\\
\Psi_{12} &= -\frac{i}{\alpha}\left(p_l^T\partial^{-1}p_j + [p_l^T\partial^{-1}p_j]^T\right)_{m\times m},\\
\Psi_{21} &= \frac{i}{\alpha}\left(q_l\partial^{-1}q_j^T + [q_l\partial^{-1}q_j^T]^T\right)_{m\times m},\\
\Psi_{22} &= -\frac{i}{\alpha}\left((\partial + q\partial^{-1}p)I_n\delta_{jl} + (q_j^T\partial^{-1}p_l^T)_{n\times n}\right)_{m\times m}.
\end{aligned} \tag{4.20}$$

It is obvious that here $\Psi_{11}, \Psi_{12}, \Psi_{21}, \Psi_{22}$ are all arranged in terms of $m\times m$ structure, with each entry being an $n\times n$ submatrix. Thus the involved indices $1 \le j, l \le m$ are used to denote that the referred $n\times n$ block is locating on the position of the (j,l)-entry. In addition, for example, p_j and q_l are used to denote the j-th row of p and l-th column of q, respectively.

We now show that the generalized multicomponent AKNS integrable hierarchy (4.18) possesses a bi-Hamiltonian structure that can be engendered through the trace identity, or in general, the variational identity. Indeed, one can compute for all $1 \le j \le m$ and $1 \le k \le n$ that

$$\begin{aligned}
&\frac{\partial U}{\partial \lambda} = \begin{bmatrix} \alpha_1 I_m & 0 \\ 0 & \alpha_2 I_n \end{bmatrix}, \quad \frac{\partial U}{\partial p_{jk}} = \begin{bmatrix} (0)_{m\times m} & (1_{jk})_{m\times n} \\ (0)_{n\times m} & (0)_{n\times n} \end{bmatrix},\\
&\frac{\partial U}{\partial q_{kj}} = \begin{bmatrix} (0)_{m\times m} & (0)_{m\times n} \\ (1_{kj})_{n\times m} & (0)_{n\times n} \end{bmatrix},
\end{aligned} \tag{4.21}$$

where (1_{jk}) denotes a matrix with all entries being 0 except only that the (j,k)-entry is 1, and also

$$\mathrm{tr}\left(W\frac{\partial U}{\partial \lambda}\right) = \alpha_1\mathrm{tr}(a) + \alpha_2\mathrm{tr}(d), \quad \mathrm{tr}\left(W\frac{\partial U}{\partial p_{jk}}\right) = c_{kj}, \quad \mathrm{tr}\left(W\frac{\partial U}{\partial q_{kj}}\right) = b_{jk}. \tag{4.22}$$

These results are now connected by the trace identity as follows:

$$\frac{\delta}{\delta u}\int[\alpha_1\mathrm{tr}(a) + \alpha_2\mathrm{tr}(d)]dx = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\begin{pmatrix} \mathrm{tr}\left(W\frac{\partial U}{\partial \bar{p}}\right) \\ \mathrm{tr}\left(W\frac{\partial U}{\partial \bar{q}}\right)\end{pmatrix} = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\begin{pmatrix} \bar{c} \\ \bar{b}\end{pmatrix}, \tag{4.23}$$

Through balancing the leading coefficients of all powers of λ, we have

$$\frac{\delta}{\delta u}\int[\alpha_1\sum_{j=1}^{m}a_{jj}^{(k+1)} + \alpha_2\sum_{j=1}^{n}d_{jj}^{(k+1)}]dx = (\gamma - k)\begin{bmatrix} \bar{c}^{(k)} \\ \bar{b}^{(k)}\end{bmatrix}, \quad k \ge 0. \tag{4.24}$$

The constant γ, since it does not vary with respect to k, can be computed by checking a particular case, say, $k = 2$; and this gives $\gamma = 0$ (so far $\gamma = 0$ is verified for all $m, n \le 4$; we believe it is 0 for all $m, n \ge 1$, but this requires a proof), thus the above variational identity becomes

$$\frac{\delta}{\delta u}\int\left(-\frac{\alpha_1\sum_{j=1}^m a_{jj}^{(k+1)}+\alpha_2\sum_{j=1}^n d_{jj}^{(k+1)}}{k}\right)dx=\begin{bmatrix}\bar{c}^{(k)}\\ \bar{b}^{(k)}\end{bmatrix},\quad k\geq 1, \tag{4.25}$$

As an immediate consequence, one can choose the Hamiltonian functionals to be

$$\mathcal{H}_k=\int\left(-\frac{\alpha_1\sum_{j=1}^m a_{jj}^{(k+1)}+\alpha_2\sum_{j=1}^n d_{jj}^{(k+1)}}{k}\right)dx,\quad k\geq 1, \tag{4.26}$$

and (4.25) can now be reformulated as

$$\frac{\delta\mathcal{H}_k}{\delta u}=G_{k-1},\quad \text{with}\quad G_{k-1}=\begin{bmatrix}\bar{c}^{(k)}\\ \bar{b}^{(k)}\end{bmatrix},\quad k\geq 1. \tag{4.27}$$

This suggests as a consequence the following bi-Hamiltonian structure for the multi-component AKNS system which was computed above:

$$u_t=K_k=JG_k=J\frac{\delta\mathcal{H}_{k+1}}{\delta u}=M\frac{\delta\mathcal{H}_k}{\delta u},\quad k\geq 1, \tag{4.28}$$

where the Hamiltonian pair thus reads

$$J=\begin{bmatrix}0 & \alpha I_{mn}\\ -\alpha I_{mn} & 0\end{bmatrix}, \tag{4.29}$$

and

$$M=J\Psi=\begin{bmatrix}M_{11} & M_{12}\\ M_{21} & M_{22}\end{bmatrix}, \tag{4.30}$$

where Ψ is given by (4.20) and thus

$$\begin{cases}M_{11}=i\left(p_l^T\partial^{-1}p_j+[p_l^T\partial^{-1}p_j]^T\right)_{m\times m},\\ M_{12}=-i\left((\partial+p^T\partial^{-1}q^T)I_n\delta_{jl}+(p_j\partial^{-1}q_l)_{n\times n}\right)_{m\times m},\\ M_{21}=-i\left((\partial+q\partial^{-1}p)I_n\delta_{jl}+(q_j^T\partial^{-1}p_l^T)_{n\times n}\right)_{m\times m},\\ M_{22}=i\left(q_l\partial^{-1}q_j^T+[q_l\partial^{-1}q_j^T]^T\right)_{m\times m}.\end{cases} \tag{4.31}$$

where the indices j,l are interpreted in a similar manner as that in (4.20). Thus, we know that the operator $\Phi=\Psi^\dagger=MJ^{-1}$ is a recursion operator for the entire hierarchy (4.28).

4.2 Riemann-Hilbert problems

We shall work on in this section a special case of the integrable hierarchy discussed above, i.e., we shall choose $m=n=2$, which corresponds to a special generalized modified KdV system with 8 potentials

(since $m = n$ in this case, the determinants $\det p$ and $\det q$, as well as the cofactors of the entries of the submatrices p and q, will make sense). By letting $\beta = \beta_1 - \beta_2$, and

$$\bar{p} = (p_{11}, p_{12}, p_{21}, p_{22})^T, \quad \bar{q} = (q_{11}, q_{21}, q_{12}, q_{22})^T, \tag{4.32}$$

the first two nonlinear integrable systems of this hierarchy (4.18) can be written down for all $1 \leq r, j \leq 2$ as:

$$p_{rj,t} = \frac{-i\beta}{\alpha^2}[p_{rj,xx} + 2(\bar{p}^T\bar{q})p_{rj} - 2C_{rj}(q)\det p], \tag{4.33a}$$

$$q_{rj,t} = \frac{i\beta}{\alpha^2}[q_{rj,xx} + 2(\bar{p}^T\bar{q})q_{rj} - 2C_{rj}(p)\det q], \tag{4.33b}$$

and

$$p_{rj,t} = -\frac{\beta}{\alpha^3}[p_{rj,xxx} + 3(\bar{p}^T\bar{q})p_{rj,x} + 3(\bar{p}_x^T\bar{q})p_{rj} - 3C_{rj}(q)(\det p)_x], \tag{4.34a}$$

$$q_{rj,t} = -\frac{\beta}{\alpha^3}[q_{rj,xxx} + 3(\bar{p}^T\bar{q})q_{rj,x} + 3(\bar{p}^T\bar{q}_x)q_{rj} - 3C_{rj}(p)(\det q)_x]. \tag{4.34b}$$

where $C_{ij}(p)$, $C_{ij}(q)$ are the (i, j)-cofactors of p, q, respectively.

Based on the results obtained in the previous section, the matrix spectral problems of the 8-component mKdV system (4.34a), (4.34b) is indeed

$$-i\phi_x = U\phi = U(u, \lambda)\phi, \quad -i\phi_t = V^{[3]}\phi = V^{[3]}(u, \lambda)\phi, \tag{4.35}$$

where the Lax pair reads

$$U = \lambda\Lambda + P, \quad V^{[3]} = \lambda^3\Omega + Q, \tag{4.36}$$

with $\Lambda = \text{diag}(\alpha_1 I_m, \alpha_2 I_n)$, $\Omega = \text{diag}(\beta_1 I_m, \beta_2 I_n)$, and

$$P = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} a^{(1)}\lambda^2 + a^{(2)}\lambda + a^{(3)} & b^{(1)}\lambda^2 + b^{(2)}\lambda + b^{(3)} \\ c^{(1)}\lambda^2 + c^{(2)}\lambda + c^{(3)} & d^{(1)}\lambda^2 + d^{(2)}\lambda + d^{(3)} \end{bmatrix}, \tag{4.37}$$

where $a^{(k)}$, $b^{(k)}$, $c^{(k)}$, $d^{(k)}$ $(1 \leq k \leq 3)$ are determined recursively from (4.14).

From now on throughout the context below in the chapter we shall concentrate our attention on this 8-component mKdV system (4.34a) & (4.34b); and in this section, we shall study the direct and inverse scatterings for this system using the Riemann-Hilbert approach. The yielded results will be called in the following section for computing soliton solutions. To make issues consistent, without loss of generality we apply the assumptions $\alpha = \alpha_1 - \alpha_2 > 0$ and $\beta = \beta_1 - \beta_2 > 0$ for sake of convenience. Also we assume that

all the potentials (i.e., all the entries in the submatrices p and q) are rapidly decreasing functions in Schwartz space, i.e., for all $1 \leq k \leq m$ and $1 \leq l \leq n$,

$$\sup_{0\leq t<\infty, x\in\mathbb{R}} |x|^{m_1}|t|^{m_2}\left(\left|\frac{\partial^{n_1}}{\partial x^{n_1}}\frac{\partial^{n_2}}{\partial t^{n_2}}p_{kl}\right| + \left|\frac{\partial^{n_1}}{\partial x^{n_1}}\frac{\partial^{n_2}}{\partial t^{n_2}}q_{lk}\right|\right) < \infty, \quad m_1, m_2, n_1, n_2 \geq 0. \tag{4.38}$$

By the above rapidly-decreasing assumption, from the matrix problems (4.35) we obtain the asymptotic behavior of $\phi \sim e^{i\lambda\Lambda x + i\lambda^3\Omega t}$ as $x, t \to \pm\infty$. Thus, through the variable transformation

$$\phi = \psi E_g, \quad E_g = e^{i\lambda\Lambda x + i\lambda^3\Omega t},$$

we achieve to have the canonical normalization $\psi \to I_4$, when $x, t \to \pm\infty$. By introducing $\tilde{P} = iP$ and $\tilde{Q} = iQ$, the matrix spectral problem (4.35) is converted equivalently into

$$\psi_x = i\lambda[\Lambda, \psi] + \tilde{P}\psi, \tag{4.39a}$$

$$\psi_t = i\lambda^3[\Omega, \psi] + \tilde{Q}\psi. \tag{4.39b}$$

Let us now formulate the corresponding RHP with respect to x. In the direct scattering problem, we first introduce the two matrix solutions ψ^+ and ψ^- of (4.39a) characterized by the asymptotic behavior

$$\psi^\pm \to I_4, \quad \text{as } x \to \pm\infty, \tag{4.40}$$

respectively. The above superscripts indicates to which end of the x-axis these boundary conditions are imposed. By the Liouville's formula in ODE theory, it follows then from (4.39a) and (4.39b) (in particular because $\tilde{P}$ and $\tilde{Q}$ are both traceless) that $\det\psi^+ = \det\psi^- \equiv 1$ everywhere on the real line. Since

$$\phi^\pm = \psi^\pm E, \quad E = e^{i\lambda\Lambda x}, \tag{4.41}$$

are both solutions of (4.35) (which is of first order), they are linearly dependent, i.e. $\psi^- E$ and $\psi^+ E$ can be associated with each other in terms of

$$\psi^- E = \psi^+ E S(\lambda), \quad \lambda \in \mathbb{R}, \tag{4.42}$$

via the so-called **scattering matrix** $S(\lambda) = (s_{jk})_{4\times 4}$. It follows immediately at this moment that $\det\psi^\pm \equiv 1$ implies $\det S(\lambda) \equiv 1$.

One derives based on (4.39a) that

$$\begin{aligned}
&\psi_x - i\lambda[\Lambda, \psi] = \tilde{P}\psi \\
&\quad \Rightarrow \mathrm{e}^{-i\lambda\Lambda x}(\psi_x - i\lambda[\Lambda, \psi])\mathrm{e}^{i\lambda\Lambda x} = \mathrm{e}^{-i\lambda\Lambda x}\tilde{P}\psi\mathrm{e}^{i\lambda\Lambda x} \\
&\quad \Rightarrow \frac{d}{dx}\big(\mathrm{e}^{-i\lambda\Lambda x}\psi\mathrm{e}^{i\lambda\Lambda x}\big) = \mathrm{e}^{-i\lambda\Lambda x}\tilde{P}\psi\mathrm{e}^{i\lambda\Lambda x},
\end{aligned}$$

from which we naturally realize that we can convert the spatial part of the matrix spectral problem (4.39a) into the Volterra integral equations for ψ^+ and ψ^-:

$$\psi^-(\lambda,x) = I_4 + \int_{-\infty}^{x} e^{i\lambda\Lambda(x-x')}\tilde{P}(x')\psi^-(\lambda,x')e^{i\lambda\Lambda(x'-x)}dx', \tag{4.43a}$$

$$\psi^+(\lambda,x) = I_4 - \int_{x}^{\infty} e^{i\lambda\Lambda(x-x')}\tilde{P}(x')\psi^+(\lambda,x')e^{i\lambda\Lambda(x'-x)}dx', \tag{4.43b}$$

where the boundary condition (4.40) here takes effect. In order to extend ψ^+ and ψ^- analytically off the real axis, one necessarily has to perform a spectral analysis for (4.43a) and (4.43b) w.r.t. λ, i.e., to study the behavior of convergence of the integrals on the right hand sides. Recalling that $\alpha > 0$ and Λ is diagonal, we can directly find that the integrands of the integral equations for the third and fourth columns of ψ^- contain only the exponential factor $e^{-\alpha\lambda_2(x-x')}$, which attenuates as long as $\lambda = \lambda_1 + i\lambda_2 \in \mathbb{D}^+$, since $x' \leq x$ in the integral of (4.43a). So do the first and second columns of ψ^+ because the involved integrands contain only the factor $\mathrm{e}^{\alpha\lambda_2(x-x')}$ and $x' \geq x$ in the integral of (4.43b). This implies that the columns mentioned above are analytically continuable to $\mathbb{D}^+$. Likewise, we conclude that the first two columns of ψ^- and the last two columns of ψ^+ are analytically continuable to $\mathbb{D}^-$.

Or to put it more exactly, the idea involved in the spectral analysis is to classify the columns or rows in ψ^+ and ψ^-, and rearrange them according to their behavior of convergence in $\mathbb{D}^+$ and $\mathbb{D}^-$. If we denote

$$\psi^+ = (\psi_1^+, \psi_2^+, \psi_3^+, \psi_4^+), \quad \psi^- = (\psi_1^-, \psi_2^-, \psi_3^-, \psi_4^-) \tag{4.44}$$

with $\psi_j^\pm$ standing for the jth column of $\psi^\pm$ ($1 \leq j \leq 4$), then the re-assembled matrix solution

$$Y^+ = Y^+(x,\lambda) = (\psi_1^+, \psi_2^+, \psi_3^-, \psi_4^-) = \psi^+ H_1 + \psi^- H_2 \tag{4.45}$$

is analytic for all $\lambda \in \mathbb{D}^+$, whereas the other re-assembled matrix solution

$$(\psi_1^-, \psi_2^-, \psi_3^+, \psi_4^+) = \psi^- H_1 + \psi^+ H_2 \tag{4.46}$$

is analytic for all $\lambda \in \mathbb{D}^-$, where

$$H_1 = \mathrm{diag}(1,1,0,0), \quad H_2 = \mathrm{diag}(0,0,1,1). \tag{4.47}$$

Moreover, from the Volterra integral equation (4.43a), (4.43b), we see that

$$Y^+(x,\lambda) \to I_4, \quad \text{when } |\lambda| \to \infty \text{ in } \mathbb{D}^+, \tag{4.48}$$

and

$$(\psi_1^-, \psi_2^-, \psi_3^+, \psi_4^+) \to I_4, \quad \text{when } |\lambda| \to \infty \text{ in } \mathbb{D}^-. \tag{4.49}$$

We still have to construct the analytic counterpart of Y^+ in $\mathbb{D}^-$ from the adjoint counterparts of the matrix spectral problems. Considering that for a matrix A depending on x,

$$(AA^{-1})_x = A_x A^{-1} + AA_x^{-1} = 0, \quad \Rightarrow \quad A_x^{-1} = -A^{-1}A_x A^{-1},$$

let now $\tilde{\phi}^{\pm} = (\phi^{\pm})^{-1}$ and $\tilde{\psi^{\pm}} = (\psi^{\pm})^{-1}$, the adjoint equation of the x-part of (4.35) and the adjoint equation of (4.39a) can be derived to be

$$i\tilde{\phi}_x = i(-\tilde{\phi}\phi_x\tilde{\phi}) = i(-\tilde{\phi}iU\phi\tilde{\phi}) = \tilde{\phi}U, \tag{4.50}$$

and

$$\begin{aligned} i\tilde{\psi}_x &= -i\tilde{\psi}\psi_x\tilde{\psi} = -i\tilde{\psi}(i\lambda[\Lambda,\psi] + \tilde{P}\psi)\tilde{\psi} \\ &= \lambda\tilde{\psi}(\Lambda\psi - \psi\Lambda)\tilde{\psi} - i\tilde{\psi}\tilde{P} \\ &= \lambda(\tilde{\psi}\Lambda - \Lambda\psi) + \tilde{\psi}P \\ &= \lambda[\tilde{\psi},\Lambda] + +\tilde{\psi}P, \end{aligned} \tag{4.51}$$

respectively.

Note that $\tilde{\phi}^{\pm}$ and $\tilde{\psi^{\pm}}$ solve (4.50) and (4.51), respectively. Upon expressing $\tilde{\psi^{\pm}}$ as

$$\tilde{\psi^{\pm}} = \begin{pmatrix} \tilde{\psi}^{\pm,1} \\ \tilde{\psi}^{\pm,2} \\ \tilde{\psi}^{\pm,3}, \\ \tilde{\psi}^{\pm,4} \end{pmatrix}, \tag{4.52}$$

with $\tilde{\psi^{\pm,j}}$ standing for the jth row of $\tilde{\psi^{\pm}}$ $(1 \leq j \leq 4)$, we can verify by similar arguments that the adjoint matrix solution

$$Y^- = \begin{pmatrix} \tilde{\psi}^{+,1} \\ \tilde{\psi}^{+,2} \\ \tilde{\psi}^{-,3}, \\ \tilde{\psi}^{-,4} \end{pmatrix} = H_1\tilde{\psi}^+ + H_2\tilde{\psi}^- = H_1(\psi^+)^{-1} + H_2(\psi^-)^{-1} \tag{4.53}$$

is analytic for all $\lambda \in \mathbb{D}^-$, and the matrix solution

$$\begin{pmatrix} \tilde{\psi}^{-,1} \\ \tilde{\psi}^{-,2} \\ \tilde{\psi}^{+,3}, \\ \tilde{\psi}^{+,4} \end{pmatrix} = H_1\tilde{\psi}^- + H_2\tilde{\psi}^+ = H_1(\psi^-)^{-1} + H_2(\psi^+)^{-1} \tag{4.54}$$

is analytic for all $\lambda \in \mathbb{D}^+$. Pretty similarly, it applies also that

$$Y^-(x,\lambda) \to I_4, \quad \text{as } |\lambda| \to \infty \text{ in } \mathbb{D}^-, \tag{4.55}$$

and

$$\begin{pmatrix} \tilde{\psi}^{-,1} \\ \tilde{\psi}^{-,2} \\ \tilde{\psi}^{+,3}, \\ \tilde{\psi}^{+,4} \end{pmatrix} \to I_4, , \quad \text{as } |\lambda| \to \infty \text{ in } \mathbb{D}^+. \tag{4.56}$$

We have so far constructed the two matrix functions, Y^+ and Y^-, which are analytic in $\mathbb{D}^+$ and $\mathbb{D}^-$, respectively. Now define for all $\lambda \in \mathbb{R}$

$$G^+(x,\lambda) = \lim_{\mu \in \mathbb{D}^+, \mu \to \lambda} Y^+(x,\mu), \quad G^-(x,\lambda) = \lim_{\nu \in \mathbb{D}^-, \nu \to \lambda} Y^-(x,\nu), \tag{4.57}$$

we can directly show that the two matrix functions G^+ and G^- are related on the real line $\mathbb{R}$ by

$$G^-(x,\lambda)G^+(x,\lambda) = G(x,\lambda), \quad \lambda \in \mathbb{R}, \tag{4.58}$$

where by (4.42), it follows that for all $\lambda \in \mathbb{R}$,

$$\begin{aligned} G(x,\lambda) &= G^-(x,\lambda)G^+(x,\lambda) \\ &= \lim_{\nu \in \mathbb{D}^-, \nu \to \lambda} Y^-(x,\nu) \cdot \lim_{\mu \in \mathbb{D}^+, \mu \to \lambda} Y^+(x,\mu) \\ &= \lim_{\nu \in \mathbb{D}^-, \nu \to \lambda} (H_1(\psi^+)^{-1} + H_2(\psi^-)^{-1}) \cdot \lim_{\mu \in \mathbb{D}^+, \mu \to \lambda} (\psi^+ H_1 + \psi^- H_2) \\ &= E(H_1 + H_2 S^{-1}(\lambda))(H_1 + S(\lambda)H_2)E^{-1} \\ &= E \begin{bmatrix} 1 & 0 & s_{13} & s_{14} \\ 0 & 1 & s_{23} & s_{24} \\ \hat{s}_{31} & \hat{s}_{32} & 1 & 0 \\ \hat{s}_{41} & \hat{s}_{42} & 0 & 1 \end{bmatrix} E^{-1}. \end{aligned} \tag{4.59}$$

Note that $S^{-1}(\lambda) = (S(\lambda))^{-1} = (\hat{s}_{jk})_{4\times 4}$. The equations (4.58) and (4.59) are precisely the associated matrix Riemann-Hilbert problems one would like to build for the 8-component mKdV system (4.34a), (4.34b). The asymptotic behavior

$$Y^{\pm}(x,\lambda) \to I_4, \quad \text{when } |\lambda| \to \infty \text{ in } \mathbb{D}^{\pm} \tag{4.60}$$

is certainly consistent with the canonical normalization conditions

$$G^{\pm}(x,\lambda), G(x,\lambda) \to I_4, \quad \text{when } \lambda \in \mathbb{R}, |\lambda| \to \infty, \tag{4.61}$$

for the Riemann-Hilbert problems constructed above.

The direct scattering transform achieves it mission by obtaining the temporal evolution of the scattering matrix. So let's work out the derivative of (4.42) with respect to t and apply to it the condition that the potentials vanish at infinity of t. It can be thus derived that the time evolution of the scattering matrix S is governed by (observing that in this system, the matrices E and Ω are commutative):

$$\begin{aligned}
(\psi^- E)_t = (\psi^+ ES)_t, \quad \text{i.e.} \quad & \psi_t^- E = \psi_t^+ ES + \psi^+ ES_t, \\
\Rightarrow\ & (i\lambda^3[\Omega, \psi^-] + \tilde{Q}\psi^-)E = (i\lambda^3[\Omega, \psi^+] + \tilde{Q}\psi^+)ES + \psi^+ ES_t, \\
\Rightarrow\ & i\lambda^3(\Omega\psi^- - \psi^-\Omega)E = i\lambda^3(\Omega\psi^+ - \psi^+\Omega)ES + \psi^+ ES_t, \\
\Rightarrow\ & -i\lambda^3\psi^- E\Omega = -i\lambda^3\psi^+ E\Omega S + \psi^+ ES_t, \\
\Rightarrow\ & -i\lambda^3\psi^+ ES\Omega = -i\lambda^3\psi^+ E\Omega S + \psi^+ ES_t, \\
\Rightarrow\ & S_t = i\lambda^3[\Omega, S].
\end{aligned} \tag{4.62}$$

From this first-order linear matrix differential equation all the individual scattering coefficients of S can be solved as:

$$\begin{cases}
s_{13} = s_{13}(\lambda, 0)e^{i\beta\lambda^3 t}, & s_{14} = s_{14}(\lambda, 0)e^{i\beta\lambda^3 t}, \\
s_{23} = s_{23}(\lambda, 0)e^{i\beta\lambda^3 t}, & s_{24} = s_{24}(\lambda, 0)e^{i\beta\lambda^3 t}, \\
s_{31} = s_{31}(\lambda, 0)e^{-i\beta\lambda^3 t}, & s_{41} = s_{41}(\lambda, 0)e^{-i\beta\lambda^3 t}, \\
s_{32} = s_{32}(\lambda, 0)e^{-i\beta\lambda^3 t}, & s_{42} = s_{42}(\lambda, 0)e^{-i\beta\lambda^3 t},
\end{cases} \tag{4.63}$$

whereas all other scattering coefficients do not vary with t and hence are functions of λ only.

4.3 Soliton solutions

It is inevitable that we often encounter Riemann-Hilbert problems in which $\det Y^+$ and $\det Y^-$ have zeros. The most effective way of handling such RHPs is to transform them into the ones without zeros. However, if the numbers of zeros of $\det Y^+$ and $\det Y^-$ are not the same, the canonical normalization conditions are no longer guaranteed in such an transformation—which implies that this RHP is not solvable. Only when $\det Y^+$ and $\det Y^-$ have the same numebrs of simple zeros which are all specified, we can confidently say that this RHP can be uniquely solved [105, 106, 107].

Based on $\det \psi^\pm = 1$, it follows from the definitions of $Y^\pm$ and $\det S = 1$ that

$$\begin{aligned}
\det Y^+ &= s_{44}(\lambda)s_{33}(\lambda) - s_{34}(\lambda)s_{43}(\lambda) = \hat{s}_{11}(\lambda)\hat{s}_{22}(\lambda) - \hat{s}_{12}(\lambda)\hat{s}_{21}(\lambda), \\
\det Y^- &= \hat{s}_{44}(\lambda)\hat{s}_{33}(\lambda) - \hat{s}_{34}(\lambda)\hat{s}_{43}(\lambda) = s_{11}(\lambda)s_{22}(\lambda) - s_{12}(\lambda)s_{21}(\lambda).
\end{aligned}$$

Let $N > 0$ be another integer and assume that $\det Y^+$ has N roots $\lambda_k \in \mathbb{D}^+$, and $\det Y^-$ has N roots $\hat{\lambda}_k \in \mathbb{D}^-, 1 \leq k \leq N$. To generate simple soliton solutions, we assume also that all the zeros, λ_k and $\hat{\lambda}_k$ $(1 \leq k \leq N)$, are simple. Therefore, every $\ker Y^+(\lambda_k)$ $(1 \leq k \leq N)$ contains merely a single basis column vector, denoted by w_k; for the same reason there is merely a single basis row vector, $\hat{w}_k$ $(1 \leq k \leq N)$, contained in every $\ker Y^-(\hat{\lambda}_k)$:

$$Y^+(\lambda_k)w_k = 0, \quad \hat{w}_k Y^-(\hat{\lambda}_k) = 0, \qquad 1 \leq k \leq N. \tag{4.64}$$

The RHPs formulated by (4.58) and (4.59), with the canonical normalization conditions in (4.61) and the zero structures in (4.64) can be explicitly solved so as to work out the potential matrix P. Note that Y^+ solves the matrix spectral problem (4.39a). Hence, by expanding Y^+ at large λ as [39, 105]

$$Y^+(x, \lambda) = I_4 + \lambda^{-1} Y_1^+(x) + \mathrm{O}(\lambda^{-2}), \quad \lambda \to \infty,$$

and substituting this series expansion into (4.39a), the comparison of the $\mathrm{O}(1)$ terms will give

$$\tilde{P} = -i[\Lambda, Y_1^+]. \tag{4.65}$$

This is equivalent to retrieving the potential matrix as:

$$P = -[\Lambda, Y_1^+] = \begin{bmatrix} 0 & 0 & -\alpha(Y_1^+)_{13} & -\alpha(Y_1^+)_{14} \\ 0 & 0 & -\alpha(Y_1^+)_{23} & -\alpha(Y_1^+)_{24} \\ \alpha(Y_1^+)_{31} & \alpha(Y_1^+)_{32} & 0 & 0 \\ \alpha(Y_1^+)_{41} & \alpha(Y_1^+)_{42} & 0 & 0 \end{bmatrix}. \tag{4.66}$$

Alternatively, the matrix P can also be retrieved by expanding $Y^-(x, \lambda)$ as

$$Y^-(x, \lambda) = I_4 + \lambda^{-1} Y_1^-(x) + \mathrm{O}(\lambda^{-2}), \quad \lambda \to \infty,$$

and then substituting it into the adjoint equation (4.51), to yield

$$P = [\Lambda, Y_1^-]. \tag{4.67}$$

That is to say, the 8 potentials in submatrices p and q can be retrieved from Y^+ or Y^- as follows:

$$\begin{aligned} p_{11} &= -\alpha(Y_1^+)_{13} = \alpha(Y_1^-)_{13}, \quad p_{12} = -\alpha(Y_1^+)_{14} = \alpha(Y_1^-)_{14}, \\ p_{21} &= -\alpha(Y_1^+)_{23} = \alpha(Y_1^-)_{23}, \quad p_{22} = -\alpha(Y_1^+)_{24} = \alpha(Y_1^-)_{24}, \\ q_{11} &= \alpha(Y_1^+)_{31} = -\alpha(Y_1^-)_{31}, \quad q_{12} = \alpha(Y_1^+)_{32} = -\alpha(Y_1^-)_{32}, \\ q_{21} &= \alpha(Y_1^+)_{41} = -\alpha(Y_1^-)_{41}, \quad q_{22} = \alpha(Y_1^+)_{42} = -\alpha(Y_1^-)_{42}. \end{aligned} \tag{4.68}$$

We make a short pause here to elucidate briefly about how an RHP is solved. First of all, if $\det Y^{\pm}$ do not have zeros in $\mathbb{D}^+$ and $\mathbb{D}^-$, the RHP is said to be **regular** (it is worthwhile to mention that in case of a regular RHP, $G^-(x,\lambda)G^+(x,\lambda) = G(x,\lambda)$ for all $\lambda \in \mathbb{R}$ can be equivalently reformulated as $\hat{G}^+(x,\lambda) = \hat{G}^-(x,\lambda)G(x,\lambda)$ for all $\lambda \in \mathbb{R}$, where $\hat{G}^+(x,\lambda) = G^+(x,\lambda)$ and $\hat{G}^-(x,\lambda) = (G^-)^{-1}(x,\lambda)$—since G^- is invertible due to the regularity). A regular RHP subjected to the canonical normalization condition is proved [104] to be uniquely solvable in the light of complex analysis (in particular the Liouville's theorem, or its alternative—the so-called **Vanishing Lemma** [24]), and the solution can be formally expressed in terms of the Plemelj formula.

If however as indicated above, in case $N \geq 1$, such an RHP is said to be **nonregular**. The idea of solving nonregular RHPs is simply to convert them into regular ones. We introduce now the matrix

$$\Gamma(\lambda) = \Gamma_N(\lambda)\Gamma_{N-1}(\lambda)\ldots\Gamma_1, \tag{4.69}$$

where

$$\Gamma_k(\lambda) = I - \frac{\lambda_k - \hat{\lambda}_k}{\lambda - \hat{\lambda}_k}P_k, \quad 1 \leq k \leq N. \tag{4.70}$$

Here $P_k = \frac{|w_k\rangle\langle\hat{w}_k|}{\langle\hat{w}_k|w_k\rangle}$ is a projection operator, where to distinguish between the column vector w_k and the row vector $\hat{w}_k$, we have used the Dirac bra-ket notation $|w_k\rangle$ and $\langle\hat{w}_k|$. It is easy to check that $P_k^2 = P_k$. It is also direct to show that

$$\Gamma^{-1}(\lambda) = \Gamma_1^{-1}(\lambda)\Gamma_2^{-1}(\lambda)\ldots\Gamma_N^{-1}(\lambda), \tag{4.71}$$

where

$$\Gamma_k^{-1}(\lambda) = I - \frac{\hat{\lambda}_k - \lambda_k}{\lambda - \lambda_k}P_k, \quad 1 \leq k \leq N. \tag{4.72}$$

Let now $\tilde{Y}^+(\lambda) = Y^+(\lambda)\Gamma^{-1}(\lambda)$ and $\tilde{Y}^-(\lambda) = \Gamma(\lambda)Y^-(\lambda)$. Using complex analysis (e.g., residue thereoem) one straightforwardly proves that $\tilde{Y}^+(\lambda)$ and $\tilde{Y}^-(\lambda)$ are analytic in $\mathbb{D}^+$ and $\mathbb{D}^-$, respectively; and $\det\tilde{Y}^+(\lambda)$ and $\det\tilde{Y}^-(\lambda)$ no longer vanish at their original zeros—since, say, the zero λ_k in $Y^+(\lambda)$ is canceled out by $\lambda - \lambda_k$ living on the denominator in $\Gamma_k^{-1}(\lambda)$; the situation is likewise for $Y^-(\lambda)$ and $\Gamma(\lambda)$.

At this moment one can readily show that the original RHP can now be reformulated into

$$\tilde{G}^-(x,\lambda)\tilde{G}^+(x,\lambda) = \tilde{G}(x,\lambda), \quad \lambda \in \mathbb{R}, \tag{4.73}$$

where for all $\lambda \in \mathbb{R}$,

$$\tilde{G}^+(x,\lambda) = \lim_{\mu\in\mathbb{D}^+,\mu\to\lambda}\tilde{Y}^+(x,\mu), \quad \tilde{G}^-(x,\lambda) = \lim_{\nu\in\mathbb{D}^-,\nu\to\lambda}\tilde{Y}^-(x,\nu), \tag{4.74}$$

and

$$\tilde{G}(x,\lambda) = \Gamma^{-1}(\lambda)G(x,\lambda)\Gamma(\lambda). \tag{4.75}$$

Based on the arguments above, $\tilde{G}^+(x,\lambda)$ and $\tilde{G}^-(x,\lambda)$ do not have zeros in $\mathbb{D}^+$ and $\mathbb{D}^-$, respectively. Therefore the original nonregular RHP is thus reduced to a regular one.

We admit now $G = I_4$ in the Riemann-Hilbert problem (4.58) in order to calculate soliton solutions. One accomplishes this by setting $s_{13} = \hat{s}_{31} = s_{14} = \hat{s}_{41} = s_{23} = \hat{s}_{32} = s_{24} = \hat{s}_{42} = 0$, which corresponds to a reflectioless scattering problem. Following the above spirits, it is already verified that the solutions to the reduced Riemann-Hilbert problem can be explicitly written down as [39, 105, 104]

$$Y^+(\lambda) = I_4 - \sum_{k,l=1}^{N} \frac{w_k (M^{-1})_{kl} \hat{w}_l}{\lambda - \hat{\lambda}_l}, \quad Y^-(\lambda) = I_4 + \sum_{k,l=1}^{N} \frac{w_k (M^{-1})_{kl} \hat{w}_l}{\lambda - \lambda_l}, \tag{4.76}$$

where the entries of the square matrix $M = (m_{kl})_{N\times N}$ are computed through

$$m_{kl} = \frac{\hat{w}_k w_l}{\lambda_l - \hat{\lambda}_k}, \quad 1 \leq k, l \leq N. \tag{4.77}$$

The fact that the zeros λ_k and $\hat{\lambda}_k$ are space and time independent helps us to work out the evolutions with respect to space and time for the kernel vectors $w_k(x,t)$ and $\hat{w}_k(x,t)$ $(1 \leq k \leq N)$ routinely. Let's take, say, the first set of equations in (4.64), and compute its x-derivative. Recalling (4.39a) and (4.64), we obtain

$$Y^+(\lambda_k, x)\left(\frac{dw_k}{dx} - i\lambda_k \Lambda w_k\right) = 0, \quad 1 \leq k \leq N. \tag{4.78}$$

It thus implies that for each $1 \leq k \leq N$, $\frac{dw_k}{dx} - i\lambda_k \Lambda w_k$ lives in the kernel of $Y^+(\lambda_k, x)$ and hence must be a constant multiple of w_k, i.e., in general,

$$\frac{dw_k}{dx} - i\lambda_k \Lambda w_k = \alpha_k(x) w_k, \quad 1 \leq k \leq N, \tag{4.79}$$

where $\alpha_k(x)$ is a scalar function depending on x only, from which one obtains the solution

$$w_k(x) = \mathrm{e}^{i\lambda_k \Lambda (x - x_0)} w_{k_0} \mathrm{e}^{\int_{x_0}^{x} \alpha_k(s) ds}, \quad v_{k_0} = v_k|_{x=x_0}.$$

One finds that once the above $w_k(x)$ goes back into the formulae (4.76) and (4.77), the term $\mathrm{e}^{\int_{x_0}^{x} \alpha_k(s) ds}$ cancels out and becomes immaterial. Hence, without loss of generality we are happy to use here the simplest case, i.e.,

$$\frac{dw_k}{dx} = i\lambda_k \Lambda w_k, \quad 1 \leq k \leq N. \tag{4.80}$$

Pretty similarly, one obtains the time dependence of w_k

$$\frac{dw_k}{dt} = i\lambda_k^3 \Omega w_k, \quad 1 \leq k \leq N, \tag{4.81}$$

by playing the same trick upon the t-part of the matrix spectral problem. We obtain therefore

$$w_k(x,t) = \mathrm{e}^{i\lambda_k \Lambda x + i\lambda_k^3 \Omega t} w_{k,0}, \quad 1 \le k \le N, \tag{4.82a}$$

$$\hat{w}_k(x,t) = \hat{w}_{k,0} \mathrm{e}^{i\lambda_k \Lambda x + i\lambda_k^3 \Omega t}, \quad 1 \le k \le N, \tag{4.82b}$$

where $w_{k,0}$ and $\hat{w}_{k,0}$ are constant column and row vectors, respectively.

Eventually, through (4.68) we conclude

$$Y_1^+ = -\sum_{k,l=1}^{N} w_k (M^{-1})_{kl} \hat{w}_l. \tag{4.83}$$

And in turn through the results we obtained above in (4.76), the N-soliton solution to our 8-component mKdV system can be represented by

$$p_{rj} = \alpha \sum_{k,l=1}^{N} w_{k,r} (M^{-1})_{kl} \hat{w}_{l,j+2}, \quad q_{rj} = -\alpha \sum_{k,l=1}^{N} w_{k,r+2} (M^{-1})_{kl} \hat{w}_{l,j}, \tag{4.84}$$

where $1 \le r, j \le 2$, $w_k = (w_{k,1}, w_{k,2}, w_{k,3}, w_{k,4})^T$ and $\hat{w}_k = (\hat{w}_{k,1}, \hat{w}_{k,2}, \hat{w}_{k,3}, \hat{w}_{k,4})$ $(1 \le k \le N)$ are given by (4.82a) and (4.82b), respectively.

4.4 Reductions

We make a specific reduction by applying the following constraints to the potential matrix P:

$$P^\dagger = CPC^{-1}, \quad C = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}, \tag{4.85}$$

where the application of $\dagger$ produces the Hermitian transpose of a matrix and $\Sigma_{1,2}$ are 2×2 constant Hermitian symmetric matrices: $\Sigma_{1,2}^\dagger = \Sigma_{1,2}$. Below we shall denote the complex conjugate of a complex number z by $\bar{z}$; and for a matrix B depending on the spectral parameter λ, we denote the Hermitian transpose of $B(\lambda)$ by $B^\dagger(\bar{\lambda})$, the inverse of $B(\bar{\lambda})$ by $B^{-1}(\bar{\lambda})$, respectively; i.e., $B^\dagger(\bar{\lambda}) = (B(\lambda))^\dagger$ and $B^{-1}(\bar{\lambda}) = (B(\bar{\lambda}))^{-1}$.

Let $\psi(\lambda)$ be a matrix eigenfunction of (4.39a). One can easily prove that both matrix adjoint eigenfunctions $C\psi^{-1}(\bar{\lambda})$ and $\psi^\dagger(\bar{\lambda})C$ solve the adjoint equation (4.51) associated with the eigenvalue $\bar{\lambda}$. Upon noticing the asymptotic behavior of $\psi^\pm$ as $\lambda \to \infty$, the uniqueness of solution therefore guarantees that for either $\psi = \psi^+$ or $\psi = \psi^-$,

$$C\psi^{-1}(\bar{\lambda}) = \psi^\dagger(\bar{\lambda})C \quad \Rightarrow \quad \psi^\dagger(\bar{\lambda}) = C\psi^{-1}(\bar{\lambda})C^{-1}. \tag{4.86}$$

Moreover, one derives also from the definitions of $Y^\pm$ that the so-called involution relation can be constructed between Y^+ and Y^-:

$$(Y^+)^\dagger(\bar{\lambda}) = CY^-(\bar{\lambda})C^{-1}, \tag{4.87}$$

as well as between the scattering matrix $S(\lambda)$ and its inverse (likewise by definition):

$$S^{\dagger}(\bar{\lambda}) = CS^{-1}(\bar{\lambda})C^{-1}. \tag{4.88}$$

For the zeros of $\det Y^{\pm}$, starting from (4.87) one can show that

$$\begin{aligned} \det(Y^{+})^{\dagger}(\bar{\lambda}) = \det CY^{-}(\bar{\lambda})C^{-1}, \quad &\Rightarrow \det(Y^{+}(\lambda))^{\dagger} = \det Y^{-}(\bar{\lambda}), \\ &\Rightarrow \overline{\det Y^{+}(\lambda)} = \det Y^{-}(\bar{\lambda}), \end{aligned}$$

from which it follows immediately that under the reduction given by (4.85), $\det Y^{+}(\lambda) = 0$ if and only if $\det Y^{-}(\bar{\lambda}) = 0$; or in another word under this reduction one fabulously associates λ_k and $\bar{\lambda}_k$ in terms of

$$\hat{\lambda}_k = \bar{\lambda}_k, \quad 1 \leq k \leq N. \tag{4.89}$$

We now compute the Hermitian transpose of the first equation in (4.64) in order to obtain the involution eigenvectors v_k and $\hat{v}_k$ with the existence of (4.87) and (4.89):

$$Y^{+}(\lambda_k)w_k = 0 \quad \Rightarrow \quad 0 = w_k^{\dagger}(Y^{+}(\lambda_k))^{\dagger} = w_k^{\dagger}CY^{-}(\bar{\lambda}_k)C^{-1}, \quad 1 \leq k \leq N. \tag{4.90}$$

It is implied in the above result that the involution eigenvectors can be written for all $1 \leq k \leq N$ as:

$$w_k(x,t) = \mathrm{e}^{i\lambda_k \Lambda x + i\lambda_k^3 \Omega t} w_{k,0}, \quad \hat{w}_k(x,t) = w_k^{\dagger}C = w_{k,0}^{\dagger}\mathrm{e}^{-i\bar{\lambda}_k \Lambda x - i\bar{\lambda}_k^3 \Omega t}C, \tag{4.91}$$

where $v_{k,0}$ are constant column vectors.

We take now

$$\Sigma_1 = \frac{1}{\sigma_1}I_2, \quad \Sigma_2 = \frac{1}{\sigma_2}I_2, \quad \sigma_1, \sigma_2 \in \mathbb{R}\backslash\{0\}, \tag{4.92}$$

from which (and (4.85)) it follows immediately that

$$q = \frac{\sigma_2}{\sigma_1}p^{\dagger}. \tag{4.93}$$

If one admits simultaneously the matrix function c in (4.12) to be

$$c = \frac{\sigma_2}{\sigma_1}b^{\dagger}, \tag{4.94}$$

it can be derived easily that all these will guarantee

$$a^{\dagger} = a, \quad d^{\dagger} = d, \tag{4.95}$$

where a and d are given by (4.12). In turn this will lead to (by consideration of (4.36) and (4.37))

$$(V^{[3]})^{\dagger}(\bar{\lambda}) = CV^{[3]}(\bar{\lambda})C^{-1}, \quad Q^{\dagger}(\bar{\lambda}) = CQ(\bar{\lambda})C^{-1}. \tag{4.96}$$

The reduction depicted by (4.92) therefore takes effect reasonably for the x-part and t-part of the matrix spectral problems (4.35). The 8-component mKdV system (4.34a), (4.34b) is thus reduced to

$$p_{rj,t} = -\frac{\beta}{\alpha^3}p_{rj,xxx} - 3\frac{\beta\sigma_2}{\alpha^3\sigma_1}\Big(\sum_{k,l=1}^{2}|p_{kl}|^2\Big)p_{rj,x} - 3\frac{\beta\sigma_2}{\alpha^3\sigma_1}\Big(\sum_{k,l=1}^{2}p_{kl,x}\bar{p}_{kl}\Big)p_{rj}$$
$$+ 3\frac{\beta\sigma_2}{\alpha^3\sigma_1}C_{rj}(p^\dagger)(\det p)_x, \qquad 1 \leq r, j \leq 2. \tag{4.97}$$

One sees that in such a reduction only the ratio σ_2/σ_1, other than the exact values of $\sigma_{1,2}$, is fundamental.

4.5 Concluding remarks

In this chapter, by starting from a generalized mKdV matrix spectral problem that has been properly treated to guarantee analytical continuation of eigenfunctions into $\mathbb{D}^+$ and $\mathbb{D}^-$, respectively, we have first computed its soliton hierarchy. We have then developed an inverse scattering transform in terms of Riemann-Hilbert formulation for one of the equations in this hierarchy. Based on this, the computation of soliton solutions has been successfully performed by reducing the Riemann-Hilbert problems to those with an identity jump matrix, i.e., those correspond to a reflectionless inverse scattering problems. Both the Riemann-Hilbert approach and the inverse scattering transform demonstrate their power in linearizing a problem which is originally nonlinear in essence.

The Riemann-Hilbert problems formulated from inverse scattering transforms has become, apart from the Hirota bilinear method, the Wronskian technique, the Bäcklund and Darboux transformations [108], another very effective tool in computing soliton solutions [104, 109, 103]. At the moment, various attempts are also being made to apply this tool in generating lump solutions [94, 110], positon and complexiton solutions [95, 111], as well as algebro-geometric solutions [112, 113]. Also recently, generalizations of this method were applied in solving some specific Cauchy problems of nonlinear integrable equations on the half real line [114], as well as on some finite intervals [115].

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