May 2018

A Hybrid Dynamic Modeling of Time-to-event Processes and Applications

Emmanuel A. Appiah
University of South Florida, emmanuel1@mail.usf.edu

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by

Emmanuel A. Appiah

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
Department of Mathematics & Statistics
College of Arts and Sciences
University of South Florida

Major Professor: Gangaram S. Ladde, Ph.D.
Leslaw Skrzypek, Ph.D.
Mohamed Elhamdadi, Ph.D.
Brian Curtin, Ph.D.

Date of Approval:
May 23, 2018

Keywords: Binary State, Invariant Sets, Modified LLGMM, Stochastic Hybrid System, Survival Principle

Dedication

To my beloved family.

Acknowledgments

First and foremost, I would like to thank God Almighty for guiding me successfully through this journey. Without His grace and manifold blessings, I would not have been able to accomplish this feat.

My profound gratitude goes to my advisor, Professor Gangaram Ladde for his advice, help, guidance, and encouragement. His patience, diligence, critical thinking, and genuine care and concern for students has greatly influenced me.

I am also grateful to Dr. Leslaw Skrzypek, Dr. Mohamed Elhamdadi, and Dr. Brian Curtin for their support and serving as members of my dissertation committee. My sincere appreciation also goes to Dr. Kirpal Bisht for taking time out of his busy schedule to chair the defense committee.

I would also like to thank Jay G. Ladde, M.D., for his insightful comments and suggestions.

I thankfully acknowledge financial support (Summer 2015 - Spring 2018) by the Mathematical Sciences Division, U.S. Army Research Office, Grant No: W911NF-15-1-0182.

My warm appreciation goes to all the faculty and staff of the mathematics and statistics department who contributed in diverse ways to make my experience in the department a memorable one.

Most importantly, I would like to express my heartfelt gratitude to my beloved family and friends for their love, support, and encouragement.

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Abstract

In the survival and reliability data analysis, parametric and nonparametric methods are used to estimate the hazard/risk rate and survival functions. A parametric approach is based on the assumption that the underlying survival distribution belongs to some specific family of closed form distributions (normal, Weibull, exponential, etc.). On the other hand, a nonparametric approach is centered around the best-fitting member of a class of survival distribution functions. Moreover, the Kaplan-Meier and Nelson-Aalen type nonparametric approach do not assume either distribution class or closed-form distributions. Historically, well-known time-to-event processes are death of living specie in populations and failure of component in engineering systems. Recently, the human mobility, electronic communications, technological changes, advancements in engineering, medical, and social sciences have further diversified the role and scope of time-to-event processes in cultural, epidemiological, financial, military, and social sciences. To incorporate extensions, generalizations and minimize scope of existing methods, we initiate an innovative alternative modeling approach for time-to-event dynamic processes. The innovative approach is composed of the following basic components: (1) development of continuous-time state of dynamic process, (2) introduction of discrete-time dynamic intervention process, (3) formulation of continuous and discrete-time interconnected dynamic system, (4) utilizing Euler-type discretized schemes, developing theoretical dynamic algorithms, and (5) introduction of conceptual and computational state and parameter estimation procedures. The presented approach is motivated by state and parameter estimation of time-to-event processes in biological, chemical, engineering, epidemiological, medical, military, multiple-markets and social dynamic processes under the influence of discrete-time intervention processes. We initiate (1) a time-to-event process to be a probabilistic dynamic process with unitary state. Action, normal, operational, radical, survival, susceptible, etc. and its complementary states, reaction, abnormal, nonoperational, non-radical, failure, infective and so on (quantitative and qualitative variables), are considered to be illustrations of a unitary state of time-to-event dynamic processes. A unitary state is measured by a probability distribution function. Employing Newtonian dynamic modeling approach and observing the definition of hazard rate as a specific rate, survival or failure probabilistic state dynamic model is developed. This dynamic model is further extended to incorporate internal or external discrete-time dynamic intervention processes acting on unitary state time-to-event processes (2). This further demanded a formulation and development of an interconnected continuous-discrete-time hybrid, and totally discrete-time dynamic models for time-to-event processes (3). Employing the developed hybrid model, Euler-type

discretized schemes, a very general fundamental conceptual analytic algorithm is outlined (4). Using the developed theoretical computational procedure in (4), a general conceptual computational data organizational and simulation schemes are presented (5) for state and parameter estimation problems in unitary state time-to-event dynamic processes. The well-known theoretical existing results in the literature are exhibited as special cases in a systematic and unified manner (6). In fact, the Kaplan-Meier and Nelson-Aalen type non-parametric estimation approaches are systematically analyzed by the developed totally discrete-time hybrid dynamic modeling process. The developed approach is applied to two data sets. Moreover, this approach does not require a knowledge of either a closed-form solution distribution or a class of distributions functions. A hazard rate need not be constant. The procedure is dynamic. In the existing literature, the failure and survival distribution functions are treated to be evolving/progressing mutually exclusively with respect to corresponding to two mutually exclusive time varying events. We refer to these two functions (failure and survival) as cumulative distributions of two mutually disjoint state output processes with respect to two mutually exclusive time-varying complementary unitary states of a time-to-event processes in any discipline of interest (7). This kind of time-to-event process can be thought of as a Bernoulli-type of deterministic/stochastic process. Corresponding to these two complementary output processes of the Bernoulli-type of stochastic process, we associate two unitary dynamic states corresponding to a binary choice options/actions (8), namely, ({action, reaction}, {normal, abnormal}, {survival, failure}, {susceptible, infective}, {operational, nonoperational}, {radical, non-radical}, and so on.) Under this consideration, we extend unitary state time-to-event dynamic model to binary state time-to-event dynamic model. Using basic tools in mathematical sciences, we initiate a Newtonian-type dynamic approach for binary state time-to-event processes in the sciences, technologies, and engineering (9). Introducing an innovative concept of "survival state dynamic principle", an innovative interconnected nonlinear non-stationary large-scale hybrid dynamic model for number of units/species and its unitary survival state corresponding to binary state time-to-event process is formulated (10). The developed model in (10) includes dynamic model (3) as a special case. The developed approach is directly applicable to binary state time-to-event dynamic processes in biological, chemical, engineering, financial, medical, physical, military, and social sciences in a coherent manner. A by-product of this is a transformed interconnected nonlinear hybrid dynamic model with a theoretical discrete-time conceptual computational dynamic process (11). Employing the transformed discrete-time conceptual computational dynamic process, we introduce notions of data coordination, state data decomposition and aggregation, theoretical conceptual iterative processes, conceptual and computational parameter estimation and simulation schemes, conceptual and computational state simulation schemes in a systematic way (12). The usefulness of the developed interconnected algorithm is validated by using three real world data sets (13). We note that the presented algorithm does not need a closed-form representation of distribution/likelihood function.

In fact, it is free from any required assumptions of the “Classical Maximum Likelihood Function Approach” in the “Survival and Reliability Analysis.”

The rapid electronic communication and human mobility processes have facilitated to transform information, knowledge, and ideas almost instantly around the globe. This indeed generates heterogeneity, and it causes to form nonlinear and non-stationary dynamic processes. Moreover, the heterogeneity, nonlinearity, non-stationarity, further generates two types of uncertainties, namely, deterministic, and stochastic. In view of this, it is obvious that nothing is deterministic. In short, the 21st century problems are highly nonlinear, non-stationary and under the influence of internal and external random perturbations. Using tools in stochastic analysis, interconnected deterministic models in (3) and (10) are extended to interconnected stochastic hybrid dynamic model for binary state time-to-event processes (14). The developed model is described by a large-scale nonlinear and non-stationary stochastic differential equations. Moreover, a stochastic version of a survival function is also introduced (15). Analytical, computational, statistical, and simulation algorithms/procedures are also extended and analyzed in a systematic and unified way (16). The presented interconnected stochastic model is motivated to initiate conceptual computational parameter and state estimation schemes for time-to-event statistical data (17). Again, stochastic version of computational algorithms are validated in the context of three real world data sets. The obtained parameter and state estimates show that the algorithm is independent of the choice of nonlinear transformation (18).

Utilizing the developed alternative innovative procedure and the recently modified deterministic version of Local Lagged Adapted Generalized Method of Moments (LLGMM) is also extended to stochastic version in a natural way (19). This approach provides a degree of measure of confidence, prediction, and planning assessments (20). In addition, it initiates a conceptual computational parameter and state estimation and simulation schemes that is suitable for the usage of mean square sub-optimal procedure (21). The usefulness and the significance of the approach is illustrated by applying to three data sets (22). The approach provides insight for investigating various type of invariant sets, namely, sustainable/unsustainable, survival/failure, reliable/unreliable (23), and qualitative properties such as sustainability versus unsustainability, reliability versus unreliability, etc. (24) Once again, the presented algorithm is independent of any form of survival distribution functions or data sets. Moreover, it does not require a closed form survival function distribution. We also note that the introduction of intervention processes provides a measure of influence and confidence for the usage of new tools/procedures/approaches in continuous-time binary state time-to-event dynamic process (25). Moreover, the presented dynamic modeling is more feasible for its usage of investigating a more complex time-to-event dynamic process (26). The developed procedure is dynamic and indeed nonparametric (27). The dynamic approach adapts with current changes and updates statistic process (28). The dynamic nature is natural rather than the existing static and single-shot techniques (29).

Chapter 1
Linear Hybrid Deterministic Dynamic Modeling for Time-to-event Processes

1.1 Introduction

In the survival and reliability data analysis, the main interest is focused on a nonnegative random variable, say T which describes a time-to-event process characterizing an occurrence of time until a certain event. Historically, well-known time-to-event processes are deaths in population dynamic and component failures in mechanical systems [25]. The human mobility, electronic communications, technological changes, advancements in engineering, medical, and social sciences have diversified the role and scope of time-to-event processes in cultural, epidemiological, financial, military and social sciences [2, 11, 32, 34, 50].

The study of survival analysis rests on the concept of time-to-event. The mathematical statistics development of time-to-event analysis is based on the probabilistic approach and the concept of hazard rate. Moreover, the time-to-event is described by the closed form expressions of survival function that is determined by the concept of hazard rate [25, 37, 39]. We note that in general, hazard rate is unknown. This leads to a problem of determining hazard rate function. This is based on a feasible approach of collecting data set for the time-to-event processes in biological, chemical, engineering, epidemiological, medical, multiple-markets and social sciences. The hazard/risk rate and survival function estimation problems in the survival and reliability analysis are centered around the idea of "right censored data" [39]. In fact, the common conventional understanding for resolving ties between censored and uncensored observations is adopted by shifting the censored observations slightly to the left of uncensored observations [51]. In short, the items/individuals/objects in a given sample are decomposed into two mutually exclusive groups, namely, (a) deaths/failure/removal/non-operational/inactive, and (b) censored/losses/withdrawals.

In the survival and reliability data analysis, parametric and nonparametric methods are applied to estimate the hazard/risk rate and survival functions [25, 37]. A parametric approach is based on the assumption that the underlying survival distribution belongs to some specific family of distributions (e.g. normal, Weibull, exponential). On the other hand, a nonparametric approach is centered around the best-fitting member of a class of survival distribution functions [26]. Moreover, Kaplan-Meier(KME) [26] and Nelson-Aalen [1, 41] type nonparametric approach do not assume either distribution class or closed-form distributions. In fact, it just depends on a data. The Kaplan-Meier and Nelson-Aalen type nonparametric estimation approaches are systematically analyzed by our totally discrete-time hybrid dynamic modeling process.

In the existing literature [25, 37], the closed-form expression for a survival function is based on the usage of probabilistic analysis approach. The closed-form representation of the survival function coupled with mathematical statistics method (parametric approach) is used to estimate both survival and hazard/risk rate functions. In fact, the parametric approach/model has advantages of simplicity, the availability of likelihood based inference procedures and the ease of use for a description, comparison, prediction, or decision [37]. In this work, we initiate an innovative alternative approach for modeling time-to-event dynamic processes. This approach leads to the development for estimating survival and hazard/risk rate functions. The presented approach is motivated by a simple observation regarding the probabilistic definition of the survival function [25]. Moreover, this approach does not require a knowledge of either a closed-form solution distribution or a class of distributions.

Historically, exponential distributions have been widely used in analyzing survival/reliability data [14, 37]. This was partly due to the mathematical simplicity and the availability of simple statistical methods. An application of the exponential model with covariates to medical survival data was initiated in Feigl and Zelen (1965). The assumption of a constant hazard/risk rate function is very restrictive. In fact, it is often violated. This is due to the fact that in some real life applications, sudden changes in the hazard rate at unknown times can be encountered due to a major maintenance in a mechanical system or a new treatment procedure in medical sciences [2]. For example, usually a machine component functions with a constant hazard/risk rate function λ_1, until it suffers a shock. After this shock, the component may continue to operate but with a different constant hazard/risk rate function λ_2. In the medical field, there is usually a high initial risk after a major operation which settles down to a lower constant long-term risk rate (Anis, 2009). This type of change could occur in multiple times. In view of this, one is often interested in detecting the locations of such changes and estimating the size of the detected changes. Recently, several authors [17, 19–21] have proposed estimators based on change point hazard models. A Bayesian approach for estimating the piecewise exponential distribution [18] and estimating the grid of time-points [15] for the piecewise exponential model are also available in the literature. In order to incorporate these types of sudden changes (intervention process) in the hazard rate function, we modify the developed continuous state dynamic model to an interconnected hybrid dynamic model that is composed of both continuous time state and discrete time state (intervention process) dynamic processes.

Employing the total time on test (TTT) for undefined censored data beyond the last observation, the idea of Piecewise Exponential Estimator (PEXE) of a survival function was introduced by [28] and applied for estimating life distribution from incomplete data. The PEXE has been modified to address the issues regarding the presence of ties in the data by Whittemore and Keller [51].

The comparison of the PEXE with the KME [27] exhibits the advantage of the PEXE over the KME. For example, the PEXE is a continuous survival function. Moreover, it exhibits the complete information that is coming from the censored data. Using a total time test and the PEXE based approach, the estimators of the hazard/risk rate and cumulative distribution functions on the left closed pairwise consecutive failure time intervals are determined in Kulasekera and White [30]. The PEXE is further extended by Malla and Mukerjee [38] with an exponential tail extension in the framework of the Kaplan and Meier [26] nonparametric estimator approach. Under the presented dynamic framework, we develop the PEXE and new PEXE of Malla and Mukerjee [38] types in a systematic and unified way. In short, the presented novel approach incorporates all the existing features such as: incomplete data, issues regarding the ties, exponential tail extensions in the framework of Kaplan and Meier [26], and so on in a coherent manner.

The organization of this chapter is as follows. In Section 1.2, recognizing the classical probabilistic analysis model of time-to-event as a dynamic process, we initiate a linear hybrid deterministic dynamic model for time-to-event processes. Moreover, a fundamental mathematical result that provides a basis for interconnected continuous-discrete-time and totally discrete-time dynamic processes, is developed. Utilizing the dynamic model and the main result developed in Section 1.2, basic conceptual analytic algorithms and its special cases for interconnected continuous-discrete-time and totally discrete-time linear hybrid dynamic models for time-to-event processes are presented in Section 1.3. In Section 1.4, we outline theoretical and computational procedures and results for parameter and state estimations for time-to-event processes. Moreover, several well-known results are exhibited as special cases. In Section 1.5, we present a very general conceptual and computational algorithm for estimating a hazard/risk rate function for multiple censoring times between consecutive failure times. These general results include the presented results in Section 1.4 as special cases.

1.2 Linear Hybrid Dynamic Modeling of Time-to-event Process

In this section, based on the probabilistic definition of the survival function, we develop a model for time-to-event dynamic processes. From the probabilistic definition of the survival function [25, 37, 39] and differential calculus [3], we recognize that

$$\lambda(t)\Delta t \approx \frac{S(t) - S(t+\Delta t)}{S(t)}\,, \tag{1.2.1}$$

where S and λ are survival and hazard/risk rate functions, respectively. Moreover, from (1.2.1) and differential calculus [3], we have

$$\mathrm{d}S = -\lambda(t)S\mathrm{d}t\,, \quad S(t_0) = S_0\,, \quad t \in [t_0, \infty)\,, \tag{1.2.2}$$

where $\mathrm{d}S$ is a differential of a survival function S. In fact, (1.2.2) is a differential equation, and it is an initial value problem (IVP) [32]. Based on continuous-time dynamic modeling [32], (1.2.2) represents a continuous-time linear dynamic model of time-to-event processes. In fact, we consider time-to-event processes to be probabilistic dynamic processes. The state of the process is represented by survival/infective/operational/radical and its complementary state, failure/removal/death/non-operational/normal, and it is measured by a probability distribution function. Employing Newtonian modeling approach, the instantaneous rate of change of survival state is directly proportional to the magnitude of the survival. The negative sign in (1.2.2) signifies that the state of survival is decaying/diminishing/decreasing. λ is a positive constant of proportionality. In general, it is a function of time. This is because of the fact that in general, the time-to-event processes are non-stationary. The solution of (1.2.2) on the interval $[t_0, \infty)$ is given by

$$S(t) = S_0 \exp\left[-\Lambda(t)\right] , \tag{1.2.3}$$

where

$$\Lambda(t) = \int_0^t \lambda(u)\mathrm{d}u \,, \tag{1.2.4}$$

and it is the cumulative hazard/risk rate function.

REMARK 1.2.1 If $\lambda(t) = \lambda$ for $t \geq 0$, $t_0 = 0\,, S(0) = 1$, then (1.2.3) reduces to the following well-known exponential distribution function:

$$S(t) = \exp[-\lambda t] \,, \quad t \in [0, \infty) \,, \tag{1.2.5}$$

and a complementary state of the survival state of time-to-event process is represented by

$$F(t) = 1 - S(t) = 1 - \exp[-\lambda t] \,, \quad t \in [0, \infty) \,,$$

and it is referred as a failure distribution function. Furthermore, we note that survival state dynamic model (1.2.2) signifies that the time-to-event process is closed (Rosen, 1970), that is, $S(t)+F(t) = 1$. It is analogous to epidemiological dynamic modeling process without removal [32, 50].

The presented motivational observation coupled with the introduction of the idea of continuous-time state dynamic process (1.2.2) operating under the discrete-time intervention processes further leads to a development of a linear hybrid dynamic model [32] for time-to-event processes. It is known [32] that many real world time-to-event dynamic processes are subject to intervention processes (internal or external). Therefore, it is natural that time-to-event dynamic processes undergo state adjustment processes. This causes a modification of the presented state dynamic processes that are described by simple state dynamic model (1.2.2). We note that the dynamic state adjustment processes are caused by periodic changes in science, technology, medicine, culture, socio-economic, environmental conditions and general behavior.

In the following, we introduce a type of hazard/risk rate function. Moreover, using dynamic approach, we present a development of PEXE [27, 28] in a systematic and unified way.

DEFINITION 1.2.1 Let $t_0 < t_1 < t_2 < \ldots < t_k < t_{k+1}$ be a given partition of a time interval $[t_0, \mathcal{T}]$, with $t_0 = 0$ and $t_{k+1} = \infty$. Let $\lambda_1, \lambda_2, \ldots, \lambda_{k+1}$ be model parameters. A hazard/risk rate function for a nonnegative random variable T that characterizes time-to-event processes, is of the following form:

$$\lambda(t) = \sum_{i=1}^{k+1} = \lambda_j I_{[t_{j-1}, t_j)}(t) \,, \quad t \in \mathbf{R}_+ = [0, \infty) \,, \tag{1.2.6}$$

where λ_j are positive real numbers for $j \in I(1, k+1)$, $(I(1,l) = \{1, 2, \ldots, l\})$; $I_{[t_{j-1}, t_j)}$ is the characteristic function with respect to $[t_{j-1}, t_j)$. Moreover, T is said to have a piecewise constant hazard function.

Definition 1.2.2 $\prod\limits_{i|t_j\le t}$ denotes the symbol for a product of objects for all positive integers $i \in I(1,\infty)$ that satisfy the conditions $t_i \le t_j$ and $t_j \le t < t_{j+1}$ for some $j \in I(1,n)$ and for $t_i, t_{j-1}, t_{j+1}, t \in [t_0, \mathfrak{T}]$.

From Definition 1.2.1, we recognize that the sudden changes in the hazard/risk rate function are encountered due to various types of intervention processes (internal or external) [32]. This causes to interrupt the current continuous-time state dynamic process (1.2.2). Following the linear hybrid dynamic model [32], a modified version of time-to-event dynamic model (1.2.2) is represented by:

$$\begin{cases} \mathrm{d}S = -\lambda(t)S\mathrm{d}t\,, \quad S(t_{j-1}) = S_{j-1}\,, \quad t \in [t_{j-1}, t_j)\,, \\ S_j = S(t_j^-, t_{j-1}, S_{j-1})\,, \quad S(t_0) = S_0\,, \quad j \in I(1, k+1)\,, \end{cases} \tag{1.2.7}$$

where $S(t_j^-) = S(t_j^-|\lambda, t_{j-1}, S_{j-1})$ describes a very simple form of intervention process generated at an intervention time t_j; t_j^- stands for $t \in [t_{j-1}, t_j)$, that is less than t_j and very close to t_j. We note that System (1.2.7) is interconnected hybrid dynamic system composed of both continuous and discrete time state dynamic systems. Imitating the procedure described in Ladde and Ladde [32], the solution process of the IVP (1.2.7) is as follows:

$$S(t, t_{j-1}, S_{j-1}|\lambda) = S_{j-1}\exp\left[-\int_{t_{j-1}}^{t} \lambda(u)\mathrm{d}u\right]\,, \text{ for all } \ t \in [t_{j-1}, t_j)\,. \tag{1.2.8}$$

Furthermore, the solution process of the overall time-to-event dynamic process (1.2.7) on $[t_0, \mathfrak{T})$ is

$$S(t, t_{j-1}, S_0|\lambda) = S_0 \prod_{m=1}^{j-1} \exp\left[-\int_{t_{m-1}}^{t_m} \lambda(u)\mathrm{d}u\right] \exp\left[-\int_{t_{j-1}}^{t} \lambda(u)\mathrm{d}u\right], \; t \in [t_0, \mathfrak{T})\,, j \in I(1, k+1)\,. \tag{1.2.9}$$

Remark 1.2.2 From (1.2.7) and (1.2.8), we note that the solution process (1.2.8) is indeed PEXE [27, 28].

In the following, we present a very simple fundamental auxiliary result that would be used, subsequently. Moreover, it exhibits an analytic unified bridge and basis for (1.2.7) and its complete discrete-time version.

Theorem 1.2.1 *Let $\{t_j\}_0^n$ be a partition of $[0, \mathfrak{T}]$ and let β be a monotonic nondecreasing function defined by*

$$\beta(t) = \begin{cases} 0, & t \in [t_{j-1}, t_j)\,, \\ 1, & t = t_j\,, \end{cases} \tag{1.2.10}$$

for each $j \in I(1,n)$. Let x be a state dynamic process in biological, engineering, epidemiological, human, medical, military, physical and social sciences under the influence of time-to-event processes. Let x be described by:

$$\begin{cases} \mathrm{d}x = [-\alpha(t)\, x + \gamma(t)]\, \mathrm{d}\beta(t), \quad t \in [t_{j-1}, t_j)\,, \\ x_j = (1-\alpha_j)x(t_j^-, t_{j-1}, x_{j-1}) + \gamma_j\,, \quad x(t_0) = x_0\,, \end{cases} \tag{1.2.11}$$

where α and γ are real-valued continuous functions defined on $[0,\infty)$; $\alpha_j = \alpha(t_j)$ and $\gamma_j = \gamma(t_j)$. Then

$$x(t) = \prod_{k|t_j \le t} (1-\alpha_k)x_0 + \sum_{i=1}^{j-1} \Phi(t,t_i)\gamma_i + \gamma_j\,, \quad \textit{for} \quad t \ge t_0\,, \tag{1.2.12}$$

where j is the largest integer so that $t_j \le t < t_{j+1}$, $t_k \le t_j$ and

$$\Phi(t,t_i) = \prod_{t_i \le t_j \le t} (1-\alpha_i)\,, \quad \Phi(t_i,t_i) = 1 \quad \text{for} \quad i \in I(0,n).$$

Proof. The theorem is proved by the principle of mathematical induction (PMI) [32]. From (1.2.11), for $j = 1$, we have

$$\mathrm{d}x = [-\alpha(t)\, x + \gamma(t)]\mathrm{d}\beta(t),\ x(t_0) = x_0,\ t \in [t_0, t_1)\ .$$

From (1.2.10) and the definition of Riemann-Stieltjes integral [4], we have

$$x(t) - x(t_0) = \int_{t_0}^{t} [-\alpha(s)\, x(s) + \gamma(s)]\mathrm{d}\beta(s) = 0, \text{ for } t \in [t_0, t_1). \tag{1.2.13}$$

We define

$$x(t) = x(t,t_0,x_0) = x_0(t,t_0,x_0)\,, \quad x_0(t_0) = x_0 \quad \text{for } t \in [t_0,t_1). \tag{1.2.14}$$

From (1.2.10), (1.2.11), (1.2.13), and $x_0(t,t_0,x_0) = x_0(t_1^-,t_0,x_0)$ for $t \in [t_0, t^-]$, we have

$$x_0(t_1) - x_0(t_0) = 0 + \int_{t_1^-}^{t} [-\alpha(s)\, x(s) + \gamma(s)]\, \mathrm{d}\beta(s), \text{for}\, t \in [t_0, t_1].$$

From this, the continuity of α and γ, the definitions of Riemann-Stieltjes integral [4] and the initial value problem [32], we have

$$\begin{aligned} x_0(t_1,t_0,x_0) &= x_0(t_0) + \beta(t_1)[-\alpha(t_1^*)x(t_1^*) + \gamma(t_1^*)] - \beta(t_1^*)[-\alpha(t_1^*)x(t_1^*) + \gamma(t_1^*)] \\ &= x_0(t_0) - \alpha_1 x_0(t_1^-,t_0,x_0) + \gamma_1\ , \end{aligned} \tag{1.2.15}$$

for $t_1^* \in [t_1^-, t_1]$. From (1.2.15) and $x_0(t_1,t_0,x_0) = x(t_1) = x_1$ and again $x(t_1^-,t_0,x_0) = x_0$, we obtain

$$\begin{aligned} x_1 &= x(t_1^-,t_0,x_0) - \alpha_1 x(t_1^-,t_0,x_0) + \gamma_1 \\ &= (1-\alpha_1)x_0 + \gamma_1. \end{aligned} \tag{1.2.16}$$

Continuing the above argument, we can establish the induction hypothesis [32] as:

$$x_j = \Phi(t_j, t_0)x_0 + \sum_{i=1}^{j} \Phi(t_j, t_i)\gamma_i \quad \text{for} \quad x(t_j) = x_j,$$

where

$$\Phi(t_j, t_i) = \prod_{k=i}^{j}(1-\alpha_k)\,, \Phi(t_i, t_i) = 1 \quad \text{for} \quad i \in I(0, n).$$

Now, we consider

$$\mathrm{d}x = [-\alpha(t)\,x + \gamma(t)]\,\mathrm{d}\beta(t), \quad x(t_j) = x_j,\, t \in [t_j, t_{j+1}).$$

From the definitions of x_j and Φ, and using the above argument, one can establish the following:

$$x_j(t) = x(t, t_j, x_j) = \prod_{k=1}^{j}(1-\alpha_k)x_0 + \sum_{i=1}^{j-1} \Phi(t_j, t_i)\gamma_i + \gamma_j \quad \text{for}\, t \in [t_j, t_{j+1}) \;. \tag{1.2.17}$$

Hence

$$\begin{cases} x(t_{j+1}^-, t_j, x_j) = \prod\limits_{k=1}^{j}(1-\alpha_k)x_0 + \sum\limits_{i=1}^{j} \Phi(t_j, t_i)\gamma_i\,, \\ x_{j+1}(t_{j+1}, t_j, x_j) = (1-\alpha_{j+1})x_j + \gamma_{j+1}\;. \end{cases} \tag{1.2.18}$$

Therefore, from (1.2.17) and (1.2.18), we have

$$\begin{aligned} x_{j+1} &= (1-\alpha_{j+1})x_j + \gamma_{j+1} \\ &= \prod_{k=1}^{j+1}(1-\alpha_k)x_0 + \sum_{i=1}^{j+1} \Phi(t_{j+1}, t_i)\gamma_i\,. \end{aligned}$$

By the application of PMI and the definition of the IVP regarding hybrid dynamic systems [32], we have

$$x(t) = \prod_{k|t_j \le t}(1-\alpha_k)x_0 + \sum_{i=1}^{j-1} \Phi(t, t_i)\gamma_i + \gamma_j\,,$$

for $t \ge t_0$ and $t \in [t_{j-1}, t_{j+1})$. This establishes the Theorem. □

Remark 1.2.3 From (1.2.10), the hybrid dynamic system (1.2.11), is equivalent to the hybrid dynamic system

$$\begin{cases} \mathrm{d}x = 0\,\mathrm{d}t\,, \quad x(t_{j-1}) = x_{j-1}\,, \quad t \in [t_{j-1}, t_i)\,, \\ x_j = (1-\alpha_j)x(t_j^-, t_{j-1}, x_{j-1}) + \gamma_j\,, \quad x(t_0) = x_0\,, \end{cases} \tag{1.2.19}$$

for $j \in I(1, n)$. The solution process of (1.2.19) is represented in (1.2.12).

In the following, we present a couple of special cases of Theorem 1.2.1. These special cases illustrate a systematic way for exhibiting the existing results in Kaplan and Meier [26], Nelson [41], Aalen [1] and Malla and Mukerjee [38] in the framework of presented innovative dynamic approach.

Corollary 1.2.1 *If functions α and γ in Theorem 1.2.1 are replaced by functions λ and $\gamma = 0$, then (1.2.12) reduces to*

$$x(t) = \prod_{j|t_j \le t} (1-\lambda_j)x_0\,, \quad t \ge t_0\ . \tag{1.2.20}$$

Corollary 1.2.2 *If $\alpha = 0 = x_0$ in Theorem 1.2.1, then the conclusion of Theorem 1.2.1 reduces to*

$$x(t) = \sum_{i|t_{j-1} \le t} \gamma_i\,, \quad t \ge t_0 \quad \textit{and} \quad t \in [t_{j-1}, t_j)\ . \tag{1.2.21}$$

In the following, we present a definition of cumulative jump process [38] in the framework of hybrid dynamic model.

Example 1.2.1 Let $T_1, T_2, \ldots, T_n$ be discrete failure times for the discrete-time event process, and $0 = a_0 < a_1 \le a_2 \le \ldots \le a_m$ be jumps of a survival function in magnitude. Then the dynamic for the cumulative jump process is as described in Corollary 1.2.2, and its solution process is exhibited in (1.2.21).
In this example, applying Corollary 1.2.2 in the context of $\gamma_0 = 0$, $\gamma_i = a_i$, the cumulative jump process is represented by

$$x(t) = \begin{cases} A_{j-1} = \sum_{i=1}^{j-1} a_i\,, & \text{for} \quad t \in [t_{j-1}, t_j)\,, \\ A_j = \sum_{i=1}^{j} a_i\,, & t = t_j\ . \end{cases} \tag{1.2.22}$$

From (1.2.22), we recognize that the cumulative jump defined in Malla and Mukerjee [38] is indeed recast as the discrete time intervention process described by the hybrid dynamic system illustrated in Corollary 1.2.2 at the discrete time t_j for $j \in I(1, m)$ with $\gamma_0 = a_0 = 0$ and $\gamma_i = a_i$.

Example 1.2.2 Under the conditions of Example 1.2.1, the magnitude of the survival function at the failure times is represented by

$$S(t) = \begin{cases} 1 - A_{j-1}\,, & \text{for} \quad t \in [t_{j-1}, t_j)\,, \\ 1 - A_j\,, & t = t_j, \quad j \in I(1, m)\,, \end{cases} \tag{1.2.23}$$

where $\gamma_0 = 1$ and $x(t_j) = A_j$. The $S(t)$ in (1.2.23) is the magnitude of the survival function determined by the cumulative jump [38] process described in Example 1.2.1.

Remark 1.2.4 We remark that the continuous-time dynamic model can be exhibited by the cumulative hazard/risk rate function. In fact, from (1.2.2), we have

$$\mathrm{d} \ln S = -\lambda(t)\mathrm{d}t\,, \quad \ln S(t_0) = S_0\ . \tag{1.2.24}$$

Based on the solution processes of (1.2.2) and (1.2.7), the solution process of (1.2.24) can be represented as:

$$-\ln\left[\frac{S(t)}{S(t_0)}\right] = \Lambda(t,t_0,S_0|\lambda) = \int_{t_0}^{t}\lambda(u)\mathrm{d}u\ . \tag{1.2.25}$$

and

$$-\ln\left[\frac{S(t)}{S(t_0)}\right] = \Lambda(t,t_0|\lambda) = \sum_{m=1}^{j-1}\int_{t_{m-1}}^{t_m}\lambda(u)\mathrm{d}u + \int_{t_{j-1}}^{t}\lambda(u)\mathrm{d}u,\ t\in[t_{j-1},t_j)\ . \tag{1.2.26}$$

respectively. Furthermore, we set $x = \ln S$, $S_0 = 1$ and $\gamma(t) = -\lambda(t)$ where S and λ are defined in (1.2.24). From Corollary 1.2.2, we have

$$\ln S(t) = -\Lambda(t)\,, \tag{1.2.27}$$

where $\Lambda(t) = \sum_{i|t_i\le t}\lambda_i$ is a cumulative hazard function.

Remark 1.2.5 We remark that if x is replaced by survival function, S in Corollary 1.2.1, and x and γ are replaced by S and λ in Corollary 1.2.2, then (1.2.20) and (1.2.21) are replaced by:

$$S(t) = \prod_{j|t_j\le t}(1-\lambda_j)S_0\,,\quad t\ge t_0 \tag{1.2.28}$$

and

$$S(t) = \sum_{i|t_i\le t}\lambda_i\,,\quad t\ge t_0\,, \tag{1.2.29}$$

respectively. Moreover, (1.2.28) is the solution process of the discrete-time dynamic system described by Corollary 1.2.1. Furthermore, dynamic system outlined in Corollary 1.2.1 provides an innovative alternative approach for finding the discrete-time survival function (Kaplan & Meier, 1958) in a systematic manner.

We utilize the above presented concepts and results in subsequent sections in a systematic and unified way.

1.3 Fundamental Results for Continuous and Discrete-Time to Event Dynamic Processes

In this section, we utilize hybrid dynamic model (1.2.7) and fundamental analytic Theorem 1.2.1 for time-to-event process to develop a general fundamental result. The developed result provides basic analytic and computational tools for estimating survival state and parameters. The presented approach also provides a systematic and unified way of estimating the parameters and survival functions.

Let $x(t)$ be the total number of units/individuals operating/alive (or survivals) at time t, for $t\in[t_0,\mathcal{T}]$. It is described by (1.2.11). Let λ and S be hazard/risk rate and survival functions of the units/patients/infectives/species/individuals, respectively. Employing a dynamic model for number of units/species/ individuals coupled with survival state dynamic model (1.2.2) or (1.2.7), we present an interconnected hybrid dynamic model below.

Following the argument used in developing dynamic models (Ladde & Ladde, 2012), we introduce the following interconnected system of differential equations:

$$\begin{cases} \mathrm{d}S = -\lambda(t)S\mathrm{d}t\,, \quad t \in [t_{j-1}, t_j)\,, \\ S_j = (1-\beta_j)S(t_j^-, t_{j-1}, S_{j-1})\,, \quad S(t_0) = 1\,, \\ dx = (-\alpha(t)x + \gamma(t))\mathrm{d}\beta(t)\,, \quad x(t_0) = x_0\,, \quad t \in [t_{j-1}, t_j)\,, \\ x_j = (1-\alpha_j)x(t_j^-, t_{j-1}, x_{j-1}) + \gamma_j\,\,, \end{cases} \tag{1.3.1}$$

Remark 1.3.1 We outline a few important observations that exhibit the role and scope of dynamic approach to illustrate the existing results [20, 26–28, 49] as special cases.

(i) Dynamic system (1.3.1) in the context of (1.2.19) (Remark 1.2.3) is reduced to

$$\begin{cases} \mathrm{d}S = -\lambda(t)S\mathrm{d}t\,, \quad t \in [t_{j-1}, t_j)\,, \\ S_j = (1-\beta_j)S(t_j^-, tj-1, S_{j-1})\,, \quad S(t_0) = 1\,, \\ dx = 0\,\mathrm{d}t\,, \quad x(t_0) = x_0\,, \quad t \in [t_{j-1}, t_j)\,, \\ x_j = (1-\alpha_j)x(t_j^-, t_{j-1}, x_{j-1}) + \gamma_j\,\,. \end{cases} \tag{1.3.2}$$

(ii) From Corollary 1.2.1 in the context of Remark 1.2.5, in particular (1.2.28), system (1.3.1) becomes:

$$\begin{cases} \mathrm{d}S = 0\,\mathrm{d}t\,, \quad t \in [t_{j-1}, t_j)\,, \\ S_j = (1-\lambda_j)S_{j-1}\,, \\ \mathrm{d}x = 0\,\mathrm{d}t\,, \quad x(t_0) = x_0\,, \\ x_j = (1-\alpha_j)x_{j-1} + \gamma_j\,\,. \end{cases} \tag{1.3.3}$$

We note that (1.3.3) is a special version of (1.3.1). In addition, we refer to system (1.3.3) as a totally discrete-time hybrid dynamic system.

Now, we are ready to present a basic result regarding continuous and discrete time interconnected dynamic of survival species or objects or thoughts operating under the time-to-event intervention processes. Prior to the formulation of the fundamental result, we introduce a concept of number of survivals.

Definition 1.3.1 Let z be a function defied by $z(t) = x(t)S(t)$, where S and x are solution process of (1.3.1) for $t \in [t_0, \mathfrak{T}]$. Moreover, for each $t \in [t_0, \mathfrak{T}]$, $z(t)$ stands for the number of survivals at t under an influence of time-to-event process.

Theorem 1.3.1 *Let (x, S) be a solution process of (1.3.1). Then the interconnected hybrid dynamic population model for time-to-event process (1.3.1) and corresponding intervention iterative process are described* *by:*

$$\begin{cases} \mathrm{d}z = -\lambda(t)z\mathrm{d}t\,, \quad z(t_{j-1}) = z_{j-1}\,, \quad \textit{for} \quad t \in [t_{j-1}, t_j)\,, \quad j \in I(1,k)\,, \\ z(t_j) = (1-\alpha_j)(1-\beta_j)z(t_j^-) + \gamma_j(1-\beta_j)\,, \end{cases} \tag{1.3.4}$$

and

$$z(t_j) = (1-\lambda(t_j)\Delta t_j)(1-\alpha_j)(1-\beta_j)z(t_{j-1}) + \gamma_j(1-\beta_j), \tag{1.3.5}$$

respectively, where z is defined in Definition 1.3.1 and $\Delta t_j = t_j - t_{j-1}$ for $j \in I(1,k)$.

Proof. For $t \in [t_{j-1}, t_j),\ j \geq 1$, from Definition 1.3.1, Remark 1.3.1 and the nature of S, we have

$$\mathrm{d}z(t) = -\lambda(t)z(t)\mathrm{d}t\ . \tag{1.3.6}$$

This establishes the continuous-time dynamic equation in (1.3.4). The proof of the discrete-time dynamic part in (1.3.4) and iterative process in (1.3.5) are outlined below. Multiplying the discrete-time iterative process in (1.3.1) by $S(t_j^-)$ and noting the fact that $S(t_j) = S(t_j^-)$, we obtain

$$x(t_j)S(t_j) = (1-\alpha_j)(1-\beta_j)x(t_j^-)S(tj^-) + \gamma_j(1-\beta_j)S(t_j^-)\ . \tag{1.3.7}$$

Moreover, using the definition of z, (1.3.7) reduces to

$$z(t_j) = (1-\alpha_j)(1-\beta_j)z(t_j^-) + \gamma_j(1-\beta_j)\ . \tag{1.3.8}$$

This establishes (1.3.4).

Applying the Euler-type numerical scheme [8] to (1.3.6) over an interval $[t_{j-1}, t_j^-]$, we obtain

$$z(t_j^-) - z(t_{j-1}) = -\lambda(t_{j-1})z(t_{j-1})\Delta t_j\ . \tag{1.3.9}$$

From (1.3.8) and (1.3.9) , we have

$$z(t_j) = (1-\lambda(t_j)\Delta t_j)(1-\alpha_j)(1-\beta_j)z(t_{j-1}) + \gamma_j(1-\beta_j)\ . \tag{1.3.10}$$

(1.3.10) exhibits the discrete time dynamic for survival process corresponding to the continuous-time dynamic process described in (1.3.4) and the discrete-time intervention process. Moreover, (1.3.10) exhibits the validity of (1.3.5). This establishes proof of Theorem 1.3.1. □

In the following, we present a few special/trivial cases that exhibit existing results in the framework of hybrid dynamic of time-to-event interconnected system.

Corollary 1.3.1 *Let us consider a very special/trivial case of Theorem 1.3.1 as follows:*

$$
\begin{cases}
\mathrm{d}S = -\lambda(t)S\mathrm{d}t\,, \quad t \geq t_0\,, \\
\mathrm{d}x = 0\,\mathrm{d}t\,, \quad t \geq t_0\,, \\
x(t_j) = x(t_j^-, t_{j-1}, x_{j-1})\,, \quad x(t_0) = x_0\,, \quad j \in I(1,k)\ .
\end{cases}
\tag{1.3.11}
$$

Applying Theorem 1.3.1 and using (1.3.4) and (1.3.5), (1.3.11) reduces to

$$
\begin{cases}
\mathrm{d}z = -\lambda(t)z\mathrm{d}t\,, \quad z(t_{j-1}) = z_{j-1}\,, \quad t \in [t_{j-1}, t_j)\,, \\
z(t_j) = z(t_j^-, t_{j-1}, z_{j-1}) = z(t_{j-1})\,, \quad j \in I(1,k)\,,
\end{cases}
\tag{1.3.12}
$$

and

$$
z(t_j) = (1 - \lambda(t_j)\Delta t_j)\, z(t_{j-1})\ . \tag{1.3.13}
$$

COROLLARY 1.3.2 *Let us consider a special case of (1.3.1) as follows:*

$$
\begin{cases}
\mathrm{d}S = -\lambda(t)S\mathrm{d}t\,, \quad S(t_{j-1}) = S_{j-1}\,, \quad t \in [t_{j-1}, t_j)\,, \\
S(t_j) = S(t_j^-, t_{j-1}, S_{j-1})\,,
\end{cases}
\tag{1.3.14}
$$

where a_j is defined in Example 1.2.1. Then applying Euler-type discretization scheme [8] on $[t_{j-1}, t_j^-]$, yields

$$
S(t_j^-) - S(t_{j-1}) = -\lambda(t_{j-1})S(t_{j-1})\Delta t_j\ . \tag{1.3.15}
$$

Moreover, from (1.3.14) and (1.3.15), we have

$$
S(t_j) - S(t_{j-1}) = -\lambda(t_j)S(t_{j-1})\Delta t_j\ . \tag{1.3.16}
$$

COROLLARY 1.3.3 *Under the assumptions of Theorem 1.3.1 in the context of Remark 1.3.1(ii), (1.3.3) becomes:*

$$
\begin{cases}
\mathrm{d}z = 0\,\mathrm{d}t\,, \quad z(t_{j-1}) = z_{j-1}\,, \quad t \in [t_{j-1}, t_j)\,, \\
z(t_j) = (1-\lambda_j)(1-\alpha_j)z_{j-1} + \gamma_j\,,
\end{cases}
\tag{1.3.17}
$$

and

$$
z(t_j) = (1-\lambda_j)(1-\alpha_j)z(t_{j-1}) + \gamma_j\ . \tag{1.3.18}
$$

This corollary is indeed a totally discrete-time version of hybrid dynamic system operating under discrete-time intervention process.

Using Definition 1.3.1 and the discrete-time iterative process (1.3.5), we introduce a couple of definitions.

DEFINITION 1.3.2 Let t_{j-1} and t_j be a pair of consecutive observation times belonging to $[0, \mathfrak{T}]$. $z(t_{j-1})$ stands for the number of survivals at the time t_{j-1} for each $j \in I(1,k)$. Moreover, $z(t_{j-1})$ is the number of survivals under observation over the sub-interval of time $[t_{j-1}, tj)$. $z(t_{j-1})\Delta t_j$ is the amount of time spent under observation/testing/evaluation by $z(t_{j-1})$ survivals over the length Δt_j of time interval $[t_{j-1}, t_j)$.

DEFINITION 1.3.3 For $j \in I(1,k)$, $z(t_{j-1}) - z(t_j)$ stands for the change in number of survivals over the interval of time $[t_{j-1}, t_j]$ of length Δt_j.

REMARK 1.3.2 The discrete-time processes (1.3.5), (1.3.13), (1.3.16) and (1.3.18) are referred as our numerical schemes with respect to interconnected hybrid dynamic models for a survival population dynamic processes. Moreover, from (1.3.5), we will introduce three more special numerical schemes, namely, time-to-event: (i) failure/death/removal/infective, (ii) censored/withdrawn, and (iii) admission/joining/susceptible/relapsed processes. We further note that the presented numerical schemes allow "ties" with deaths/failure or censored/quiting process. In addition, the population under the presented observation/supervision process includes the patient/objects population as a special case.

(i) For each $j \in I(1,k)$, let us assume that either t_{j-1} and t_j are consecutive failure/death/removal/infective times of individual/machine/species, or t_{j-1} and t_j are censored and failure times, respectively. For $\alpha_j = \gamma_j = \beta_j = 0$, the numerical scheme (1.3.5) for failure/death/removal/infective/etc process data set is described by

$$z(t_j) = (1 - \lambda(t_j)\Delta t_j)z(t_{j-1})\,, \tag{1.3.19}$$

and hence

$$z(t_j) - z(t_{j-1}) = -\lambda(t_j)z(t_{j-1})\Delta t_j\,, \tag{1.3.20}$$

where t_{j-1} is either the failure or censored time.

Moreover, $\alpha_j = \gamma_j = \beta_j = 0$ in (1.3.5) coupled with (1.3.9) is equivalent to the Kaplan and Meier (1958) assumption, namely,

$$x(t_j^-) - x(t_j) = \text{the number of deaths at } t_j\ .$$

That is

$$z(t_{j-1}) - z(t_j^-) = 0 \quad \text{and} \quad z(t_j) = z(t_j^+)\ .$$

This implies that $z(t)$ is left discontinuous and right continuous at t_j.

(ii) Let us assume that either t_{j-1} and t_j are consecutive censored times, or t_{j-1} and t_j are failure and censored times, respectively. For $\alpha_j = \beta_j = 0$, and γ_j^c stands for the number of censored objects/infectives/etc at a time t_j. The numerical scheme (1.3.5) for censored/listed/identified process data set is described by

$$z(t_j) = (1 - \lambda(t_j)\Delta t_j)\, z(t_{j-1}) - \gamma_j^c\,, \tag{1.3.21}$$

where t_{j-1} is either a failure or censored time.

Thus

$$z(t_j) - z(t_{j-1}) = -\lambda(t_j)z(t_{j-1})\Delta t_j - \gamma_j^c \tag{1.3.22}$$

Again, we note that $\alpha_j = \beta_j = 0, \gamma_j^c$, in the context of (1.3.9) is equivalent to the Kaplan and Meier (1958) assumption, namely,

$$z(t_j) = z(t_j^-) \quad \text{and} \quad z(t_j) - z(t_j^+) = \gamma_j^c \ .$$

This implies that $z(t)$ is left continuous and right discontinuous at t_j.

(iii) Let us assume that t_{j-1} is either failure or censored time, and t_j is a joining/admitting/relapsing time. For $\alpha_j = 0$ and γ_j^a denoting the number of objects/infectives that joined the observation process at time t_j. The numerical scheme (1.3.5) for admission/joining/sustainable/recruiting/relapsing process is

$$z(t_j) = (1 - \lambda(t_j)\Delta t_j)\, z(t_{j-1}) + \gamma_j^a \ . \tag{1.3.23}$$

The scheme determined by $\alpha_j = 0$ in (1.3.5) with (1.3.9) and the addition γ_j^a in (1.3.23) is equivalent to $z(t_j) - z(t_j^-) = \gamma_j^a$ and $z(t_j) = z(t_j^+)$.

(iv) Remarks (i), (ii) and (iii) remain valid for the iterative processes (1.3.5), (1.3.13) and (1.3.18).

(I) For $\alpha_j = 0 = \beta_j = \gamma_j$ in (1.3.5), (1.3.16) reduces to (1.3.20); for $\alpha_j = 0 = \beta_j = \gamma_j$, (1.3.18) reduces to $z(t_j) = (1 - \lambda_j)z(t_{j-1})$.

(II) For $\alpha_j = 0 = \beta_j$ and $\gamma_j = -\gamma_j^c$ in (1.3.5), (1.3.5) reduces to (1.3.22); for $\alpha_j = 0 = \lambda_j$ and $\gamma_j = -\gamma_j^c$, (1.3.18) becomes

$$z(t_j) - z(t_{j-1}) = (1 - \lambda_j)z(t_{j-1}) - \gamma_j^c \ . \tag{1.3.24}$$

(III) For $\alpha_j = 0 = \beta_j$ and $\gamma_j = \gamma_j^a$ in (1.3.5), and $\alpha_j = 0$ and $\gamma_j = \gamma_j^a$ in (1.3.18), (1.3.5) reduces to (1.3.23), and (1.3.18) reduces to

$$z(t_j) - z(t_{j-1}) = (1 - \lambda_j)z(t_{j-1}) + \gamma_j^a. \tag{1.3.25}$$

1.4 Estimations of Risk Rate and Survival Functions

Now, we are ready to find an estimate for the hazard/risk rate and survival functions for interconnected continuous and discrete-time survival state dynamic processes. For the sake of completeness and clarity, we first introduce a couple of definitions.

Definition 1.4.1 For $j \in I(1,k)$, let t_{j-1} and t_j be consecutive change times under continuous-time state survival dynamic process. The parameter estimate at t_j is defined by the quotient of change of objects over the consecutive time change interval $[t_{j-1}, t_j)$ and the total time spent by the objects under observation over the time interval of length Δt_j.

Definition 1.4.2 For $j \in I(1,k)$, let t_{j-1} and t_j be consecutive change times for discrete-time state survival dynamic process. The parameter estimate at t_j is defined by the quotient of the relative frequency of the change in the number of survival state over the consecutive time change interval $[t_{j-1}, t_j)$ and the number of objects at the immediate past time, that is, either the change time or the censored time.

Remark 1.4.1 We observe that the Definitions 1.4.1 and 1.4.2 are consistent with each other. This statement can be justified in the context of discrete-time iterative scheme (1.3.10) and the continuous and discrete-time hybrid-type descriptions of survival state dynamic model (1.3.2) and totally discrete-time hybrid dynamic system (1.3.3).

Now, we are ready to present a main result regarding parameter and survival state estimation problems. This result includes several existing results as special cases. In the following, we simply state a conceptual computational algorithm. The detailed proof is given in the supplementary section.

Theorem 1.4.1 *Let us assume that the conditions of Theorem 1.3.1 in the context of Remarks 1.3.1 and 1.3.2(i),(ii) are satisfied.*

(a) For $j \in I(1,k)$, if t_{j-1} and t_j are consecutive risk/failure/removal/death/non-operational times in $[t_0, \mathfrak{T}]$ then an estimate for the hazard/risk rate function at t_j is determined by:

$$\hat{\lambda}(t_j) = \frac{z(t_{j-1}) - z(t_j)}{z(t_{j-1})\Delta t_j}\,, \tag{1.4.1}$$

and an estimate for the hazard/risk rate function is

$$\hat{\lambda}(t) = \hat{\lambda}(t_j)\,, \quad \textit{for} \quad t \in [t_{j-1}, t_j) \quad \textit{and} \quad j \in I(1,k). \tag{1.4.2}$$

(b) For $j \in I(1,k)$, if $t_{j-1} < t_j^c < t_j$, and t_j^c is censored time between a pair of consecutive failure times t_{j-1} and t_j in $[t_0, \mathfrak{T})$, then

(i) a change in the number of items/subjects/thoughts that are under observation over the subinterval $[t_{j-1}, t_j)$ of the time interval of study $[t_0, \mathfrak{T}]$ is

$$z(t_{j-1}) - z(t_j) - \gamma_j^c; \tag{1.4.3}$$

(ii) a total amount of time spent under the observation/testing/evaluation of $z(t_{j-1}) - z(t_j) - \gamma_j^c$ items/patients/infectives/radicals/subjects over the time interval $[t_{j-1}, t_j)$ is

$$z(t_{j-1})\Delta t_j^c + z(t_j^c)\Delta t_{jc}\,, \quad \Delta t_{jc} = t_j - t_j^c. \tag{1.4.4}$$

(iii) an estimate for the hazard/risk rate function at t_j is defined as:

$$\hat{\lambda}(t_j) = \frac{z(t_{j-1}) - z(t_j) - \gamma_j^c}{z(t_{j-1})\Delta t_j^c + z(t_j^c)\Delta t_{jc}}\,, \tag{1.4.5}$$

and an estimate for the hazard/risk rate function is

$$\hat{\lambda}(t) = \hat{\lambda}(t_j)\,, \quad \textit{for} \quad t \in [t_{j-1}, t_j) \quad \textit{and} \quad j \in I(1,k)\ . \tag{1.4.6}$$

(iv) Moreover, an estimate for the survival function in (1.3.1) *is*

$$\hat{S}(t) = S_0 \exp\left[\sum_{m=1}^{j-1} \hat{\lambda}_m (t_m - t_{m-1}) + \hat{\lambda}_j\,(t - t_{j-1})\right],\ t \in [t_{j-1}, t_j). \tag{1.4.7}$$

Proof.

(a) Using the discrete-time iterative scheme (1.3.5), Remark 1.3.2(i) and Definitions 1.3.2, 1.3.3 and 1.4.1, we have

$$\lambda(t) = \hat{\lambda}(t_j) = \frac{z(t_{j-1}) - z(t_j)}{z(t_{j-1})\Delta t_j}$$

for $t \in [t_{j-1}, t_j)$ and $j \in I(1,k)$. This establishes (a).

(b) Let t_j^c be a censoring time between two consecutive risk/failure times, t_{j-1} and t_j. We consider a partition of $[t_{j-1}, t_j]$: $t_{j-1} < t_j^c < t_j$.

Employing iterative processes in (1.3.22) and (1.3.20) on respective subintervals $[t_{j-1}, t_j^c]$ and $[t_j^c, t_j]$, we have

$$\begin{aligned} z(t_j) - z(t_{j-1}) &= z(t_j^c) - z(t_{j-1}) + z(t_j) - z(t_j^c) \\ &= -\lambda(t_{j-1})\Delta t_j^c - \gamma_j^c - \lambda(t_j) z(t_j^c)\Delta t_{jc} \\ &= -\lambda(t_j)\left[z(t_{j-1})\Delta t_j^c + z(t_j^c)\Delta t_{jc}\right] - \gamma_j^c\ . \end{aligned} \tag{1.4.8}$$

From (1.4.8), we obtain:

$$z(t_{j-1}) - z(t_j) - \gamma_j^c = \lambda(t_j)\left[z(t_{j-1})\Delta t_j^c + z(t_j^c)\Delta t_{jc}\right]\ . \tag{1.4.9}$$

From (1.4.9) and knowing that $\lambda(t_j)$ is the hazard/risk rate of change per unit time per unit object/subject, we conclude that $z(t_{j-1}) - z(t_j) - \gamma_j^c$ is the number of failure/non-operating objects and $z(t_{j-1})\Delta t_j^c + z(t_j^c)\Delta t_{jc}$ denotes the total amount of time spent by $z(t_{j-1}) - z(t_j) - \gamma_j^c$ over the the interval $[t_{j-1}, t_j)$. This establishes (i) and (ii).

To complete the proofs of (iii) and (iv), we utilize Definition 1.4.1 and (1.4.9), and obtain

$$\hat{\lambda}(t_j) = \frac{z(t_{j-1}) - z(t_j) - \gamma_j^c}{z(t_{j-1})\Delta t_j^c + z(t_j^c)\Delta t_{jc}} \quad \text{for} \quad j \in I(1,k) \ .$$

and hence

$$\lambda(t) = \hat{\lambda}(t_j), \quad t \in [t_{j-1}, t_j)\,, \quad j \in I(1,k) \ .$$

This establishes proof of the theorem.

□

Remark 1.4.2 We note that if $t_j^c = t_j$ in Theorem 1.4.1(b), then we have "ties" between censored and failure times. In this case, $\Delta t_j^c = \Delta t_j$ and $\Delta t_{jc} = 0$. From this, (1.4.4) and (1.4.5) reduce to

$$z(t_{j-1})\Delta t_j \,, \tag{1.4.10}$$

and

$$\hat{\lambda}(t_j) = \frac{z(t_{j-1}) - z(t_j) - \gamma_j^c}{z(t_{j-1})\Delta t_j} \quad \text{for} \quad j \in I(1,k). \tag{1.4.11}$$

This observation justifies Remark 1.3.2 regarding the mixed "ties."

In the following, we exhibit the role and scope of Theorem 1.4.1. This is achieved by presenting the well-known hazard/risk rate and survival functions as special cases.

Corollary 1.4.1 *Assume that conditions of Corollary 1.3.3 in the context of Remark 1.3.2(iv)(I) are satisfied.*

(a) For $j \in I(1,k)$, if t_{j-1} and t_j are consecutive risk/failure times in $[t_0, \mathfrak{T}]$, then employing Definitions 1.3.2, 1.3.3 and 1.4.2, an estimate for the risk/hazard rate function at t_j is determined by:

$$\hat{\lambda}(t_j) = \frac{z(t_{j-1}) - z(t_j)}{z(t_{j-1})}\,, \tag{1.4.12}$$

and

$$\lambda(t) = \hat{\lambda}(t_j)\,, \quad t \in [t_{j-1}, t_j). \tag{1.4.13}$$

Substituting (1.4.12) into (1.2.28), an estimate for the survival function is obtained as:

$$\begin{aligned} S(t) = \prod_{i|t_{j-1}\le t} \left(1 - \hat{\lambda}_i\right) &= \prod_{i|t_{j-1}\le t} \left(1 - \frac{z(t_{i-1}) - z(t_i)}{z(t_{i-1})}\right) \\ &= \prod_{i|t_{j-1}\le t} \left(1 - \frac{d_i}{z(t_{i-1})}\right), \quad t \ge t_0\,, \end{aligned} \tag{1.4.14}$$

where $d_i = z(t_{i-1}) - z(t_i)$ *is the number of deaths over the consecutive risk/failure time interval* $[t_{i-1}, t_i)$, $t_i \leq t_{j-1} \leq t < t_j$ *for some* $j \in I(1,k)$.

(b) *For* $j \in I(1,k)$*, if* $t_{j-1} < t^c_j < t_j$*, and* t^c_j *is censored time between a pair of consecutive risk/failure times* t_{j-1} *and* t_j *in* $[t_0, \mathfrak{T})$*, then, employing Definitions 1.3.2, 1.3.3 and 1.4.2, an estimate for the risk/hazard rate function at* t_j *is determined by:*

$$\hat{\lambda}(t_j) = \frac{z(t_{j-1}) - z(t_j) - \gamma^c_j}{z(t^c_j)} , \tag{1.4.15}$$

and

$$\lambda(t) = \hat{\lambda}(t_j) , \quad t \in [t_{j-1}, t_j) . \tag{1.4.16}$$

Substituting (1.4.15) into (1.2.28), an estimate for the survival function when t^c_j *is a censored time between consecutive failure times,* t_{j-1} *and* t_j *is given by:*

$$\begin{aligned} S(t) = \prod_{i|t_{j-1} \leq t} \left(1 - \hat{\lambda}_i\right) &= \prod_{i|t_{j-1} \leq t} \left(1 - \frac{z(t_{i-1}) - z(t_i) - \gamma^c_i}{z(t^c_i)}\right) \\ &= \prod_{i|t_{j-1} \leq t} \left(1 - \frac{d_i}{z(t^c_i)}\right) , \quad t \geq t_0 , \end{aligned} \tag{1.4.17}$$

where i *runs over the positive integers for which* $t_i \leq t_{j-1}$*,* $t_{j-1} \leq t < t$ *for some* $j \in I(1,k)$*;* t_{i-1}, t_i *are consecutive failure times for* $i \in I(1,j)$*, and* $d_i = z(t_{i-1}) - z(t_i) - \gamma^c_i$ *is the number of deaths over the consecutive failure time interval* $[t_{j-1}, t_j)$*.*

Remark 1.4.3 (a) We remark that (1.4.14) and (1.4.17) are indeed the Kaplan and Meier (1958)-type survival estimate functions.

(b) In the literature [25, 37], the numbers in the denominator of (1.4.14) and (1.4.17) are referred to as the number of individuals at risk at t_{j-1} and t^c_j respectively. Denoting this by n_j, we can write both (1.4.14) and (1.4.17) as:

$$S(t) = \prod_{i|t_{j-1} \leq t} \left(\frac{n_i - d_i}{n_i}\right) . \tag{1.4.18}$$

This is the well-known formula cited in the literature [25, 37].

(c) From Remark 1.2.4, we obtain

$$\hat{\Lambda}(t) = \sum_{t_j \leq t} \hat{\lambda}_j = \sum_{t_j \leq t} \frac{d_j}{n_j} , \quad t \geq t_0 , \tag{1.4.19}$$

where

$$n_j = \begin{cases} z(t_{j-1}) & \text{if there are no censors in} \quad [t_{j-1}, t_j)\,, \\ z(t_j^c) \quad \text{if} \quad t_j^c & \text{is a censored time in} \quad [t_{j-1}, t_j)\ . \end{cases} \tag{1.4.20}$$

This is the estimator introduced by Nelson [41] and [1]. These special cases exhibit the role and scope of the presented innovative alternative dynamic approach.

In the following, we state a corollary that further illustrates the role and scope of our dynamic approach. Further details regarding the proof is outlined in the supplementary section.

Corollary 1.4.2 *Let us assume that the conditions of Corollary 1.3.2 and Example 1.2.2 in the context of Remark 1.3.2(iii) are satisfied. For $j \in I(1, n)$, if t_{j-1} and t_j are consecutive risk/failure times in $[t_0, \mathfrak{T}]$, then employing Definitions 1.3.2, 1.3.3 and 1.4.2, an estimate for the risk/hazard rate function at t_j is determined by:*

$$\hat{\lambda}(t_j) = \frac{a_j}{(1 - A_{j-1})\Delta t_j}\,, \tag{1.4.21}$$

and

$$\hat{\lambda}(t) = \hat{\lambda}(t_j)\,, \quad t \in [t_{j-1}, t_j)\,, \tag{1.4.22}$$

where a_j and A_{j-1} are defined in Example 1.2.1.

Moreover, an estimate for the survival function is represented by

$$\hat{S}(t) = S_{j-1} \exp\left[-\hat{\lambda}_j (t - t_{j-1})\right] \quad \textit{for} \quad t \in [t_{j-1}, t_j)\ . \tag{1.4.23}$$

Proof. Under the conditions of Example 1.2.1 and using the relationship between S, the cumulative jumps in Example 1.2.2, Corollary 1.3.2(in particular (1.3.16)), an estimate for the risk/hazard rate function at t_j is obtained as:

$$\hat{\lambda}(t_j) = \frac{a_j}{(1 - A_{j-1})\Delta t_j}\,, \tag{1.4.24}$$

and an estimate for the risk/hazard rate function is

$$\hat{\lambda}(t) = \hat{\lambda}(t_j)\,, \quad \text{for} \quad t \in [t_{j-1}, t_j) \quad \text{and} \quad j \in I(1, m) \tag{1.4.25}$$

From (1.3.14), using (1.2.8) and (1.4.25), an estimate for the survival function is given by:

$$\hat{S}(t) = \exp(-\Lambda_{j-1}) \exp\left(\frac{-a_j (t - t_{j-1})}{(1 - A_{j-1})(t_j - t_{j-1})}\right)\,, \quad t_{j-1} \le t < t_j\,, \tag{1.4.26}$$

where

$$\Lambda_j = \sum_{i=1}^{j} \frac{a_i}{1 - A_{i-1}},\ 1 \le j \le m,\ \Lambda_0 := 0,$$

and Λ_j is the cumulative hazard function. This establishes the proof of the corollary. $\square$

REMARK 1.4.4 The PEXE of Kitchin et al. [28], as well as Kim and Proschan [27] is undefined beyond the last observed failure time. To rectify that, Malla and Mukerjee [38] provided the following exponential tail hazard/risk rate estimate:

$$\hat{\lambda}_{\text{tail}} = \frac{\exp(-\hat{\Lambda}_m)}{\sum_{i=1}^{m}(I_j - J_j)}, \tag{1.4.27}$$

where

$$I_j = \int_{t_{j-1}}^{t_j} \hat{S}^{KM}(t)\mathrm{d}t = (1 - A_{j-1})(t_j - t_{j-1})$$

and

$$J_j = \int_{t_{j-1}}^{t_j} \hat{S}^{MN}(t) = \exp(-\hat{\Lambda}_{j-1})\frac{(1 - A_{j-1})(t_j - t_{j-1})}{a_j}\left[1 - \exp\left(-\frac{a_j}{1 - A_{j-1}}\right)\right].$$

Thus, under the following assumptions: (i) no ties among the failure times, (ii) the last observation is uncensored, a new PEXE of Malla and Mukerjee [38] is given by

$$S(t) = \begin{cases} \exp(-\Lambda_{j-1})\exp\left(\frac{-a_j(t-t_{j-1})}{(1-A_{j-1})(t_j-t_{j-1})}\right), & t_{j-1} \le t < t_j, \quad j \in I(1,m) \\ \exp(-\hat{\Lambda}_m)\exp(-\hat{\lambda}_{\text{tail}}(t - t_m)), & t_m \le t < \infty . \end{cases} \tag{1.4.28}$$

We further note that the presented dynamic approach does not require the failure function to be invertible.

1.5 Multiple Censored Times Between Consecutive Failure Times

In this section, we further apply the conceptual dynamic results developed in Sections 1.2 and 1.3 to multiple censored times between consecutive failure times. We present a result that provides a very general algorithm for estimating a hazard rate function for multiple censoring times between consecutive failure times t_{j-1} and t_j with $t_{j-1}, t_j \in [t_0, \mathcal{T})$. We further note that the presented results in this section extend the results of Section 1.4 in a systematic and unified manner.

THEOREM 1.5.1 *Let the hypotheses of Theorem 1.3.1 in the context of Remarks 1.3.1, 1.3.2(i) and 1.3.2(ii) be satisfied. For each $j \in I(1,m)$, let t_{j-1} and t_j be consecutive failure times. Let $\{t_{j-1l}\}_{l=1}^{k_j}$ be a finite sequence of censored time observations over a time interval $[t_{j-1}, t_j]$. Let γ_j^l be the number of objects censored at time t_{j-1l}, for $l \in I(1,k_j)$ and $\{\gamma_j^l\}_{l=1}^{k_j}$ be a corresponding sequence of observed number of objects/species/-patients/etc. Then*

1. *$z(t_{j-1}) - z(t_j) - \sum_{l=1}^{k_j} \gamma_j^l$ is a change in the number of items/subjects that is under the observation over the sub-interval $[t_{j-1}, t_j]$ of the time interval of study $[t_0, \mathcal{T}]$.*
2. *$\sum_{l=1}^{k_j+1} z(t_{j-1l-1})\Delta(t_{j-1l})$ is a total amount of time spent under the observation/testing/evaluation/monitoring of $z(t_{j-1l-1})$*

items/patients/ infectives/subjects on the interval $[t_{j-1l-1}, t_{j-1l})$ *for* $l \in I(1, k_j))$ *and* $j \in I(1, n)$.

3. *an estimate for the hazard rate function at* t_j *is determined by*

$$\hat{\lambda}(t_j) = \frac{z(t_{j-1}) - z(t_j) - \sum_{l=1}^{k_j} \gamma_j^l}{\sum_{l=1}^{k_j+1} z(t_{j-1l-1})\Delta(t_{j-1l})}, \tag{1.5.1}$$

and an estimate for the hazard rate function is

$$\hat{\lambda}(t) = \hat{\lambda}(t_j), \quad \textit{for} \quad t \in [t_{j-1}, t_j) \quad \textit{and} \quad j \in I(1, n). \tag{1.5.2}$$

Proof. For each $j \in I(1, n)$ and $t_{j-1}, t_j \in \mathcal{P}_0^{\mathcal{T}}$, objects/subjects are censored k_j times over a partition of $[t_{j-1}, t_j]$ of consecutive failure times. Let $\mathcal{P}_j$ be a partition corresponding to a given finite sequence of censored times over the failure time interval $[t_{j-1}, t_j)$, and let it be represented by

$$\mathcal{P}_j : t_{j-1} = t_{j-10} < t_{j-11} < \ldots < t_{j-1l-1} < t_{j-1l} < \ldots < t_{j-1k_{j-1}} < t_{j-1k_j} \;. \tag{1.5.3}$$

where $\mathcal{P}_j$ is a partition of $[t_{j-1}, t_j]$.

For each $j \in I(1, n)$, using the iterative schemes (1.3.20) and (1.3.22) we have

$$\begin{aligned} z(t_j) - z(t_{j-1}) &= \sum_{l=1}^{k_j} [z(t_{j-1l}) - z(t_{j-1l-1})] + [z(t_j) - z(t_{j-1k_j})] \\ &= -\lambda(t_j) \left[\sum_{l=1}^{k_j+1} z(t_{j-1l-1}) \Delta t_{j-1l} \right] - \sum_{l=1}^{k_j} \gamma_j^l \,, \end{aligned} \tag{1.5.4}$$

and hence

$$z(t_{j-1}) - z(t_j) - \sum_{l=1}^{k_j} \gamma_j^l = \lambda(t_j) \sum_{l=1}^{k_j+1} z(t_{j-1l-1}) \Delta(t_{j-1l}) \;. \tag{1.5.5}$$

Thus, $z(t_{j-1}) - z(t_j) - \sum_{l=1}^{k_j} \gamma_j^l$ is a change in the number of items/subjects that are under observation over the subinterval $[t_{j-1}, t_j]$, and $\sum_{l=1}^{k_j+1} z(t_{j-1l-1})\Delta(t_{j-1l})$ is a total amount of time spent under the observation/testing/evaluation/monitoring of $z(t_{j-1l})$ items/patients/infectives/subjects on the interval $[t_{j-1l-1}, t_{j-1l})$ for $l \in I(1, k_j))$ and $j \in I(1, n)$. These statements establish conclusions 1 and 2 of Theorem 1.5.1.

Finally, from Definition 1.4.1, we obtain an estimate for a hazard rate function at $t_j \in [t_0, \mathcal{T})$ as:

$$\hat{\lambda}(t_j) = \frac{z(t_{j-1}) - z(t_j) - \sum_{l=1}^{k_j} \gamma_j^l}{\sum_{l=1}^{k_j+1} z(t_{j-1l-1})\Delta(t_{j-1l})} \;.$$

This establishes (1.5.1).

Moreover,

$$\hat{\lambda}(t) = \hat{\lambda}(t_j)\,, \quad \text{for} \quad t \in [t_{j-1}, t_j) \quad \text{and} \quad j \in I(1,n)\ . \tag{1.5.6}$$

This completes the proof of the theorem. □

Corollary 1.5.1 *Under the conditions of Theorem 1.5.1 and assumptions of Corollary 1.3.3 in the context of Remark 1.3.2(iv), an estimate for the hazard rate function at t_j is determined by*

$$\hat{\lambda}(t_j) = \frac{z(t_{j-1}) - z(t_j) - \sum_{l=1}^{k_j} \gamma_j^l}{z(t_{j-1k_j})}\,, \tag{1.5.7}$$

and an estimate for the hazard rate function is $\hat{\lambda}(t) = \hat{\lambda}(t_j)$, for $t \in [t_{j-1}, t_j)$ and $j \in I(1,n)$. An estimate for the survival function is thus given by

$$\hat{S}(t) = \prod_{i|t_{j-1}<t} (1 - \hat{\lambda}(t_i)),\, t \geq t_0,\, t_i \leq t_{j-1} \leq t < t_j \textit{ for some} j \in I(1,n). \tag{1.5.8}$$

Corollary 1.5.2 *Under the conditions of Theorem 1.5.1 and estimate for the cumulative hazard/risk rate and survival functions are respectively represented by:*

$$\hat{\Lambda}(t, t_0) = \sum_{m=1}^{j-1} \hat{\lambda}_m (t_m - t_{m-1}) + \hat{\lambda}_j\,(t - t_{j-1})\,,\, t \in [t_{j-1}, t_j)$$

and

$$\hat{S}(t, t_0) = S_0 \exp\left[\sum_{m=1}^{j-1} \hat{\lambda}_m (t_m - t_{m-1}) + \hat{\lambda}_j\,(t - t_{j-1})\right],\, t \in [t_{j-1}, t_j)$$

for $t \geq t_0$, $t_{j-1} \leq t < t_j$ for some $j \in I(1,n)$.

Remark 1.5.1 (a) We remark that the innovative dynamic approach for the development of computational parameter estimation algorithm (1.5.1) is an alternative approach for the algorithm proposed Kim and Proschan [27].

(b) The estimates (1.5.1) in the context of (1.2.26) yields the estimate obtained by Kulasekera and White [30] as special cases.

(c) For continuous-time interconnected hybrid state survival dynamic process, if $k_j = 0$, for some $j \in I(1,n)$, then $l = 0$ and $\gamma_j^0 = 0$ and (1.5.1) reduces to (1.4.1). On the other hand, if $k_j = 1$ for some $j \in I(1,n)$, then $l = 0$ and $\gamma_j^1 = \gamma_j^c$ and (1.5.1) implies (1.4.5).

(d) For discrete-time interconnected hybrid state survival dynamic process, if $k_j = 0$, for some $j \in I(1,n)$, then $l = 0$ and $\gamma_j^0 = 0$ and (1.5.7) reduces to (1.4.12). On the other hand, if $k_j = 1$, for some $j \in I(1,n)$, then $l = 0$ and $\gamma_j^1 = \gamma_j^c$ and (1.5.7) implies (1.4.15).

The presented innovative approach of parameter and state estimation includes the Thaler [49]-type hazard rate estimation problem as a particular case. To justify this statement, we first introduce a concept of hazard/risk rate function for responder and non-responder states. In addition, we state a corollary of Theorem 1.5.1 without its proof. The proof is outlined in the supplementary section.

DEFINITION 1.5.1 For $i \in I(0,1)$, Let $\lambda_0(t)$ and $\lambda_1(t)$ represent the hazard/risk rate functions in the non-responder and responder states, respectively, at time t [49] .

COROLLARY 1.5.3 *Let us assume that the conditions of Corollary 1.3.1 in the context of Remark 1.3.2(i) are satisfied. For $j \in I(1,n_0)$, let t_{j-1} and t_j be consecutive risk/failure times in state 0. For $j' \in (1,n_1)$, let $t_{j'-1}$ and $t_{j'}$ be consecutive failure times in state 1. Let $z_0(t_j)$ be the number of survivals at t_j in state* 0. *Let $z_1(t_{j'})$ be the number of survivals at $t_{j'}$ in state* 1. *Then an estimate for the hazard/risk rate function at t_j is determined by:*

$$\hat{\lambda}_0(t_j) = \frac{\sum\limits_{m=1}^{j}\left[z_0(t_{m-1}) - z_0(t_m)\right]}{\sum\limits_{m=1}^{j} z_0(t_{m-1})\Delta t_m} = \frac{\sum\limits_{m=1}^{j} d_{0j}}{\sum\limits_{m=1}^{j} z_0(t_{m-1})\Delta t_m}\,, \tag{1.5.9}$$

where d_{0j} is the number of deaths/failures at the jth distinct failure time in state i, and an estimate for the hazard rate function is

$$\hat{\lambda_0}(t) = \hat{\lambda}_0(t_j)\,, \quad \textit{for} \quad t \in [t_{j-1}, t_j) \quad \textit{and} \quad j \in I(1,n_0)\ . \tag{1.5.10}$$

An estimate for the hazard/risk rate function at $t_{j'}$ is determined by:

$$\hat{\lambda}_1(t_{j'}) = \frac{\sum\limits_{m=1}^{j'}\left[z_1(t_{m-1}) - z_1(t_m)\right]}{\sum\limits_{m=1}^{j'} z_1(t_{m-1})\Delta t_m} = \frac{\sum\limits_{m=1}^{j} d_{1j'}}{\sum\limits_{m=1}^{j} z_1(t_{m-1})\Delta t_m}\,, \tag{1.5.11}$$

where $d_{1j'}$ is the number of deaths/failures at the j'th distinct failure time in state 1, *and an estimate for the hazard rate function is*

$$\hat{\lambda_1}(t) = \hat{\lambda}_1(t_{j'})\,, \quad \textit{for} \quad t \in [t_{j'-1}, t_{j'}) \quad \textit{and} \quad j' \in I(1,n_1)\ . \tag{1.5.12}$$

The hazard/risk ratio rate function estimate is given by: $\frac{\hat{\lambda}_0(t_j)}{\hat{\lambda}_1(t_{j'})}$. The corresponding estimate of the log *hazard/risk rate ratio function for patients currently in a response compared to a nonresponse state is given by:*

$$\hat{\rho}(t) = \ln\left[\frac{\hat{\lambda}_0(t_j)}{\hat{\lambda}_1(t_{j'})}\right] \ \text{for}\,, t_{j-1} < t \leq t_j \,\text{and}\, t_{j'-1} \leq t < t_{j'}\ . \tag{1.5.13}$$

Proof. Let $t_0 < t_1 < \ldots < t_{m-1} < t_m < \ldots < t_{j-1} < t_j < \ldots < t_n = \mathcal{T}$ be a partition of $[t_0, \mathcal{T}]$. Using

(1.3.13), for fixed $i = 0$ and $j \in I(1, n_0)$, we have

$$z_0(t_m) - z_0(t_{m-1}) = -\lambda_0(t_m) z_0(t_{m-1}) \Delta t_m \ . \tag{1.5.14}$$

Summing (1.5.14) from $m = 1$ to j, we obtain

$$\begin{aligned} \sum_{m=1}^{j} [z_0(t_m) - z_0(t_{m-1})] &= \sum_{m=1}^{j} -\lambda_0(t_m) z_0(t_{m-1}) \Delta_m \\ &= -\lambda_0(t_j) \sum_{m=1}^{j} z_0(t_{m-1}) \Delta t_m \ . \end{aligned} \tag{1.5.15}$$

Rearranging (1.5.15) establishes (1.5.9). The proof of (1.5.11) is similar to the proof of (1.5.9). (1.5.13) is obtained by taking the natural log of the ratio of (1.5.9) and (1.5.11) . This establishes the proof of the corollary. □

Remark 1.5.2 We remark that (1.5.9), (1.5.11) and (1.5.13) are identical to the result obtained in Thaler [49]. Moreover, the estimates in (1.5.9), (1.5.11) and (1.5.13) were obtained in the framework of an innovative dynamic approach.

In the following, we state a general theorem that provides a theoretical estimate for the hazard/risk rate function between two successive change point times, t_{j-1} and t_j.

Theorem 1.5.2 *Let the hypothesis of Theorem 1.5.1 be satisfied. Let $\{T_i^j\}_{i=1}^n$ be a sequence of times(failure/ censor/arrival) that fall between the change point times t_{j-1} and t_j for $j = I(1, k)$. Then an estimate for the hazard rate function at t_j is determined by*

$$\hat{\lambda}(t_j) = \frac{z(t_{j-1}) - z(t_j) - \sum\limits_{m=1}^{l} \eta_m^j}{\sum\limits_{m=1}^{l+1} z(T_m^j) \Delta(T_m^j)} \ , \ j \in I(1, k+1) \ . \tag{1.5.16}$$

where

$$\eta_m^j = \begin{cases} 0 & \text{if } \ T_m^j \text{ is failure time} \\ \gamma_m^{jc} & \text{if } \ T_m^j \text{ is censored time} \ ; \\ -\gamma_m^{ja} & \text{if } \ T_m^j \text{ is arrival time} \end{cases} \tag{1.5.17}$$

γ_m^{jc} is the number of objects/items/individuals censored at time T_m^j; γ_m^{ja} is the number of objects/items/individuals joining/arriving at time T_m^j, and an estimate for the hazard rate function is $\lambda(t) = \hat{\lambda}(t_j)$ for $t \in [t_{j-1}, t_j)$.

Proof. Let $0 = t_0 < t_1 < t_2 < \ldots < t_{j-1} < t_j < \ldots < t_k$ be the partition of $[t_0, \mathfrak{T})$ corresponding to change point times.

For $j = 1, 2, \ldots, k$, we consider a partition of $[t_{j-1}, t_j]$ as follows:

$$\mathcal{P}_j^t : t_{j-1} = T_0^j < T_1^j < T_2^j < T_3^j < \ldots < T_{l-1}^j < T_l^j < \ldots < T_{n-1}^j < T_n^j < T_{n+1}^j = t_j \ . \tag{1.5.18}$$

Imitating the proof of Theorem 1.5.1, we have

$$\begin{aligned} z(t_j) - z(t_{j-1}) &= \sum_{m=1}^{l} \left[z(T_m^j) - z(T_{m-1}^j) \right] + [z(t_j) - z(T_l^j)] \\ &= \sum_{m=1}^{l} \left[-\lambda(T_{m-1}^j) z(T_{m-1}^j) \Delta T_m^j - \eta_m^j \right] + [-\lambda(T_l^j) z(T_l^j) \Delta t_j] \\ &- \lambda(t_j) \left[\sum_{m=1}^{l} z(T_{m-1}^j) \Delta T_m^j \right] - \sum_{m=1}^{l} \eta_m^j - \lambda(t_j) z(t_l^j) \Delta t_j \\ &= -\lambda(t_j) \left[\sum_{m=1}^{l+1} z(T_{m-1}^j) \Delta T_m^j \right] - \sum_{m=1}^{l} \eta_m^j \,, \end{aligned} \tag{1.5.19}$$

and hence

$$z(t_{j-1}) - z(t_j) - \sum_{m=1}^{l} \eta_m^j = \lambda(t_j) \sum_{m=1}^{l+1} z(T_{m-1}^j) \Delta T_m^j \tag{1.5.20}$$

Thus, $z(t_{j-1}) - z(t_j) - \sum_{m=1}^{l} \eta_m^j$ is a change in the number of items/subjects that is under the observation over the subinterval $[t_{j-1}, t_j]$ of the time interval of study $[t_0, \mathcal{T}]$ and $\sum_{m=1}^{l+1} z(T_m^j) \Delta T_m^j$ is a total amount of time spent under the observation/testing/evaluation of $z(T_m^j)$ items/patients/infectives/subjects on the interval $[T_{m-1}^j, T_m^j)$ for $m \in I(1, l))$ and $j \in I(1, k)$. Finally, from Definition 1.4.1, we obtain an estimate for a hazard rate function at $t_j \in [t_0, \mathcal{T})$ as:

$$\hat{\lambda}(t_j) = \frac{z(t_{j-1}) - z(t_j) - \sum_{m=1}^{l} \eta_m^j}{\sum_{m=1}^{l+1} z(T_{m-1}^j) \Delta T_m^j} \,,$$

Moreover,

$$\hat{\lambda}(t) = \hat{\lambda}(t_j) \,, \quad \text{for} \quad t \in [t_{j-1}, t_j) \quad \text{and} \quad j \in I(1, k) \ . \tag{1.5.21}$$

This establishes the proof of the theorem.

□

Chapter 2
Conceptual Computational Algorithms

2.1 Introduction

In this chapter, we outline very general conceptual computational, data organizational and simulation schemes. The computational and simulation algorithms are based on the fundamental theoretical result (Theorem 1.5.1) developed in Section 1.5. In Section 2.2, conceptual computational parameter and state estimation schemes are developed. Conceptual and computational simulation algorithms are given in Section 2.3. The developed computational schemes are applied time-to-event datasets to estimate hazard/risk rate and survival functions in a systematic and unified way in Section 2.4.

2.2 Conceptual Computational Parameter and State Estimation Scheme

The theoretical computational algorithm for interconnected continuous-time hybrid dynamic process (1.3.1), is as follows:

$$z(t_{j-1}) - z(t_j) - \sum_{l=1}^{k_j} \gamma_j^l = \hat{\lambda}(t_j) \sum_{l=1}^{k_j+1} z(t_{j-1l-1})\Delta(t_{j-1l}), \tag{2.2.1}$$

and the conceptual computational algorithm for totally discrete-time hybrid dynamic process (1.3.3) is

$$z(t_{j-1}) - z(t_j) - \sum_{l=1}^{k_j} \gamma_j^l = \hat{\lambda}(t_j) z(t_{j-1k_j})\,. \tag{2.2.2}$$

Here $\mathcal{P}_0^{\mathcal{T}} : t_0 < t_1 < \ldots < t_{j-1} < t_j < \ldots < t_n$ is a partition of failure times over the time interval $[0, \mathcal{T})$. Let $\mathcal{P}_j$ be a partition corresponding to a given finite sequence of censored times over the failure time interval $[t_{j-1}, t_j)$, and let it be represented by

$$\mathcal{P}_j : t_{j-1} = t_{j-10} < t_{j-11} < \ldots < t_{j-1l-1} < t_{j-1l} < \ldots < t_{j-1k_{j-1}} < t_{j-1k_j}\ . \tag{2.2.3}$$

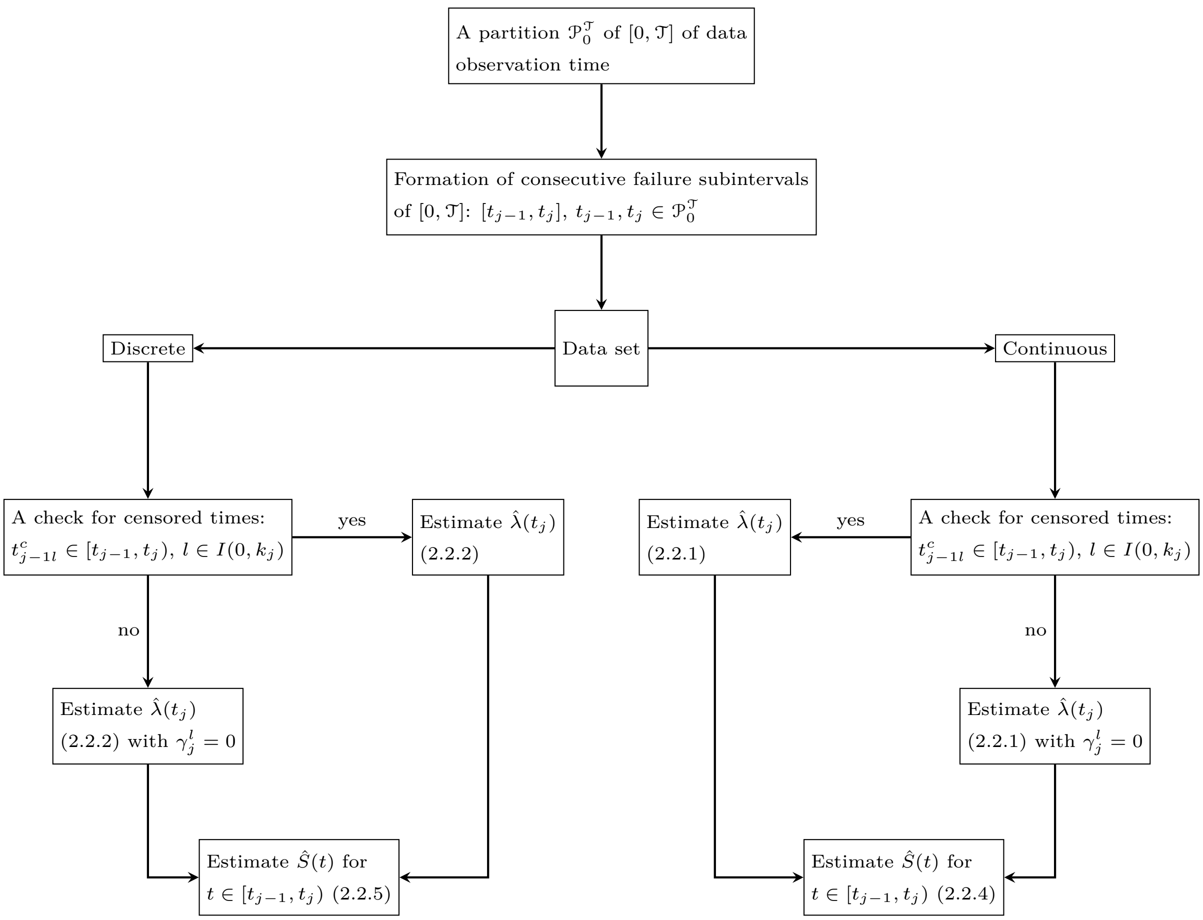
For $j \in I(1, n), \lambda$ is the hazard rate function; $z(t)$ stands for the number of survivals at time t; γ_j^l denotes the number of objects censored at the time $t_{j-1l}, j \in I(1, m)$ and $l \in I(0, k_j)$, $k_j \in I(0, \infty)$. For the continuous-time hybrid dynamic process (1.3.1), an estimate of the survival function is represented by

$$\hat{S}(t, t_0) = S_0 \exp\left[\sum_{m=1}^{j-1} \hat{\lambda}_m(t_m - t_{m-1}) + \hat{\lambda}_j\,(t - t_{j-1})\right]\ ,\ t \in [t_{j-1}, t_j) \text{ for } t \geq t_0\,. \tag{2.2.4}$$

For the totally discrete-time hybrid dynamic process (1.3.3), an estimate of the survival function is represented by

$$\hat{S}(t) = \prod_{i|t_{j-1}<t} (1 - \hat{\lambda}(t_i)),\ t \geq t_0. \tag{2.2.5}$$

First, we construct a detailed flowchart for the general conceptual computational algorithm developed in Section 1.5.



Flowchart 1.: Conceptual Computational Algorithm

We observe that the conceptual computational algorithm (Flowchart 1) is composed of two sub-conceptual computational algorithms, namely, continuous-time and discrete-time hybrid dynamic processes.

2.3 Conceptual and Computational Simulation Algorithms

A pseudocode for a simulation scheme for both interconnected continuous-time and totally discrete-time hybrid dynamic processes are outlined below:

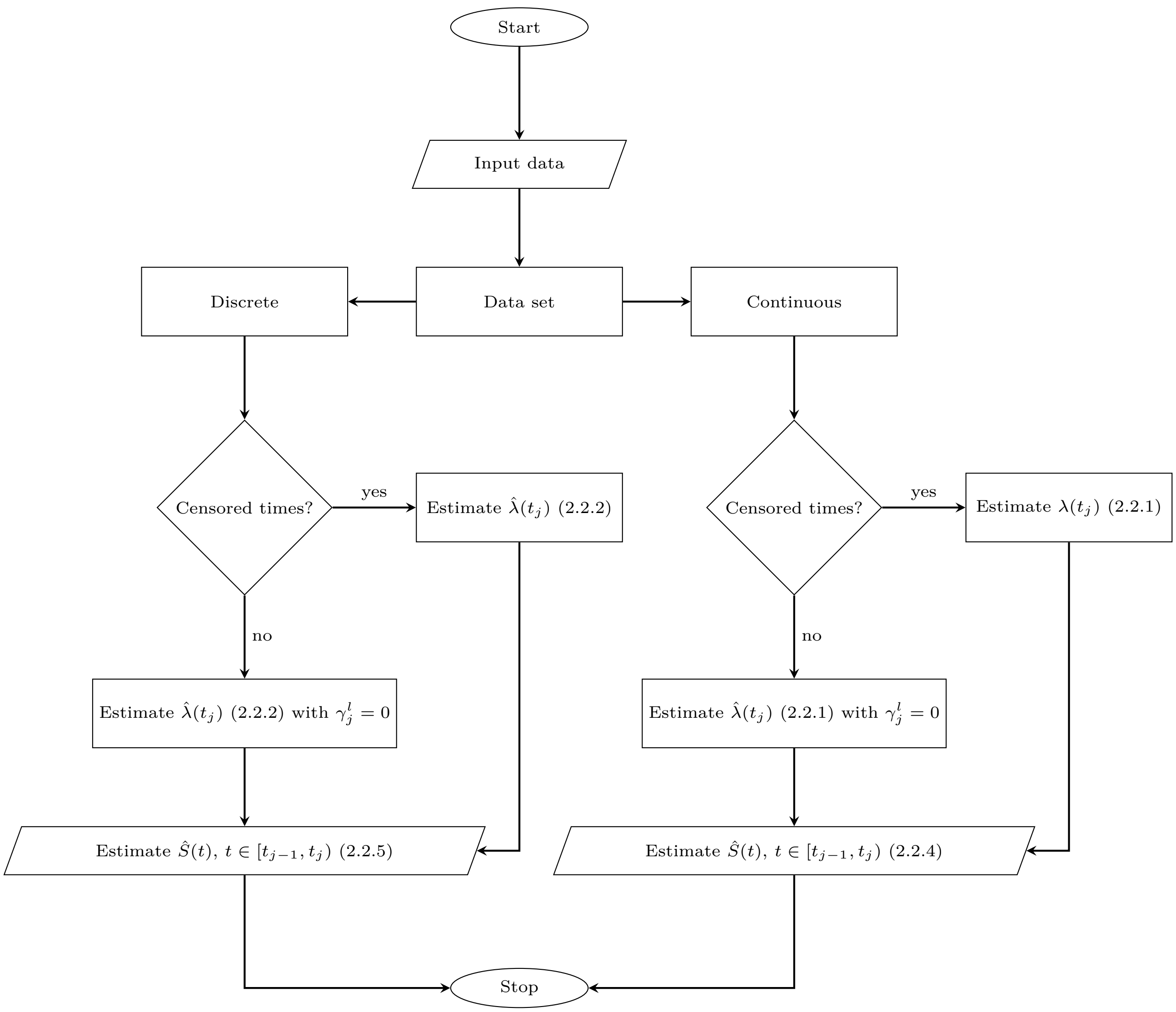
for $j = 1$ to N **do**
Compute k_j, $z(t_{j-1}), z(t_j)$
    if $k_j = 0$ **then**
Compute $z(t_{j-1})\Delta t_j$
    else
Compute $\sum_{l=1}^{k_j} \gamma_j^l, \ \sum_{l=1}^{k_j+1} z(t_{j-1l-1})\Delta(t_{j-1l})$
    end if
Compute $\hat{\lambda}(t_j), \ \hat{S}(t)$
end for

Simulation Scheme 2a.: Pseudocode for interconnected continuous-time hybrid dynamic process

for $j = 1$ to N **do**
Compute k_j, $z(t_{j-1}), z(t_j)$
    if $k_j = 0$ **then**
Compute $z(t_{j-1})$
    else
Compute $\sum_{l=1}^{k_j} \gamma_j^l, \ z(t_{j-1k_j})$
    end if
Compute $\hat{\lambda}(t_j), \ \hat{S}(t)$
end for

Simulation Scheme 2b.: Pseudocode for totally discrete-time hybrid dynamic process

Moreover, a flowchart for the simulation algorithm for parameter and state estimation problems for interconnected continuous-time (1.3.1) and discrete-time (1.3.3) hybrid dynamic processes are provided in Flowchart 3.



Flowchart 3.: Simulation Algorithm for interconnected hybrid dynamic processes

We note that flowchart for simulation algorithm (Flowchart 3) is composed of two sub-simulation algorithms, namely, continuous-time and totally discrete-time hybrid dynamic processes.

2.4 Applications to Time-to-event Datasets

In the following, using the conceptual computational algorithm, we exemplify our theoretical procedure by estimating hazard rate and survival functions of two data sets in a systematic and unified way. The first data set can be found in [26].

ILLUSTRATION 2.4.1 Suppose that out of a sample of 8 items the following are observed:

Table 1: Dataset used by Kaplan and Meier [26]

Order of Observation	Time of Cessation of Observation	Cause of Cessation	Time Notation
1	0.8	Failure	t_1
2	1.0	Censored	t_{11}
3	2.7	Censored	t_{12}
4	3.1	Failure	t_2
5	5.4	Failure	t_3
6	7.0	Censored	t_{31}
7	9.2	Failure	t_4
8	12.1	Censored	

We note that the data set in Table 1 is for the totally discrete-time hybrid time-to-event dynamic process (1.3.3). In view of this, we apply the totally discrete-time parameter and state estimation schemes (2.2.2) and (2.2.5). In short, we utilize the discrete-time conceptual computational sub-algorithm (Simulation Scheme 2b) "pseudocode" and simulation sub-algorithm (Flowchart 3).

For $t \in [t_0, t_1)$, there are no censored times between $[t_0, t_1)$. Therefore, $k_j = 0$, and from Remark 1.5.1(d) and hence using (2.2.2) we have

$$\hat{\lambda}(t_1) = \hat{\lambda}_1 = \frac{z(t_0) - z(t_1)}{z(t_0)} = \frac{1}{8} \; .$$

Utilizing (2.2.5), the corresponding survival function is given by

$$\hat{S}(t) = \begin{cases} 1\,, & \text{for} \quad t \in [t_0, t_1)\,, \\ 1 - \lambda_1 = \frac{7}{8}\,, & \text{for} \quad t = t_1 \; . \end{cases}$$

For $t \in [t_1, t_2)$, we note that there are two censored times between t_1 and t_2. So, $k_j = k_2 = 2$. Hence

$$\sum_{l=1}^{2} \gamma_2^l = \gamma_2^1 + \gamma_2^2 = 1 + 1 = 2 \ .$$

Also, $z(t_{j-1k_j}) = z(t_{12}) = 5$. Thus, from Remark 1.5.1(d) and hence applying (2.2.2), we have

$$\hat{\lambda}(t_2) = \hat{\lambda}_2 = \frac{z(t_1) - z(t_2) - \sum_{l=1}^{2} \gamma_2^l}{z(t_{12})} = \frac{1}{5} \ .$$

Utilizing (2.2.5), the corresponding survival function is thus given by

$$\hat{S}(t) = \begin{cases} \frac{7}{8} \,, & \text{for} \quad t \in [t_1, t_2) \,, \\ \prod_{k|t_j \leq t} (1 - \hat{\lambda}_j) = \prod_{j=1}^{2} (1 - \hat{\lambda}_j) = \frac{7}{10} \,, & \text{for} \quad t = t_2 \ . \end{cases}$$

There is no censoring time between the interval $[t_2, t_3) = [3.1, 5.4)$. Therefore, $k_j = 0$, and from Remark 1.5.1(d) and hence using (2.2.2) we obtain

$$\hat{\lambda}(t_3) = \frac{z(t_2) - z(t_3)}{z(t_2)} = \frac{1}{4} \ .$$

Once again, utilizing (2.2.5), the corresponding survival function is thus given by

$$\hat{S}(t) = \begin{cases} \frac{7}{10} \,, & \text{for} \quad t \in [t_2, t_3) \,, \\ \prod_{j=1}^{3} (1 - \hat{\lambda}_j) = \frac{21}{40} \,, & \text{for} \quad t = t_3 \ . \end{cases}$$

Continuing in this manner, we record the estimates for hazard rate and survival functions in the following table with the last column exhibiting the survival function estimate as obtained by Kaplan and Meier [26].

Table 2: Kaplan and Meier Survival estimates for data set given in [26].

Failure Times t_j	Survivals $z(t_j)$	Hazard Rate Function $\hat{\lambda}(t_j)$	Survival Function $\hat{S}(t_j)$
0.8	7	1/8	7/8
3.1	4	1/5	7/10
5.4	3	1/4	21/40
9.2	1	1/2	21/80
(12.1)	0	1/2	21/80

Using the dataset in [27] and theoretical computational algorithm, Theorem 1.5.1, we illustrate the estimation of hazard rate and survival functions, systematically.

Illustration 2.4.2 Suppose that seven items (new) are put on test at time 0. Each item is observed until it fails or until it is withdrawn, whichever occurs first. The resulting set of observation [27] is shown in Table 3 in order of occurrence.

Table 3: Data from Kim and Proschan [27]

Order of Observation	Time of Cessation of Observation	Cause of Cessation	Time Notation	Finite sequence of censored Time	Size of sequence	Number of Censored
0	0					
1	2.0	Failure	$t_1 = t_{01} = t_{10}$			
2	3.5	Censored	t_{11}	$\{t_{j-1l}\}_{l=1}^{2}$	$k_2 = 2$	$\{\gamma_2^l\}_{l=1}^{2}$
3	4.5	Censored	t_{12}			
4	6.2	Failure	$t_2 = t_{13} = t_{20}$			
5	8.0	Censored	t_{21}	$\{t_{j-1l}\}_{l=1}^{1}$	$k_3 = 1$	$\{\gamma_3^l\}_{l=1}^{2}$
6	8.8	Failure	$t_3 = t_{22}$			
7	11.3	Failure	t_4			

The data set in Table 3 is for the interconnected continuous-time hybrid dynamic time-to-event dynamic process (1.3.1). In view of this, we apply the continuous-time parameter and state estimation schemes (2.2.1) and (2.2.4). In short, we utilize the continuous-time conceptual computational sub-algorithm (Simulation Scheme 2a) "pseudocode" and simulation sub-algorithm (Flowchart 3).

For $[0, t_1)$, since there are no censored times in between $[0, t_1)$, $k_j = k_1 = 0$. Thus from Remark 1.5.1(c) and using (2.2.1) we have

$$\hat{\lambda}(t_1) = \frac{z(t_0) - z(t_1)}{z(t_0)(t_{01} - t_0)} = \frac{1}{14}.$$

Thus $\hat{\lambda}(t) = \frac{1}{14} \approx 0.0714$ for $t \in [t_0, t_1) = [0, 2.0)$.

For the estimate on $[t_1, t_2) = [2.0, 6.2)$, we note that there are two censoring times between $[t_1, t_2)$, hence $k_j = k_2 = 2$ and

$$\sum_{l=1}^{2} \gamma_2^l = \gamma_2^1 + \gamma_2^2 = 1 + 1 = 2.$$

Thus from Remark 1.5.1(c) and thus applying (2.2.1), we have

$$\hat{\lambda}(t_2) = \frac{z(t_1) - z(t_2) - \sum_{l=1}^{k_2} \gamma_2^l}{\sum_{l=1}^{k_2+1} z(t_{1l-1})\Delta t_{1l}} = \frac{z(t_1) - z(t_2) - \sum_{l=1}^{2} \gamma_2^l}{\sum_{l=1}^{3} z(t_{1l-1})\Delta t_{1l}} = \frac{1}{20.8}.$$

Thus, $\hat{\lambda}(t) = \frac{1}{20.8}$, for $t \in [2.0, 6.2)$.

On the interval $[t_2, t_3) = [6.2, 8.8)$, we have only one censoring time in between the two failure times. So, $k_j = k_3 = 1$. Thus from Remark 1.5.1(c) and hence, using (1.5.1), we obtain

$$\hat{\lambda}(t_3) = \frac{z(t_2) - z(t_3) - \sum_{l=1}^{1} \gamma_3^l}{\sum_{l=1}^{2} z(t_{2l-1})\Delta t_{2l}} = \frac{3 - 1 - 1}{z(t_{20})\Delta t_{21} + z(t_{21})\Delta t_{22}} = \frac{1}{7}.$$

Hence, $\hat{\lambda}(t) = \frac{1}{7}$, for $t \in [6.2, 8.0)$.

There is no censoring in the interval $[t_3, t_4)$. Thus,

$$\hat{\lambda}(t_4) = \frac{z(t_3) - z(t_4)}{z(t_3)\Delta t_4} = \frac{1}{2.5} \,,$$

which implies that $\hat{\lambda}(t) = \frac{1}{2.5} = 0.4$, for $t \in [8.0, 11.3)$. Following this estimation procedure we have

$$\hat{\lambda}(t) = \begin{cases} 0.0714 & 0 \le t < t_1 = 2 \\ 0.0481 & t_1 \le t < t_2 = 6.2 \\ 0.1429 & t_2 \le t < t_3 = 8.8 \\ 0.4 & t_3 \le t < t_4 = 11.3 \,. \end{cases} \tag{2.4.1}$$

To obtain the estimate of survival function, we use (2.2.4) or we apply the solution process described in Section 1.2 regarding (1.2.7) and obtain exponential pieces on successive intervals between failure times that are joined to form a continuous function. Thus,

$$\hat{S}(t) = \begin{cases} \exp(-0.0714t) \,, & 0 \le t < 2 \\ \exp\left[-0.1429 - 0.0481(t-2)\right] \,, & 2 \le t < 6.2 \\ \exp\left[0.3448 - 0.1429(t-6.2)\right] \,, & 6.2 \le t < 8.8 \\ \exp\left[0.4591 - 0.4(t-8.8)\right] \,, & 8.8 \le t < 11.3 \\ \text{no estimator,} & t \ge 11.3 \end{cases} \tag{2.4.2}$$

Remark 2.4.1 These are the same results obtained by using the method proposed by Kim and Proschan [27].

Chapter 3

Interconnected Nonlinear Hybrid Dynamic Modeling for Time-to-event Processes

3.1 Introduction

In survival and reliability analysis, parametric methods are often applied to estimate the hazard/risk rate and survival functions [37]. A parametric approach is based on the assumption that the underlying survival distribution belongs to some specific family of distributions (e.g. Weibull, log-logistic, exponential etc). Mostly, classical likelihood-based models, methods and its extensions/generalizations are developed and utilized [9, 25, 36, 37].

The log-logistic distribution [9, 12, 24, 36, 43] has played a significant role in the survival data analysis. In this chapter, we present an alternative approach for modeling nonlinear time-to-event processes in biological, chemical, engineering, epidemiological, medical, military, multiple-markets and social dynamic processes. This approach does not require any knowledge of either a closed form solution distribution or a class of distributions. Our innovative approach leads to development of a nonlinear dynamic model for time-to-event processes.

The human mobility, electronic communications, technological changes, advancements in engineering, medical, and social sciences have diversified and extended the role and scope of time-to-event processes in biological, cultural, epidemiological, financial, military and social sciences [2, 11, 33, 34, 50]. It is known that sudden changes in the hazard rate/risk at unspecified or specified times are frequently encountered in engineering and medical sciences [2]. These changes could occur multiple times. As a result of this, investigators [17, 19, 21] are often interested in (a) detecting the location of the changes, and (b) estimating the sizes of the detected changes. For incorporating intervention processes, we transform a continuous nonlinear state dynamic model into an interconnected nonlinear hybrid dynamic model composed of both continuous-time and discrete-time state (intervention) dynamic processes. The presented approach is motivated by parameter and state estimation problems of continuous-time time-to-event processes. The developed approach us directly applicable to time-to-event dynamic processes in biological, chemical, engineering, financial, medical, physical, military and social sciences. A by-product of the transformed interconnected nonlinear hybrid dynamic model is derivation of theoretical discrete-time conceptual computational dynamic process. Employing the transformed discrete-time conceptual computational dynamic process, we introduce notions of data coordination, state data decomposition and aggregation, theoretical conceptual iterative processes, conceptual and computational parameter estimation and simulation schemes, conceptual and computational state simulation schemes.

The organization of the presented work in this chapter is as follows. A few basic existing concepts and observations are outlined in Section 3.2. Recognizing the rapid growth and increased efficiency and speed in communication, science and technology in the 21^{st} century, we develop a nonlinear dynamic model for time-to-event process in Section 3.3. Fundamental theoretical results for nonlinear hybrid dynamic processes are outlined in Section 3.4. In fact, interconnected transformed nonlinear hybrid dynamic survival state system and transformed discrete-time conceptual computational interconnected dynamic algorithm are developed. The approach is motivated by the preliminary work initiated in [5]. In Section 3.5, we develop very general theoretical and computational procedures and results for parameter and state estimations for the time-to-event dynamic process.

3.2 Basic Existing Concepts and Observations

For the better understanding of the development of nonlinear and non-stationary dynamic algorithm of time-to-event data analysis, we outline a few existing features and ideas in the theory of survival analysis, as well as make some observations.

Historically, it is known [25] that the study of time-to-event processes is centered around the medical and engineering sciences. Mostly, classical likelihood based models, methods and its extensions/generalizations are developed and utilized [25]. The study is based on the concepts in the theory of probability and stochastic processes. In particular, probabilistic concepts of hazard rate function λ and survival/failure probability distributions of a random time variable T form a core of concepts. We note that for $t \in \mathbf{R}$, $F(t)$ is a cumulative probability distribution of T, and $S(t)$ is a survival function of time-to-event process. Moreover, $S(t)+F(t) = 1$. In the existing literature, these probabilistic functions are treated to be evolving/progressing mutually exclusively corresponding to two mutually exclusive time varying events. We refer to S and F as cumulative distributions of two mutually disjoint output processes with respect to two mutually exclusive time-varying events of a random dynamic process in any discipline. This kind of random dynamic process can be thought of as the Bernoulli-type of stochastic process. Corresponding to these two output processes of the Bernoulli-type of stochastic process, we associate two dynamic states of a binary choice/option/action. Indeed, a stochastic binary-state dynamic process ({ action, reaction}, {normal, abnormal}, {survival, failure}, {susceptible, infective}, {operational, non-operational}, {radical, non-radical}, and so on) exhibits abstractions and generalizations of Newton's 3^{rd} law of dynamic motion process ({reaction}).

A Logistic-type survival distribution function has been introduced through a random time transformation. Moreover, the logistic distribution was introduced by recognizing the properties of the solution of logistic population dynamic model in the literature [25, 37]. We further note that the hazard rate function satisfies the conditions: $\lambda \geq 0$, and $\lim_{t\to\infty}\left[\int_0^t \lambda(s)\mathrm{d}s\right] = \infty$. This is a very restrictive assumption. In the following, using basic tools in mathematical sciences, we initiate a Newtonian-type dynamic approach for time-to-event processes in sciences, technologies, and engineering.

3.3 Motivations and Model Formulation

We recognize the rapid growth and increased efficiency and speed [2, 33, 34, 45, 46, 50] in communication, science, engineering and technology in the 21^{st} century. Under continuous advancements in science and technology, the study of time-to-event processes in medical and engineering sciences have been significantly improved, and can be easily extended to other disciplines that are conceptually similar but apparently different. In fact, the scientific and technological changes are playing a role for extension to dynamic processes in business, economic, management, military and social sciences [11, 33, 34, 45, 46, 50]. It is known that classical likelihood based models and methods of time-to-event models are very restrictive. For example, most of the time-to-event processes studied in the literature [25, 37] are focused on exclusively either failure or survival state dynamic of time-to-event processes. In fact, in economic/financial/social sciences, the group of human beings are interacting with a fellow human consumer/associate or a user of similar goods/services/information/knowledge/background/entities easily and more frequently for making a decision choice. Recently [46], introducing the concept of network externality process and its dynamic principle, the consumer group network influence has led to the definition of network externality value. Moreover, network good value is determined by a current market share/size. It has been further remarked that the collection of network externality functions includes sub-classes of survival/failure functions with finite domain of operation. We associate two mutually time-to-events in sciences and technologies with respect to two mutually exclusive dynamic states operating/functioning in the sciences, engineering and technologies to develop a dynamic model.

In this chapter, we initiate a nonlinear dynamic model for time-to-event processes in biological, medical, business, economic, management, military and social sciences as a binary-state probabilistic dynamic process interacting or influencing simultaneously instead of mutually exclusively (isolated manner). Let survival/operating/susceptible/action/normal and failure/non-operating/infective/ inaction/abnormal be probabilistic states of a time-to-event dynamic process in sciences, engineering, financial, medical, military, technological and social disciplines. Let us denote the probabilistic measures of these two dynamic states by S and F, respectively.

For this purpose, we introduce a dynamic principle for a binary state time-to-event process as:

Survival Principle: A specific survival state probability measure differential rate over an interval of time $[t, t + \Delta t]$ of a time-to-event binary-state dynamic process is directly proportional to the product of failure state probability measure and the length of the interval Δt:

$$\frac{\mathrm{d}S}{S} \propto F\mathrm{d}t\,,$$

that is

$$\begin{aligned} \mathrm{d}S &= -\lambda(t)SF\mathrm{d}t \\ &= -\lambda(t)S(1-S)\mathrm{d}t\,, \end{aligned} \tag{3.3.1}$$

where λ is a nonnegative function of proportionality; $\mathrm{d}S$ stands for a differential of survival state probability measure over an interval of length $\Delta t \equiv \mathrm{d}t$; $\frac{\mathrm{d}S}{S}$ denotes a specific survival state probability measure differential rate over the length of time interval Δt; negative sign in (3.3.1) signifies that survival state probability decreases as t increases; and $1-S$ represents a potential of failure; in addition, $1-S$ characterizes instantaneous effects of the failure state on the dynamic of survival state. Moreover, the differential of S in (3.3.1) is directly proportional to the product of the variance SF of binary state dynamic of time-to-event process and time Δt. The function of proportionality may depend on time, probabilistic measure states of Bernoulli-type dynamic process, and parameters of time-to-event process.

The development of nonlinear survival state dynamic model (3.3.1) motivates to study a very general survival state dynamic model of time-to-event process described by

$$\mathrm{d}S = -S\lambda(t,S)\,\mathrm{d}t\,, \quad S(t_0) = S_0\,, \tag{3.3.2}$$

where λ is a continuous function defined on $\mathbb{R}\times\mathbb{R}$ into $\mathbb{R}$, and it is smooth enough to assure the existence, uniqueness, and the non-negativity of solution process of (3.3.2) with $0 \le S \le 1$, whenever $0 \le S_0 \le 1$. Moreover, the solution process $S(t,t_0,S_0)$ is increasing in S_0 for each $(t,t_0) \in \mathbb{R}\times\mathbb{R}$.

In the following, we present an example that exhibits the role and scope of the presented dynamic modeling approach.

Example 3.3.1 We consider the following very simple dynamic model for the binary state time-to-event dynamic process. We consider

$$\begin{cases} \mathrm{d}S = (-\beta_s S + \alpha_s)\mathrm{d}t,\ S(t_0) = S_0,\ 0 < S_0 < 1, \\ \mathrm{d}F = (-\beta_F F + \alpha_F)\mathrm{d}t,\ F(t_0) = F_0,\ 0 < F_0 < 1, \end{cases} \tag{3.3.3}$$

where $\beta_s, \alpha_s, \beta_F$ and α_F are positive real numbers; these positive parameters satisfy the following conditions: $0 < \alpha_s < \beta_s$ and $\alpha_F < \beta_F$. $S(t) = \exp[-\beta_s(t-t_0)]S_0 + \frac{\alpha_s}{\beta_s}(1-\exp[-\beta_s(t-t_0)])$ and $F(t) = \exp[-\beta_F(t-t_0)]F_0 + \frac{\alpha_F}{\beta_F}(1-\exp[-\beta_F(t-t_0)])$ are solution processes of (3.3.3). Moreover, $0 < F(t) \le 1$ and $0 < S(t) \le 1$. In addition, $F(t) + S(t) = 1$, provided $\beta \equiv \beta_s = \beta_F$ and $\alpha_s + \alpha_F = \beta$.

Remark 3.3.1 As of now, we do not have any real world data to justify the validity of its usage. In fact, this opens a new avenue to undertake a study of time-to-event process. We note that this example provides a theoretical illustration for the measure of sustainability/unsustainability, stability/unstability, sustainable/unsustainable invariant sets, and attainable/unattainable sets.

Remark 3.3.2 Let (t_0, S_0) be a given initial condition. The initial data (t_0, S_0) together with (3.3.1) is referred to as the initial value problem (IVP)[33]. Employing an elementary technique, the initial value problem

$$\mathrm{d}S = -\lambda(t)S(1-S)\,\mathrm{d}t\,, \quad S(t_0) = S_0 \quad t \in [t_0, \infty)\,, \tag{3.3.4}$$

has a unique non-negative solution.

Moreover, the closed form solution process of (3.3.4) is represented by

$$S(t) = \frac{S(t_0)\exp\left[-\int_{t_0}^{t}\lambda(s)\,\mathrm{d}s\right]}{1 - S(t_0) + S(t_0)\exp\left[-\int_{t_0}^{t}\lambda(s)\,\mathrm{d}s\right]}\ . \tag{3.3.5}$$

The solution representation in (3.3.5) can be rewritten as

$$S(t) = \frac{1}{1+\exp\left[H(t)-\alpha(t_0)\right]}\,, \quad S(t_0) = \frac{1}{1+\exp[-\alpha(t_0)]}\,, \tag{3.3.6}$$

where $H(t) = H(t_0) + \int_{t_0}^{t}\lambda(s)\mathrm{d}s$ and $\alpha(t_0) = H(t_0) - \ln\left[\frac{1-S(t_0)}{S(t_0)}\right]$.

From (3.3.6), we further note that

$$F(t) = \frac{1}{1+\exp\left[\alpha(t_0)-H(t)\right]}\ . \tag{3.3.7}$$

F in (3.3.7) can be referred as a generalized logistic distribution.

In the following, we exhibit a well-known log-logistic distribution as a special case of (3.3.4).

Example 3.3.2 Let us consider a transformation,

$$Y = \ln T = \alpha + \sigma X \tag{3.3.8}$$

where $\alpha \in \mathbb{R}, \sigma > 0$, and a random variable X has the standard logistic cumulative distribution [25]. Under the transformation (3.3.8), (3.3.4) reduces to

$$\mathrm{d}S = -\frac{1}{\sigma t}S(1-S)\mathrm{d}t\,,\ S(t_0) = S_0\,,$$

with $\lambda = \frac{1}{\sigma t}$, $H(t) = \frac{\ln t}{\sigma}$ and $\alpha(t_0) = -\ln\left[\frac{1-S_0}{S_0}\right] + \frac{\ln t_0}{\sigma}$.

The nonlinear survival dynamic model described by (3.3.2) is too restrictive. It does not address the problems of external intervention processes generated by the usage of modern scientific, engineering, medical and technological tools/products/procedures/etc. In order to incorporate updated tools for the betterment of services/results/benefits, dynamic model (3.3.2) needs to be modified. For this purpose, we introduce a definition and modify dynamic model (3.3.2).

DEFINITION 3.3.1 Let $t_0 < t_1 < t_2 < \ldots < t_k < t_{k+1}$ be a given partition $(\mathcal{P})$ of a time interval $[t_0, \mathcal{T}]$, and $t_{k+1} \leq \infty$. Let $\lambda_1, \lambda_2, \ldots, \lambda_{k+1}$ be model parameters. We associate a finite increasing sequence $\{t_{j-1}\}_{j=1}^{k+1}$ of intervention process corresponding to the partition $(\mathcal{P})$ of the overall time interval $[t_0, \mathcal{T}]$ of study. Moreover, we decompose $[t_0, \mathcal{T}]$ by the finite sequence of subintervals $\{[t_{j-1}, t_j)\}_{j=1}^{k+1}$ of $[t_0, \mathcal{T}]$. A hazard/risk rate function for a nonnegative random variable T that characterizes time-to-event processes is of the following form:

$$\lambda(t) = \begin{cases} \lambda_1 & 0 \leq t < t_1 \\ \lambda_2 & t_1 \leq t < t_2 \\ \vdots & \\ \lambda_{k+1} & t \geq t_k \ , \end{cases} \tag{3.3.9}$$

where λ_j are positive real numbers for $j \in I(1, k+1)$, $(I(1, l) = \{1, 2, \ldots, l\})$.

From Definition 3.3.1, we recognize that the sudden changes in $\lambda(t)$ are encountered due to various types of intervention processes (internal or external) [33]. It is known [33] that many real world time-to-event dynamic processes undergo state adjustment processes, periodically. Due to constant changes in science, technology, medicine, cultural, environmental, educational, financial and socio-economic changes/behavior, continuous-time dynamic processes are frequently interrupted by discrete-time events. This results in a modification of (3.3.2) under the influence of intervention process. Following the nonlinear hybrid dynamic model [33], a modified version of the time-to-event dynamic model (3.3.2) is described by

$$\begin{cases} \mathrm{d}S = -S\lambda(t, S)\mathrm{d}t\,, \quad S(t_{j-1}) = S_{j-1}\,, \quad t \in [t_{j-1}, t_j)\,, \\ S_j = \Lambda(t_j^-, S(t_j^-, t_{j-1}, S_{j-1}))\,, \quad S(t_0) = S_0, \quad j \in I(1, k)\,, \end{cases} \tag{3.3.10}$$

where λ is defined in (3.3.2); Λ is a Borel-measurable survival state discrete-time intervention rate function; $S(t_j^-) = S(t_j^-, t_{j-1}, S_{j-1})$ represents the left-hand limit of survival state function at time t_j. We note that System (3.3.10) is an interconnected nonlinear hybrid dynamic system composed of both continuous and discrete time survival state dynamic systems.

REMARK 3.3.3 The hybrid dynamic model corresponding to (3.3.4) is as:

$$\begin{cases} \mathrm{d}S = -\lambda(t)S(1-S)\mathrm{d}t\,, \quad S(t_{j-1}) = S_{j-1}\,, \quad t \in [t_{j-1}, t_j)\,, \\ S_j = S(t_j^-, t_{j-1}, S_{j-1})\,, \quad S(t_0) = S_0, \quad j \in I(1, k)\,. \end{cases} \tag{3.3.11}$$

Imitating the procedure described in [33], the solution process of the initial value problem (IVP) (3.3.11) is as follows:

$$S(t, t_{j-1}, S_{j-1}) = \frac{1}{1 + \frac{1-S_{j-1}}{S_{j-1}} \exp\left[\int_{t_{j-1}}^{t} \lambda(s)\, \mathrm{d}s\right]}, \quad t \in [t_{j-1}, t_j). \tag{3.3.12}$$

Furthermore, the solution process of the overall time-to-event dynamic process (3.3.11) on $[t_0, \mathcal{T}]$ is

$$S(t, t_{j-1}, S_{j-1}) = \frac{1}{1 + \frac{1-S_{j-1}}{S_{j-1}} \exp\left[\int_{t_{j-1}}^{t} \lambda(s)\, \mathrm{d}s\right]}, \quad t \in [t_0, \mathcal{T}), \tag{3.3.13}$$

where

$$S_{j-1} = \frac{1}{1 + \frac{1-S_0}{S_0} \prod_{m=1}^{j-1} \exp\left[\int_{t_{m-1}}^{t_m} \lambda(s)\, \mathrm{d}s\right]}, \quad \text{for} \quad j \in I(1, k). \tag{3.3.14}$$

Moreover, from (3.3.13), we obtain that

$$\ln\left[\frac{1 - S(t, t_{j-1}, S_{j-1})}{S(t, t_{j-1}, S_{j-1})}\right] = \ln\left[\frac{1 - S_{j-1}}{S_{j-1}}\right] + \int_{t_{j-1}}^{t} \lambda(s)\, \mathrm{d}s\,,\ t \in [t_0, \mathcal{T}], \tag{3.3.15}$$

is the log odds of survival at time t.

In the following, we develop basic theoretical results that lay down a foundation for the development of an innovative approach for state and parameter estimation of time-to-event dynamic process. Most of the parameter estimation methods in the survival analysis literature are centered around the closed form representation of likelihood functions, whereby, the entire data set has been utilized to estimate the parameters on the overall interval $[t_0, \mathcal{T}]$ of study.

3.4 Fundamental Results for Nonlinear Hybrid Dynamic Process

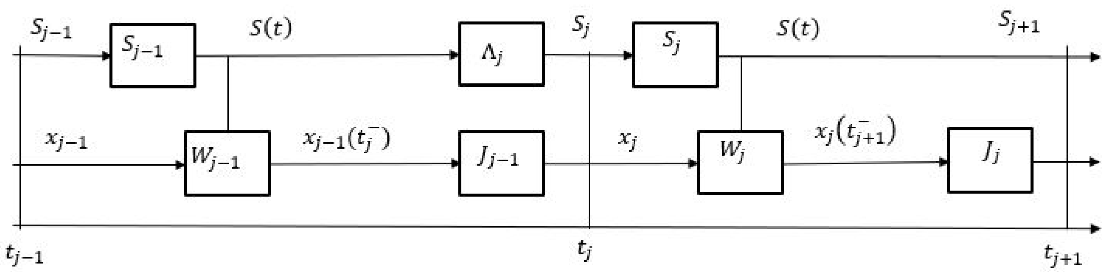
In this section, we employ dynamic model (3.3.10) and Euler-type discretization scheme [8] to develop a fundamental theoretical results. The presented analytic results provide the basis for conceptual computational tools for survival state and parameter estimation problems in time-to-event data analysis processes.

Let $x(t)$ be total number of units/individuals operating/alive (or survivals) at time t for $t \in [t_0, \mathcal{T})$. Let λ and S be the hazard rate and survival state functions of units/patients/infectives/species/individuals described by (3.3.2), respectively. Using a dynamic model for number of units/species/individuals/infectives coupled with hybrid survival state dynamic model (3.3.10) that forms a large-scale dynamic system, we present an interconnected nonlinear hybrid dynamic model of time-to-event process (INHDMTTEP).

Following the argument outlined in developing dynamic models in [5, 33], we introduce the following systems of nonlinear and non-stationary differential equations:

$$
\begin{cases}
dx = W(t, Sx)\mathrm{d}\eta(t)\,, \quad x(t_0) = x_0\,, \quad t \in [t_{j-1}, t_j)\,, \\
x_j = x_{j-1} + J(t_j^-, S(t_j^-, t_{j-1}, S_{j-1})x(t_j^-, t_{j-1}, x_{j-1}), x_{j-1})\,, \\
\mathrm{d}S = -S\lambda(t, S)\mathrm{d}t, \quad t \geq 0, \quad S(t_0) = S_0\,, \\
S_j = S_{j-1} + \Lambda(t_j^-, S(t_j^-, t_{j-1}, S_{j-1})), S(t_0) = S_0\,,
\end{cases}
\tag{3.4.1}
$$

where S is a survival state function;the finite sequence of subintervals $\{[t_{j-1}, t_j)\}_{j=1}^{k+1}$ is defined in Definition 3.3.1; λ is defined in (3.3.2); W is a continuous function defined on $[t_{j-1}, t_j) \times \mathbb{R}$ into $\mathbb{R}$ for $j \in I(1, k)$; $J(t_j^-, S(t_j^-, t_{j-1}, S_{j-1})x(t_j^-, t_{j-1}, x_{j-1}), x_{j-1}) = \eta_j^- W(t_j^-, S(t_j^-, t_{j-1}, S_{j-1})x(t_j^-, t_{j-1}, S_{j-1})) - \eta_{j-1}^+ W(t_{j-1}, S_{j-1}x_{j-1})$; η_j^- and η_{j-1}^+ are positive constants; η is a function of bounded variation defined on $[t_{j-1}, t_j)$ into $\mathbb{R}$; Λ is defined in (3.3.10). In addition, it is assumed that (3.4.1) has a solution process [33]. It is denoted by (x, S). The Flowchart-4 exhibits the structural and operational dynamic of INHDMTTEP.



Flowchart 4.: Structural and Operational Dynamic of INHDMTTEP

Remark 3.4.1 In addition to the conditions on (3.4.1), if W and λ are non-negative functions (i.e. $W, \lambda \geq 0$), and if

$$
\eta(t) = \begin{cases} 0, & t \in [t_{j-1}, t_j)\,, \\ 1, & t = t_j\,, \end{cases}
$$

then (3.4.1) reduces to a partially discrete-time interconnected nonlinear hybrid dynamic system:

$$
\begin{cases}
dx = 0\,\mathrm{d}t\,, \quad x(t_0) = x_0\,, \quad t \in [t_{j-1}, t_j)\,, \\
x_j = x_{j-1} + J(t_j^-, S(t_j^-, t_{j-1}, S_{j-1})x(t_j^-, t_{j-1}, x_{j-1}), x_{j-1})\,, \\
\mathrm{d}S = -S\lambda(t, S)\mathrm{d}t\,, \quad t \in [t_{j-1}, t_j)\,, \\
S_j = S_{j-1} + \Lambda(t_j^-, S(t_j^-, t_{j-1}, S_{j-1})), S(t_0) = S_0\,.
\end{cases}
\tag{3.4.2}
$$

Example 3.4.1 $S\lambda(t, S) = \lambda(t)S(1 - S)$ is an admissible function in (3.4.1) and (3.4.2).

Employing the interconnected hybrid dynamic model for time-to-event process described in (3.4.1), we present a fundamental result regarding continuous and discrete-time dynamic of survival species or operating objects or thoughts and survival state. Prior to this result, we introduce a few concepts that will be utilized, subsequently.

Definition 3.4.1 Let z be a function defined by $z(t) = x(t)S(t)$, where S and x are solution processes of (3.4.1) for $t \in [t_0, \mathcal{T})$. Moreover, for each $t \in [t_0, \mathcal{T})$, $z(t)$ stands for the number of survivals/operating units at t.

Definition 3.4.2 The sequence $\{t_{j-1}\}_{j=1}^{k}$ defined in Definition 3.3.1 is referred to as the conceptual data collection/observation/intervention sequence over the interval of time $[t_0, \mathcal{T})$, and sequence of subinterval $\{[t_{j-1}, t_j)\}_{j=1}^{k}$ is called a continuous-time hybrid system operating subinterval sequence with its right-end-point as a conceptual data observation time.

Now, we are ready to present a fundamental theoretical result. The presented result provides a foundation for the development of survival data analysis of time-to-event processes in any field of interest that are conceptually similar but apparently different [33].

Theorem 3.4.1 *Let (x, S) be a solution process of (3.4.1), and let t_{j-1} and t_j be any pair of consecutive conceptual data observation times in a given interval of time $[t_0, \mathcal{T})$. Then the transformed interconnected nonlinear hybrid dynamic model of survival species and state of time-to-event dynamic process described by (3.4.1) is reduced to:*

$$\begin{cases} \mathrm{d}z = -z\lambda(t, S)\mathrm{d}t + SW(t, z)\mathrm{d}\eta(t)\,, \quad z(t_{j-1}) = z_{j-1}\,, \quad \textit{for } t \in [t_{j-1}, t_j)\ , \textit{and } j \in I(1, k), \\ \mathrm{d}S = -S\lambda(t, S)\mathrm{d}t,\ S(t_0) = S_0, \\ z_j = z_{j-1} + x_{j-1}\Lambda(t_j^-, S(t_j^-, t_{j-1}, S_{j-1})) + S_j J(t_j^-, z(t_j^-, t_{j-1}, x_{j-1}), x_{j-1})\,, \quad z(t_0) = z_0\,, \end{cases} \tag{3.4.3}$$

and corresponding transformed discrete-time conceptual computational interconnected dynamic algorithm

$$\begin{cases} z(t_j) = z(t_{j-1}) - \lambda(t_{j-1},\ S(t_{j-1}))z(t_{j-1})\Delta t_j + \gamma_j\ \ , z(t_0) = z_0\,, \\ S(t_j) = S(t_{j-1}) - \lambda(t_{j-1}, S(t_{j-1}))S(t_{j-1})\Delta t_j,\ S(t_0) = S_0,\ j \in I(1, k), \end{cases} \tag{3.4.4}$$

where z is defined in Definition 3.4.1; $\gamma_j = S(t_j^-)W(t_j^-, z_j^-)) - S(t_{j-1})W(t_{j-1}, z_{j-1})$, and it represents change in survivals due to either failure/censored/admitted or change-point process; and $\Delta t_j = t_j - t_{j-1}$ for $j \in I(1, k)$.

Proof.

For $t \in [t_{j-1}, t_j),\ j \geq 1$, from Definition 3.4.1 and the nature of S, we have

$$\begin{aligned} \mathrm{d}z(t) &= x(t)\mathrm{d}S + S(t)\mathrm{d}x(t) \\ &= x(t)\left[-S(t)\lambda(t, S(t))\mathrm{d}t\right] + S(t)W(t, S(t)x)\mathrm{d}\eta(t) \\ &= -z(t)\lambda(t, S(t))\mathrm{d}t + S(t)W(t, z(t))\mathrm{d}\eta(t). \end{aligned} \tag{3.4.5}$$

This establishes the continuous-time dynamic subsystem in (3.4.3). The proofs of the discrete-time dynamic subsystem in (3.4.3) and iterative process (3.4.4) are outlined below.

From the discrete-time dynamic of population/species state x and survival state intervention process in (3.4.1), we have

$$z_j = z_{j-1} + x_{j-1}\Lambda(t_j^-, S(t_j^-, t_{j-1}, S_{j-1})) + S_j J(t_j^-, z(t_j^-, t_{j-1}, x_{j-1}), x_{j-1}) \tag{3.4.6}$$

This establishes the discrete-time dynamic subsystem in (3.4.3).

Now, applying the Euler-type numerical scheme [8] to (3.4.5) over an interval $[t_{j-1}, t_j]$, we obtain

$$z(t_j) - z(t_{j-1}) = -\lambda(t_{j-1},\, S(t_{j-1}))z(t_{j-1})\Delta t_j + \int_{t_{j-1}}^{t_j} S(s)W(s, z(s))\mathrm{d}\eta(s). \tag{3.4.7}$$

By applying the Riemann-Stieltjes integral property [4], we approximate (3.4.7) as:

$$z(t_j) - z(t_{j-1}) = -\lambda(t_{j-1}, S(t_{j-1}))z(t_{j-1})\Delta t_j + S(t_j^-)W(t_j^-, z(t_j^-)) - S(t_{j-1})W(t_{j-1}, z_{j-1}). \tag{3.4.8}$$

From (3.4.8), we have

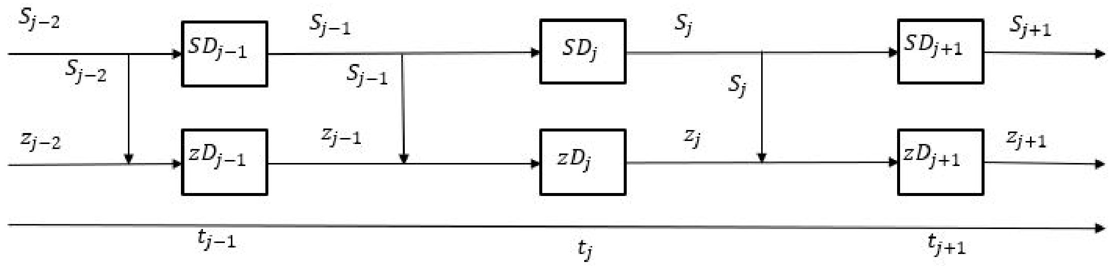
$$z(t_j) = [1 - \lambda(t_{j-1},\, S(t_{j-1}))\Delta t_j]\, z(t_{j-1}) + \gamma_j \text{ , for } j \in I(1, k)\,, \tag{3.4.9}$$

where $\gamma_j = S(t_j^-)W(t_j^-, z(t_j^-)) - S(t_{j-1})W(t_{j-1}, z_{j-1})$ is a jump at t_j, and it represents change in survivals due to an intervention process. Applying the Euler numerical scheme to the continuous-time dynamic in (3.4.1) over the interval $[t_{j-1}, t_j]$ yields

$$S(t_j) = S(t_{j-1}) - \lambda(t_{j-1}, S(t_{j-1}))S(t_{j-1})\Delta t_j \tag{3.4.10}$$

(3.4.9) and (3.4.10) establishes the discrete time conceptual theoretical dynamic for joint survival state process in the context of joint continuous-time interconnected nonlinear dynamic and the discrete-time intervention component processes (3.4.3). Moreover, (3.4.9) and (3.4.10) exhibits the derivation of (3.4.4). This establishes the proof of Theorem 3.4.1. Furthermore (3.4.4) is an approximation of transformed intervention process in (3.4.3). □

Remark 3.4.2 The transformed theoretical discrete-time computational dynamic process (3.4.4) provides a basis for the discrete-time conceptual computational and simulation dynamic processes. The Flowchart-4 exhibits the structural and discrete-time operational dynamic of interconnected discrete-time algorithm of time-to-event data statistic.



Flowchart 5.: Structural and Operational Dynamic of IDATTEDS

Now, using (3.4.2), we present a result that is jointly totally discrete-time interconnected nonlinear hybrid system.

Corollary 3.4.1 *Let us consider a very special case of* (3.4.2) *as follows:*

$$\begin{cases} dx = 0\,\mathrm{d}t\,, \quad x(t_0) = x_0\,, \quad t \in [t_{j-1}, t_j)\,, \\ x_j = x_{j-1} + J(t_j^-, S(t_j^-, t_{j-1}, S_{j-1})x(t_j^-, t_{j-1}, x_{j-1}), x_{j-1})\,, \\ \mathrm{d}S = 0, \quad t \in [t_{j-1}, t_j)\,, \\ S_j = S_{j-1} + \Lambda(t_j^-, S(t_j^-, t_{j-1}, S_{j-1})), S(t_0) = S_0\,. \end{cases} \tag{3.4.11}$$

Then under the assumptions of Theorem 1.3.1, (3.4.11) *reduces to*

$$\begin{cases} \mathrm{d}z = 0\,\mathrm{d}t\,, \quad z(t_{j-1}) = z_{j-1}, \quad t \in [t_{j-1}, t_j), \\ z_j = z_{j-1} + x_{j-1}\Lambda(t_j^-, S(t_j^-, t_{j-1}, S_{j-1})) + S_j J(t_j^-, z(t_j^-, t_{j-1}, x_{j-1}), x_{j-1})\,, \quad z(t_0) = z_0,\,, \end{cases} \tag{3.4.12}$$

and

$$\begin{cases} z(t_j) = z(t_{j-1}) - \lambda(t_{j-1},\, S(t_{j-1}))z(t_{j-1}) + \gamma_j \;\;, z(t_0) = S_0 x_0\,, \\ S(t_j) = S(t_{j-1}) - \lambda(t_{j-1}, S(t_{j-1}))S(t_{j-1}),\;\; S(t_0) = S_0,\;\; j \in I(1,k) \end{cases} \tag{3.4.13}$$

We remark that this corollary is transformed totally discrete-time version of nonlinear hybrid dynamic system operating under discrete-time intervention component processes.

In the following section, we establish theoretical discrete-time conceptual computational parameter and state estimation algorithms.

3.5 Theoretical/Conceptual Parameter and State Estimations

Using Definition 3.4.1 and the transformed theoretical discrete-time iterative process (3.4.4), we develop conceptual computational parameter dynamic estimation algorithms. In addition, parameter and state estimations are determined conceptually. For this purpose, we introduce a few definitions and notations.

Definition 3.5.1 Let t_{j-1} and t_j be a pair of consecutive conceptual data collection/observation times on $[t_0, \mathfrak{T})$, and let $z(t)$ be as defined in Definition 3.4.1. $z(t_{j-1})$ stands for the number of survivals at the time t_{j-1} for each $j \in I(1,k)$. Moreover, the number of survivals $z(t_{j-1})$ are under observation/supervision over the sub-interval of time $[t_{j-1}, t_j)$ of length Δt_j. $z(t_{j-1})\Delta t_j$ is the amount of time spent by $z(t_{j-1})$ survivals under observation/testing/evaluation over the length Δt_j of time interval $[t_{j-1}, t_j)$.

Definition 3.5.2 For $j \in I(1,k)$, let t_{j-1} and t_j be consecutive data observation/supervision times of joint population/objects/entities and state survival dynamic process. The parameter estimate at t_j is defined by the quotient of change of entities/objects over the consecutive change time subinterval $[t_{j-1}, t_j)$ and the total time spent by the entities/objects under observation/supervision over the subinterval $[t_{j-1}, t_j)$ of length Δt_j.

Definition 3.5.3 Let $\{z_{j-1}\}_{j=1}^{k}$ be an overall sequence of transformed conceptual state data set with respect to the conceptual state data collection/observation time sequence $\{t_{j-1}\}_{j=1}^{k}$, and let $\{t^f_{j-1i-1}\}_{i=1}^{k_f}$, $\{t^c_{j-1l-1}\}_{l=1}^{k_c}$ and $\{t^a_{j-1m-1}\}_{m=1}^{k_a}$ be overall conceptual failure, censored and admitted increasing subsequences of the overall conceptual data collection time sequence $\{t_{j-1}\}_{j=1}^{k}$, respectively. Three subsequences of the overall conceptual state data sequence $\{z_{j-1}\}_{j=1}^{k}$ associated with the three overall conceptual subsequences of failure, censored and admitted time subsequences are represented by:

$$\{z^f_{j-1i-1}\}_{i=1}^{k_f}, \quad \{z^c_{j-1l-1}\}_{l=1}^{k_c}, \quad \text{and} \quad \{z^a_{j-1m-1}\}_{m=1}^{k_a}, \tag{3.5.1}$$

respectively. These conceptual state data subsequences are called conceptual failure, censored and admitted state subsequences of $\{z_{j-1}\}_{j=1}^{k}$, respectively. We note that $k_f + k_c + k_a = k$.

Definition 3.5.4 The union of the boundary point set of the interval $[t_0, \mathfrak{T})$ and the range of the overall failure, subsequence $\{t^f_{j-1i-1}\}_{i=1}^{k_f+1}$ constitutes a partition of the interval $[t_0, \mathfrak{T}), \mathfrak{T} \leq \infty$. This partition of $[t_0, \mathfrak{T}), \mathfrak{T} \leq \infty$ is termed as overall conceptual failure-time partition of $[t_0, \mathfrak{T})$, and it is denoted by (P^f). Moreover, $P^f \subseteq P$ in Definition 3.3.1.

Definition 3.5.5 For $j \in I(1,k)$ and any consecutive pair $(t^f_{j-1i-1}, t^f_{j-1i})$ of conceptual failure-times for $i \in I(1,k_f)$ under the notations $t^f_{j-100} = t^f_{j-1}$ for $i = 1$ and either $l = 1$ or $m = 1$; furthermore, $t^f_{000} = t_0$ if $i = j = 1$; either $t^f_{j-1ik_{c_i}+1} = t^f_{j-1i-1l} = t^f_{j-1i}$ or $t^f_{j-1i-1m} = t^f_{j-1ik_{a_i}+1} = t^f_{j-1i}$ depending on whether $l = k_{c_i} + 1$ or $m = k_{a_i} + 1$; a ji-th consecutive conceptual failure-time subinterval is $[t^f_{j-1i-1}, t^f_{j-1i})$ for $i \in I(1,k_f)$; $t^f_{j-1k_f}$. In addition, the conceptual transformed state data associated with the consecutive conceptual initial failure-times is denoted by $z^f_{j-100} = z^f_{j-1}$ and for $j = 1$, $z^f_{1-10} = z^f_{000} = z^f_0$.

Definition 3.5.6 Let $\{z^c_{j-1l-1}\}_{l=1}^{k_c}$ and $\{z^a_{j-1m-1}\}_{m=1}^{k_a}$ be overall censored and admitted conceptual transformed state data subsequences defined in Definition 3.5.3. Let $\{t^c_{j-1i-1p}\}_{p=1}^{k_{c_i}}$ and $\{t^a_{j-1i-1q}\}_{q=1}^{k_{a_i}}$ be conceptual subsequences restricted to the $j-1i$-th consecutive conceptual failure-time subinterval $[t^f_{j-1i-1}, t^f_{j-1i})$ of overall conceptual censored and admitted subsequences $\{t^c_{j-1l-1}\}_{l=1}^{k_c}$ and $\{t^a_{j-1m-1}\}_{m=1}^{k_a}$ of times of the overall sequence $\{t_{j-1}\}_{j=1}^{k}$ of times, respectively. Moreover, the union of the boundary points of $[t^f_{j-1i-1}, t^f_{j-1i})$

and the range of subsequences $\{t^c_{j-1i-1p}\}_{p=1}^{k_{c_i}}$ and $\{t^c_{j-1i-1q}\}_{q=1}^{k_{a_i}}$ form a sub-partition P^f_{j-1} of P^f and the partition of $j-1$-th subinterval $[t^f_{j-1i-1}, t^f_{j-1i})$. Two subsequences of the overall censored and/or admitted conceptual transformed state data subsequences $\{z^c_{j-1l-1}\}_{l=1}^{k_c}$ and/or $\{z^a_{j-1m-1}\}_{m=1}^{k_a}$ with respect to the two overall conceptual censored and admitted time subsequences of the overall sequence of times $\{[t_{j-1}, t_j)\}_{j=1}^{k}$ restricted to the $j-1i$-th consecutive conceptual failure-time subinterval $[t^f_{j-1i-1}, t^f_{j-1i})$ are represented by:

$$\{z^c_{j-1i-1p-1}\}_{p=1}^{k_{c_i}} \quad \text{and} \quad \{z^a_{j-1i-1q-1}\}_{q=1}^{k_{a_i}}\,, \tag{3.5.2}$$

respectively. These conceptual transformed state data subsequences are called subsequences of the overall censored and admitted conceptual state data subsequences $\{z^c_{j-1l-1}\}_{l=1}^{k_c}$ and $\{z^a_{j-1m-1}\}_{l=1}^{k_a}$ of the overall conceptual sequence $\{z_{j-1}\}_{j=1}^{k}$ of data set, respectively. We note that $k_c = \sum_{l=1}^{k_c} k_{c_l}$ and $k_a = \sum_{m=1}^{k_a} k_{a_m}$. Moreover, for $p = 1$ and $q = 1$, (3.5.2) reduces to $z^c_{j-1i-10} = z^c_{j-1i-1}$ and $z^a_{j-1i-10} = z^a_{j-1i-1}$ respectively; for $p = k_{c_i} + 2$, and $q = k_{a_i} + 2$, we have $z^c_{j-1i-1k_{c_i}+1} = z^c_{ji}$ and $z^a_{j-1i-1k_{a_i}+1} = z^a_{ji}$ respectively.

Remark 3.5.1 The transformed discrete-time dynamic process (3.4.4) is referred as conceptual computational interconnected dynamic algorithm for time-to-event data statistic (IDATTEDS). Moreover, from (3.4.4), we introduce three more special transformed theoretical numerical dynamic schemes for time-to-event dynamic processes, namely: (i) abnormal/failure/death/removal/infective/etc species or objects, (ii) censored/quitting/withdrawn/etc species or objects, and (iii) admitted/joining/relapsed/susceptible/etc species or objects. We further note that the presented numerical dynamic schemes allow "ties" with deaths/failure or censored/quiting or admitted/susceptible process. In addition, the population/species under the presented observation/supervision process includes the abnormal/species/patient/objects/infectives population as a special case.

(i) For each $j \in I(1,k)$, let t^{fca}_{j-1} be either failure, censored or admitting time at t_{j-1}. For $\gamma^f_j = 0$, the transformed discrete-time dynamic component (3.4.4) at t^f_j for failure/death/removal/infective/etc process data set is described by

$$z(t^f_j) = \left[1 - \lambda(t^{fca}_{j-1},\, S(t^{fca}_{j-1}))\Delta t^f_j\right] z(t^{fca}_{j-1}) \quad \text{for} \quad j \in I(1,k)\,. \tag{3.5.3}$$

This together with (3.4.4), one obtains

$$\begin{cases} z(t^f_j) - z(t^{fca}_{j-1}) = -\lambda(t_{j-1},\, S(t_{j-1}))z(t^{fca}_{j-1})\Delta t^f_j,\, z(t_0) = z_0\,, \\ S(t_{j-1}) = S(t_{j-2}) - \lambda(t^f_{j-2}, S(t^f_{j-2}))S(t^f_{j-2})\Delta t^f_{j-1}\,, S(t_0) = S_0\,, \end{cases} \tag{3.5.4}$$

where a pair (t^{fca}_{j-1}, t^f_j) stands for either (t^f_{j-1}, t^f_j), or (t^c_{j-1}, t^f_j) or (t^a_{j-1}, t^f_j); t^f_j, t^c_{j-1} and t^a_{j-1} stand for failure, censored and admitting times, respectively; $\Delta t^f_j = t^f_j - t^{fca}_{j-1}$.

(ii) For each $j \in I(1,k)$, let t^{caf}_{j-1} be either censored, admitting or failure time at t_{j-1}. γ^c_j stands for the conceptual number of censored objects/infectives/quitting/withdrawn/etc at a time t_j^c. The transformed discrete-time component (3.4.4) at t_j^c for censored/listed/identified process data set is reduced to

$$z(t_j^c) = \left[1 - \lambda(t_{j-1}^{caf}, S(t_{j-1}^{caf}))\Delta t_j^c\right] z(t_{j-1}^{caf}) - \gamma_j^c \quad \text{for} \quad j \in I(1,k)\,, \tag{3.5.5}$$

where a pair (t_{j-1}^{caf}, t_j^c) stands for either $(t_{j-1}^c, t_j^c), (t_{j-1}^a, t_j^c)$ or (t_{j-1}^f, t_j^c); $\Delta t_j^c = t_j^c - t_{j-1}^{caf}$. Thus

$$\begin{cases} z(t_j^c) - z(t_{j-1}^{caf}) = -\lambda(t_{j-1}, S(t_{j-1}))z(t_{j-1}^{caf})\Delta t_j^c - \gamma_j^c,\ z(t_0) = z_0\,, \\ S(t_{j-1}) = S(t_{j-2}) - \lambda(t_{j-2}^f, S(t_{j-2}^f))S(t_{j-2}^f)\Delta t_{j-1}^f\,, S(t_0) = S_0\,. \end{cases} \tag{3.5.6}$$

(iii) For each $j \in I(1,k)$, let t_{j-1}^{acf} be either admitting, censored or failure time at t_{j-1}. γ_j^a stands for the conceptual number of objects/infectives/etc arriving/joining at a time t_j^a. The transformed discrete-time dynamic component (3.4.4) at t_j^a for admitting/joining/sustainable/recruiting/etc process data set is represented by

$$z(t_j^a) = \left[1 - \lambda(t_{j-1}^{acf}, S(t_{j-1}^{acf}))\Delta t_j^a\right] z(t_{j-1}^{acf}) + \gamma_j^a \quad \text{for} \quad j \in I(1,k)\,, \tag{3.5.7}$$

where a pair (t_{j-1}^{acf}, t_j^a) belongs to a set: $(t_{j-1}^{acf}, t_j^a) \in \{(t_{j-1}^a, t_j^a), (t_{j-1}^c, t_j^a), (t_{j-1}^f, t_j^a)\}$; $\Delta t_j^a = t_j^a - t_{j-1}^{acf}$.

Hence

$$\begin{cases} z(t_j^a) - z(t_{j-1}^{acf}) = -\lambda(t_{j-1}, S(t_{j-1}))z(t_{j-1}^{acf})\Delta t_j^a + \gamma_j^a,\ z(t_0) = z_0\,, \\ S(t_{j-1}) = S(t_{j-2}) - \lambda(t_{j-2}^f, S(t_{j-2}^f))S(t_{j-2}^f)\Delta t_{j-1}^f\,, S(t_0) = S_0\,. \end{cases} \tag{3.5.8}$$

(iv) Remarks (i), (ii) and (iii) remain valid for the iterative process (3.4.13).

(I) For $\gamma_j^f = 0$, (3.4.13) reduces to

$$\begin{cases} z(t_j^f) - z(t_{j-1}^{fca}) = -\lambda(t_{j-1}, S(t_{j-1}))z(t_{j-1}^{fca}),\ z(t_0) = z_0\,, \\ S(t_{j-1}) = S(t_{j-2}) - \lambda(t_{j-2}^f, S(t_{j-2}^f))S(t_{j-2}^f)\Delta t_{j-1}^f,\ S(t_0) = S_0\,. \end{cases} \tag{3.5.9}$$

(II) For $\gamma_j = \gamma_j^c$ in (3.4.13), (3.4.13) reduces to

$$\begin{cases} z(t_j^c) - z(t_{j-1}^{caf}) = -\lambda(t_{j-1}, S(t_{j-1}))z(t_{j-1}^{caf}) - \gamma_j^c,\ z(t_0) = z_0\,, \\ S(t_{j-1}) = S(t_{j-2}) - \lambda(t_{j-2}^f, S(t_{j-2}^f))S(t_{j-2}^f)\Delta t_{j-1}^f,\ S(t_0) = S_0\,. \end{cases} \tag{3.5.10}$$

(III) For $\gamma_j = \gamma_j^a$ in (3.4.13), (3.4.13) reduces to

$$\begin{cases} z(t_j^a) - z(t_{j-1}^{acf}) = -\lambda(t_{j-1}, S(t_{j-1}))z(t_{j-1}^{acf}) + \gamma_j^a,\ z(t_0) = z_0\,, \\ S(t_{j-1}) = S(t_{j-2}) - \lambda(t_{j-2}^f, S(t_{j-2}^f))S(t_{j-2}^f)\Delta t_{j-1}^f,\ S(t_0) = S_0\,. \end{cases} \tag{3.5.11}$$

In the following, we present very simple result that provides an insight for the understanding of the discrete-time dynamic of state and parameter estimation problems. Moreover, the result provides one of the assumptions of the Principle of Mathematical Induction.

Theorem 3.5.1 *Assume that the conditions of Theorem 3.4.1 in the context of Remarks 3.5.1(i),(ii) and (iii) and Definitions 3.5.5 and 3.5.6 are satisfied.*

(a) For $j \in I(1,k)$, if t^f_{j-1} and t^f_j are consecutive risk/failure/removal/death/non-operational times in $[t_0, \mathcal{T}), \mathcal{T} \leq \infty$. Then the theoretical/computational estimation algorithm and parameter estimation for $\lambda(t, S(t))$ at t^f_j are described by (i) and (ii) below.

(i)

$$\begin{cases} z(t^f_j) = z(t^f_{j-1}) - \lambda(t^f_{j-1}, S(t^f_{j-1}))z(t^f_{j-1})\Delta t^f_j,\ z(t_0) = z_0\,, \\ S(t^f_{j-1}) = S(t_{j-2}) - \lambda(t^f_{j-2}, S(t^f_{j-2}))S(t^f_{j-2})\Delta t^f_{j-1},\ S(t_0) = S_0\,. \end{cases} \tag{3.5.12}$$

(ii)

$$\hat{\lambda}(t^f_{j-1}, S(t^f_{j-1})) = \frac{z(t^f_{j-1}) - z(t^f_j)}{z(t^f_{j-1})\Delta t^f_j}\,, \quad \Delta t^f_j = t^f_j - t^f_{j-1}\,. \tag{3.5.13}$$

Moreover an overall conceptual computational estimate for $z(t), S(t)$ and $\lambda(t, S(t))$ on the time-interval of study $[t_0, \mathcal{T}), \mathcal{T} \leq \infty$ is

$$\begin{cases} \hat{\lambda}(t, \hat{S}(t_{j-1})) = \hat{\lambda}(t^f_{j-1}, \hat{S}(t^f_{j-1}))\,, \quad \textit{for} \quad t \in [t^f_{j-1}, t^f_j) \quad \textit{and} \quad j \in I(1,k), \\ \hat{S}(t, t_{j-1}, \hat{S}_{j-1}),\ \hat{S}(t_{j-1}) = \hat{S}_{j-1}, \\ \hat{z}(t, t_{j-1}, \hat{z}_{j-1}),\ \hat{z}(t_{j-1}) = \hat{z}_{j-1}. \end{cases} \tag{3.5.14}$$

(b) For $j \in I(1,k)$, if $t^f_{j-1} < t^c_j < t^f_j$, and t^c_j is censored time between a pair of consecutive failure times t^f_{j-1} and t^f_j in $[t_0, \mathcal{T}), \mathcal{T} \leq \infty$. Then the theoretical/computational estimation algorithm and parameter estimation for $\lambda(t, S(t))$ at t^f_j are respectively determined by :

(i)

$$\begin{cases} z(t^f_j) = z(t^f_{j-1}) - \lambda(t^f_{j-1}, S(t^f_{j-1}))\left[z(t^f_{j-1})\Delta t^{cf}_j + z(t^c_j)\Delta t^{fc}_j\right] - \gamma^c_j,\ z(t_0) = z_0\,, \\ S(t^f_{j-1}) = S(t_{j-2}) - \lambda(t^f_{j-2}, S(t^f_{j-2}))S(t^f_{j-2})\Delta t^f_{j-1},\ S(t_0) = S_0\,. \end{cases} \tag{3.5.15}$$

(ii)

$$\hat{\lambda}(t_{j-1}, \hat{S}(t_{j-1})) = \frac{z(t^f_{j-1}) - z(t^f_j) - \gamma^c_j}{\left[z(t^f_{j-1})\Delta t^{fc}_j + z(t^c_j)\Delta t^{cf}_j\right]}\,, \tag{3.5.16}$$

where $\Delta t^{fc}_j = t^c_j - t^f_{j-1}$, $\Delta t^{cf}_{j1} = t^f_j - t^c_j$. Moreover an overall conceptual computational estimate *for $z(t), S(t)$ and $\lambda(t, S(t))$ on the time-interval of study $[t_0, \mathcal{T}), \mathcal{T} \leq \infty$ is*

$$\begin{cases} \hat{\lambda}(t, \hat{S}(t_{j-1})) = \hat{\lambda}(t^f_{j-1}, \hat{S}(t^f_{j-1}))\,, \quad \text{for} \quad t \in [t^f_{j-1}, t^f_j) \quad \text{and} \quad j \in I(1,k), \\ \hat{S}(t, t_{j-1}, \hat{S}_{j-1}),\ \hat{S}(t_{j-1}) = \hat{S}_{j-1}, \\ \hat{z}(t, t_{j-1}, \hat{z}_{j-1}),\ \hat{z}(t_{j-1}) = \hat{z}_{j-1}. \end{cases} \tag{3.5.17}$$

(c) For $j \in I(1,k)$, if $t^f_{j-1} < t^a_j < t^f_j$, and t^a_j is joining/admitting time between a pair of consecutive failure times t^f_{j-1} and t^f_j in $[t_0, \mathcal{T}), \mathcal{T} \leq \infty$. Then the theoretical/computational estimation algorithm and parameter estimation for $\lambda(t, S(t))$ at t^f_j are determined by

(i)

$$\begin{cases} z(t^f_j) = z(t^f_{j-1}) - \lambda(t^f_{j-1}, S(t^f_{j-1}))\left[z(t^f_{j-1})\Delta t^{af}_j + z(t^{fa}_j)\Delta t^{af}_j\right] + \gamma^a_j\,, z(t_0) = z_0 \\ S(t^f_{j-1}) = S(t_{j-2}) - \lambda(t^f_{j-2}, S(t^f_{j-2}))S(t^f_{j-2})\Delta t^f_{j-1}\,, S(t_0) = S_0 \end{cases} \tag{3.5.18}$$

and

(ii)

$$\hat{\lambda}(t^f_{j-1}, \hat{S}(t^f_{j-1})) = \frac{z(t^f_{j-1}) - z(t^f_j) + \gamma^a_j}{\left[z(t^f_{j-1})\Delta t^{fa}_{j1} + z(t^{af}_{j1})\Delta t^{af}_{j1}\right]}\,, \tag{3.5.19}$$

where $\Delta t^{af}_j = t^a_j - t^f_{j-1}$, $\Delta t^{fa}_j = t^f_j - t^a_j$. Moreover an overall conceptual computational estimate for $z(t), S(t)$ and $\lambda(t, S(t))$ on the time-interval of study $[t_0, \mathcal{T}), \mathcal{T} \leq \infty$ is

$$\begin{cases} \hat{\lambda}(t, \hat{S}(t_{j-1})) = \hat{\lambda}(t^f_{j-1}, \hat{S}(t^f_{j-1}))\,, \quad \text{for} \quad t \in [t^f_{j-1}, t^f_j) \quad \text{and} \quad j \in I(1,k), \\ \hat{S}(t, t_{j-1}, \hat{S}_{j-1}),\ \hat{S}(t_{j-1}) = \hat{S}_{j-1}, \\ \hat{z}(t, t_{j-1}, \hat{z}_{j-1}),\ \hat{z}(t_{j-1}) = \hat{z}_{j-1}. \end{cases} \tag{3.5.20}$$

Proof. (a) Let t^f_{j-1} and t^f_j be two consecutive conceptual failure times. In this case, $k_{c_i} = k_{a_i} = 0$. From Definition 3.5.5, here $i = 1$, therefore, for the subinterval $[t^f_{j-1i-1l-1}, t^f_{j-1i}), l = i = 1$, and $t^f_{j1} = t^f_j; t^f_{j-1} = t^f_{j-100}$. Using the theoretical discrete-time iterative scheme (3.4.4) and Remark 3.5.1(i)(1.3.20), we have

$$\begin{cases} z(t^f_j) = z(t^f_{j-1}) - \lambda(t^f_{j-1}, S(t^f_{j-1}))z(t^f_{j-1})\Delta t^f_j\ , z(t_0) = z_0 \\ S(t^f_{j-1}) = S(t_{j-2}) - \lambda(t^f_{j-2}, S(t^f_{j-2}))S(t^f_{j-2})\Delta t^f_{j-1}\,, S(t_0) = S_0 \end{cases}$$

This establishes a(i). For the validity of a(ii), from Definition 3.5.1, backward substitution, and using

Definition 3.5.2, we obtain

$$\begin{cases} \hat{\lambda}(t,\hat{S}(t_{j-1})) = \hat{\lambda}(t^f_{j-1}, S(t^f_{j-1})) = \frac{z(t^f_{j-1})-z(t^f_j)}{z(t^f_{j-1})\Delta t^f_j}\,, \quad \Delta t^f_j = t^f_j - t^f_{j-1}\,, \\ \hat{S}(t,t_{j-1},\hat{S}_{j-1}),\ \hat{S}(t_{j-1}) = \hat{S}_{j-1}, \\ \hat{z}(t,t_{j-1},\hat{z}_{j-1}),\ \hat{z}(t_{j-1}) = \hat{z}_{j-1}. \end{cases}$$

for $t \in [t^f_{j-1}, t^f_j)$ and $j \in I(1,k)$. This establishes (a)(ii). This completes the proof of (a).

(b) Let t^c_j be a censoring time between two consecutive conceptual risk/failure times, t^f_{j-1} and t^f_j. We consider a partition of subinterval $[t^f_{j-1}, t^f_j)$ to be $P^f_{ji} = [t^f_{j-1}, t^f_j] : t_{j-1} < t^c_{j-1} < t_j$. In addition, from Definitions 3.5.5 and 3.5.6, $k_{a_i} = 0, k_{c_i} = 1$, and $0 + k_{c_i} + 2 = 3$. Thus, the size of P^f_{ji} is 3. We note that $i = 1$, since $t^f_{j-1} = t^f_{j-10}$ and $t^f_j = t^f_{j2} = t_{j-1k_{c_i}+1}$.

Employing Remark 3.5.1(ii) in the context of $[t^f_{j-1}, t^c_j)$ and $[t^c_j, t^f_j)$, respectively, and algebraic simplifications, we have

$$z(t^c_j) - z(t^f_{j-1}) = -\lambda(t^f_{j-1}, S(t^f_{j-1}))z(t^f_{j-1})\Delta t^{cf}_{j-1} - \gamma^c_j$$

and

$$z(t^f_j) - z(t^c_{j-1}) = -\lambda(t^c_{j-1}, S(t^c_{j-1}))z(t^c_{j-1})\Delta t^{fc}_{j-1} = -\lambda(t^f_{j-1}, S(t^f_{j-1}))z(t^f_{j-1})\Delta t^{fc}_{j-1}\,.$$

Adding and simplifying, we obtain

$$z(t^f_j) - z(t^f_{j-1}) = -\lambda(t^f_{j-1}, S(t^f_{j-1}))\left[z(t^f_{j-1})\Delta t^{cf}_{j-1} + z(t^c_{j-1})\Delta t^{fc}_{j-1}\right] - \gamma^c_j\,,$$

and hence

$$\begin{cases} z(t^f_j) = z(t^f_{j-1}) - \lambda(t^f_{j-1}, S(t^f_{j-1}))\left[z(t^f_{j-1})\Delta t^{cf}_{j-1} + z(t^c_{j-1})\Delta t^{fc}_{j-1}\right] - \gamma^c_j,\ z(t_0) = z_0\,, \\ S(t^f_{j-1}) = S(t_{j-2}) - \lambda(t^f_{j-2}, S(t^f_{j-2}))S(t^f_{j-2})\Delta t^f_{j-1},\ S(t_0) = S_0. \end{cases} \tag{3.5.21}$$

This establishes (b)(i).

From (3.5.21) and the backward substitution, we conclude that $z(t^f_{j-1}) - z(t^f_j) - \gamma^c_j$ is the number of failure/non-operating objects and $z(t^f_{j-1})\Delta t^{cf}_{j-1} + z(t^c_j)\Delta t^{fc}_{j1}$ denotes the total amount of time spent by $z(t^f_{j-1}) - z(t^f_j) - \gamma^c_j$ over the the interval $[t_{j-1}, t_j)$. Hence, solving for $\lambda(t^f_{j-1}, S(t^f_{j-1}))$ establishes (b)(ii).

(c) The proof of (c) can be constructed by slightly modifying the argument for the proof of (b). This establishes proof of the theorem. □

In the following, we extend Theorem 3.5.1, for multiple censored and admitting times between two consecutive failure times.

Theorem 3.5.2 *Let the hypotheses of Theorem 1.3.1 in the context of Remarks 3.5.1(i), 3.5.1(ii), and*

3.5.1(iii) and Definitions 3.5.5 and 3.5.6 be satisfied. For each $j \in I(1,k)$, and each $i \in I(1,k_f)$, let t^f_{j-1i-1} and t^f_{j-1i} be consecutive failure times. Let $\{t^c_{j-1i-1p-1}\}_{p=1}^{k_{c_i}+1}$, $\{t^a_{j-1i-1q-1}\}_{q=1}^{k_{a_i}+1}$ be a finite subsequences of censored and admitted time observations, respectively, over a consecutive failure-time subinterval $[t^f_{j-1i-1}, t^f_{j-1i})$, where k_{c_i} is the total number of censored objects/species/infective/quitting covered over the subinterval $[t^f_{j-1i-1}, t^f_{j-1i})$; k_{a_i} is the the total number of admitting/entering/joining/susceptible/etc covered over the subinterval $[t^f_{j-1i-1}, t^f_{j-1i})$. Then the theoretical transformed/computational estimation algorithm and parameter estimation for $\lambda(t,S(t))$ at t^f_{j-1i} are respectively determined by :

(i)

$$\begin{cases} z(t^f_{j-1i}) = z(t^f_{j-1i-1}) - \lambda(t^f_{j-1i-1}, S(t^f_{j-1i-1}))\left[\sum_{l=1}^{k_{b_i}+1}, z(t^{c/a}_{j-1i-1l-1})\Delta(t^{c/a}_{j-1i-1l})\right] - k_{c_i} + k_{a_i}\,,\ z(t_0) = z_0\,, \\ S(t^f_{j-1i-1}) = S(t_{j-2}) - \lambda(t^f_{j-2i-2}, S(t^f_{j-2i-2}))S(t^f_{j-2i-2})\Delta t^f_{j-1i-2},\ S(t_0) = S_0. \end{cases} \tag{3.5.22}$$

for $i \in I(1,k_f), j \in I(1,k)$ and

(ii)

$$\hat{\lambda}(t^f_{j-1i-1}, \hat{S}(t^f_{j-1i-1})) = \frac{z(t^f_{j-1i-1}) - z(t^f_{j-1i}) - k_{c_i} + k_{a_i}}{\sum_{l=1}^{k_{b_i}+1} z(t^{c/a}_{j-1i-1l-1})\Delta(t^{c/a}_{j-1i-1l})}\,,\ t \in [t^f_{j-1i-1}, t^f_{j-1i})\,, \tag{3.5.23}$$

where $k_{b_i} = k_{c_i} + k_{a_i}$.

Moreover an overall conceptual parameter estimate for $z(t), S(t)$ and $\lambda(t,S(t))$ on the time-interval of study $[t_0, \mathcal{T})$ are determined by

$$\begin{cases} \hat{\lambda}(t, \hat{S}(t^f_{j-1i-1})) = \hat{\lambda}(t^f_{j-1i-1}, \hat{S}(t^f_{j-1i-1})) \quad \text{for } t \in [t^f_{j-1i-1}, t^f_{j-1i}),\ j \in I(1,k) \text{ and } i \in I(1,k_f)\,, \\ \hat{S}(t) = \hat{S}(t, t^f_{j-1i-1}, \hat{S}(t^f_{j-1i-1})),\ \ \hat{S}(t^f_{j-1i-1}) = S_{j-1i-1}, \\ \hat{z}(t) = \hat{z}(t, t^f_{j-1i-1}, \hat{z}(t^f_{j-1i-1})). \end{cases} \tag{3.5.24}$$

Proof. From Definitions 3.5.5 and 3.5.6, $l = p = j = i = 1, t^f_{000} = t_0$ and $t^f_{0i-1k_{b_i}+1} = t^f_{01}$ and the application of Theorem 3.5.1, we note that one of the fundamental assumptions of the Principle of Mathematical Induction(PMI) [33] is satisfied. For the validity of the application of PMI, we assume that (3.5.22) *is valid for $j-1 \in I(1,k)$, and then need to show that* (3.5.22) *is satisfied for $j \in I(1,k)$. For this purpose, we note that for $j \in I(1,k)$, each $i \in I(1,k_f)$, and $t^f_{j-1i-1}, t^f_{j-1i} \in [t_0, \mathcal{T}]$, k_{c_i} and k_{a_i} objects/species/subjects are censored and admitted over the subinterval $[t^f_{j-1i-1}, t^f_j]$ of consecutive failure times, respectively. Let $\mathcal{P}^f_{ji}$ be a partition corresponding to the union of the range of two finite subsequences of censored and admitted times over the consecutive failure-time subinterval $[t^f_{j-1i-1}, t^f_{ji})$, and let it be represented by*

$$\mathcal{P}^f_{j-1i} : t^f_{j-1i-11-1} = t^f_{j-1i-10} = t^f_{j-1i-1} < t^{c/a}_{j-1i-11} < \ldots < t^{c/a}_{j-1i-1l-1} < t^{c/a}_{j-1i-1l} < \ldots \\ < t^{c/a}_{j-1i-1k_{b_i}} < t^{c/a}_{j-1i-1k_{b_i}+1} = t^f_{j-1i}\,. \tag{3.5.25}$$

In short, $\mathcal{P}^f_{ji}$ *is a partition of* $[t^f_{j-1i-1}, t^f_{j-1i}]$ *with the size of the partition* $k_{b_i}+2$, *and* $k_{b_i}=k_{c_i}+k_{a_i}$.

For $j\in I(1,k)$ *and* $i\in I(1,k_f)$, *using the iterative schemes (1.3.20), (3.5.6) and* (3.5.8) *and noting the nature of the process* $\lambda(t^{c/a}_{j-1i-1l-1}, S(t^{c/a}_{j-1i-1l-1}))=\lambda(t^f_{j-1i-i}, S(t^f_{j-1i-1}))$ *in the context of Definitions 3.5.5 and 3.5.6 for* $l\in I(1,k_{b_i})$, *we have*

$$
\begin{aligned}
z(t^f_{j-1i})-z(t^f_{j-1i-1}) =& -\lambda(t^f_{j-1i-1}, S(t^f_{j-1i-1}))z(t^{f\,c/a}_{j-1i-1})\Delta t^{f\,c/a}_{j-1i-1}+\gamma^{c/a}_{j-1i-1}\\
&-\sum_{m=2}^{k_{b_i}}\left[\lambda(t^{c/a}_{j-1i-1m-1}, S(t^{c/a}_{j-1i-1m-1}))z(t^{c/a}_{j-1i-1m-1})\Delta t^{c/a}_{j-1i-1m}+\gamma^{c/a}_{j-1i-1m-1}\right]\\
&+\lambda(t^{c/a}_{j-1i-1k_{b_i}}, S(t^{c/a}_{j-1i-1k_{b_i}}))z(t^{c/a}_{j-1i-1k_{b_i}})\Delta t^f_{jik_{b_i}+1}\\
&=-\lambda(t^f_{j-1i-1}, S(t^f_{j-1i-1}))\left[\sum_{l=1}^{k_{b_i}+1} z(t^{c/a}_{j-1i-1l-1})\Delta t^{c/a}_{j-1i-1l}\right]-k_{b_i}\ .
\end{aligned}
$$

Hence,

$$
\begin{cases}
z(t^f_{j-1i})=z(t^f_{j-1i-1})-\lambda(t^f_{j-1i-1}, S(t^f_{j-1i-1}))\left[\sum\limits_{l=1}^{k_{b_i}+1} z(t^{c/a}_{j-1i-1l-1})\Delta t^{c/a}_{j-1i-1l}\right]-k_{c_j}+k_{a_j},\ z(t_0)=z_0\\
S(t^f_{j-1i-1})=S(t^f_{j-2i-2})-\lambda(t^f_{j-2i-2}, S(t^f_{j-2i-2i-2}))S(t^f_{j-2i-2})\Delta t^f_{j-1i-1}\ , S(t_0)=S_0.
\end{cases}
\tag{3.5.26}
$$

This establishes (i).

From (3.5.26), *we note that* $z(t^f_{j-1i-1})-z(t^f_{ji})-k_{c_i}+k_{a_i}$ *is a change in the number of items/subjects that are under observation over the subinterval* $[t^f_{j-1i-1}, t^f_{ji}]$, *and* $\sum\limits_{l=1}^{k_{b_i}+1} z(t^{c/a}_{j-1i-1l-1})\Delta(t^{c/a}_{j-1i-1l})$ *is a total amount of time spent under the observation/testing/evaluation/monitoring of* $z(t^{c/a}_{ji-1l})$ *items/patients/infectives/-subjects on the interval* $[t^{c/a}_{j-1i-1l-1}, t^{c/a}_{j-1i-1l})$ *for* $l\in I(1,k_{b_j})$, $j\in I(1,n)$ *and* $i\in I(1,k_f)$. *From this and Definition 3.5.2, and the backward substitution, we obtain*

$$
\hat\lambda(t^f_{j-1i-1}, \hat S(t^f_{j-1i-1}))=\frac{z(t^f_{j-1i-1})-z(t^f_{j-1i})-k_{c_j}+k_{a_j}}{\sum\limits_{l=1}^{k_{b_j}+1} z(t^{c/a}_{j-1i-1l-1})\Delta(t^{c/a}_{j-1i-1l})}\ , t\in[t^f_{j-1i-1}, t^f_{ji})\, for\ i\in I(1,k_f) and, j\in I(1,k).
$$

This establishes (3.5.23). Moreover,

$$
\begin{cases}
\hat\lambda(t,\hat S(t^f_{j-1i-1}))=\hat\lambda(t^f_{j-1i-1},\hat S(t^f_{j-1i-1})),\ for\ t\in[t^f_{j-1i-1}, t^f_{j-1i}),\ j\in I(1,k)\ and\ i\in I(1,k_f)\\
\hat S(t)=\hat S(t, t^f_{j-1i-1}, \hat S(t^f_{j-1i-1})),\ \ \hat S(t^f_{j-1i-1})=S_{j-1i-1},\\
\hat z(t)=\hat z(t, t^f_{j-1i-1}, \hat z(t^f_{j-1i-1})).
\end{cases}
$$

This completes the proof of the theorem.

□

In the following, we present a special case, when $\lambda(t,S)$ takes a specific form.

EXAMPLE 3.5.1 For $\lambda(t,S)=\lambda(t)(1-S)$, (3.5.23) reduces to

$$\hat{\lambda}(t^f_{j-1i-1})=\frac{z(t^f_{j-1i-1})-z(t^f_{j-1i})-k_{c_i}+k_{a_i}}{(1-S(t^f_{j-1i-1}))\left[\sum_{l=1}^{k_{b_i}+1} z(t^{c/a}_{j-1i-1l-1})\Delta(t^{c/a}_{j-1i-1l})\right]}\,,\ t\in[t^f_{j-1i-1},t^f_{ji})\,, \tag{3.5.27}$$

for $i\in I(1,k_f)$ and $j\in I(1,k)$.

EXAMPLE 3.5.2 Let $\lambda(t)=\frac{1}{\sigma t}$, where σ is a parameter to be estimated from empirical data. Then applying Theorem 3.5.2, we obtain

$$\frac{1}{\hat{\sigma}(t^f_{j-1i-1})}=\frac{z(t^f_{j-1i-1})-z(t^f_{j-1i})-k_{c_i}+k_{a_i}}{(1-S(t^f_{j-1i-1}))\left[\sum_{l=1}^{k_{b_i}+1} z(t^{c/a}_{j-1i-1l-1})\frac{\Delta(t^{c/a}_{j-1i-1l})}{t^{c/a}_{j-1i-1l-1}}\right]}\,,\ t\in[t^f_{j-1i-1},t^f_{ji})\,, \tag{3.5.28}$$

for $i\in I(1,k_f)$ and $j\in I(1,k)$.

In the following, we present a few results that are very special cases of Theorem 3.5.2.

COROLLARY 3.5.1 *Let the hypotheses of Theorem 3.5.2 be satisfied except $k_a=0$. Then the theoretical/-conceptual estimation algorithm and parameter estimation for $\lambda(t,S(t))$ at t^f_{ji} are respectively determined by:*

(i)

$$\begin{cases} z(t^f_{ji})=z(t^f_{j-1i-1})-\lambda(t^f_{j-1i-1},S(t^f_{j-1i-1}))\left[\sum_{p=1}^{k_{c_i}+1},z(t^c_{j-1i-1p-1})\Delta(t^c_{j-1i-1p})\right]-k_{c_i},\ z(t_0)=z_0\,,\\ S(t^f_{j-1i-1})=S(t^f_{j-2i-2})-\lambda(t^f_{j-2i-2},S(t^f_{j-2i-2}))S(t^f_{j-2i-2})\Delta t^f_{j-1i-1},\ S(t_0)=S_0\,. \end{cases} \tag{3.5.29}$$

(ii)

$$\hat{\lambda}(t^f_{j-1i-1},\hat{S}(t^f_{j-1i-1}))=\frac{z(t^f_{j-1i-1})-z(t^f_{j-1i})-k_{c_i}}{\sum_{p=1}^{k_{c_i}+1} z(t^c_{j-1i-1p-1})\Delta(t^c_{j-1i-1p})}\,,\ t\in[t^f_{j-1i-1},t^f_{j-1i}) \tag{3.5.30}$$

for $i\in I(1,k_f)$ and $j\in I(1,k)$. Moreover an overall conceptual computational estimate for $z(t),S(t)$ and $\lambda(t,S(t))$ on the time-interval of study $[t_0,\mathcal{T})$ is

$$\begin{cases} \hat{\lambda}(t,\hat{S}(t^f_{j-1i-1}))=\hat{\lambda}(t^f_{j-1i-1},\hat{S}(t^f_{j-1i-1})),\ \textit{for}\ t\in[t^f_{j-1i-1},t^f_{j-1i}),\ j\in I(1,k)\ \textit{and}\ i\in I(1,k_f)\\ \hat{S}(t)=\hat{S}(t,t^f_{j-1i-1},\hat{S}(t^f_{j-1i-1})),\ \ \hat{S}(t^f_{j-1i-1})=S_{j-1i-1},\\ \hat{z}(t)=\hat{z}(t,t^f_{j-1i-1},\hat{z}(t^f_{j-1i-1})). \end{cases}$$

Corollary 3.5.2 *Let the hypotheses of Theorem 3.5.2 be satisfied except* $k_c = 0$. *Then the theoretical/-conceptual estimation algorithm and parameter estimation for* $\lambda(t, S(t))$ *at* t^f_{j-1i} *are respectively determined by:*

(i)

$$\begin{cases} z(t^f_{ji}) = z(t^f_{j-1i-1}) - \lambda(t^f_{j-1i-1}, S(t^f_{j-1i-1})) \left[\sum_{p=1}^{k_{a_i}+1}, z(t^a_{j-1i-1q-1})\Delta(t^a_{j-1i-1q}) \right] + k_{a_i},\ z(t_0) = z_0 \\ S(t^f_{j-1i-1}) = S(t^f_{j-2i-2}) - \lambda(t^f_{j-2i-2}, S(t^f_{j-2i-2}))S(t^f_{j-2i-2})\Delta t^f_{j-1i-1},\ S(t_0) = S_0. \end{cases} \tag{3.5.31}$$

and

(ii)

$$\hat{\lambda}(t^f_{j-1i-1}, \hat{S}(t^f_{j-1i-1})) = \frac{z(t^f_{j-1i-1}) - z(t^f_{j-1i}) + k_{a_i}}{\sum_{q=1}^{k_{a_i}+1} z(t^a_{j-1i-1q-1})\Delta(t^a_{j-1i-1q})},\ t \in [t^f_{j-1i-1}, t^f_{ji}) \tag{3.5.32}$$

for $i \in I(1, k_f)$ *and* $j \in I(1, k)$. *Moreover an overall conceptual computational estimate for* $z(t), S(t)$ *and* $\lambda(t, S(t))$ *on the time-interval of study* $[t_0, \mathfrak{T}]$ *is*

$$\begin{cases} \hat{\lambda}(t, \hat{S}(t^f_{j-1i-1})) = \hat{\lambda}(t^f_{j-1i-1}, \hat{S}(t^f_{j-1i-1})),\ \textit{for } t \in [t^f_{j-1i-1}, t^f_{j-1i}),\ j \in I(1,k) \textit{ and } i \in I(1, k_f), \\ \hat{S}(t) = \hat{S}(t, t^f_{j-1i-1}, \hat{S}(t^f_{j-1i-1})),\ \ \hat{S}(t^f_{j-1i-1}) = S_{j-1i-1}, \\ \hat{z}(t) = \hat{z}(t, t^f_{j-1i-1}, \hat{z}(t^f_{j-1i-1})). \end{cases}$$

The following special case of Theorem 3.5.2 is with respect to the totally discrete-time hybrid dynamic model for time-to-event dynamic process.

Corollary 3.5.3 *Let us assume that the conditions of Corollary* (3.4.1) *in the context of Definitions 3.5.5 and 3.5.6 and Remarks 3.5.1(iv) (I),(II), and (III) are satisfied. For each* $j \in I(1, k)$, *and each* $i \in I(1, k_f)$, *let* t^f_{j-1i-1} *and* t^f_{j-1i} *be consecutive failure times. Let* $\{t^c_{j-1i-1p}\}_{p=1}^{k_{c_j}}$, $\{t^a_{j-1i-1q}\}_{q=1}^{k_{a_i}}$ *be a finite subsequences of censored and admitted time observations, respectively, over a consecutive failure-time subinterval* $[t^f_{j-1i-1}, t^f_{j-1i})$, *where* k_{c_i} *is the total number of censored objects/species/infective/quitting covered over the subinterval* $[t^f_{j-1i-1}, t^f_{j-1i})$; k_{a_i} *is the the total number of admitting/entering/joining/susceptible/etc covered over the subinterval* $[t^f_{j-1i-1}, t^f_{j-1i})$. *Then the theoretical/conceptual estimation algorithm and parameter estimation for* $\lambda(t, S(t))$ *at* t^f_{j-1i} *are determined by :*

(i)

$$\begin{cases} z(t^f_{j-1i}) = z(t^f_{j-1i-1}) - \lambda(t^f_{j-1i-1}, S(t^f_{j-1i-1})) \left[\sum_{l=1}^{k_{b_i}+1} z(t^{c/a}_{j-1i-1l-1}) \right] - k_{c_i} + k_{a_i}\, , z(t_0) = z_0 \\ S(t^f_{j-1i-1}) = S(t^f_{j-2i-2}) - \lambda(t^f_{j-2i-2}, S(t^f_{j-2i-2}))S(t^f_{j-2i-2})\, , S(t_0) = S_0. \end{cases} \tag{3.5.33}$$

and

(ii)

$$\hat{\lambda}(t^f_{j-1i-1}, \hat{S}(t^f_{j-1i-1})) = \frac{z(t^f_{j-1i-1}) - z(t^f_{j-1i}) - k_{c_i} + k_{a_i}}{\sum_{l=1}^{k_{b_i}+1} z(t^{c/a}_{j-1i-1l-1})}, \ t \in [t^f_{j-1i-1}, t^f_{j-1i}) \tag{3.5.34}$$

respectively for $i \in I(1, k_f)$ *and* $j \in I(1, k)$.

Moreover an overall conceptual computational estimate $z(t), S(t)$ *and for* $\lambda(t, S(t))$ *on the time-interval of study* $[t_0, \mathcal{T}]$ *is*

$$\begin{cases} \hat{\lambda}(t, \hat{S}(t^f_{j-1i-1})) = \hat{\lambda}(t^f_{j-1i-1}, \hat{S}(t^f_{j-1i-1})), \ \textit{for}\ t \in [t^f_{j-1i-1}, t^f_{j-1i}),\ j \in I(1,k)\ \textit{and}\ i \in I(1, k_f), \\ \hat{S}(t) = \hat{S}(t, t^f_{j-1i-1}, \hat{S}(t^f_{j-1i-1})),\ \ \hat{S}(t^f_{j-1i-1}) = S_{j-1i-1}, \\ \hat{z}(t) = \hat{z}(t, t^f_{j-1i-1}, \hat{z}(t^f_{j-1i-1})). \end{cases} \tag{3.5.35}$$

Now, we state a very general theorem that provides a theoretical estimate for $\lambda(t, S)$ between two consecutive change point times, t^{cp}_{j-1r-1} and t^{cp}_{j-1r}.

Theorem 3.5.3 *Let the hypotheses of Theorem 1.3.1 in the context of Definitions 3.5.5 and 3.5.6 and Remarks 3.4.1, 3.5.1(i), 3.5.1(ii), and 3.5.1(iii) be satisfied. For each* $j \in I(1, k)$ *and each* $r \in I(1, n)$, *let* t^{cp}_{j-1r-1} *and* t^{cp}_{j-1r} *be consecutive change point times. Let* $\{t^f_{j-1r-1i-1}\}^{k_{f_r}}_{i=1}$, $\{t^c_{j-1r-1p-1}\}^{k_{c_r}}_{p=1}$, *and* $\{t^a_{j-1r-1q-1}\}^{k_{a_r}}_{q=1}$ *be the a sequence of failure, censored and admission times respectively in the interval* $[t^{cp}_{j-1r-1}, t^{cp}_{j-1r})$. k_{f_r}, k_{c_r}, *and* k_{a_r} *are respectively, the total number of failures, censored and admitting items/objects/species/etc in the consecutive change-point subinterval* $[t^{cp}_{j-1r-1}, t^{cp}_{j-1r})$. *Then the theoretical/conceptual estimation algorithm and parameter estimation for* $\lambda(t, S(t))$ *at* t^{cp}_{j-1r} *are determined by:*

(i)

$$\begin{cases} z(t^{cp}_{j-1r}) = z(t^{cp}_{j-1r-1}) - \lambda(t^{cp}_{j-1r-1}, S(t^{cp}_{j-1r-1})) \left[\sum_{l=1}^{k_{b_r}+1} z(t^{f/c/a}_{j-1r-1l-1}) \Delta(t^{f/c/a}_{j-1r-1l})\right] \\ \qquad\qquad -k_{f_r} - k_{c_r} + k_{a_r},\ z(t_0) = z_0, \\ S(t^{cp}_{j-1r-1}) = S(t^{cp}_{j-2r-2}) - \lambda(t^{cp}_{j-2r-2}, S(t^{cp}_{j-2r-2})) S(t^{cp}_{j-2r-2}) \Delta t^{cp}_{j-1r-1}, S(t_0) = S_0. \end{cases} \tag{3.5.36}$$

and

$$\hat{\lambda}(t^{cp}_{j-1r-1}, \hat{S}(t^{cp}_{j-1r-1})) = \frac{z(t^{cp}_{j-1r-1}) - z(t^{cp}_{j-1r}) - k_{f_r} - k_{c_r} + k_{a_r}}{\sum_{l=1}^{k_{b_r}+1} z(t^{f/c/a}_{j-1r-1l-1}) \Delta(t^{f/c/a}_{j-1r-1l})}, \ t \in [t^{cp}_{j-1r-1}, t^{cp}_{j-1r}), \tag{3.5.37}$$

respectively for $r \in I(1, n)$ *and* $j \in I(1, k)$. $k_{b_r} = k_{f_r} + k_{c_r} + k_{a_r}$. *Moreover an overall conceptual estimate* *for* $z(t), S(t)$ *and* $\lambda(t, S(t))$ *on the time-interval of study* $[t_0, \mathcal{T})$ *is*

$$\begin{cases} \hat{\lambda}(t, \hat{S}(t^{cp}_{j-1r-1})) = \hat{\lambda}(t^{cp}_{j-1r-1}, \hat{S}(t^{cp}_{j-1r-1})), \ \textit{for } t \in [t^{cp}_{j-1r-1}, t^{cp}_{j-1r}), \ r \in I(1,n) \textit{ and } j \in I(1,k), \\ \hat{S}(t) = \hat{S}(t, t^{cp}_{j-1r-1}, \hat{S}(t^{cp}_{j-1r-1})), \ \hat{S}(t^{cp}_{j-1i-1}) = S_{j-1i-1}, \\ \hat{z}(t) = \hat{z}(t, t^{cp}_{j-1r-1}, \hat{z}(t^{cp}_{j-1r-1})). \end{cases} \tag{3.5.38}$$

Proof. Imitating the proof of Theorem 3.5.2, one can establish the proof of the Theorem 3.5.3. □

REMARK 3.5.2 Corollaries parallel to Corollaries 3.5.1 and 3.5.2 can be formulated.

The following special case of Theorem 3.5.3 is with respect to the totally discrete-time hybrid dynamic model for time-to-event dynamic process.

COROLLARY 3.5.4 *Let us assume that all conditions of Corollary* (3.4.1) *in the context of Definitions 3.5.5 and 3.5.6 and Remarks 3.5.1(iv) (I),(II), and (III) are satisfied. For each* $j \in I(1,k)$ *and each* $r \in I(1,n)$*, let* t^{cp}_{j-1r-1} *and* t^{cp}_{jr} *be consecutive change point times. Let* $\{t^f_{j-1r-1i-1}\}^{k_{f_r}}_{i=1}$*,* $\{t^c_{j-1r-1p-1}\}^{k_{c_r}}_{p=1}$*, and* $\{t^a_{j-1r-1q-1}\}^{k_{a_r}}_{q=1}$ *be the a sequence of failure, censored and admission times respectively in the interval* $[t^{cp}_{j-1r-1}, t^{cp}_{jr})$. k_{f_r}, k_{c_r}*, and* k_{a_r} *are respectively, the total number of failures, censored and admitting items/objects/species/etc in the consecutive change-point subinterval* $[t^{cp}_{j-1r-1}, t^{cp}_{jr})$*. Then the theoretical/conceptual estimation algorithm and parameter estimation for* $\lambda(t, S(t))$ *at* t^{cp}_{jr} *are determined by:*

(i)

$$\begin{cases} z(t^{cp}_{j-1r}) = z(t^{cp}_{j-1r-1}) - \lambda(t^{cp}_{j-1r-1}, S(t^{cp}_{j-1r-1})) \left[\sum_{l=1}^{k_{b_r}+1} z(t^{f/c/a}_{j-1r-1l-1}) \right] - k_{f_r} - k_{c_r} + k_{a_r}, \ z(t_0) = z_0, \\ S(t^{cp}_{j-1r-1}) = S(t^{cp}_{j-2r-2}) - \lambda(t^{cp}_{j-2r-2}, S(t^{cp}_{j-2r-2})) S(t^{cp}_{j-2r-2}), \ S(t_0) = S_0 \, , \end{cases} \tag{3.5.39}$$

and

$$\hat{\lambda}(t^{cp}_{j-1r-1}, \hat{S}(t^{cp}_{j-1r-1})) = \frac{z(t^{cp}_{j-1r-1}) - z(t^{cp}_{j-1r}) - k_{f_r} - k_{c_r} + k_{a_r}}{\sum_{l=1}^{k_{b_r}+1} z(t^{f/c/a}_{j-1r-1l-1})} \, , \ t \in [t^{cp}_{j-1r-1}, t^{cp}_{j-1r}) \, , \tag{3.5.40}$$

respectively. Moreover an overall conceptual estimate for $z(t), S(t)$ *and* $\lambda(t, S(t))$ *on the time-interval of study* $[t_0, \mathcal{T})$ *is*

$$\begin{cases} \hat{\lambda}(t, \hat{S}(t^{cp}_{j-1r-1})) = \hat{\lambda}(t^{cp}_{j-1r-1}, \hat{S}(t^{cp}_{j-1r-1})), \ \textit{for } t \in [t^{cp}_{j-1r-1}, t^{cp}_{j-1r}), \ r \in I(1,n) \textit{ and } j \in I(1,k), \\ \hat{S}(t) = \hat{S}(t, t^{cp}_{j-1r-1}, \hat{S}(t^{cp}_{j-1r-1})), \ \hat{S}(t^{cp}_{j-1i-1}) = S_{j-1i-1}, \\ \hat{z}(t) = \hat{z}(t, t^{cp}_{j-1r-1}, \hat{z}(t^{cp}_{j-1r-1})). \end{cases} \tag{3.5.41}$$

Chapter 4
Conceptual Computational and Simulation Algorithms

4.1 Introduction

In this chapter, we outline a conceptual computational dynamic algorithm that includes both (a) survival state and (b) change point survival state and parameter estimation problems in a systematic and unified way. For the undertaking of this task, we need to conceptually coordinate the data collection, numerical scheme and simulation times with theoretical discrete-time dynamic algorithm. In addition, it is essential to decompose, to reorganize, and re-aggregate a given overall data set in a suitable manner to meet the overall goal(s). Prior to the development of the scheme, we define, introduce notations and reorganize the observed data set for the usage of a conceptual computational dynamic algorithm in Sections 4.2 and 4.3. We outline conceptual computational dynamical algorithms for survival state and change-point survival state and parameter estimation problems in Sections 4.4 and 4.5. The developed computational algorithms are then applied to three data sets in Section 4.6. In Section 4.7, the recently developed LLGMM method [44, 45] is extended and applied to three data sets and results are compared. In fact, LLGMM method provides the measure of confidence, prediction and planning assessments.

4.2 Data Collection Coordination with Iterative Processes

Without loss of generality, we assume that the real data observation/collection schedule is indeed a finite sequence $\{t_{j-1}\}_{j=1}^{k}$ corresponding to the partition P of $[t_0, \mathfrak{T})$ defined in Section 3.3. Moreover, the real world data set and its data observation/collection times are coordinated with conceptual data set sequence and data collection sequence of times.

4.3 Data Decomposition, Reorganization and Aggregation

Based on our research, we recognize that there are two major problems of interests in a time-to-event dynamic process, namely: (1) Survival state and (2) change point state estimation analysis problems. For the study of these problems, we decompose, reorganize and re-aggregate the original real world data set in a respective framework of (1) Survival state and (2) change point study in a time-to-event process. The original data is coordinated, decomposed, reorganized, and aggregated with reference to the conceptual data coordination, decomposition, reorganization and aggregation in the manner analogous to Definitions 3.5.3–3.5.6.

4.4 Conceptual Computational Parameter and State Estimations Scheme

For the conceptual computational parameter estimation, we use nonlinear discrete-time conceptual computational interconnected dynamic algorithm (3.4.4) for time-to-event data statistic (Flowchart- 1b). The original state data subsequences are associated with conceptual data set. The decomposition of the original real world data set into three types of subsequences of data is as defined in the context of Definition 3.5.3. We consider the original data set as the real data set. For $i \in (1, k_f)$, conceptual computational dynamic estimation algorithms in (3.5.22) and (3.5.33) are used for continuous and totally discrete-time real world data sets, respectively. The parameter and state estimates at t^f_{j-1i} are determined using (3.5.23) and (3.5.34) for continuous and totally discrete-time real world data sets, respectively. Finally, employing the Principle of Mathematical Induction [33], an overall parameter and state estimations for $z(t), S(t)$ and $\lambda(t, S(t))$ over the time interval $[t_0, \mathfrak{T})$ of study are determined from (3.5.24) and (3.5.35).

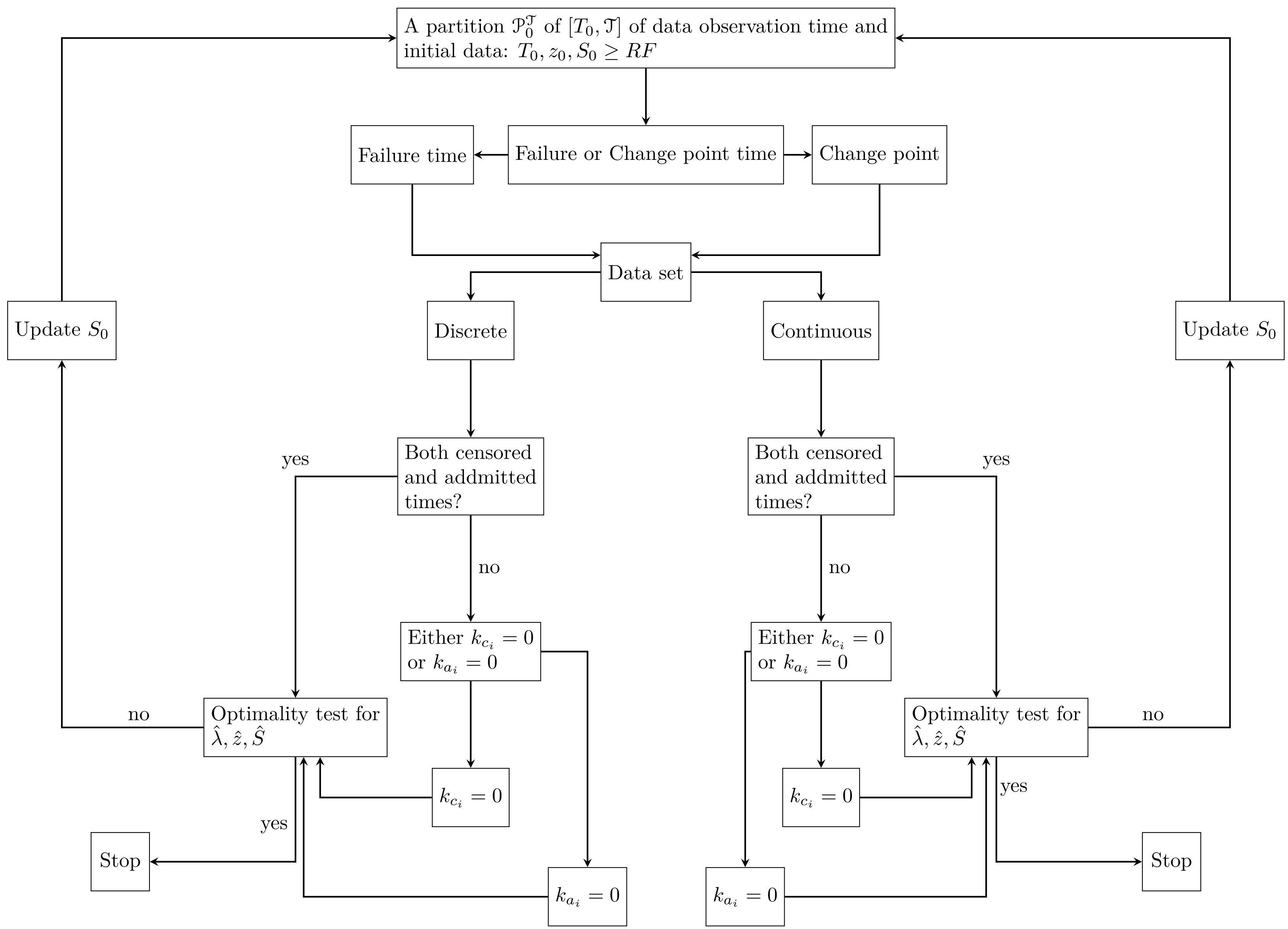
4.5 Conceptual Computational State Simulation Scheme

We utilize the common sense ideas, namely, range of finite sequence of data collection time, the initial relative frequency of the survival and the range of relative frequency. In addition, we employ the fundamental properties of solution process of initial value problems in the theory of differential equations [33], in particular, the continuous dependence of solutions with respect to initial data and other properties. We identify the initial data (t_0, S_0, z_0) for various choices of S_0. The best estimates are obtained when near optimal convergence is achieved for a particular choice of initial survival state, S_0. In summary, the Conceptual Computational Algorithm consists of three-step nested processes.

4.5.1 Change Point Data Analysis Problem

In this subsection, we address the usage of the study of time-to-event process. A Change-point process in the time-to-event process measures the effects of intervention process. Here, again the overall pair of sequence of discrete-time interconnected state dynamic data set is characterized by single right-end point data set with two consecutive change point dynamic process. A sequence of two consecutive change point times is assumed to be a single subsequence of overall sequence $\{t_{j-1}\}_{j=1}^{k}$ of conceptual state data observation times. The sequence of two consecutive change point times is denoted by $\{t^{cp}_{j-1r-1}\}_{r=1}^{n}$ for $r \in I(1, n)$ with $n \leq k$. Generally, using the time-to-event state dynamic data set, the change point sequence of times is estimated. A change point process in the time-to-event process measures the effects of intervention process. The rest of the data collection coordination with conceptual iterative process is parallel to the survival state problem, except notations. Except for notational changes (for example, replacing $[t^f_{j-1i-1}, t^f_{j-1i})$ by $[t^{cp}_{j-1r-1}, t^{cp}_{j-1r}))$, entire conceptual computational procedure regarding the survival state data analysis problem is imitated for the change-point problem analogously. For $i \in I(1, n)$ the conceptual computational dynamic algorithms in (3.5.36) and (3.5.39) are used for continuous and totally discrete-time real world data sets, respectively.

The parameter and state estimates at t^{cp}_{j-1r} are determined using (3.5.37) and (3.5.40) for continuous and totally discrete-time real world data sets, respectively. Finally, employing the Principle of Mathematical Induction, an overall parameter and state estimation for $z(t), S(t)$ and $\lambda(t, S(t))$ over the time interval $[t_0, \mathcal{T})$ of study are determined from (3.5.38) and (3.5.41). In summary, the Conceptual Computational Algorithm is outlined in Flowchart 6.



Flowchart 6.: Conceptual Computational Algorithm

We present an algorithm and a flowchart for the simulation schemes described above.

Given t_0, S_0 and z_0

for $j = 1$ to k **do**

if Failure time **then**

for $i = 1$ to k_f **do**

Compute k_{c_i}, k_{a_i}, $z(t^f_{j-1i-1}), z(t^f_{ji})$

if Continuous **then**

Compute $\sum_{l=1}^{k_{b_i}+1} z(t^{c/a}_{j-1i-1l-1})\Delta(t^{c/a}_{j-1i-1l})$

else

Compute $\sum_{l=1}^{k_{b_i}+1} z(t^{c/a}_{j-1i-1l-1})$

end if

Compute $\hat{\lambda}, \hat{z}$ and $\hat{S}$

end for

else

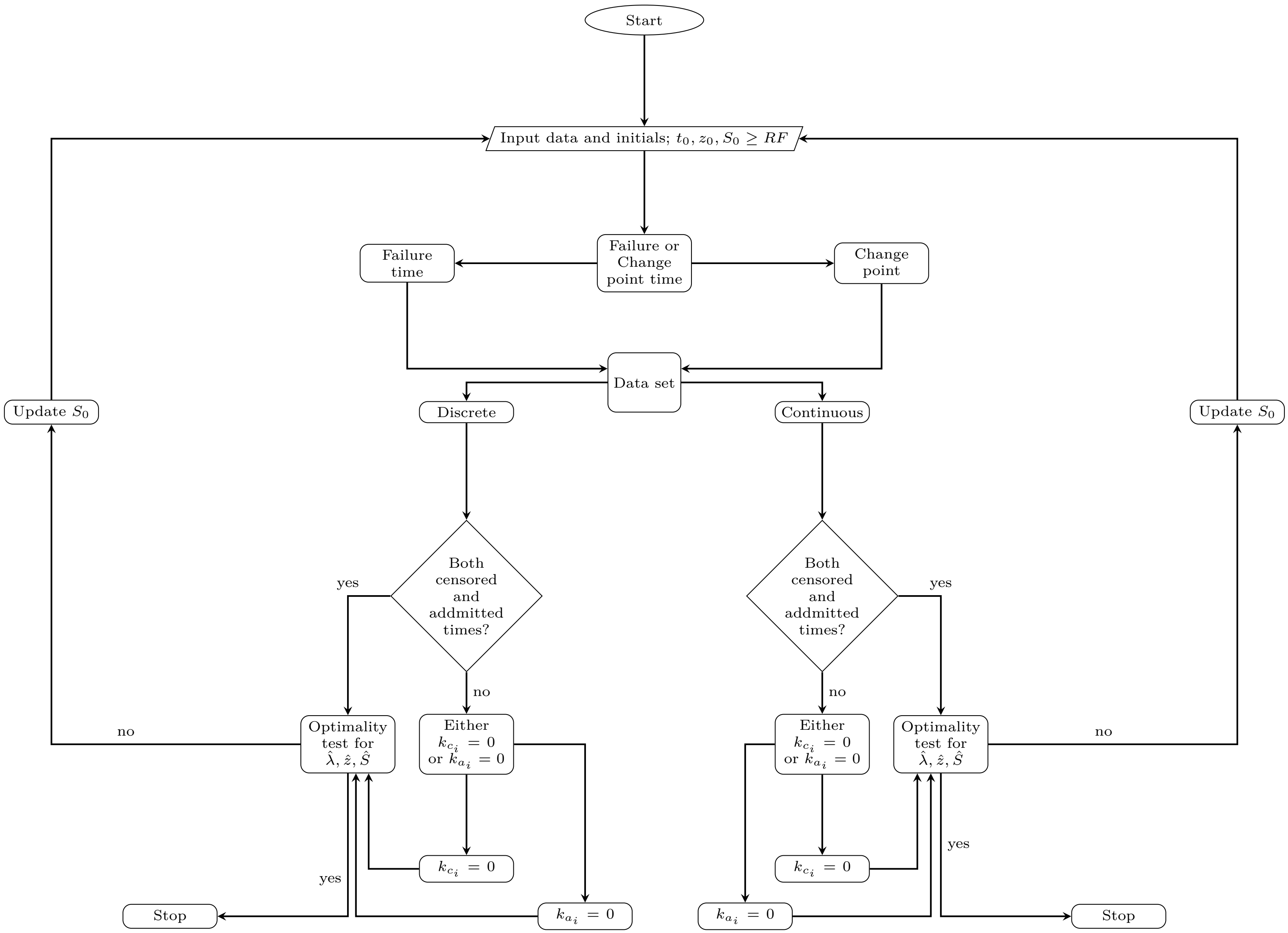
Change point analysis

⋮

end if

end for

Algorithm 7.: Simulation Scheme



Flowchart 8.: Simulation Algorithm for Survival and Change point Data Analysis Problems

4.6 Applications to Time-to-event Datasets

In this section, using the conceptual computational algorithm, we exemplify our theoretical algorithms and procedures for estimating parameters and survival state for three datasets. 96 locomotive control failure data set in number of thousand miles [37, 47], a follow-up time and vital status of 100 subjects in the Worcester heart attack study [23] and data set describing time (in months) of death and losses of a sample of 8 items found in [26] that was analyzed by[38].

Illustration 4.6.1 The data in Table 4 was discussed in [37, 47] regarding the number of thousand miles at which different locomotive controls failed in a life test involving ninety-six controls. The test was terminated after $135,000$ miles, by which time thirty-seven failures had occurred. Fifty-nine locomotive controls were censored at $135,000$ miles.
We apply the developed conceptual computational algorithm. Employing (3.5.28) with $k_a = 0$, we demonstrate our innovative alternative approach for finding parameter and survival function estimates on consecutive failure time intervals.

Table 4: Locomotive control Life-test Dataset [37, 47]

Data Observation per 1000 miles	Failure/ Censor Time	Frequency of Failure or Censors at t_i	Survival or Operating units at t_i: $z(t_i)$	Data Observation per 1000 miles	Failure or Censor Time	Frequency of Failure or Censors at t_i	Survival or Operating units at t_i: $z(t_i)$
$t_0 = 1.0$	Initial		96	$t_{20} = 91.5$	Failure	1	76
$t_1 = 22.5$	Failure	1	95	$t_{21} = 93.5$	Failure	1	75
$t_2 = 37.5$	Failure	1	94	$t_{22} = 102.5$	Failure	1	74
$t_3 = 46.5$	Failure	1	93	$t_{23} = 107.0$	Failure	1	73
$t_4 = 48.5$	Failure	1	92	$t_{24} = 108.5$	Failure	1	72
$t_5 = 51.5$	Failure	1	91	$t_{25} = 112.5$	Failure	1	71
$t_6 = 53.5$	Failure	1	90	$t_{26} = 113.5$	Failure	1	70
$t_7 = 54.5$	Failure	1	89	$t_{27} = 116.0$	Failure	1	69
$t_8 = 57.5$	Failure	1	88	$t_{28} = 117.0$	Failure	1	68
$t_9 = 66.5$	Failure	1	87	$t_{29} = 118.5$	Failure	1	67
$t_{10} = 68.0$	Failure	1	86	$t_{30} = 119.0$	Failure	1	66
$t_{11} = 69.5$	Failure	1	85	$t_{31} = 120.0$	Failure	1	65
$t_{12} = 76.5$	Failure	1	84	$t_{32} = 122.5$	Failure	1	64
$t_{13} = 77.0$	Failure	1	83	$t_{33} = 123.0$	Failure	1	63
$t_{14} = 78.5$	Failure	1	82	$t_{34} = 127.5$	Failure	1	62
$t_{15} = 80.0$	Failure	1	81	$t_{35} = 131.0$	Failure	1	61
$t_{16} = 81.5$	Failure	1	80	$t_{36} = 132.5$	Failure	1	60
$t_{17} = 82.5$	Failure	1	79	$t_{37} = 134.0$	Failure	1	59
$t_{18} = 83.0$	Failure	1	78	$t_{38} = 135.0$	Censored	59	0
$t_{19} = 84.0$	Failure	1	77				

We utilize the range of finite sequence of data collection time. We note the initial relative frequency of the survival locomotive control to be $\frac{95}{96}$. In fact the range of relative frequency is $[0.6146, 0.9896]$. We chose initial survival probability to be $S_0 = 0.985, 0.989, 0.99, 0.999, 0.9999, 0.99999, 0.999999$ and applied the conceptual computational simulation algorithm for consecutive failure-time subintervals. The results are recorded in Table 5. The simulation results exhibits the almost optimal convergence of survival state probability estimates for $S_0 = 0.99999$. We then conclude that the best survival state estimate is for $S_0 = 0.99999$ for the locomotive control data set. This was further reaffirmed by the application of the modified version of LLGMM method that assures a certain degree of confidence in the survival state estimates. In addition, the modified version of LLGMM method provides a test for the best optimality of state and parameter estimates. Moreover, it provides a confidence interval for the survival state estimates. Furthermore, it also provides the measure of significance for the usage of new procedures/tools/etc.

Table 5: Estimates $\hat{\sigma}(t_{j-1i}) \equiv \hat{\sigma}_{j-1i}$ and $\hat{S}(t_{j-1i-1}) \equiv \hat{S}_{j-1i-1}$ using $S_0 = 0.985$, 0.98900, 0.99000, 0.99900, 0.99990, 0.99999, 0.999999 using (3.5.28) with $k_a = 0$ and the procedure outlined in Chapter 4

Consecutive Failure time interval, $[t_{j-1i-1}, t_{j-1i})$	$S_0 = 0.985$		$S_0 = 0.98900$		$S_0 = 0.99000$		$S_0 = 0.99900$		$S_0 = 0.9999$		$S_0 = 0.99999$		$S_0 = 0.999999$	
	$\hat{\sigma}_{j-1i}$	$\hat{S}_{j-1i-1}$	$\hat{\sigma}_{j-1i}$	$\hat{S}_{j-1i-1}$	$\hat{\sigma}_{j-1i}$	$\hat{S}_{j-1}$	$\hat{\sigma}_{j}$	$\hat{S}_{j-1i-1}$	$\hat{\sigma}_{j-1i}$	$\hat{S}_{j-1i-1}$	$\hat{\sigma}_{j-1i}$	$\hat{S}_{j-1i-1}$	$\hat{\sigma}_{j-1i}$	$\hat{S}_{j-1i-1}$
$[1, 22.5)$	30.9600	0.9850	22.7040	0.9890	20.6400	0.9900	2.0640	0.9990	0.2064	0.9999	0.0206	0.99999	0.0021	0.999999
$[22.5, 37.5)$	1.5998	0.9747	1.3491	0.9787	1.2865	0.9797	0.7224	0.9886	0.6660	0.9895	0.6603	0.9896	0.6598	0.9896
$[37.5, 46.5)$	0.8014	0.9645	0.7130	0.9684	0.6909	0.9694	0.4921	0.9782	0.4722	0.9791	0.4702	0.9792	0.4700	0.9792
$[46.5, 48.5)$	0.1831	0.9542	0.1676	0.9581	0.1637	0.9591	0.1289	0.9678	0.1254	0.9687	0.1250	0.9687	0.1250	0.9687
$[48.5, 51.5)$	0.3189	0.9440	0.2971	0.9478	0.2916	0.9488	0.2426	0.9574	0.2377	0.9582	0.2372	0.9583	0.2371	0.9583
$[51.5, 53.5)$	0.2343	0.9337	0.2209	0.9375	0.2176	0.9384	0.1874	0.9470	0.1844	0.9478	0.1841	0.9479	0.1841	0.9479
$[53.5, 54.5)$	0.1288	0.9234	0.1225	0.9272	0.1209	0.9281	0.1067	0.9366	0.1053	0.9374	0.1052	0.9375	0.1051	0.9375
$[54.5, 57.5)$	0.4254	0.9132	0.4072	0.9169	0.4026	0.9178	0.3618	0.9262	0.3577	0.9270	0.3573	0.9271	0.3572	0.9271
$[57.5, 66.5)$	1.3372	0.9029	1.2867	0.9066	1.2741	0.9075	1.1605	0.9158	1.1491	0.9166	1.1480	0.9167	1.1478	0.9167
$[66.5, 68.0)$	0.2107	0.8927	0.2035	0.8963	0.2018	0.8972	0.1858	0.9053	0.1842	0.9062	0.1840	0.9062	0.1840	0.9062
$[68.0, 69.5)$	0.2231	0.8824	0.2163	0.8860	0.2146	0.8869	0.1993	0.8949	0.1978	0.8957	0.1976	0.8958	0.1976	0.8958
$[69.5, 76.5)$	1.0947	0.8721	1.0643	0.8757	1.0568	0.8766	0.9885	0.8845	0.9817	0.8853	0.9810	0.8854	0.9810	0.8854
$[76.5, 77.0)$	0.0758	0.8619	0.0739	0.8654	0.0734	0.8663	0.0691	0.8741	0.0687	0.8749	0.0686	0.8750	0.0686	0.8750
$[77.0, 78.5)$	0.2399	0.8516	0.2343	0.8551	0.2329	0.8559	0.2204	0.8637	0.2191	0.8645	0.2190	0.8646	0.2190	0.8646
$[78.5, 80.0)$	0.2486	0.8414	0.2432	0.8448	0.2419	0.8456	0.2298	0.8533	0.2286	0.8541	0.2285	0.8542	0.2285	0.8542
$[80.0, 81.5)$	0.2565	0.8311	0.2514	0.8345	0.2501	0.8353	0.2386	0.8429	0.2374	0.8437	0.2373	0.8437	0.2373	0.8437
$[81.5, 82.5)$	0.1759	0.8208	0.1726	0.8242	0.1718	0.8250	0.1644	0.8325	0.1637	0.8332	0.1636	0.8333	0.1636	0.8333
$[82.5, 83.0)$	0.0907	0.8106	0.0891	0.8139	0.0887	0.8147	0.0852	0.8221	0.0848	0.8228	0.0848	0.8229	0.0848	0.8229
$[83.0, 84.0)$	0.1877	0.8003	0.1846	0.8036	0.1838	0.8044	0.1770	0.8117	0.1763	0.8124	0.1762	0.8125	0.1762	0.8125
$[84.0, 91.5)$	1.4434	0.7901	1.4213	0.7933	1.4158	0.7941	1.3662	0.8013	1.3612	0.8020	1.3607	0.8021	1.3607	0.8021
$[91.5, 93.5)$	0.3658	0.7798	0.3606	0.7830	0.3592	0.7838	0.3474	0.7909	0.3462	0.7916	0.3461	0.7917	0.3461	0.7917
$[93.5, 102.5)$	1.6638	0.7695	1.6413	0.7727	1.6356	0.7734	1.5849	0.7805	1.5798	0.7812	1.5793	0.7812	1.5792	0.7812
$[102.5, 107.0)$	0.7821	0.7593	0.7721	0.7624	0.7696	0.7631	0.7470	0.7701	0.7448	0.7708	0.7445	0.7708	0.7445	0.7708

continued on next page

Table 5 – continued from previous page

Consecutive Failure time interval, $[t_{j-1i-1}, t_{j-1i})$	$S_0 = 0.985$		$S_0 = 0.98900$		$S_0 = 0.99000$		$S_0 = 0.99900$		$S_0 = 0.9999$		$S_0 = 0.99999$		$S_0 = 0.999999$	
	$\hat{\sigma}_{j-1i}$	$\hat{S}_{j-1i-1}$	$\hat{\sigma}_{j-1i}$	$\hat{S}_{j-1i-1}$	$\hat{\sigma}_{j-1i}$	$\hat{S}_{j-1}$	$\hat{\sigma}_{j}$	$\hat{S}_{j-1i-1}$	$\hat{\sigma}_{j-1i}$	$\hat{S}_{j-1i-1}$	$\hat{\sigma}_{j-1i}$	$\hat{S}_{j-1i-1}$	$\hat{\sigma}_{j-1i}$	$\hat{S}_{j-1i-1}$
$[107.0, 108.5)$	0.2569	0.7490	0.2537	0.7521	0.2530	0.7528	0.2460	0.7597	0.2453	0.7603	0.2452	0.7604	0.2452	0.7604
$[108.5, 112.5)$	0.6935	0.7388	0.6855	0.7417	0.6835	0.7425	0.6656	0.7493	0.6638	0.7499	0.6636	0.7500	0.6636	0.7500
$[112.5, 113.5)$	0.1714	0.7285	0.1695	0.7314	0.1690	0.7322	0.1648	0.7388	0.1644	0.7395	0.1644	0.7396	0.1644	0.7396
$[113.5, 116.0)$	0.4344	0.7182	0.4300	0.7211	0.4288	0.7219	0.4187	0.7284	0.4177	0.7291	0.4176	0.7292	0.4176	0.7292
$[116.0, 117.0)$	0.1737	0.7080	0.1720	0.7108	0.1716	0.7116	0.1677	0.7180	0.1673	0.7187	0.1673	0.7187	0.1673	0.7187
$[117.0, 118.5)$	0.2635	0.6977	0.2611	0.7005	0.2604	0.7013	0.2549	0.7076	0.2543	0.7083	0.2543	0.7083	0.2543	0.7083
$[118.5, 119.0)$	0.0884	0.6874	0.0876	0.6902	0.0874	0.6909	0.0856	0.6972	0.0854	0.6978	0.0854	0.6979	0.0854	0.6979
$[119.0, 120.0)$	0.1790	0.6772	0.1775	0.6799	0.1771	0.6806	0.1737	0.6868	0.1734	0.6874	0.1733	0.6875	0.1733	0.6875
$[120.0, 122.5)$	0.4510	0.6669	0.4474	0.6696	0.4465	0.6703	0.4382	0.6764	0.4374	0.6770	0.4373	0.6771	0.4373	0.6771
$[122.5, 123.0)$	0.0897	0.6567	0.0890	0.6593	0.0888	0.6600	0.0872	0.6660	0.0871	0.6666	0.0871	0.6667	0.0871	0.6667
$[123.0, 127.5)$	0.8150	0.6464	0.8089	0.6490	0.8074	0.6497	0.7938	0.6556	0.7925	0.6562	0.7923	0.6562	0.7923	0.6562
$[127.5, 131.0)$	0.6193	0.6361	0.6149	0.6387	0.6138	0.6394	0.6039	0.6452	0.6029	0.6458	0.6028	0.6458	0.6028	0.6458
$[131.0, 132.5)$	0.2613	0.6259	0.2595	0.6284	0.2591	0.6291	0.2551	0.6348	0.2547	0.6354	0.2547	0.6354	0.2547	0.6354
$[132.5, 134.0)$	0.2611	0.6156	0.2594	0.6181	0.2590	0.6188	0.2551	0.6244	0.2548	0.6249	0.2547	0.6250	0.2547	0.6250
(134)		0.6054		0.6078		0.6084		0.6140		0.6145		0.6146		0.6146

In the following illustration, we apply the developed algorithm to a follow-up time and vital status of 100 patients in the Worcester heart attack study [23].

ILLUSTRATION 4.6.2 The data in Table 6 below shows follow-up time and vital status(failure or censored) for 100 subjects in the Worcester Heart Attack Study [23]. We note that there are multiple censored times occurring between any two consecutive failure times unlike the data set in Table 4, where all censored times occurred after the last failure time. We note that the initial relative frequency of the survival of patient data is 0.98. In fact the range of relative frequency is $[0.51, 0.98]$. Here also, we chose initial survival probability to be $S_0 = 0.985, 0.989, 0.99, 0.999, 0.9999, 0.99999, 0.999999$ and applied the conceptual computational simulation algorithm (3.5.28) with $k_a = 0$ for consecutive failure-time subintervals. The results are recorded in Table 7. The simulation results exhibits the optimal convergence of survival state probability estimates for $S_0 = 0.99999$. We conclude that the almost best survival state estimate is for $S_0 = 0.99999$ for the Worcester Heart Attack Study data set. This was also confirmed by the application of the modified version of LLGMM method that assures a certain degree of confidence in the survival state estimates.

Table 6: A follow-up time of 100 Worcester Heart Attack study Dataset [23].

Data Observation	Failure or Censor Time	Frequency of Failure/ Censors at t_i	Survival or Operating units at t_i: $z(t_i)$	Data Observation	Failure or Censor Time	Frequency of Failure/ Censors at t_i	Survival or Operating units at t_i: $z(t_i)$
$t_0 = 1.0$	Initial		100	$t_{48} = 1879$	Censored	1	49
$t_1 = 6$	Failure	2	98	$t_{49} = 1883$	Censored	1	48
$t_2 = 14$	Failure	1	97	$t_{50} = 1889$	Censored	1	47
$t_3 = 44$	Failure	1	96	$t_{51} = 1907$	Failure	1	46
$t_4 = 62$	Failure	1	95	$t_{52} = 1912$	Censored	1	45
$t_5 = 89$	Failure	1	94	$t_{53} = 1916$	Censored	1	44
$t_6 = 98$	Failure	1	93	$t_{54} = 1922$	Censored	1	43
$t_7 = 104$	Failure	1	92	$t_{55} = 1923$	Censored	1	42
$t_8 = 107$	Failure	1	91	$t_{56} = 1929$	Censored	1	41
$t_9 = 114$	Failure	1	90	$t_{57} = 1934$	Censored	1	40
$t_{10} = 123$	Failure	1	89	$t_{58} = 1939$	Censored	2	38
$t_{11} = 128$	Failure	1	88	$t_{59} = 1969$	Censored	1	37
$t_{12} = 148$	Failure	1	87	$t_{60} = 1984$	Censored	1	36
$t_{13} = 182$	Failure	1	86	$t_{61} = 1993$	Censored	1	35
$t_{14} = 187$	Failure	1	85	$t_{62} = 2003$	Censored	1	34
$t_{15} = 189$	Failure	1	84	$t_{63} = 2012$	Failure	1	33
$t_{16} = 274$	Failure	2	82	$t_{64} = 2013$	Censored	1	32
$t_{17} = 302$	Failure	1	81	$t_{65} = 2031$	Failure	1	31
$t_{18} = 363$	Failure	1	80	$t_{66} = 2052$	Censored	1	30
$t_{19} = 374$	Failure	1	79	$t_{67} = 2054$	Censored	1	29
$t_{20} = 451$	Failure	1	78	$t_{68} = 2061$	Censored	1	28
$t_{21} = 461$	Failure	1	77	$t_{69} = 2065$	Failure	1	27
$t_{22} = 492$	Failure	1	76	$t_{70} = 2072$	Censored	1	26
$t_{23} = 538$	Failure	1	75	$t_{71} = 2074$	Censored	1	25
$t_{24} = 774$	Failure	1	74	$t_{72} = 2084$	Censored	1	24
$t_{25} = 841$	Failure	1	73	$t_{73} = 2114$	Censored	1	23
$t_{26} = 936$	Failure	1	72	$t_{74} = 2124$	Censored	1	22
$t_{27} = 1002$	Failure	1	71	$t_{75} = 2137$	Censored	2	20
$t_{28} = 1011$	Failure	1	70	$t_{76} = 2145$	Censored	1	19
$t_{29} = 1048$	Failure	1	69	$t_{77} = 2157$	Censored	1	18
$t_{30} = 1054$	Failure	1	68	$t_{78} = 2173$	Censored	1	17
$t_{31} = 1172$	Failure	1	67	$t_{79} = 2174$	Censored	1	16
$t_{32} = 1205$	Failure	1	66	$t_{80} = 2183$	Censored	1	15
$t_{33} = 1278$	Failure	1	65	$t_{81} = 2190$	Censored	1	14
$t_{34} = 1401$	Failure	1	64	$t_{82} = 2201$	Failure	1	13
$t_{35} = 1497$	Failure	1	63	$t_{83} = 2421$	Failure	1	12
$t_{36} = 1557$	Failure	1	62	$t_{84} = 2573$	Censored	1	11
$t_{37} = 1577$	Failure	1	61	$t_{85} = 2574$	Censored	1	10
$t_{38} = 1624$	Failure	1	60	$t_{86} = 2578$	Censored	1	9
$t_{39} = 1669$	Failure	1	59	$t_{87} = 2595$	Censored	1	8
$t_{40} = 1806$	Failure	1	58	$t_{88} = 2610$	Censored	1	7
$t_{41} = 1836$	Censored/Alive	2	56	$t_{89} = 2613$	Censored	1	6
$t_{42} = 1846$	Censored/Alive	1	55	$t_{90} = 2624$	Failure	1	5
$t_{43} = 1859$	Censored/Alive	1	54	$t_{91} = 2631$	Censored	1	4
$t_{44} = 1860$	Censored/Alive	1	53	$t_{92} = 2638$	Censored	1	3
$t_{45} = 1870$	Censored/Alive	1	52	$t_{93} = 2641$	Censored	1	2
$t_{46} = 1874$	Failure	1	51	$t_{94} = 2710$	Failure	1	1
$t_{47} = 1876$	Censored/Alive	1	50	$t_{95} = 2719$	Censored	1	0

Table 7: Estimates $\hat{\sigma}(t_{j-1i}) \equiv \hat{\sigma}_{j-1i}$ and $\hat{S}(t_{j-1i-1}) \equiv \hat{S}_{j-1i-1}$ using $S_0 = 0.985$, 0.98900, 0.99000, 0.99900, 0.99990, 0.99999, 0.999999 using (3.5.28) with $k_a = 0$ and the procedure outlined in Chapter 4

Consecutive Failure time interval, $[t_{j-1i-1}, t_{j-1i})$	$S_0 = 0.985$		$S_0 = 0.98900$		$S_0 = 0.99000$		$S_0 = 0.99900$		$S_0 = 0.9999$		$S_0 = 0.99999$		$S_0 = 0.999999$	
	$\hat{\sigma}_{j-1i}$	$\hat{S}_{j-1i-1}$	$\hat{\sigma}_{j-1i}$	$\hat{S}_{j-1i-1}$	$\hat{\sigma}_{j-1i}$	$\hat{S}_{j-1}$	$\hat{\sigma}_{j}$	$\hat{S}_{j-1i-1}$	$\hat{\sigma}_{j-1i}$	$\hat{S}_{j-1i-1}$	$\hat{\sigma}_{j-1i}$	$\hat{S}_{j-1i-1}$	$\hat{\sigma}_{j-1i}$	$\hat{S}_{j-1i-1}$
[1.0, 6.0]	3.7500	0.9850	2.7500	0.9890	2.5000	0.9900	0.2500	0.9990	0.0250	0.9999	0.0025	0.99999	0.0003	0.999999
[6.0, 14.0)	4.5341	0.9653	4.0219	0.9692	3.8939	0.9702	2.7414	0.9790	2.6261	0.9799	2.6146	0.9800	2.6135	0.9800
[14.0, 44.0)	9.2600	0.9554	8.4535	0.9593	8.2519	0.9603	6.4373	0.9690	6.2559	0.9699	6.2377	0.9700	6.2359	0.9700
[44.0, 62.0)	2.1364	0.9456	1.9856	0.9494	1.9479	0.9504	1.6086	0.9590	1.5747	0.9599	1.5713	0.9600	1.5709	0.9600
[62.0, 89.0)	2.6581	0.9357	2.5009	0.9396	2.4616	0.9405	2.1079	0.9490	2.0725	0.9499	2.0689	0.9500	2.0686	0.9500
[89.0, 98.0)	0.7044	0.9259	0.6686	0.9297	0.6597	0.9306	0.5793	0.9391	0.5712	0.9399	0.5704	0.9400	0.5703	0.9400
[98.0, 104.0)	0.4780	0.9160	0.4568	0.9198	0.4515	0.9207	0.4039	0.9291	0.3991	0.9299	0.3986	0.9300	0.3986	0.9300
[104.0, 107.0)	0.2489	0.9062	0.2392	0.9099	0.2367	0.9108	0.2147	0.9191	0.2126	0.9199	0.2123	0.9200	0.2123	0.9200
[107.0, 114.0)	0.6171	0.8963	0.5954	0.9000	0.5900	0.9009	0.5412	0.9091	0.5363	0.9099	0.5358	0.9100	0.5358	0.9100
[114.0, 123.0)	0.8064	0.8865	0.7809	0.8901	0.7745	0.8910	0.7169	0.8991	0.7112	0.8999	0.7106	0.9000	0.7105	0.9000
[123.0, 128.0)	0.4463	0.8766	0.4334	0.8802	0.4302	0.8811	0.4012	0.8891	0.3983	0.8899	0.3980	0.8900	0.3980	0.8900
[128.0, 148.0)	1.8315	0.8668	1.7831	0.8703	1.7710	0.8712	1.6621	0.8791	1.6512	0.8799	1.6501	0.8800	1.6500	0.8800
[148.0, 182.0)	2.8591	0.8569	2.7895	0.8604	2.7721	0.8613	2.6156	0.8691	2.6000	0.8699	2.5984	0.8700	2.5983	0.8700
[182.0, 187.0)	0.3612	0.8471	0.3531	0.8505	0.3511	0.8514	0.3328	0.8591	0.3310	0.8599	0.3308	0.8600	0.3308	0.8600
[187.0, 189.0)	0.1480	0.8372	0.1449	0.8407	0.1441	0.8415	0.1371	0.8491	0.1364	0.8499	0.1364	0.8500	0.1364	0.8500
[189.0, 274.0)	3.2602	0.8274	3.1968	0.8308	3.1809	0.8316	3.0381	0.8392	3.0238	0.8399	3.0224	0.8400	3.0222	0.8400
[274.0, 302.0)	1.6114	0.8077	1.5839	0.8110	1.5770	0.8118	1.5152	0.8192	1.5090	0.8199	1.5084	0.8200	1.5083	0.8200
[302.0, 363.0)	3.3074	0.7978	3.2544	0.8011	3.2411	0.8019	3.1218	0.8092	3.1099	0.8099	3.1087	0.8100	3.1086	0.8100
[363.0, 374.0)	0.5139	0.7880	0.5062	0.7912	0.5042	0.7920	0.4868	0.7992	0.4850	0.7999	0.4849	0.8000	0.4849	0.8000
[374.0, 451.0)	3.6083	0.7781	3.5569	0.7813	3.5441	0.7821	3.4284	0.7892	3.4169	0.7899	3.4157	0.7900	3.4156	0.7900
[451.0, 461.0)	0.4007	0.7683	0.3953	0.7714	0.3940	0.7722	0.3818	0.7792	0.3806	0.7799	0.3805	0.7800	0.3805	0.7800
[461.0, 492.0)	1.2507	0.7584	1.2348	0.7615	1.2308	0.7623	1.1949	0.7692	1.1913	0.7699	1.1910	0.7700	1.1909	0.7700
[492.0, 538.0)	1.7864	0.7486	1.7648	0.7516	1.7594	0.7524	1.7108	0.7592	1.7059	0.7599	1.7054	0.7600	1.7054	0.7600

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Table 7 – continued from previous page

Consecutive Failure time interval, $[t_{j-1i-1}, t_{j-1i})$	$S_0 = 0.985$		$S_0 = 0.98900$		$S_0 = 0.99000$		$S_0 = 0.99900$		$S_0 = 0.9999$		$S_0 = 0.99999$		$S_0 = 0.999999$	
	$\hat{\sigma}_{j-1i}$	$\hat{S}_{j-1i-1}$	$\hat{\sigma}_{j-1i}$	$\hat{S}_{j-1i-1}$	$\hat{\sigma}_{j-1i}$	$\hat{S}_{j-1}$	$\hat{\sigma}_{j}$	$\hat{S}_{j-1i-1}$	$\hat{\sigma}_{j-1i}$	$\hat{S}_{j-1i-1}$	$\hat{\sigma}_{j-1i}$	$\hat{S}_{j-1i-1}$	$\hat{\sigma}_{j-1i}$	$\hat{S}_{j-1i-1}$
[538.0, 774.0)	8.5950	0.7387	8.4963	0.7418	8.4717	0.7425	8.2496	0.7492	8.2274	0.7499	8.2252	0.7500	8.2249	0.7500
[774.0, 841.0)	1.7366	0.7289	1.7176	0.7319	1.7129	0.7326	1.6702	0.7393	1.6660	0.7399	1.6655	0.7400	1.6655	0.7400
[841.0, 936.0)	2.3168	0.7190	2.2927	0.7220	2.2867	0.7227	2.2325	0.7293	2.2271	0.7299	2.2265	0.7300	2.2265	0.7300
[936.0, 1002.0)	1.4764	0.7092	1.4617	0.7121	1.4581	0.7128	1.4252	0.7193	1.4219	0.7199	1.4216	0.7200	1.4215	0.7200
[1002.0, 1011.0)	0.1917	0.6993	0.1899	0.7022	0.1895	0.7029	0.1854	0.7093	0.1850	0.7099	0.1849	0.7100	0.1849	0.7100
[1011.0, 1048.0)	0.7954	0.6895	0.7883	0.6923	0.7865	0.6930	0.7703	0.6993	0.7687	0.6999	0.7686	0.7000	0.7685	0.7000
[1048.0, 1054.0)	0.1266	0.6796	0.1255	0.6824	0.1252	0.6831	0.1227	0.6893	0.1225	0.6899	0.1225	0.6900	0.1225	0.6900
[1054.0, 1172.0)	2.5138	0.6698	2.4931	0.6725	2.4879	0.6732	2.4413	0.6793	2.4366	0.6799	2.4362	0.6800	2.4361	0.6800
[1172.0, 1205.0)	0.6415	0.6599	0.6365	0.6626	0.6352	0.6633	0.6238	0.6693	0.6227	0.6699	0.6226	0.6700	0.6226	0.6700
[1205.0, 1278.0)	1.3990	0.6501	1.3885	0.6527	1.3858	0.6534	1.3621	0.6593	1.3597	0.6599	1.3595	0.6600	1.3594	0.6600
[1278.0, 1401.0)	2.2505	0.6402	2.2343	0.6429	2.2302	0.6435	2.1936	0.6493	2.1900	0.6499	2.1896	0.6500	2.1896	0.6500
[1401.0, 1497.0)	1.6209	0.6304	1.6096	0.6330	1.6068	0.6336	1.5816	0.6394	1.5790	0.6399	1.5788	0.6400	1.5788	0.6400
[1497.0, 1557.0)	0.9581	0.6205	0.9518	0.6231	0.9502	0.6237	0.9359	0.6294	0.9344	0.6299	0.9343	0.6300	0.9343	0.6300
[1557.0, 1577.0)	0.3100	0.6107	0.3081	0.6132	0.3076	0.6138	0.3031	0.6194	0.3027	0.6199	0.3026	0.6200	0.3026	0.6200
[1577.0, 1624.0)	0.7257	0.6008	0.7212	0.6033	0.7201	0.6039	0.7101	0.6094	0.7091	0.6099	0.7090	0.6100	0.7090	0.6100
[1624.0, 1669.0)	0.6800	0.5910	0.6760	0.5934	0.6750	0.5940	0.6660	0.5994	0.6651	0.5999	0.6650	0.6000	0.6650	0.6000
[1669.0, 1806.0)	2.0285	0.5811	2.0171	0.5835	2.0142	0.5841	1.9885	0.5894	1.9859	0.5899	1.9857	0.5900	1.9856	0.5900
[1806.0, 1874.0)	0.8921	0.5713	0.8873	0.5736	0.8861	0.5742	0.8752	0.5794	0.8741	0.5799	0.8740	0.5800	0.8740	0.5800
[1874.0, 1907.0)	0.3686	0.5610	0.3667	0.5632	0.3662	0.5638	0.3619	0.5689	0.3614	0.5694	0.3614	0.5695	0.3614	0.5695
[1907.0, 2012.0)	0.9364	0.5492	0.9317	0.5514	0.9306	0.5520	0.9202	0.5570	0.9191	0.5575	0.9190	0.5576	0.9190	0.5576
[2012.0, 2031.0)	0.1408	0.5346	0.1401	0.5368	0.1400	0.5374	0.1385	0.5422	0.1383	0.5427	0.1383	0.5428	0.1383	0.5428
[2031.0, 2065.0)	0.2424	0.5180	0.2414	0.5201	0.2411	0.5206	0.2387	0.5253	0.2385	0.5258	0.2385	0.5258	0.2385	0.5258
[2065.0, 2201.0)	0.6657	0.5007	0.6630	0.5027	0.6623	0.5033	0.6562	0.5078	0.6556	0.5083	0.6555	0.5083	0.6555	0.5083
[2201.0, 2421.0)	0.6809	0.4760	0.6784	0.4779	0.6778	0.4784	0.6721	0.4827	0.6716	0.4832	0.6715	0.4832	0.6715	0.4832
[2421.0, 2624.0)	0.5114	0.4394	0.5097	0.4411	0.5093	0.4416	0.5057	0.4456	0.5053	0.4460	0.5053	0.4460	0.5053	0.4461

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Table 7 – continued from previous page

Consecutive Failure time interval, $[t_{j-1i-1}, t_{j-1i})$	$S_0 = 0.985$		$S_0 = 0.98900$		$S_0 = 0.99000$		$S_0 = 0.99900$		$S_0 = 0.9999$		$S_0 = 0.99999$		$S_0 = 0.999999$	
	$\hat{\sigma}_{j-1i}$	$\hat{S}_{j-1i-1}$	$\hat{\sigma}_{j-1i}$	$\hat{S}_{j-1i-1}$	$\hat{\sigma}_{j-1i}$	$\hat{S}_{j-1}$	$\hat{\sigma}_{j}$	$\hat{S}_{j-1i-1}$	$\hat{\sigma}_{j-1i}$	$\hat{S}_{j-1i-1}$	$\hat{\sigma}_{j-1i}$	$\hat{S}_{j-1i-1}$	$\hat{\sigma}_{j-1i}$	$\hat{S}_{j-1i-1}$
$[2624.0, 2710.0)$	0.0479	0.3990	0.0477	0.4006	0.0477	0.4010	0.0474	0.4046	0.0474	0.4050	0.0474	0.4050	0.0474	0.4050
(2710.0)		0.2348		0.2357		0.2360		0.2381		0.2383		0.2384		0.2384

Table 8: Data set describing time(in months) to death(failure) and losses(censored) [38]

Data Observation	Failure or Censor Time	Frequency of Failure/ Censors at t_i	Survival or Operating units at t_i: $z(t_i)$
$t_0 = 0$	Initial		8
$t_1 = 0.8$	Failure	1	7
$t_2 = 1.0$	Censored	1	6
$t_3 = 2.7$	Censored	1	5
$t_4 = 3.1$	Failure	1	4
$t_5 = 5.4$	Failure	1	3
$t_6 = 7.0$	Censored	1	2
$t_7 = 9.2$	Failure	1	1
$t_8 = 12.1$	Censored	1	0

In the following illustration, we apply the developed alternative innovative algorithm to a data set used by [38].

ILLUSTRATION 4.6.3 The data set in Table 8 is originally from [26]. Malla et al. used the data set to exemplify their approach. Malla et al. assumed that the largest observation 12.1 is uncensored. They also assumed that $0 = a_0 \leq a_1 \leq a_2 < \ldots \leq a_m$ are jumps of the Kaplan Meier [26] survival estimator in magnitude, and thus obtained $a_1 = 0.125, a_2 = 0.175, a_3 = 0.175, a_4 = 0.2625, a_5 = 0.2625$. They then proceeded to calculate the hazard rate function using the following:

$$\hat{\lambda}(t) = \frac{a_k}{1 - A_{k-1} \cdot \Delta d_k}\ , \tag{4.6.1}$$

where d_k is distinct failure time and $A_0 = 0$, $A_k = \sum_{i=1}^{k} a_i$ for $d_{k-1} \leq t < d_k, 1 \leq k \leq m$. The survival estimate on $[0, d_m]$ was defined as follows:

$$\hat{S}(t) = \hat{S}(d_{k-1}) \exp\left[-\int_{d_{k-1}}^{t} \frac{a_k}{(1 - A_{k-1})\Delta d_k}\, \mathrm{d}u\right], \quad d_{k-1} \leq t < d_k, \quad 1 \leq k \leq m. \tag{4.6.2}$$

Utilizing (4.6.1) and (4.6.2), the following estimates summarized in Table 9 were obtained.

Table 9: Estimates $\hat{\lambda}(t_j)$ and $\hat{S}(t_{j-1})$ using sing the procedure outlined in [38]

Consecutive Failure time interval, $[t_{j-1i-1}, t_{j-1i})$	$\hat{\lambda}(t_{j-1i})$	$\hat{S}(t_{j-1i-1})$
$[0, 0.8)$	0.1563	1.0000
$[0.8, 3.1)$	0.0870	0.8824
$[3.1, 5.4)$	0.1087	0.7224
$[5.4, 9.2)$	0.1316	0.5626
$[9.2, 12.1)$	0.3448	0.3412

In the following, we apply our innovative alternative algorithm to the data set in Table 8. Specifically, we used (3.5.27) in Example 3.5.1 with $k_{a_i} = 0$ for all i for parameter estimation. Additionally, survival state estimates at the failure times were estimated using the Euler scheme:

$$S(t^f_{ji}) = S(t^f_{j-1i-1}) - \hat{\lambda}(t^f_{j-1i-1})S(t^f_{j-1i-1})(1 - S(t^f_{j-1i-1}))\Delta t^f_{ji} \ . \tag{4.6.3}$$

We used initial survival probability to be $S_0 = 0.999, 0.9999, 0.99999, 0.999999$ and applied the conceptual computational simulation algorithm (3.5.27) for consecutive failure-time subintervals. Optimal convergence of survival state probability estimates was obtained for $S_0 = 0.9999$. Thus, we conclude that the best survival state estimate is for $S_0 = 0.9999$ for the data set in Table 8. The results are summarized in Table 10. Again, these results were confirmed by the application of the modified version of LLGMM method that assures a certain degree of confidence in the survival state estimates as compared to the estimates obtained in Table 9.

Table 10: Estimates $\hat{\lambda}(t_{j-1i})$ and $\hat{S}(t_{j-1i-1})$ using S_0 = 0.99900, 0.99990, 0.99999, 0.999999

Consecutive Failure time interval, $[t_{j-1i-1}, t_{j-1i-1})$	$S_0 = 0.99900$		$S_0 = 0.9999$		$S_0 = 0.99999$		$S_0 = 0.999999$	
	$\hat{\lambda}(t_{j-1i})$	$\hat{S}(t_{j-1i-1})$	$\hat{\lambda}(t_{j-1i})$	$\hat{S}(t_{j-1i-1})$	$\hat{\lambda}(t_{j-1i})$	$\hat{S}(t_{j-1i-1})$	$\hat{\lambda}(t_{j-1i})$	$\hat{S}(t_{j-1i-1})$
$[0, 0.8)$	156.25	0.9990	1562.5	0.9999	15625.0	0.99999	156250.0	0. 999999
$[0.8, 3.1)$	0.5841	0.8741	0.5878	0.8749	0.5882	0.8750	0.5882	0.8750
$[3.1, 5.4)$	0.3971	0.7263	0.3981	0.7269	0.3982	0.7270	0.3982	0.7270
$[5.4, 9.2)$	0.2387	0.5447	0.2390	0.5452	0.2390	0.5453	0.2390	0.5453
$[9.2, 12.1)$	0.5069	0.3197	0.5071	0.3200	0.5071	0.3200	0.5071	0.3200

4.7 Statistical Comparative Analysis with Existing Methods

In the following we exhibit an innovative alternative procedure for finding the parameter and state estimates at each failure points by using a modification of the Local Lagged Adapted adapted Generalized Method of Moments (LLGMM) [44].

4.7.1 Modified LLGMM Parameter and State Estimation

In this section, we develop a modified version of the Local Lagged Adapted adapted Generalized Method of Moments (LLGMM) [44]. This is achieved by by utlizing the developed alternative procedure in Section 3.5 and the LLGMM method. We also make an attempt to coordinate and compare the developed innovative approach for parameter and state estimation of time-to-event process with recently developed LLGMM approach. We note that the transformed conceptual computational interconnected dynamic algorithm for time-to-event data statistic (IDATTEDS) is local. It is centered around each consecutive pair of failure or change time ordered subinterval $[t^{f}_{j-1i-1}, t^{f}_{j-1i})$ or $[t^{cp}_{j-1i-1}, t^{cp}_{j-1i})$ with its right-end-point data observation/collection process for $i \in I(1, k_f)$ or $i \in I(1, k_{cp})$. Moreover, parameter and state estimation of the time-to-event process is relative to each consecutive pair of failure or change time subinterval operation of the time-to-event dynamic process. This type of parameter and state estimation problem in time-to-event processes can be characterized by the local single-shot procedure identified by the right-end point of the $j-1i$-th consecutive failure or change point subinterval for each $i \in I(1, k_f)$ or $i \in (1, k_{cp})$.

These observations motivates to extend this single-shot parameter and state estimation problem to a finite multi-choice local lagged consecutive failure or change time subintervals with right-end-point data observation/collection process. For this, we introduce a couple of definitions that form a bridge to connect IDATTEDS approach with the LLGMM approach. From Definitions 3.5.1-3.5.6, we recall that $\{t^{f}_{j-1i-1}\}_{i=1}^{k_f}$, $\{[t^{f}_{j-1i-1}, t^{f}_{j-1i})\}_{i=1}^{k_f}$, P^{f}_{j-1}, $\{z_{j-1i-1}\}_{i=1}^{k_f}$, $\{P^{f}_{j-1i}\}_{i=1}^{k_f}$, are increasing sequences of overall consecutive failure-times, consecutive failure-time subintervals, failure-time partition of $[t_0, \mathfrak{T})$, conceptual data sequence at failure-time, sequence of sub-partition of consecutive time subinterval $[t^{f}_{j-1i-1}, t^{f}_{j-1i})$, respectively for $i \in I(1, k_f)$.

Definition 4.7.1 For each $i \in I(1, k_f)$ and each $m_i \in I(1, i)$, a partition of closed interval $[t^{f}_{j-1i-m_i}, t^{f}_{j-1i}]$ is called local at a failure-time t^{f}_{j-1i} , and it is defined by

$$P^{f}_{j-1i-m_i} := t^{f}_{j-1i-m_i} < t^{f}_{j-1i-m_i+1} < \ldots < t^{f}_{j-1i-1} < t^{f}_{j-1i} \ . \tag{4.7.1}$$

A m_i-size consecutive failure time subinterval subsequence $\{[t^{f}_{j-1i+l}, t^{f}_{j-1i+l+1})\}_{l=-m_i}^{-1}$ of the overall consecutive failure time subinterval sequence $\{[t^{f}_{j-1i-1}, t^{f}_{j-1i})\}_{i=1}^{k_f}$ is called local lagged moving failure-time subsequence at t^{f}_{j-1i} that is a cover of $[t^{f}_{j-1i-m_i}, t^{f}_{j-1i})$:

$$\bigcup_{l=-m_i}^{-1} [t^{f}_{j-1i+l}, t^{f}_{j-1i+l+1}) = [t^{f}_{j-1i-m_i}, t^{f}_{j-1i}) \ . \tag{4.7.2}$$

$P^{f}_{j-1i-m_i}$ is a sub-partition of the partition P^{f}.

Definition 4.7.2 For each $i \in I(1, k_f)$ and each $m_i \in I_1(1, i)$, a local lagged moving consecutive failure time subsequence of subintervals, $\{[t^{f}_{j-1i+l}, t^{f}_{j-1i+l+1})\}_{l=-m_i}^{-1}$ at failure time t^{f}_{j-1i} of the size m_i is identified by the restriction of overall failure time state data subsequence $\{z_{j-1i-1}\}_{i=1}^{k_f}$ to $P^{f}_{j-1i-m_i}$ in (4.7.1), and it is defined by

$$s_{m_i,j-1i} := \{F^{l} z_{j-1i}\}_{l=-m_i}^{0} \ . \tag{4.7.3}$$

Here F is a forward-shift operator, and $F^{-1} = B$, B is the backward shift operator. m_i varies from 1 to i; the corresponding local sequence $s_{m_i,i}$ at t^{f}_{j-1i} varies from $\{F^{l} z_{j-1i}\}_{l=-1}^{0}$ to $\{F^{l} z_{j-1i}\}_{l=-i+1}^{0}$. As a result of this, the sequence defined in (4.7.3) is also called a m_i-local moving sequence of failure-time state data associated with m_i-local lagged finite sequence of subintervals at a failure-time t^f_{j-1i} for each $i \in I(1, k_f)$.

In the following, we outline computational scheme for the survival state data analysis problem. Using the concept of m_i-moving sequence of failure-time state data at a failure time t^f_{j-1i}, computational schemes for the change point problem can be developed analogously.

Hereafter, we utilize Definitions 4.7.1 and 4.7.2, and recast the LLGMM algorithm [44, 45]. For each $m_i \in I(1, i-1)$, using (3.5.23) and $l \in I(-m_i, -1)$, we determine estimates of λ at each failure time t^f_{j-1i} for the special case of $S\lambda(t, S) = S\lambda(t)(1 - S)$ (without loss of generality), as follows:

$$\hat{\lambda}_{m_i,i} = \frac{\sum_{l=-m_i}^{-1} \left[z(t_{j-1i+l}) - z(t_{j-1i+l+1}) - k_{c_{i+l}} + k_{a_{i+l}}\right]}{\sum_{l=-m_i}^{-1} (1 - F^l S(t^f_{j-1i})) \sum_{n=1}^{k_{b_{i+l}}+1} z(t^{c/a}_{j-1i+ln-1}) \Delta t^{c/a}_{j-1i+1n}}, \tag{4.7.4}$$

where $\lambda(t, S) = \lambda(t)(1 - S)$; $m_i \in I(1, i-1)$; $k_{c_{i+l}}$ is the total number of censored objects/species/infective/quitting covered over the subinterval $[t^f_{j-1i+l}, t^f_{j-1i+l+1})$; $k_{a_{i+l}}$ is the the total number of admitting/entering/joining/susceptible/etc covered over the subinterval $[t^f_{j-1i+l}, t^f_{j-1i+l+1})$; $k_{b_{i+l}} = k_{c_{i+l}} + k_{a_{i+l}}$.

Remark 4.7.1 For the special case of $\lambda(t) = \frac{1}{\sigma t}$, (4.7.4) reduces to

$$\hat{\sigma}_{m_i,i} = \frac{\sum_{l=-m_i}^{-1} (1 - F^l S(t_{j-1i})) \sum_{n=1}^{k_{b_{i+l}}+1} z(t^{c/a}_{j-1i+ln-1}) \frac{\Delta t^{c/a}_{j-1i+ln}}{t^{c/a}_{j-1i+ln-1}}}{\sum_{l=-m_i}^{-1} \left[z(t_{j-1i+1}) - z(t_{j-1i+l+1}) - k_{c_{i+l}} + k_{a_{i+l}}\right]}. \tag{4.7.5}$$

In short, the usage of the transformed continuous-time deterministic dynamic hybrid model for time-to-event process, and discrete-time interconnected hybrid dynamic algorithm of local sample mean lead to an innovative alternative method for parameter and state estimation problems for continuous-time dynamic models described by both linear and nonlinear deterministic differential equations.

4.7.2 Computational Algorithm

The numerical approximation and simulation processes need to be synchronized with the existing data collection process in the context of the partition of $[t_0, \mathcal{T}]$. For each $i \in I(1, k_f)$, we assume that t^f_{j-1i} is the scheduled time clock for the $j-1i$-th collected data of the state of the system under investigation. The iterative and simulation time processes are both t^f_{j-1i}. For each $m_i \in OS_{j-1i} = I(1, i-1)$ at t^f_{j-1i}, from Definition 4.7.2, we pick a m_i local admissible sequence $\{F^l z_{j-1i}\}^0_{l=-m_i}$. Using the terms of this sequence and (4.7.4), we compute the state and parameter estimates of the continuous-time dynamic equation. These estimates form a local finite sequence of parameter estimates at t^f_{j-1i} corresponding to $AS_{j-1i} = \{s_{m_i,j-1i}:$

$m_i \in I(1,i)\}$ for each $i \in I(1,k_f)$. The Principle of Mathematical Induction is employed for the development of a conceptual computational scheme.

For each admissible sequence in AS_{j-1i}, let $z^s_{m_i,j-1i}$ be a simulated value of $s_{m_i,j-1i}$ at t^f_{j-1i}. This engenders an m_i local sequence of simulated data $\{z^s_{m_i,j-1i}\}_{m_i \in OS_{j-1,i}}$. The simulated $z^s_{m_i,j-1i}$ satisfies the following scheme:

$$z^s_{j-1i} = z^s_{j-1i-1} - \hat{\lambda}_{j-1i-1}\, z^s_{j-1i-1}(1 - S^s_{j-1i-1})\Delta t_{j-1i} - k_{c_i} + k_{a_i}. \tag{4.7.6}$$

To find the best estimate of $z(t_{j-1i})$, let us define

$$\Xi_{m_i,j-1i,z_{j-1i}} = \left|z(t_{j-1i}) - z^s_{m_i,j-1i}\right| \tag{4.7.7}$$

to be the absolute error of $z(t^f_{j-1i})$ relative to each member of the term of local admissible sequences $\{z^s_{m_i,j-1i}\}_{m_i \in OS_{j-1,i}}$ of simulated values. For any preassigned arbitrary small positive number ϵ and for each time t^f_{j-1i}, to find the best estimate from admissible simulated values, we determine the following sub-optimal admissible set of data at t^f_{j-1i} as:

$$\mathcal{M}_{j-1i} = \{m_i : \Xi_{m_i,j-1i,z_{j-1i}} < \epsilon \quad \text{for} \quad m_i \in OS_{j-1i}\}. \tag{4.7.8}$$

Among these collected sub-optimal set of values, the value that gives the minimum $\Xi_{m_i,j-1i,z_{j-1i}}$ is recorded as $\hat{m}_i$. The parameters corresponding to $\hat{m}_i$ is referred as the ϵ-level sub-optimal estimates of the true parameters. These sub-optimal estimates are estimated at time t^f_{j-1i} with $\hat{m}_i$. The simulated value $z^s_{\hat{m}_i,j-1i}$ at t^f_{j-1i} corresponding to $\hat{m}_i$ is recored as the best estimate for $z(t_{j-1i})$ at t^f_{j-1i}. Having obtained the best estimate for λ, we then proceed to find the optimal/best estimate for the survival function at t^f_{j-1i} via the following:

$$\hat{S}(t_{j-1i}) = \hat{S}(t_{j-1i-1}) - \hat{\lambda}(t_{j-1i},\hat{m}_i)\hat{S}(t_{j-1i-1})(1 - \hat{S}(t_{j-1i-1}))\Delta t_{j-1i}\ . \tag{4.7.9}$$

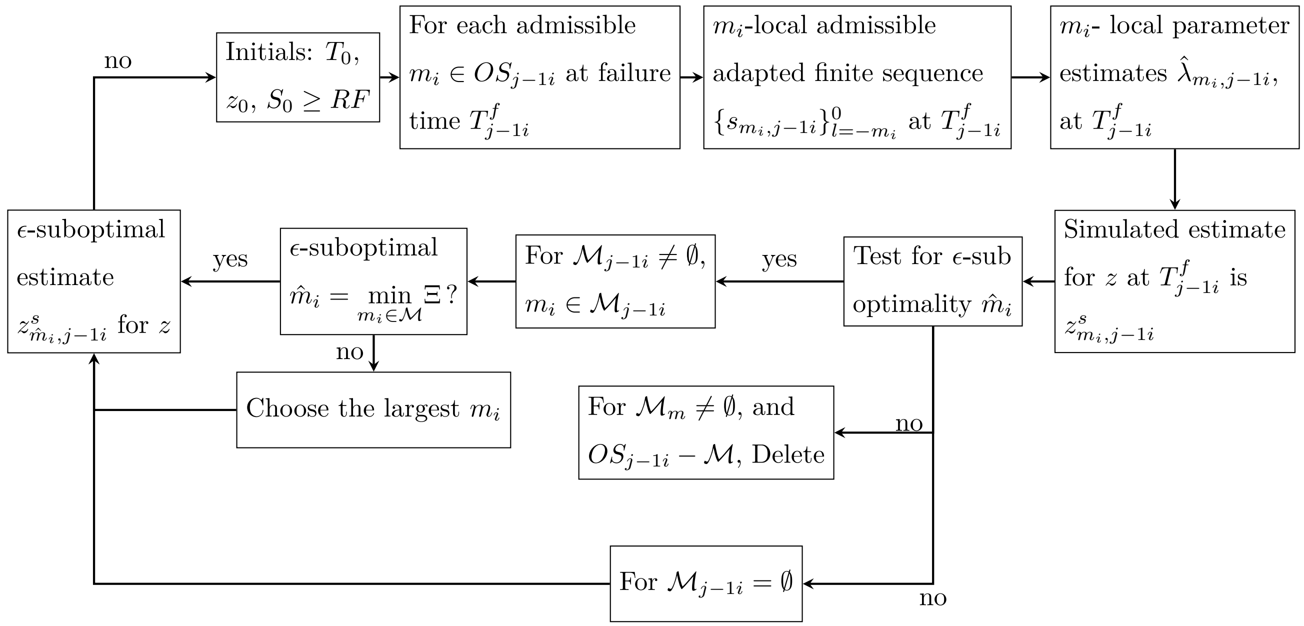
Finally, an estimate of $S_{\hat{m}_i,j-1i}$ at t^f_{j-1i} corresponding to $\hat{m}_i$ is also recorded as the best estimate for $S(t_{j-1i})$ at t^f_{j-1i}. Moreover, to summarize the computation, a modified LLGMM Conceptual Computational Algorithm is outlined in Flowchart 9.

Remark 4.7.2 Equation (4.7.6) specializes to

$$z^s_{m_i j-1i} = z^s_{m_{i-1}j-1i-1} - \frac{1}{\hat{\sigma}_{m_{i-1}j-1i-1}}\, z^s_{m_{i-1}j-1i-1}(1 - S^s_{m_{i-1}j-1i-1})\frac{\Delta t_{j-1i}}{t_{j-1i-1}} - k_{c_i} + k_{a_i}\ , \tag{4.7.10}$$

and (4.7.9) reduces to

$$\hat{S}(t_{j-1i}) = \hat{S}(t_{j-1i-1}) - \frac{1}{\hat{\sigma}(t_{j-1i},\hat{m}_i)}\hat{S}(t_{j-1i-1})(1 - \hat{S}(t_{j-1i-1}))\frac{\Delta t_{j-1i}}{t_{j-1i-1}}\ . \tag{4.7.11}$$



Flowchart 9.: Modified LLGMM Conceptual Computational Algorithm

We present an algorithm and flowchart for the simulation scheme described above.

Given initials t_0, S_0, z_0, ϵ,

for $i = 1$ to k_f **do**

  for $m_i = 1$ to i **do**

Compute $\hat{\lambda}_{m_i,j-1i}$

    for $m_i = 0$ to i **do**

Compute $z^s_{m_i,j-1i}$, $\Xi_{m_i,j-1i,z_{j-1i}}$

    end for

  end for

end for

if $\Xi_{m_i,i,z_{j-1i}} < \epsilon$ **then**

Save $\hat{m}_i$

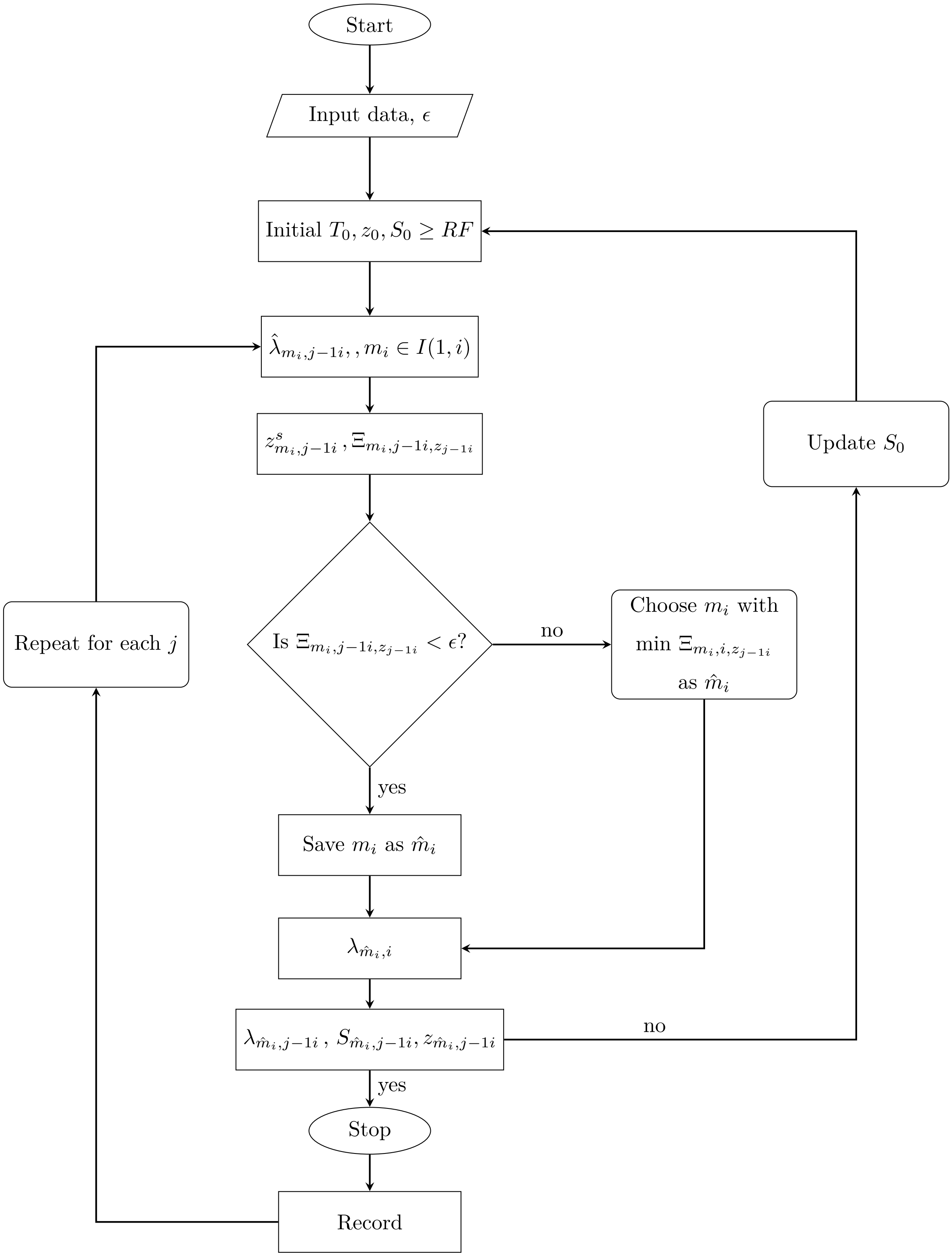
else

Find $\hat{m}_i$ that minimizes $\Xi_{m_i,j-1i,z_{j-1i}}$

end if

Compute $\lambda_{\hat{m}_i,j-1i}$, $z^s_{\hat{m}_i,j-1i}$, $S_{\hat{m}_i,j-1i}$.

Algorigthm 10.: Simulation scheme



Flowchart 11.: Modified LLGMM Simulation Algorithm

We note that the above presented innovative algorithm is valid for state and parameter estimation problems for continuous-time dynamic models described by linear hybrid deterministic differential equations for time-to-event processes. We further note that algorithm also allows for the admission/joining of individuals/items.

Remark 4.7.3 We remark that intervention processes provide a measure of influence of new tools/procedures/approaches in continuous time states of time-to-event dynamic process. In particular, it generates a measure of the degree of sustainability, survivability, reliability of the system. This further leads to sustainable/unsustainable, survivable/failure, reliable/unreliable binary state invariant sets. In addition, intervention processes provides the comparison between the past and currently used tools/procedures/approaches/attitudes/etc.

Illustration 4.7.1 [Application of LLGMM-type Conceptual Computational Algorithm to the datasets in Tables 4, 6 and 8]

We apply the above procedure to the three datasets in Tables 4, 6 and 8 by utilizing (4.7.5), (4.7.10), and (4.7.11) with $\epsilon = 0.001$. The results are summarized in Tables 33, 34 and 11, respectively.

Table 11: LLGMM Based Estimates using $S_0 = 0.99900, 0.99990, 0.99999, 0.999999$ using procedure outlined in Subsection 4.7.2

		$S_0 = 0.99900$		$S_0 = 0.9999$		$S_0 = 0.99999$		$S_0 = 0.999999$	
t^f_{j-1i}	$\hat{m}_i$	$\lambda_{j-1i,\hat{m}_i}$	$S_{j-1,\hat{m}_i}$	$\lambda_{j-1i,\hat{m}_i}$	$S_{j-1,\hat{m}_i}$	$\lambda_{j-1i,\hat{m}_i}$	$S_{j-1,\hat{m}_i}$	$\lambda_{j-1i,\hat{m}_i}$	$S_{j-1,\hat{m}_i}$
0.8	1	156.25	0.8741	1562.5	0.8749	15625.0	0.8750	156250.0	0.8750
3.1	1	0.5841	0.7263	0.5878	0.7269	0.5882	0.7270	0.5882	0.7270
5.4	1	0.3971	0.5447	0.3981	0.5452	0.3982	0.5453	0.3982	0.5453
9.2	1	0.2387	0.3197	0.2390	0.3200	0.2390	0.3200	0.2390	0.3200
12.1	1	0.5069	0.0000	0.5071	0.0000	0.5071	0.0000	0.5071	0.0000

Remark 4.7.4 We remark that using the LLGMM-type estimation approach yields the almost close simulation results as the estimation procedure outlined in Illustrations 4.6.1 and 4.6.2 with the added bonus of survival estimates at the last failure time for both data sets in Tables 4 and 6.

In the following, we compare the IDATTEDS and modified LLGMM results with the existing methods, namely, Maximum Likelihood and Kaplan-Meier approach.

4.7.3 Overall Statistical Comparison with Existing Approaches

In this subsection, the presented simulation results is compared with the existing methods, namely, Maximum Likelihood [25] and Kaplan-Meier [26] estimates. The simulation results are recored in Tables 12 and 13. In Table 14, we compare our results with Kaplan-Meier and Malla et al. estimates.

Table 12: Comparison of survival function estimates for data set in Table 4

Failure Time: t_{j-1i}	IDATTEDS $\hat{S}(t_{j-1i})$	LLGMM Based $S_{j-1i}, \hat{m}_i$	Maximum Likelihood Method: $\hat{S}_{ML}(t_{j-1i})$	Kaplan-Meier-type Estimate $\hat{S}_{KM}(t_{j-1i})$	Failure Time: t_{j-1i}	IDATTEDS $\hat{S}(t_{j-1i})$	LLGMM Based $S_{j-1i}, \hat{m}_i$	Maximum Likelihood Method: $\hat{S}_{ML}(t_{j-1i})$	Kaplan-Meier-type Estimate $\hat{S}_{KM}(t_{j-1i})$
22.5	0.9896	0.9896	0.9941	0.9896	91.5	0.7917	0.7917	0.8139	0.7917
37.5	0.9792	0.9792	0.9781	0.9792	93.5	0.7812	0.7812	0.8052	0.7813
46.5	0.9687	0.9687	0.9623	0.9686	102.5	0.7708	0.7708	0.7650	0.7708
48.5	0.9583	0.9583	0.9581	0.9583	107.0	0.7604	0.7604	0.7442	0.7604
51.5	0.9479	0.9479	0.9513	0.9473	108.5	0.7500	0.7500	0.7373	0.7500
53.5	0.9375	0.9375	0.9465	0.9375	112.5	0.7396	0.7396	0.7186	0.7396
54.5	0.9271	0.9271	0.9440	0.9271	113.5	0.7292	0.7292	0.7139	0.7292
57.5	0.9167	0.9167	0.9362	0.9167	116.0	0.7187	0.7187	0.7022	0.7188
66.5	0.9062	0.9062	0.9094	0.9063	117.0	0.7083	0.7083	0.6975	0.7083
68.0	0.8958	0.8958	0.9045	0.8958	118.5	0.6979	0.6979	0.6905	0.6979
69.5	0.8854	0.8854	0.8995	0.8854	119.0	0.6875	0.6875	0.6881	0.6875
76.5	0.8750	0.8750	0.8746	0.8750	120.0	0.6771	0.6771	0.6834	0.6771
77.0	0.8646	0.8646	0.8727	0.8646	122.5	0.6667	0.6667	0.6717	0.6667
78.5	0.8542	0.8542	0.8670	0.8542	123.0	0.6562	0.6562	0.6694	0.6563
80.0	0.8437	0.8437	0.8612	0.8438	127.5	0.6458	0.6458	0.6483	0.6458
81.5	0.8333	0.8333	0.8553	0.8333	131.0	0.6354	0.6354	0.6321	0.6354
82.5	0.8229	0.8229	0.8514	0.8229	132.5	0.6250	0.6250	0.6252	0.6250
83.0	0.8125	0.8125	0.8494	0.8125	134.0	0.6146	0.6146	0.6183	0.6146
84.0	0.8021	0.8021	0.8453	0.8021					

Table 13: Comparison of survival function estimates for data set in Table 6

Failure Time: t_{j-1i}	IDATTEDS $\hat{S}(t_{j-1i})$	LLGMM Based $S_{j-1i}, \hat{m}_i$	Maximum Likelihood Method: $\hat{S}_{ML}(t_{j-1i})$	Kaplan-Meier-type Estimate $\hat{S}_{KM}(t_{j-1i})$	Failure Time: t_{j-1i}	IDATTEDS $\hat{S}(t_{j-1i})$	LLGMM Based $S_{j-1i}, \hat{m}_i$	Maximum Likelihood Method: $\hat{S}_{ML}(t_{j-1i})$	Kaplan-Meier-type Estimate $\hat{S}_{KM}(t_{j-1i})$
6.0	0.9800	0.9800	0.9928	0.98	936.0	0.7200	0.7200	0.6753	0.72
14.0	0.9700	0.9700	0.9856	0.97	1002.0	0.7100	0.7100	0.6627	0.71
44.0	0.9600	0.9600	0.9636	0.96	1011.0	0.7000	0.7000	0.6611	0.70
62.0	0.9500	0.9500	0.9521	0.95	1048.0	0.6900	0.6900	0.6543	0.69
89.0	0.9400	0.9400	0.9364	0.94	1054.0	0.6800	0.6800	0.6533	0.68
98.0	0.9300	0.9300	0.9314	0.93	1172.0	0.6700	0.6700	0.6330	0.67
104.0	0.9200	0.9200	0.9282	0.92	1205.0	0.6600	0.6600	0.6276	0.66
107.0	0.9100	0.9100	0.9266	0.91	1278.0	0.6500	0.6500	0.6161	0.65
114.0	0.9000	0.9000	0.9230	0.90	1401.0	0.6400	0.6400	0.5979	0.64
123.0	0.8900	0.8900	0.9183	0.89	1497.0	0.6300	0.6300	0.5846	0.63
128.0	0.8800	0.8800	0.9158	0.88	1557.0	0.6200	0.6200	0.5766	0.62
148.0	0.8700	0.8700	0.9060	0.87	1577.0	0.6100	0.6100	0.5740	0.61
182.0	0.8600	0.8600	0.8903	0.86	1624.0	0.6000	0.6000	0.5681	0.60
187.0	0.8500	0.8500	0.8881	0.85	1669.0	0.5900	0.5900	0.5625	0.59
189.0	0.8400	0.8400	0.8872	0.84	1806.0	0.5800	0.5800	0.5463	0.58
274.0	0.8200	0.8200	0.8524	0.82	1874.0	0.5696	0.5699	0.5386	0.5688
302.0	0.8100	0.8100	0.8420	0.81	1907.0	0.5576	0.5580	0.5350	0.5566
363.0	0.8000	0.8000	0.8205	0.80	2012.0	0.5428	0.5459	0.5239	0.5402
374.0	0.7900	0.7900	0.8169	0.79	2031.0	0.5258	0.5288	0.5220	0.5233
451.0	0.7800	0.7800	0.7924	0.78	2065.0	0.5083	0.5112	0.5185	0.5046
461.0	0.7700	0.7700	0.7894	0.77	2201.0	0.4832	0.4927	0.5053	0.4685
492.0	0.7600	0.7600	0.7802	0.76	2421.0	0.4461	0.4548	0.4855	0.4325
538.0	0.7500	0.7500	0.7672	0.75	2624.0	0.4050	0.4164	0.4688	0.3604
774.0	0.7400	0.7400	0.7089	0.74	2710.0	0.2384	0.3871	0.46208	0.1802
841.0	0.7300	0.7300	0.6945	0.73					

Table 14: Comparison of survival function estimates for data set in Table 6

Failure Time:	IDATTEDS	LLGMM Based	Maximum Likelihood Method:	Kaplan-Meier-type Estimate
t_{j-1i}	$\hat{S}(t_{j-1i})$	$S_{j-1i}, \hat{m}_i$	$\hat{S}_{ML}(t_{j-1i})$	$\hat{S}_{KM}(t_{j-1i})$
0.8	0.8741	0.8741	0.8824	0.8750
3.1	0.7263	0.7623	0.7224	0.7000
5.4	0.5447	0.5447	0.5626	0.525
9.2	0.3197	0.3197	0.3412	0.2625
12.1	0.0000	0.0000	0.126	0.2625

Chapter 5

Stochastic Hybrid Dynamic Modeling for Time-to-event Processes

5.1 Introduction

Parametric and nonparametric methods are often applied to estimate the hazard/risk rate and survival functions in the study of survival and reliability data analysis [25, 37]. A parametric approach is based on the assumption that an underlying survival distribution function belongs to some specific family of distributions (e.g. exponential, loglogistic, lognormal, Weibull, etc). Mostly, classical likelihood based models, methods and its extensions/generalizations are developed and utilized [9, 25, 37]. On the other hand, a nonparametric approach is centered around the best-fitting member of a class of survival distribution functions [26]. Moreover, Kaplan [26] and Nelson-Aalen [1, 41] type nonparametric approaches assume neither distribution class nor closed-form distributions.

The human mobility, electronic communications, technological changes, advancements in engineering, medical, and social sciences have diversified and extended the role and scope of time-to-event processes in biological, cultural, epidemiological, financial, military and social sciences [2, 11, 33, 34, 50]. It is known that sudden changes in the hazard rate/risk at unspecified or specified times are frequently encountered in engineering and medical sciences [2]. These changes could occur multiple times. As a result of this, investigators [17, 19, 21] are often interested in (a) detecting the location of the changes, and (b) estimating the sizes of the detected changes. For incorporating intervention processes, we transform a continuous state dynamic model into an interconnected hybrid dynamic model composed of both continuous-time and discrete-time state(intervention) dynamic processes.

In this work, we present an alternative approach for modeling time-to-event processes in biological, chemical, engineering, epidemiological, medical, military, multiple-markets, and social dynamic processes. This approach does not require any knowledge of either a closed-form solution distribution or a class of distributions. Our innovative approach leads to the development of a stochastic dynamic model for time-to-event processes.

The developed approach is directly applicable to time-to-event dynamic processes in biological, chemical, engineering, financial, medical, physical, military and social sciences. A by-product of the transformed interconnected stochastic hybrid dynamic model is a mixture of theoretical continuous-discrete-time conceptual computational dynamic process. Employing the transformed discrete-time conceptual computational dynamic process, we introduce notions of data coordination, state data decomposition and aggregation, theoretical conceptual iterative processes, conceptual and computational parameter estimation and simulation schemes, conceptual and computational state simulation schemes.

The organization of the presented work is as follows. Recognizing the rapid growth, increased efficiency and speed in communication, science and technology in the 21^{st} century, we develop a stochastic dynamic model for time-to-event process in Section 5.2. Fundamental theoretical results for stochastic hybrid dynamic processes are also presented in Section 5.3. In fact, interconnected transformed stochastic hybrid survival state dynamic system and transformed discrete-time conceptual computational interconnected dynamic algorithm are developed. The approach is a continuation of the recently initiated work in [5, 6]. In Section 5.4, we present very general theoretical and computational procedures and results for parameter and state estimations for a time-to-event dynamic process.

5.2 Motivation and Model Development

The rapid electronic communication and human mobility processes have facilitated to transform the information, knowledge and ideas almost instantly around the globe. This indeed generates heterogeneity that engenders nonlinear and non-stationary dynamic processes. Moreover, the heterogeneity, nonlinearity, non-stationarity, further generate uncertainties both deterministic and stochastic. In view of this, it is obvious that nothing is deterministic. In short, the 21^{st} century problems are highly nonlinear, non-stationary and under the influence of internal and external random perturbations.

The mathematical models of dynamic processes under randomly varying environmental perturbations are described by two major approaches: (a) Newtonian mechanics and (b) random flow characterized by probabilistic models [31]. The random flow approach under a probabilistic law leads to deterministic differential equation known as Kolmogorov's backward (master) equation. The Newtonian approach generates stochastic/random differential equations. Using these methods, one determines distribution and moment functions [25]. In general, the flows are described by explanatory or covariate variables or functions of explanatory/-covariate variables. Dynamic flows are described by dynamic equations. Certain flows depend on either its deterministic or random parameters that may be subject to vary by explanatory variables. The dynamic flows can be visualized by either a family of curves or a single unique curve. For a covariate dependent parameter varying smooth dynamic flow $u(\alpha(t,x))$, the rate of $Du(\alpha(t,x))$ in the direction of covariate variate variables (t,x) is described by $\frac{\mathrm{d}u}{\mathrm{d}t}\left[\frac{\partial}{\partial t}\alpha(t,x)+\frac{\partial}{\partial x}\alpha(t,x)\right]$.

In the following, we present an illustration that motivates to develop dynamic models of time-to-event processes in engineering, medical, economic, social and technological sciences. This dynamic model can be considered as stochastic and deterministic parametric variation of a flow described by $u\left(\int_0^t \alpha(s)\mathrm{d}s, \int_0^t \sigma(s)\mathrm{d}w(s)\right)$. Moreover, the rate of u in the direction of $\Lambda(t) = \int_0^t \alpha(s)\mathrm{d}t$ and $\mathcal{E} = \int_0^t \sigma(s)\mathrm{d}w(s)$ is represented by $\mathrm{d}u = \frac{\partial}{\partial \Lambda}u(\Lambda,\mathcal{E})\alpha(t)\mathrm{d}t + \frac{\partial}{\partial \mathcal{E}}u(\Lambda,\mathcal{E})\sigma(t)\mathrm{d}w$. We present a few illustrations to exhibit this idea.

Illustration 5.2.1 Let us consider a linear stochastic differential equation of Itô-Doob -type [33]

$$\mathrm{d}x = -\alpha(t)x\,\mathrm{d}t + \sigma(t)x\,\mathrm{d}w,\; x(t_0) = x_0, \tag{5.2.1}$$

where x is a generic state of dynamic process; α and σ are univariate dynamic rate parameters that are referred to as drift and diffusion time-varying rate functions. w is a standard Wiener (Brownian motion) process. For a detailed justification, see [33].

We note that the following stochastic process [33],

$$x(t,t_0,x_0) = x_0 \exp\left[\int_{t_0}^{t} -\left[\alpha(s) + \frac{1}{2}\sigma^2(s)\right]\mathrm{d}s + \int_{t_0}^{t}\sigma(s)\mathrm{d}w(s)\right] \tag{5.2.2}$$

is the unique solution process of (5.2.1) for the given initial data (t_0, x_0), that is, $x(t, t_0, x_0)$ satisfies stochastic differential equation (5.2.1). We further note that the solution process (5.2.2) is non-negative, whenever the initial state $x_0 = x(t_0, t_0, x_0)$ is a non-negative random variable defined on a complete probability space, $(\Omega, \mathcal{F}, P)$ that is independent of the Wiener process. In addition, if $0 \le x_0 \le 1$ and α is a positive function, then

$$0 \le x_0 \exp\left[\int_{t_0}^{t} -\left[\alpha(s) + \frac{1}{2}\sigma^2(s)\right]\mathrm{d}s + \int_{t_0}^{t}\sigma(s)\mathrm{d}w(s)\right] \le 1, \quad \text{for} \quad t \ge t_0\,. \tag{5.2.3}$$

Under the above specified conditions, we have

$$\lim_{\Delta\to 0}\frac{1}{\Delta t}\,\mathbb{E}[x(t+\Delta t) - x(t)|\,\mathcal{F}_t] = -\alpha(t)x(t) \le 0 \quad \text{for} \quad t \ge t_0\,, \tag{5.2.4}$$

where $\mathcal{F}_t$ is a sub-σ-algebra of $\mathcal{F}$ under which $x(t)$ is measurable. Hence $(x(t), \mathcal{F}_t)$ is non-negative supermartingale [35, 42]. Furthermore, for $x_0 = 0$, $x(t, t_0, x_0) \equiv 0$, for all $t \ge t_0$; for $x_0 = 1$, $x(t, t_0, x_0) = \exp\left[\int_{t_0}^{t} -\left[\alpha(s) + \frac{1}{2}\sigma^2(s)\right]\mathrm{d}s + \int_{t_0}^{t}\sigma(s)\mathrm{d}w(s)\right]$. We further assume that, as $t \to \infty$,

$$\left[\int_{t_0}^{t}\left(\frac{1}{2}\sigma^2(s) + \alpha(s)\right)\mathrm{d}s\right] \to \infty\,. \tag{5.2.5}$$

We note that

$$y(t) = \left(1 - x_0 \exp\left[\int_{t_0}^{t} -\left[\alpha(s) + \frac{1}{2}\sigma^2(s)\right]\mathrm{d}s + \int_{t_0}^{t}\sigma(s)\mathrm{d}w(s)\right]\right)\,, \quad y(t_0) = 1 - x_0, \tag{5.2.6}$$

for $t \ge t_0$, and $0 \le x_0 \le 1$; Moreover, y in (5.2.6) is a solution process [33] of the following differential equation:

$$\mathrm{d}y = \alpha(t)(1 - y(t))\mathrm{d}t - \sigma(t)(1 - y(t))\mathrm{d}w, \quad y(t_0) = 1 - x_0\,. \tag{5.2.7}$$

From (5.2.2) and (5.2.7), we have

$$x(t) + y(t) = 1 \quad \text{for} \quad t \ge t_0. \tag{5.2.8}$$

From (5.2.3) and (5.2.8), we conclude that $x(t)$ and $y(t)$ are indeed stochastic versions of survival and failure functions, respectively. Furthermore, it is known [33] that $x(t,t_0,x_0)$ has log-normal probability distribution function with mean, $\mathbb{E}\left[\ln x(t,t_0,x_0)\right] = \mathbb{E}\left[\ln x_0\right] + \int_{t_0}^{t} -[\alpha(s)]\mathrm{d}s$ and variance, $\mathrm{Var}\left[\ln x(t,t_0,x_0)\right] = \mathrm{Var}(\ln x_0) + \int_{t_0}^{t} \sigma^2(t)\mathrm{d}s$ for each $t \geq t_0$.

From (5.2.7), we define a differential of hazard rate function as:

$$\mathrm{d}(\lambda(t,w(t))) = \frac{\mathrm{d}y}{1-y} = \alpha(t)\mathrm{d}t - \sigma(t)\mathrm{d}w(t)\,. \tag{5.2.9}$$

We present another illustration that provides a stochastic version of survival function.

Illustration 5.2.2 We consider a following stochastic differential equation

$$\mathrm{d}x = \alpha x(1-x)\mathrm{d}t + \sigma x \mathrm{d}w\,, \quad x(t_0) = x_0. \tag{5.2.10}$$

The solution process of (5.2.10) for constant functions α and σ is

$$x(t,t_0,x_0) = \frac{\frac{x_0}{\Phi(t,t_0)}}{1 + x_0\int_{t_0}^{t}\Phi^{-1}(s,t_0)\mathrm{d}s} = \frac{x_0\exp\left[(\alpha-\frac{1}{2}\sigma^2)(t-t_0)+\sigma(w(t)-w(t_0))\right]}{1+\alpha x_0\int_{t_0}^{t}\exp\left[(\alpha-\frac{1}{2}\sigma^2)(s-t_0)+\sigma(w(s)-w(t_0))\right]\,\mathrm{d}s}\,, \tag{5.2.11}$$

where

$$\Phi(t,t_0) = \exp\left[-\left(\alpha - \frac{1}{2}\sigma^2\right)(t-t_0) - \sigma(t-t_0)\right]\,, \tag{5.2.12}$$

and

$$\Phi^{-1}(t,t_0) = \exp\left[\left(\alpha - \frac{1}{2}\sigma^2\right)(t-t_0) + \sigma(t-t_0)\right]. \tag{5.2.13}$$

For $\alpha > 0$, if $x_0 > 0$, then $x(t) = x(t,t_0,x_0) > 0$, for $t \geq t_0$. Moreover, if $\alpha < \frac{1}{2}\sigma^2$, then $0 \leq x(t) \leq 1$, and $x(t) + y(t) = 1$, for $t \geq t_0$ whenever

$$y(t) = 1 - \frac{x_0\Phi^{-1}(t,t_0)}{1+\alpha x_0\int_{t_0}^{t}\exp\left[(\alpha-\frac{1}{2}\sigma^2)(s-t_0)+\sigma(w(s)-w(t_0))\right]\,\mathrm{d}s}\,. \tag{5.2.14}$$

Further, we note that $y(t)$ satisfies:

$$\mathrm{d}y = \alpha y(1-y)\mathrm{d}t - \sigma(1-y)\mathrm{d}w(t)\,,\ y_0 = 1 - x_0\,. \tag{5.2.15}$$

This justifies that x determined by (5.2.10) with $0 < x_0 < 1$ is a stochastic version of the survival function.

Let us assume that there are k individuals/items under a study having independent random failure times. Let T_j be a time to failure of the j-th subject/entity, $j = 1,\ldots,k$. In general, failure times $T_1,\ldots,T_k$ are not completely observable. In fact, one only observes $(\tilde{T}_j,\delta_j), j \in \{1,\ldots,k\} = I(1,k)$, where δ_j is a censoring indicator, describing whether T_j or only a lower bound to T_j is observed. Thus

$$\begin{cases} T_j = \tilde{T}_j & \text{if} \quad \delta_j = 1\,, \\ T_j > \tilde{T}_j & \text{if} \quad \delta_j = 0\,,\ j \in I(1,k). \end{cases} \tag{5.2.16}$$

We remark that each $\tilde{T}_j$ is a random time. At $\tilde{T}_j$, the value of the corresponding δ_j is available. In addition, and we know whether the corresponding event is either a failure or a censoring.

In the following, we imitate the argument used in developing dynamic models in [5, 6, 33]. We then introduce an interconnected stochastic hybrid dynamic model of a time-to-event process described by following a large-scale nonlinear and non-stationary stochastic differential equations:

$$\begin{cases} \mathrm{d}x = xW(t^-, Sx)\mathrm{d}\eta(t)\,,\ x(T_0) = x_0, \quad t \in [T_{j-1}, T_j),\ j \in I(1,k)\,, \\ x_j = x(T_j^-) + \int_{T_j^-}^{T_j^+} x(u)W(u^-, S(u)x(u))\mathrm{d}\eta(u), \quad x(T_j) = x_j, \\ \mathrm{d}S = -S\lambda(t,S)\mathrm{d}t + S\sigma(t,S)\mathrm{d}w(t), \quad t \geq 0, \quad S(T_0) = S_0, \\ S_j = S(T_j^-, T_{j-1}, S_{j-1})\,, \end{cases} \tag{5.2.17}$$

where $x(t)$ is the total number of units/individuals operating/living under the study at time t, for $t \in [T_0, \mathfrak{T}]$; t^- and t^+ stand for $t^- < t < t^+$ and they are very close to t; T_{j-1}, T_j are consecutive observation/study/evaluation times in $[T_0, \mathfrak{T})$; S is a survival state function; $x(T_j^-)$ and $S(T_j^-)$ stands for $x(T_j^-, T_{j-1}, x_{j-1})$ and $S(T_j^-, T_{j-1}, S_{j-1})$ respectively; for each $j \in I(1,k), T_j, (T_j, \delta_j)$, x_0 and w are independent stochastic processes defined on a complete probability space $(\Omega, \mathcal{F}, P)$ and $\mathcal{F}_t$ measurable ; w is a standard Wiener process, and $\mathcal{F}_t$ is a sub-σ-algebra of $\mathcal{F}$; λ is a continuous function defined on $\mathbb{R}_+ \times \mathbb{R}$ into $\mathbb{R}$; η is a function of bounded variation; W is defined on $\mathbb{R}_+ \times \mathbb{R}$ into $\mathbb{R}$, and is a continuous function on (t_{j+1}, t_j), where $t_{j-1}, t_j \in [t_0, \mathfrak{T})$, and are consecutive points of discontinuities of η; W satisfies conditions at t_j's so that the initial value problem of Riemann-Stieltjes differential equation has unique solution [48]. It is assumed that (5.2.17) has a solution process [33].

5.3 Fundamental Results for Stochastic Hybrid Dynamic Process

In this section, we develop a fundamental theoretical results. The presented analytic results provide the basis for conceptual computational tools for survival state and parameter estimation problems in time-to-event data analysis processes.

Definition 5.3.1 Let z be a stochastic process defined by $z(t) = x(t)S(t)$, where S and x are solution process of (5.2.17) for $t \in [t_0, \mathfrak{T})$. Moreover, for each $t \in [t_0, \mathfrak{T})$, $z(t)$ stands for the number of survivals/operating units/entities at t.

In the following, imitating the definition given in [25], we define an appropriate conditioning event using the concept of history or filtration.

Definition 5.3.2 [7] Let the history process $(\mathcal{G}_t)$ be defined by $\mathcal{G}_t = \{(T_j, \delta_j) : T_j \leq t\}$. Then $\mathcal{G}_t \subseteq \mathcal{F}_t$. This means that all $\mathcal{G}_t$ measurable processes are $\mathcal{F}_t$ measurable and independent of intervention/observation processes. Furthermore, $\mathcal{G}_t = \mathcal{G}_{T_{j-1}}$ for all $T_{j-1} \leq t < T_j^+$. This implies that $\mathcal{G}_{t^-} = \mathcal{G}_{T_{j-1}}$ for $T_{j-1} < t \leq T_j$.

For easy reference, we present a couple of results that provides a basis for the development of theoretical and computational dynamic results, subsequently.

Lemma 5.3.1 *[33] Let V be a function defined on $\mathbb{R}_+ \times \mathbb{R}$, and suppose $\frac{\partial V}{\partial t}$, $\frac{\partial V}{\partial y}$ and $\frac{\partial^2 V}{\partial y^2}$ exist and are continuous for $(t, y) \in \mathbb{R}_+ \times \mathbb{R}$ into $\mathbb{R}$. Let us consider a system of stochastic differential equations:*

$$\mathrm{d}y = g(t, y)\mathrm{d}t + \Lambda(t, y)\mathrm{d}w\,. \tag{5.3.1}$$

Then

$$\mathrm{d}V(t, y) = LV(t, y)\mathrm{d}t + \frac{\partial}{\partial y} V(t, y)\Lambda(t, y)\mathrm{d}w\,, \tag{5.3.2}$$

where

$$L(t, V) = \frac{\partial}{\partial t} V(t, y) + g(t, y)\frac{\partial}{\partial y} V(t, y) + \frac{1}{2}\mathrm{tr}\left(\frac{\partial^2}{\partial y^2} V(t, y)\Lambda(t, y)\Lambda^T(t, y)\right)\,. \tag{5.3.3}$$

In the following, we present a result that provides a foundation for the development of the study of time-to-event dynamic processes in any field of interest. We present a general result that sheds light and insight on the solution process of Riemann-Stieltjes type ordinary differential equation [48].

Lemma 5.3.2 *Let t_{k-1} and t_k be a pair of ordered consecutive points of discontinuities of η in the time interval $[t_0, \mathcal{T})$. Let us assume that the initial value problem described by the Riemann-Stieltjes type ordinary differential equation in (5.2.17) has a solution, $x(t) \equiv x(t, t_0, x_0)$ for $t \geq t_0$. Then the structure of solution has a following representation:*

$$\begin{cases} \mathrm{d}x = xW(t^-, Sx)\,\mathrm{d}\alpha(t)\,, \\ x(t, t_0, x_0) = x(t, t_{k-1}, x_{k-1}) \quad \text{for} \quad t \in [t_{k-1}, t_k)\,, k \in I(1, \infty) \\ x_k = x(t_k^-, t_{k-1}, x_{k-1}) + \gamma_k, \quad \text{for} \quad t = t_k\,,\ x(t_0) = x_0\,, \end{cases} \tag{5.3.4}$$

where $\gamma_k = x(t_k^+) - x(t_k^-, t_{k-1}, x_{k-1})$ is a jump size of $x(t, t_{k-1}, x_{k-1})$ at t_k for $k \in I(1, \infty), x(t_0) = x_0$; α is a continuous function of finite variation, and $\eta = \alpha + s$; s is a Saltus function [4, 22, 40].

Proof. We recall that in the theory of differential equations, the solution of initial value problem is right-hand continuous at an initial time, t_0. Furthermore, every function of bounded variation [4, 22, 40]:

(a) is the difference of two monotonic increasing functions (α, β), and also the sum of its Saltus function, s, and continuous function of bounded variation, α: $\eta = \alpha + s$;

(b) is differentiable almost everywhere; its derivative is integrable; its points of discontinuities are of the following type: $\eta(t^+) = \lim_{u\to t^+}\eta(u)$, $\eta(t^-) = \lim_{u\to t^-}\eta(u)$, and a jump of η at t is $\Delta\eta(t) = \eta(t^+) - \eta(t^-)$.

In view of the above observations, in general, the qualitative behavior of initial value problem (5.3.4) at t_0 is described by

$$x(t_0^+) = x(t_0) + \lim_{t\to t_0^+}\int_{t_0}^{t} x(u)W(u, x(u))\mathrm{d}\eta(u)\,.$$

Hence

$$x(t_0^+) = \begin{cases} x_0 + x(t_0^+)W(t_0^+, x_0(t_0^+))[\eta(t_0^+) - \eta(t_0)], & \text{if jump } \Delta\eta(t_0^+) \neq 0\,, \\ x_0, \quad \text{if jump } \Delta\eta(t_0^+) = 0\,. \end{cases} \tag{5.3.5}$$

We set $x_0 = x(t_0) = x(t_0^-)$. From this, we rewrite (5.3.5) as:

$$\begin{cases} x(t_0^+) = x(t_0^-) + x(t_0^-)W(t_0^-, x(t_0^-))\Delta\eta(t_0), & \Delta\eta(t_0) \neq 0\,, \\ x(t_0^+) = x(t_0^-), & \text{jump } \Delta\eta(t_0) = 0\,. \end{cases} \tag{5.3.6}$$

Obviously, $x(t_0^+) = x(t_0^-) + \Delta x(t_0)$, where $\Delta x(t_0) = x(t_0^-)W(t_0^-, x(t_0^-))\Delta\eta(t_0^-)$ is a jump of x at t_0. Moreover, $x(t_0^+)$ is considered to be an initial value of x at $t = t_0$, and it depends on an immediate past knowledge/history of x. In the light of this observation, the initial value problem in (5.3.4) can be considered as initial value problem of generalized ordinary functional differential equations [48].

Using this initial data in (5.3.6), we define a solution as follows:

$$\begin{cases} \mathrm{d}x = xW(t^-, Sx)\mathrm{d}\alpha(t)\,, \\ x(t, t_0, x_0) = x(t, t_0, x_0) \quad \text{for} \quad t \in [t_0, t_1)\,, \\ x_1 = x(t_1^-, t_0, x_0) + \gamma_1, \quad \text{for} \quad t = t_1\,,\ x(t_0) = x_0\,, \end{cases} \tag{5.3.7}$$

where $\gamma_1 = x(t_1^+) - x(t_1^-, t_0, x_0)$ is a jump of x at $t = t_1$. We continue this process, and then apply the principle of mathematical induction [32] to conclude that (5.3.4) is valid for any $k \in (1, \infty)$. □

Corollary 5.3.1 *Let T_j and T_{j-1} be a pair of consecutive data observation/collection/failure/censored times; let $\{t_{jk_l}\}_{l=1}^{\infty}$ be a subsequence of the sequence $\{t_k\}_{k=1}^{\infty}$ in Lemma 5.3.2, and $t_{jk_l} \in [T_{j-1}, T_j]$ for $j \in I(1, n)$ and $l \in I(1, \infty)$. Then from Lemma 5.3.2, we have*

$$\begin{cases} \mathrm{d}x = xW(t^-, Sx)\mathrm{d}\alpha(t)\,, \\ x(t, T_0, x_0) = x(t, T_0, x_{j-1}) \quad \textit{for} \quad t \in [T_{j-1}, T_j),\ j \in I(1, n)\,, \\ x_j = x(T_j^-, T_{j-1}, x_{j-1}) + \sum_{l=1}^{\infty}\gamma_{jk_l} + \gamma_j^o, \quad \textit{for} \quad t = T_j\,,\ x(T_0) = x_0\,. \end{cases} \tag{5.3.8}$$

where γ_j^o denotes jump size at observation/study time T_j, and $\gamma_j^{no} = \sum_{l=1}^{\infty} \gamma_{jk_l}$ stands for total jump size currently not under observable/study time sub-sequence $\{T_{jk_l}\}_{l=1}^{\infty}$ over a jth-consecutive pair of observation time interval $[T_{j-1}, T_j)$; moreover, due to the finite variation nature of η and the nature of W, γ_j^o and γ_j^{no} are finite over the interval $[T_{j-1}, T_j]$.

Remark 5.3.1 We remark that under the assumptions of Lemma 5.3.2, for $k \in I(1,\infty)$, $t_k^- < t_k < t_k^+$, we have left-hand $x(t_k) - s(t_k^-, t_{k-1}, x_{k-1})$ and right-hand jumps of solution process $x(t, t_{k-1}, x_{k-1})$ at t_k. Moreover, if a solution process $x(t, t_{k-1}, x_{k-1})$ is continuous from the left or/and right, then $x(t_k, t_{k-1}, x_{k-1}) = x(t_k^-, t_{k-1}, x_{k-1})$ or/and $x(t_k^+, t_k, x_k) = x(t_k, t_k, x_k)$, respectively.

Remark 5.3.2 We further remark that the Riemann-Stieltjes type ordinary differential equation in (5.2.17) can be reformulated by the following system of hybrid dynamic system:

$$\begin{cases} \mathrm{d}x = xW_{k-1}(t^-, Sx)\mathrm{d}\alpha(t), \quad t \in [t_{k-1}, t_k),\, k \in I(1,\infty)\,, \\ x_k = x_{k-1} + x(t_k^-)W_{k-1}(t_k^-, S(t_k^-)x(t_k^-))\Delta\alpha(t_k) + \gamma_k\,, \end{cases} \tag{5.3.9}$$

where $\eta_k = \alpha_k + s_k$, γ_k and α_k are defined in Lemma 5.3.2, accordingly; W_{k-1} is a rate function corresponding to a jump time t_{k-1}.

Remark 5.3.3 A few additional features of Riemann-Stieltjes integrals with respect to a function of bounded/finite variation η [4, 22, 40] are outlined . For $t_k^- < t_k < t_k^+$, $k \in I(0,\infty)$; $\Delta\eta(t_k^+) = \eta(t_k^+) - \eta(t_k)$, $\Delta\eta(t_k^-) = \eta(t_k) - \eta(t_k^-)$,

$$x(t_k^+) = x_k + x(t_k^+)W(t_k^+, x(t_k^+))\Delta\eta(t_k^+) \iff x(t_k^+) - x_k = x(t_k^+)W(t_k^+, x(t_k^+))\Delta(t_k^+) = \text{right-hand jump at } t_k$$

$$\begin{aligned} x_k \equiv x(t_k) = x(t_k^-) + x(t_k^-)W(t_k^-, x(t_k^-))\Delta\eta(t_k^-) \iff x_k - x(t_k^-) &= x(t_k^-)W(t_k^-, x(t_k^-))\Delta\eta(t_k) \\ &= \text{left-hand-jump at } t_k. \end{aligned}$$

Adding the above right-hand and left-hand jumps of x at t_k, we obtain an overall jump size at t_k in the context of W and η:

$$x(t_k^+) - x(t_k^-) = x(t_k^+)W(t_k^+, x(t_k^+))\Delta\eta(t_k^+) + x(t_k^-)W(t_k^-, x(t_k^-))\Delta\eta(t_k^-) = \text{overall jump at } t_k. \tag{5.3.10}$$

From (5.3.10), we draw a few special cases:

(1) $x(t_k^+) = x(t_k^-) + \gamma_k$, where γ_k stands for the jump at t_k, and $\gamma_k = W(t_k^+, x(t_k^+))\Delta\eta(t_k^+) + W(t_k^-, x(t_k^-))\Delta\eta(t_k^-)$. We note that for $k \in I(0,\infty)$, $x(t_k^-) = x(t_k^-, t_{k-1}, x_{k-1}), t^- \in [t_{k-1}, t_k)$.

(2) $x_k = x_{k-1} + \int_{t_{k-1}}^{t_k^-} x(s)W(s, x(s))\mathrm{d}\alpha(s) + \gamma_k$ for $t = t_k$;

(3) $\gamma_k = [x(t_k^+)W(t_k^+, x(t_k^+))\eta(t_k^+) - x(t_k^-)W(t_k^-, x(t_k^-))\eta(t_k^-)] + [x(t_k^-)W(t_k^-, x(t_k^-)) - x(t_k^+)W(t_k^+, x(t_k^+))]\eta(t_k)$;

(4) $\gamma_k = [x(t_k^+)W(t_k^+, x(t_k^+))\eta(t_k^+) - x(t_k^-)W(t_k^-, x(t_k^-))\eta(t_k^-)] - [x(t_k^+)W(t_k^+, x(t_k^+)) - x(t_k^-)W(t_k^-, x(t_k^-))]\eta(t_k)$.
We note that (3) and (4) are identical.

Furthermore, we observe that

(i) If W and η have left and right-hand limits and are discontinuous at (t_k, x), then (3) is valid. This leads to the development of a discrete-time iterative dynamic process at t_k, for $k \in I(1, \infty)$. This iterative process is called as "discrete-time intervention process."

(ii) For $k \in I(0, \infty)$, if W is either left or/and right continuous on $[t_{k-1}, t_k) \times \mathbb{R}$ and has both left and right-hand limits in x at t_k, then (3) remains valid. Here, the jump is due to the discontinuity of W in x. Again, this discrete-time dynamic process is referred as impulse type response/impulsive process [48]. The folowing two cases of (ii) are of great interest in the study of time-to-event dynamic processes:

1. Kaplan and Meier [26] type assumption: For $k \in I(0, \infty)$, if W is left discontinuous (W has left-hand limit that is different from its value) and right-continuous.
2. Kaplan and Meier [26] type assumption: For $k \in I(0, \infty)$, if W is right discontinuous (W has right-hand limit that is different from its value) and left-continuous.

(iii) If W is continuous in $(t_k, x) \in \{t_k\} \times \mathbb{R}, \eta$ is discontinuous at t_k, then (3) remains valid. Moreover, $\gamma_k = x(t_k)W(t_k, x(t_k)\Delta\eta(t_k))$.

(iv) If η has either left or right continuity at t_k and W has left-and right-hand limits at x, then x is left-hand continuous at t_k, and

$$\begin{aligned}(\text{overall jump size at}\, t_k) = (\text{either right or left-hand jump at}\, t_k) &= x(t_k^{\pm})W(t_k^{\pm}, x(t_k^{\pm}))\Delta\eta(t_k^{\pm}) \\ &= x(t_k^{\pm})W(t_k^{\pm}, x(t_k^{\pm}))(\eta(t_k^{\pm}) - \eta(t_k)) \\ &= \gamma_k.\end{aligned}$$

Moreover, if η is continuous at t_k, then W is discontinuous at (t_k, x).

(v) If η is continuous from the right at t_k, then x has right continuity at t_k, and
$(\text{overall jump size at}\;\, t_k) = (\text{left-hand jump at}\, t_k) = x(t_k^-)W(t_k^-, x(t_k^-))(\eta(t_k) - \eta(t_k^-)) = \gamma_k.$

Now, we are ready to present a fundamental result in the theory of time-to-event dynamic processes.

Theorem 5.3.1 *Let (x, S) be a solution process of* (5.2.17)*, and let T_{j-1} and T_j be any pair of consecutive conceptual data observation times in a given interval of time $[T_0, \mathfrak{T})$. Let z be defined in Definition 5.3.1. Then the transformed interconnected hybrid dynamic models of survival species and state of time-to-event dynamic process described in* (5.2.17) *are as:*

$$
\begin{cases}
\mathrm{d}z = -z\lambda(t,S)\mathrm{d}t + z\sigma(t,S)\mathrm{d}w + zW(t,z)\mathrm{d}\alpha\,,\ z(T_{j-1}) = z_{j-1}\,,\\
z_j = z(T_j^-) + z_j^{no} + z_j^{o}\,, \quad z(T_0) = z_0\,,\\
\mathrm{d}V(t,z) = LV(t,z)\mathrm{d}t + z\sigma(t,S)\frac{\partial}{\partial z}V(t,z)\mathrm{d}w + L^{\alpha}V(t,z)\mathrm{d}\alpha\,,\\
V(T_j,z_j) = V(T_j^-, z(T_j^-,T_{j-1},z_{j-1})) + \overline{\frac{\partial V}{\partial z}}V(T_j^-, z(T_j^-),\Delta z(T_j))\Delta z(T_j)\,,\\
\mathrm{d}S = -S\lambda(t,S)\mathrm{d}t + S\sigma(t,S)\mathrm{d}w,\ S(T_0) = S_0,\ t\in[T_{j-1},T_j)\ ,\ j\in I(1,k),
\end{cases}
\tag{5.3.11}
$$

where $z(T_j^-) = z(T_j^-, T_{j-1}, z_{j-1})$ *and*

$$
\begin{cases}
\overline{\frac{\partial V}{\partial z}}(T_j^-, z(T_j^-),\Delta z) = \int_0^1 \frac{\partial}{\partial z}V(T_j^-, z(T_j^-) + \theta\Delta z(T_j))\mathrm{d}\theta \quad \textit{and} \quad \Delta z(T_j) = z(T_j^+) - z(T_j^-)\,,\\
LV(t,z) = L^dV(t,z) + \frac{1}{2}z^2\sigma^2(t,S)\frac{\partial^2}{\partial z^2}V(t,z)\,,\\
L^dV(t,z) = \frac{\partial}{\partial t}V(t,z) - z\lambda(t,S)\frac{\partial}{\partial z}V(t,z)\,,\\
L^{\alpha}V(t,z) = zW(t,z)\frac{\partial}{\partial z}V(t,z)\ .
\end{cases}
\tag{5.3.12}
$$

Proof. For $t\in[T_{j-1},T_j),\ j\geq 1$, from Definition 5.3.1, the nature of S and x in (5.2.17), and applying the Itô-Doob stochastic differential formula to z [33] and Corollary 5.3.1, we have

$$
\begin{aligned}
\mathrm{d}z &= \mathrm{d}(xS) = x\mathrm{d}S + S\mathrm{d}x + (\mathrm{d}x)(\mathrm{d}S)\\
&= x\left[-S\lambda(t^-,S)\mathrm{d}t + S\sigma(t,S)\mathrm{d}w\right] + SxW(t^-,Sx)\mathrm{d}\eta + xW(t^-,(Sx))\mathrm{d}\eta[-S\lambda(t,S)\mathrm{d}t + S\sigma(t,S)\mathrm{d}w]\\
&= -z\lambda(t,S)\mathrm{d}t + z\sigma(t,S)\mathrm{d}w + zW(t^-,z)\mathrm{d}\alpha,\ \text{for}\,(t,z)\in[T_{j-1},T_j)\times\mathbb{R}.
\end{aligned}
\tag{5.3.13}
$$

This establishes the first component of the continuous-time dynamic subsystems in (5.3.11). The proof of the iterative processes z in (5.3.11) is outlined below.

Employing Definition 5.3.1, Remark 5.3.2, and (5.3.8), we have $z(T_j^-) = x(T_j^-, T_{j-1}, z_{j-1})S(T_j^-, T_{j-1}, S_{j-1})$ and $z(T_j^+) = S(T_j^+)x(T_j^+)$. $x(T_j^-)$ and $S(T_j^-)$ are as defined in (5.2.17). From the discrete-time dynamic of population/species x, (5.3.10), survival state process S in (5.2.17) and its continuity together with $S(T_j^-)\approx S(u)\approx S(T_j^+)$ for $T_j^-\leq u\leq T_j^+$, we have

$$
\begin{aligned}
x_jS_j &= S(T_j^-)\left[x(T_j^-) + \gamma_j^{no} + \gamma_j^{o}\right]\\
&= z(T_j^-) + \gamma_j^{no} + \gamma_j^{o}.
\end{aligned}
\tag{5.3.14}
$$

Using Lemma 5.3.1, the proofs of continuous and discrete-time dynamic process z in (5.3.11), the proofs of continuous and discrete time generalized transformed dynamic processes $V(t,z)$ in (5.3.11) can be formulated, analogously [31].

This completes the proof of Theorem 5.3.1. □

Example 5.3.1 For $V(t,z)=z^2$, (5.3.11) reduces to:

$$\begin{cases} \mathrm{d}(z^2) = -[2z^2\lambda(t,S) - z^2\sigma^2(t,S)]\mathrm{d}t + 2z^2\sigma(t,S)\mathrm{d}w + z^2W(t^-,z)\mathrm{d}\alpha\,, \\ z_j^2 = (z(T_j^-,T_{j-1},z_{j-1}))^2 + \overline{\frac{\partial V}{\partial z}}(T_j^-,z(T_j^-),\Delta z(T_j))\Delta z(T_j)\,, \end{cases} \tag{5.3.15}$$

where $\overline{\frac{\partial V}{\partial z}}(T_j^-,z(T_j^-),\Delta z(T_j)) = 2(z(T_j^-)+\frac{1}{2}\Delta z(T_j))$.

Example 5.3.2 For $V(t,z)=\ln z$, (5.3.11) becomes:

$$\begin{cases} \mathrm{d}(\ln z) = [-\lambda(t,S) - \frac{1}{2}\sigma^2(t,S)]\mathrm{d}t + \sigma(t,S)\mathrm{d}w + W(t^-,z)\mathrm{d}\alpha\,, \\ \ln z_j = \ln z(T_j^-,T_{j-1},z_{j-1}) + \overline{\frac{\partial V}{\partial z}}(T_j^-,z(T_j),\Delta z(T_j))\Delta z(T_j^-)\,, \end{cases} \tag{5.3.16}$$

where $\overline{\frac{\partial V}{\partial z}}(T_j^-,z(T_j^-),\Delta z(T_j)) = \int_0^1 \frac{\mathrm{d}\theta}{z(T_j^-)+\theta\Delta z(T_j)}$.

In the following, we develop a very general result that provides a theoretical computational tool to determine theoretical algebraic observation equations for a conceptual computation of state and parameter estimates. The proof of the result follows by using the standard mathematical reasoning [34, 35, 42].

Theorem 5.3.2 *Let us assume that the conditions of Theorem 5.3.1 are satisfied. Then transformed discrete-time interconnected theoretical computational dynamic algorithm is described by:*

$$\begin{cases} \Delta z_j = -z_{j-1}\lambda(T_{j-1},S_{j-1})\Delta T_j + z_{j-1}\sigma(T_{j-1},S_{j-1})\Delta w(T_j) + \Gamma_j^{no} + \gamma_j^o\,, z(T_0)=z_0\,, \\ \Delta V(T_j,z_j) = LV(T_{j-1},z_{j-1})\Delta T_j + z(T_{j-1})\sigma(T_{j-1},S_{j-1})\frac{\partial}{\partial z}V(T_{j-1},z_{j-1})\Delta w(T_j) + \Gamma_j^{nov} + \gamma_j^{ov}\,, \\ \Delta S_j = -S_{j-1}\lambda(T_{j-1},S_{j-1})\Delta T_j + S_{j-1}\sigma(T_{j-1},S_{j-1})\Delta w(T_j),\ S(T_0)=S_0,\ j\in I(1,k)\,, \end{cases} \tag{5.3.17}$$

and moreover

$$\begin{cases} \mathbb{E}(\Delta z_j \mid \mathcal{G}_{j-1}) = -z_{j-1}\lambda(T_{j-1},S_{j-1})\Delta T_j + \Gamma_j^{no} + \gamma_j^o\,, \quad z(T_0)=z_0\,, \\ \mathbb{E}\left[(\Delta z_j - \mathbb{E}(\Delta z_j \mid \mathcal{G}_{j-1}))^2 \mid \mathcal{G}_{j-1}\right] = \sigma^2(T_{j-1},S_{j-1})z_{j-1}^2\Delta T_j\,, \\ \mathbb{E}[\Delta V(T_j,z_j) \mid \mathcal{G}_{j-1}] = LV(T_{j-1},z_{j-1})\Delta T_j + \Gamma_j^{nov} + \gamma_j^{ov}\,, \\ \mathbb{E}\left[(\Delta V(T_j,z_j) - \mathbb{E}(\Delta V(T_j,z_j) \mid \mathcal{G}_{j-1}))^2 \mid \mathcal{G}_{j-1}\right] = \sigma_{j-1}^2 z_{j-1}^2\left(\frac{\partial}{\partial z}V(T_{j-1},z_{j-1})\right)^2\Delta T_j\,, \end{cases} \tag{5.3.18}$$

where $\Delta z_j = z_j - z_{j-1}$; $\Delta z(T_{jk_l}) = z(T_{jk_l}^+) - z(T_{jk_l}^-) = \gamma_{jk_l}^{no}$ *and* $\Delta z(T_j) = z(T_j^+) - z(T_j^-) = \gamma_j^{no} + \gamma_j^o$ *are jumps at* T_{jk_l} *for* $T_{jk_l} \in (T_{j-1},T_j]$ *and* T_j*, respectively; the total jump* $\sum_{l=1}^{\infty}\Delta z(T_{jk_l})$ *and a continuous-time change of survivals,* $z_{j-1}W(T_{j-1},z_{j-1})\Delta\alpha(T_j)$ *over the* j*-th interval of observation* $[T_{j-1},T_j)$ *are given by:*

$$\Gamma_j^{no} + \gamma_j^o = \int_{T_{j-1}}^{T_j^-} z(s)W(s,z(s))\mathrm{d}\alpha(s) + \gamma_j^{no} + \gamma_j^o = \sum_{l=1}^{\infty}\Delta z(T_{jk_l}) + z_{j-1}W(T_{j-1},z_{j-1})\Delta\alpha(T_j), \tag{5.3.19}$$

where $\gamma_{jk_l}^{no}$, Γ_j^{no} denote number of survivals not currently under observation; γ_j^{o} stands for number of survivals under observation; moreover, $\Gamma_j^{no}+\gamma_j^{o}$ represents a change in survival state due to either censored/admitted/birth/natural death/ immigration/emigration process and their combinations; for $T_{jk_l}\in(T_{j-1},T_j)$, $\Delta V(T_{jk_l}) = V(T_{jk_l}^{+}, z(T_{jk_l}^{+})) - V(T_{jk_l}^{-}, z(T_{jk_l}^{-})) = \gamma_{jk_l}^{nov}$ and $\Delta V(T_j) = V(T_j^{+}, z(T_j^{+})) - V(T_j^{-}, z(T_j^{-})) = \gamma_j^{ov}$ stand for a jumps of V at T_{jk_l} and T_j, respectively; the overall jump of V and a continuous-time change of survivals on the j-th interval of observation $[T_{j-1},T_j)$ is as:

$$\Gamma_j^{nov}+\gamma_j^{nov} = \int_{T_j^-1}^{T_j} L^{\alpha}V(s,z(s))\mathrm{d}\alpha(s) + \sum_{l=1}^{\infty}\gamma_{jk_l}^{nov} + \gamma_j^{ov} = \sum_{l=1}^{\infty}\Delta V(T_{jk_l}) + L^{\alpha}(T_{j-1},z_{j-1})\Delta\alpha(T_j), \quad (5.3.20)$$

where $\gamma_{jk_l}^{nov}$ and γ_j^{ov} stand for number of survivals not observed under the transformation V and γ_j^{ov} denotes the number of observed survivals; furthermore, $\Gamma_j^{nov}+\gamma_j^{ov}$ represents the transformed change in survival state due to either censored/admitted/birth/natural death/immigration /emigration process and their combinations; $\Delta T_j = T_j - T_{j-1}$, $\Delta w(T_j) = w(T_j) - w(T_{j-1})$ for $j \in I(1,k)$; $\mathcal{G}_{T_{j-1}} = \mathcal{G}_{j-1}$ is the joint filtration of dynamic process up to time T_{j-1} and intervention/observation processes at T_j^{+}.

Proof. Using The Euler-Maruyama-type numerical schemes [29] for survival state interconnected large-scale dynamic system (5.3.11) over the j-th observation time interval $[T_{j-1},T_j)$, Corollary 5.3.1 and employing the standard arguments, we obtain

$$\begin{cases} \Delta z_j = -z_{j-1}\lambda(T_{j-1},S_{j-1})\Delta T_j + z_{j-1}\sigma(T_{j-1},S_{j-1})\Delta w(T_j) + z_{j-1}W(T_{j-1},z_{j-1})\Delta\alpha(T_j) + \gamma_j^{no} + \gamma_j^{o}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad z(T_0) = z_0\,, \\ \Delta V(T_j,z_j) = LV(T_{j-1},z_{j-1})\Delta T_j + z(T_{j-1})\sigma(T_{j-1},S_{j-1})\frac{\partial}{\partial z}V(T_{j-1},z_{j-1})\Delta w(T_j) \\ \qquad\qquad\qquad +L^{\alpha}(T_{j-1},z_{j-1})\Delta\alpha(T_j) + \gamma_j^{nov} + \gamma_j^{ov}\,, \\ \Delta S_j = -S_{j-1}\lambda(T_{j-1},S_{j-1})\Delta T_j + S_{j-1}\sigma(T_{j-1},S_{j-1})\Delta w(T_j),\ S(T_0) = S_0,\ j \in I(1,k)\,. \end{cases} \quad (5.3.21)$$

From (5.3.19) and (5.3.20), (5.3.21) reduces to (5.3.17). Moreover, from (5.3.17), (5.3.18) follows, immediately. This completes the proof of the theorem.

□

In the following, we apply Theorem 5.3.2 to Examples 5.3.1 and 5.3.2. The developed results will be used, subsequently.

EXAMPLE 5.3.3 For V in Example 5.3.1, using (5.3.15), the discrete-time system (5.3.18) reduces to:

$$
\begin{cases}
\mathbb{E}(\Delta z_j|\,\mathcal{G}_{j-1}) = -z_{j-1}\lambda(T_{j-1},S_{j-1})\Delta T_j + \Gamma_j^{no} + \gamma_j^{o} \ \ , z(T_0) = z_0\,, \\
\mathbb{E}\left[(\Delta z_j - \mathbb{E}(\Delta z_j \mid \mathcal{G}_{j-1}))^2 \mid \mathcal{G}_{j-1}\right] = \sigma^2(T_{j-1},S_{j-1})z_{j-1}^2\Delta T_j\,, \\
\mathbb{E}[\Delta(z_j^2) \mid \mathcal{G}_{j-1}] = \left[-2\lambda(T_{j-1},S_{j-1}) + \sigma^2(T_{j-1},S_{j-1})\right] z_{j-1}^2\Delta T_j + \Gamma_j^{nov} + \gamma_j^{ov}\,, \\
\mathbb{E}\left[\left(\Delta z_j^2 - \mathbb{E}(\Delta z_j^2) \mid \mathcal{G}_{j-1})\right)^2 \mid \mathcal{G}_{j-1}\right] = 4\sigma^2(T_{j-1},S_{j-1})z_{j-1}^4\Delta T_j\,,
\end{cases}
\tag{5.3.22}
$$

where $\Gamma_j^{nov}, \Gamma_j^{no}$, γ_j^{ov}, and γ_j^{o} are defined in (5.3.19) and (5.3.20) in the context of V in Example 5.3.1.

Example 5.3.4 For V in Example 5.3.2, the system of observation equations in (5.3.18) becomes:

$$
\begin{cases}
\mathbb{E}(\Delta z_j \mid \mathcal{G}_{j-1}) = -z_{j-1}\lambda(T_{j-1},S_{j-1})\Delta T_j + \Gamma_j^{no} + \gamma_j^{o}\,, \quad z(T_0) = z_0\,, \\
\mathbb{E}\left[(\Delta z_j - \mathbb{E}(\Delta z_j \mid \mathcal{G}_{j-1}))^2 \mid \mathcal{G}_{j-1}\right] = \sigma^2(T_{j-1},S_{j-1})z_{j-1}^2\Delta T_j\,, \\
\mathbb{E}[\Delta\ln(z_j) \mid \mathcal{G}_{j-1}] = -\left[\lambda(T_{j-1},S_{j-1}) + \frac{1}{2}\sigma^2(T_{j-1},S_{j-1})\right]\Delta T_j + \Gamma_j^{nov} + \gamma_j^{ov}\,, \\
\mathbb{E}\left[(\Delta\ln(z_j) - \mathbb{E}(\Delta\ln(z_j) \mid \mathcal{G}_{j-1}))^2 \mid \mathcal{G}_{j-1}\right] = \sigma^2(T_{j-1},S_{j-1})\Delta T_j\,,
\end{cases}
\tag{5.3.23}
$$

where $\Gamma_j^{nov}, \Gamma_j^{no}$, γ_j^{ov}, and γ_j^{o} are determined by (5.3.19) and (5.3.20) in the context of V in Example 5.3.2.

Remark 5.3.4 (i) In order to identify and illustrate the role and scope of our presented study, we specify the following structure of Riemann-Stieltjes ordinary differential equation in (5.2.17):

$$
\mathrm{d}z = zW_a(t,z)\mathrm{d}\eta_a + zW_b(t,z)\mathrm{d}\eta_b + zW_i(t,z)\mathrm{d}\eta_i + zW_d(t,z)\mathrm{d}\eta_d + zW_e(t,z)\mathrm{d}\eta_e + zW_l(t,z)\mathrm{d}\eta_l + zW_o(t,z)\mathrm{d}\eta_o\,,
\tag{5.3.24}
$$

where a, b, i, d, e, l, and o stand for arrivals/admitted, natural birth, immigration, natural death, emigration, leaving and observation, respectively; $W_a, W_b, W_i, W_d, W_e, W_l$, and W_o are corresponding rate functions; $\eta_a, \eta_b, \eta_i, \eta_d, \eta_e, \eta_l$, and η_o are corresponding cumulative probability distribution or increasing functions. Under this type of structural considerations, the structure of γ_j in Lemma 5.3.2 and in general under transformation γ_j^v are represented by $\gamma_j = \gamma_j^a + \gamma_j^b + \gamma_j^i - \gamma_j^d - \gamma_j^e - \gamma_j^l - \gamma_j^o$ and $\gamma_j^v = \gamma_j^{av} + \gamma_j^{bv} + \gamma_j^{iv} - \gamma_j^{dv} - \gamma_j^{ev} - \gamma_j^{lv} - \gamma_j^{ov}$, where $\gamma_j^a, \gamma_j^b, \gamma_j^i, \gamma_j^d, \gamma_j^e, \gamma_j^l, \gamma_j^o, \gamma_j^{av}, \gamma_j^{bv}, \gamma_j^{iv}, \gamma_j^{dv}, \gamma_j^{ev}, \gamma_j^{lv}$, and γ_j^{ov} are non-negative integers.
(ii) Moreover, for the comparison of the presented approach with the existing methods in the time-to-event statistical data analysis, we further represent the structure of γ_j and γ_j^v as follows: $\gamma_j = \gamma_j^{no} + \gamma_j^o$ and $\gamma_j^v = \gamma_j^{nov} + \gamma_j^{vo}$, where $\gamma_j^a = \gamma_j^{na} + \gamma_j^{oa}; \gamma_j^{av} = \gamma_j^{nav} + \gamma_j^{oav}$; γ_j^{no} and γ_j^{nov} denote the total number of data sizes that are not under the observation/study corresponding to the overall γ_j and γ_j^v data sizes, respectively; γ_j^a and γ_j^{av} are composed of non-observed γ_j^{na}, γ_j^{nov} and observed $\gamma_j^{oa}, \gamma_j^{oav}$ data, respectively; $\gamma_j^{oo} = \gamma_j^{oa} - \gamma_j^o$ and $\gamma_j^{oov} = \gamma_j^{oav} - \gamma_j^{ov}$; γ_j^o is composed of failure or right-censored data representing of number of failure γ_j^f and censored γ_j^c data. It is hoped that in the 21-st century and beyond, this type of structural representation would play a very significant role in studying time-to-event dynamic processes. In fact, this representation allows one to investigate the effectiveness, efficiency, measure, change, etc of treatments, and taking administrative actions or making intervention processes.

In the following section, we establish theoretical discrete-time conceptual computational parameter and state estimation algorithms.

5.4 Theoretical/Conceptual Parameter and State Estimations

For the sake of completeness, we recall a few definitions [6]. These definitions will be utilized for developing the conceptual parameter and state estimations. The presented work is not limited to a particular pool of objects/subjects in time-to-event dynamic processes in biological, chemical, engineering, medical, economic, financial, and social sciences. Moreover, the current study of time-to-event dynamic processes is treated as open dynamic processes. This allows us to expand the role and scope of time-to-event dynamic processes beyond the processes in engineering and medical sciences. In the light of this, the population under consideration of study is grouped into two categories, namely, (1) the sub-population under study/observation/supervision, and (2) a remaining part of population not currently considered under study/observations. The study allows the members of these sub-population groups to move from one group into the other. It is assumed that the overall size of the population of time-to-event dynamic process is $n = n_0 + n_n$, where n_o and n_n stand for the total overall sizes of the sub-populations under observation/study and not under observation/study at an initial time T_0, respectively. The study is considered to be over an interval of time $[T_0, \mathfrak{T})$.

Now, for the sake of completeness, we outline a few definitions [6] that will be used, subsequently.

Definition 5.4.1 For $j \in I(1, k)$, let T_{j-1} and T_j be consecutive data observation/supervision times of joint population/objects/entities and state survival dynamic process. A parameter estimate at T_j is defined by the quotient of change of entities/objects over the consecutive change time subinterval $[T_{j-1}, T_j)$ and the total time spent by the entities/objects under observation/supervision over the subinterval $[T_{j-1}, T_j)$ of length $\Delta T_j = T_j - T_{j-1}$.

Definition 5.4.2 Let $\{z_{j-1}\}_{j=1}^{k}$ be an overall sequence of transformed conceptual state data set with respect to the conceptual state data collection/observation time sequence $\{T_{j-1}\}_{j=1}^{k}$, and let $\{T^{f}_{j-1i-1}\}_{i=1}^{k_f}$, $\{T^{c}_{j-1l-1}\}_{l=1}^{k_c}$ and $\{T^{a}_{j-1m-1}\}_{m=1}^{k_a}$ be overall increasing conceptual failure, censored and admitted subsequences of the overall conceptual data collection time sequence $\{T_{j-1}\}_{j=1}^{k}$, respectively. Three subsequences of the overall conceptual state data sequence $\{z_{j-1}\}_{j=1}^{k}$ associated with the three overall conceptual subsequences of failure, censored and admitted time subsequences are represented by:

$$\{z^{f}_{j-1i-1}\}_{i=1}^{k_f}, \quad \{z^{c}_{j-1l-1}\}_{l=1}^{k_c}, \quad \text{and} \quad \{z^{a}_{j-1m-1}\}_{m=1}^{k_a}, \tag{5.4.1}$$

respectively. These conceptual state data subsequences are called conceptual failure, censored and admitted state subsequences of $\{z_{j-1}\}_{j=1}^{k}$, respectively. We note that $k_f + k_c + k_a = k$.

Definition 5.4.3 The union of the boundary point set of the interval $[t_0, \mathfrak{T})$ and the range of the overall failure subsequence $\{T^f_{j-1i-1}\}_{i=1}^{k_f+1}$ constitutes a partition of the interval $[t_0, \mathfrak{T}), \mathfrak{T} \le \infty$. This partition of $[t_0, \mathfrak{T}), \mathfrak{T} \le \infty$ is termed as the overall conceptual failure-time partition of $[t_0, \mathfrak{T})$, and it is denoted by (P^f).

Definition 5.4.4 For $j \in I(1,k)$ and any consecutive pair $(T^f_{j-1i-1}, T^f_{j-1i})$ of conceptual failure-times for $i \in I(1,k_f)$ under the notations $T^f_{j-100} = T^f_{j-1}$ for $i = 1$ and either $l = 1$ or $m = 1$; furthermore, $T^f_{000} = T_0$ if $i = j = 1$; either $T^f_{j-1k_{c_i}} = T^f_{j-1i}$ for $l = 1+k_{c_i}, i = 2$ or $T^f_{j-1k_{a_i}} = T^f_{j-1i}$ depending on whether $l = k_{c_i}+1$ and $i = 2$ or $m = k_{a_i}+1$ and $i = 2$; a ji-th consecutive conceptual failure-time subinterval is $[T^f_{j-1i-1}, T^f_{j-1i})$ for $i \in I(1,k_f)$. In addition, the conceptual transformed state data associated with the consecutive conceptual initial failure-times is denoted by $z^f_{j-100} = z^f_{j-1}$ and for $j = 1$, $z^f_{1-10} = z^f_{000} = z^f_0$.

Definition 5.4.5 Let $\{z^c_{j-1l-1}\}_{l=1}^{k_c}$ and $\{z^a_{j-1m-1}\}_{m=1}^{k_a}$ be overall censored and admitted conceptual transformed state data subsequences defined in Definition 5.4.2. Let $\{T^c_{j-1i-1p}\}_{p=1}^{k_{c_i}}$ and $\{T^a_{j-1i-1q}\}_{q=1}^{k_{a_i}}$ be conceptual subsequences restricted to the $j-1i$-th consecutive conceptual failure-time subinterval $[T^f_{j-1i-1}, T^f_{j-1i})$ of overall conceptual censored and admitted subsequences $\{T^c_{j-1l-1}\}_{l=1}^{k_c}$ and $\{T^a_{j-1m-1}\}_{m=1}^{k_a}$ of times of the overall sequence $\{T_{j-1}\}_{j=1}^{k}$ of times, respectively. Moreover, the union of the boundary points of $[T^f_{j-1i-1}, T^f_{j-1i})$ and the range of subsequences $\{T^c_{j-1i-1p}\}_{p=1}^{k_{c_i}}$ and $\{t^c_{j-1i-1q}\}_{q=1}^{k_{a_i}}$ form a sub-partition P^f_{j-1} of P^f and the partition of $j-1i$-th subinterval $[T^f_{j-1i-1}, T^f_{j-1i})$. Two subsequences of the overall censored and/or admitted conceptual transformed state data subsequences $\{z^c_{j-1l-1}\}_{l=1}^{k_c}$ and/or $\{z^a_{j-1m-1}\}_{m=1}^{k_a}$ with respect to the two overall conceptual censored and admitted time subsequences of the overall sequence of times $\{[T_{j-1}, T_j)\}_{j=1}^{k}$ restricted to the $j-1i$-th consecutive conceptual failure-time subinterval $[T^f_{j-1i-1}, T^f_{j-1i})$ are represented by:

$$\{z^c_{j-1i-1p-1}\}_{p=1}^{k_{c_i}} \quad \text{and} \quad \{z^a_{j-1i-1q-1}\}_{q=1}^{k_{a_i}}, \tag{5.4.2}$$

respectively. These conceptual transformed state data subsequences are called subsequences of the overall censored and admitted conceptual state data subsequences $\{z^c_{j-1l-1}\}_{l=1}^{k_c}$ and $\{z^a_{j-1m-1}\}_{l=1}^{k_a}$ of the overall conceptual sequence $\{z_{j-1}\}_{j=1}^{k}$ of data set, respectively. We note that $k_c = \sum_{l=1}^{k_c} k_{c_l}$ and $k_a = \sum_{m=1}^{k_a} k_{a_m}$. Moreover, for $p = 1$ and $q = 1$, (5.4.2) reduces to $z^c_{j-1i-10} = z^c_{j-1i-1}$ and $z^a_{j-1i-10} = z^a_{j-1i-1}$, respectively; for $p = k_{c_i} + 2$, and $q = k_{a_i} + 2$, we have $z^c_{j-1i-1k_{c_i}+1} = z^c_{ji}$ and $z^a_{j-1i-1k_{a_i}+1} = z^a_{ji}$, respectively.

In the following, we outline a very general fundamental conceptual results for the development of state data observation system. Observation of dynamic systems are in the frame-work of right-censored data observation process conceptual setting [25, 37].

Lemma 5.4.1 *Let the hypotheses of Theorem 5.3.2 and Remark 5.3.4 be satisfied. From* (5.3.17)*, the transformed discrete-time dynamic observation components are developed below:*

(a) For each $j \in I(1,k)$*, let* T^{fca}_{j-1} *be either failure, censored or admitting time, and* T^f_j *is the failure/death/re-*

moval/ infective/etc observation time. Then $\gamma_j^{oa} = \gamma_j^{o} = \gamma_j^{ov} = \gamma_j^{oav} = 0$*(that is* $\gamma_j^{oo} = 0$*), and*

$$\begin{cases} \mathbb{E}(\Delta z_j | \mathcal{G}_{j-1}) = -z_{j-1}\lambda(T_{j-1}^{fca}, S_{j-1})\Delta T_j^f + \Gamma_j^{no}\,,\; z(T_0) = z_0\,,\; j \in I(1,k)\,, \\ \mathbb{E}\left[(\Delta z_j - \mathbb{E}(\Delta z_j \mid \mathcal{G}_{j-1}))^2 \mid \mathcal{G}_{j-1}\right] = \sigma^2(T_{j-1}^{fca}, S_{j-1}) z_{j-1}^2 \Delta T_j^f\,, \\ \mathbb{E}[\Delta V(T_j^f, z_j) \mid \mathcal{G}_{j-1}] = LV(T_{j-1}^f, z_{j-1})\Delta T_j^f + \Gamma_j^{nov}\,, \\ \mathbb{E}\left[\left(\Delta V(T_j^f, z_j) - \mathbb{E}(\Delta V(T_j^f, z_j) \mid \mathcal{G}_{j-1})\right)^2 \mid \mathcal{G}_{j-1}\right] = \sigma^2(T_{j-1}^{fca}, S_{j-1}) z_{j-1}^2 \left(\frac{\partial}{\partial z} V(T_{j-1}, z_{j-1})\right)^2 \Delta T_j^f\,, \\ \Delta S_j = -S_{j-1}\lambda(T_{j-1}^f, S_{j-1})\Delta T_j^f + S_{j-1}\sigma(T_{j-1}^f, S_{j-1})\Delta w(T_j^f)\,, S(T_0) = S_0\,, \end{cases} \tag{5.4.3}$$

where a pair (T_{j-1}^{fca}, T_j^f) *stands for either* (T_{j-1}^f, T_j^f)*, or* (T_{j-1}^c, T_j^f) *or* (T_{j-1}^a, T_j^f)*;* T_j^f*,* T_{j-1}^c *and* T_{j-1}^a *stand for failure, censored and admitting observation times, respectively;* $\Delta T_j^f = T_j^f - T_{j-1}^{fca}; \Delta w(T_j^f) = w(T_j^f) - w(T_{j-1}^{fca})$*;*

(b) *For each* $j \in I(1,k)$*, let* T_{j-1}^{caf} *be either censored, admitting or failure observation time, and* T_j^c *is a censored/listed observation time. Then* $\gamma_j^{oo} = \gamma_j^c$*,* $\gamma_j^{oov} = \gamma_j^{cv}$*, and*

$$\begin{cases} \mathbb{E}(\Delta z_j | \mathcal{G}_{j-1}) = -z_{j-1}\lambda(T_{j-1}^{caf}, S_{j-1})\Delta T_j^c + \Gamma_j^{no} - \gamma_j^c\,,\; z(T_0) = z_0\,, \\ \mathbb{E}\left[(\Delta z_j - \mathbb{E}(\Delta z_j \mid \mathcal{G}_{j-1}))^2 \mid \mathcal{G}_{j-1}\right] = (\sigma(T_{j-1}^{caf}, S_{j-1}) z_{j-1})^2 \Delta T_j^c\,, \\ \mathbb{E}[\Delta V(T_j^c, z_j) \mid \mathcal{G}_{j-1}] = LV(T_{j-1}^{caf}, z_{j-1})\Delta T_j^c + \Gamma_j^{nov} - \gamma_j^{cv}\,, \\ \mathbb{E}\left[\left(\Delta V(T_j^c, z_j) - \mathbb{E}(\Delta V(T_j^c, z_j)) \mid \mathcal{G}_{j-1})\right)^2 \mid \mathcal{G}_{j-1}\right] = \sigma^2(T_{j-1}^{caf}, S_{j-1}) z_{j-1}^2 \left(\frac{\partial}{\partial z} V(T_{j-1}, z_{j-1})\right)^2 \Delta T_j^c\,, \\ \Delta S_j = -S_{j-1}\lambda(T_{j-1}^f, S_{j-1})\Delta T_j^f + S_{j-1}\sigma(T_{j-1}^f, S_{j-1})\Delta w(T_j^f)\,, S(T_0) = S_0\,, \end{cases} \tag{5.4.4}$$

where a pair (T_{j-1}^{caf}, T_j^c) *stands for either* $(T_{j-1}^c, T_j^c), (T_{j-1}^a, T_j^c)$ *or* (T_{j-1}^f, T_j^c)*;* $\Delta T_j^c = T_j^c - T_{j-1}^{caf}$*;* γ_j^c *stands for the number of censored objects/infectives/quitting/withdrawn/etc observation time* T_j^c*;*

(c) *For each* $j \in I(1,k)$*, let* T_{j-1}^{acf} *be either admitting, censored or failure observation time, and* T_j^a *is a admitting/joining/ recruiting/etc observation time. Then* $\gamma_j^{oo} = \gamma_j^{oa}, \gamma_j^{oov} = \gamma_j^{oav}$*, and*

$$\begin{cases} \mathbb{E}(\Delta z_j | \mathcal{G}_{j-1}) = -z_{j-1}\lambda(T_{j-1}^{acf}, S_{j-1})\Delta T_j^a + \Gamma_j^{no} + \gamma_j^{oa}\,,\; z(T_0) = z_0\,, \\ \mathbb{E}\left[(\Delta z_j - \mathbb{E}(\Delta z_j \mid \mathcal{G}_{j-1}))^2 \mid \mathcal{G}_{j-1}\right] = (\sigma(T_{j-1}^{acf}, S_{j-1}) z_{j-1})^2 \Delta T_j^a\,, \\ \mathbb{E}[\Delta V(T_j^a, z_j) \mid \mathcal{G}_{j-1}] = LV(T_{j-1}^{acf}, z_{j-1})\Delta T_j^c + \Gamma_j^{nov} + \gamma_j^{oav}\,, \\ \mathbb{E}\left[\left(\Delta V(T_j^a, z_j) - \mathbb{E}(\Delta V(T_j^a, z_j) \mid \mathcal{G}_{j-1})\right)^2 \mid \mathcal{G}_{j-1}\right] = \sigma^2(T_{j-1}^{acf}, S_{j-1}) z_{j-1}^2 \left(\frac{\partial}{\partial z} V(T_{j-1}, z_{j-1})\right)^2 \Delta T_j^a\,, \\ \Delta S_j = -S_{j-1}\lambda(T_{j-1}^f, S_{j-1})\Delta T_j^f + S_{j-1}\sigma(T_{j-1}^f, S_{j-1}))\Delta w(T_j^f)\,, S(T_0) = S_0\,, \end{cases} \tag{5.4.5}$$

where a pair (T_{j-1}^{acf}, T_j^a) *belongs to a set:* $(T_{j-1}^{acf}, T_j^a) \in \{(T_{j-1}^a, T_j^a), (T_{j-1}^c, T_j^a), (T_{j-1}^f, T_j^a)\}$*;* $\Delta T_j^a = T_j^a - T_{j-1}^{acf}$*;* γ_j^a *stands for the conceptual number of objects/infectives/etc arriving/joining observation time* T_j^a*.*

Proof. Employing (5.3.17), (5.3.18) and Remark 5.3.4 in the context of right-censored data collection process, the proofs of (a), (b), and (c) can be easily constructed. The details are left to the reader. □

Using Examples 5.3.1 and 5.3.2, the developed conceptual results in Lemma 5.4.1 are illustrated.

EXAMPLE 5.4.1 For V in Example 5.3.1, the systems of observation equations (5.4.3), (5.4.4), and (5.4.5) reduces to:

$$\begin{cases} \mathbb{E}(\Delta z_j | \mathcal{G}_{j-1}) = -z_{j-1}\lambda(T^{fca}_{j-1}, S_{j-1})\Delta T^f_j + \Gamma^{no}_j \ \ , z(T_0) = z_0 \,, \\ \mathbb{E}\left[(\Delta z_j - \mathbb{E}(\Delta z_j \mid \mathcal{G}_{j-1}))^2 \mid \mathcal{G}_{j-1}\right] = \sigma^2(T^{fca}_{j-1}, S_{j-1}) z^2_{j-1}\Delta T^f_j \,, \\ \mathbb{E}[\Delta z^2_j \mid \mathcal{G}_{j-1}] = \left[-2\lambda(T^{fca}_{j-1}, S_{j-1}) + \sigma^2(T^{fca}_{j-1}, S_{j-1})\right] z^2_{j-1}\Delta T^f_j + \Gamma^{nov}_j \,, \\ \mathbb{E}\left[\left(\Delta z^2_j - \mathbb{E}(\Delta z^2_j \mid \mathcal{G}_{j-1})\right)^2 \mid \mathcal{G}_{j-1}\right] = 4\sigma^2(T^{fca}_{j-1}, S_{j-1}) z^4_{j-1}\Delta T^f_j \,, \\ \Delta S_j = -S_{j-1}\lambda(T^f_{j-1}, S_{j-1})\Delta T^f_j + S_{j-1}\sigma(T^f_{j-1}, S_{j-1})\Delta w(T^f_j)\,, S(T_0) = S_0 \,, \end{cases} \tag{5.4.6}$$

$$\begin{cases} \mathbb{E}(\Delta z_j | \mathcal{G}_{j-1}) = -z_{j-1}\lambda(T^{caf}_{j-1}, S_{j-1})\Delta T^c_j + \Gamma^{no}_j - \gamma^c_j \ \ , z(T_0) = z_0 \,, \\ \mathbb{E}\left[(\Delta z_j - \mathbb{E}(\Delta z_j \mid \mathcal{G}_{j-1}))^2 \mid \mathcal{G}_{j-1}\right] = \sigma^2(T^{caf}_{j-1}, S_{j-1}) z^2_{j-1}\Delta T^c_j \,, \\ \mathbb{E}[\Delta z^2_j \mid \mathcal{G}_{j-1}] = \left[-2\lambda(T^{caf}_{j-1}, S_{j-1}) + \sigma^2(T^{caf}_{j-1}, S_{j-1})\right] z^2_{j-1}\Delta T^c_j + \Gamma^{nov}_j - \gamma^{cv}_j \,, \\ \mathbb{E}\left[\left(\Delta z^2_j - \mathbb{E}(\Delta z^2_j \mid \mathcal{G}_{j-1})\right)^2 \mid \mathcal{G}_{j-1}\right] = 4\sigma^2(T^{caf}_{j-1}, S_{j-1}) z^4_{j-1}\Delta T^c_j \,, \\ \Delta S_j = -S_{j-1}\lambda(T^f_{j-1}, S_{j-1})\Delta T^f_j + S_{j-1}\sigma(T^f_{j-1}, S_{j-1})\Delta w(T^f_j)\,, S(T_0) = S_0 \,, \end{cases} \tag{5.4.7}$$

and

$$\begin{cases} \mathbb{E}(\Delta z_j | \mathcal{G}_{j-1}) = -z_{j-1}\lambda(T^{acf}_{j-1}, S_{j-1})\Delta T^c_j + \Gamma^{no}_j + \gamma^{oa}_j \,, \ z(T_0) = z_0 \,, \\ \mathbb{E}\left[(\Delta z_j - \mathbb{E}(\Delta z_j \mid \mathcal{G}_{j-1}))^2 \mid \mathcal{G}_{j-1}\right] = \sigma^2(T^{acf}_{j-1}, S_{j-1}) z^2_{j-1}\Delta T^a_j \,, \\ \mathbb{E}[\Delta z^2_j \mid \mathcal{G}_{j-1}] = \left[-2\lambda(T^{acf}_{j-1}, S_{j-1}) + \sigma^2(T^{acf}_{j-1}, S_{j-1})\right] z^2_{j-1}\Delta T^a_j + \Gamma^{nov}_j + \gamma^{oav}_j \,, \\ \mathbb{E}\left[\left(\Delta z^2_j - \mathbb{E}(\Delta z^2_j) \mid \mathcal{G}_{j-1})\right)^2 \mid \mathcal{G}_{j-1}\right] = 4\sigma^2(T^{acf}_{j-1}, S_{j-1}) z^4_{j-1}\Delta T^a_j \,, \\ \Delta S_j = -S_{j-1}\lambda(T^f_{j-1}, S_{j-1})\Delta T^f_j + S_{j-1}\sigma(T^f_{j-1}, S_{j-1})\Delta w(T^f_j)\,, S(T_0) = S_0 \,, \end{cases} \tag{5.4.8}$$

respectively.

EXAMPLE 5.4.2 For V in Example 5.3.2, the systems of observation equations (5.4.3), (5.4.4), and (5.4.5) becomes

$$\begin{cases}\mathbb{E}(\Delta z_j \mid \mathcal{G}_{j-1}) = -z_{j-1}\lambda(T^{fca}_{j-1}, S_{j-1})\Delta T^f_j + \Gamma^{no}_j\,, \quad z(T_0) = z_0\,,\\ \mathbb{E}\left[(\Delta z_j - \mathbb{E}(\Delta z_j \mid \mathcal{G}_{j-1}))^2 \mid \mathcal{G}_{j-1}\right] = \sigma^2(T^{fca}_{j-1}, S_{j-1}) z^2_{j-1}\Delta T^f_j\,,\\ \mathbb{E}[\Delta \ln(z_j) \mid \mathcal{G}_{j-1}] = -\left[\lambda(T^{fca}_{j-1}, S_{j-1}) + \frac{1}{2}\sigma^2(T^{fca}_{j-1}, S_{j-1})\right]\Delta T^f_j + \Gamma^{nov}_j\,,\\ \mathbb{E}\left[(\Delta \ln(z_j) - \mathbb{E}(\Delta \ln(z_j) \mid \mathcal{G}_{j-1}))^2 \mid \mathcal{G}_{j-1}\right] = \sigma^2(T^{fca}_{j-1}, S_{j-1})\Delta T^f_j\,,\\ \Delta S_j = -S_{j-1}\lambda(T^f_{j-1}, S_{j-1})\Delta T^f_j + S_{j-1}\sigma(T^f_{j-1}, S_{j-1})\Delta w(T^f_j)\,,\; S(T_0) = S_0\,,\end{cases} \tag{5.4.9}$$

$$\begin{cases}\mathbb{E}(\Delta z_j \mid \mathcal{G}_{j-1}) = -z_{j-1}\lambda(T^{caf}_{j-1}, S_{j-1})\Delta T^c_j + \Gamma^{no}_j - \gamma^c_j\,, \quad z(T_0) = z_0\,,\\ \mathbb{E}\left[(\Delta z_j - \mathbb{E}(\Delta z_j \mid \mathcal{G}_{j-1}))^2 \mid \mathcal{G}_{j-1}\right] = \sigma^2(T^{caf}_{j-1}, S_{j-1}) z^2_{j-1}\Delta T^c_j\,,\\ \mathbb{E}[\Delta \ln(z_j) \mid \mathcal{G}_{j-1}] = -\left[\lambda(T^{caf}_{j-1}, S_{j-1}) + \frac{1}{2}\sigma^2(T^{caf}_{j-1}, S_{j-1})\right]\Delta T^c_j + \Gamma^{nov}_j - \gamma^{cv}_j\,,\\ \mathbb{E}\left[(\Delta \ln(z_j) - \mathbb{E}(\Delta \ln(z_j) \mid \mathcal{G}_{j-1}))^2 \mid \mathcal{G}_{j-1}\right] = \sigma^2(T^{caf}_{j-1}, S_{j-1})\Delta T^c_j\,,\\ \Delta S_j = -S_{j-1}\lambda(T^f_{j-1}, S_{j-1})\Delta T^f_j + S_{j-1}\sigma(T^f_{j-1}, S_{j-1})\Delta w(T^f_j)\,,\; S(T_0) = S_0\,,\end{cases} \tag{5.4.10}$$

and

$$\begin{cases}\mathbb{E}(\Delta z_j \mid \mathcal{G}_{j-1}) = -z_{j-1}\lambda(T^{acf}_{j-1}, S_{j-1})\Delta T^a_j + \Gamma^{no}_j + \gamma^{oa}_j\,, \quad z(T_0) = z_0\,,\\ \mathbb{E}\left[(\Delta z_j - \mathbb{E}(\Delta z_j \mid \mathcal{G}_{j-1}))^2 \mid \mathcal{G}_{j-1}\right] = \sigma^2(T^{acf}_{j-1}, S_{j-1}) z^2_{j-1}\Delta T^a_j\,,\\ \mathbb{E}[\Delta \ln(z_j) \mid \mathcal{G}_{j-1}] = -\left[\lambda(T^{acf}_{j-1}, S_{j-1}) + \frac{1}{2}\sigma^2(T^{acf}_{j-1}, S_{j-1})\right]\Delta T^a_j + \Gamma^{nov}_j + \gamma^{oav}_j\,,\\ \mathbb{E}\left[(\Delta \ln(z_j) - \mathbb{E}(\Delta \ln(z_j) \mid \mathcal{G}_{j-1}))^2 \mid \mathcal{G}_{j-1}\right] = \sigma^2(T^{acf}_{j-1}, S_{j-1})\Delta T^a_j\,,\\ \Delta S_j = -S_{j-1}\lambda(T^f_{j-1}, S_{j-1})\Delta T^f_j + S_{j-1}\sigma(T^f_{j-1}, S_{j-1})\Delta w(T^f_j)\,,\; S(T_0) = S_0\,,\end{cases} \tag{5.4.11}$$

respectively.

On the basis of the above discussions, we present a very simple result that provides an insight for the understanding of the development of discrete-time conceptual computational dynamic of state and parameter estimation problems. Moreover, the results provide a systematic mathematical basis for the usage of the assumptions of the Principle of Mathematical Induction [32].

LEMMA 5.4.2 *Assume that the conditions of Lemma 5.4.1 are satisfied and let T^f_{j-1} and T^f_j be a pair of consecutive failure/risk/death/etc observation times.*

(a) For $j \in I(1,k)$, T^f_{j-1} and T^f_j are consecutive risk/failure/removal/death/non-operational observation times in $[T_0, \mathfrak{T}), \mathfrak{T} \le \infty$. Then the theoretical/computational parameter estimation algorithm is given by

$$\begin{cases}\mathbb{E}(\Delta z_j | \mathcal{G}_{j-1}) = -z_{j-1}\lambda(T^f_{j-1}, S_{j-1})\Delta T^f_j + \Gamma^{no}_j \;\,, z(T_0) = z_0\,,\\ \mathbb{E}[\Delta V(T^f_j, z_j \mid \mathcal{G}_{j-1}] = \left[L^d V(T^f_{j-1}, z_{j-1}) + \frac{1}{2} z^2_{j-1}\sigma^2(T^f_{j-1}, S_{j-1})\frac{\partial^2}{\partial z^2} V(T^f_{j-1}, z_{j-1})\right]\Delta T^f_j + \Gamma^{nov}_j\\ \Delta S_j = -S_{j-1}\lambda(T^f_{j-1}, S_{j-1})\Delta T^f_j + S_{j-1}\sigma(T^f_{j-1}, S_{j-1})\Delta w(T^f_j)\,,\; S(T_0) = S_0\,;\end{cases} \tag{5.4.12}$$

parameter estimations at T_j^f are determined by:

$$\begin{cases} \hat{\lambda}(T_{j-1}^f, S_{j-1}) = \dfrac{-\mathbb{E}(\Delta z_j|\,\mathcal{G}_{j-1}) + \Gamma_j^{no}}{z_{j-1}\Delta T_j^f}\,, \quad \Delta T_j^f = T_j^f - T_{j-1}^f\,, \\ \\ \hat{\sigma}^2(T_{j-1}^f, S_{j-1}) = 2\left[\dfrac{\mathbb{E}[\Delta V(T_j^f,\, z_j) \mid \mathcal{G}_{j-1}] - \Gamma_j^{nov} - L^d V(T_{j-1}^f, z_{j-1})\Delta T_j^f}{z_{j-1}^2 \frac{\partial^2}{\partial z^2} V(T_{j-1}, z_{j-1})\Delta T_j^f}\right], \end{cases} \tag{5.4.13}$$

where $L^dV(t,z)$ is defined in (5.3.12).

Using parameter estimates in (5.4.13)*, local state estimations on $[T_{j-1}^f, T_j^f)$ are determined by:*

$$\begin{cases} \Delta S_j = -\hat{S}_{j-1}\hat{\lambda}(T_{j-1}^f, \hat{S}_{j-1})\Delta T_j^f + \hat{S}_{j-1}\hat{\sigma}(T_{j-1}^f, \hat{S}_{j-1})\Delta w(T_j^f)\,,\ \hat{S}(T_0) = \hat{S}_0\,, \quad j \in I(1,k), \\ \Delta z_j = -\hat{z}_{j-1}\hat{\lambda}(T_{j-1}^f, \hat{S}_{j-1})\Delta T_j^f + \hat{z}_{j-1}\hat{\sigma}(T_{j-1}^f, \hat{S}_{j-1})\Delta w(T_j^f) + \gamma_j\,,\ \hat{z}(T_0) = \hat{z}_0\,. \end{cases} \tag{5.4.14}$$

Moreover, estimate of solution process (S,z) of interconnected dynamic system (5.3.11) *is represented by:*

$$\begin{cases} \hat{S}(t, T_{j-1}, \hat{S}_{j-1}),\ \hat{S}(T_{j-1}) = \hat{S}_{j-1}, \hat{S}_0 = S(T_0) \quad \text{for } t \in [T_{j-1}^f, T_j^f)\,, \\ \hat{z}(t, T_{j-1}, \hat{z}_{j-1}),\ \hat{z}(T_{j-1}) = \hat{z}_{j-1}, \hat{z}_0 = z(T_0). \end{cases} \tag{5.4.15}$$

(b) *For $j \in I(1,k)$ and $T_{j-1}^f < T_j^c < T_j^f$, where T_j^c is censored time between a pair of consecutive failure observation times T_{j-1}^f and T_j^f in $[T_0, \mathfrak{T}), \mathfrak{T} \le \infty$. Then the theoretical/computational parameter estimation algorithm is described by:*

$$\begin{cases} \mathbb{E}(\Delta z_j|\,\mathcal{G}_{j-1}) = -\lambda(T_{j-1}^f, S_{j-1})\left[z_{j-1}\Delta T_j^{fc} + z(T_j^c)\Delta T_j^{cf}\right] + \Gamma_j^{no} - \gamma_j^c\,, z(T_0) = z_0\,, \\ \mathbb{E}[\Delta V(T_j^f, z_j) \mid \mathcal{G}_{j-1}] = L^dV(T_{j-1}^f, z_{j-1})\Delta T_j^{fc} + L^dV(T_j^c, z(T_j^c))\Delta T_j^{cf} + \Gamma_j^{nov} - \gamma_j^{cv} + \\ \qquad \frac{1}{2}\sigma^2(T_{j-1}, S_{j-1})\left[z_{j-1}^2 \frac{\partial^2}{\partial z^2}V(T_{j-1}^f, z_{j-1})\Delta T_j^{fc} + z^2(T_j^c)\frac{\partial^2}{\partial z^2}V(T_j^c, z(T_j^c))\Delta T_j^{cf}\right], \\ \Delta S_j = -S_{j-1}\lambda(T_{j-1}^f, S_{j-1})\Delta T_j^f + S_{j-1}\sigma(T_{j-1}^f, S_{j-1})\Delta w(T_j^f)\,, S(T_0) = S_0\,; \end{cases} \tag{5.4.16}$$

parameter estimations at T_j^f are described by;

$$\begin{cases} \hat{\lambda}(T_{j-1}^f, S_{j-1}) = \dfrac{-\mathbb{E}(\Delta z_j|\,\mathcal{G}_{j-1}) + \Gamma_j^{no} - \gamma_j^c}{\left[z_{j-1}\Delta T_j^{fc} + z(T_j^c)\Delta T_j^{cf}\right]}\,, \\ \\ \hat{\sigma}^2(T_{j-1}^f, S_{j-1}) = \\ \qquad 2\left[\dfrac{\mathbb{E}[\Delta V(T_j^f, z_j) \mid \mathcal{G}_{j-1}] - \left(L^dV(T_{j-1}^f, z_{j-1})\Delta T_j^{fc} + L^dV(T_j^c, z(T_j^c))\Delta T_j^{cf} + \Gamma_j^{nov} - \gamma_j^{cv}\right)}{z_{j-1}^2\frac{\partial^2}{\partial z^2}V(T_{j-1}, z_{j-1})\Delta T_j^{cf} + z^2(T_j^c)\frac{\partial^2}{\partial z^2}V(T_j^c, z(T_j^c))\Delta T_j^{fc}}\right], \end{cases} \tag{5.4.17}$$

where $\Delta T_j^{fc} = T_j^c - T_{j-1}^f$, $\Delta T_j^{cf} = T_j^f - T_j^c$; $L^dV(t,z)$ *is defined in* (5.3.12).

Using the local parameter estimates in (5.4.17)*, local state estimates of* (5.4.14) *on* $[T_{j-1}^f, T_j^f)$ *are determined. Again, solution process* (S,z) *of* (5.3.11) *are represented as in* (5.4.15).

(c) *For* $j \in I(1,k)$ *and* $T_{j-1}^f < T_j^a < T_j^f$, *where* T_j^a *is joining/admitting time between a pair of consecutive failure observation times* T_{j-1}^f *and* T_j^f *in* $[t_0, \mathfrak{T}), \mathfrak{T} \leq \infty$. *Then the theoretical/computational parameter estimation algorithm is given by:*

$$\begin{cases} \mathbb{E}(\Delta z_j | \mathcal{G}_{j-1}) = -\lambda(T_{j-1}^f, S_{j-1}) \left[z_{j-1}\Delta T_j^{af} + z(T_j^a)\Delta T_j^{fa} \right] + \Gamma_j^{no} + \gamma_j^{oa} \, , z(T_0) = z_0 \, , \\ \mathbb{E}[\Delta V(T_j^f, z_j) \mid \mathcal{G}_{j-1}] = L^dV(T_{j-1}^f, z_{j-1})\Delta T_j^{fa} + L^dV(T_j^a, z(T_j^a))\Delta T_j^{af} + \Gamma_j^{nov} + \gamma_j^{oav} + \\ \qquad \frac{1}{2}\sigma^2(T_{j-1}, S_{j-1}) \left[z_{j-1}^2 \frac{\partial^2}{\partial z^2} V(T_{j-1}^f, z_{j-1})\Delta T_j^{fa} + z^2(T_j^a) \frac{\partial^2}{\partial z^2} V(T_j^a, z(T_j^a))\Delta T_j^{af} \right], \\ \Delta S_j = -S_{j-1}\lambda(T_{j-1}^f, S_{j-1})\Delta T_j^f + S_{j-1}\sigma(T_{j-1}^f, S_{j-1})\Delta w(T_j^f) \, , S(T_0) = S_0 \, ; \end{cases} \tag{5.4.18}$$

parameter estimation are given below:

$$\begin{cases} \hat{\lambda}(T_{j-1}^f, S_{j-1}) = \dfrac{-\mathbb{E}\left[\Delta z(T_j^a) | \mathcal{G}_{j-1}\right] + \Gamma_j^{no} + \gamma_j^{oa}}{\left[z_{j-1}\Delta T_j^{fa} + z(T_j^a)\Delta T_j^{af} \right]} \, , \\ \hat{\sigma}^2(T_{j-1}^f, S_{j-1}) = \\ \qquad 2\left[\dfrac{\mathbb{E}[\Delta V(T_j^f, z(T_j^f)) \mid \mathcal{G}_{j-1}] - \left(L^dV(T_{j-1}^f, z_{j-1})\Delta T_j^{fa} + L^dV(T_j^a, z(T_j^a))\Delta T_j^{af} + \Gamma_j^{nov} + \gamma_j^{oav} \right)}{z_{j-1}^2 \frac{\partial^2}{\partial z^2} V(T_{j-1}^f, z_{j-1})\Delta T_j^{fa} + z^2(T_j^a) \frac{\partial^2}{\partial z^2} V(T_j^a, z(T_j^a))\Delta T_j^{af}} \right], \end{cases} \tag{5.4.19}$$

where $\Delta T_j^{af} = T_j^a - T_{j-1}^f$, $\Delta T_j^{fa} = T_j^f - T_j^a$;$L^dV(t,z)$ *is defined in* (5.3.12).

Using the parameter estimates in (5.4.19)*, local state estimates on* $[T_{j-1}^f, T_j^f)$ *are computed from* (5.4.14).

In addition, state estimates are as described in (5.4.15)*:*

Proof.

(a) Let T_{j-1}^f and T_j^f be two consecutive conceptual failure times. In this case, $k_{c_i} = k_{a_i} = 0$. From Definition 5.4.4, here $i = 1$. Therefore, for the subinterval $[T_{j-1i-1l-1}^f, T_{j-1i}^f), l = i = 1$, and $T_{j-11}^f = T_j^f; T_{j-1}^f = T_{j-100}^f$. Using the theoretical discrete-time iterative scheme (5.3.17), (5.3.12) and (5.4.3), we have

$$\begin{cases} \mathbb{E}(\Delta z_j | \mathcal{G}_{j-1}) = -z_{j-1}\lambda(T_{j-1}^f, S_{j-1})\Delta T_j^f + \Gamma_j^{no} \, , z(T_0) = z_0 \, , \\ \mathbb{E}[\Delta V(T_j^f, z_j) \mid \mathcal{G}_{j-1}] = \left[L^dV(T_{j-1}^f, z_{j-1}) + \frac{1}{2} z_{j-1}^2 \sigma^2(T_{j-1}^f, S_{j-1}) \frac{\partial^2}{\partial z^2} V(T_{j-1}^f, z_{j-1}) \right] \Delta T_j^f + \Gamma_j^{nov} \\ \Delta S_j = -S_{j-1}\lambda(T_{j-1}^f, S_{j-1})\Delta T_j^f + S_{j-1}\sigma(T_{j-1}^f, S_{j-1})\Delta w(T_j^f) \, , S(T_0) = S_0 \, . \end{cases} \tag{5.4.20}$$

From Definition 5.4.1, the validity of (5.4.12) is then established. Solving for λ and using backward substitution process, the validity of (5.4.13) follows immediately.

Now, we use $\lambda = \hat{\lambda}$ and $\sigma = \hat{\sigma}$ determined by (5.4.13) to solve the system in (5.4.14). Moreover, the solution processes S and z in (5.3.11) are estimated by using an initial data and estimated parameters (5.4.15). This completes the proof of (a).

(b) Let T_j^c be a censoring time between two consecutive conceptual risk/failure times, T_{j-1}^f and T_j^f. We consider a partition of a subinterval $[T_{j-1}^f, T_j^f)$ to be $P_{ji}^f = [T_{j-1}^f, T_j^f] : T_{j-1} < T_{j-1}^c < T_j$. In addition, from Definitions 5.4.4 and 5.4.5,$k_{a_i} = 0, k_{c_i} = 1$, and $0 + k_{c_i} + 2 = 3$. Thus, the size of P_{ji}^f is 3. We note that $i = 1$, since $T_{j-1}^f = T_{j-10}^f$ and $T_j^f = T_{j2}^f = T_{j-1k_{c_i}+1}$. Employing Lemma 5.4.1(b) and (a) in the context of $[T_{j-1}^f, T_j^c)$ and $[T_j^c, T_j^f)$, respectively. We note the fact that $[T_{j-1}^f, T_j^f) = [T_{j-1}^f, T_j^c) \cup [T_j^c, T_j^f)$, we have

$$\mathbb{E}(\Delta z_j^{fc} | \mathcal{G}_{j-1}) + \mathbb{E}(\Delta z_j^{cf} | \mathcal{G}_{j-1}) = -z_{j-1}\lambda(T_{j-1}^f, S_{j-1})\Delta T_j^{fc} + \Gamma_j^{no} - \gamma_j^c - \lambda(T_{j-1}^c, S_{j-1}) z(T_j^c)\Delta T_j^{cf}$$

$$\mathbb{E}(\Delta z_j | \mathcal{G}_{j-1}) = -\lambda(T_{j-1}^f, S_{j-1}) \left[z_{j-1}\Delta T_j^{fc} + z(T_j^c)\Delta T_j^{cf} \right] + \Gamma_j^{no} - \gamma_j^c. \tag{5.4.21}$$

By repeating the above argument and using Lemma 5.4.1 (b) and (a), we obtain

$$\begin{aligned}\mathbb{E}[\Delta V(T_j^f, z_j) \mid \mathcal{G}_{j-1}] = &\left[L^d V(T_{j-1}^f, z_{j-1})\Delta T_j^{fc} + L^d V(T_j^c, z(T_j^c))\Delta T_j^{cf} \right] + \\ &\frac{1}{2}\sigma^2(T_{j-1}, S_{j-1}) \left[z_{j-1}^2 \frac{\partial^2}{\partial z^2} V(T_{j-1}^f, z_{j-1})\Delta T_j^{fc} + z^2(T_j^c)\frac{\partial^2}{\partial z^2} V(T_j^c, z(T_j^c))\Delta T_j^{cf} \right] \\ &+ \Gamma_j^{nov} - \gamma_j^{cv} \ .\end{aligned} \tag{5.4.22}$$

Hence

$$\begin{cases} \mathbb{E}(\Delta z_j | \mathcal{G}_{j-1}) = -\lambda(T_{j-1}^f, S_{j-1}) \left[z_{j-1}\Delta T_j^{fc} + z(T_j^c)\Delta T_j^{cf} \right] + \Gamma_j^{no} - \gamma_j^c \,, z(T_0) = z_0 \,, \\ \mathbb{E}[\Delta V(T_j^f, z_j) \mid \mathcal{G}_{j-1}] = L^d V(T_{j-1}^f, z_{j-1})\Delta T_j^{fc} + L^d V(T_j^a, z(T_j^c))\Delta T_j^{cf} + \Gamma_j^{nov} - \gamma_j^{cv} + \\ \qquad \tfrac{1}{2}\sigma^2(T_{j-1}, S_{j-1}) \left[z_{j-1}^2 \tfrac{\partial^2}{\partial z^2} V(T_{j-1}^f, z_{j-1})\Delta T_j^{fc} + z^2(T_j^c)\tfrac{\partial^2}{\partial z^2} V(T_j^c, z(T_j^c))\Delta T_j^{cf} \right] \\ \Delta S_j = -S_{j-1}\lambda(T_{j-1}^f, S_{j-1})\Delta T_j^f + S_{j-1}\sigma(T_{j-1}^f, S_{j-1})\Delta w(T_j^f) \,, S(T_0) = S_0 \,. \end{cases} \tag{5.4.23}$$

First, solving for λ and then using backward substitution process, we determine σ^2. Hence, this establishes (5.4.17). Now, substituting the estimates of λ and σ into the third equation in (5.4.23), the survival state estimate is obtained. This establishes (b). Moreover, using parameters in (5.4.17), solution process (S, z) of (5.3.11) are estimated.

(c) The proof of (c) can be constructed by emulating the proof of (b) with slight modifications.

This establishes proof of the theorem. □

Example 5.4.3 For $V(t,z) = z^2$, (5.4.13), (5.4.17) and (5.4.19) reduce to

$$\begin{cases} \hat{\lambda}(T^f_{j-1}, S_{j-1}) = \dfrac{-\,\mathbb{E}(\Delta z_j \,|\, \mathcal{G}_{j-1}) + \Gamma^{no}_j}{z_{j-1}\Delta T^f_j}\,, \quad \Delta T^f_j = T^f_j - T^f_{j-1}\,, \\ \\ \hat{\sigma}^2(T^f_{j-1}, S_{j-1}) = \dfrac{\mathbb{E}[\Delta(z_j^2) \mid \mathcal{G}_{j-1}] - \Gamma^{nov}_j}{z^2_{j-1}\Delta T^f_j} + 2\hat{\lambda}(T^f_{j-1}, S_{j-1})\,, \end{cases} \tag{5.4.24}$$

$$\begin{cases} \hat{\lambda}(T_{j-1}, S_{j-1}) = \dfrac{-\,\mathbb{E}(\Delta z_j \,|\, \mathcal{G}_{j-1}) + \Gamma^{no}_j - \gamma^c_j}{\left[z_{j-1}\Delta T^{fc}_j + z(T^c_j)\Delta T^{cf}_j\right]}\,, \\ \\ \hat{\sigma}^2(T^f_{j-1}, S_{j-1}) = \dfrac{\mathbb{E}[\Delta(z_j^2) \mid \mathcal{G}_{j-1}] - \Gamma^{nov}_j + \gamma^{cv}_j}{\left[z^2_{j-1}\Delta T^{cf}_j + z^2(T^c_j)\Delta T^{fc}_j\right]} + 2\hat{\lambda}(T_{j-1}, S_{j-1})\,, \end{cases} \tag{5.4.25}$$

and

$$\begin{cases} \hat{\lambda}(T^f_{j-1}, S_{j-1}) = \dfrac{-\,\mathbb{E}\left[\Delta z_j \,|\, \mathcal{G}_{j-1}\right] + \Gamma^{no}_j + \gamma^{oa}_j}{\left[z_{j-1}\Delta T^{fa}_j + z(T^{af}_j)\Delta T^{af}_j\right]}\,, \\ \\ \hat{\sigma}^2(T^f_{j-1}, S_{j-1}) = \dfrac{\mathbb{E}[\Delta(z_j^2) \mid \mathcal{G}_{j-1}] - \Gamma^{nov}_j - \gamma^{oav}_j}{\left[z^2_{j-1}\Delta T^{fa}_j + z^2(T^{af}_j)\Delta T^{af}_j\right]} + 2\hat{\lambda}(T_{j-1}, S_{j-1})\,, \end{cases} \tag{5.4.26}$$

respectively. We note that the parameter estimates in (5.4.24) to (5.4.26) are valid under an approximation assumption of $\overline{\frac{\partial V}{\partial z}}(t^-, z, \Delta z) \approx z$.

Example 5.4.4 For $V(t,z) = \ln z$, (5.4.13), (5.4.17), and (5.4.19) reduce to

$$\begin{cases} \hat{\lambda}(T^f_{j-1}, S_{j-1}) = \dfrac{-\,\mathbb{E}(\Delta z_j \,|\, \mathcal{G}_{j-1}) + \Gamma^{no}_j}{z_{j-1}\Delta T^f_j}\,, \quad \Delta T^f_j = T^f_j - T^f_{j-1}\,, \\ \\ \hat{\sigma}^2(T_{j-1}, S_{j-1}) = -2\left[\dfrac{\mathbb{E}[\Delta \ln(z_j) \mid \mathcal{G}_{j-1}] - \Gamma^{nov}_j}{\Delta T_j} + \hat{\lambda}(T^f_{j-1}, S_{j-1})\right]\,, \end{cases} \tag{5.4.27}$$

$$\begin{cases} \hat{\lambda}(T_{j-1}, \hat{S}(T_{j-1})) = \dfrac{-\,\mathbb{E}(\Delta z_j \,|\, \mathcal{G}_{j-1}) + \Gamma^{no}_j - \gamma^c_j}{\left[z_{j-1}\Delta T^{fc}_j + z(T^c_j)\Delta T^{cf}_j\right]}\,, \\ \\ \hat{\sigma}^2(T_{j-1}, \hat{S}(T_{j-1}) = -2\left[\dfrac{\mathbb{E}[\Delta \ln(z_j) \mid \mathcal{G}_{j-1}] - \Gamma^{nov}_j + \gamma^{cv}_j}{\Delta T^f_j} + \hat{\lambda}(T^f_{j-1}, \hat{S}(T^f_{j-1}))\right]\,, \end{cases} \tag{5.4.28}$$

and

$$\begin{cases} \hat{\lambda}(T^f_{j-1}, S_{j-1}) = \dfrac{-\mathbb{E}\left[\Delta \ln z_j | \mathcal{G}_{j-1}\right] + \Gamma^{no}_j + \gamma^{oa}_j}{\left[z_{j-1}\Delta T^{fa}_j + z(T^{af}_j)\Delta T^{af}_j\right]}\,, \\ \\ \hat{\sigma}^2(T_{j-1}, S_{j-1}) = -2\left[\dfrac{\mathbb{E}[\Delta \ln(z_j) \mid \mathcal{G}_{j-1}] - \Gamma^{nov}_j - \gamma^{oav}_j}{\Delta T^f_j} + \hat{\lambda}(T^f_{j-1}, S_{j-1})\right], \end{cases} \tag{5.4.29}$$

respectively. Again, we note that the parameter estimates in (5.4.27) to (5.4.29) are valid under an approximation assumption of $\frac{\overline{\partial V}}{\partial z}(t^-, z, \Delta z) = \frac{1}{\Delta z}\ln(z + \frac{\Delta z}{z}) \approx \frac{1}{z}$.

In the following, we extend Lemma 5.4.2, for multiple censoring and admitting times between two consecutive failure times.

THEOREM 5.4.1 *Let the hypotheses of Lemma 5.4.2 be satisfied. For each $j \in I(1,k)$, and each $i \in I(1,k_f)$, let T^f_{j-1i-1} and T^f_{j-1i} be consecutive failure times. Let $\{T^c_{j-1i-1p-1}\}_{p=1}^{k_{c_i}+1}$, $\{T^a_{j-1i-1q-1}\}_{q=1}^{k_{a_i}+1}$ be a finite subsequences of censored and admitted time observations, respectively, over a consecutive failure-time observation subinterval $[T^f_{j-1i-1}, T^f_{j-1i})$, where k_{c_i} is the total number of censored objects/species/infective/quitting over the subinterval $[T^f_{j-1i-1}, T^f_{j-1i})$; k_{a_i} is the total number of admitting/entering/ joining/susceptible/etc over the subinterval $[T^f_{j-1i-1}, T^f_{j-1i})$. Γ^{no}_{ji} is the total number of objects/entities not under observation in the study over the subinterval $[T^f_{j-1i-1}, T^f_{j-1i})$. Then the theoretical transformed/computational estimation algorithm and parameter estimation for $\lambda(t, S(t))$ and $\sigma^2(t, S(t))$ at T^f_{j-1i} are determined by :*

$$\begin{cases} \mathbb{E}\left[\Delta z_{j-1i} \mid \mathcal{G}_{j-1i-1}\right] = -\lambda(T^f_{j-1i-1}, S_{j-1i-1})\left[\sum\limits_{l=1}^{k_{b_i}+1} z(T^{c/a}_{j-1i-1l-1})\Delta T^{c/a}_{j-1i-1l}\right] + \Gamma^{no}_{ji} - k_{c_i} + k_{a_i}\,,\ z(t_0) = z_0\,, \\ \mathbb{E}[\Delta V(T^f_{j-1i}, z_{j-1i}) \mid \mathcal{G}_{j-1i-1}] = \sum\limits_{l=1}^{k_{b_i}+1} \frac{\partial}{\partial t} V(T^{c/a}_{j-1i-1l-1}, z(T^{c/a}_{j-1i-1l-1}))\Delta T^{c/a}_{j-1i-1l} - \\ \qquad \lambda(T^f_{j-1i-1}, S_{j-1i-1})\left[\sum\limits_{l=1}^{k_{b_i}+1} z(T^{c/a}_{j-1i-1l-1})\frac{\partial}{\partial z} V(T^{c/a}_{j-1i-1l-1}, z(T^{c/a}_{j-1i-1l-1}))\Delta T^{c/a}_{j-1i-1l}\right] + \\ \qquad \frac{1}{2}\sigma^2(T^f_{j-1i-1}, S_{j-1i-1})\left[\sum\limits_{l=1}^{k_{b_i}+1} z^2(T^{c/a}_{j-1i-1l-1})\frac{\partial^2}{\partial z^2} V(T^{c/a}_{j-1i-1l-1}, z(T^{c/a}_{j-1i-1l-1}))\Delta T^{c/a}_{j-1i-1l}\right] + \\ \qquad \Gamma^{nov}_{ji} - k^{cv}_{c_i} + k^{av}_{a_i}\,, \\ \Delta S_{j-1i} = -S_{j-1i-1}\lambda(T^f_{j-1i-1}, S_{j-1i-1})\Delta T^f_{j-1i} + S_{j-1i-1}\sigma(T^f_{j-1i-1}, S_{j-1i-1})\Delta w(T^f_{j-1i})\,, S(T_0) = S_0\,, \end{cases} \tag{5.4.30}$$

for $i \in I(1,k_f), j \in I(1,k)$;

parameter estimates are represented as:

$$
\begin{cases}
\hat{\lambda}(T^{f}_{j-1i-1}, S_{j-1i-1}) = \dfrac{-\,\mathbb{E}\left[\Delta z_{j-1i} \middle| \mathcal{G}_{j-1i-1}\right] + \Gamma^{no}_{j-1i} - k_{c_i} + k_{a_i}}{\sum\limits_{l=1}^{k_{b_i}+1} z(T^{c/a}_{j-1i-1l-1})\Delta T^{c/a}_{j-1i-1l}}\,,\; t \in [T^{f}_{j-1i-1}, T^{f}_{j-1i}) \\[2em]
\hat{\sigma}^2(T^{f}_{j-1i-1}, S_{j-1i-1}) = \\
2\left[\dfrac{\begin{array}{r}\mathbb{E}[\Delta V(T^{f}_{j-1i},\, z_{j-1i}) \mid \mathcal{G}_{j-1i-1}] - \Gamma^{nov}_{j-1i} + k^{v}_{c_i} - k^{v}_{a_i} - \sum\limits_{l=1}^{k_{b_i}+1} \frac{\partial}{\partial t} V(T^{c/a}_{j-1i-1l-1}, z(T^{c/a}_{j-1i-1l-1}))\Delta T^{c/a}_{j-1i-1l} \\ -\,\lambda(T^{f}_{j-1i-1}, S_{j-1i-1})\left[\sum\limits_{l=1}^{k_{b_i}+1} z(T^{c/a}_{j-1i-1l-1})\frac{\partial}{\partial z} V(T^{c/a}_{j-1i-1l-1}, z(T^{c/a}_{j-1i-1l-1}))\Delta T^{c/a}_{j-1i-1l}\right]\end{array}}{\sum\limits_{l=1}^{k_{b_i}+1} z^2(T^{c/a}_{j-1i-1l-1})\frac{\partial^2}{\partial z^2} V(T^{c/a}_{j-1i-1l-1}, z(T^{c/a}_{j-1i-1l-1}))\Delta T^{c/a}_{j-1i-1l}}\right],
\end{cases}
\tag{5.4.31}
$$

where $k_{b_i} = k_{c_i} + k_{a_i}$.

Moreover an overall conceptual state and parameter estimates for $z(t), S(t)$, $\lambda(t, S(t))$ *and* $\sigma(t, S(t))$ *in* (5.3.11) *on the time-interval of study* $[t_0, \mathfrak{T})$ *are determined by*

$$
\begin{cases}
\hat{\lambda}(t, \hat{S}_{j-1i-1}) = \hat{\lambda}(T^{f}_{j-1i-1}, \hat{S}_{j-1i-1}),\; \textit{for } t \in [T^{f}_{j-1i-1}, T^{f}_{j-1i}),\; j \in I(1,k) \textit{ and } i \in I(1,k_f)\,, \\
\hat{\sigma}(t, \hat{S}_{j-1i-1}) = \hat{\sigma}(T^{f}_{j-1i-1}, \hat{S}_{j-1i-1})\,, \\
\hat{S}(t) = \hat{S}(t, T_{j-1i-1}, \hat{S}_{j-1i-1})\,, \quad S(T_{j-1i-1}) = \hat{S}_{j-1i-1}\,, \\
\hat{z}(t) = \hat{z}(t, T^{f}_{j-1i-1}, \hat{z}(T^{f}_{j-1i-1})).
\end{cases}
\tag{5.4.32}
$$

Proof. From Definitions 5.4.4 and 5.4.5, $l = p = j = i = 1, T^{f}_{000} = T_0$ and $T^{f}_{0i-1k_{b_i}+1} = T^{f}_{01} = T^{f}_{1}$, for $i = 1$, and the application of Lemma 5.4.2, we note that one of the fundamental assumptions of the Principle of Mathematical Induction(PMI) [33] is satisfied. For the validity of the application of PMI, we assume that (5.4.30) and (5.4.31) are valid for some $j-1 \in I(1,k)$. We need to justify the induction hypothesis, that is (5.4.30) and (5.4.31) are satisfied for $j \in I(1,k)$. For this purpose, we note that for $j \in I(1,k)$, each $i \in I(1,k_f)$, and $T^{f}_{j-1i-1}, T^{f}_{j-1i} \in [T_0, \mathfrak{T}]$ with k_{c_i} and k_{a_i} being number of censored and admitted objects/species/subjects over the subinterval $[T^{f}_{j-1i-1}, T^{f}_{j}]$ of consecutive failure times, respectively. Let $\mathcal{P}^{f}_{ji}$ be a partition of $[T^{f}_{j-1i-1}, T^{f}_{j}]$ corresponding to the union of the range of two finite subsequences (censored and admitted times) over the consecutive failure-time subinterval $[T^{f}_{j-1i-1}, T^{f}_{ji})$. These subsequences are represented by

$$
\begin{aligned}
\mathcal{P}^{f}_{j-1i} : T^{f}_{j-1i-11-1} = T^{f}_{j-1i-10} = T^{f}_{j-1i-1} < T^{c/a}_{j-1i-11} < \ldots < T^{c/a}_{j-1i-1l-1} < T^{c/a}_{j-1i-1l} < \ldots \\
< T^{c/a}_{j-1i-1k_{b_i}} < T^{c/a}_{j-1i-1k_{b_i}+1} = T^{f}_{j-1i}\,.
\end{aligned}
\tag{5.4.33}
$$

In short, $\mathcal{P}^f_{ji}$ is a partition of $[T^f_{j-1i-1}, T^f_{j-1i}]$ with the size of the partition $k_{b_i}+2$, and $k_{b_i} = k_{c_i} + k_{a_i}$. For $j \in I(1,k)$ and $i \in I(1,k_f)$, using the iterative schemes (5.4.12), (5.4.16), and (5.4.18) and noting the nature of the processes $\lambda(T^{c/a}_{j-1i-1l-1}, S(T^{c/a}_{j-1i-1l-1})) = \lambda(T^f_{j-1i-i}, S_{j-1i-1})$, $\sigma^2(T^{c/a}_{j-1i-1l-1}, S(T^{c/a}_{j-1i-1l-1})) = \sigma^2(T^f_{j-1i-i}, S_{j-1i-1})$ in the context of Definitions 5.4.4 and 5.4.5 for $l \in I(1,k_{b_i})$, we have

$$\begin{aligned}
\mathbb{E}\left[\Delta z_{j-1i} \mid \mathcal{G}_{j-1}\right] &= -\lambda(T^f_{j-1i-1}, S_{j-1i-1}) z(T^f_{j-1i-1}) \Delta T^{f\,c/a}_{j-1i-10} + \Gamma^{no}_{j-10_i} \mp \gamma^{c/a}_{j-1i-10} \\
&\quad - \sum_{m=2}^{k_{b_i}} \left[\lambda(T^{c/a}_{j-1i-1m-1}, S(T^{c/a}_{j-1i-1m-1})) z(T^{c/a}_{j-1i-1m-1}) \Delta T^{c/a}_{j-1i-1m}\right] + \sum_{l=1}^{k_{b_i}} \Gamma^{no}_{j-10_i} \mp \sum_{l=1}^{k_{b_i}} \gamma^{c/a}_{j-1i-10} \\
&\quad - \lambda(T^{c/a}_{j-1i-1k_{b_i}}, S(T^{c/a}_{j-1i-1k_{b_i}})) z(T^{c/a}_{j-1i-1k_{b_i}}) \Delta T^f_{jik_{b_i}+1} \\
&= -\lambda(T^f_{j-1i-1}, S_{j-1i-1}) \left[\sum_{l=1}^{k_{b_i}+1} z(T^{c/a}_{j-1i-1l-1}) \Delta T^{c/a}_{j-1i-1l}\right] + \sum_{l=1}^{k_{b_i}} \Gamma^{no}_{j-1i-1l-1} \mp \sum_{l=1}^{k_{b_i}} \gamma^{c/a}_{j-1i-1l-1} \\
&= -\lambda(T^f_{j-1i-1}, S_{j-1i-1}) \left[\sum_{l=1}^{k_{b_i}+1} z(T^{c/a}_{j-1i-1l-1}) \Delta T^{c/a}_{j-1i-1l-1}\right] + \Gamma^{no}_{j-1i} \mp \gamma^{c/a}_{j-1i} \\
&= -\lambda(T^f_{j-1i-1}, S_{j-1i-1}) \left[\sum_{l=1}^{k_{b_i}+1} z(T^{c/a}_{j-1i-1l-1}) \Delta T^{c/a}_{j-1i-1l-1}\right] + \Gamma^{no}_{j-1i} - k_{c_i} + k_{a_i}\ .
\end{aligned}$$

Similarly, we find that

$$\begin{aligned}
&\mathbb{E}[\Delta V(T^f_{j-1i}, z_{j-1i}) \mid \mathcal{G}_{j-1i-1}] = \sum_{l=1}^{k_{b_i}+1} \frac{\partial}{\partial t} V(T^{c/a}_{j-1i-1l-1}, z(T^{c/a}_{j-1i-1l-1})) \Delta T^{c/a}_{j-1i-1l} - \\
&\lambda(T^f_{j-1i-1}, S_{j-1i-1}) \left[\sum_{l=1}^{k_{b_i}+1} z(T^{c/a}_{j-1i-1l-1}) \frac{\partial}{\partial z} V(T^{c/a}_{j-1i-1l-1}, z(T^{c/a}_{j-1i-1l-1})) \Delta T^{c/a}_{j-1i-1l}\right] + \\
&\frac{1}{2}\sigma^2(T^f_{j-1i-1}, S_{j-1i-1}) \left[\sum_{l=1}^{k_{b_i}+1} z^2(T^{c/a}_{j-1i-1l-1}) \frac{\partial^2}{\partial z^2} V(T^{c/a}_{j-1i-1l-1}, z(T^{c/a}_{j-1i-1l-1})) \Delta T^{c/a}_{j-1i-1l}\right] + \\
&\sum_{l=1}^{k_{b_i}} \Gamma^{nov}_{j-1i-1l-1} \mp \sum_{l=1}^{k_{b_i}} \gamma^{c/a}_{j-1i-1l-1}\ .
\end{aligned}$$

Hence,

$$\begin{cases}\mathbb{E}\left[\Delta z_{j-1i} \mid \mathcal{G}_{j-1i-1}\right] = -\lambda(T^f_{j-1i-1}, S_{j-1i-1})\left[\sum_{l=1}^{k_{b_i}+1} z(T^{c/a}_{j-1i-1l-1})\Delta T^{c/a}_{j-1i-1l}\right] + \Gamma^{no}_{j_i} - k_{c_i} + k_{a_i}\,,\ z(t_0) = z_0\,,\\ \mathbb{E}[\Delta V(T^f_{j-1i}, z_{j-1i}) \mid \mathcal{G}_{j-1i-1}] = \sum_{l=1}^{k_{b_i}+1} \frac{\partial}{\partial t}V(T^{c/a}_{j-1i-1l-1}, z(T^{c/a}_{j-1i-1l-1}))\Delta T^{c/a}_{j-1i-1l} - \\ \qquad \lambda(T^f_{j-1i-1}, S_{j-1i-1})\left[\sum_{l=1}^{k_{b_i}+1} z(T^{c/a}_{j-1i-1l-1})\frac{\partial}{\partial z}V(T^{c/a}_{j-1i-1l-1}, z(T^{c/a}_{j-1i-1l-1}))\Delta T^{c/a}_{j-1i-1l}\right] + \\ \qquad \frac{1}{2}\sigma^2(T^f_{j-1i-1}, S_{j-1i-1})\left[\sum_{l=1}^{k_{b_i}+1} z^2(T^{c/a}_{j-1i-1l-1})\frac{\partial^2}{\partial z^2}V(T^{c/a}_{j-1i-1l-1}, z(T^{c/a}_{j-1i-1l-1}))\Delta T^{c/a}_{j-1i-1l}\right] + \\ \qquad \Gamma^{nov}_{j_i} - k^{cv}_{c_i} + k^{av}_{a_i}\,,\\ \Delta S_{j-1i} = -S_{j-1i-1}\lambda(T^f_{j-1i-1}, S_{j-1i-1})\Delta T^f_{j-1i} + S_{j-1i-1}\sigma(T^f_{j-i-1}, S_{j-1i-1})\Delta w(T^f_{j-1i})\,, S(T_0) = S_0\,,\end{cases} \tag{5.4.34}$$

This establishes (5.4.30).

Using the backward substitution approach and solving for λ and σ^2 establishes (5.4.31). Moreover,

$$\begin{cases}\hat{\lambda}(t, S_{j-1i-1}) = \hat{\lambda}(T^f_{j-1i-1}, \hat{S}_{j-1i-1}),\ \text{for } t \in [T^f_{j-1i-1}, T^f_{j-1i}),\ j \in I(1,k) \text{ and } i \in I(1,k_f)\,,\\ \hat{\sigma}(t, \hat{S}_{j-1i-1}) = \hat{\sigma}(T^f_{j-1i-1}, \hat{S}_{j-1i-1})\,,\\ \hat{S}(t) = (t, T^f_{j-1i-1}, \hat{S}_{j-1i-1})\,, \quad S(T^f_{j-1i-1}) = \hat{S}_{j-1i-1}\,,\\ \hat{z}(t) = \hat{z}(t, T^f_{j-1i-1}, \hat{z}_{j-1i-1}).\end{cases} \tag{5.4.35}$$

This concludes the proof of the theorem. □

COROLLARY 5.4.1 *Let the hypotheses of Theorem 5.4.1 be satisfied except $k_a = 0 = k_c$. Then the theoretical/conceptual estimation algorithm, parameters, $\lambda(t, S(t))$, $\sigma^2(t, S(t))$, state and solution process estimates are determined by* (5.4.12), (5.4.13), (5.4.14) *and* (5.4.15) *respectively, as a special case of Theorem 5.4.1.*

EXAMPLE 5.4.5 From Lemma 5.4.2, Examples 5.4.1 and 5.4.3, the theoretical transformed/computational estimation algorithms, parameter and state estimations determined by :

$$\begin{cases}\mathbb{E}\left[\Delta z_{j-1i} \mid \mathcal{G}_{j-1i-1}\right] = -\lambda(T^f_{j-1i-1}, S_{j-1i-1})\left[\sum_{l=1}^{k_{b_i}+1} z(T^{c/a}_{j-1i-1l-1})\Delta T^{c/a}_{j-1i-1l}\right] + \Gamma^{no}_{ji} - k_{c_i} + k_{a_i}\,,\ z(t_0) = z_0\,,\\ \mathbb{E}[\Delta(z^2_{j-1i}) \mid \mathcal{G}_{j-1i-1}] = \left[-2\lambda(T^f_{j-1i-1}, S_{j-1i-1}) + \sigma^2(T^f_{j-1i-1}, S_{j-1i-1})\right]\left[\sum_{l=1}^{k_{b_i}+1} z^2(T^{c/a}_{j-1i-1l-1})\Delta T^{c/a}_{j-1i-1l}\right] \\ \qquad + \Gamma^{nov}_{ji} - k^{cv}_{c_i} + k^{av}_{a_i}\,,\\ \Delta S_{j-1i} = -S_{j-1i-1}\lambda(T^f_{j-1i-1}, S_{j-1i-1})\Delta T^f_{j-1i} + S_{j-1i-1}\sigma(T^f_{j-1i-1}, S_{j-1i-1})\Delta w(T^f_{j-1i})\,, S(T_0) = S_0\,,\end{cases} \tag{5.4.36}$$

for $i \in I(1,k_f), j \in I(1,k)$;

parameter estimates are given by:

$$\begin{cases} \hat{\lambda}(T^f_{j-1i-1}, \hat{S}_{j-1i-1}) = \dfrac{-\,\mathbb{E}\left[\Delta z_{j-1i} \middle| \mathcal{G}_{j-1}\right] + \Gamma^{no}_{ji} - k_{c_i} + k_{a_i}}{\sum\limits_{l=1}^{k_{b_i}+1} z(T^{c/a}_{j-1i-1l-1})\Delta T^{c/a}_{j-1i-1l}}\,,\; t \in [T^f_{j-1i-1}, T^f_{j-1i})\,, \\ \\ \hat{\sigma}^2(T^f_{j-1i-1}, \hat{S}_{j-1i-1}) = \dfrac{\mathbb{E}[\Delta(z^2_{j-1i}) \mid \mathcal{G}_{j-1i-1}] + \Gamma^{nov}_{ji} + k^{cv}_{c_i} - k^{av}_{a_i}}{\sum\limits_{l=1}^{k_{b_i}+1} z^2(T^{c/a}_{j-1i-1l-1})\Delta T^{c/a}_{j-1i-1l}} + 2\hat{\lambda}(T^f_{j-1i-1}, \hat{S}_{j-1i-1})\,,\; t \in [T^f_{j-1i-1}, T^f_{j-1i})\,. \end{cases} \tag{5.4.37}$$

Moreover, if $k_a = 0 = k_c$ then the theoretical/conceptual estimation algorithm and parameter estimation for $\lambda(t, S(t))$ and $\sigma(t, S(t))$ at $T^f_{ji} = T^f_j$ reduces to (5.4.12) and (5.4.24) as special cases. An overall conceptual parameter estimate for $z(t), S(t), \lambda(t, S(t))$and $\sigma(t, S(t))$ on the time-interval of study $[T_0, \mathcal{T})$ are determined by (5.4.35).

Example 5.4.6 From Lemma 5.4.2 and Examples 5.4.2 and 5.4.4, the theoretical transformed/computational estimation algorithm and parameter estimation for $\lambda(t, S(t))$ and $\sigma(t, S(t))$ at T^f_{j-1i} are determined by :

$$\begin{cases} \mathbb{E}\left[\Delta z_{j-1i} \mid \mathcal{G}_{j-1i-1}\right] = -\lambda(T^f_{j-1i-1}, S_{j-1i-1}) \left[\sum\limits_{l=1}^{k_{b_i}+1} z(T^{c/a}_{j-1i-1l-1})\Delta T^{c/a}_{j-1i-1l}\right] + \Gamma^{no}_{j_i} - k_{c_i} + k_{a_i}\,,\; z(T_0) = z_0\,, \\ \mathbb{E}[\Delta \ln(z_{j-1i}) \mid \mathcal{G}_{j-1i-1}] = \left[\lambda(T^f_{j-1}, S_{j-1i-1}) - \frac{1}{2}\sigma^2(T^f_{j-1}, S_{j-1i-1})\right] \left[\sum\limits_{l=1}^{k_{b_i}+1} \Delta T^{c/a}_{j-1i-1l}\right] \\ \qquad\qquad + \Gamma^{nov}_{ji} - k^{cv}_{c_i} + k^{av}_{a_i}\,, \\ \Delta S_{j-1i} = -S_{j-1i-1}\lambda(T^f_{j-1i-1}, S_{j-1i-1})\Delta T^f_{j-1i} + S_{j-1i-1}\sigma(T^f_{j-1i-1}, S_{j-1i-1})\Delta w(T^f_{j-1i})\,,\; S(T_0) = S_0\,, \end{cases} \tag{5.4.38}$$

and parameter estimates are as:

$$\begin{cases} \hat{\lambda}(T^f_{j-1i-1}, \hat{S}_{j-1i-1}) = -\dfrac{\mathbb{E}\left[\Delta z_{j-1i} \middle| \mathcal{G}_{j-1}\right] + \Gamma^{nov}_{ji} + k_{c_i} - k_{a_i}}{\sum\limits_{l=1}^{k_{b_i}+1} z(T^{c/a}_{j-1i-1l-1})\Delta(T^{c/a}_{j-1i-1l})}\,,\; T \in [T^f_{j-1i-1}, T^f_{j-1i})\,, \\ \\ \hat{\sigma}^2(T_{j-1i-1}, \hat{S}_{j-1i-1}) = -2\left[\dfrac{\mathbb{E}[\Delta \ln(z_{j-1i}) \mid \mathcal{G}_{j-1i-1}] + \Gamma^{nov}_{ji} + k^{cv}_{c_i} - k^{av}_{a_i}}{\sum\limits_{l=1}^{k_{b_i}+1} \Delta T^{c/a}_{j-1i-1l}} + \hat{\lambda}(T^f_{j-1i-1}, \hat{S}_{j-1i-1})\right]\,, \end{cases} \tag{5.4.39}$$

for $i \in I(1, k_f), j \in I(1, k)$, and $t \in [T^f_{j-1i-1}, T^f_{j-1i})$, where $k_{b_i} = k_{c_i} + k_{a_i}$. Moreover, if $k_a = 0 = k_c$. Then the theoretical/conceptual estimation algorithm and parameter estimation for λ and σ at $T^f_{ji} = T^f_j$ reduces to (5.4.12) and (5.4.27). An overall conceptual parameter estimate for $z(t), S(t), \lambda(t, S(t))$and $\sigma(t, S(t))$ on the time-interval of study $[T_0, \mathcal{T})$ are determined by (5.4.35).

Now, we state a very general theorem that provides a theoretical estimate for $\lambda(t,S)$ and $\sigma(t,S)$ between two consecutive change point times, T^{cp}_{j-1r-1} and T^{cp}_{j-1r}.

THEOREM 5.4.2 *Let the hypotheses of Lemmas 5.4.1 and 5.4.2 be satisfied. For each* $j \in I(1,k)$ *and each* $r \in I(1,n)$, *let* T^{cp}_{j-1r-1} *and* T^{cp}_{j-1r} *be consecutive change point times. Let* $\{T^{f}_{j-1r-1i-1}\}_{i=1}^{k_{f_r}}$, $\{T^{c}_{j-1r-1p-1}\}_{p=1}^{k_{c_r}}$, *and* $\{T^{a}_{j-1r-1q-1}\}_{q=1}^{k_{a_r}}$ *be the a sequence of failure, censored and admission times, respectively, in the* $j-1r$*-th change point time interval* $[T^{cp}_{j-1r-1}, t^{cp}_{j-1r})$. k_{f_r}, k_{c_r}, *and* k_{a_r} *are the total number of failures, censored and admitting items/objects/species/etc in the consecutive change-point subinterval* $[T^{cp}_{j-1r-1}, T^{cp}_{j-1r})$, *respectively.* Γ^{no}_{jr} *is the total number of objects/entities not under observation in the study over the subinterval* $[T^{cp}_{j-1r-1}, T^{cp}_{j-1r})$. *Then the theoretical transformed/computational estimation algorithm and parameter estimation for* $\lambda(t, S(t))$ *and* $\sigma^2(t,S)$ *at* T^{cp}_{j-1r} *are determined by:*

$$\begin{cases}
\mathbb{E}\left[\Delta z_{j-1r} \mid \mathcal{G}_{j-1r-1}\right] = -\lambda(T^{cp}_{j-1r-1}, S_{j-1r-1})\left[\sum_{l=1}^{k_{b_r}+1} z(T^{f/c/a}_{j-1r-1l-1})\Delta T^{f/c/a}_{j-1r-1l}\right] + \Gamma^{no}_{jr} - k_{f_r} - k_{c_r} + k_{a_r}, \\
\qquad z(T_0) = z_0\,, \\
\mathbb{E}[\Delta V(T^{cp}_{j-1r}, z_{j-1r}) \mid \mathcal{G}_{j-1r-1}] = \sum_{l=1}^{k_{b_r}+1} \frac{\partial}{\partial t} V(T^{f/c/a}_{j-1r-1l-1}, z(T^{f/c/a}_{j-1r-1l-1}))\Delta T^{f/c/a}_{j-1r-1l} - \\
\qquad \lambda(T^{cp}_{j-1r-1}, S_{j-1r-1})\left[\sum_{l=1}^{k_{b_r}+1} z(T^{f/c/a}_{j-1r-1l-1})\frac{\partial}{\partial z} V(T^{f/c/a}_{j-1r-1l-1}, z(T^{f/c/a}_{j-1r-1l-1}))\Delta T^{f/c/a}_{j-1i-1l}\right] + \\
\qquad \frac{1}{2}\sigma^2(T^{cp}_{j-1r-1}, S_{j-1r-1})\left[\sum_{l=1}^{k_{b_r}+1} z^2(T^{c/a}_{j-1r-1l-1})\frac{\partial^2}{\partial z^2} V(T^{f/c/a}_{j-1i-1l-1}, z(T^{f/c/a}_{j-1r-1l-1}))\Delta T^{f/c/a}_{j-1r-1l}\right] + \\
\qquad \Gamma^{nov}_{jr} - k^{fv}_{f_r} - k^{cv}_{c_r} + k^{av}_{a_r} \\
\Delta S_{j-1r} = -S_{j-1r-1}\lambda(T^{cp}_{j-1r-1}, S_{j-1r-1})\Delta T^{cp}_{j-1r} + S_{j-1r-1}\sigma(T^{cp}_{j-1r-1}, S_{j-1r-1})\Delta w(T^{cp}_{j-1r})\,, S(T_0) = S_0\,,
\end{cases} \tag{5.4.40}$$

for $r \in I(1,r), j \in I(1,k)$ *; and parameter estimates are as follows:*

$$
\begin{cases}
\hat{\lambda}(T^{cp}_{j-1r-1}, \hat{S}_{j-1r-1}) = -\dfrac{\mathbb{E}\left[\Delta z_{j-1i} \middle| \mathcal{G}_{j-1i-1}\right] + \Gamma^{no}_{jr} + k_{c_r} - k_{a_r}}{\sum\limits_{l=1}^{k_{b_i}+1} z(T^{f/c/a}_{j-1i-1l-1}) \Delta T^{f/c/a}_{j-1i-1l}}\,,\ t \in [T^{cp}_{j-1r-1}, T^{cp}_{j-1r})\,, \\
\\
\hat{\sigma}^2(T^{cp}_{j-1r-1}, S_{j-1r-1}) = \\
2\left[\dfrac{\begin{array}{l}\mathbb{E}[\Delta V(T^{cp}_{j-1r},\, z_{j-1r}) \mid \mathcal{G}_{j-1r-1}] - \Gamma^{nov}_{jr} + k^{fv}_{f_r} + k^{cv}_{c_r} - k^{av}_{a_r} \\ \quad - \sum\limits_{l=1}^{k_{b_r}+1} \frac{\partial}{\partial t} V(T^{f/c/a}_{j-1r-1l-1}, z(T^{f/c/a}_{j-1r-1l-1}))\Delta T^{f/c/a}_{j-1r-1l} \\ \qquad - \lambda(T^{cp}_{j-1r-1}, S_{j-1r-1}) \left[\sum\limits_{l=1}^{k_{b_r}+1} z(T^{f/c/a}_{j-1r-1l-1}) \frac{\partial}{\partial z} V(T^{f/c/a}_{j-1i-1l-1}, z(T^{f/c/a}_{j-1r-1l-1}))\Delta T^{f/c/a}_{j-1i-1l}\right]\end{array}}{\sum\limits_{l=1}^{k_{b_r}+1} z^2(T^{f/c/a}_{j-1r-1l-1})\, \frac{\partial^2}{\partial z^2} V(T^{f/c/a}_{j-1r-1l-1}, z(T^{f/c/a}_{j-1r-1l-1}))\Delta T^{f/c/a}_{j-1r-1l}}\right],
\end{cases}
\tag{5.4.41}
$$

for $t \in [T^{cp}_{j-1r-1}, T^{cp}_{j-1r})$ *, where* $k^v_{b_r} = k^v_{f_r} + k^v_{c_r} + k^v_{a_r}$*.*

Moreover an overall conceptual parameter estimate for $z(t), S(t),\ \lambda(t, S(t))$ *and* $\sigma(t, S)$ *in* (5.3.11) *on the time-interval of study* $[t_0, \mathcal{T})$ *are determined by*

$$
\begin{cases}
\hat{\lambda}(t, \hat{S}_{j-1r-1}) = \hat{\lambda}(T^{cp}_{j-1r-1}, \hat{S}_{j-1r-1}),\ \textit{for}\ t \in [T^{cp}_{j-1r-1}, T^{cp}_{j-1r}),\ j \in I(1,k)\ \textit{and}\ r \in I(1,n)\,, \\
\hat{\sigma}(t, \hat{S}_{j-1r-1}) = \hat{\sigma}(T^{cp}_{j-1r-1}, \hat{S}_{j-1r-1})\,, \\
\hat{S}(t) = \hat{S}(t, T^{cp}_{j-1r-1}, \hat{S}_{j-1r-1}),\ \ \hat{S}(T^{cp}_{j-1r-1}) = S_{j-1r-1}, \\
\hat{z}(t) = \hat{z}(t, T^{cp}_{j-1r-1}, \hat{z}_{j-1r-1}).
\end{cases}
\tag{5.4.42}
$$

Proof. The proof of the theorem follows from the proof of Theorem 5.4.2 with appropriate modifications. □

Chapter 6
Conceptual Computational Algorithms

6.1 Introduction

In this chapter, we outline a conceptual computational dynamic algorithm that includes both (a) survival state and (b) change point state and parameter estimation problems in a systematic and unified way. We develop conceptual computational dynamic algorithms for survival state and parameter estimation problems. Prior to the development of the scheme, we define, introduce notations and reorganize the observed data set for the usage of a conceptual computational dynamic algorithm in Sections 6.2 and 6.3. We outline conceptual computational dynamical algorithms for survival state and change-point survival state and parameter estimation problems in Section 6.4. The developed computational algorithms are illustrated by applying to three real world data sets in Section 6.5. In Section 6.6, the recently developed LLGMM method [44, 45] is extended and applied to three time-to-event data sets. The computational results are compared with existing methods in Section 6.7. The modified LLGMM method provides the measure of confidence, prediction and planning assessments in Section 6.8.

6.2 Data Collection Coordination with Iterative Processes

Without loss of generality, we assume that the real data observation/collection schedule is indeed a finite sequence $\{T_{j-1}\}_{j=1}^{k}$ corresponding to a partition P of $[T_0, \mathcal{T})$. Moreover, the real world data set and its data observation/collection times are coordinated with conceptual data set sequence and data collection sequence of times.

6.3 Data Decomposition, Reorganization and Aggregation

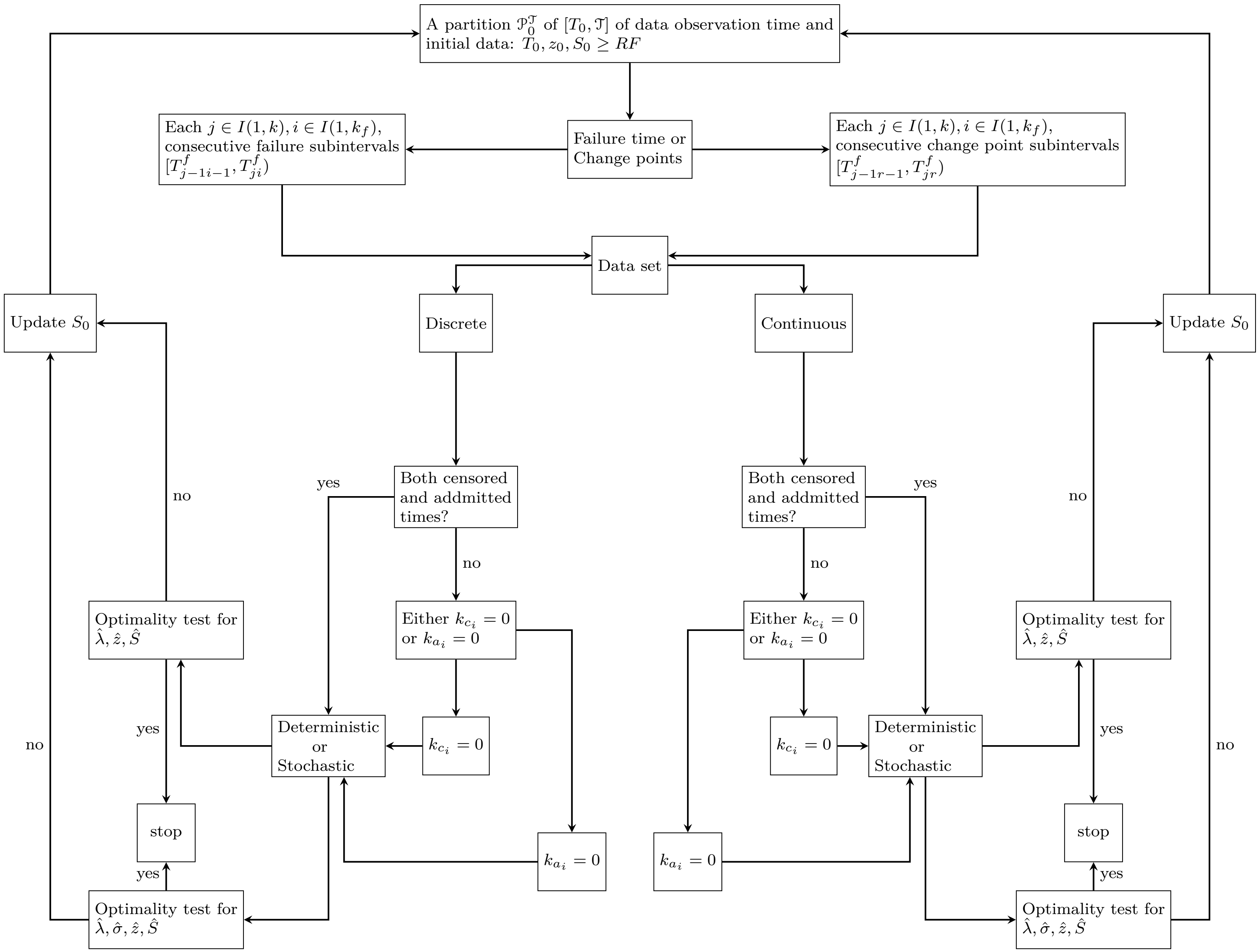
Based on our research [5, 6], we recognize and present tools for solving two major problems of interests in a time-to-event dynamic process, namely: (1) survival state and (2) change point state estimation analysis. For the study of these problems, we decompose, reorganize and re-aggregate the original real world data set in a respective framework for (1) survival state and (2) change point study in a time-to-event process. The original data is coordinated, decomposed, reorganized, and aggregated with reference to the conceptual data coordination, decomposition, reorganization and aggregation in the manner analogous to Definitions 5.4.2–5.4.5 and earlier work [6].

6.4 Conceptual Computational Parameter and State Estimations Scheme

For the conceptual computational parameter estimation, we use discrete-time conceptual computational interconnected dynamic algorithms (5.3.17) and (5.3.18) for time-to-event data statistic. The original state dynamic data subsequences are associated with conceptual data set. The decomposition of the original real world data set into three types of subsequences of data is reorganized in the context of Definition 5.4.2. We consider the original dynamic data set as the real data set, and organize/coordinate in the context of conceptual data set. For $i \in (1, k_f)$, conceptual computational dynamic estimation algorithms in (5.4.30) are used for continuous and discrete-time real world data sets, respectively. The parameter and state estimates at T^f_{j-1i} are determined using (5.4.31) for continuous and discrete-time real world data sets and a choice of initial value $S(T_0) = S_0$. Knowing the continuous dependence of solution process of continuous-time dynamic system (5.3.11) and using an initial relative frequency of a given data set, a choice of initial time and initial value S_0 is made. In fact, the solution of (5.3.11) is increasing with respect to S_0. In view of this, the optimal choice of initial value S_0 is based on the stability of the mean-square deviation of the states corresponding to the choice of the closest two initial values S_0. Finally, employing the Principle of Mathematical Induction [32], an overall parameter and state estimations of $z(t), S(t)$, $\lambda(t, S(t))$ and σ over the time interval $[t_0, \mathcal{T})$ of study are determined from (5.4.32).

6.4.1 Change Point Data Analysis Problem

In this subsection, we address the scope of the study of a time-to-event process. A change-point process in the time-to-event process measures the effects of intervention process. Here, again the overall pair of sequence of discrete-time interconnected state dynamic data set is characterized by single right-end point data set with two consecutive change point dynamic process. A sequence of two consecutive change point times is assumed to be a single subsequence of overall sequence $\{T_{j-1}\}_{j=1}^{k}$ of conceptual state dynamic data observation times. The sequence of two consecutive change point times is denoted by $\{T^{cp}_{j-1r-1}\}_{r=1}^{n}$ for $r \in I(1, n)$ with $n \leq k$. Generally, using the time-to-event state dynamic data set, the change point sequence of times is estimated. A change point process in the time-to-event process measures the effects of intervention process. The rest of the data collection coordination, decomposition/aggregation and organization with conceptual iterative process is parallel to the survival state problem, except notations. Except for notational changes (for example, replacing $[T^f_{j-1i-1}, T^f_{j-1i})$ by $[T^{cp}_{j-1r-1}, T^{cp}_{j-1r}))$, entire conceptual computational procedure regarding the survival state data analysis problem is imitated for the change-point problem, analogously. For $i \in I(1, n)$, the conceptual computational dynamic algorithms in (5.4.40) are used for continuous and discrete-time real world data sets. The parameter and state estimates at T^{cp}_{j-1r} are determined using (5.4.41) for continuous and totally discrete-time real world data sets, respectively. Finally, employing the Principle of Mathematical Induction, an overall parameter and state estimation for $z(t), S(t)$, $\lambda(t, S(t))$ and $\sigma(t, S(t))$ over the time interval $[t_0, \mathcal{T})$ of study are determined from (5.4.32) and (5.4.42). In summary, a flowchart that depicts the estimation procedure is exhibited in Flowchart 12. Here, we choose T_0, S_0, and z_0 so that $S_0 \geq RF$, where RF denotes a relative frequency at the initial time T_0. A similar flowchart incorporates the study of the change-point problem.



Flowchart 12.: Conceptual Computational Algorithm

Given T_0, S_0 and z_0

for $j = 1$ to k **do**

if Failure time **then**

for $i = 1$ to k_f **do**

Compute k_{c_i}, k_{a_i}, $z(T^f_{j-1i-1}), z(T^f_{ji})$

if Continuous **then**

Compute $\sum_{l=1}^{k_{b_i}+1} z(T^{c/a}_{j-1i-1l-1})\Delta(T^{c/a}_{j-1i-1l})$

else

Compute $\sum_{l=1}^{k_{b_i}+1} z(T^{c/a}_{j-1i-1l-1})$

end if

Compute $\hat{\lambda}, \hat{z}$, $\hat{\sigma^2}$ and $\hat{S}$

end for

else

Change point analysis

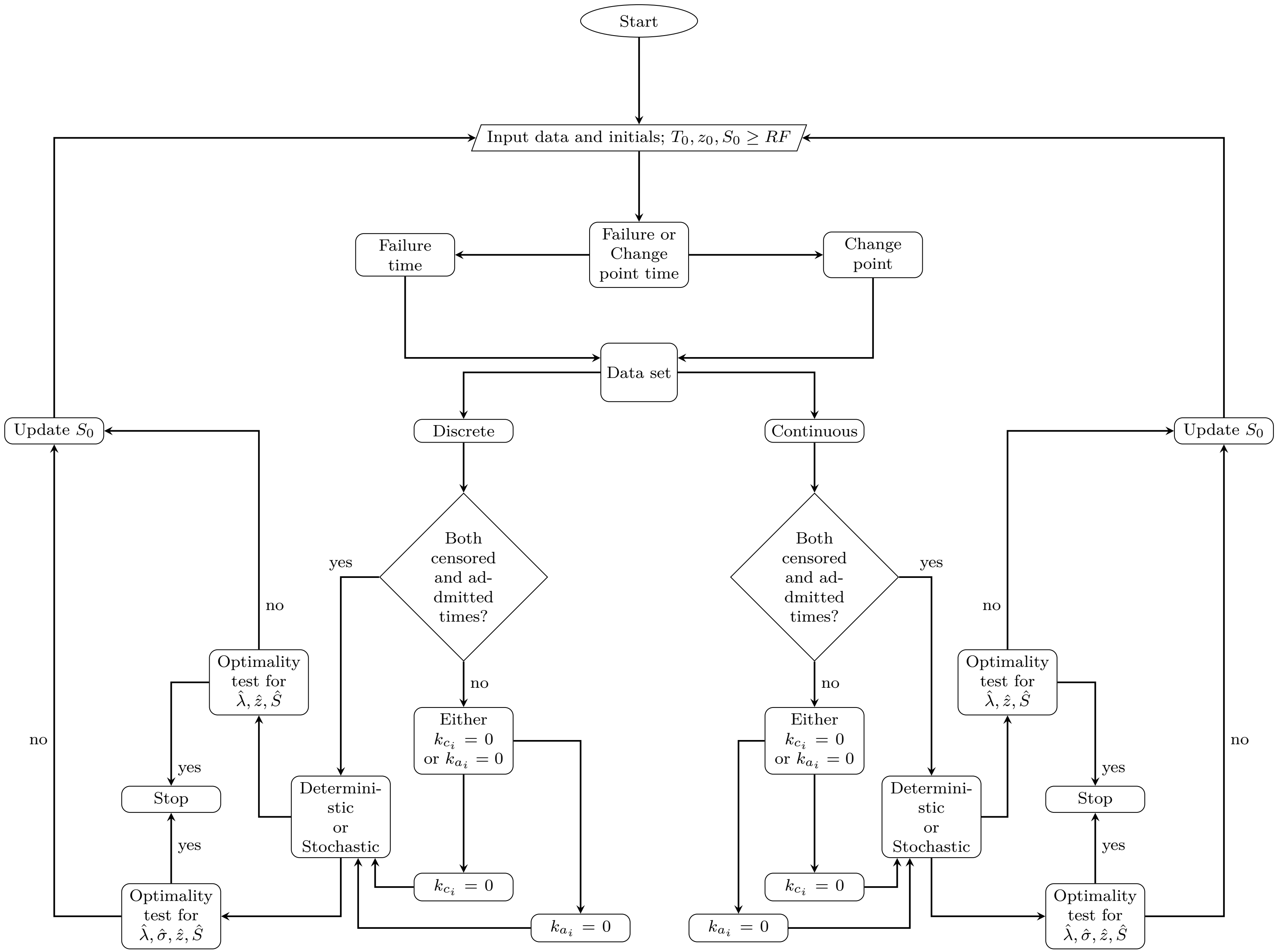
$\vdots$

end if

end for

Algorithm 13.: Simulation Scheme

We present an algorithm for the simulation schemes described above.



Flowchart 14.: Simulation Algorithm for Survival and Change point Data Analysis Problems

6.5 Illustrations

In this section, using the conceptual computational algorithm, we exemplify our theoretical algorithms and procedures for estimating parameters and survival state for three data sets: (i) the number of million revolutions failure times for each of 23 ball bearings [37], (ii) the length of remission in weeks for control group of leukemia patients, and (iii) the length or remission in weeks for the treated group of leukemia patients. The leukemia control and treated groups of patients were analyzed by Cox in his original proportional hazards paper [13]. This was based on the method of proportional hazards.

ILLUSTRATION 6.5.1 The data below show the length of remission in weeks for control group of leukemia patients that was analyzed by Cox in his original proportional hazards paper [13].

Table 15: Control Group Dataset [13]

Data Observation in weeks	Failure/ Censor Time	Frequency of Failure/ Censors at t_i	Survival/ Operating units at t_i: $z(t_i)$
$t_0 = 0$	Initial		21
$t_1 = 1$	Failure	2	19
$t_2 = 2$	Failure	2	17
$t_3 = 3$	Failure	1	16
$t_4 = 4$	Failure	2	14
$t_5 = 5$	Failure	2	12
$t_6 = 8$	Failure	4	8
$t_7 = 11$	Failure	2	6
$t_8 = 12$	Failure	2	4
$t_9 = 15$	Failure	1	3
$t_{10} = 17$	Failure	1	2
$t_{11} = 22$	Failure	1	1
$t_{12} = 23$	Failure	1	0

We note that data set has no censored or arrival times. Thus, $k_a = 0 = k_c$. We demonstrate our innovative alternative approach for finding parameter and survival function estimates on consecutive failure time intervals (locally) by employing computational scheme outlined in Section 6.2. We note the initial relative frequency of the survival locomotive control to be $\frac{19}{21}$. Employing the initial relative survival state frequency, we chose an initial survival probability to be $S_0 = 0.99, 0.999, 0.9999, 0.99999, 0.999999$. First, we choose $V(t, z) = z^2$. We then apply conceptual computational algorithms (5.4.24). The simulation/computational results are recoded in Table 16. Second, making a choice of $V(t, z) = \ln z$, we apply conceptual computational algorithms (5.4.27) for consecutive failure time intervals. The computational results are exhibited in Table 17 . The simulation results in Tables 16 and 17 show that the estimates are stabilized for $S_0 \geq 0.9999$. This justifies the almost certain optimal convergence of survival state probability estimates for $S_0 \geq 0.9999$. Thus for the leukemia data set, we conclude that the best survival state estimate is assured at $S_0 = 0.99999$. Moreover, the results in Tables 16 and 17 indicate that our innovative approach is independent of the choice of nonlinear transformation $V(t, z)$ so far as the obtained system of algebraic equations can be solved.

Table 16: Estimates using $S_0 = 0.99000, 0.99900, 0.99990, 0.99999, 0.999999$ by employing conceptual computational algorithm (5.4.24)

Consecutive Failure time interval, $[T_{j-1}, T_j)$	$S_0 = 0.99000$			$S_0 = 0.99900$			$S_0 = 0.9999$			$S_0 = 0.99999$			$S_0 = 0.999999$		
	$\hat{\lambda}_j$	$\hat{\sigma}_j$	$\hat{S}_{j-1}$	$\hat{\lambda}_j$	$\hat{\sigma}_j$	$\hat{S}_{j-1}$	$\hat{\lambda}_j$	$\hat{\sigma}_j$	$\hat{S}_{j-1}$	$\hat{\lambda}_j$	$\hat{\sigma}_j$	$\hat{S}_{j-1}$	$\hat{\lambda}_j$	$\hat{\sigma}_j$	$\hat{S}_{j-1}$
$[0, 1)$	0.0952	0.0952	0.9900	0.0952	0.0952	0.9990	0.0952	0.0952	0.9999	0.0952	0.0952	0.99999	0.0952	0.0952	0.999999
$[1, 2)$	0.1053	0.1053	0.8958	0.1053	0.1053	0.9040	0.1053	0.1053	0.9048	0.1053	0.1053	0.9049	0.1053	0.1053	0.9049
$[2, 3)$	0.0588	0.0588	0.8017	0.0588	0.0588	0.8090	0.0588	0.0588	0.8097	0.0588	0.0588	0.8098	0.0588	0.0588	0.8098
$[3, 4)$	0.1250	0.1250	0.7546	0.1250	0.1250	0.7614	0.1250	0.1250	0.7621	0.1250	0.1250	0.7622	0.1250	0.1250	0.7622
$[4, 5)$	0.1429	0.1429	0.6604	0.1429	0.1429	0.6664	0.1429	0.1429	0.6670	0.1429	0.1429	0.6670	0.1429	0.1429	0.6670
$[5, 8)$	0.1111	0.1925	0.5662	0.1111	0.1925	0.5713	0.1111	0.1925	0.5718	0.1111	0.1925	0.5719	0.1111	0.1925	0.5719
$[8, 11)$	0.0833	0.1443	0.3776	0.0833	0.1443	0.3810	0.0833	0.1443	0.3814	0.0833	0.1443	0.3814	0.0833	0.1443	0.3814
$[11, 12)$	0.3333	0.3333	0.2833	0.3333	0.3333	0.2858	0.3333	0.3333	0.2861	0.3333	0.3333	0.2861	0.3333	0.3333	0.2861
$[12, 15)$	0.0833	0.1443	0.1890	0.0833	0.1443	0.1907	0.0833	0.1443	0.1909	0.0833	0.1443	0.1909	0.0833	0.1443	0.1909
$[15, 17)$	0.1667	0.2357	0.1418	0.1667	0.2357	0.1430	0.1667	0.2357	0.1432	0.1667	0.2357	0.1432	0.1667	0.2357	0.1432
$[17, 22)$	0.1000	0.2236	0.0945	0.1000	0.2236	0.0954	0.1000	0.2236	0.0955	0.1000	0.2236	0.0955	0.1000	0.2236	0.0955
$[22, 23)$	1.0000	1.0000	0.0473	1.0000	1.0000	0.0477	1.0000	1.0000	0.0478	1.0000	1.0000	0.0478	1.0000	1.0000	0.0478
(23)			0.0001			0.0001			0.0001			0.0001			0.0001

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Table 17: Estimates using $S_0 = 0.99000, 0.99900, 0.99990, 0.99999, 0.999999$ by employing conceptual computational algorithm (5.4.27)

Consecutive Failure time interval, $[T_{j-1}, T_j)$	$S_0 = 0.99000$			$S_0 = 0.99900$			$S_0 = 0.9999$			$S_0 = 0.99999$			$S_0 = 0.999999$		
	$\hat{\lambda}_j$	$\hat{\sigma}_j$	$\hat{S}_{j-1}$	$\hat{\lambda}_j$	$\hat{\sigma}_j$	$\hat{S}_{j-1}$	$\hat{\lambda}_j$	$\hat{\sigma}_j$	$\hat{S}_{j-1}$	$\hat{\lambda}_j$	$\hat{\sigma}_j$	$\hat{S}_{j-1}$	$\hat{\lambda}_j$	$\hat{\sigma}_j$	$\hat{S}_{j-1}$
$[0, 1]$	0.0952	0.0952	0.9900	0.0952	0.0952	0.9990	0.0952	0.0952	0.9999	0.0952	0.0952	1.0000	0.0952	0.0952	1.0000
$[1, 2)$	0.1053	0.1053	0.8958	0.1053	0.1053	0.9040	0.1053	0.1053	0.9048	0.1053	0.1053	0.9049	0.1053	0.1053	0.9049
$[2, 3)$	0.0588	0.0588	0.8017	0.0588	0.0588	0.8090	0.0588	0.0588	0.8097	0.0588	0.0588	0.8098	0.0588	0.0588	0.8098
$[3, 4)$	0.1250	0.1250	0.7546	0.1250	0.1250	0.7614	0.1250	0.1250	0.7621	0.1250	0.1250	0.7622	0.1250	0.1250	0.7622
$[4, 5)$	0.1429	0.1429	0.6604	0.1429	0.1429	0.6664	0.1429	0.1429	0.6670	0.1429	0.1429	0.6670	0.1429	0.1429	0.6671
$[5, 8)$	0.1111	0.1925	0.5662	0.1111	0.1925	0.5713	0.1111	0.1925	0.5718	0.1111	0.1925	0.5719	0.1111	0.1925	0.5719
$[8, 11)$	0.0833	0.1443	0.3776	0.0833	0.1443	0.3810	0.0833	0.1443	0.3814	0.0833	0.1443	0.3814	0.0833	0.1443	0.3814
$[11, 12)$	0.3333	0.3333	0.2833	0.3333	0.3333	0.2859	0.3333	0.3333	0.2861	0.3333	0.3333	0.2861	0.3333	0.3333	0.2862
$[12, 15)$	0.0833	0.1443	0.1890	0.0833	0.1443	0.1907	0.0833	0.1443	0.1909	0.0833	0.1443	0.1909	0.0833	0.1443	0.1909
$[15, 17)$	0.1667	0.2357	0.1418	0.1667	0.2357	0.1431	0.1667	0.2357	0.1432	0.1667	0.2357	0.1432	0.1667	0.2357	0.1432
$[17, 22)$	0.1000	0.2236	0.0946	0.1000	0.2236	0.0954	0.1000	0.2236	0.0955	0.1000	0.2236	0.0955	0.1000	0.2236	0.0955
$[22, 23)$	1.0000	1.0000	0.0473	1.0000	1.0000	0.0478	1.0000	1.0000	0.0478	1.0000	1.0000	0.0478	1.0000	1.0000	0.0478
(23)			0.0001			Inf			Inf			Inf			Inf

Illustration 6.5.2 The data below are number of million revolutions failure times for each of 23 ball bearings. The data was analyzed in Lawless[37].

Table 18: Ball Bearings Dataset [37]

Data Observation in weeks	Failure/ Censor Time	Frequency of Failure/ Censors at t_i	Survival/ Operating units at t_i: $z(t_i)$
$t_0 = 0$	Initial		23
$t_1 = 17.88$	Failure	1	22
$t_2 = 28.92$	Failure	1	21
$t_3 = 33.00$	Failure	1	20
$t_4 = 41.52$	Failure	1	19
$t_5 = 42.12$	Failure	1	18
$t_6 = 45.60$	Failure	1	17
$t_7 = 48.40$	Failure	1	16
$t_8 = 51.84$	Failure	1	15
$t_9 = 51.96$	Failure	1	14
$t_{10} = 54.12$	Failure	1	13
$t_{11} = 55.56$	Failure	1	12
$t_{12} = 67.80$	Failure	1	11
$t_{13} = 68.64$	Failure	2	9
$t_{14} = 68.88$	Failure	1	8
$t_{15} = 84.12$	Failure	1	7
$t_{16} = 93.12$	Failure	1	6
$t_{17} = 98.64$	Failure	1	5
$t_{18} = 105.12$	Failure	1	4
$t_{19} = 105.84$	Failure	1	3
$t_{20} = 127.92$	Failure	1	2
$t_{21} = 128.04$	Failure	1	1
$t_{22} = 173.40$	Failure	1	0

Again, we note that data set has no censored or arrival times. Thus, $k_a = 0 = k_c$. We also note that the initial relative frequency of the survival of ball bearing data is 0.9565. Using the initial relative frequency of ball bearing dataset, we chose initial survival probability to be $S_0 = 0.99, 0.999, 0.9999, 0.99999, 0.999999$. We demonstrate our approach by picking two choices of $V(t, z)$ to construct observation equations. First, choosing $V(t, z) = z^2$ and applying the conceptual computational simulation algorithms (5.4.24) for consecutive failure-time intervals, the simulation results are summarized in Table 19. Choosing $V(t, z) = \ln z$ and then applying conceptual computational simulation algorithms (5.4.27) for consecutive failure-time subintervals. The results are recored in Table 20. The simulation results in Tables 19 and 20 show that estimates are stabilized for $S_0 \geq 0.9999$. In other words, optimal convergence of survival state probability estimates are reached for $S_0 \geq 0.9999$. We then conclude that the almost best survival state estimate is for $S_0 = 0.99999$ for the ball bearings data set. Moreover, the results in Tables 19 and 20 also confirm the parameter and survival state estimates are independent of the choice of $V(t, z)$.

Table 19: Estimates using S_0 = 0.99000, 0.99900, 0.99990, 0.99999, 0.999999 by employing conceptual computational simulation algorithm (5.4.24)

Consecutive Failure time interval, $[T_{j-1}, T_j)$	$S_0 = 0.99000$			$S_0 = 0.99900$			$S_0 = 0.9999$			$S_0 = 0.99999$			$S_0 = 0.999999$		
	$\hat{\lambda}_j$	$\hat{\sigma}_j$	$\hat{S}_{j-1}$	$\hat{\lambda}_j$	$\hat{\sigma}_j$	$\hat{S}_{j-1}$	$\hat{\lambda}_j$	$\hat{\sigma}_j$	$\hat{S}_{j-1}$	$\hat{\lambda}_j$	$\hat{\sigma}_j$	$\hat{S}_{j-1}$	$\hat{\lambda}_j$	$\hat{\sigma}_j$	$\hat{S}_{j-1}$
$[0, 17.88)$	0.0024	0.0103	0.9900	0.0024	0.0103	0.9990	0.0024	0.0103	0.9999	0.0024	0.0103	0.99999	0.0024	0.0103	0.999999
$[17.88, 28.92)$	0.0041	0.0137	0.9470	0.0041	0.0137	0.9556	0.0041	0.0137	0.9564	0.0041	0.0137	0.9565	0.0041	0.0137	0.9565
$[28.92, 33.00)$	0.0117	0.0236	0.9039	0.0117	0.0236	0.9122	0.0117	0.0236	0.9130	0.0117	0.0236	0.9131	0.0117	0.0236	0.9131
$[33.00, 41.52)$	0.0059	0.0171	0.8609	0.0059	0.0171	0.8688	0.0059	0.0171	0.8695	0.0059	0.0171	0.8696	0.0059	0.0171	0.8696
$[41.52, 42.12)$	0.0877	0.0679	0.8179	0.0877	0.0679	0.8253	0.0877	0.0679	0.8261	0.0877	0.0679	0.8262	0.0877	0.0679	0.8262
$[42.12, 45.60)$	0.0160	0.0298	0.7749	0.0160	0.0298	0.7820	0.0160	0.0298	0.7827	0.0160	0.0298	0.7827	0.0160	0.0298	0.7828
$[45.60, 48.40)$	0.0210	0.0352	0.7319	0.0210	0.0352	0.7386	0.0210	0.0352	0.7392	0.0210	0.0352	0.7393	0.0210	0.0352	0.7393
$[48.40, 51.84)$	0.0182	0.0337	0.6889	0.0182	0.0337	0.6951	0.0182	0.0337	0.6958	0.0182	0.0337	0.6958	0.0182	0.0337	0.6958
$[51.84, 51.96)$	0.5556	0.1925	0.6459	0.5556	0.1925	0.6517	0.5556	0.1925	0.6523	0.5556	0.1925	0.6524	0.5556	0.1925	0.6524
$[51.96, 54.12)$	0.0331	0.0486	0.6030	0.0331	0.0486	0.6084	0.0331	0.0486	0.6090	0.0331	0.0486	0.6091	0.0331	0.0486	0.6091
$[54.12, 55.56)$	0.0534	0.0641	0.5599	0.0534	0.0641	0.5650	0.0534	0.0641	0.5655	0.0534	0.0641	0.5656	0.0534	0.0641	0.5656
$[55.56, 67.80)$	0.0068	0.0238	0.5169	0.0068	0.0238	0.5216	0.0068	0.0238	0.5221	0.0068	0.0238	0.5221	0.0068	0.0238	0.5221
$[67.80, 68.64)$	0.2165	0.1984	0.4739	0.2165	0.1984	0.4782	0.2165	0.1984	0.4786	0.2165	0.1984	0.4786	0.2165	0.1984	0.4786
$[68.64, 68.88)$	0.4630	0.2268	0.3878	0.4630	0.2268	0.3913	0.4630	0.2268	0.3917	0.4630	0.2268	0.3917	0.4630	0.2268	0.3917
$[68.88, 84.12)$	0.0082	0.0320	0.3448	0.0082	0.0320	0.3480	0.0082	0.0320	0.3483	0.0082	0.0320	0.3483	0.0082	0.0320	0.3483
$[84.12, 93.12)$	0.0159	0.0476	0.3018	0.0159	0.0476	0.3045	0.0159	0.0476	0.3048	0.0159	0.0476	0.3048	0.0159	0.0476	0.3048
$[93.12, 98.64)$	0.0302	0.0709	0.2587	0.0302	0.0709	0.2610	0.0302	0.0709	0.2613	0.0302	0.0709	0.2613	0.0302	0.0709	0.2613
$[98.64, 105.12)$	0.0309	0.0786	0.2156	0.0309	0.0786	0.2175	0.0309	0.0786	0.2177	0.0309	0.0786	0.2178	0.0309	0.0786	0.2178
$[105.12, 105.84)$	0.3472	0.2946	0.1725	0.3472	0.2946	0.1741	0.3472	0.2946	0.1742	0.3472	0.2946	0.1742	0.3472	0.2946	0.1742
$[105.84, 127.92)$	0.0151	0.0709	0.1294	0.0151	0.0709	0.1306	0.0151	0.0709	0.1307	0.0151	0.0709	0.1307	0.0151	0.0709	0.1307
$[127.92, 128.04)$	4.1667	1.4434	0.0863	4.1667	1.4434	0.0871	4.1667	1.4434	0.0872	4.1667	1.4434	0.0872	4.1667	1.4434	0.0872
$[128.04, 173.40)$	0.0220	0.1485	0.0433	0.0220	0.1485	0.0437	0.0220	0.1485	0.0437	0.0220	0.1485	0.0438	0.0220	0.1485	0.0438
(173.40)			0.0000			0.0000			0.0000			0.0000			0.0000

Table 20: Estimates using $S_0 = 0.99000$, 0.99900, 0.99990, 0.99999, 0.999999 using conceptual computational simulation algorithm (5.4.27)

Consecutive Failure time interval, $[T_{j-1}, T_j)$	$S_0 = 0.99000$			$S_0 = 0.99900$			$S_0 = 0.9999$			$S_0 = 0.99999$			$S_0 = 0.999999$		
	$\hat{\lambda}_j$	$\hat{\sigma}_j$	$\hat{S}_{j-1}$	$\hat{\lambda}_j$	$\hat{\sigma}_j$	$\hat{S}_{j-1}$	$\hat{\lambda}_j$	$\hat{\sigma}_j$	$\hat{S}_{j-1}$	$\hat{\lambda}_j$	$\hat{\sigma}_j$	$\hat{S}_{j-1}$	$\hat{\lambda}_j$	$\hat{\sigma}_j$	$\hat{S}_{j-1}$
$[0, 17.88)$	0.0024	0.0103	0.9900	0.0024	0.0103	0.9990	0.0024	0.0103	0.9999	0.0024	0.0103	1.0000	0.0024	0.0103	1.0000
$[17.88, 28.92)$	0.0041	0.0137	0.9470	0.0041	0.0137	0.9556	0.0041	0.0137	0.9564	0.0041	0.0137	0.9565	0.0041	0.0137	0.9565
$[28.92, 33.00)$	0.0117	0.0236	0.9039	0.0117	0.0236	0.9122	0.0117	0.0236	0.9130	0.0117	0.0236	0.9131	0.0117	0.0236	0.9131
$[33.00, 41.52)$	0.0059	0.0171	0.8609	0.0059	0.0171	0.8688	0.0059	0.0171	0.8695	0.0059	0.0171	0.8696	0.0059	0.0171	0.8696
$[41.52, 42.12)$	0.0877	0.0679	0.8179	0.0877	0.0679	0.8253	0.0877	0.0679	0.8261	0.0877	0.0679	0.8262	0.0877	0.0679	0.8262
$[42.12, 45.60)$	0.0160	0.0298	0.7749	0.0160	0.0298	0.7820	0.0160	0.0298	0.7827	0.0160	0.0298	0.7827	0.0160	0.0298	0.7828
$[45.60, 48.40)$	0.0210	0.0352	0.7319	0.0210	0.0352	0.7386	0.0210	0.0352	0.7392	0.0210	0.0352	0.7393	0.0210	0.0352	0.7393
$[48.40, 51.84)$	0.0182	0.0337	0.6889	0.0182	0.0337	0.6952	0.0182	0.0337	0.6958	0.0182	0.0337	0.6958	0.0182	0.0337	0.6958
$[51.84, 51.96)$	0.5556	0.1925	0.6459	0.5556	0.1925	0.6517	0.5556	0.1925	0.6523	0.5556	0.1925	0.6524	0.5556	0.1925	0.6524
$[51.96, 54.12)$	0.0331	0.0486	0.6030	0.0331	0.0486	0.6085	0.0331	0.0486	0.6090	0.0331	0.0486	0.6091	0.0331	0.0486	0.6091
$[54.12, 55.56)$	0.0534	0.0641	0.5599	0.0534	0.0641	0.5650	0.0534	0.0641	0.5655	0.0534	0.0641	0.5656	0.0534	0.0641	0.5656
$[55.56, 67.80)$	0.0068	0.0238	0.5169	0.0068	0.0238	0.5216	0.0068	0.0238	0.5221	0.0068	0.0238	0.5221	0.0068	0.0238	0.5221
$[67.80, 68.64)$	0.2165	0.1984	0.4739	0.2165	0.1984	0.4782	0.2165	0.1984	0.4786	0.2165	0.1984	0.4786	0.2165	0.1984	0.4786
$[68.64, 68.88)$	0.4630	0.2268	0.3878	0.4630	0.2268	0.3914	0.4630	0.2268	0.3917	0.4630	0.2268	0.3917	0.4630	0.2268	0.3918
$[68.88, 84.12)$	0.0082	0.0320	0.3449	0.0082	0.0320	0.3480	0.0082	0.0320	0.3483	0.0082	0.0320	0.3483	0.0082	0.0320	0.3483
$[84.12, 93.12)$	0.0159	0.0476	0.3018	0.0159	0.0476	0.3045	0.0159	0.0476	0.3048	0.0159	0.0476	0.3048	0.0159	0.0476	0.3048
$[93.12, 98.64)$	0.0302	0.0709	0.2587	0.0302	0.0709	0.2610	0.0302	0.0709	0.2613	0.0302	0.0709	0.2613	0.0302	0.0709	0.2613
$[98.64, 105.12)$	0.0309	0.0786	0.2156	0.0309	0.0786	0.2176	0.0309	0.0786	0.2177	0.0309	0.0786	0.2178	0.0309	0.0786	0.2178
$[105.12, 105.84)$	0.3472	0.2946	0.1725	0.3472	0.2946	0.1741	0.3472	0.2946	0.1742	0.3472	0.2946	0.1742	0.3472	0.2946	0.1742
$[105.84, 127.92)$	0.0151	0.0709	0.1294	0.0151	0.0709	0.1306	0.0151	0.0709	0.1307	0.0151	0.0709	0.1308	0.0151	0.0709	0.1308
$[127.92, 128.04)$	4.1667	1.4434	0.0863	4.1667	1.4434	0.0871	4.1667	1.4434	0.0872	4.1667	1.4434	0.0872	4.1667	1.4434	0.0872
$[128.04, 173.40)$	0.0220	0.1485	0.0434	0.0220	0.1485	0.0438	0.0220	0.1485	0.0438	0.0220	0.1485	0.0438	0.0220	0.1485	0.0438
(173.40)			Inf			Inf			Inf			Inf			0.0000

In the following illustration, we apply our innovative alternative algorithm to a data set consisting of multiple censored times between consecutive failure times.

Illustration 6.5.3 The data in Table 21 below show the length of remission in weeks leukemia patients under the influence of treatment study [13]. We note that there are multiple censored times occurring between any two consecutive failure times unlike the data sets in Tables 15 and 18. Here also, we exemplify our approach by picking two choices of $V(t, z)$ to construct observation equations. First, we choose $V(t, z) = z^2$ and apply (5.4.37) with $k_a = 0$ for consecutive failure time intervals. The results are recorded in Table 22. Choosing $V(t, z) = \ln z$ and applying conceptual computational simulation algorithm (5.4.39), we obtain estimates that are summarized in Table 23. The computational results in Tables 22 and 23 show that the estimates are stabilized for $S_0 \geq 0.9999$. Thus for the data set in Table 21, we conclude that the best survival state estimate is attained at the initial value for $S_0 = 0.99999$. In addition, the results in Tables 22 and 23 indicate that our alternative approach is independent of the choice of nonlinear transformation $V(t, z)$ provided that the obtained system of algebraic equations can be solved.

Table 21: Treated Group Dataset [13]

Data Observation in weeks	Failure/ Censor Time	Frequency of Failure/ Censors at t_i	Survival/ Operating units at t_i: $z(t_i)$
$t_0 = 0$	Initial		21
$t_1 = 6$	Failure	3	18
$t_2 = 6$	Censored	1	17
$t_3 = 7$	Failure	1	16
$t_4 = 9$	Censored	1	15
$t_5 = 10$	Failure	1	14
$t_6 = 10$	Censored	1	13
$t_7 = 11$	Failure	1	12
$t_8 = 13$	Censored	1	11
$t_9 = 16$	Failure	1	10
$t_{10} = 17$	Censored	1	9
$t_{11} = 19$	Censored	1	8
$t_{12} = 20$	Censored	1	7
$t_{13} = 22$	Failure	1	6
$t_{14} = 23$	Failure	1	5
$t_{15} = 25$	Censored	1	4
$t_{16} = 32$	Censored	2	2
$t_{17} = 34$	Censored	1	1
$t_{18} = 35$	Censored	1	0

Table 22: Estimates using S_0 = 0.99000, 0.99900, 0.99990, 0.99999, 0.999999 by employing conceptual computational algorithm (5.4.37)

Consecutive Failure time interval, $[T_{j-1}, T_j)$	$S_0 = 0.99000$			$S_0 = 0.99900$			$S_0 = 0.9999$			$S_0 = 0.99999$			$S_0 = 0.999999$		
	$\hat{\lambda}_j$	$\hat{\sigma}_j$	$\hat{S}_{j-1}$	$\hat{\lambda}_j$	$\hat{\sigma}_j$	$\hat{S}_{j-1}$	$\hat{\lambda}_j$	$\hat{\sigma}_j$	$\hat{S}_{j-1}$	$\hat{\lambda}_j$	$\hat{\sigma}_j$	$\hat{S}_{j-1}$	$\hat{\lambda}_j$	$\hat{\sigma}_j$	$\hat{S}_{j-1}$
$[0, 6)$	0.0238	0.0583	0.9900	0.0238	0.0583	0.9990	0.0238	0.0583	0.9999	0.0238	0.0583	1.0000	0.0238	0.0583	1.0000
$[6, 7)$	0.0588	0.0588	0.8486	0.0588	0.0588	0.8564	0.0588	0.0588	0.8571	0.0588	0.0588	0.8572	0.0588	0.0588	0.8572
$[7, 10)$	0.0213	0.0566	0.7988	0.0213	0.0566	0.8061	0.0213	0.0566	0.8068	0.0213	0.0566	0.8069	0.0213	0.0566	0.8069
$[10, 11)$	0.0769	0.0769	0.7479	0.0769	0.0769	0.7547	0.0769	0.0769	0.7553	0.0769	0.0769	0.7554	0.0769	0.0769	0.7554
$[11, 16)$	0.0175	0.0532	0.6904	0.0175	0.0532	0.6967	0.0175	0.0532	0.6973	0.0175	0.0532	0.6974	0.0175	0.0532	0.6974
$[16, 22)$	0.0200	0.0966	0.6299	0.0200	0.0966	0.6356	0.0200	0.0966	0.6362	0.0200	0.0966	0.6363	0.0200	0.0966	0.6363
$[22, 23)$	0.0204	0.3065	0.5544	0.0204	0.3065	0.5594	0.0204	0.3065	0.5599	0.0204	0.3065	0.5600	0.0204	0.3065	0.5600
(23)			0.5450			0.5499			0.5504			0.5505			0.5505

Table 23: Estimates using S_0 = 0.99000, 0.99900, 0.99990, 0.99999, 0.999999 by employing conceptual computational algorithm (5.4.39)

Consecutive Failure time interval, $[T_{j-1}, T_j)$	$S_0 = 0.99000$			$S_0 = 0.99900$			$S_0 = 0.9999$			$S_0 = 0.99999$			$S_0 = 0.999999$		
	$\hat{\lambda}_j$	$\hat{\sigma}_j$	$\hat{S}_{j-1}$	$\hat{\lambda}_j$	$\hat{\sigma}_j$	$\hat{S}_{j-1}$	$\hat{\lambda}_j$	$\hat{\sigma}_j$	$\hat{S}_{j-1}$	$\hat{\lambda}_j$	$\hat{\sigma}_j$	$\hat{S}_{j-1}$	$\hat{\lambda}_j$	$\hat{\sigma}_j$	$\hat{S}_{j-1}$
$[0, 6)$	0.0238	0.0583	0.9900	0.0238	0.0583	0.9990	0.0238	0.0583	0.9999	0.0238	0.0583	1.0000	0.0238	0.0583	1.0000
$[6, 7)$	0.0588	0.0588	0.8487	0.0588	0.0588	0.8564	0.0588	0.0588	0.8571	0.0588	0.0588	0.8572	0.0588	0.0588	0.8572
$[7, 10)$	0.0213	0.0566	0.7988	0.0213	0.0566	0.8061	0.0213	0.0566	0.8068	0.0213	0.0566	0.8069	0.0213	0.0566	0.8069
$[10, 11)$	0.0769	0.0769	0.7479	0.0769	0.0769	0.7547	0.0769	0.0769	0.7554	0.0769	0.0769	0.7554	0.0769	0.0769	0.7554
$[11, 16)$	0.0175	0.0532	0.6904	0.0175	0.0532	0.6967	0.0175	0.0532	0.6973	0.0175	0.0532	0.6974	0.0175	0.0532	0.6974
$[16, 22)$	0.0200	0.0966	0.6299	0.0200	0.0966	0.6356	0.0200	0.0966	0.6362	0.0200	0.0966	0.6363	0.0200	0.0966	0.6363
$[22, 23)$	0.0204	0.3065	0.5544	0.0204	0.3065	0.5594	0.0204	0.3065	0.5600	0.0204	0.3065	0.5600	0.0204	0.3065	0.5600
(23)			0.5450			0.5499			0.5505			0.5505			0.5505

6.6 Modified LLGMM Parameter and State Estimation

In this section, we develop a modified version of the Local Lagged Adapted Generalized Method of Moments(LLGMM) [44, 45]. This is achieved by utilizing the developed alternative procedure in Section 5.4 and the LLGMM method. We note that the transformed conceptual computational interconnected dynamic algorithm for time-to-event data statistic process is local. It is centered around each consecutive pair of ordered failure time subinterval $[T^f_{j-1i-1}, T^f_{j-1i})$ with its right-end-point data observation/collection process for $i \in I(1, k_f)$, and $j \in I(1, n)$. Moreover, parameter and state estimations of the time-to-event process is relative to each consecutive pair of ordered failure or change time subinterval of operation of the time-to-event dynamic process. This type of parameter and state estimation problem in time-to-event processes can be characterized by the local single-shot procedure identified by the right-end point of the $j-1i$-th consecutive failure or change point subinterval for each $i \in I(1, k_f)$.

These observations motivate the extension of the presented local single-shot innovative parameter and state dynamic estimation procedure developed in Section 5.4 to a finite multi-choice local lagged consecutive failure or change time subintervals with right-end-point data observation/collection process. For this, we recall [6] a couple of definitions that form a bridge to connect our developed innovative approach with the LLGMM approach. For easy reference, we present some of the useful definitions.

Definition 6.6.1 For each $i \in I(1, k_f)$ and each $m_i \in I(1, i)$, a partition of closed interval $[T^f_{j-1i-m_i}, T^f_{j-1i}]$ is called local lagged at a failure-time T^f_{j-1i}, and it is defined by:

$$P^f_{j-1i-m_i} := T^f_{j-1i-m_i} < T^f_{j-1i-m_i+1} < \ldots < T^f_{j-1i-1} < T^f_{j-1i} \ . \tag{6.6.1}$$

A m_i-size consecutive ordered failure time subinterval subsequence $\{[T^f_{j-1i+l}, T^f_{j-1i+l+1})\}_{l=-m_i}^{-1}$ of the overall consecutive ordered failure time subinterval sequence $\{[T^f_{j-1i-1}, T^f_{j-1i})\}_{i=1}^{k_f}$ is called local lagged moving failure-time subinterval subsequence at T^f_{j-1i} that forms a cover [16] of $[T^f_{j-1i-m_i}, T^f_{j-1i})$:

$$\bigcup_{l=-m_i}^{-1} [T^f_{j-1i+l}, T^f_{j-1i+l+1}) = [T^f_{j-1i-m_i}, T^f_{j-1i}) \ . \tag{6.6.2}$$

$P^f_{j-1i-m_i}$ is a sub-partition of the overall partition P^f in Definition 5.4.3.

Definition 6.6.2 For each $i \in I(1, k_f)$ and each $m_i \in I_1(1, i)$, a local lagged moving consecutive ordered failure time subsequence of subintervals, $\{[T^f_{j-1i+l}, T^f_{j-1i+l+1})\}_{l=-m_i}^{-1}$ at failure time T^f_{j-1i} of the size m_i is identified by the restriction of overall failure time state data subsequence $\{z_{j-1i-1}\}_{i=1}^{k_f}$ with $P^f_{j-1i-m_i}$ in (6.6.1), and it is defined by:

$$s_{m_i,j-1i} := \{F^l z_{j-1i}\}_{l=-m_i}^{0} \ . \tag{6.6.3}$$

Here F is a forward-shift operator, and $F^{-1} = B$, where B is the backward shift operator [10]. m_i varies from 1 to i, so also the corresponding local sequence $s_{m_i,i}$ at T^f_{j-1i} in (6.6.3) varies from $\{F^l z_{j-1i}\}^0_{l=-1}$ to $\{F^l z_{j-1i}\}^0_{l=-i+1}$. As a result of this, the sequence defined in (6.6.3) is also called a m_i-local moving sequence of consecutive failure-time state data associated with m_i-local lagged finite sequence of subintervals at a failure-time T^f_{j-1i} for each $i \in I(1, k_f)$.

In the following, we outline computational scheme for the survival state data analysis problems. Using the concept of m_i-moving sequence of failure-time state data at a failure time T^f_{j-1i}, computational schemes for the change point problem can also be formulated and developed, analogously.
Hereafter, we utilize Definitions 6.6.1 and 6.6.2, and recast the LLGMM algorithm [44, 45]. For each $m_i \in I(1, i+1)$, and $l \in I(-m_i, -1)$, using (5.4.31) we determine estimates of λ and σ^2 at each failure time T^f_{j-1i} as follows:

$$\begin{cases} \hat{\lambda}_{i,m_i} = \dfrac{\sum\limits_{l=-m_i}^{-1} \Big[-\mathbb{E}(\Delta z_{j-1i+l+1} \,|\, \mathcal{G}_{j-1i+l}) + \Gamma^{no}_{j-1i+l} - k_{c_{i+l}} + k_{a_{i+l}} \Big]}{\sum\limits_{l=-m_i}^{-1} \sum\limits_{n=1}^{k_{b_{i+l}}+1} z(T^{c/a}_{j-1i+ln-1}) \Delta T^{c/a}_{j-1i+ln}}, \\[2ex] \hat{\sigma}^2_{i,m_i} = \\ 2 \left[\dfrac{\sum\limits_{l=-m_i}^{-1} \left(\begin{array}{l} \mathbb{E}[\Delta V(T^f_{j-1i+l+1}, z_{j-1i+l+1}) \mid \mathcal{G}_{j-1i+l}] + \Gamma^{nov}_{j-1i+l} + k^v_{b_i} \\ \quad - \sum\limits_{n=1}^{k_{b_{i+l}}+1} \frac{\partial}{\partial t} V(T^{c/a}_{j-1i+ln-1}, z(T^{c/a}_{j-1i-1+ln-1})) \Delta T^{c/a}_{j-1i+ln} \\ \qquad - \lambda(T^f_{j-1i-1}, S_{j-1i-1}) \left[\sum\limits_{n=1}^{k_{b_i}+1} z(T^{c/a}_{j-1i+ln-1}) \frac{\partial}{\partial z} V(T^{c/a}_{j-1i+ln-1}, z(T^{c/a}_{j-1i-1+ln-1})) \Delta T^{c/a}_{j-1i+ln} \right] \end{array} \right)}{\sum\limits_{l=-m_i}^{-1} \sum\limits_{n=1}^{k_{b_i}+1} z^2(T^{c/a}_{j-1i+ln-1}) \frac{\partial^2}{\partial z^2} V(T^{c/a}_{j-1i+ln-1}, z(T^{c/a}_{j-1i-1+ln-1})) \Delta T^{c/a}_{j-1i+ln}} \right], \end{cases} \tag{6.6.4}$$

where $k^v_{b_i} = k^{cv}_{c_{i+l}} - k^{oav}_{a_{i+l}}$; $m_i \in I(1, i-1)$; $k_{c_{i+l}}$ stands for the total number of censored objects/species/infective/quitting covered over the subinterval $[T^f_{j-1i+l}, T^f_{j-1i+l+1})$; $k_{a_{i+l}}$ denotes the total number of admitting/entering/joining/susceptible/etc covered over the subinterval $[T^f_{j-1i+l}, T^f_{j-1i+l+1})$; $k_{f_{i+l}}$ is the total number of failures covered over the subinterval $[T^f_{j-1i+l}, T^f_{j-1i+l+1})$; $k_{b_{i+l}} = k_{c_{i+l}} + k_{a_{i+l}}$.

Remark 6.6.1 In the case where $k_b = 0$, Then (6.6.4) reduces to

$$\begin{cases} \hat{\lambda}_{j,m_j} = \dfrac{\sum\limits_{l=-m_i}^{-1} \Big[- \mathbb{E}(\Delta z_{j+l+1} | \mathcal{G}_{j+l}) + \Gamma^{no}_{j-1i+l}\Big]}{\sum\limits_{l=-m_j}^{-1} z_{j+l} \Delta T^f_{j+1+1}}, \\ \\ \hat{\sigma}^2_{j,m_j} = 2 \left[\dfrac{\sum\limits_{l=-m_i}^{-1} \begin{pmatrix} \mathbb{E}[\Delta V(T^f_{j+l+1}, z_{j+l+1}) \mid \mathcal{G}_{j+l}] + \Gamma^{nov}_{j-1i+l} - \frac{\partial}{\partial t} V(T^f_{j+l}, z_{j+l}) \Delta T^f_{j+l+1} + \\ z_{j+l} \lambda_{j,m_j} \frac{\partial}{\partial z} V(T^f_{j+l}, z_{j+l}) \Delta T^f_{j+l+1} \end{pmatrix}}{\sum\limits_{l=-m_i}^{-1} z^2_{j+l} \frac{\partial^2}{\partial z^2} V(T^f_{j+l}, z_{j+l}) \Delta T^f_{j+l+1}} \right]. \end{cases} \tag{6.6.5}$$

In short, the usage of the transformed continuous-time stochastic dynamic hybrid model for time-to-event process (5.3.11) and discrete-time interconnected hybrid dynamic algorithms of local sample mean lead to an innovative alternative method for parameter and state estimation problems for continuous-time dynamic models described by both linear and nonlinear stochastic differential equations.

Example 6.6.1 Using the parameter estimates in Example 5.4.5, (6.6.4) becomes:

$$\begin{cases} \hat{\lambda}_{i,m_i} = \dfrac{\sum\limits_{l=-m_i}^{-1} \Big[- \mathbb{E}(\Delta z_{j-1i+l+1} | \mathcal{G}_{j-1i+l}) + \Gamma^{no}_{j-1i+l} - k_{c_{i+l}} + k_{a_{i+l}}\Big]}{\sum\limits_{l=-m_i}^{-1} \sum\limits_{n=1}^{k_{b_{i+l}}+1} z(T^{c/a}_{j-1i+ln-1}) \Delta T^{c/a}_{j-1i+ln}}, \\ \\ \hat{\sigma}^2_{j,m_j} = \dfrac{\sum\limits_{l=m_i}^{-1} \Big[\mathbb{E}(\Delta z^2_{j+l+1} | \mathcal{G}_{j+l}) - \Gamma^{no}_{j-1i+l} + k_{c_{i+l}} - k_{a_{i+l}}\Big]}{\sum\limits_{l=-m_i}^{-1} z(T^{c/a}_{j+l+1})^2 \Delta T^{c/a}_{j+l+1}} + 2\hat{\lambda}_{j,m_j}. \end{cases} \tag{6.6.6}$$

If in addition, $k_b = 0$, then (6.6.6) reduces to:

$$\begin{cases} \hat{\lambda}_{j,m_j} = -\dfrac{\sum\limits_{l=-m_i}^{-1} \Big[\mathbb{E}(\Delta z_{j+l+1} | \mathcal{G}_{j+l}) + \Gamma^{no}_{j-1i+l}\Big]}{\sum\limits_{l=-m_j}^{-1} z_{j+l} \Delta T^f_{j+1+1}}, \\ \\ \hat{\sigma}^2_{j,m_j} = \dfrac{\sum\limits_{l=-m_i}^{-1} \Big[\mathbb{E}(\Delta z^2_{j+l+1} | \mathcal{G}_{j+l}) - \Gamma^{nov}_{j-1i+l}\Big]}{\sum\limits_{l=-m_i}^{-1} z^2_{j+l+1} \Delta T^f_{j+l+1}} + 2\hat{\lambda}_{j,m_j}. \end{cases} \tag{6.6.7}$$

EXAMPLE 6.6.2 Employing the parameter estimates in Example 5.4.6, (6.6.4) reduces to:

$$\begin{cases} \hat{\lambda}_{i,m_i} = \dfrac{\sum\limits_{l=-m_i}^{-1} \left[-\mathbb{E}(\Delta z_{j-1i+l+1} | \mathcal{G}_{j-1i+l}) + \Gamma^{no}_{j-1i+l} - k_{c_{i+l}} + k_{a_{i+l}} \right]}{\sum\limits_{l=-m_i}^{-1} \sum\limits_{n=1}^{k_{b_{i+l}}+1} z(T^{c/a}_{j-1i+ln-1}) \Delta T^{c/a}_{j-1i+ln}}, \\ \\ \hat{\sigma}^2_{j,m_j} = -2 \left[\hat{\lambda}_{j,m_j} + \dfrac{\sum_{l=-m_j}^{-1} \left[\mathbb{E}[\Delta \ln(\Delta z_{j+l+1}) \mid \mathcal{G}_{j+l}] - \Gamma^{nov}_{j-1i+l} + k_{c_{i+l}} - k_{a_{i+l}} \right]}{\sum_{l=-m_j}^{-1} \Delta T^{c/a}_{j+l+1}} \right]. \end{cases} \quad (6.6.8)$$

If in addition, $k_b = 0$, then (6.6.8) becomes:

$$\begin{cases} \hat{\lambda}_{j,m_j} = \dfrac{\sum\limits_{l=-m_i}^{-1} \left[-\mathbb{E}(\Delta z_{j+l+1} | \mathcal{G}_{j+l}) + \Gamma^{no}_{j-1i+l} \right]}{\sum\limits_{l=-m_j}^{-1} z(T^{f}_{j+l}) \Delta T_{j+1+1}}, \\ \\ \hat{\sigma}^2_{j,m_j} = -2 \left[\hat{\lambda}_{j,m_j} + \dfrac{\sum_{l=-m_j}^{-1} \left[\mathbb{E}[\Delta \ln(\Delta z_{j+l+1}) \mid \mathcal{G}_{j+l}] - \Gamma^{nov}_{j-1i+l} \right]}{\sum_{l=-m_j}^{-1} \Delta T^{f}_{j+l+1}} \right]. \end{cases} \quad (6.6.9)$$

6.6.1 Computational Algorithm

The numerical approximation and simulation processes need to be synchronized with the existing data collection schedule process in the context of the partition of $[t_0, \mathfrak{T}]$. For each $i \in I(1, k_f)$ and $j \in I(1, n)$, we assume that T^f_{j-1i} is a failure scheduled time clock for the $j-1i$-th collected data of the failure state of a system under investigation. From Definition 6.6.2, for each $m_i \in OS_{j-1i} = I(1, i)$ at T^f_{j-1i}, we pick a m_i local admissible sequence $\{F^l z_{j-1i}\}^0_{l=-m_i}$. Using the terms of this sequence and (6.6.4), we compute the state and parameter estimates of the continuous-time dynamic model (5.3.11) for a choice of initial values $S(T_0) = S_0$ specified in Sub-section 6.4. These estimates form a local finite sequence of parameter estimates at T^f_{j-1i} corresponding to $AS_{j-1i} = \{z_{m_i,j-1i} : m_i \in I(1, i)\}$ for each $i \in I(1, k_f)$. The Principle of Mathematical Induction [33] is employed for the development of a conceptual computational scheme.

For each admissible sequence in AS_{j-1i}, let $z^s_{m_i,j-1i}$ be a simulated value of $z_{m_i,j-1i}$ at T^f_{j-1i}. This engenders an m_i local sequence of simulated data $\{z^s_{m_i,j-1i}\}_{m_i \in OS_{j-1,i}}$. The simulated $z^s_{m_i,j-1i}$ value satisfies the following scheme:

$$z^s_{j-1i} = z^s_{j-1i-1} - \hat{\lambda}_{j-1i-1}\, z^s_{j-1i-1} \Delta T_{j-1i} + \hat{\sigma}_{j-1i-1}\, z^s_{j-1i-1} \Delta w_{j-1i} - k_{c_i} + k_{a_i}. \quad (6.6.10)$$

To find the best estimate of $z(T^f_{j-1i})$ with a best choice of initial state (Section 6.4), let us define a mean-square estimate error of $z(T^f_{j-1i})$ to be

$$\Xi_{m_i,j-1i,z_{j-1i}} = \Big(z(T^f_{j-1i}) - z^s_{m_i,j-1i}\Big)^2 \tag{6.6.11}$$

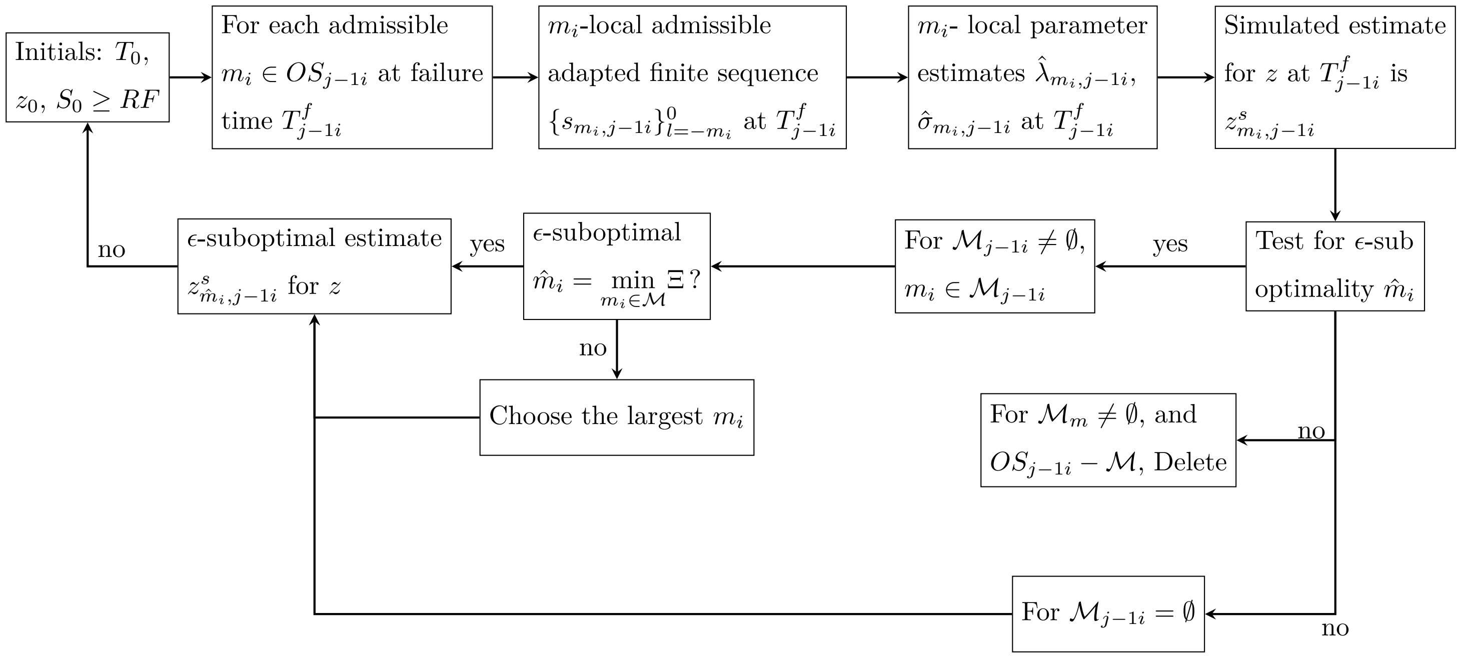
relative to each member of the term of local admissible sequences $\{z^s_{m_i,j-1i}\}_{m_i \in OS_{j-1,i}}$ of simulated values. For any preassigned arbitrary small positive number ϵ and for each failure time T^f_{j-1i}, we find the best estimate from admissible simulated values. We determine the following sub-optimal admissible set of size of moving average at T^f_{j-1i} as:

$$\mathcal{M}_{j-1i} = \{m_i : \Xi_{m_i,j-1i,z_{j-1i}} < \epsilon \quad \text{for} \quad m_i \in OS_{j-1i}\}. \tag{6.6.12}$$

Among these collected sub-optimal set of values, the value that gives the minimum $\Xi_{m_i,j-1i,z_{j-1i}}$ is recorded as $\hat{m}_i$. The parameters corresponding to $\hat{m}_i$ is referred as the ϵ-level sub-optimal estimates of the true parameters. These sub-optimal estimates are estimated at time T^f_{j-1i} with $\hat{m}_i$. The simulated value $z^s_{\hat{m}_i,j-1i}$ at T^f_{j-1i} corresponding to $\hat{m}_i$ is recored as the best sub-optimal estimate for dynamic state $z(T_{j-1i})$ at T^f_{j-1i}. Having obtained the best estimate for λ and σ^2, we then proceed to find the best sub-optimal estimate for the survival state function at T^f_{j-1i} via the following discrete-time simulation dynamic process:

$$\hat{S}(T_{j-1i}) = \hat{S}(T_{j-1i-1}) - \hat{S}(T_{j-1i-1})\hat{\lambda}(T_{j-1i-1}, S_{j-1i-1})\Delta T_{j-1i} + \hat{S}(T_{j-1i})\hat{\sigma}(T_{j-1i-1}, S_{j-1i-1})\Delta w(T_{j-1i}) \tag{6.6.13}$$

Finally, an estimate of $S_{\hat{m}_i,j-1i}$ at T^f_{j-1i} corresponding to $\hat{m}_i$ is also recorded as the best estimate for survival state $S(T_{j-1i})$ at T^f_{j-1i}. Moreover, a conceptual computational modified LLGMM algorithm is outlined in Flowchart 15. Here, we choose T_0, S_0, and z_0 so that $S_0 \geq RF$, where RF denotes a relative frequency at the initial time T_0.



Flowchart 15.: LLGMM-type Conceptual Computational Algorithm

We present an algorithm and flowchart for the simulation scheme described above.

Given initials T_0, S_0, z_0, ϵ,

for $i = 1$ to k_f **do**

for $m_i = 1$ to i **do**

Compute $\hat{\lambda}_{m_i,j-1i}, \hat{\sigma}_{m_i,j-1i}$

for $m_i = 0$ to i **do**

Compute $z^s_{m_i,j-1i}$, $\Xi_{m_i,j-1i,z_{j-1i}}$

end for

end for

end for

if $\Xi_{m_i,i,z_{j-1i}} < \epsilon$ **then**

Save $\hat{m}_i$

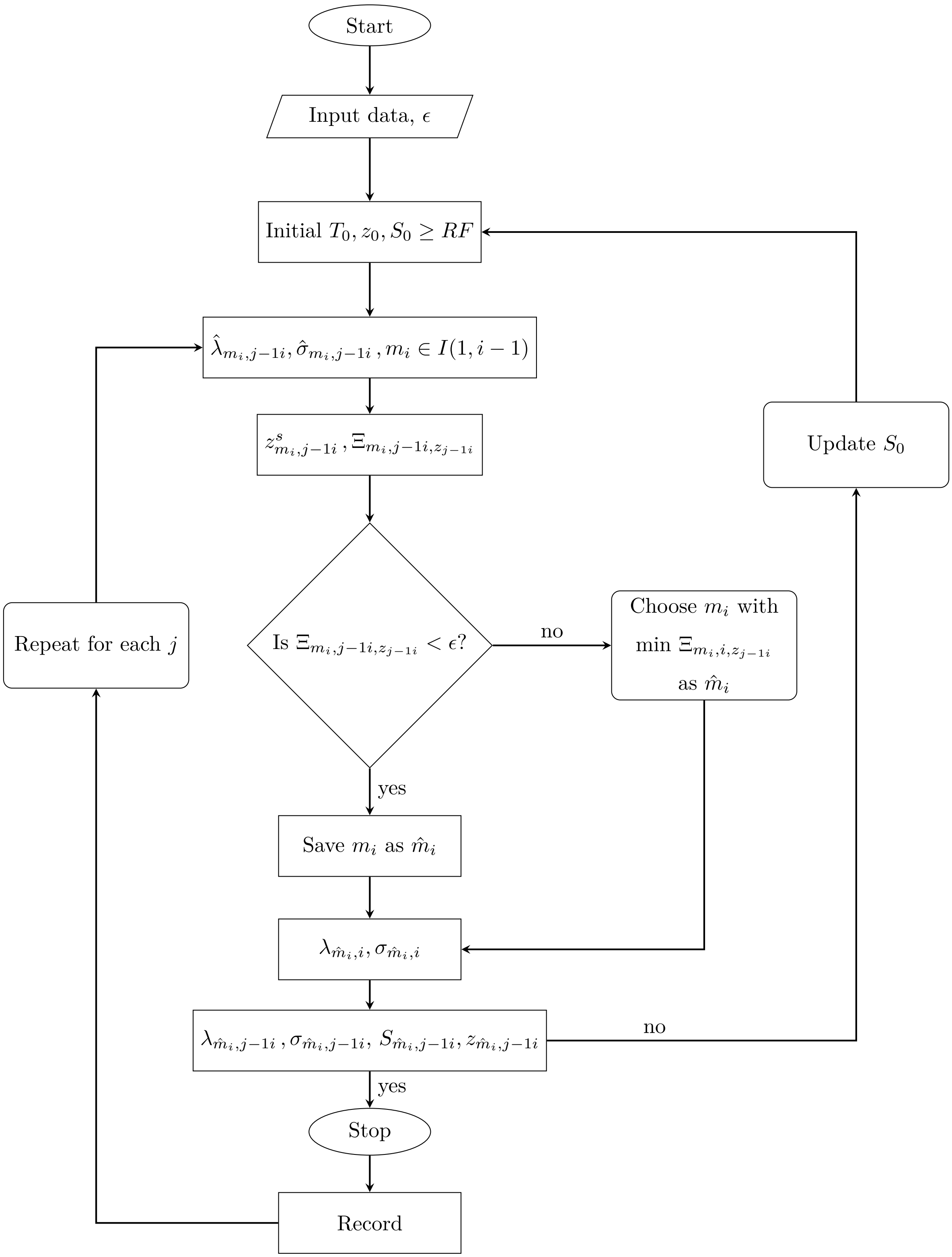
else

Find $\hat{m}_i$ that minimizes $\Xi_{m_i,j-1i,z_{j-1i}}$

end if

Compute $\lambda_{\hat{m}_i,j-1i}$, $\sigma_{\hat{m}_i,j-1i}$, $z^s_{\hat{m}_i,j-1i}$, $S_{\hat{m}_i,j-1i}$.

Algorigthm 16.: Simulation scheme



Flowchart 17.: LLGMM-type Simulation Algorithm

In the following, we give illustrations on how to apply modified LLGMM method to three data sets in Tables 15, 18 and 21.

Illustration 6.6.1 [Application of LLGMM-type Conceptual Computational Algorithm to the datasets in Table 15]
We apply the modified LLGMM procedure to the dataset in Table 15. Using (6.6.7), (6.6.10), and (6.6.13) with $\epsilon = 0.001$, the results are summarized in Table 24. Utilizing (6.6.9), (6.6.10), and (6.6.13) with $\epsilon = 0.001$, the results are exhibited in Table 25.

Illustration 6.6.2 [Application of LLGMM-type Conceptual Computational Algorithm to the datasets in Table 18]
We apply the above procedure to the dataset in Table 18. Employing (6.6.7), (6.6.10), and (6.6.13) with $\epsilon = 0.001$, the results are summarized in Table 26. Utilizing (6.6.9), (6.6.10), and (6.6.13) with $\epsilon = 0.001$, the simulation results are recorded in Table 27.

Illustration 6.6.3 [Application of LLGMM-type Conceptual Computational Algorithm to the datasets in Table 21]
We apply the above procedure to the dataset in Table 21. Using (6.6.6), (6.6.10), and (6.6.13) with $\epsilon = 0.001$, the results are recorded in Table 28. Utilizing (6.6.8), (6.6.10), and (6.6.13) with $\epsilon = 0.001$, the simulation results are summarized in Table 29 .

Table 24: Modified LLGMM Based Estimates using $S_0 = 0.99000$, 0.99900, 0.99990, 0.99999, 0.999999 by by utilizing (6.6.7), (6.6.10), and (6.6.13) with $\epsilon = 0.001$

		$S_0 = 0.99000$			$S_0 = 0.99900$			$S_0 = 0.9999$			$S_0 = 0.99999$			$S_0 = 0.999999$		
T_j^f	$\hat{m}_j$	$\lambda_{j,\hat{m}_j}$	$\sigma_{j,\hat{m}_j}$	$S_{j,\hat{m}_j}$	$\lambda_{j,\hat{m}_j}$	$\sigma_{j,\hat{m}_j}$	$S_{j,\hat{m}_j}$	$\lambda_{j,\hat{m}_j}$	$\sigma_{j,\hat{m}_j}$	$S_{j,\hat{m}_j}$	$\lambda_{j,\hat{m}_j}$	$\sigma_{j,\hat{m}_j}$	$S_{j,\hat{m}_j}$	$\lambda_{j,\hat{m}_j}$	$\sigma_{j,\hat{m}_j}$	$S_{j,\hat{m}_j}$
0		0.0952	0.0952	0.9900	0.0952	0.0952	0.9990	0.0952	0.0952	0.9999	0.0952	0.0952	0.99999	0.0952	0.0952	0.999999
1.00	1	0.0952	0.0952	0.8958	0.0952	0.0952	0.9040	0.0952	0.0952	0.9048	0.0952	0.0952	0.9049	0.0952	0.0952	0.9049
2.00	1	0.1053	0.1053	0.8017	0.1053	0.1053	0.8090	0.1053	0.1053	0.8097	0.1053	0.1053	0.8098	0.1053	0.1053	0.8098
3.00	1	0.0588	0.0588	0.7546	0.0588	0.0588	0.7614	0.0588	0.0588	0.7621	0.0588	0.0588	0.7622	0.0588	0.0588	0.7622
4.00	1	0.1250	0.1250	0.6604	0.1250	0.1250	0.6664	0.1250	0.1250	0.6670	0.1250	0.1250	0.6670	0.1250	0.1250	0.6670
5.00	1	0.1429	0.1429	0.5662	0.1429	0.1429	0.5713	0.1429	0.1429	0.5718	0.1429	0.1429	0.5719	0.1429	0.1429	0.5719
8.00	1	0.1111	0.1925	0.3776	0.1111	0.1925	0.3810	0.1111	0.1925	0.3814	0.1111	0.1925	0.3814	0.1111	0.1925	0.3814
11.00	1	0.0833	0.1443	0.2833	0.0833	0.1443	0.2858	0.0833	0.1443	0.2861	0.0833	0.1443	0.2861	0.0833	0.1443	0.2861
12.00	1	0.3333	0.3333	0.1890	0.3333	0.3333	0.1907	0.3333	0.3333	0.1909	0.3333	0.3333	0.1909	0.3333	0.3333	0.1909
15.00	1	0.0833	0.1443	0.1418	0.0833	0.1443	0.1430	0.0833	0.1443	0.1432	0.0833	0.1443	0.1432	0.0833	0.1443	0.1432
17.00	1	0.1667	0.2357	0.0945	0.1667	0.2357	0.0954	0.1667	0.2357	0.0955	0.1667	0.2357	0.0955	0.1667	0.2357	0.0955
22.00	1	0.1000	0.2236	0.0473	0.1000	0.2236	0.0477	0.1000	0.2236	0.0478	0.1000	0.2236	0.0478	0.1000	0.2236	0.0478
23.00	1	1.0000	1.0000	0.0001	1.0000	1.0000	0.0001	1.0000	1.0000	0.0001	1.0000	1.0000	0.0001	1.0000	1.0000	0.0001

Table 25: Modified LLGMM Based Estimates using $S_0 = 0.99000$, 0.99900, 0.99990, 0.99999, 0.999999 by utilizing (6.6.9), (6.6.10), and (6.6.13) with $\epsilon = 0.001$

		$S_0 = 0.99000$			$S_0 = 0.99900$			$S_0 = 0.9999$			$S_0 = 0.99999$			$S_0 = 0.999999$		
T_j^f	$\hat{m}_j$	$\lambda_{j,\hat{m}_j}$	$\sigma_{j,\hat{m}_j}$	$S_{j,\hat{m}_j}$	$\lambda_{j,\hat{m}_j}$	$\sigma_{j,\hat{m}_j}$	$S_{j,\hat{m}_j}$	$\lambda_{j,\hat{m}_j}$	$\sigma_{j,\hat{m}_j}$	$S_{j,\hat{m}_j}$	$\lambda_{j,\hat{m}_j}$	$\sigma_{j,\hat{m}_j}$	$S_{j,\hat{m}_j}$	$\lambda_{j,\hat{m}_j}$	$\sigma_{j,\hat{m}_j}$	$S_{j,\hat{m}_j}$
0		0.0952	0.0984	0.9900	0.0952	0.0984	0.9990	0.0952	0.0984	0.9999	0.0952	0.0984	0.99999	0.0952	0.0984	0.999999
1.00	1	0.0952	0.0984	0.8958	0.0952	0.0984	0.9040	0.0952	0.0984	0.9048	0.0952	0.0984	0.9049	0.0952	0.0984	0.9049
2.00	1	0.1053	0.1092	0.8017	0.1053	0.1092	0.8090	0.1053	0.1092	0.8097	0.1053	0.1092	0.8098	0.1053	0.1092	0.8098
3.00	1	0.0588	0.0600	0.7546	0.0588	0.0600	0.7614	0.0588	0.0600	0.7621	0.0588	0.0600	0.7622	0.0588	0.0600	0.7622
4.00	1	0.1250	0.1306	0.6604	0.1250	0.1306	0.6664	0.1250	0.1306	0.6670	0.1250	0.1306	0.6670	0.1250	0.1306	0.6670
5.00	1	0.1429	0.1503	0.5662	0.1429	0.1503	0.5713	0.1429	0.1503	0.5718	0.1429	0.1503	0.5719	0.1429	0.1503	0.5719
8.00	1	0.1111	0.2193	0.3776	0.1111	0.2193	0.3810	0.1111	0.2193	0.3814	0.1111	0.2193	0.3814	0.1111	0.2193	0.3814
11.00	1	0.0833	0.1585	0.2833	0.0833	0.1585	0.2859	0.0833	0.1585	0.2861	0.0833	0.1585	0.2861	0.0833	0.1585	0.2861
12.00	1	0.3333	0.3798	0.1890	0.3333	0.3798	0.1907	0.3333	0.3798	0.1909	0.3333	0.3798	0.1909	0.3333	0.3798	0.1909
15.00	1	0.0833	0.1585	0.1418	0.0833	0.1585	0.1431	0.0833	0.1585	0.1432	0.0833	0.1585	0.1432	0.0833	0.1585	0.1432
17.00	1	0.1667	0.2686	0.0946	0.1667	0.2686	0.0954	0.1667	0.2686	0.0955	0.1667	0.2686	0.0955	0.1667	0.2686	0.0955
22.00	1	0.1000	0.2780	0.0473	0.1000	0.2780	0.0478	0.1000	0.2780	0.0478	0.1000	0.2780	0.0478	0.1000	0.2780	0.0478
23.00	1	1.0000	Inf	Inf	1.0000	Inf	Inf	1.0000	Inf	Inf	1.0000	Inf	Inf	1.0000	Inf	Inf

Table 26: Modified LLGMM Based Estimates using $S_0 = 0.99000, 0.99900, 0.99990, 0.99999, 0.999999$ by utilizing (6.6.7), (6.6.10), and (6.6.13) with $\epsilon = 0.001$

		$S_0 = 0.99000$			$S_0 = 0.99900$			$S_0 = 0.9999$			$S_0 = 0.99999$			$S_0 = 0.999999$		
T_j^f	$\hat{m}_j$	$\lambda_{j,\hat{m}_j}$	$\sigma_{j,\hat{m}_j}$	$S_{j,\hat{m}_j}$	$\lambda_{j,\hat{m}_j}$	$\sigma_{j,\hat{m}_j}$	$S_{j,\hat{m}_j}$	$\lambda_{j,\hat{m}_j}$	$\sigma_{j,\hat{m}_j}$	$S_{j,\hat{m}_j}$	$\lambda_{j,\hat{m}_j}$	$\sigma_{j,\hat{m}_j}$	$S_{j,\hat{m}_j}$	$\lambda_{j,\hat{m}_j}$	$\sigma_{j,\hat{m}_j}$	$S_{j,\hat{m}_j}$
0		0.0024	0.0103	0.99	0.0024	0.0103	0.999	0.0024	0.0103	0.9999	0.0024	0.0103	0.999999	0.0024	0.0103	0.999999
17.88	1	0.0024	0.0103	0.9470	0.0024	0.0103	0.9556	0.0024	0.0103	0.9564	0.0024	0.0103	0.9565	0.0024	0.0103	0.9565
28.92	1	0.0041	0.0137	0.9039	0.0041	0.0137	0.9122	0.0041	0.0137	0.9130	0.0041	0.0137	0.9131	0.0041	0.0137	0.9131
33.00	1	0.0117	0.0236	0.8609	0.0117	0.0236	0.8688	0.0117	0.0236	0.8695	0.0117	0.0236	0.8696	0.0117	0.0236	0.8696
41.52	1	0.0059	0.0171	0.8179	0.0059	0.0171	0.8253	0.0059	0.0171	0.8261	0.0059	0.0171	0.8262	0.0059	0.0171	0.8262
42.12	1	0.0877	0.0679	0.7749	0.0877	0.0679	0.7820	0.0877	0.0679	0.7827	0.0877	0.0679	0.7827	0.0877	0.0679	0.7827
45.60	1	0.0160	0.0298	0.7319	0.0160	0.0298	0.7386	0.0160	0.0298	0.7392	0.0160	0.0298	0.7393	0.0160	0.0298	0.7393
48.40	1	0.0210	0.0352	0.6889	0.0210	0.0352	0.6951	0.0210	0.0352	0.6958	0.0210	0.0352	0.6958	0.0210	0.0352	0.6958
51.84	1	0.0182	0.0337	0.6459	0.0182	0.0337	0.6517	0.0182	0.0337	0.6523	0.0182	0.0337	0.6524	0.0182	0.0337	0.6524
51.96	1	0.5556	0.1925	0.6030	0.5556	0.1925	0.6084	0.5556	0.1925	0.6090	0.5556	0.1925	0.6091	0.5556	0.1925	0.6091
54.12	1	0.0331	0.0486	0.5599	0.0331	0.0486	0.5650	0.0331	0.0486	0.5655	0.0331	0.0486	0.5656	0.0331	0.0486	0.5656
55.56	1	0.0534	0.0641	0.5169	0.0534	0.0641	0.5216	0.0534	0.0641	0.5221	0.0534	0.0641	0.5221	0.0534	0.0641	0.5221
67.80	1	0.0068	0.0238	0.4739	0.0068	0.0238	0.4782	0.0068	0.0238	0.4786	0.0068	0.0238	0.4786	0.0068	0.0238	0.4786
68.64	1	0.2165	0.1984	0.3878	0.2165	0.1984	0.3913	0.2165	0.1984	0.3917	0.2165	0.1984	0.3917	0.2165	0.1984	0.3917
68.88	1	0.4630	0.2268	0.3448	0.4630	0.2268	0.3480	0.4630	0.2268	0.3483	0.4630	0.2268	0.3483	0.4630	0.2268	0.3483
84.12	1	0.0082	0.0320	0.3018	0.0082	0.0320	0.3045	0.0082	0.0320	0.3048	0.0082	0.0320	0.3048	0.0082	0.0320	0.3048
93.12	1	0.0159	0.0476	0.2587	0.0159	0.0476	0.2610	0.0159	0.0476	0.2613	0.0159	0.0476	0.2613	0.0159	0.0476	0.2613
98.64	1	0.0302	0.0709	0.2156	0.0302	0.0709	0.2175	0.0302	0.0709	0.2177	0.0302	0.0709	0.2178	0.0302	0.0709	0.2178
105.12	1	0.0309	0.0786	0.1725	0.0309	0.0786	0.1741	0.0309	0.0786	0.1742	0.0309	0.0786	0.1742	0.0309	0.0786	0.1742
105.84	1	0.3472	0.2946	0.1294	0.3472	0.2946	0.1306	0.3472	0.2946	0.1307	0.3472	0.2946	0.1307	0.3472	0.2946	0.1307
127.92	1	0.0151	0.0709	0.0863	0.0151	0.0709	0.0871	0.0151	0.0709	0.0872	0.0151	0.0709	0.0872	0.0151	0.0709	0.0872
128.04	1	4.1667	1.4434	0.0433	4.1667	1.4434	0.0437	4.1667	1.4434	0.0437	4.1667	1.4434	0.0438	4.1667	1.4434	0.0438
173.40	1	0.0220	0.1485	0.0000	0.0220	0.1485	0.0000	0.0220	0.1485	0.0000	0.0220	0.1485	0.0000	0.0220	0.1485	0.0000

Table 27: Modified LLGMM Based Estimates using $S_0 = 0.99000, 0.99900, 0.99990, 0.99999, 0.999999$ by employing (6.6.9), (6.6.10), and (6.6.13) with $\epsilon = 0.001$

		$S_0 = 0.99000$			$S_0 = 0.99900$			$S_0 = 0.9999$			$S_0 = 0.99999$			$S_0 = 0.999999$		
T_j^f	$\hat{m}_j$	$\lambda_{j,\hat{m}_j}$	$\sigma_{j,\hat{m}_j}$	$S_{j,\hat{m}_j}$	$\lambda_{j,\hat{m}_j}$	$\sigma_{j,\hat{m}_j}$	$S_{j,\hat{m}_j}$	$\lambda_{j,\hat{m}_j}$	$\sigma_{j,\hat{m}_j}$	$S_{j,\hat{m}_j}$	$\lambda_{j,\hat{m}_j}$	$\sigma_{j,\hat{m}_j}$	$S_{j,\hat{m}_j}$	$\lambda_{j,\hat{m}_j}$	$\sigma_{j,\hat{m}_j}$	$S_{j,\hat{m}_j}$
0		0.0024	0.0104	0.99	0.0024	0.0104	0.9990	0.0024	0.0104	0.9999	0.0024	0.0104	0.99999	0.0024	0.0104	0.999999
17.88	1	0.0024	0.0104	0.9470	0.0024	0.0104	0.9556	0.0024	0.0104	0.9564	0.0024	0.0104	0.9565	0.0024	0.0104	0.9565
28.92	1	0.0041	0.0139	0.9039	0.0041	0.0139	0.9122	0.0041	0.0139	0.9130	0.0041	0.0139	0.9131	0.0041	0.0139	0.9131
33.00	1	0.0117	0.0240	0.8609	0.0117	0.0240	0.8688	0.0117	0.0240	0.8695	0.0117	0.0240	0.8696	0.0117	0.0240	0.8696
41.52	1	0.0059	0.0174	0.8179	0.0059	0.0174	0.8253	0.0059	0.0174	0.8261	0.0059	0.0174	0.8262	0.0059	0.0174	0.8262
42.12	1	0.0877	0.0692	0.7749	0.0877	0.0692	0.7820	0.0877	0.0692	0.7827	0.0877	0.0692	0.7827	0.0877	0.0692	0.7827
45.60	1	0.0160	0.0304	0.7319	0.0160	0.0304	0.7386	0.0160	0.0304	0.7392	0.0160	0.0304	0.7393	0.0160	0.0304	0.7393
48.40	1	0.0210	0.0359	0.6889	0.0210	0.0359	0.6952	0.0210	0.0359	0.6958	0.0210	0.0359	0.6958	0.0210	0.0359	0.6958
51.84	1	0.0182	0.0344	0.6459	0.0182	0.0344	0.6517	0.0182	0.0344	0.6523	0.0182	0.0344	0.6524	0.0182	0.0344	0.6524
51.96	1	0.5556	0.1969	0.6030	0.5556	0.1969	0.6085	0.5556	0.1969	0.6090	0.5556	0.1969	0.6091	0.5556	0.1969	0.6091
54.12	1	0.0331	0.0498	0.5599	0.0331	0.0498	0.5650	0.0331	0.0498	0.5655	0.0331	0.0498	0.5656	0.0331	0.0498	0.5656
55.56	1	0.0534	0.0658	0.5169	0.0534	0.0658	0.5216	0.0534	0.0658	0.5221	0.0534	0.0658	0.5221	0.0534	0.0658	0.5221
67.80	1	0.0068	0.0245	0.4739	0.0068	0.0245	0.4782	0.0068	0.0245	0.4786	0.0068	0.0245	0.4786	0.0068	0.0245	0.4786
68.64	1	0.2165	0.2119	0.3878	0.2165	0.2119	0.3914	0.2165	0.2119	0.3917	0.2165	0.2119	0.3917	0.2165	0.2119	0.3917
68.88	1	0.4630	0.2358	0.3449	0.4630	0.2358	0.3480	0.4630	0.2358	0.3483	0.4630	0.2358	0.3483	0.4630	0.2358	0.3483
84.12	1	0.0082	0.0335	0.3018	0.0082	0.0335	0.3045	0.0082	0.0335	0.3048	0.0082	0.0335	0.3048	0.0082	0.0335	0.3048
93.12	1	0.0159	0.0501	0.2587	0.0159	0.0501	0.2610	0.0159	0.0501	0.2613	0.0159	0.0501	0.2613	0.0159	0.0501	0.2613
98.64	1	0.0302	0.0753	0.2156	0.0302	0.0753	0.2176	0.0302	0.0753	0.2177	0.0302	0.0753	0.2178	0.0302	0.0753	0.2178
105.12	1	0.0309	0.0845	0.1725	0.0309	0.0845	0.1741	0.0309	0.0845	0.1742	0.0309	0.0845	0.1742	0.0309	0.0845	0.1742
105.84	1	0.3472	0.3235	0.1294	0.3472	0.3235	0.1306	0.3472	0.3235	0.1307	0.3472	0.3235	0.1308	0.3472	0.3235	0.1308
127.92	1	0.0151	0.0808	0.0863	0.0151	0.0808	0.0871	0.0151	0.0808	0.0872	0.0151	0.0808	0.0872	0.0151	0.0808	0.0872
128.04	1	4.1667	1.7942	0.0434	4.1667	1.7942	0.0438	4.1667	1.7942	0.0438	4.1667	1.7942	0.0438	4.1667	1.7942	0.0438
173.40	1	0.0220	Inf	Inf	0.0220	Inf	Inf	0.0220	Inf	Inf	0.0220	Inf	Inf	0.0220	Inf	Inf

Table 28: Modified LLGMM Based Estimates using $S_0 = 0.99000, 0.99900, 0.99990, 0.99999, 0.999999$ by by utilizing (6.6.6), (6.6.10), and (6.6.13) with $\epsilon = 0.001$

		$S_0 = 0.99000$			$S_0 = 0.99900$			$S_0 = 0.9999$			$S_0 = 0.99999$			$S_0 = 0.999999$		
T_j^f	$\hat{m}_j$	$\lambda_{j,\hat{m}_j}$	$\sigma_{j,\hat{m}_j}$	$S_{j,\hat{m}_j}$	$\lambda_{j,\hat{m}_j}$	$\sigma_{j,\hat{m}_j}$	$S_{j,\hat{m}_j}$	$\lambda_{j,\hat{m}_j}$	$\sigma_{j,\hat{m}_j}$	$S_{j,\hat{m}_j}$	$\lambda_{j,\hat{m}_j}$	$\sigma_{j,\hat{m}_j}$	$S_{j,\hat{m}_j}$	$\lambda_{j,\hat{m}_j}$	$\sigma_{j,\hat{m}_j}$	$S_{j,\hat{m}_j}$
0		0.0238	0.0583	0.9900	0.0238	0.0583	0.9990	0.0238	0.0583	0.9999	0.0238	0.0583	0.99999	0.0238	0.0583	0.999999
6.00	1	0.0238	0.0583	0.8486	0.0238	0.0583	0.8564	0.0238	0.0583	0.8571	0.0238	0.0583	0.8572	0.0238	0.0583	0.8572
7.00	1	0.0588	0.0588	0.7988	0.0588	0.0588	0.8061	0.0588	0.0588	0.8068	0.0588	0.0588	0.8069	0.0588	0.0588	0.8069
10.00	1	0.0213	0.0566	0.7479	0.0213	0.0566	0.7547	0.0213	0.0566	0.7553	0.0213	0.0566	0.7554	0.0213	0.0566	0.7554
11.00	1	0.0769	0.0769	0.6904	0.0769	0.0769	0.6967	0.0769	0.0769	0.6973	0.0769	0.0769	0.6974	0.0769	0.0769	0.6974
16.00	1	0.0175	0.0532	0.6299	0.0175	0.0532	0.6356	0.0175	0.0532	0.6362	0.0175	0.0532	0.6363	0.0175	0.0532	0.6363
22.00	2	0.0187	0.0759	0.5593	0.0187	0.0759	0.5644	0.0187	0.0759	0.5649	0.0187	0.0759	0.5650	0.0187	0.0759	0.5650
23.00	6	0.0258	0.0842	0.5450	0.0258	0.0842	0.5499	0.0258	0.0842	0.5504	0.0258	0.0842	0.5505	0.0258	0.0842	0.5505

Table 29: Modified LLGMM Based Estimates using $S_0 = 0.99000, 0.99900, 0.99990, 0.99999, 0.999999$ by by utilizing (6.6.8), (6.6.10), and (6.6.13) with $\epsilon = 0.001$

		$S_0 = 0.99000$			$S_0 = 0.99900$			$S_0 = 0.9999$			$S_0 = 0.99999$			$S_0 = 0.999999$		
T_j^f	$\hat{m}_j$	$\lambda_{j,\hat{m}_j}$	$\sigma_{j,\hat{m}_j}$	$S_{j,\hat{m}_j}$	$\lambda_{j,\hat{m}_j}$	$\sigma_{j,\hat{m}_j}$	$S_{j,\hat{m}_j}$	$\lambda_{j,\hat{m}_j}$	$\sigma_{j,\hat{m}_j}$	$S_{j,\hat{m}_j}$	$\lambda_{j,\hat{m}_j}$	$\sigma_{j,\hat{m}_j}$	$S_{j,\hat{m}_j}$	$\lambda_{j,\hat{m}_j}$	$\sigma_{j,\hat{m}_j}$	$S_{j,\hat{m}_j}$
0		0.0238	0.0614	0.9900	0.0238	0.0614	0.9990	0.0238	0.0614	0.9999	0.0238	0.0614	0.99999	0.0238	0.0614	0.999999
6.00	1	0.0238	0.0614	0.8487	0.0238	0.0614	0.8564	0.0238	0.0614	0.8571	0.0238	0.0614	0.8572	0.0238	0.0614	0.8572
7.00	1	0.0588	0.0600	0.7988	0.0588	0.0600	0.8061	0.0588	0.0600	0.8068	0.0588	0.0600	0.8069	0.0588	0.0600	0.8069
10.00	1	0.0213	0.0587	0.7479	0.0213	0.0587	0.7547	0.0213	0.0587	0.7554	0.0213	0.0587	0.7554	0.0213	0.0587	0.7554
11.00	1	0.0769	0.0790	0.6904	0.0769	0.0790	0.6967	0.0769	0.0790	0.6973	0.0769	0.0790	0.6974	0.0769	0.0790	0.6974
16.00	1	0.0175	0.0551	0.6299	0.0175	0.0551	0.6356	0.0175	0.0551	0.6362	0.0175	0.0551	0.6363	0.0175	0.0551	0.6363
22.00	2	0.0187	0.0893	0.5593	0.0187	0.0893	0.5644	0.0187	0.0893	0.5649	0.0187	0.0893	0.5650	0.0187	0.0893	0.5650
23.00	6	0.0258	0.1548	0.5450	0.0258	0.1548	0.5500	0.0258	0.1548	0.5505	0.0258	0.1548	0.5505	0.0258	0.1548	0.5505

Remark 6.6.2 We remark that using the LLGMM-type estimation approach yields the almost close simulation results as the estimation procedure outlined in Illustrations 6.5.1, 6.5.2, and 6.5.3 for both data sets in Tables 15, 18, and 21.

In the following, we compare alternative innovative approach and modified LLGMM results with the well-known existing methods, namely, Maximum Likelihood and Kaplan-Meier approach.

6.7 Statistical Comparative Analysis with Existing Methods

In this subsection, the presented simulation results (with optimal initial data choice $S_0 = 0.99999$) is compared with the existing methods, namely, Maximum Likelihood [25] (by fitting a lognormal distribution to the data sets) and Kaplan-Meier [26] estimates. The simulation results are recored in Tables 30, 31, and 32.

Table 30: Comparison of survival function estimates for leukemia data set in Table 15

Failure Time: T_j	Innovative Approach $\hat{S}(T_j)$	Modified LLGMM $S_{j,\hat{m}_j}$	Maximum Likelihood Method: $\hat{S}_{ML}(T_j)$	Kaplan-Meier-type Estimate $\hat{S}_{KM}(T_j)$
0	0.99999	0.99999	1	1
1	0.9049	0.9049	0.9783	0.9048
2	0.8098	0.8098	0.8950	0.8095
3	0.7622	0.7622	0.7894	0.7619
4	0.6670	0.6670	0.6865	0.6667
5	0.5719	0.5719	0.5943	0.5714
8	0.3814	0.3814	0.3891	0.3810
11	0.2861	0.2861	0.2629	0.2857
12	0.1909	0.1909	0.2325	0.1905
15	0.1432	0.1432	0.1641	0.1429
17	0.0955	0.0955	0.1321	0.0952
22	0.0478	0.0478	0.0805	0.0476
23	0.0001	0.0001	0.0734	0.0000

Table 31: Comparison of survival function estimates for ball bearings data set in Table 18

Failure Time: T_j	Innovative Approach $\hat{S}(T_j)$	Modified LLGMM $S_{j,\hat{m}_j}$	Maximum Likelihood Method: $\hat{S}_{ML}(T_j)$	Kaplan-Meier-type Estimate $\hat{S}_{KM}(T_j)$
0.00	0.99999	0.99999	1	1
17.88	0.9565	0.9565	0.9924	0.9565
28.92	0.9131	0.9131	0.9938	0.9130
33.00	0.8696	0.8696	0.8947	0.8696
41.52	0.8262	0.8262	0.7916	0.8261
42.12	0.7827	0.7827	0.7836	0.7826
45.60	0.7393	0.7393	0.7364	0.7391
48.40	0.6958	0.6958	0.6978	0.6975
51.84	0.6524	0.6524	0.6500	0.6522
51.96	0.6091	0.6091	0.6489	0.6087
54.12	0.5656	0.5656	0.6195	0.5652
55.56	0.5221	0.5221	0.6002	0.5217
67.80	0.4786	0.4786	0.4493	0.4783
68.64	0.3917	0.3917	0.4399	0.3913
68.88	0.3483	0.3483	0.4373	0.3478
84.12	0.3048	0.3048	0.2940	0.3043
93.12	0.2613	0.2613	0.2310	0.2609
98.64	0.2178	0.2178	0.1988	0.2174
105.12	0.1742	0.1742	0.1666	0.1739
105.84	0.1307	0.1307	0.1634	0.1304
127.92	0.0872	0.0872	0.0895	0.0870
128.04	0.0438	0.0438	0.0892	0.0435
173.40	0.0000	0.0000	0.0270	0.0000

Table 32: Comparison of survival function estimates for leukemia data set in Table 21

Failure Time: T_j	Innovative Approach $\hat{S}(T_j)$	Modified LLGMM $S_{j,\hat{m}_j}$	Maximum Likelihood Method: $\hat{S}_{ML}(T_j)$	Kaplan-Meier-type Estimate $\hat{S}_{KM}(T_j)$
0	0.99999	0.99999	1	1
6	0.8572	0.8572	0.9228	0.8571
7	0.8069	0.8069	0.8978	0.8067
10	0.7554	0.7554	0.8184	0.7529
11	0.6974	0.6974	0.7920	0.6950
16	0.6363	0.6363	0.6684	0.6318
22	0.5650	0.5650	0.5456	0.5416
23	0.5505	0.5505	0.5278	0.4513

6.8 Forecasting

In this section, we sketch an outline of a forecasting problem. An ϵ-sub-optimal simulated value $S^s_{j,\hat{m}_j}$ at time T^f_j is used to define a forecast $S^f_{j,\hat{m}_j}$ for S_j at a time T^f_j.

Imitating the computational procedure outlined in Section 6.6, we find the estimate of the forecast $S^f_{j,\hat{m}_j}$ at time T_j as follows:

$$S^f_{j,\hat{m}_j} = S^s_{j-1,\hat{m}_{j-1}} - \lambda_{j-1,\hat{m}_{j-1}} S^s_{j-1,\hat{m}_{j-1}} \Delta T_j + \sigma_{j-1,\hat{m}_{j-1}} S^s_{j-1,\hat{m}_{j-1}} \Delta W_j \;, \tag{6.8.1}$$

where the estimates $\lambda_{j-1,\hat{m}_{j-1}}$ and $\sigma^2_{j-1,\hat{m}_{j-1}}$ are determined by using (6.6.5), respectively. We note that $S^f_{j,\hat{m}_j}$ is the ϵ-sub estimate for S_j at time T_j.

To determine $S^f_{j+1,\hat{m}_{j+1}}$, we need $\lambda_{j,\hat{m}_j}$ and $\sigma^2_{j,\hat{m}_j}$. The forecasted estimate $S^f_{j,\hat{m}_j}$ is used as the estimate of S_j at time T^f_j and also to estimate $\lambda_{j,\hat{m}_j}$ and $\sigma^2_{j,\hat{m}_j}$. Hence, we write $\lambda_{j,\hat{m}_j} \equiv \lambda_{S_{j-\hat{m}_{j+1}}, S_{j-\hat{m}_{j+2}}, \ldots, S_{j-1}, S^f_{j,\hat{m}_j}, \hat{m}_j}$. Similarly, we write $\sigma^2_{j,\hat{m}_j} \equiv \sigma^2_{S_{j-\hat{m}_{j+1}}, S_{j-\hat{m}_{j+2}}, \ldots, S_{j-1}, S^f_{j,\hat{m}_j}, \hat{m}_j}$. To find $S^f_{j+1,\hat{m}_j}$, we use the following estimates:

$$\begin{aligned}
\lambda_{j+1,\hat{m}_{j+1}} &\equiv \lambda_{S_{j-\hat{m}_j+2}, S_{j-\hat{m}_j+3}, \ldots S_{j-1}, S^f_{j,\hat{m}_j}, S^f_{j+1,\hat{m}_{j+1}}, \hat{m}_{j+1}} \\
\sigma^2_{j+1,\hat{m}_{j+1}} &\equiv \sigma^2_{S_{j-\hat{m}_j+2}, S_{j-\hat{m}_j+3}, \ldots S_{j-1}, S^f_{j,\hat{m}_j}, S^f_{j+1,\hat{m}_{j+1}}, \hat{m}_{j+1}} \;.
\end{aligned}$$

Continuing this process in this manner, we use the estimates

$$\begin{aligned}
\lambda_{j+n-1,\hat{m}_{j+n-1}} &\equiv \lambda_{S_{j-\hat{m}_j+n}, S_{j-\hat{m}_j+n+1}, \ldots S_{j-1}, S^f_{j,\hat{m}_j}, S^f_{j+1,\hat{m}_{j+1}}, \ldots S^f_{j+n-1,\hat{m}_{j+1}} \hat{m}_{j+1}} \;, \\
\sigma^2_{j+n-1,\hat{m}_{j+n-1}} &\equiv \sigma^2_{S_{j-\hat{m}_j+n}, S_{j-\hat{m}_j+n+1}, \ldots S_{j-1}, S^f_{j,\hat{m}_j}, S^f_{j+1,\hat{m}_{j+1}}, \ldots S^f_{j+n-1,\hat{m}_{j+1}} \hat{m}_{j+1}} \;.
\end{aligned}$$

to estimate $S^f_{j+n,\hat{m}_{j+n}}$.

6.8.1 Prediction/Confidence Intervals

To be able to assess the future uncertainty, we now discuss the prediction/confidence interval. We define the $100(1-\alpha)\%$ confidence interval for the forecast of the state $S^f_{j,\hat{m}_j}$ at time T_j as $S^f_{j,\hat{m}_j} \pm z_{1-\alpha/2} \sigma_{j-1,\hat{m}_{j-1}} S^f_{j-1,\hat{m}_{j-1}}$. The 95% confidence interval for the forecast at time T^f_j is given by

$$\left(S^f_{j,\hat{m}_j} - 1.96 \sigma_{j-1,\hat{m}_{j-1}} S^f_{j-1,\hat{m}_{j-1}},\; S^f_{j,\hat{m}_j} + 1.96 \sigma_{j-1,\hat{m}_{j-1}} S^f_{j-1,\hat{m}_{j-1}} \right) \;, \tag{6.8.2}$$

where the lower end denotes the lower bound of the state estimate and the upper end denotes the upper bound of the state estimate.

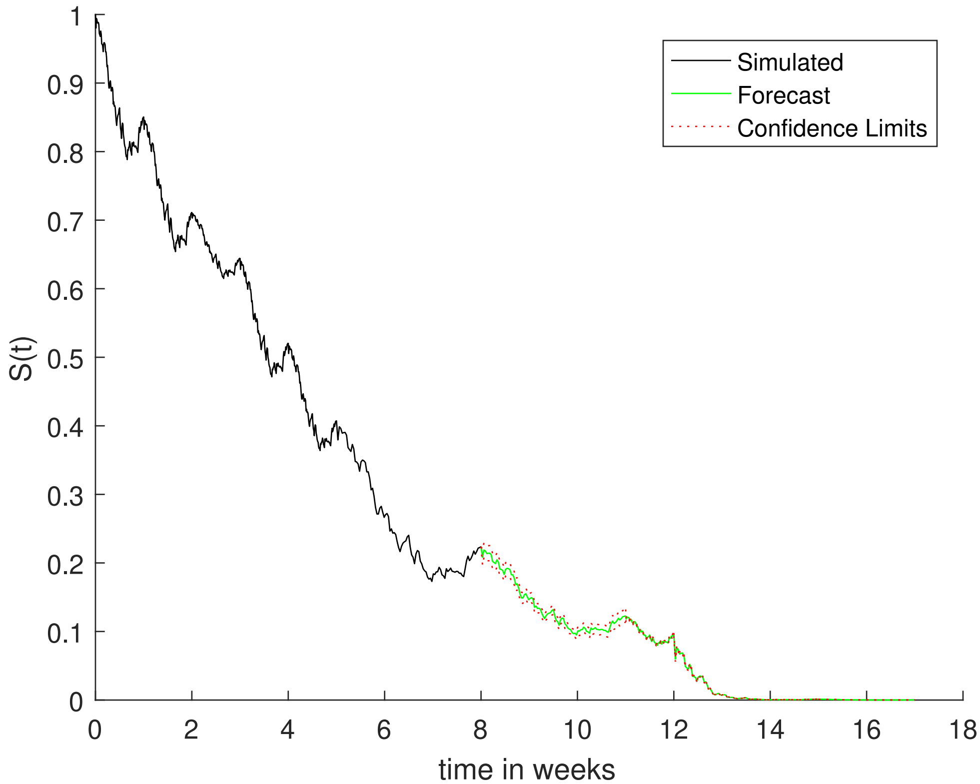


Figure 18.: Simulated and forecasted survival function estimates for Table 15

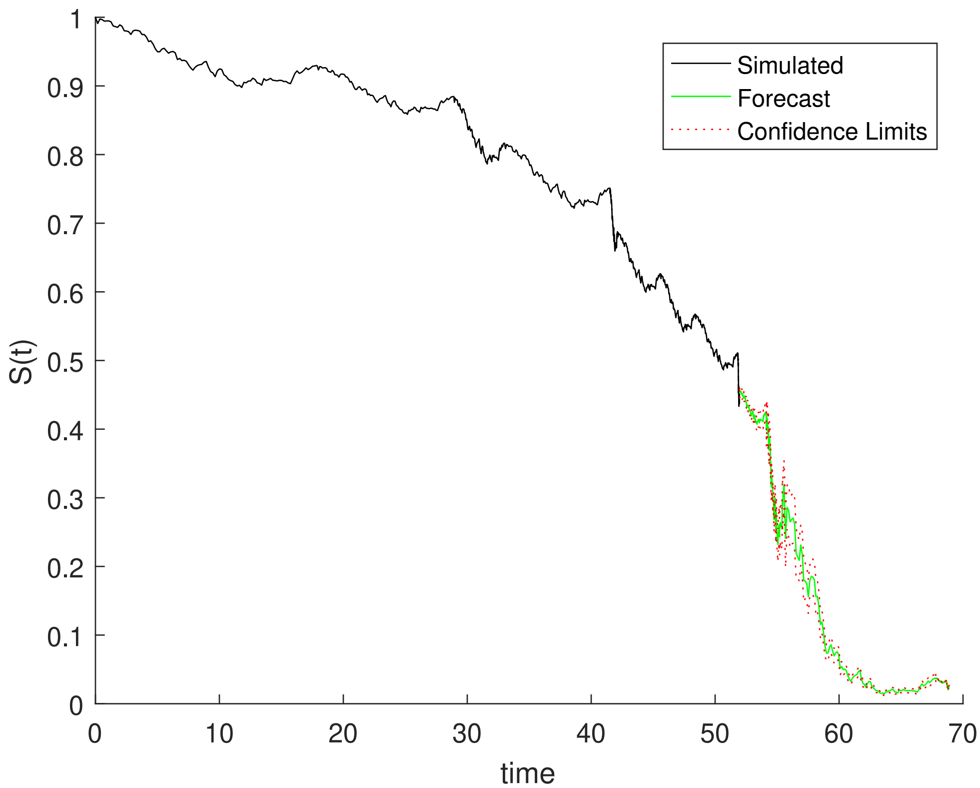


Figure 19.: Simulated and forecasted survival function estimates for Table 18

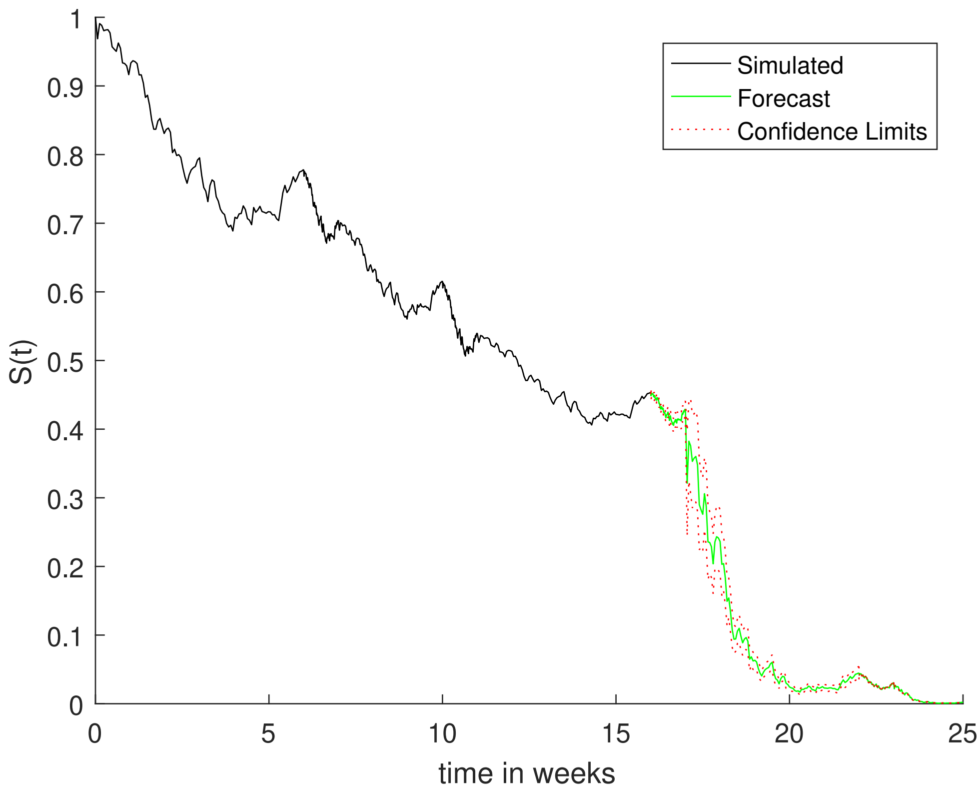


Figure 20.: Simulated and forecasted survival function estimates for Table 21

Chapter 7
Conclusions and Future work

In the area of survival/reliability analysis, most of the research work is centered around the probabilistic analysis approach. In general, a closed-form probability distribution is not feasible. The presented dynamic modeling is more appropriate for complex and more diversified time-to-event processes. This alternative approach does not require knowledge of either a closed-form probability distribution or a class of distributions. It does not require restrictive conditions on hazard rate functions. The time domain of a survival function need not be positively infinite. The influence of human mobility, rapid electronic communication devices, frequent technological changes, the rapidly growing knowledge, tools and procedures, advancements in biological, engineering, medical, military, physical and social sciences have generated a greater influence for the expansion of time-to-event processes beyond engineering and medical sciences. Naturally, these ideas motivated to initiate, formulate and develop an innovative interconnected dynamic modeling approach for generalized version of time-to-event processes under randomly varying environments in biological, chemical, engineering, epidemiological, medical, multiple-markets and social dynamic processes through discrete-time intervention processes under deterministic perturbations. The presented innovative alternative modeling approach enhances our motivation to develop parameter and state estimation procedures. Moreover, the parameter and state estimation approach is dynamic. The dynamic nature is more natural rather than the existing static and single-shot approach. Moreover, it is a nonparametric approach. The dynamic approach adapts with current changes and updates the statistic process. This plays a very significant role in parameter and state estimation problems in a systematic and unifying way. Recently developed LLGMM approach is extended to the problems in the time-to-event dynamic processes in a systematic and unified way. On the other hand, the MLE is centered on the parameter and state estimates using the entire data. In addition, the LLGMM stabilizes the parameter and state estimation procedure with a finite and small size data set. On the contrary, the MLE, does not have this flexibility. Intervention processes provide a measure of influence of new tools/procedures/approaches in continuous-time states of time-to-event dynamic process. In particular, it generates a measure of the degree of sustainability, survivability, reliability of the system. This further leads to sustainable/unsustainable, survivable/failure, reliable/unreliable binary state invariant sets. Moreover, intervention processes provide the comparison between the past and currently used tools/procedures/approaches/attitudes/etc. In fact, the full force of the role and scope of our innovative modeling approach for time-to-event processes is currently under investigation.

The procedures developed in this work provides insights, tips, and tools for undertaking similar tasks in context of stochastic framework. In fact, it allows to have a time-varying covariate state influence on the dynamic of a complex survival/reliability of systems. This is the basis for future work in modeling time-to-event processes. Moreover, the parameter and state estimation approach is dynamic. The dynamic nature rather than the existing algebraic approach plays a very significant role in state and parameter estimation problems in a systematic and unifying way. In the future, we plan to introduce time dependent covariates(external and internal) in the developed models and consider more complex time-to-event dynamic studies. Furthermore, we also plan to extend the developed models and algorithms to include recurrent events and competing risks events.

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Appendix A

Modified LLGMM Estimates Corresponding to Datasets in Tables 4 and 6

Table 33: LLGMM Based Estimates using $S_0 = 0.985,\ 0.98900,\ 0.99000,\ 0.99900,\ 0.99990,\ 0.99999,\ 0.999999$ using using procedure outlined in Subsection 4.7.2.

		$S_0 = 0.985$		$S_0 = 0.98900$		$S_0 = 0.99000$		$S_0 = 0.99900$		$S_0 = 0.9999$		$S_0 = 0.99999$		$S_0 = 0.999999$	
t^f_{j-1i}	$\hat{m}_i$	$\sigma_{j-1i}, \hat{m}_i$	$S_{j-1i}, \hat{m}_i$	$\sigma_{j-1i}, \hat{m}_i$	$St_{j-1i}, \hat{m}_i$	$\sigma_{j-1i}, \hat{m}_i$	$S_{j-1i}, \hat{m}_i$	$\sigma_{j-1i}, \hat{m}_i$	$S_{j-1i}, \hat{m}_i$	$\sigma_{j-1i}, \hat{m}_i$	$S_{j-1i}, \hat{m}_i$	$\sigma_{j-1i}, \hat{m}_i$	$S_{j-1i}, \hat{m}_i$	$\sigma_{j-1i}, \hat{m}_i$	$S_{j-1i}, \hat{m}_i$
22.5	1	30.9600	0.9747	30.9600	0.9787	20.6400	0.9797	2.0640	0.9886	0.2064	0.9895	0.0206	0.9896	0.0021	0.9896
37.5	1	1.5998	0.9645	1.5998	0.9684	1.2865	0.9694	0.7224	0.9782	0.6660	0.9791	0.6603	0.9792	0.6598	0.9792
46.5	1	0.8013	0.9542	0.8013	0.9581	0.6909	0.9591	0.4921	0.9678	0.4722	0.9687	0.4702	0.9687	0.4700	0.9687
48.5	1	0.1831	0.9440	0.1831	0.9478	0.1637	0.9488	0.1289	0.9574	0.1254	0.9582	0.1250	0.9583	0.1250	0.9583
51.5	1	0.3189	0.9337	0.3189	0.9375	0.2916	0.9384	0.2426	0.9470	0.2377	0.9478	0.2372	0.9479	0.2371	0.9479
53.5	1	0.2343	0.9234	0.2343	0.9272	0.2176	0.9281	0.1874	0.9366	0.1844	0.9374	0.1841	0.9375	0.1841	0.9375
54.5	1	0.1288	0.9132	0.1288	0.9169	0.1209	0.9178	0.1067	0.9262	0.1053	0.9270	0.1052	0.9271	0.1051	0.9271
57.5	1	0.4254	0.9029	0.4254	0.9066	0.4026	0.9075	0.3618	0.9158	0.3577	0.9166	0.3573	0.9167	0.3572	0.9167
66.5	1	1.3372	0.8927	1.3372	0.8963	1.2741	0.8972	1.1605	0.9053	1.1491	0.9062	1.1480	0.9062	1.1478	0.9062
68.0	1	0.2107	0.8824	0.2107	0.8860	0.2018	0.8869	0.1858	0.8949	0.1842	0.8957	0.1840	0.8958	0.1840	0.8958
69.5	1	0.2231	0.8721	0.2231	0.8757	0.2146	0.8766	0.1993	0.8845	0.1978	0.8853	0.1976	0.8854	0.1976	0.8854
76.5	1	1.0947	0.8619	1.0947	0.8654	1.0568	0.8663	0.9885	0.8741	0.9817	0.8749	0.9810	0.8750	0.9810	0.8750
77.0	1	0.0758	0.8516	0.0758	0.8551	0.0734	0.8559	0.0691	0.8637	0.0687	0.8645	0.0686	0.8646	0.0686	0.8646
78.5	1	0.2399	0.8414	0.2399	0.8448	0.2329	0.8456	0.2204	0.8533	0.2191	0.8541	0.2190	0.8542	0.2190	0.8542
80.0	1	0.2486	0.8311	0.2486	0.8345	0.2419	0.8353	0.2298	0.8429	0.2286	0.8437	0.2285	0.8437	0.2285	0.8437
81.5	1	0.2565	0.8208	0.2565	0.8242	0.2501	0.8250	0.2386	0.8325	0.2374	0.8332	0.2373	0.8333	0.2373	0.8333
82.5	1	0.1759	0.8106	0.1759	0.8139	0.1718	0.8147	0.1644	0.8221	0.1637	0.8228	0.1636	0.8229	0.1636	0.8229
83.0	1	0.0907	0.8003	0.0907	0.8036	0.0887	0.8044	0.0852	0.8117	0.0848	0.8124	0.0848	0.8125	0.0848	0.8125
84.0	1	0.1877	0.7901	0.1877	0.7933	0.1838	0.7941	0.1770	0.8013	0.1763	0.8020	0.1762	0.8021	0.1762	0.8021
91.5	1	1.4434	0.7798	1.4434	0.7830	1.4158	0.7838	1.3662	0.7909	1.3612	0.7916	1.3607	0.7917	1.3607	0.7917
93.5	1	0.3658	0.7695	0.3658	0.7727	0.3592	0.7734	0.3474	0.7805	0.3462	0.7812	0.3461	0.7812	0.3461	0.7812
102.5	1	1.6638	0.7593	1.6638	0.7624	1.6356	0.7631	1.5849	0.7701	1.5798	0.7708	1.5793	0.7708	1.5792	0.7708
107.0	1	0.7821	0.7490	0.7821	0.7521	0.7696	0.7528	0.7470	0.7597	0.7448	0.7603	0.7445	0.7604	0.7445	0.7604
108.5	1	0.2569	0.7388	0.2569	0.7417	0.2530	0.7425	0.2460	0.7493	0.2453	0.7499	0.2452	0.7500	0.2452	0.7500
112.5	1	0.6935	0.7285	0.6935	0.7314	0.6835	0.7322	0.6656	0.7388	0.6638	0.7395	0.6636	0.7396	0.6636	0.7396
113.5	1	0.1714	0.7182	0.1714	0.7211	0.1690	0.7219	0.1648	0.7284	0.1644	0.7291	0.1644	0.7292	0.1644	0.7292
116.0	1	0.4344	0.7080	0.4344	0.7108	0.4288	0.7116	0.4187	0.7180	0.4177	0.7187	0.4176	0.7187	0.4176	0.7187
117.0	1	0.1737	0.6977	0.1737	0.7005	0.1716	0.7013	0.1677	0.7076	0.1673	0.7083	0.1673	0.7083	0.1673	0.7083
118.5	1	0.2635	0.6874	0.2635	0.6902	0.2604	0.6909	0.2549	0.6972	0.2543	0.6978	0.2543	0.6979	0.2543	0.6979
119.0	1	0.0884	0.6772	0.0884	0.6799	0.0874	0.6806	0.0856	0.6868	0.0854	0.6874	0.0854	0.6875	0.0854	0.6875
120.0	1	0.1790	0.6669	0.1790	0.6696	0.1771	0.6703	0.1737	0.6764	0.1734	0.6770	0.1733	0.6771	0.1733	0.6771
122.5	1	0.4510	0.6567	0.4510	0.6593	0.4465	0.6600	0.4382	0.6660	0.4374	0.6666	0.4373	0.6667	0.4373	0.6667
123.0	1	0.0897	0.6464	0.0897	0.6490	0.0888	0.6497	0.0872	0.6556	0.0871	0.6562	0.0871	0.6562	0.0871	0.6562
127.5	1	0.8150	0.6361	0.8150	0.6387	0.8074	0.6394	0.7938	0.6452	0.7925	0.6458	0.7923	0.6458	0.7923	0.6458
131.0	1	0.6193	0.6259	0.6193	0.6284	0.6138	0.6291	0.6039	0.6348	0.6029	0.6354	0.6028	0.6354	0.6028	0.6354
132.5	1	0.2613	0.6156	0.2613	0.6181	0.2591	0.6188	0.2551	0.6244	0.2547	0.6249	0.2547	0.6250	0.2547	0.6250
134.0	1	0.2611	0.6054	0.2611	0.6078	0.2590	0.6084	0.2551	0.6140	0.2548	0.6145	0.2547	0.6146	0.2547	0.6146

Table 34: LLGMM Based Estimates using S_0 = 0.985, 0.98900, 0.99000, 0.99900, 0.99990, 0.99999, 0.999999 using procedure outlined in Subsection 4.7.2

		$S_0 = 0.985$		$S_0 = 0.98900$		$S_0 = 0.99000$		$S_0 = 0.99900$		$S_0 = 0.9999$		$S_0 = 0.99999$		$S_0 = 0.999999$	
t^f_{j-1i}	$\hat{m}_i$	$\sigma_{j-1i}, \hat{m}_i$	$S_{j-1i}, \hat{m}_i$	$\sigma_{j-1i}, \hat{m}_i$	$S_{j-1i}, \hat{m}_i$	$\sigma_{j-1i}, \hat{m}_i$	$S_{j-1i}, \hat{m}_i$	$\sigma_{j-1i}, \hat{m}_i$	$S_{j-1i}, \hat{m}_i$	$\sigma_{j-1i}, \hat{m}_i$	$S_{j-1i}, \hat{m}_i$	$\sigma_{j-1i}, \hat{m}_i$	$S_{j-1i}, \hat{m}_i$	$\sigma_{j-1i}, \hat{m}_i$	$S_{j-1i}, \hat{m}_i$
		$S_0 = 0.985$		$S_0 = 0.98900$		$S_0 = 0.99000$		$S_0 = 0.99900$		$S_0 = 0.9999$		$S_0 = 0.99999$		$S_0 = 0.999999$	
t^f_{j-1i}	$\hat{m}_i$	$\sigma_{j-1i}, \hat{m}_i$	$S_{j-1i}, \hat{m}_i$	$\sigma_{j-1i}, \hat{m}_i$	$St_{j-1i}, \hat{m}_i$	$\sigma_{j-1i}, \hat{m}_i$	$S_{j-1i}, \hat{m}_i$	$\sigma_{j-1i}, \hat{m}_i$	$S_{j-1i}, \hat{m}_i$	$\sigma_{j-1i}, \hat{m}_i$	$S_{j-1i}, \hat{m}_i$	$\sigma_{j-1i}, \hat{m}_i$	$S_{j-1i}, \hat{m}_i$	$\sigma_{j-1i}, \hat{m}_i$	$S_{j-1i}, \hat{m}_i$
6.0	1	3.7500	0.9653	2.7500	0.9692	2.5000	0.9702	0.2500	0.9790	0.0250	0.9799	0.0025	0.9800	0.0003	0.9800
14.0	1	4.5341	0.9554	4.0219	0.9593	3.8939	0.9603	2.7414	0.9690	2.6261	0.9699	2.6146	0.9700	2.6135	0.9700
44.0	1	9.2600	0.9456	8.4535	0.9494	8.2519	0.9504	6.4373	0.9590	6.2559	0.9599	6.2377	0.9600	6.2359	0.9600
62.0	1	2.1364	0.9357	1.9856	0.9396	1.9479	0.9405	1.6086	0.9490	1.5747	0.9499	1.5713	0.9500	1.5709	0.9500
89.0	1	2.6581	0.9259	2.5009	0.9297	2.4616	0.9306	2.1079	0.9391	2.0725	0.9399	2.0689	0.9400	2.0686	0.9400
98.0	1	0.7044	0.9160	0.6686	0.9198	0.6597	0.9207	0.5793	0.9291	0.5712	0.9299	0.5704	0.9300	0.5703	0.9300
104.0	1	0.4780	0.9062	0.4568	0.9099	0.4515	0.9108	0.4039	0.9191	0.3991	0.9199	0.3986	0.9200	0.3986	0.9200
107.0	1	0.2489	0.8963	0.2392	0.9000	0.2367	0.9009	0.2147	0.9091	0.2126	0.9099	0.2123	0.9100	0.2123	0.9100
114.0	1	0.6171	0.8865	0.5954	0.8901	0.5900	0.8910	0.5412	0.8991	0.5363	0.8999	0.5358	0.9000	0.5358	0.9000
123.0	1	0.8064	0.8766	0.7809	0.8802	0.7745	0.8811	0.7169	0.8891	0.7112	0.8899	0.7106	0.8900	0.7105	0.8900
128.0	1	0.4463	0.8668	0.4334	0.8703	0.4302	0.8712	0.4012	0.8791	0.3983	0.8799	0.3980	0.8800	0.3980	0.8800
148.0	1	1.8315	0.8569	1.7831	0.8604	1.7710	0.8613	1.6621	0.8691	1.6512	0.8699	1.6501	0.8700	1.6500	0.8700
182.0	1	2.8591	0.8471	2.7895	0.8505	2.7721	0.8514	2.6156	0.8591	2.6000	0.8599	2.5984	0.8600	2.5983	0.8600
187.0	1	0.3612	0.8372	0.3531	0.8407	0.3511	0.8415	0.3328	0.8491	0.3310	0.8499	0.3308	0.8500	0.3308	0.8500
189.0	1	0.1480	0.8274	0.1449	0.8308	0.1441	0.8316	0.1371	0.8392	0.1364	0.8399	0.1364	0.8400	0.1364	0.8400
274.0	1	3.2602	0.8077	3.1968	0.8110	3.1809	0.8118	3.0381	0.8192	3.0238	0.8199	3.0224	0.8200	3.0222	0.8200
302.0	1	1.6114	0.7978	1.5839	0.8011	1.5770	0.8019	1.5152	0.8092	1.5090	0.8099	1.5084	0.8100	1.5083	0.8100
363.0	1	3.3074	0.7880	3.2544	0.7912	3.2411	0.7920	3.1218	0.7992	3.1099	0.7999	3.1087	0.8000	3.1086	0.8000
374.0	1	0.5139	0.7781	0.5062	0.7813	0.5042	0.7821	0.4868	0.7892	0.4850	0.7899	0.4849	0.7900	0.4849	0.7900
451.0	1	3.6083	0.7683	3.5569	0.7714	3.5441	0.7722	3.4284	0.7792	3.4169	0.7799	3.4157	0.7800	3.4156	0.7800
461.0	1	0.4007	0.7584	0.3953	0.7615	0.3940	0.7623	0.3818	0.7692	0.3806	0.7699	0.3805	0.7700	0.3805	0.7700
492.0	1	1.2507	0.7486	1.2348	0.7516	1.2308	0.7524	1.1949	0.7592	1.1913	0.7599	1.1910	0.7600	1.1909	0.7600
538.0	1	1.7864	0.7387	1.7648	0.7418	1.7594	0.7425	1.7108	0.7492	1.7059	0.7499	1.7054	0.7500	1.7054	0.7500
774.0	1	8.5950	0.7289	8.4963	0.7319	8.4717	0.7326	8.2496	0.7393	8.2274	0.7399	8.2252	0.7400	8.2249	0.7400
841.0	1	1.7366	0.7190	1.7176	0.7220	1.7129	0.7227	1.6702	0.7293	1.6660	0.7299	1.6655	0.7300	1.6655	0.7300
936.0	1	2.3168	0.7092	2.2927	0.7121	2.2867	0.7128	2.2325	0.7193	2.2271	0.7199	2.2265	0.7200	2.2265	0.7200
1002.0	1	1.4764	0.6993	1.4617	0.7022	1.4581	0.7029	1.4252	0.7093	1.4219	0.7099	1.4216	0.7100	1.4215	0.7100
1011.0	1	0.1917	0.6895	0.1899	0.6923	0.1895	0.6930	0.1854	0.6993	0.1850	0.6999	0.1849	0.7000	0.1849	0.7000
1048.0	1	0.7954	0.6796	0.7883	0.6824	0.7865	0.6831	0.7703	0.6893	0.7687	0.6899	0.7686	0.6900	0.7685	0.6900
1054.0	1	0.1266	0.6698	0.1255	0.6725	0.1252	0.6732	0.1227	0.6793	0.1225	0.6799	0.1225	0.6800	0.1225	0.6800
1172.0	1	2.5138	0.6599	2.4931	0.6626	2.4879	0.6633	2.4413	0.6693	2.4366	0.6699	2.4362	0.6700	2.4361	0.6700
1205.0	1	0.6415	0.6501	0.6365	0.6527	0.6352	0.6534	0.6238	0.6593	0.6227	0.6599	0.6226	0.6600	0.6226	0.6600
1278.0	1	1.3990	0.6402	1.3885	0.6429	1.3858	0.6435	1.3621	0.6493	1.3597	0.6499	1.3595	0.6500	1.3594	0.6500
1401.0	1	2.2505	0.6304	2.2343	0.6330	2.2302	0.6336	2.1936	0.6394	2.1900	0.6399	2.1896	0.6400	2.1896	0.6400

continued on next page

Table 34 – continued from previous page

		$S_0 = 0.985$		$S_0 = 0.98900$		$S_0 = 0.99000$		$S_0 = 0.99900$		$S_0 = 0.9999$		$S_0 = 0.99999$		$S_0 = 0.999999$	
t^f_{j-1i}	$\hat{m}_i$	$\sigma_{j-1i}, \hat{m}_i$	$S_{j-1i}, \hat{m}_i$	$\sigma_{j-1i}, \hat{m}_i$	$St_{j-1i}, \hat{m}_i$	$\sigma_{j-1i}, \hat{m}_i$	$S_{j-1i}, \hat{m}_i$	$\sigma_{j-1i}, \hat{m}_i$	$S_{j-1i}, \hat{m}_i$	$\sigma_{j-1i}, \hat{m}_i$	$S_{j-1i}, \hat{m}_i$	$\sigma_{j-1i}, \hat{m}_i$	$S_{j-1i}, \hat{m}_i$	$\sigma_{j-1i}, \hat{m}_i$	$S_{j-1i}, \hat{m}_i$
1497.0	1	1.6209	0.6205	1.6096	0.6231	1.6068	0.6237	1.5816	0.6294	1.5790	0.6299	1.5788	0.6300	1.5788	0.6300
1557.0	1	0.9581	0.6107	0.9518	0.6132	0.9502	0.6138	0.9359	0.6194	0.9344	0.6199	0.9343	0.6200	0.9343	0.6200
1577.0	1	0.3100	0.6008	0.3081	0.6033	0.3076	0.6039	0.3031	0.6094	0.3027	0.6099	0.3026	0.6100	0.3026	0.6100
1624.0	1	0.7257	0.5910	0.7212	0.5934	0.7201	0.5940	0.7101	0.5994	0.7091	0.5999	0.7090	0.6000	0.7090	0.6000
1669.0	1	0.6800	0.5811	0.6760	0.5835	0.6750	0.5841	0.6660	0.5894	0.6651	0.5899	0.6650	0.5900	0.6650	0.5900
1806.0	1	2.0285	0.5713	2.0171	0.5736	2.0142	0.5742	1.9885	0.5794	1.9859	0.5799	1.9857	0.5800	1.9856	0.5800
1874.0	6	0.9324	0.5614	0.9269	0.5637	0.9255	0.5643	0.9131	0.5694	0.9119	0.5699	0.9118	0.5699	0.9118	0.5699
1907.0	1	0.3682	0.5496	0.3663	0.5519	0.3658	0.5524	0.3615	0.5574	0.3611	0.5579	0.3610	0.5576	0.3610	0.5580
2012.0	19	1.1562	0.5378	1.1472	0.5400	1.1449	0.5405	1.1247	0.5454	1.1226	0.5458	0.1342	0.5580	1.1224	0.5459
2031.0	1	0.1398	0.5211	0.1392	0.5231	0.1390	0.5237	0.1376	0.5283	0.1374	0.5288	1.1224	0.5459	0.1374	0.5288
2065.0	1	0.2409	0.5037	0.2398	0.5057	0.2396	0.5062	0.2372	0.5107	0.2370	0.5112	0.2370	0.5112	0.2370	0.5112
2201.0	13	0.9086	0.4856	0.9031	0.4875	0.9017	0.4880	0.8894	0.4922	0.8881	0.4927	0.8880	0.4927	0.8880	0.4927
2421.0	1	0.6684	0.4482	0.6660	0.4500	0.6653	0.4504	0.6598	0.4544	0.6592	0.4548	0.6592	0.4548	0.6592	0.4548
2624.0	8	0.5512	0.4106	0.5488	0.4122	0.5481	0.4126	0.5426	0.4161	0.5420	0.3961	0.5420	0.4164	0.5420	0.4164
2710.0	2	0.2751	0.3818	0.2742	0.3832	0.2740	0.3836	0.2721	0.3868	0.2719	0.3871	0.2719	0.3871	0.2719	0.3872

Appendix B

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Title: Nonparametric Estimation from Incomplete Observations
Author: E. L. Kaplan, Paul Meier
Publication: Journal of the American Statistical Association
Publisher: Taylor & Francis
Date: Jun 1, 1958

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Title: Piecewise exponential estimator of the survivor function

Author: J.S. Kim

Publication: Reliability, IEEE Transactions on

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Date: June 1991

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