May 2018

Non-equilibrium Phase Transitions in Interacting Diffusions

Wael Al-Sawai
University of South Florida, walsawai@usf.edu

Non-equilibrium Phase Transitions in Interacting Diffusions

by

Wael Al-Sawai

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
Department of Mathematics & Statistics
College of Arts and Sciences
University of South Florida

Major Professor: Razvan Teodorescu, Ph.D.
Sherwin Kouchekian, Ph.D.
Seung-Yeop Lee, Ph.D.
Leslaw A. Skrzypek, Ph.D.

Date of Approval:
April 6, 2018

Keywords: Phase Transition, Kuramoto Model, Synchronization, Geometric Deformation, Dynamical Systems

Dedication

To my family,
Amanda and Sami.

Acknowledgments

This thesis would not have been possible without the support, encouragement, and guidance of my supervisor Dr. Razvan Teodorescu, whose help from the beginning to the end enabled me to develop and understand the subject. I would like to thank the Mathematics and Statistics Department at the University of South Florida for the support I received to complete this work. I owe my deepest gratitude to Dr. Yuncheng You, Dr. Shrewin Koucheckian, Dr. Manoug Manougian, Dr. Seung-Yeop Lee, Dr. Scott Rimbey, Dr. Andrei Barbos, and Dr. Leslaw A. Skrzypek. I offer my regards and blessings to all of those who supported me in any respect during the completion of the project. Last but not the least, I am heartily thankful to my wife and my son for unconditional love and support.

Table of Contents

List of Figures

Abstract

The theory of thermodynamic phase transitions has played a central role both in theoretical physics and in dynamical systems for several decades. One of its fundamental results is the classification of various physical models into equivalence classes with respect to the scaling behavior of solutions near the critical manifold. From that point of view, systems characterized by the same set of critical exponents are equivalent, regardless of how different the original physical models might be. For non equilibrium phase transitions, the current theoretical framework is much less developed. In particular, an equivalent classification criterion is not available, thus requiring a specific analysis of each model individually. In this thesis, we propose a potential classification method for time-dependent dynamical systems, namely comparing the possible deformations of the original problem, and identifying dynamical systems which share the same deformation space. The specific model on which this procedure is developed is the Kuramoto model for interacting, disordered oscillators. Studied in the mean-field limit by a variety of methods, its associated synchronization phase transition appears as an appropriate model for cooperative phenomena ranging from coupled Josephson junctions to self-ordering patterns in biological and social systems. We investigate the geometric deformation of the dynamical system into the space of univalent maps of the unit disk, related to the Douady-Earle extension and the Denjoy-Wolff theory, and separately the algebraic deformation into the space of nonlinear sigma models for unitary operators. The results indicate that the Kuramoto model is representative for a large class of non equilibrium synchronization models, with a rich phase-space diagram.

Chapter 1
The Kuramoto Model

1.1 Introduction

Synchronization of a large population of mutually coupled oscillators is an ubiquitous phenomenon in the universe. It is observed in many complex biological, chemical, physical, and sociological systems with different origins of periodical activity and different mechanisms of coupling. This phenomenon brings up many mathematical and physical challenges to our understanding of collective phenomena, as they emerge in complex systems, either at equilibrium or dynamical. The Kuramoto model [1] is successful in describing how coherency emerges in complex systems. The model is based on several assumptions, including, that the oscillators are coupled, that they are identical or closely identical, and that the interactions depend sinusoidally on the phase difference between each pair of the oscillators. Note that, depending on the field, the term oscillator may refer to different systems, such as a neuron in the neural system, a cell in yeast cells, a Cooper pair in superconducting Josephson junctions, etc [2]. The model can describe many synchronization phenomena. Also, it has proved to be useful in designing artificially networked systems capable of self-organization in the absence of any centralized control mechanism, such as wireless sensor networks, and smart power grids [3–6].

1.2 Kuramoto's Model

The Kuramoto model [1] consists of N coupled oscillators, $\theta_i,\ i = 1, 2, ..., N$, with natural frequencies $\omega_i \in \mathbb{R}$ and whose dynamics is given by the system of coupled, ordinary differential equations

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i),\ \text{where } i = 1, 2, ..., N. \tag{1.1}$$

Kuramoto assumed that the frequencies, ω_i, are distributed according to some probability distribution density $g(\omega)$. He further assumed that $g(\omega)$ is unimodal and symmetric about its mean frequency Ω, i.e. $g(\Omega-\omega)=g(\Omega+\omega)$ where the domain of definition of g, is symmetric with respect to Ω [7]. Due to the rotational symmetry of the model, we can set the mean frequency $\Omega=0$ by redefining $\theta_i \to \theta_i + \Omega t \ \ \forall i$, this is equivalent to a rotating frame with frequency Ω which leaves $g(\omega)=g(-\omega)$.

1.2.1 Mean-Field Approach

Eq(1.1) can be rewritten in a more convenient way by introducing the order parameter $r(t)$ defined as

$$r(t)\exp i\psi(t) = \frac{1}{N}\sum_{j=1}^{N}\exp{(i\theta_j)}. \tag{1.2}$$

where $\psi(t)$ is the phase of the complex order parameter. Multiplying both sides by $\exp{(-i\theta_i)}$ we get

$$r(t)\exp i(\psi(t)-\theta_i) = \frac{1}{N}\sum_{j=1}^{N}\exp{(i(\theta_j-\theta_i))}, \tag{1.3}$$

equating the imaginary parts, one gets

$$r(t)\sin{(\psi(t)-\theta_i)} = \frac{1}{N}\sum_{j=1}^{N}\sin{((\theta_j-\theta_i))}, \tag{1.4}$$

therefore we can rewrite Eq(1.1) as

$$\dot{\theta}_i = \omega_i + Kr(t)\sin(\psi(t)-\theta_i), \tag{1.5}$$

where Kr is the effective coupling, and r by itself is proportional to the coherency of the oscillators. We can assume without loss of generality that the average phase, ψ, is equal to zero, therefore Eq(1.5) can be written as

$$\dot{\theta}_i = \omega_i - Kr(t)\sin(\theta_i), \tag{1.6}$$

where the mean field signature is superficial, and the collective effect of the coupled oscillators is represented by two parameters r and ψ. Each oscillator appears to be uncoupled from the other oscillators. Notice that r is a measure of the coherency of the system. The interplay between the coupling and coherency creates a self-driven process, meaning, as the oscillators become more coherent, r grows and the effective $Kr(t)$ coupling increases which leads to more oscillators to join the synchronized oscillators. If the coherence is increased by a new oscillator, the process will continue, or it becomes self-limiting [8].

1.2.2 Kuramoto's Analysis

We can write the order parameter equation in Eq(1.2) as

$$r\exp{(i\psi)} = \int_{-\pi}^{\pi} \exp(i\theta) \frac{\sum_{j=1}^{N} \delta(\theta-\theta_j)}{N} d\theta, \tag{1.7}$$

one way of thinking about the Dirac delta function is that it is is a Gaussian random variable centered at θ_j, with infinitely small standard deviation. Then, in the limit $N \to \infty$, the order parameter amplitude $r(t)$ and phase $\psi(t)$, defined by Eq(1.2), can be written as

$$r\exp{(i\psi)} = \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} \mathrm{e}^{i\theta} \rho(\theta, \omega, t) g(\omega) d\theta d\omega. \tag{1.8}$$

This equation explicitly shows that the order parameter r is a measure of the oscillator synchronization (phase coherence), and the interaction between oscillators of different frequencies occurs solely through the order parameter. Note that, when $K \to 0$, Eq(1.5) yields $\theta_i = \omega t + \theta_i(0)$ which means the oscillators rotate at angular frequency equal to their own natural frequencies. On other hand, if $\theta = \omega t$ then, in Eq(1.8) the integral $\int_{-\infty}^{\infty} \mathrm{e}^{i\theta} \rho(\theta, \omega, t) d\omega \to 0$ as $t \to \infty$ by Riemann-Lebesgue lemma (the extension of this result to the case $K > 0$ still questionable), therefore when $r \to 0$ the oscillators become less and less synchronized [9]. In the case of strong coupling, $K \to \infty$, the oscillators are synchronized to their average phase and Eq(1.8) implies $r \to 1$. Now the question is when the oscillators start to synchronize? In other words, at what value of the coupling parameter, K, the system started to experience phase transition from the completely random to partially synchronized oscillators? In Eq(1.1) the oscillators density, $\rho(\theta, \omega, t)$ can be found by noting that the oscillators rotate with angular velocity $\dot{\theta}_i$. Therefore, the one-oscillator density must satisfy the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \theta}\Big([\omega + Kr\sin(\psi - \theta)]\rho \Big) = 0, \tag{1.9}$$

subject to normalization condition

$$\int_{-\pi}^{\pi} \rho(\theta, \omega, t) d\theta = 1, \ \forall \ \omega, \ t. \tag{1.10}$$

In the long term, the system converges to a steady state system. Therefore, the first term in Eq(1.9) vanishes, and we get

$$\rho(\theta,\omega) = \frac{C}{|\omega - Kr\sin(\theta)|}, \tag{1.11}$$

which is the density of incoherent oscillators, conventionally called drift group. Eq(1.10) determines the normalization constant

$$C = \frac{1}{2\pi}\sqrt{\omega^2 - (Kr)^2}. \tag{1.12}$$

Also, it follows from Eq(1.6) that the dynamics of oscillators with $|\omega| \leq Kr$ approaches $\omega_i = Kr\sin(\theta_i)$ as $t \to \infty$, where $|\theta_i| \leq \frac{\pi}{2}$. This group of oscillators is "locked" or synchronized, and has distribution

$$\rho(\theta,\omega) = \delta[Kr\sin(\theta) - \omega)]H(\cos(\theta)) \quad \text{where} \quad |\omega| < Kr, \tag{1.13}$$

where $H(x)$ is Heaviside step function. By using Eq(1.8,1.11& 1.13) we can calculate the order parameter r. Using Dirac's bra-ket notation, we can rewrite Eq(1.8) as

$$< \exp i\theta >= r\exp i\psi =< \exp i\theta >_{lock} + < \exp i\theta >_{drift}, \tag{1.14}$$

the drift group term

$$< \exp i\theta >_{drift}= \int_{-\pi}^{\pi}\int_{|\omega|>Kr} \rho(\theta,\omega)g(\omega)d\omega d\theta = 0$$

vanishes, since $g(\omega) = g(-\omega)$. From Eq(1.11), we have $\rho(\theta,\omega) = \rho(\theta+\pi,-\omega)$. In the lock term, or synchronization term, the imaginary part disappears; since $\rho(\theta,\omega) = \rho(-\theta,-\omega)$ and $g(\omega) = g(-\omega)$,

$$< \exp(i\theta) >_{lock}= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\infty}^{\infty} \cos(\theta)\delta[\omega - Kr\sin(\theta)]g(\omega)d\theta d\omega,$$

$$r = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\theta)g(Kr\sin(\theta))Kr\cos(\theta)d\theta, \tag{1.15}$$

this equation has the trivial solution at $r = 0$ valid for any value of K, corresponding to incoherent phase with

$$\rho(\theta,\omega) = \frac{1}{2\pi} \ \forall\ \theta \text{ and } \omega, \tag{1.16}$$

which has a second branch of solutions, when $r \neq 0$, corresponding to partially synchronized phase Eq(1.13), therefore

$$1 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} K\cos^2(\theta) g(Kr\sin(\theta))d\theta. \tag{1.17}$$

This solution bifurcates continuously from $r = 0$ at the value $K = K_c$ obtained by setting $r \to 0^+$ in Eq(1.17), thus

$$1 = Kg(0)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(\theta)d\theta,$$

and

$$K_c = \frac{2}{\pi g(0)}. \tag{1.18}$$

This formula and the arguments leading to it were suggested by Kuramoto [1]. The system when $K < K_c$ is in incoherent state in which the oscillators exhibit independent oscillations, while when $K > K_c$ is in coherent state in which part of oscillators population is synchronized. By expanding the integral in Eq(1.17) with respect to r,

$$1 = K\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(\theta)\Big(g(0) + \frac{g^{'}(0)Kr\sin\theta}{1!} + \frac{g^{''}(0)(Kr\sin\theta)^2}{2!} + ...\Big)d\theta, \tag{1.19}$$

after taking the integral,

$$1 \simeq K\Big(\frac{1}{K_c} + \frac{g^{''}(0)(K_c r)^2\pi}{16}\Big), \tag{1.20}$$

we can rearrange the terms,

$$\frac{K_c - K}{K_c} = \mu \simeq \frac{g^{''}(0)K_c^3 r^2\pi}{16}. \tag{1.21}$$

For the Lorentzian distribution (smooth, unimodal, and even) densities, $g(\omega) = \frac{\gamma^2}{\pi(\omega^2+\gamma^2)}$, $g^{'}(0) = 0$ and $g^{''}(0) = -\frac{16}{\pi K_c^3} < 0$. For all $K > K_c = 2\gamma$ we get

$$r \simeq \sqrt{\mu} = \sqrt{\frac{K - K_c}{K}}. \tag{1.22}$$

Thus, the system bifurcation is supper-critical for $K > K_c$ if $g^{''}(0) < 0$ and sub-critical for $K < K_c$ if $g^{''}(0) > 0$.

1.3 Stability of Solutions and Open Problems

Notice that Kuramoto's calculations for partially synchronized phase does not indicate whether this phase is stable, either globally or locally. The linear stability theory of incoherence has been investigated by Strogatz [8].

1.3.1 Synchronization as N Approaches Infinity

Strogatz [10–12], presents the first rigorous stability analysis of the incoherent solution for the infinite oscillators system. When the order parameter $r = 0$ the system is incoherent, linearly stable, and non-unique (there is an infinite number of $K's$ that satisfy Eq(1.15)). The state is neutrally stable if $K < K_c$ and has equiprobability Eq(1.16). When $K = K_c$ a new stationary solution (the partially synchronized state) bifurcates from Eq(1.16). If the coupling exceeds the critical value, $K > K_c$, the incoherent state becomes unstable and a synchronization state bifurcates from it [13].

1.3.2 Synchronization at Finite N

The finite size effect is an issue with a kinetic equation that describes populations of infinitely many elements, which exists in Kuramoto's model. The Lyapunov function argument was used to point out that a population of finitely many Kuramoto oscillators reach a stationary state as $t \to \infty$ [14]. In this work, we present a rigorous analysis for large finite-N of Eq(1.1), and then prove the convergence as $N \to \infty$. However, [7, 15, 16] have investigated the problem using computer simulation and physical arguments. It appears that the fluctuations are indeed $O(N^{-\frac{1}{2}})$ except very close to K_c. To the best of our knowledge, no progress has been made in this problem at the time of writing this paper.

1.4 Noisy Kuramoto Models

Kuramoto's analysis of the infinite-N limit is successful in many ways, but it has peculiarities that have been discussed earlier. The Kuramoto model does not have phase transition such as phase transition occurs in thermodynamics or statistical physics in which fluctuations play a major rule. However, for noisy dynamics, a phase transition has its usual meaning as in thermodynamics [17, 18]. The effect of noise on the collective properties of phase oscillators can be modeled by adding stochastic fluctuations (white noise to each oscillator) to Eq(1.6) [19],

$$d\theta_i = (\omega_i + Kr\sin(\theta_i))dt + dW_i(t) \tag{1.23}$$

where $dW_i = \xi_i(t)dt$ is the increment of Wiener process and $<\xi_i(t)>= 0$, $<\xi_i(t)\xi_i(t')>= 2D\delta(t-t')$, where D is the noise strength. Such noise can be interpreted as thermal fluctuations or rapid fluctuations of interstice, closely spaced, frequencies of the oscillators [13]. This model was suggested first by [19] and later used by Strogatz ([10]) to explain anomalous properties of Kuramoto model. The probability density of the process Eq(1.23) is a solution of one-dimensional Fokker-Planck equation:

$$\frac{\partial\rho}{\partial t} = -\frac{\partial}{\partial\theta}\Big((\omega_i + Kr\sin(\theta_i))\rho\Big) + D\frac{\partial^2\rho}{\partial\theta^2}. \tag{1.24}$$

Note that Eq(1.8, 1.24) govern the evolution of the density $\rho(\theta,t,\omega)$. The stationary solution of Eq(1.24) satisfying the periodic boundary condition $\rho(\theta,\omega) = \rho(\theta+2\pi,\omega)$ is given by [19]

$$\begin{gathered}\rho(\theta,\omega) = \exp\big(\frac{-Kr + \omega\theta + Kr\cos\theta}{D}\big)\rho(0;\omega) \\ \left(1 + \frac{(\mathrm{e}^{-2\pi\omega/D} - 1)\int_0^\theta \mathrm{e}^{(-\omega\theta' - Kr\cos\theta')/D}d\theta'}{\int_0^{2\pi}\mathrm{e}^{(-\omega\theta' - Kr\cos\theta')/D}d\theta'}\right),\end{gathered} \tag{1.25}$$

where $\rho(0,\omega)$ is determined by the normalization condition Eq(1.10). Substituting of Eq(1.25) into Eq(1.8) we obtain

$$r = \int_{-\infty}^{\infty} g(\omega)d\omega \int_{-\pi}^{\pi}\rho(\theta,\omega)\exp i\theta d\theta, \tag{1.26}$$

to find the critical coupling K_c and a small amplitude solution near K_c. Since $g(\omega)$ is symmetric about $\omega = 0$, the imaginary part on the right-hand side of Eq(1.26) is always zero. The real part on the right-hand side may be expanded in powers of Kr/D as [19]

$$\begin{aligned}r = \frac{Kr}{2D}\Big(\tfrac{1}{2}\int_{-\infty}^{\infty}\frac{g(\omega)}{1+\omega^2/D^2} - \frac{K^2r^2}{2D^2}\int_{-\infty}^{\infty}d\omega\frac{1 - 2\omega^2/D^2}{(1+\omega^2/D^2)^2(4+\omega^2/D^2)}g(\omega) + \\ O(r^4)\Big),\end{aligned} \tag{1.27}$$

according to the implicit function theorem, $r = 0$, the critical strength coupling as a function of D is determined from Eq(1.27), and we obtain [19]

$$K_c = 2\Big(\int_{-\infty}^{\infty} g(\omega)\frac{d\omega}{\omega^2+1}\Big)^{-1}. \tag{1.28}$$

As K increases, a nontrivial solution branches off the trivial zero solution at $K = K_c$.

1.4.1 Incoherent Solution

We would like to analyze the evolution of the probability density $\rho(\theta, t, \omega)$ in the neighborhood of the incoherent solution, namely

$$\rho_0 = \frac{1}{2\pi} \quad \forall\ \theta,\ t, \text{and } \omega, \tag{1.29}$$

This solution corresponds to a state in which, for each ω, all the oscillators are uniformly distributed around the circle. Note that, this solution is a solution to Fokker-Planck Eq(1.29).

1.4.2 Linear Stability Analysis of the Incoherent Stationary State

The stability analysis of the incoherent state Eq(1.29) is done by investigating the linearized Fokker-Planck equation obtained from Eq(1.24) as

$$\rho(\theta, \omega, t) = \frac{1}{2\pi} + \delta\eta(\theta, \omega, t); |\delta| \ll 1, \tag{1.30}$$

the normalization condition Eq(1.10) suggests that $\eta(\theta, \omega, t)$ satisfies

$$\int_{-\pi}^{\pi} \eta(\theta, \omega, t) d\theta = 0;\ \forall\ \omega, t, \tag{1.31}$$

and Fokker-Planck Eq(1.24) implies

$$\delta\frac{\partial\eta}{\partial t} = \delta D\frac{\partial^2\eta}{\partial\theta^2} - \frac{\partial}{\partial\theta}\Big((\tfrac{1}{2\pi} + \delta\eta)v_i\Big), \tag{1.32}$$

where $v_i = \omega_i + Kr\sin\theta_i$. Thus, the order parameter becomes

$$r\exp i\phi = \delta\int_{-\pi}^{\pi}\int_{-\infty}^{\infty} \mathrm{e}^{i\theta}\eta(\theta, \omega, t)g(\omega)d\theta d\omega = \delta r'\exp i\phi \tag{1.33}$$

where

$$r' \exp i\phi = \int_{-\pi}^{\pi}\int_{-\infty}^{\infty} \mathrm{e}^{i\theta}\eta(\theta,\omega,t)g(\omega)d\theta d\omega, \tag{1.34}$$

substitute Eqs(1.5,1.30, and 1.34) into Eq(1.9), we obtain its linearized form

$$\frac{\partial\eta}{\partial t} = \frac{\partial^2\eta}{\partial\theta^2} - \omega\frac{\partial\eta}{\partial\theta} + \frac{Kr'\cos(\phi-\theta)}{2\pi} \tag{1.35}$$

to analyze Eq(1.35) it is convenient to use Fourier series. Since the function $\eta(\theta,\omega,t)$ is real and 2π-periodic in θ, we look for a solution of the form

$$\eta(\theta,\omega,t) = c(\omega,t)\exp(i\theta) + c^*(\omega,t)\exp(-i\theta) + \eta^{\perp}(\theta,\omega,t), \tag{1.36}$$

where $\eta^{\perp}(\theta,\omega,t)$ represents the higher Fourier harmonics. By combining Eq(1.34) and Eq(1.36), one gets

$$r' \exp i(\phi-\theta) = \exp(-i\theta)\int_{-\pi}^{\pi}\int_{-\infty}^{\infty} \mathrm{e}^{ix}\eta(x,\omega,t)g(\omega)d\omega dx = 2\pi\mathrm{e}^{-i\theta}\int_{-\infty}^{\infty} c^*(\omega,t)g(\omega)d\omega, \tag{1.37}$$

similarly

$$r' \exp i(\phi+\theta) = 2\pi\mathrm{e}^{i\theta}\int_{-\infty}^{\infty} c(\omega,t)g(\omega)d\omega. \tag{1.38}$$

Combining Eq(1.37) and Eq(1.38) we get

$$r'\cos(\phi-\theta) = \pi\Big(\mathrm{e}^{-i\theta}\int_{-\infty}^{\infty} c^*(\omega,t)g(\omega)d\omega + \mathrm{e}^{i\theta}\int_{-\infty}^{\infty} c(\omega,t)g(\omega)d\omega\Big), \tag{1.39}$$

the amplitude equation for $c(\omega,t)$ is obtained by substituting Eq(1.36), Eq(1.37) into Eq(1.35), and comparing the coefficients of term $\mathrm{e}^{i\theta}$, one gets

$$\frac{\partial c(\omega,t)}{\partial t} = -(D+i\omega)c(\omega,t) + \frac{K}{2}\int_{-\infty}^{\infty} c(v,t)g(v)dv. \tag{1.40}$$

Note that, c^* is the complex conjugate of Eq(1.40), and $r(t)$ is determined by c via Eq(1.38).

1.4.3 Discrete Spectrum

Eq(1.40) has both discrete and continuous spectra. To find the discrete spectrum, we can seek a type of solution of the form

$$c(t,\omega) = b(\omega)\mathrm{e}^{\lambda t}, \tag{1.41}$$

where λ is independent of ω, then Eq(1.41) becomes

$$\lambda b(\omega) = -(D + i\omega)b(\omega) + \frac{K}{2}\int_{-\infty}^{\infty} b(v)g(v)dv. \tag{1.42}$$

Eq(1.42) can be solved in a self-consistent way. Let $A = \frac{K}{2}\int_{-\infty}^{\infty} b(v)g(v)dv$, solving Eq(1.42) for $b(\omega)$ we get

$$b(\omega) = \frac{A}{\lambda + D + i\omega}, \tag{1.43}$$

substituting this back into the expression of A, we obtain

$$1 = \frac{K}{2}\int_{-\infty}^{\infty} \frac{g(v)}{\lambda + D + iv}dv. \tag{1.44}$$

Note that Eq(1.44) relates λ to the coupling strength K. Then Eq(1.44) can be transformed into

$$1 = \frac{K}{2}\int_{-\infty}^{\infty} \frac{\lambda + D}{(\lambda + D)^2 + v^2}g(v)dv, \tag{1.45}$$

which shows how the eigenvalue λ depends on the noise strength D, the coupling strength K, and the frequency density $g(\omega)$. If $g(\omega)$ is even, we further assume that $g(\omega)$ is non-increasing on $[0, \infty)$ in the sense that $g(\omega) \leq g(v) \quad \forall\omega \geq v$ which holds for Gaussian, Lorentzian, and uniform distributions. Therefore Eq(1.44) has at most one solution for $\lambda > -D$, and if such a solution exists, it must be real [20]. Obviously, K_c correspond to $\lambda = 0$. When $\lambda > 0$ the fundamental mode is unstable, and the coherence grows like $r(t) \approx r_o \mathrm{e}^{\lambda t}$[10]. By using Eq(1.27) the critical condition $r = 0$, implies

$$K_c = 2\Big(\int_{-\infty}^{\infty} \frac{D}{D^2 + v^2}g(v)dv\Big)^{-1}, \tag{1.46}$$

Eq(1.46) is the same as Eq(1.28), thus K_c corresponding to λ. For the noise-free case, the incoherent solution goes unstable for $K > K_c = 2/(\pi g(0))$ as suggested by Kuramoto. To prove this, let us consider $D = 0$ in Eq(1.46) and take the limit $\lambda \to 0^+$. The kernel function $\lambda/(\lambda^2 + v^2)$ becomes more and more sharply peaked about $v = 0$, simply we can write $\pi\delta(0) = \lim_{\lambda\to 0^+} \lambda/(\lambda^2 + v^2)$. Thus Eq(1.46) becomes $K_c = 2/(\pi g(0))$ so we recover the results in Eq(1.18).

1.5 Continuous Spectrum

To find the continuous spectra we apply the operator $\mathcal{L}$ to the Eq(1.41) as follows

$$\mathcal{L}b = -(D + i\omega)b + \tfrac{K}{2}\int_{-\infty}^{\infty} b(v)g(v)dv, \tag{1.47}$$

the continuous spectra of $\mathcal{L}$ is defined as the set of complex numbers λ such that the operator $\mathcal{L} - \lambda I$ is not surjective, i.e. $\det|\mathcal{L} - \lambda I| = 0$. Now, adding $-\lambda b$ at each side of the equality we get

$$-(D + \lambda + i\omega)b + \tfrac{K}{2}\int_{-\infty}^{\infty} b(v)g(v)dv = f(\omega), \tag{1.48}$$

where $f(\omega)$ is an arbitrary function that satisfies $(\mathcal{L} - \lambda I) = f(\omega)$. If $\lambda + D + i\omega = 0$ for ω in the support of $g(\omega)$, then the equation is not solvable in general [10]. Hence, the continuous spectra contains the set

$$\{-D - i\omega : \omega \in \text{support}(g(\omega))\}, \tag{1.49}$$

the last set is all of the continuous spectra just assuming that λ is not in the support of $g(\omega)$. Then Eq(1.48) is solvable

$$b(\omega) = \frac{A - f(\omega)}{\lambda + D + i\omega}, \tag{1.50}$$

where A is the integral in Eq(1.48) isolating $b(\omega)$ and replacing again in the A equation we get

$$A\left(\frac{K}{2}\int_{-\infty}^{\infty} \frac{g(\omega)}{\lambda + D + i\omega}d\omega\right) = \frac{K}{2}\int_{-\infty}^{\infty} \frac{g(\omega)f(\omega)}{\lambda + D + i\omega}d\omega. \tag{1.51}$$

By assumption, λ is not in the discrete spectrum, and $A \neq 0$ (we do not consider the trivial solution). Thus, Eq(1.51) can be solved for A. Hence, the set considered before is the continuous spectrum. We notice that for $D = 0$, noise-free case, the spectrum lies in the imaginary axis, the fundamental mode for $K > K_c$ is unstable and for $K > K_c$ is neutrally stable. Continuous and discrete spectrum for the linear operator Eq(1.47), for the noisy case $D > 0$, can be summarized as follows: when $K > K_c$ the fundamental mode is unstable since $\lambda > 0$, and the continuous spectrum lies in the left-plane. When $K = K_c$, we are at the critical point, so $\lambda = 0$. When $K^* < K < K_c$[1], the fundamental mode is stable since $\lambda < 0$. When $K^* \leq K$, the discrete value is absorbed by the continuous spectrum.

[1] K^* is the value of K when $\lambda = -D$

1.6 Non-Uniform Coupling Constant

Daido has considered the general mean-field model [21]

$$\dot{\theta}_i = \omega_i + \sum_{i=1}^{N} K_{ij} \sin(\theta_j - \theta_i + A_{ij}) + \xi_i(t), \quad i = 1, ..., N, \tag{1.52}$$

where N is the size of the system, $A_{ij} \in [-\pi \; \pi]$ is a random phase shift, disorder factor, acting like potential vector differences between sites which is assumed to produce frustration. $\omega_i' s$ have $g(\omega)-$distribution and ξ is Gaussian noise. $K_{ij} = K_{ji}$ are independent random variables with normal distribution, $N(0, K^2/N)$ where K is control parameter, denoted by $P(K_{ij})$ and is given by

$$P(K_{ij}) = (\frac{2\pi K^2}{N})^{-\frac{1}{2}} \exp(-\frac{NK_{ij}^2}{2K^2}). \tag{1.53}$$

The interest of this model lies in the fact that the coupling term in Eq(1.53) vanishes when $\theta_i - \theta_j \neq 0$, i.e. the system is in an incoherent state which is called frustration. The model when $A_{ij} = 0$ in Eq(1.53) has been subject to a recent work [21, 22]. The model equation is

$$\dot{\theta}_i = \omega_i + \sum_{i=1}^{N} K_{ij} \sin(\theta_j - \theta_i) + \xi_i(t), \quad i = 1, ..., N, \tag{1.54}$$

where K_{ij} are given by Eq(1.53). Kirkpatrick & Sherrington [23] have studied the model, Eq(1.54), when $\omega_i = 0$ to mimic the behavior of frustrated magnets. Most studies were done without noise and for Gaussian frequency distribution $g(\omega)$. In two different studies, Daido [21] and Stiller [24] have conducted numerical analysis but, unfortunately, with contradicting results.

1.7 Numerical Simulations

In part for pedagogical reasons we have conducted a numerical investigation for Kuramoto model. The language used is C++ [1]. A sample of 1500 oscillators is implemented in this numerical simulation. The oscillators frequencies are randomized using a function that generates normally distributed random numbers from a built-in uniform random generator, centered at zero with a user-defined standard deviation. The oscillators phases are initialized with uniform phases over a unit circle. A numerical integration of equation Eq(1.5) has been performed by implementing Euler's method. The order parameter versus coupling constant are recorded and plotted in Fig(1),

[1] Please see Appendix

which shows a clear signature of phase transition, below $K = 3$ no significant synchronization is noticed, and the predicted critical coupling is $K_c = 4$.

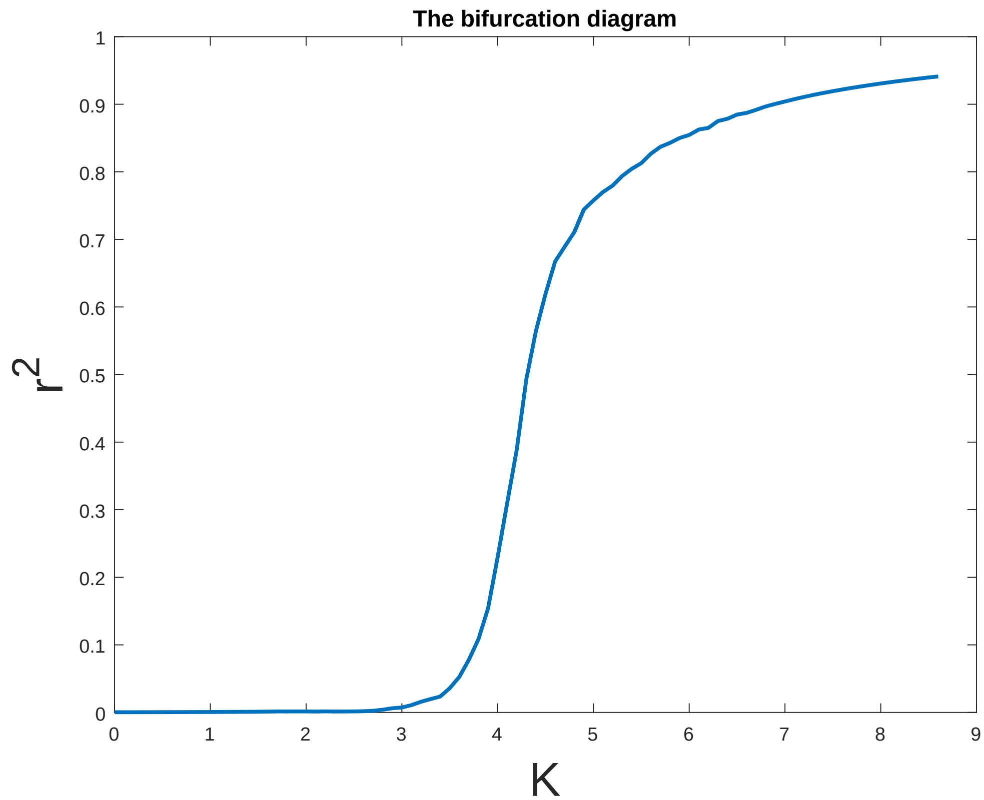

Figure 1.: The bifurcation diagram of Kuramoto model with 1500 oscillators

Chapter 2
Stochastic Dynamics

2.1 Brownian Motion

[1] A probabilistic model is often used when the problem of interest lacks sufficient information to determine how the system behaves, or the system is so complex that an exact description of it becomes impossible. Many important real-world systems are subject to random events, which could be referred to as noise or fluctuations caused by the interaction between the system and its environment. Such systems are best understood in the context of Stochastic Dynamics. In the real world there is no noise-free system. Deterministic dynamics works very well if the noise scale is negligible compared to the scale of the system. Yet the interactions that are eliminated from large-scale (macroscopic) models make themselves felt in other ways: The most famous example of observable fluctuations in a physical system is Brownian motion where a continuous random meandering of a pollen grain suspended in a fluid. In 1827, R. Brown discovered under the microscope the continuous and irregular motion of small pollen particles suspended in water. He also remarked that small mineral particles behave exactly in the same way (such an observation is important since it excludes the biological nature of the motion). In a general way, a particle in suspension in a fluid performs a Brownian motion when its mass is much larger than the mass of one of the fluid's molecules. The idea according to which the motion of a Brownian particle is a result of the motion of the lighter molecules of the surrounding fluid became popular during the second half of the nineteenth century. This explanation was introduced by A. Einstein in 1905, which marked the beginning of the theory of stochastic processes.

2.2 Basic Concepts on Stochastic Processes

A stochastic process is a term that refers to any collection of random variables $\{X(t,\omega)\}$[1] depending on time t, defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For fixed $\omega \in \Omega$, $X(t,.)$ is a sample

[1]The discussion of this chapter follows the chapters (1-4)[25], (1-5)[26],(1-3)[27], and (1-6)[28].
[1]$X(t,\omega)$ and $X_t(\omega)$ will be used in the text interchangeably

path of the process. At a fixed time t, properties of the random variable $X(.,\omega)$ are described by the probability distribution of $X(.,\omega)$.

DEFINITION 2.2.1 [Stochastic Process] Suppose that for each $t \in \mathbb{R}^+$ there is a random variable $X_t : \Omega \to \mathbb{R}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$. The function $X_t : T \times \Omega \to \mathbb{R}$ defined by $X(t,\omega) = X_t(\omega)$ is called a stochastic process with indexing set t and written as $X = \{X_t, t \in T\}$.

2.2.1 Gaussian Process

A stochastic process determines a probability distribution of the form

$$\mathbb{P}(X_{t_1}(\omega) \leq x_1, X_{t_2}(\omega) \leq x_2, ..., X_{t_n}(\omega) \leq x_n) \tag{2.1}$$

where $n \in \mathbb{N}$, $t_1 < t_2, ..., < t_n$, and $x_1, x_2, ..., x_n \in \mathbb{R}$. If the probability distribution is Gaussian (multivariate distributions), then the process is called a Gaussian process with mean given by $m(t) = \mathbb{E}(X_t)$ and covariance function $\gamma(t,s) = cov\{X_t, X_s\}$.

DEFINITION 2.2.2 The stochastic process is Gaussian $\{X_t\}_{t\geq 0}$ if it is Gaussian for any choice of $\{t_i\}$.

2.3 Brownian Motion

Brownian motion is the most fundamental stochastic process in continuous space and time. Much of the stochastic dynamics Mathematics was developed for studying Brownian motion. It is merely a simple mathematical illustration and formalism that provides a direct and concrete connection to physical reality. However, in its own right is not physical [2] but it can be used to model physical systems by employing assumptions that suited for each phenomenon. The limitations of the various assumptions, employed in the modeling of physical phenomena, are made obvious due to simplicity of the Brownian motion. The first mathematical construction for Brownian motion was proposed by N. Wiener in 1923. He used a random Fourier series to construct Brownian motion. Our treatment follows later ideas of Lévy and Kolmogorov. We start by giving a formal definition of the stochastic process.

[2]Since the energy of such system diverges as the time goes to infinity.

Definition 2.3.1 [Wiener Process] Let $W = (W_t)_{t \geq 0}$ be a stochastic process in $\mathbb{R}^N$. We say that **W** is a Wiener process in $\mathbb{R}^N$ if

- $W(0) = 0$ a.s.
- W has independent increment for any finite time sequence $0 \leq t_0 < t_1 < t_2 < ... < t_n$, the increments $W_{t_1} - W_{t_0}, W_{t_2} - W_{t_1}, W_{t_3} - W_{t_2}, ..., W_{t_n} - W_{t_{n-1}}$ are independent random variables.
- $W_t - W_s \sim N(0, (t-s))$, for all $s < t$.
- The sample path $W_t(\omega)$ are a.s. continuous for $t \geq 0$

Theorem 2.3.1 (Wiener Theorem) *There exists a Brownian motion on some probability space.*

We will show that such a process exists by explicitly constructing one.

2.4 Mathematical Construction of Brownian Motion

2.4.1 The $\mathbb{L}_2-$space Theory

We need to show that Brownian motion âĂŸexistsâĂŹ in the sense that we have a Gaussian process $W(t)$[1] with the right covariance function. Let $\{\phi_i\}$ be a complete orthonormal basis of $\mathbb{L}_2[0,1]$ and $X_1, X_2, ...$ be a sequence of independent identically distributed random functions defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with $X_i \sim N(0,1)$. For $n = 1, 2, ...$ define

$$W_t^n = \sum_{i=1}^{n} X_i \int_0^t \phi_i(s) ds \tag{2.2}$$

Theorem 2.4.1 *For each* t, W_t^n *is a Cauchy sequence in* $\mathbb{L}_2(\Omega, \mathcal{F}, \mathbb{P})$ *whose limit,* W_t*, is a normal random variable with mean zero and variance* t*. For any two times* $t, s, \mathbb{E}[W_t W_s] = t \wedge s$*, where* $t \wedge s \equiv$ $\min(t, s)$.

Proof. Define

$$I_t(s) = \begin{cases} 1 & , \ x < t \\ 0 & , \ s \geq t \end{cases}$$

Then

$$\int_0^t \phi_i(s) ds = < I_t, \phi_i >, \ I_t = \sum_{i}^{\infty} < I_t, \phi_i > \phi_i \text{and} \ \ ||I_t||^2 = \sum_{1}^{\infty} < I_t, \phi_i >^2 . \tag{2.3}$$

[1] $W(t)$ and W_t are used interchangeably.

since ϕ_i is a complete orthonormal basis. Thus for $n > m$

$$\mathbb{E}(W_t^n - W_t^m)^2 = \mathbb{E}\Big(\sum_{i=m+1}^{n} X_i \int_0^t \phi_i(s)ds\Big)^2 = \sum_{i=m+1}^{n} < I_t, \phi_i >^2 \to 0 \text{ as } m, n \to \infty$$

Thus W_t^n is a Cauchy sequence in $\mathbb{L}_2(\Omega, \mathcal{F}, \mathbb{P})$. From Eq(2.3) we obtain

$$\mathrm{Var}(W(t)) = \lim_{n\to\infty} \mathrm{Var}(W_t^n) = t.$$

We have

$$\mathbb{E}[W_t W_s] = \sum_{1}^{\infty} < I_t, \phi_i >< I_s, \phi_i > = < I_t, I_s > = t \wedge s.$$

□

2.5 Properties of Brownian Motion

The following properties of Brownian motion will be used a lot

- Continuous-time Brownian motion is a martingale.
- Any sample path of a Brownian motion is nowhere differentiable.
- Law of asymptotic sub-linear limit, $\lim_{t\to\infty} \dfrac{W(t)}{t} = 0$ almost surely.
- Law of iterated logarithms $\limsup_{t\to\infty} \dfrac{W(t)}{\sqrt{2t \ln \ln t}} = 1$.a.s. $\liminf_{t\to\infty} \dfrac{W(t)}{\sqrt{2t \ln \ln t}} = -1$ almost surely.
- Local Hölder continuity for any $0 < \alpha < \frac{1}{2}$, $\sup_{n\le t,s\le n+1} \dfrac{|W(t) - W(s)|}{|t-s|^\alpha} < \infty, n = 0, 1, 2, ...$
- N-dimensional Brownian Motion $W(t) \sim N(0, tI_N)$, the probability density function the density of the Gaussian random vector $\mathbf{W}(t) - \mathbf{W}(s)$ is

$$P(\mathbf{x}, t) = \frac{1}{(2\pi t))^{N/2}} \exp\Big(-\frac{x_1^2 + x_2^2 + ... + x_N^2}{2t}\Big). \tag{2.4}$$

- A Brownian motion is almost surely not a path-wise monotone on any time interval.

Let us step back and look at some technical points. We have defined Brownian motion as a stochastic process $W(t) : t \ge 0$ which is merely a collection of uncountably many random variables $\omega \longmapsto$

$W(t,\omega)$ defined on a probability space (ΩÂŋ; $\mathcal{F}$; $\mathbb{P}$). At the same time, a stochastic process can also be interpreted as a random function, normally called a sample path defined by $t \longmapsto W(t,\omega)$. The sample path properties of a stochastic process are the properties of these random functions.

DEFINITION 2.5.1 If $W(t)_{t\geq 0}$ is a Wiener process, fixing $\omega \in \Omega$, we get a function of time $X_t(\omega) = X(t,\omega)$, called a sample path of the process.

2.5.1 W_t is Gaussian

If the process is started at x, then $W_t \sim N(x,t)$. This can be written as

$$P_x(W(t) \in (a,b)) = \int_a^b \frac{1}{\sqrt{2\pi t}} \mathrm{e}^{-\frac{(y-x)^2}{2t}} dy,$$

consequently, Brownian motion is a Gaussian process where $\mathrm{cov}(W_t - W_s) = t \wedge s$ for $s,t \geq 0$. Now we need to show that W_t is normal. Note that W_t^n is a finite sum of normal random variables and is therefore normal, with variance $\sigma^2 = \sum_1^n < I_t, \phi_i >^2$. Hence the characteristic function of W_t^n is $\chi_n(u) = \mathbb{E}(\exp(iuW_t^n)) = \exp(-\sigma^2 u^2/2)$, which converges as $n \to \infty$ to $\chi(u) \equiv \exp($âĹŠ$tu^2/2)$. Now $W_t^n \to W_t$ in $\mathbb{L}_2$ implies that there is a sub-sequence $W_t^{n_k}$ such that $W_t^{n_k} \to W_t$ almost surely as $k \to \infty$. It follows from the bounded convergence theorem that $\mathbb{E}(\exp(iuW_t^{n_k})) \to \mathbb{E}(\exp(iuW_t))$ and hence that $\mathbb{E}(\exp(iuW_t)) = \chi(u)$. Thus $W_t \sim N(0,t)$.

2.5.2 W_t is Continuous

Our next step to construct the Brownian motion is to define a special orthogonal normal basis. We will make use of Haar wavelets to construct Wiener process. The idea is to construct a standard Brownian motion on $[0,1]$, so that for each $0 \leq t < \infty$, we can get $W(t)$ by setting

$$W(t) = W_{t-n}^{(n+1)} + \sum_{k=1}^{n} W^{(k)}(1) \text{ for } t \in [n, n+1).$$

We define the Haar functions as

$$\begin{aligned} H_0(x) &= 1, \\ H_1(x) &= \begin{cases} 1, & 0 \leq x < \frac{1}{2} \\ -1, & \frac{1}{2} < x \leq 1, \end{cases} \end{aligned}$$

if $2^n \leq k < 2^{n+1}$, where $n = 1, 2, ...$, then, we get

$$H_k(t) = \begin{cases} 2^{n/2}, & \frac{k-2^n}{2^n} \leq t < \frac{k-2^2+1/2}{2^n} \\ -2^{n/2}, & \frac{k-2^n+1/2}{2^n} < t \leq \frac{k-2^n+1/2}{2^n} \\ 0, & \text{elsewhere,} \end{cases} \tag{2.5}$$

then from this $H(t)$, we define a sequence of functions

$$\psi_{j,k}(t) = H(2^j t - k) \text{ for } 0 \leq j, 0 \leq k < 2^j.$$

The sequence $\psi_{j,k}$ is called Haar wavelet and from the wavelet we define the Haar function

$$\begin{cases} H_0(t) = 1 \\ H_n(t) = 2^{\frac{j}{2}} \psi_{j,k}(t), n = 2^j + k \text{ where } j \geq 0 \text{ and } 0 \leq k < 2^j. \end{cases}$$

The set $\{H_n\}$ forms a complete set in $\mathbb{L}_2$, and we are going to use this fact to construct the Brownian motion.

THEOREM 2.5.1 *The Haar functions are a complete orthonormal basis in* $\mathbb{L}_2[0, 1]$.

First, we prove that the set is orthonormal and then we prove that it is complete.

Proof. Let $[t^j, t^j_*]$ be the interval on which $H_j(x)$ is nonzero. For $j < i$ the interval $[t^i, t^i_*]$ is either disjoint from $[t^i, t^i_*]$ which implies $H_j(x)H_i(x) = 0$ or contained in it, which is equal to constant $H_j(x)$, thus

$$\int_0^1 H_i(x)H_j(x)dx = 0$$

when $i = j$ we get

$$\int_0^1 H_i(x)H_j(x)dx = 2^n(\frac{1}{2^{n+1}} + \frac{1}{2^{n+1}}) = 1.$$

□

Thus, the Haar function forms an orthonormal system of functions. Now we have to show that they are complete, i.e. that if $f \in \mathbb{L}_2[0, 1]$ we have $< f, H_k >= 0$ for all k then $f = 0$ almost everywhere. Suppose f satisfies these conditions,

Proof. If $n = 0$, we have $\int_0^1 f dx = 0$. Let $n = 1$, $\int_0^1 f H_k(x)dx = 0$. Then $\int_0^{1/2} f dx = \int_{1/2}^1 f dx$ and both are equal to zero, since $\int_0^1 f dx = \int_0^{1/2} f dx + \int_{1/2}^1 f dx = 0$. Continuing in this way, we deduce $\int_{\frac{k}{2^{n+1}}}^{\frac{k+1}{2^{n+1}}} f dx = 0$ for all $0 \le k < 2^{n+1}$. Thus $\int_s^r f dx = 0$ for all dyadic rationals $0 \le s \le r \le 1$. Since for any real number r there is a sequence r_n of dyadic rational numbers r_n converging to r, $\int_s^r f dx = 0$ for all real numbers r, s. This completes the proof. □

We define another sequence of functions, $\{\Psi_{j,k}(t)\}$ by

$$\Psi_{j,k}(t) = \int_0^t \psi_{j,k}(s) ds$$

similar to the way we constructed $\{\psi_{j,k}(t)\}$, this sequence can also be represented as the wavelet by defining a tent wavelet and constructing another function from the tent wavelet. Let us denote the tent function $\Psi(t)$ defined as

$$\Psi(t) = \begin{cases} 2t \quad 0 \le \ t < \ \frac{1}{2}, \\ 2(1-t) \quad \frac{1}{2} \le \ t \ \le \ 1, 0 \quad \text{elsewhere} \end{cases}$$

and from the tent wavelet function we define another sequence of functions

$$\Psi_{j,k}(t) = \Psi(2^j t - k) \ \text{ for } \ 0 \le j, 0 \le k < 2^j$$

Now, let us define $\{\Delta_n(t)\}$ as $\Delta_{2j+k}(t) = \Psi_{j,k}(t)$ and $\{\lambda_n\}$ as

$$\begin{cases} \lambda_0 = 1 \\ \lambda_n = \frac{2^{-j/2}}{2} \ \text{ where } \ n \ge 1 \ \text{ and } \ n = 2^j + k \ \text{ with } \ 0 \le k < 2. \end{cases}$$

Then we can define,

Definition 2.5.2 [Schauder function] For $k = 0, 1, 2, ...,$ $s_n(t) = \lambda_n \Delta_n = \int_0^t H_n(s) ds$.

The Schauder functions are "little tents" of height $\max_{0 \le t \le 1} |s_k(t)| = 2^{-(n+2)/2}$, lying above the interval $[\frac{k-2^n}{2^n}, \frac{k-2^n}{2^n}, \frac{k-2^n+1}{2^n}]$, as shown in Figure (2). Now, we have all necessary background material to construct a standard Brownian motion for $t \in [0,1]$. The idea is to show that $W_t^n \to W_t$ uniformly almost surely when we take the orthonormal basis to be the Haar functions.

Lemma 2.5.1 *Suppose that, for $n = 1, 2, ..., f_n : [0,1] \to \mathbb{R}$ is a continuous function, and that $f_n(t)$ converges uniformly to a function f i.e. given $\epsilon > 0$ there is a number N such that $n \ge N$ implies $|f_n(t) - f(t)| < \epsilon$ for any $t \in [0,1]$. Then f is continuous function.*

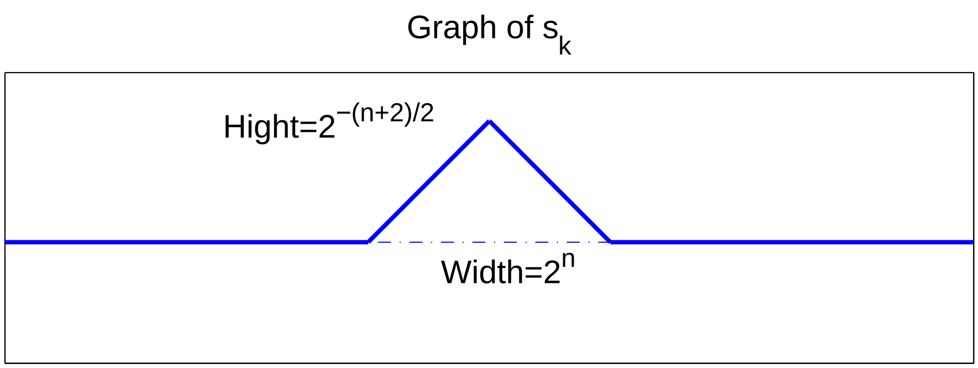

Figure 2.: Schrauder Function

Proof. For any $t, s \in [0,1]$ we can write $|f(t)-f(s)| \leq |f(t)-f_n(t)|+|f_n(t)-f_n(s)|+|f_n(s)-f(s)|$. Given $\epsilon > 0$ we can find n such that the first and third terms on the right are each less that $\epsilon/3$ (whatever t, s). Now f_n is continuous, so for fixed t we can choose δ so that the second term is less than $\epsilon/3$ for all s such that $|t-s| < \delta$. Consequently, f is continuous at t. $\square$

We define

$$W(t) = \sum_{n=0}^{\infty} \lambda_n Z_n s_n(t)$$

for $t \in [0,1]$, where the coefficients $\{Z_n\}_{n=0}^{\infty}$ are independent, normally distributed, $N(0,1)$, random variables defined on some probability space. We will prove the lemma by showing that $W(t)$ satisfies the required properties of a standard Brownian motion. First, we have to check whether this series converges. To do that, we first prove the following lemma,

Lemma 2.5.2 *Let $\{Z_n : 0 \leq n < \infty\}$ be a sequence of independent Gaussian random variables with mean 0 and variance 1, then there is a random variable C which is finite with probability one and $|Z_n| \leq C\sqrt{\log n}$ for all $n \geq 2$.*

Proof. For all $x > 0$ and $n \geq 2$, we have

$$P(|Z_n| \geq x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} \exp(-\frac{u^2}{2})du \leq \sqrt{\frac{2}{\pi}} \int_x^{\infty} u \exp(-\frac{u^2}{2})du = \exp(-\frac{x^2}{2})\sqrt{\frac{2}{\pi}}$$

Thus, for any $\alpha > 1$, we have

$$P(|Z_n| \geq \sqrt{2\alpha \log n}) \leq \exp(-\alpha \log n)\sqrt{\frac{2}{\pi}} = n^{-\alpha}\sqrt{\frac{2}{\pi}}$$

Note that for $\alpha > 1$, we have

$$\sum_{n=1}^{\infty} n^{-\alpha} < \infty$$

Using Borel-Cantelli lemma, we get

$$P(|Z_n| \geq \sqrt{2\alpha \log n} \quad \text{i.o.}) = 0.$$

Therefore, the random variable defined by

$$\sup_{2 \leq n < \infty} \frac{|Z_n|}{\log n} = C$$

is finite with probability one. □

Now we are ready to prove that $W(t)$ converges uniformly on $[0, 1]$ with probability one. Notice that for $n \in [2^j,\ 2^{j+1}]$ the function $s_n(t)$ has disjoint support and $\log n < j+1$. From Lemma(2.5.2) $|Z_n| \leq C\sqrt{\log n}$ where C is a finite random variable and $n \geq 2$, therefore, for any $J \geq 1$, if we let $M \geq 2^J$, we obtain

$$\sum_{n=M}^{\infty} \lambda_n |Z_n| s_n(t) \leq C \sum_{n=M}^{\infty} \lambda_n \sqrt{\log n} s_n(t) \leq C \sum_{j=J}^{\infty} \sum_{k=0}^{2^j - 1} \frac{2^{-j/2}}{2} \sqrt{j+1} s_{2^j+k}(t) \leq$$

$$C \sum_{j=J}^{\infty} \frac{2^{-j/2}}{2} \sqrt{j+1},$$

and note that $\sum_{j=1}^{\infty} \frac{2^{-j/2}}{2} \sqrt{j+1} < \infty$. Therefore, we have

$$\lim_{J \to \infty} C \sum_{j=J}^{\infty} \frac{2^{-j/2}}{2} \sqrt{j+1} = 0.$$

Since $\lambda_n s_n(t)$ is a bounded continuous function on $[0, 1]$, $W(t)$ converges uniformly on $[0, 1]$ with probability one, and since $s_n(t)$ is a continuous function then $W(t)$ is also continuous with probability one. We have proven that $W(t)$ is continuous and also we have proven that the Brownian motion does exist by constructing it. Now, we need to prove that $W(t)$ has independent increments. We begin by proving the following Lemma,

LEMMA 2.5.3 $\sum_{n=0}^{\infty} \lambda_n^2 s_n(s) s_n(t) = t \wedge s$.

Proof. Let $s \in [0,1]$,

$$\phi_s(\tau) = \begin{cases} 1 & 0 \le \tau \le s \\ 0 & s < \tau \le 1. \end{cases}$$

Then if $s \le t$, the completeness and the orthonormality of Haar function, Theorem (2.5.1), implies

$$s = \int_0^1 \phi_t \phi_s d\tau = \sum_{k=0}^{\infty} a_k b_k,$$

and, if $t \le s$

$$t = \int_0^1 \phi_t \phi_s d\tau = \sum_{k=0}^{\infty} a_k b_k,$$

Then, we have

$$\sum_{n=0}^{\infty} \lambda_n^2 s_n(s) s_n(t) = t \wedge s,$$

where

$$a_k = \int_0^1 \phi_t H_k d\tau = \int_0^t H_k d\tau = s_k(t), \quad b_k = \int_0^1 \phi_s H_k d\tau = s_k(s)$$

□

Next will show that $W(t)$ has independent increments by proving that $W(t)$ satisfies

$$\text{cov}(W(t), W(s)) = t \wedge s \quad \text{for all } 0 \le s, t \le T.$$

LEMMA 2.5.4 *If a process $\{W(t), 0 \le t \le T\}$ is Gaussian and has $\mathbb{E}(W(t)) = 0$ for all $0 \le t \le T$ and if cov$(W(t), W(s)) = s \wedge t$ for all $0 \le s, t \le T$, then $\{W(t)\}$ has independent increments, and if this process has continuous paths and $W(0) = 0$, then it is a standard Brownian motion on $[0, T]$.*

Proof.

$$\mathbb{E}(W(t)W(s)) = \mathbb{E}\Big(\sum_{n=0}^{\infty} \lambda_n Z_n s_n(t) \sum_{n=0}^{\infty} \lambda_m Z_m s_m(t)\Big) = \sum_{n=0}^{\infty} \lambda_n^2 s_n(t) s_n(s) = t \wedge s,$$

We make use of Lemma (2.5.3) and Theorem (2.5.1) for the second part of above equation. It is sufficient to show that the characteristic function of the multivariate $(X_{t_1}, X_{t_2}, X_{t_3}, ..., X_{t_n})$ matches the characteristic function of a multivariate Gaussian with mean zero and covariance matrix, $\Sigma = \min(t_i, t_j)$

$$\mathbb{E}\big(\exp(i\sum_{j=1}^{n}\theta_j W(t_j))\big) = \mathbb{E}\big(\exp(i\sum_{j=1}^{n}\theta_j \sum_{k=0}^{\infty}\lambda_k Z_k s_k(t_j))\big)$$

$$= \prod_{k=0}^{\infty}\mathbb{E}\big[\exp(i\lambda_k Z_k \sum_{j=1}^{n}\theta_j s_k(t_j))\big] = \prod_{k=0}^{\infty}\exp(-\frac{1}{2}\lambda_k^2(\sum_{j=1}^{n}\theta_j s_k(t_j))^2)$$

$$= \exp(-\frac{1}{2}\sum_{k=0}^{\infty}\lambda_k^2(\sum_{j=1}^{n}\theta_j s_k(t_j))^2) = \exp(-\frac{1}{2}\sum_{k=0}^{\infty}\lambda_k^2\sum_{i=1}^{n}\sum_{j=1}^{n}\theta_j\theta_i s_k(t_i)s_k(t_j))$$

Using Lemma (2.5.3) we get

$$\mathbb{E}\big(\exp(i\sum_{j=1}^{n}\theta_j W(t_j))\big) = \exp\Big(-\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\theta_i\theta_j\min(t_j, t_k)\Big)$$

and the last expression is the characteristic function of a multivariate function of a multivariate Gaussian with mean zero and covariance matrix, $\Sigma = (\min(t_i, t_j))$, and, therefore, by uniqueness of characteristic functions, $W(t)$ is indeed a standard Brownian motion. □

2.6 Properties of Brownian motion

2.6.1 Invariance Properties

LEMMA 2.6.1 (Scaling Lemma) *If W_t is a Brownian motion and $c > 0$ then $X_t = \frac{1}{c}W(\frac{1}{c^2}t)$, for $t \geq 0$, is a Brownian motion.*

Proof. X_t is a continuous function of a Brownian motion; obviously, it has continuous paths. $\mathbb{E}(X_t) = \frac{1}{c}\mathbb{E}(W(c^2t)) = 0$ since W is a Brownian motion. Let $s < t$, then

$$\mathbb{E}(X_s X_t) = \mathbb{E}(\frac{1}{c}W(c^2s)\frac{1}{c}W(c^2t)) = \frac{1}{c^2}\mathbb{E}(W(c^2s)W(c^2t)) = \frac{1}{c^2}c^2 t\wedge s = t,$$

$$\sum_{k=1}^{N}\lambda_k X_{t_k} = \sum_{k=1}^{N}\frac{\lambda_k}{c}W(c^2t_k)$$

which is a sum of Gaussian random variables, hence it is a Gaussian. □

LEMMA 2.6.2 (Time Inversion) *If W_t is a standard Brownian motion then the process*

$$X_t = \begin{cases} tW(\frac{1}{t}) & t \neq 0 \\ 0 & t = 0 \end{cases}$$

is also a standard Brownian motion.

Proof. For any finite $0 \le t_1 < t_2, ... < t_n$, the marginal random variable $W(t_1), W(t_2), ..., W(t_n)$ is a Gaussian multivariate random variable with

$$\mathbb{W}(t_i) = 0 \text{ and } \mathbb{E}(W(t_i)W(t_j)) = \text{cov}(W(t_i)W(t_j)) = t_i \wedge t_j.$$

We can check $X(0) = 0, X(t) - X(s)$ is still a normal random variable with mean zero and variance $\mathbb{E}|X(t)-X(t+h)|^2 = \mathbb{E}|X(t)|^2+\mathbb{E}|X(t+h)|^2-2\mathbb{E}|X(t)X(t+h)| = t^2(\frac{1}{t})+(t+h)^2\frac{1}{t+h}-2t(t+h)\frac{1}{t+h} = 2t + h - 2t = h$ for any $t \ge 0$ and $h > 0$. Moreover, the independent increments condition is also satisfied by $X(t)$ and the sample paths are continuous on $(0, \infty)$ almost surely. Finally we need to show that $\lim_{t \downarrow 0} X(t) = 0$ almost surely. This follows from the following asymptotic limit property that $\lim_{t\to\infty} \frac{W(t)}{t} = 0$ almost surely. □

LEMMA 2.6.3 (Translation Invariance Lemma) *For any fixed $t_0 \ge 0$, the process*

$$\tilde{W}(t) = W(t + t_0) - W(t_0)$$

is also a Brownian motion.

Proof. $\tilde{W}(t+s) - \tilde{W}(s) = W(t+s+t_0) - W(t_0) - W(s+t_0) + W(t_0) = W(t+s+t_0) - W(s+t_0)$ which is by definition normally distributed with mean 0 and variance t. $\tilde{W}(t_{j+1}) - \tilde{W}(t_j) = W(t_{j+1} + t_0) - W(t_j + t_0)$ are independent for all $j = 0, 1, ...,$ by the property of independence of disjoint increment of $W(t)$. $\tilde{W}(0) = W(t_0) - W(t_0) = 0$, as the composition and difference of continuous functions, $\tilde{W}$ is continuous. The proof is completed. □

2.6.2 Asymptotic Limit Properties

THEOREM 2.6.1 (Law of iterated logarithm) *For a Brownian motion $W(t)$ satisfies*

$$\lim_{t\to\infty} \sup \frac{W(t)}{\sqrt{2t \log \log t}} = 1$$

almost surely.

Before proving the theorem, we state an elementary lemma

LEMMA 2.6.4 *Let $X \sim N(0,1)$ be standard normally distributed. Then, for any $x > 0$,*

$$\frac{1}{\sqrt{2\pi}} \frac{1}{x + \frac{1}{x}} e^{-\frac{x^2}{2}} \leq \mathbb{P}[X \geq x] \leq \frac{1}{x} e^{-\frac{x^2}{2}} \tag{2.6}$$

Proof. Let $\phi(t) = \frac{1}{\sqrt{2\pi} e^{\frac{x^2}{2}}}$ be the density of the standard normal distribution. Partial integration yields the second inequality in Lemma (2.6.4),

$$\mathbb{P}[X \geq x] = \int_x^\infty \frac{1}{t}(t\phi(t))dt = -\frac{1}{t}\phi(t)|_x^\infty - \int_x^\infty \frac{1}{t^2}\phi(t)dt \leq \frac{1}{x}\phi(x).$$

Similarly,

$$\mathbb{P}[X \geq x] \geq \frac{1}{x}\phi(x) - \frac{1}{x^2}\int_x^\infty \phi(t)dt = \frac{1}{x}\phi(x) - \frac{1}{x^2}\mathbb{P}[X \geq x].$$

This implies the first inequality in Lemma 2.6.4. □

LEMMA 2.6.5 (Reflection Principle) *For $m \geq 0$ we have that $\mathbb{P}(\sup_{s\leq t} W(s) > m) = 2\mathbb{P}(W(t) \geq m)$.*

Proof. Let $\{\sup_{s\leq t} W(s) \geq m\}$ be the event that the Brownian motion exceeds m before time t. The sets $\{W(t) > m\}, \{W(t) = m\}, \{W(t) < m\}$ from a partition so

$$\mathbb{P}(\{\sup_{s\leq t} W(s) \geq m\}) =$$

$$\mathbb{P}(\{\sup_{s\leq t} W(s)\} \geq m\} \cap \{W(t) > m\}) + \mathbb{P}(\{\sup_{s\leq t} W(s)\} \geq m\} \cap \{W(t) = m\}) +$$

$$\mathbb{P}(\{\sup_{s\leq t} W(s) \geq m\} \cap \{W(t) < m\})$$

But $\mathbb{P}(\{\sup_{s\leq t} W(s)\} \geq m \cap \{W(t) = m\}) = 0$ and $\mathbb{P}(\{W(t) > m\}|\{\sup_{s\leq t} W(s)\} \geq m) = \mathbb{P}(\{W(t) < m\}|\{\sup_{s\leq t} W(s)\} \geq m)$ since there are the same number of paths ending above m as there are below m, this gives us $\mathbb{P}(\{\sup_{s\leq t} W(s)\} \geq m \cap \{W(t) > m\}) = \mathbb{P}(\{\sup_{s\leq t} W(s) \geq m\} \cap \{W(t) < m\})$ so indeed we have that

$$\mathbb{P}(\{\sup_{s\leq t} W(s) \geq m\}) = 2\mathbb{P}(W(t) \geq m).$$

□

Proof. [Law of the iterated logarithm]

Let $\psi(t) = \sqrt{2t\log(\log(t))}$, by symmetry of W_t it suffices to show just the first limit $\lim_{t\to\infty}\sup \frac{W_t}{\psi(t)} = 1$. We will show that $\lim_{t\to\infty}\sup \frac{W_t}{\psi} \leq 1$ and then $\lim_{t\to\infty} \frac{W_t}{\psi(t)} \geq 1$.

Step 1: $\lim_{t\to\infty}\sup \frac{W_t}{\psi(t)} \leq 1$.

This equivalent to say that for $\epsilon > 0$ we have $\frac{W(t)}{\psi(t)} \leq 1+\epsilon$ for sufficiently large t. By using Lemma (2.6.3) we have

$$\mathbb{P}(W(t)) > (1+\epsilon)\psi(t)) = \int_{(1+\epsilon)\psi(t)}^{\infty} \frac{\mathrm{e}^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx$$

$$\leq \frac{\mathrm{e}^{-(1+\epsilon)^2\log(\log(t))}}{(1+\epsilon)\sqrt{4\pi\log(\log(t))}}$$

Let $\alpha > 1$, and define $t_n = \alpha^n$for $n \in \mathbb{N}$.

$$\mathbb{P}(W(\alpha^n) > (1+\epsilon)\psi(\alpha^n) \leq \frac{\mathrm{e}^{-(1+\epsilon)\log(\log(\alpha^n))}}{(1+\epsilon)\sqrt{4\pi\log(\log(\alpha^n))}} = \frac{\mathrm{e}^{-(1+\epsilon)\log(n\log(\alpha))}}{(1+\epsilon)\sqrt{4\pi\log(\log(n\alpha))}}$$

$$\leq C(\alpha,\epsilon)n^{-(1+\epsilon)^2},$$

where C is some constant depending on α, ϵ, $\sum_{n=1}^{\infty} C(\alpha,\epsilon)n^{-(1+\epsilon)} < \infty$, by Borel-Cantelli we have that $\mathbb{P}(W(\alpha^n) > (1+\epsilon)\psi(\alpha^n) i.o.) = 0$, hence the Brownian motion will almost surely reach a last n such that at α^n it exceeds the bound. We need to show that the process will not exceed the bound between α^n, α^{n+1} for sufficiently large n.

$$\mathbb{P}(\sup_{s\leq\alpha^n} W(s) > (1+\epsilon)\psi(\alpha^n)) = 2\mathbb{P}(W(\alpha^n) > (1+\epsilon)\psi(\alpha^n)) \leq 2C(\alpha,\epsilon)n^{-(1+\epsilon)^2}$$

So by Borel-Cantelli there are almost surely only finitely many intervals $[\alpha^n, \alpha^{n+1})$, for which the Brownian exceeds the bound. We therefore have for $t \in [\alpha^n, \alpha^{n+1})$:

$$\frac{W(t)}{\psi(t)} \leq (1+\epsilon)\frac{\psi(\alpha^{n+1})}{\psi(\alpha^n)} = (1+\epsilon)\sqrt{\frac{\alpha^{n+1}\log((n+1)\log(\alpha))}{\alpha^n\log(n\log(\alpha))}} = (1+\epsilon)\sqrt{\alpha}\sqrt{\frac{\log((n+1)\log(\alpha))}{\log(n\log(\alpha))}}$$

and $\lim_{n\to\infty}\sqrt{\frac{\log((n+1)\log(\alpha))}{\log(n\log(\alpha))}} = 1$, we can choose ϵ arbitrarily small and α arbitrarily close to 1 hence indeed we have that $\dfrac{W(t)}{\psi(t)} \leq 1$ almost surely.

Step 2: We need to show that $\lim_{t\to\infty}\dfrac{W(t)}{\psi(t)} \geq 1$, similar to what we have done in step 1, the inequality is equivalent to saying that for $\epsilon > 0$ we have $\frac{W(t)}{\psi(t)} \geq 1-\epsilon$ for sufficiently large t. Again let $t_n = \alpha^n, \alpha > 1$ then

$$\mathbb{P}(W(\alpha^n)(1-\epsilon) \geq C(\alpha, \epsilon) n^{-(1-\epsilon)^2}.$$

Let $A_n = \{W(\alpha^n) - W(\alpha^{n-1}) \geq \psi(\alpha^n - \alpha^{n-1})\}$, which are independent by construction, also for sufficiently large n

$$\mathbb{P}(A_n) = \mathbb{P}(Z \geq \frac{\alpha^n - \alpha^{n-1}}{\sqrt{\alpha^n - \alpha^{n-1}}}) \geq \frac{\mathrm{e}^{-\log(\log(\alpha^n - \alpha^{n-1}))}}{2\log(\log(\alpha^n - \alpha^{n-1}))} > \frac{1}{n\log(n)}$$

and, therefore, $\sum_{n=1}^{\infty} \mathbb{P}(A_n)$ diverges, so for infinitely many n

$$W(\alpha^n) \geq W(\alpha^{n-1}) + \psi(\alpha^n - \alpha^{n-1})$$

and from the upper bound (step 1) $W(\alpha^{n-1}) \leq 2\psi(\alpha^{n-1})$ and symmetry of the standard Brownian motion, we get $W(\alpha^{n-1}) \geq -2\psi(\alpha^{n-1})$, therefore we can rewrite the above inequality as

$$W(\alpha^n) \geq W(\alpha^{n-1}) + \psi(\alpha^n - \alpha^{n-1}) \geq -2\psi(\alpha^{n-1}) + \psi(\alpha^n - \alpha^{n-1}).$$

Thus, almost surely, for infinitely many n

$$\frac{W(\alpha^n)}{\psi(\alpha^n)} \geq \frac{-2\psi(\alpha^{n-1}) + \psi(\alpha^n - \alpha^{n-1})}{\psi(\alpha^n)} \geq -\frac{2}{\sqrt{\alpha}} + \frac{\alpha^n - \alpha^{n-1}}{\alpha^n} = 1 - \frac{2}{\sqrt{\alpha}} - \frac{1}{\alpha}$$

as $\frac{\psi(t)}{\sqrt{t}}$ is increasing in t for sufficiently large t, but $\frac{\psi(t)}{t}$ is decreasing, and, therefore, we have

$$\lim_{t\to\infty} \frac{W(t)}{\psi(t)} \geq 1 - \frac{2}{\sqrt{\alpha}} - \frac{1}{\alpha}$$

and since our choice of $\alpha > 1$ was arbitrary, we get the almost sure lower bound $\limsup_{t\to\infty} \frac{W(t)}{\psi(t)} \geq 1$, and combining it with the upper bound, we know that

$$\limsup_{t\to\infty} \frac{W(t)}{\psi(t)} = 1.$$

$\square$

2.6.3 Nowhere Differentiability

Almost every sample path $W(t), 0 \leq t \leq T$ is not differentiable at any point and this can be shown by proving the following theorem,

THEOREM 2.6.2 *For every t_0,*

$$\limsup_{t \to t_0} \left| \frac{W(t) - W(t_0)}{t - t_0} \right| = \infty \quad \textit{almost surely}$$

which implies that for any t_0, almost every sample path $W(t)$ is not differentiable at this point.

Proof. Without loss of generality, we assume $t_0 = 0$. If one considers the event

$$A(h, \omega) = \{ \sup_{0 < s \leq h} \left| \frac{W(s)}{s} \right| > D \},$$

where D is constant, then for any sequence $\{h_n\}$ decreasing to 0, we have

$$A(h_n, \omega) \supset A(h_{n+1}, \omega)$$

and

$$A(h_n, \omega) \supset \{ \left| \frac{W(h_n)}{h_n} \right| > D \}.$$

So,

$$\mathbb{P}(A(h_n)) \geq \mathbb{P}(\left| \frac{W(h_n)}{\sqrt{h_n}} \right| > D\sqrt{h_n}) = \mathbb{P}(|W(1)|) > D\sqrt{h_n}) \to 1 \quad \text{as} \quad n \to \infty$$

Hence,

$$\mathbb{P}(\cap_{n=1}^{\infty} A(h_n)) = \lim_{n \to \infty} \mathbb{P}(A(h_n)) = 1$$

It follows that

$$\sup_{0 < s \leq h_n} \left| \frac{W(s)}{s} \right| \geq D \quad \text{almost surely for all n and D} > 0$$

Hence

$$\limsup_{t \to t_0} \left| \frac{W(t) - W(t_0)}{t - t_0} \right| = \infty \quad \text{almost surely}$$

□

It is worth to mention that the nowhere differentiability implies that the Brownian motion is not monotone in any interval, no matter how small the interval is.

2.7 Stochastic Calculus

2.7.1 The Itô Integral

Suppose that $g \in \mathcal{L}^2[a,b](\Omega) = \mathbb{L}_2([a,b], \mathbb{L}_2(\Omega, \mathcal{F}, \mathbb{P}))$, which is a Hilbert space with the $\mathbb{L}_2-$norm,

$$||g||_{\mathbb{L}_2} = \Big(\int_a^b \mathbb{E}|g(t,\omega)|^2 dt \Big)^{1/2}$$

Note that $\int_a^b g(s)dW_s, g(s,\omega),$ and the integrator W_t are stochastic processes. In order to define $\int_0^T g(t,\omega)dW(t)$, we approximate $g(t,\omega)$ by simple processes.

Definition 2.7.1 [Simple Stochastic Processes] A simple stochastic process is defined by

$$g(t,\omega) = \sum_{k=1}^{n} \xi_{k-1}(\omega) 1_{[t_{k-1}, t_k]}$$

where ξ_k is $\mathcal{F}_{t_k}$ measurable and $\mathbb{E}|\xi_k|^2 < \infty$.

The stochastic integral of the simple stochastic process is given by

$$I(g_n) = \int_a^b g(t,\omega)dW_t = \sum_{k=1}^{n} \xi_{k-1}(\omega) \Big(W_{t_k} - W_{t_{k-1}} \Big) \tag{2.7}$$

Lemma 2.7.1 *The integral (2.7) has the following properties*

$$\mathbb{E}(\int_a^b g(t,\omega)dW(t)) = 0$$

$$\int_a^b g(t,\omega)dW(t) \quad \textit{is} \quad \mathcal{F}_b - \textit{measurable random variable}$$

$$\mathbb{E}| \int_a^b g(t,\omega)dW(t)|^2 = ||g||^2_{\mathcal{L}^2} = \int_a^b \mathbb{E}|g(t,\omega)|^2 dt.$$

The last property is often called Itô isometric identity.

Proof. The first property:

By using the tower property of conditional expectation, we get

$$\mathbb{E}(\xi_{k-1}(W(t_k) - W(t_{k-1})) = \mathbb{E}\Big(\mathbb{E}(\xi_{k-1}(W(t_k) - W(t_{k-1})|\mathcal{F}_{t_{k-1}})\Big)$$

and implementing the product property of conditional expectation

$$\mathbb{E}(\xi_{k-1}(W(t_k) - W(t_{k-1})) = \mathbb{E}\Big(\xi_{k-1}\mathbb{E}((W(t_k) - W(t_{k-1})|\mathcal{F}_{t_{k-1}})\Big)$$

using the independent increment property of the Brownian motion

$$\mathbb{E}(\xi_{k-1}(W(t_k) - W(t_{k-1})) = \mathbb{E}\Big(\xi_{k-1}\mathbb{E}((W(t_k) - W(t_{k-1}))\Big) = \mathbb{E}(\xi_{k-1}) - 0.$$

The second property:

$$I(g) = \int_a^b g(t,\omega)dW(t) = \sum_{k=1}^{n} \xi_{k-1}(\omega)\Big(W(t_k - W(t_{k-1})\Big)$$

for $k < n, \xi_k W(t_k)$ is $\mathcal{F}_{t_n} = \mathcal{F}_t$ measurable. Thus $I(g)$ is $\mathcal{F}_t$ measurable.

The third property:

$$\mathbb{E}|\int_a^b g(t,\omega)dW(t)|^2 = \sum_{k=1}^{n}\sum_{j=1}^{n} \mathbb{E}(\xi_{k-1}\xi_{j-1}(W(t_k) - W(t_{k-1}))(W(t_j) - W(t_{j-1})))$$

for any $k < j$, by the tower and product properties of the conventional expectation, and by the mean-zero property of the Brownian motion as treated before, $\mathbb{E}(\xi_{k-1}\xi_{j-1}(W(t_k) - W(t_{k-1})(W(t_j) - W(t_{j-1})) = 0$

$$\sum_{k=1}^{n} \mathbb{E}(\xi_{k-1}\xi_{j-1}(W(t_k) - W(t_{k-1}))^2) = \sum_{k=1}^{n} \mathbb{E}(\xi_{k-1})(t_k - t_{k-1}) = \int_a^b \mathbb{E}|g(t,\omega)|^2 dt$$

□

We denote by $\mathcal{S}_T^2$ the subset of all step functions in $\mathbb{L}_2((0,T), \mathbb{L}_2(\Omega, \mathcal{F}, \mathbb{P})) = \mathcal{L}_T^2$, which is a Hilbert space with the $\mathcal{L}^2-$norm

$$||g||_{2,T} = ||g||_{\mathcal{L}_T^2} = \sqrt{\int_0^T \mathbb{E}(f(t,\omega)^2)dt}. \tag{2.8}$$

Therefore, we can approximate any function in $\mathcal{L}_T^2$ by step functions in $\mathcal{S}_T^2$ to any desired degree of accuracy. Thus, it is possible to choose a sequence $g_n(t,\omega)$ of simple processes such that as $n \to \infty$ these processes converge to continuously varying $g(t,\omega) \in \mathcal{L}_T^2$.

Lemma 2.7.2 $\mathcal{S}_T^2$ *is dense in* $(\mathcal{L}_T^2, ||.||_{\mathcal{L}_T^2})$

Proof. Let us consider the partition of $[0,T]$ of the form $0 = t_0 < t_1 < ... < t_n = T$ with $t_j - t_{j-1} \to 0$ as $n \to \infty$. when $\mathbb{E}(f(t,\omega))^2$ is mean-square continuous, we define a sequence of step functions f_n by $f_n(t,\omega) = f(t_n,\omega), w.p.1$, in $t_{n_j} \leq t \leq t_{n_{j+1}}$ for $j = 1,2,...,n$ and $n = 1,2,3,....$ Clearly then $f_n \in \mathcal{S}_T^2$ for each $n = 1,2,3,...$ and

$$\mathbb{E}\Big(|f_n(t,\omega) - f(t,\omega)|^2\Big) \to 0 \text{ as } n \to \infty$$

for each $t \in [0,T]$. Hence by the Lebesgue Dominated Convergence Theorem[4] applied to $L_1([0,T],\mathbb{F},\mathbb{P})$ we have

$$\int_0^T \mathbb{E}\Big(|f_n(t,\omega) - f(t,\omega|^2\Big)dt \to 0 \text{ as } n \to \infty$$

In general, since $f \in \mathcal{L}_T^2$ is not mean-square continuous, we can approximate it arbitrarily closely in the norm Eq(2.8). We approximate g by a bounded function $g_N \in \mathcal{L}_T^2$ defined by

$$g_N(t,\omega) = \max\{-N, \min\{g(t,\omega), N\}\}$$

for some $N > 0$. $g_n(t,\omega) = g(t,\omega)$ when $g_N(t,\omega) = g(t,\omega)$. Moreover

$$\int_0^T \mathbb{E}(|g_N(t,\omega) - g(t,\omega)|^2)dt \leq 4\int_0^T \mathbb{E}(|g(t,\omega)|^2)dt < \infty,$$

so by the Dominated Convergence Theorem applied to the function $\mathbb{E}(|g_N(t,\omega) - g(t,\omega)|^2) \in \mathbb{L}_1([0,\ T],\mathbb{F},\mathbb{P})$ consequently

$$\int_0^T \mathbb{E}(|g_N(t,\omega) - g(t,\omega)|^2)dt \to 0 \text{ as } N \to \infty$$

[4] Suppose that $f,g \in L_1(\Omega,\mathcal{F},\mathbb{P})$ where $\mathbb{P} < \infty$ and that $f_1, f_2, ...$ is a sequence of $\mathcal{F}-$ measurable functions with $|f_n(\omega)| \leq |g(\omega)|$ for almost all $\omega \in \Omega$ and n=1,2,3,....Then $\lim_{n\to\infty} \int_\Omega f_n(\omega)d\mathbb{P} = \int_\Omega f d\mathbb{P}$ if $\lim_{n\to\infty} f_n = f(\omega)$ for almost all $\omega \in \Omega$ and $n = 1,2,3,....$

Then for such an $f_k(t,\omega) = k\mathrm{e}^{-kt}\int_0^t \mathrm{e}^{ks} g_N(s,\omega)ds$ from the integral above it follows that f_k is jointly measurable $\mathbb{F}\times\mathcal{F}$, and that $f(t,.)$ is $\mathcal{F}_t-$measurable for each $t \in [0,\ T]$, then

$$|f_k(t,\omega)| \leq k\mathrm{e}^{-kt}\int_0^T \mathrm{e}^{ks}|g_N(s,\omega)|dt \leq Nk\mathrm{e}^{-kt}\int_0^T \mathrm{e}^{ks}dt$$

Thus,

$$|f_k(t,\omega)| \leq N(1-\mathrm{e}^{-kt})$$

therefore $\mathbb{E}(f_k(t,\omega)^2) < \infty$ and integrable over $0 < t < T$; hence $g_k \in \mathcal{L}_T^2$. It is straightforward to see that

$$|f_k(t,\omega) - f_k(s,\omega)| \leq 2Nk|t-s|$$

which implies that g_k is continuous. In fact this bound also implies that $\mathbb{E}(f_k(t,\omega))$ is continuous. Therefore we can approximate it by step function $g_n \in \mathcal{S}_T^2$. Thus, for any $\epsilon > 0$ we can choose g_N, f_k and g_n successively so that

$$||g-g_N||_{\mathcal{L}_T^2} < \frac{\epsilon}{3}, \qquad ||g_N - f_k||_{\mathcal{L}_T^2} < \frac{\epsilon}{3}$$

$$||f_k - g_n||_{\mathcal{L}_T^2} < \frac{\epsilon}{3}$$

Then by the triangle inequality we have

$$||g-g_n||_{\mathcal{L}_T^2} < \epsilon$$

what was required to prove. □

Thus, Lemma (2.7.2) provides a sequence of step function, $g_n \in \mathcal{L}_{step}^2$ dense in $\mathcal{L}_{(0,T)}^2$ such that:

$$\lim_{n\to\infty}\mathbb{E}\int_0^T |g(t,\omega) - g_n(t,\omega)|^2 = 0$$

Note that for any given $g \in \mathcal{L}_T^2$, by the Itô isometry, the simple stochastic process $\{g_n\}_n^\infty$ as an approximate of g on $\mathcal{L}_T^2(\Omega)$ are such that $\{\int_0^T g_n(t)dW(t)\}_0^\infty$ is a Cauchy sequence in the Hilbert space $\mathbb{L}_2(\Omega, \mathcal{F}, \mathbb{P})$. Therefore, we define the Itô integral

$$\int_0^T g(t)dW(t) = \lim_{n\to\infty} \int_0^T g_n(t)dW(t)$$

which is well-defined.

2.7.2 Stochastic Integral Properties

THEOREM 2.7.1 *For any $g \in \mathcal{L}^2_{(loc)}(\Omega)$, the Itô integral driven by the standard Brownian motion,*

$$X(t) = \int_0^T g(s)dW(s), T \geq 0$$

is a mean-zero stochastic process and a martingale with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. The Itô isometric identity holds,

$$\mathbb{E}|X(t)|^2 = \int_0^t \mathbb{E}|X(s)|^2 ds = ||g||^2_{\mathcal{L}^2_{(0,t)}(\Omega)}, \quad t \geq 0. \tag{2.9}$$

Here $\mathcal{L}^2_{loc}(\Omega) = \mathbb{L}_{2(loc)}([0,\infty), \mathbb{L}_2(\Omega, \mathbb{F}, \mathbb{P}))$. Linear properties and additivity hold for the integral.

Proof. Let $\{g_m\}$ be sn approximate sequence of simple process so that $X_m(t) = \int_0^t g_m(s)dW(s) \to X(t)$ in $\mathbb{L}_2(\Omega, \mathbb{F}, \mathbb{P})$. Then Eq(2.9) holds because

$$||X(t)||_{\mathbb{L}_2(\Omega)^2} = \lim_{m\to\infty} ||X_m(t)||^2_{\mathbb{L}_2(\Omega)} = \lim_{m\to\infty} ||g_m||^2_{\mathcal{L}^2_{loc}(\Omega)} = ||g||^2_{\mathcal{L}^2_{loc}(\Omega)}.$$

□

To prove that $\{X(t) = \int_0^t g(s)dW(s)\}_{t\geq 0}$ is a martingale, we can verify three conditions 1- $X(t)$ is adapted to $\mathcal{F}_t, t \geq 0$, because $X_m(t) = \int_0^t g_m(s)dW(s)$ is adapted to $\mathcal{F}_t$ and $\sigma-$algebra $\mathbb{F}$ is closed with respect to the pointwise limit operation. Hence $X_m(t) \to X(t)$ in $\mathbb{L}_2(\Omega, \mathbb{F}, \mathbb{P})$ implies that there is a subsequence of $X_m(t)$, which converges to $X(t)$ pointwise with probability almost everywhere in Ω.

2-$\mathbb{E}|X(t)| < \infty$ meaning $X(t)$ is integrable due to the Cauchy inequality,

$$\mathbb{E}|X(t)| \leq \sqrt{\mathbb{E}(1)\mathbb{E}|X(t)|^2} = (\mathbb{E}|X(t)|^2)^{\frac{1}{2}} < \infty$$

3-The martingale property: $\mathbb{E}(X(t)|\mathcal{F}_s) = X(s)$, almost surely for $0 \leq s < t$, $I(s) = \mathbb{E}(\int_0^t g(u)dW(u)|\mathcal{F}_s)$ for $s < t, I(s) = 0$ and $I(s) = \mathbb{E}(\int_0^s g(u)dW(u)|\mathcal{F}_s) = \mathbb{E}(\int_0^s g(u)dW(u)) = X(s)$ for $s > t$. Then, we can define the Itô integral for the continuously varying integrand $g(t,\omega)$

$$\int_0^t g(s,\omega)dW(s) = \lim_{n\to\infty} \int_0^t g_n(u,\omega)dW(u).$$

This integral will have the same properties of Itô integral of simple process.

Definition 2.7.2 Let $W(t), t \geq 0$, be a Brownian motion, and let $\mathcal{F}(t), t \geq 0$ be an associate filtration. An Itô process is a stochastic process of the form

$$X(t) = X(0) + \int_0^t \mu(s)ds + \int_0^t \sigma(s)dW(s)$$

where $X(0)$ is nonrandom and μ and σ are adapted stochastic processes, and unique almost surely.

Proof.

$$X(t) = x_0 + \int_0^t \mu(s)ds + \int_0^t \sigma(s)dW(s) = x_0' + \int_0^t \mu'(s)ds + \int_0^t \sigma'(s)dW(s)$$

since $x_0 = x_0'$ we get

$$\int_0^t (\mu(s) - \mu'(s)ds = \int_0^t (\sigma(s) - \sigma'(s))dW(s)$$

let $M(t) = \int_0^t (\mu(s) - \mu'(s))ds$. It follows that M is a martingale with finite variation since

$$\sum_{i=1}^{2^n} |M(t_i^n) - M(t_{i-1}^n)| \leq \int_0^T |\mu'(s)|ds + \int_0^T |\mu^(s)|ds < \infty$$

where $t_i^n = \frac{(i-1)t}{2^n}, i = 1, 2, ..., 2^n$. Note that for $0 \leq s < t < \infty$

$$\mathbb{E}[(M(t) - M(s))^2] = \mathbb{E}[M^2(t)] - 2\mathbb{E}[M(s)M(t)] + \mathbb{E}[M^2(s)]$$

$$= \mathbb{E}[M^2(t)] - 2\mathbb{E}[M(s)M(t)|\mathcal{F}_s] + \mathbb{E}[M^2(s)] = \mathbb{E}[M^2(t)] - \mathbb{E}[M^2(s)]$$

using the last result and the monotone convergence,

$$\mathbb{E}[\lim_{n\to\infty}\sum_{i=1}^{\infty}|M(t_i^n)-M(t_{i-1}^2)|^2],$$

$$=\lim_{n\to\infty}\sum_{i=1}^{\infty}\mathbb{E}(|M(t_i^n)-M(t_{i-1}^2)|^2),$$

$$=\lim_{n\to\infty}\sum_{i=1}^{\infty}\mathbb{E}(M(t_i^n))^2-\mathbb{E}(M(t_{i-1}^2))^2=\mathbb{E}[M^2(T)]=0,$$

it follows that $M(T)=0$ almost surely and $M(t)=\mathbb{E}(M(T)|\mathcal{F}_t)=0$ almost everywhere for all t. So $\mu=\mu'$ almost everywhere and it follows that $\int_0^t(\sigma(s)-\sigma'(s))dW(s)=0$ for all t. Hence

$$\mathbb{E}\Big[\Big(\int_0^t(\sigma(s)-\sigma'(s))dW(s)\Big)^2\Big]=\int_0^t\mathbb{E}(\sigma(s)-\sigma'(s))^2ds$$

and this implies $\sigma=\sigma'$. □

2.7.3 Itô Formula in Stochastic Calculus

If $\{X(t),t\geq 0\}$ is a real valued process, describing the state of a system at anytime. The stochastic differential equation (SDE) governing the time evolution of this process X_t is given by

$$dX(t)=\mu(X(t),t)dt+\sigma(X(t),t)dW(t) \tag{2.10}$$

or stochastic integral equation (SIE)

$$X(t)=\int_0^t\mu(X(s),s)ds+\int_0^t\sigma(X(s),s)dW(s) \tag{2.11}$$

here the first integral is pathwise Lebesgue integral and the second integral is interpreted as the itô integral. Now our goal is to find a way to evaluate an SDE for a stochastic process $f(X(t),t)$ and if we have an SDE how to find the corresponding $f(X(t),t)$ solution. The definition of Itô integrals is not very useful when we try to evaluate a given integral. That is similar to the situation for ordinary Riemann integrals, where we do not use the basic definition but rather the fundamental theorem of calculus and the chain rule in the explicit continuations. We don't have differentiation theory, only integration theory. However, it turns out that it is possible to construct an Itô integral version of the chain rule, called the Itô formula.

LEMMA 2.7.3 *Let $f : [0,\ T] \times \mathbb{R} \to \mathbb{R}$ have continuous partial derivatives $\frac{\partial f}{\partial t}, \frac{\partial f}{\partial x}$, and$\frac{\partial^2 f}{\partial x^2}$. Then for any $t, t+\Delta t \in [0,\ T]$ and $x, x+\Delta x \in \mathbb{R}$*

$$f(t, t+\Delta t, x, x+\Delta x) - f(t,x) = \frac{\partial f}{\partial t}(t,x)\Delta t + \frac{\partial f}{\partial x}(t,x)\Delta x + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\Delta x)^2 \tag{2.12}$$

Proof. The proof of the lemma is a direct result of Taylor's expansion and $\frac{\partial f}{\partial t}, \frac{\partial^2 f}{\partial x^2}$ is continuous w.p.1. □

THEOREM 2.7.2 (The 1-dimensional Itô formula) *Let X_t be an Itô process given by Eq(2.10). Let $f(t, X_t) \in C^{1,2}([0,\infty) \times \mathbb{R})$. Then*

$$df(t,X_t) = \frac{\partial f}{\partial t}(t,X_t)dt + \frac{\partial f}{\partial x}(t,X_t)dX(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,X_t)(t,X_t)(dX_t)^2$$

where $(dX_t)^2$ is computed according to the rules $dt^2 = dtdW(t) = 0, dW(t)^2 = dt$, therefore, we can rewrite Itô formula for an autonomous system as follows

$$df(t,X_t) = \Big(\frac{\partial f}{\partial t}(t,X_t) + \mu(X_t)\frac{\partial^2 f}{\partial x}(t,X_t) + \frac{1}{2}\sigma^2(X_t)\frac{\partial f}{\partial x^2}(t,X_t)\Big)dt + \sigma(X_t)\frac{\partial f}{\partial x}dW_t$$

w.p.1 for any $0 \leq s \leq t \leq T$,$\mu(X_t) \in L^1_T$ and $\sigma(X_t) \in \mathcal{L}^2_T$, X_t is a separable, jointly measurable version of $X_t - X_s$ with almost surely continuous sample paths.

Proof.

Let $s = t_1 < t_2 < ... < t_{n+1} = t$ with $\Delta t_j = t_{j+1} - t_j$. Then $f(t,X_t) - f(t,X_s) = \sum_{j=1}^{n} \Delta f_j$ where $\Delta f_j = f(t_{j+1}, X_{t_{j+1}}) - f(t_j, X_{t_j})$ for $j = 1, 2, ..., n$. Applying Lemma 2.7.3 on each time interval, we obtain $\Delta f_j = \frac{\partial f}{\partial t}(t_j, x_j)\Delta t_j + \frac{\partial f}{\partial x_j}(t_j, x_j)\Delta x_j + \frac{1}{2}\frac{\partial^2 f}{\partial x_j^2}(\Delta x_j)^2$ w.p.1, where $\Delta X_j = X_{t_{j+1}} - X_{t_j}$ and $\Delta W_j = W_{t_{j+1}} - W_{t_j}$. As $n \to \infty$ and using the rules $(\Delta t)^2 = 0, \Delta W_t \Delta t = 0, \text{and} (\Delta W_t)^2 = \Delta t$, we get

$$df(t,X_t) = \Big(\frac{\partial f}{\partial t}(t,X_t) + \mu(X_t)\frac{\partial^2 f}{\partial x}(t,X_t) + \frac{1}{2}\sigma^2(X_t)\frac{\partial f}{\partial x^2}(t,X_t)\Big)dt + \sigma(X_t)\frac{\partial f}{\partial x}dW_t$$

□

2.7.4 Itô formula in Vector Case

Let $\mathbf{W} = \mathbf{W}_t = (W_t^1, W_2^2, ..., W_t^m) \geq 0$, where $m = 1, 2, ...,$ with independent components associated with increasing family of σ-algebra $\{\mathcal{F}_t, t \geq 0\}$. Thus each W_t^i is $\mathcal{F}_t$-measurable with,

$$\mathbb{E}(W_t^i|\mathcal{F}_0) = 0, \qquad \mathbb{E}(W_t^j - W_s^j|\mathcal{F}_s) = 0$$

w.p.1, for $0 \leq s \leq t$ and $j = 1, 2, ..., m$. In addition,

$$\mathbb{E}((W_t^i - W_s^i)(W_t^j - W_t^j)|\mathcal{F}_t) = (t - s)\delta_{i,j}$$

w.p.1, for $0 \leq s \leq t$ and $i, j = 1, 2, ..., m,$. Let $\mathbf{b} : [0, T] \times \Omega \rightarrow \mathbb{R}^d$ be d-dimensional vector function and $b^i \in \mathcal{L}_T^2$, and $\sigma : [0, T] \times \Omega \rightarrow \mathbb{R}^{d \times m}$ with components $\sigma^{i,j} \in \mathcal{L}_T^2$. Thus a multidimensional SDE in $\mathbb{R}^n$ can be written as

$$dX(t) = \mathbf{b}(X)dt + \sigma(X)d\mathbf{W}. \tag{2.13}$$

Let $f(t, x)$ be a give $C^{1,2}$ function. Let the Hessian matrix be denoted by

$$D^2(f) = (\frac{\partial^2 f}{\partial x_i \partial x_j})_{n \times n}$$

the generator of the solution process of 2.13 is then $\mathcal{A} : \mathcal{D}(A) = C_0^2(\mathbb{R}^n) \rightarrow C_b(\mathbb{R}^n)$,

$$\mathcal{A}f = \sum_{i=1}^n b_i(x)\frac{\partial f}{\partial x} + \frac{1}{2}\sum_{i,j=1}^n (\sigma(x)\sigma^T(x))_{i,j}\frac{\partial^2 f}{\partial x_i \partial x_j} \tag{2.14}$$

$$= \mu(x).\nabla f + \frac{1}{2}Tr(\sigma(x)\sigma^T(x)D^2 f), \tag{2.15}$$

the multidimensional Itô formula

$$df(t, X_t) = \left(\frac{\partial f}{\partial t} + b(X_t).\nabla f + \frac{1}{2}Tr(\sigma(X_t)\sigma^T(X_t)D^2 f\right)dt + \nabla f.\sigma(X_t)dW_t$$

$$df(t, X_t) = \left(\frac{\partial f}{\partial t} + \mathcal{A}f(t, X_t)\right)dt + \nabla f(t, X_t).\sigma(X_t)dW_t.$$

2.7.5 The Stochastic Rule and the Stochastic Integration by Parts

An application for Itô formula is integration by parts. Let X_t, Y_t be respectively solutions of two scalar SDE. Applying the vector Itô formula to $g(x, y) = xy$, we get the stochastic product rule

$$d(X_tY_t) = \frac{\partial g}{\partial x}dX_t + \frac{\partial g}{\partial y}dY_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(dX_t)^2 + \frac{\partial^2 g}{\partial x \partial y}dX_t dY_t + \frac{1}{2}\frac{\partial^2 g}{\partial y^2}(dY_t)^2$$
$$= X_t dY_t + Y_t dX_t + dX_t dY_t,$$

then, the corresponding integral by parts is the in the form

$$\int_0^T X_t dY_t = X(T)Y(T) - X(0)Y(0) - \int_0^T Y_t dX_t - \int_0^T dX_t dY_t$$

Chapter 3

Complex Deformations of The Kuramoto Model

3.1 Complex Deformations and Embedding of Dynamical Systems

This section is concerned with the core result of our investigation, namely the embedding of the original stochastic Kuramoto model (regarded simply as a system of coupled nonlinear stochastic differential equations) into a larger class of dynamical systems, for the purpose of a more complete characterization of the nonequilibrium synchronization phase transition.

We start from the first-order Kuramoto model with uniform coupling and generic frequency distribution in the presence of external driving (including the stochastic case). The (classical) dynamical system is given by:

$$d\theta_k = \omega_k dt + 2\lambda \sum_{j=1}^{n} \sin(\theta_j - \theta_k)dt + dW_k, \quad \lambda \in \mathbb{R},\ \theta_k \in \mathbb{T},\ k = 1, 2, \ldots, n, \tag{3.1}$$

where the proper frequencies ω_k are characterized by a probability distribution $g(\omega)$, and the external driving $W_k(t)$ can be chosen to be either deterministic or stochastic. In the latter case, it is assumed to consist of n independent, identical Wiener processes, with correlation functions $\mathbb{E}[\eta_j(t)\eta_k(t')] \sim \sigma^2 \delta_{jk}\delta(t - t')$, where $dW_k = \eta_k dt$. The (complex) order parameter of the model is provided by the collective mode

$$r(t) \equiv \frac{1}{n} \sum_{k=1}^{n} e^{i\theta_k}, \tag{3.2}$$

with the fully-synchronized state and the unsynchronized state corresponding to $|r| = O(1)$ and $|r| = O(1/n)$, respectively. Characterizing the phase transition $|r|(n, \lambda, \sigma^2, g)$ beyond the mean-field approximation is the main goal of this study.

We introduce the complex variables

$$z_k(t) \equiv r_k(t) e^{i\theta_k(t)}, \tag{3.3}$$

and $\Re(z_k) \equiv q_k, \Im(z_k) = p_k$, with the obvious constraints $r_k(t) = 1 = p_k^2 + q_k^2$, $k = 1, 2, \ldots, n$. The system (3.1) can then be written as

$$dz_k = i(\omega_k + \eta_k)z_k dt + i\lambda \sum_{j=1}^{n} [z_k^2 \bar{z}_j - z_j]dt. \tag{3.4}$$

As explained in the preceding chapters, when taking the limit $n \to \infty, \lambda \to \lambda_c$, the onset of the phase transition is signaled by the non-analytic behavior of the function $|r|/n(\lambda)$. However, for a full characterization of the critical behavior, what is truly required is the proper limit of the counting measure

$$dm_t \equiv \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \delta_{z_k(t)}, \tag{3.5}$$

where $\delta_{z_k(t)}(z)$ is the singleton supported at $z_k(t) \in \partial\mathbb{D}$. Then for any properly chosen function $f : \mathbb{D} \to \mathbb{C}$, we have

$$\mathbb{E}_{m_t}(f) = \int_{\partial\mathbb{D}} f(z) dm_t(z) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(z_k(t)),$$

in particular $r(t) = \mathbb{E}_{m_t}(z)$ is the first moment of the measure m_t.

We mention here an observation which will be discussed in much more detail later in this section. Denote by $\mathcal{B}(\mathbb{T})$ the set of probability measures defined on $\mathbb{T} = \partial\mathbb{D}$. Then to each element $\mu \in \mathcal{B}(\mathbb{T})$ we can associate a vector field V_μ on $\mathbb{D}$, defined by

$$V_\mu(\zeta) = (1 - |\zeta|^2) \int_{\mathbb{T}} \frac{z - \zeta}{\bar{z} - \bar{\zeta}} \frac{d\mu(z)}{z},$$

so that $V_\mu(0) = \mathbb{E}_\mu(z)$. The *conformal barycenter* of the measure μ, $B(\mu)$, is then defined by $V_\mu(B(\mu)) = 0$. Douady and Earle have shown [29] that if μ has no *strong atoms* (i.e. no singletons with mass at least 1/2), then the conformal barycenter is uniquely defined.

Using this notion, Douady and Earle [29] discovered a naturally conformal extension of any circle homeomorphism $f : \mathbb{T} \to \mathbb{T}$ to a disk homeomorphism $\Phi : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$, which is furthermore real-analytic on $\mathbb{D}$. The Douady-Earle map $f \to \Phi$ is conformally natural and preserves quasi-conformal extensions of the circle homemorphism.

The Douady-Earle extension is given through the conformal barycenter of the harmonic measure associated to the homomorphism f, $\mu_z[f](A) = \omega_z(f^{-1}(A))$, for any Borel set $A \subset \mathbb{T}$, where $z \in \mathbb{D}$ and ω_z is the harmonic measure with source at z. The disk homomorphism is given by

$$\Phi[f](z) \equiv B(\mu_z[f]).$$

The connection to the original Kuramoto model has its origin in the iterative algorithm developed by Milnor and (independently) by Abikoff and Ye [30], [31], known as the MAY algorithm, which computes the conformal barycenter by iterative compositions of self-maps of the unit disk. In recent works, Jacimovic [32] and Chen *et. al* [33] have linked the infinite-size limit of the Kuramoto model to fixed points of iterative compositions of maps under hyperbolic geometry of the unit disk. The limit behavior of this iterative scheme is described by the classification of iterative compositions of unit disk univalent maps, and rests on the theorem of Denjoy and Wolff. We present a summary of this theory in the next section, which follows closely the exposition given in [34].

3.2 Evolution Families and Herglotz Vector Fields

We start with the notion of an *evolution family.* Let us consider a semigroup $\mathcal{P}$ of conformal univalent maps from the unit disk $\mathbb{D}$ into itself with superposition as a semigroup operation. This makes $\mathcal{P}$ a topological semigroup with respect of the topology of local uniform convergence on $\mathbb{D}$.

Definition 3.2.1 An evolution family of order $d \in [1, +\infty]$ is a two-parameter family $(\phi_{s,t})_{0 \le s \le t < +\infty}$ of holomorphic endomorphisms of the unit disk from $\mathcal{P}$, such that the following three conditions are satisfied.

- $\phi_{s,s} = id_{\mathbb{D}}$;
- $\phi_{s,t} = \phi_{u,t} \circ \phi_{s,u}$ for all $0 \le s \le u \le t < +\infty$;
- for any $z \in \mathbb{D}$ and $T > 0$ there is a function $k_{z,T} \in L^d([0,T], \mathbb{R})$ such that

$$|\phi_{s,u}(z) - \phi_{s,t}(z)| \le \int_u^t k_{z,T}(\xi) d\xi,$$

  for all $0 \le s \le u \le t \le T$.

An infinitesimal description of an evolution family is given in terms of a *Herglotz vector field.*

Definition 3.2.2 A (generalized) Herglotz vector field of order d is a function $G : \mathbb{D} \times [0, +\infty) \to \mathbb{C}$ satisfying the following conditions:

- the function $[0, +\infty) \ni t \mapsto G(z,t)$ is measurable for all $z \in \mathbb{D}$;
- the function $z \mapsto G(z,t)$ is holomorphic in the unit disc for $t \in [0, +\infty)$;

- for any compact set $K \subset \mathbb{D}$ and for all $T > 0$ there exists a non-negative function $k_{K,T} \in L^d([0,T],\mathbb{R})$ such that
$$|G(z,t)| \leq k_{K,T}(t)$$
for all $z \in K$ and almost every $t \in [0,T]$;

- for almost every $t \in [0,+\infty)$ the vector field $G(\cdot,t)$ is semicomplete.

By semicompleteness we mean that the solution to the problem

$$\begin{cases} \dfrac{dx(\tau)}{d\tau} = G(x(\tau),t), \\ x(s) = z \end{cases}$$

is defined for all times $\tau \in [s,+\infty)$, for any fixed $s \geq 0$, fixed $t \geq 0$ and fixed $z \in \mathbb{D}$.

An important result of general Löwner-Kufarev theory is the fact that the evolution families can be put into a one-to-one correspondence with the Herglotz vector fields by means of the so-called generalized Löwner-Kufarev ODE. This can be formulated as the following theorem.

THEOREM 3.2.1 ([35]) *For any evolution family $(\phi_{s,t})$ of order $d \geq 1$ in the unit disk there exists an essentially unique Herglotz vector field $G(z,t)$ of order d, such that for all $z \in \mathbb{D}$ and for almost all $t \in [0,+\infty)$*
$$\frac{\partial \phi_{s,t}(z)}{\partial t} = G(\phi_{s,t}(z),t).$$
Conversely, for any Herglotz vector field $G(z,t)$ of order $d \geq 1$ in the unit disk there exists a unique evolution family of order d, such that the equation above is satisfied.

Herglotz vector fields admit a convenient representation using so-called Herglotz functions.

DEFINITION 3.2.3 A Herglotz function of order $d \in [1,+\infty)$ is a function $p : \mathbb{D} \times [0,+\infty) \to \mathbb{C}$ such that

- the function $t \mapsto p(z,t)$ belongs to $L^d_{loc}([0,+\infty),\mathbb{C})$ for all $z \in \mathbb{D}$;

- the function $z \mapsto p(z,t)$ is holomorphic in $\mathbb{D}$ for each fixed $t \in [0,+\infty)$;

- $\Re p(z,t) \geq 0$ for all $z \in \mathbb{D}$ and for all $t \in [0,+\infty)$.

Now, the representation of Herglotz vector fields is given in the following theorem.

THEOREM 3.2.2 ([35, Theorem 1.2]) *Given a Herglotz vector field of order $d \geq 1$ in the unit disk, there exists an essentially unique (i.e., defined uniquely for almost all t for which $G(\cdot, t) \neq 0$) measurable function $\tau : [0, +\infty) \to \overline{\mathbb{D}}$ and a Herglotz function $p(z, t)$ of order d, such that for all $z \in \mathbb{D}$ and almost all $t \in [0, +\infty)$*

$$G(z, t) = (z - \tau(t))(\overline{\tau(t)}z - 1)p(z, t). \tag{3.6}$$

Conversely, given a measurable function $\tau : [0, +\infty) \to \overline{\mathbb{D}}$ and a Herglotz function $p(z, t)$ of order $d \geq 1$, the vector field defined by the formula above is a Herglotz vector field of order d.

According to Theorem 3.2.1, to every evolution family $(\phi_{s,t})$ one can associate an essentially unique Herglotz vector field $G(z, t)$. The pair of functions (p, τ) representing the vector field $G(z, t)$ is called the *Berkson-Porta data* of the evolution family $(\phi_{s,t})$.

To explain the geometrical meaning of the function $\tau(t)$ we need first to remind of the notion of the Denjoy-Wolff point of a unit disk endomorphism.

A classical result by Denjoy and Wolff states that for a holomorphic self-map f of the unit disk $\mathbb{D}$ other than a (hyperbolic) rotation, there exists a unique fixed point τ in the closure of $\mathbb{D}$, such that the sequence of iterates $(f_n(z))$ converges locally uniformly on $\mathbb{D}$ to τ as $n \to \infty$. This point τ is called the Denjoy-Wolff point of f and it is also characterized as the only fixed point of f satisfying $f'(\tau) \in \mathbb{D}$. In other words, τ is the only attractive fixed point of f in the above multiplier sense. It follows from the hyperbolic metric principle that, if f is not the identity, there can be no other fixed points in $\mathbb{D}$ except the Denjoy-Wolff point but, nevertheless, f can have many other repulsive or non-regular boundary fixed points.

If $\tau \in \mathbb{D}$, then the endomorphism f is called *elliptic*. Otherwise, the angular limit $\angle \lim_{z \to \tau} f(z) = \tau$ exists as well as the angular derivative $\angle \lim_{z \to \tau} f'(z) = \alpha_f$. If the value $\alpha_f \in (0, 1]$, then the map f in this case is said to be either *hyperbolic* (if $\alpha_f < 1$) or *parabolic* (if $\alpha_f = 1$) (for details and proofs see, e. g., [36]).

Now, let $(\phi_{s,t})$ be an evolution family with Berkson-Porta data (p, τ). In the simplest case when neither p, nor τ changes in time (i. e., the corresponding Herglotz vector field $G(z, t)$ is time-independent), τ turns out to be precisely the Denjoy-Wolff point of every endomorphism in the family $(\phi_{s,t})$. Moreover, for any $0 \leq s < +\infty$, we have that $\phi_{s,t}(z) \to \tau$ uniformly on compacts subsets of $\mathbb{D}$, as $t \to +\infty$. By this reason, we call τ the *attractive point* of the evolution family $(\phi_{s,t})$.

In the case when the Herglotz field $G(z,t)$ is time-dependent, the meaning of τ is explained in the following theorem.

THEOREM 3.2.3 ([35, Theorem 6.7]) *Let $(\phi_{s,t})$ be an evolution family of order $d \geq 1$ in the unit disk, and let $G(z,t) = (z-\tau(t))(\overline{\tau(t)}z-1)p(z,t)$ be the corresponding Herglotz vector field. Then for almost every $s \in [0,+\infty)$, such that $G(z,s) \neq 0$, there exists a decreasing sequence $\{t_n(s)\}$ converging to s, such that $\phi_{s,t_n(s)} \neq id_{\mathbb{D}}$ and*

$$\tau(s) = \lim_{n\to\infty} \tau(s,n),$$

where $\tau(s,n)$ denotes the Denjoy-Wolff point of $\phi_{s,t_n(s)}$.

3.2.1 Generalization of Löwner chains and Löwner-Kufarev PDE

We follow now the exposition [37] of the generalization of the classical notion of Löwner chains.

DEFINITION 3.2.4 A family $(f_t)_{0\leq t<+\infty}$ of holomorphic maps of the unit disk is called a Löwner chain of order $d \in [1,+\infty]$ if

- each function $f_t : \mathbb{D} \to \mathbb{C}$ is univalent,
- $f_s(\mathbb{D}) \subset f_t(\mathbb{D})$ for $0 \leq s < t < +\infty$,
- for any compact set $K \subset \mathbb{D}$ and all $T > 0$ there exists a non-negative function $k_{K,T} \in L^d([0,T],\mathbb{R})$ such that
$$|f_s(z) - f_t(z)| \leq \int_s^t k_{K,T}(\xi)d\xi$$
for all $z \in K$ and all $0 \leq s \leq t \leq T$.

Every Löwner chain $(f_t)_{0\leq t<+\infty}$ of order d generates an evolution family $\phi_{s,t}$ of the same order d defined by $\phi_{s,t} = f_t^{-1} \circ f_s$. This correspondence is, however, not one-to one, there may be many different Löwner chains associated to the given evolution family. Fortunately, they are unique up to normalization and composition with a univalent function, as the following theorem states.

THEOREM 3.2.4 ([37, Theorems 1.6-1.7]) *For any evolution family $(\phi_{s,t})$ of order d, there exists a unique Löwner chain (f_t) of the same order d, such that*

(i) $\phi_{s,t} = f_t^{-1} \circ f_s$ *for any* $0 \leq s \leq t$;

(ii) $f(0) = 0$ *and* $f'(0) = 1$;

(iii) $\Omega := \cup_{t\geq 0} f_t(\mathbb{D}) = \{z : |z| < R\}$, *where* $R \in (0, +\infty]$.

Any other Löwner chain satisfying the condition (i) is of the form $(g_t) = (F \circ f_t)$, *where* $F : \Omega \to \mathbb{C}$ *is univalent.*

The number R *is equal to* $1/\beta_0$, *where*

$$\beta_0 = \lim_{t\to+\infty} \frac{|\phi'_{0,t}(0)|}{1 - |\phi_{0,t}(z)|^2}.$$

It was also shown [37] that every Löwner chain (f_t) of order d satisfies the generalized Löwner PDE

$$\frac{\partial f_s(z)}{\partial s} = -G(z, s) f'_s(z) \quad \text{(for almost all } s \geq 0),$$

where $G(z, s)$ is the Herglotz vector field generating the associated evolution family $(\phi_{s,t})$.

3.2.2 Generalized Löwner-Kufarev Stochastic Evolution

In order to extend this formalism to the case of stochastic maps, we consider a setup [38] in which the sample paths are represented by the trajectories of a point (e.g., the origin) in the unit disk $\mathbb{D}$ evolving randomly under the generalized Löwner equation. The driving mechanism differs from the famous Stochastic Löwner Equation (SLE). In the SLE case the Denjoy-Wolff attracting point (∞ in the chordal case or a boundary point of the unit disk in the radial case) is fixed. In our case, the attracting point is the driving mechanism and the Denjoy-Wolff point is different from it.

Let us consider the generalized Löwner evolution driven by a Brownian particle on the unit circle. In other words, we study the following initial value problem.

$$\begin{cases} \frac{d}{dt}\phi_t(z,\omega) = \frac{(\tau(t,\omega)-\phi_t(z,\omega))^2}{\tau(t,\omega)} p(\phi_t(z,\omega), t, \omega), \\ \phi_0(z,\omega) = z, \end{cases} \quad t \geq 0,\ z \in \mathbb{D},\ \omega \in \Omega. \tag{3.7}$$

The function $p(z, t, \omega)$ is a Herglotz function for each fixed $\omega \in \Omega$. In order for $\phi_t(z, \omega)$ to be an Itô process adapted to the Brownian filtration, we require that the function $p(z, t, \omega)$ is adapted to the Brownian filtration for each $z \in \mathbb{D}$. Even though the driving mechanism in our case differs from that of SLE, the generated families of conformal maps still possess the important time-homogeneous Markov property.

For each fixed $\omega \in \Omega$, equation (3.7) similarly to SLE, may be considered as a deterministic generalized Löwner equation with the Berkson-Porta data $(\tau(\cdot,\omega), p(\cdot,\cdot,\omega))$. In particular, the solution $\phi_t(z,\omega)$ exists, is unique for each $t > 0$ and $\omega \in \Omega$, and moreover, is a family of holomorphic self-maps of the unit disk.

In order to give an explicitly solvable example let $p(z,t,\omega) = \frac{\tau(t,\omega)}{\tau(t,\omega)-z} = \frac{e^{ikB_t(\omega)}}{e^{ikB_t(\omega)}-z}$. It makes equation (3.7) linear:

$$\frac{d}{dt}\phi_t(z,\omega) = e^{ikB_t(\omega)} - \phi_t(z,\omega),$$

and a well-known formula from the theory of ordinary differential equation yields

$$\phi_t(z,\omega) = e^{-t}\left(z + \int_0^t e^s e^{ikB_s(\omega)} ds\right).$$

Taking into account the fact that $\mathbb{E}e^{ikB_t(\omega)} = e^{-\frac{1}{2}tk^2}$, we can also write the expression for the mean function $\mathbb{E}\phi_t(z,\omega)$

$$\mathbb{E}\phi_t(z,\omega) = \begin{cases} e^{-t}(z+t), & k^2 = 2, \\ e^{-t}z + \frac{e^{-tk^2/2} - e^{-t}}{1-k^2/2}, & \text{otherwise.} \end{cases} \tag{3.8}$$

Thus, in this example all maps ϕ_t and $\mathbb{E}\phi_t$ are affine transformations (compositions of a scaling and a translation).

In general, solving the random differential equation (3.7) is much more complicated than solving its deterministic counterpart, mostly because of the fact that for almost all ω the function $t \mapsto \tau(t,\omega)$ is nowhere differentiable.

If we assume that the Herglotz function has the form $p(z,t,\omega) = \tilde{p}(z/\tau(t,\omega))$, then it turns out that the process $\phi_t(z,\omega)$ has an important invariance property.

Let $s > 0$ and introduce the notation

$$\tilde{\phi}_t(z) = \frac{\phi_{s+t}(z)}{\tau(s)}.$$

Then $\tilde{\phi}_t(z)$ is the solution to the initial-value problem

$$\begin{cases} \frac{d}{dt}\tilde{\phi}_t(z,\omega) = \frac{\left(\tilde{\tau}(t,\omega) - \tilde{\phi}_t(z,\omega)\right)^2}{\tilde{\tau}(t,\omega)} \tilde{p}\left(\tilde{\phi}_t(z,\omega)/\tilde{\tau}(t)\right), \\ \tilde{\phi}_0(z,\omega) = \phi_s(z,\omega)/\tilde{\tau}(s), \end{cases}$$

where $\tilde{\tau}(t) = \tau(s+t)/\tau(s) = e^{ik(B_{s+t}-B_s)}$ is again a Brownian motion on $\mathbb{T}$ (because $\tilde{B}_t = B_{s+t} - B_s$ is a standard Brownian motion). In other words, the conditional distribution of $\tilde{\phi}_t$ given ϕ_r, $r \in [0, s]$ is the same as the distribution of ϕ_t.

By the complex Itô formula, the process $\frac{1}{\tau(t,\omega)} = e^{-ikB_t}$ satisfies the equation

$$de^{-ikB_t} = -ike^{-ikB_t}dB_t - \frac{k^2}{2}e^{-ikB_t}dt.$$

Let us denote $\frac{\phi_t(z,\omega)}{\tau(t,\omega)}$ by $\Psi_t(z, \omega)$. Applying the integration by parts formula to Ψ_t, we arrive at the following initial value problem for the Itô stochastic differential equation

$$\begin{cases} d\Psi_t = -ik\Psi_t dB_t + \left(-\frac{k^2}{2}\Psi_t + (\Psi_t - 1)^2 p(\Psi_t e^{ikB_t(\omega)}, t, \omega)\right) dt, \\ \Psi_0(z) = z. \end{cases} \tag{3.9}$$

Analyzing the process $\frac{\phi_t(z,\omega)}{\tau(t,\omega)}$ instead of the original process $\phi_t(z, \omega)$ is in many ways similar to one of the approaches used in SLE theory.

The image domains $\Psi_t(\mathbb{D}, \omega)$ differ from $\phi_t(\mathbb{D}, \omega)$ only by rotation. Due to the fact that $|\Psi_t(z, \omega)| = |\phi_t(z, \omega)|$, if we compare the processes $\phi_t(0, \omega)$ and $\Psi_t(0, \omega)$, we note that their first hit times of the circle $\mathbb{T}_r$ with radius $r < 1$ coincide, i. e.,

$$\inf\{t \geq 0, |\phi_t(0, \omega)| = r\} = \inf\{t \geq 0, |\Psi_t(0, \omega)| = r\}.$$

In other words, the answers to probabilistic questions about the expected time of hitting the circle $\mathbb{T}_r$, the probability of exit from the disk $\mathbb{D}_r = \{z : |z| < r\}$, etc. are the same for $\phi_t(0, \omega)$ and $\Psi_t(0, \omega)$.

If the Herglotz function has the form $p(z, t, \omega) = \tilde{p}(z/\tau(t, \omega))$, then the equation (3.9) becomes

$$\begin{cases} d\Psi_t = -ik\Psi_t dB_t + \left(-\frac{k^2}{2}\Psi_t + (\Psi_t - 1)^2 \tilde{p}(\Psi_t)\right) dt, \\ \Psi_0(z) = z, \end{cases} \tag{3.10}$$

and may be regarded as an equation of a 2-dimensional time-homogeneous real diffusion written in complex form. This implies, in particular, that Ψ_t is a time-homogeneous strong Markov process. By construction, $\Psi_t(z)$ always stays in the unit disk.

We now have a family of stochastic dynamical processes of maps on the unit disk, which is the proper setting for an embedding of the original Kuramoto model, in the limit $n \to \infty$. Before describing this embedding, we make a few considerations on possible Poisson structures compatible with the model.

3.3 Constrained Hamiltonian Structure of the Kuramoto Model

As a dynamical system, the Kuramoto model can be represented as the nonlinear restriction of a quadratic Hamiltonian system. For the purpose of characterizing stochastic perturbations of the deterministic model, this allows a canonical approach and provides a purely geometric interpretation of the synchronized state, in the infinite-time limit.

Consider the Hamiltonian dynamical system on $\mathbb{R}^{2n}$, given by the Hamilton function

$$H(\{p_k, q_k\}) \equiv -\frac{1}{2}\sum_{k=1}^{n}\left\{\omega_k z_k \bar{z}_k + \lambda \sum_{j\neq k}\left[(\bar{z}_j \bar{z}_k^{-1} + z_j \bar{z}_k) + c.c.\right]\right\}, \tag{3.11}$$

for which the Hamilton equations

$$\dot{q}_k = \frac{\partial H}{\partial p_k}, \quad \dot{p}_k = -\frac{\partial H}{\partial q_k} \tag{3.12}$$

are equivalent to

$$\dot{z}_k = -2i\frac{\partial H}{\partial \bar{z}_k}, \quad \dot{\bar{z}}_k = 2i\frac{\partial H}{\partial z_k}. \tag{3.13}$$

Obviously, the system (3.4) is equivalent to (3.13), subject to the nonlinear constraints $z_k \bar{z}_k = 1$, and in the presence of Langevin forces driven by the stochastic terms $dW_k(t) = \eta_k(t)dt$.

3.4 Embedding Into the Boundary of the Polydisk and Collective Variables

The mechanical state of the original Kuramoto model, $\{z_k(t)\}_{k=1}^{n}$ is a point on the $n-$dimensional torus $\mathbb{T}^n$, or the boundary of the polydisk $\partial\mathbb{D}^n$. Topologically, this is equivalent to the direct product of compact groups $U(1)^n \subset U(n)$. In this section, we investigate the embedding of the Kuramoto model into the unitary group $U(n)$, and formulate the synchronization phase transition through symmetric homogenous polynomials of eigenvalues of a matrix $\mathcal{U} \in U(n)$.

Introducing the matrix-valued function $\mathcal{U}(t)$, with diagonal elements $\mathcal{U}_{jj}(t) = z_j(t)$ and vanishing off-diagonal elements, and denoting by D the diagonal $n \times n$ "gauge" matrix $D_{jj} = \omega_j + \eta_j$, the

Kuramoto model (3.1) can be expressed as

$$\nabla \mathcal{U} = i\lambda[\mathcal{U}^2 \mathrm{Tr}(\bar{\mathcal{U}}) - \mathrm{Tr}(\mathcal{U})\mathbb{I}], \quad \nabla = \mathbb{I}\frac{d}{dt} - iD, \tag{3.14}$$

Introduce the homogenous symmetric variables

$$\phi_k \equiv \mathrm{Tr}\mathcal{U}^k, \ k \in \mathbb{Z}, \tag{3.15}$$

and use (3.14) to derive the evolution equations in collective variables space:

$$-i\frac{d\phi_k}{dt} = k[D \star \phi_k + \lambda(\phi_{-1}a^\dagger - \phi_1 a)\phi_k] \equiv [\mathcal{H}, \phi_k]_D, \ k \in \mathbb{N}, \tag{3.16}$$

where $D \star \phi_k \equiv \mathrm{Tr}\,(D \cdot \mathcal{U}^k)$, $\mathcal{H}$ is a Hamilton operator (generator of time shifts), and $a, a^\dagger$ are the lowering and raising shift operators,

$$a\phi_k \equiv \phi_{k-1}, \quad a^\dagger \phi_k \equiv \phi_{k+1}. \tag{3.17}$$

Equations (3.15) provide an embedding of the original model into the algebra of homogenous trigonometric polynomials on $\mathbb{T}^1$. The dynamical system is governed by a Hamilton operator and commutator as shown in (3.16). In this formulation, full synchronization is equivalent to

$$\frac{d}{dt}[\phi_1^{-k}\phi_k] = 0, \ \forall\, k \in \mathbb{Z}. \tag{3.18}$$

3.4.1 Time Evolution and Generators of Möbius Group

Using the fact that $\bar{z}_k = z_k^{-1}$, we obtain for any differentiable function $f(\{z_k(t)\})$ the following form of the generator of time evolution:

$$-i\frac{df}{dt} = \sum_{k=1}^{n}[\omega_k L_0^{(k)} + \lambda(\bar{r}L_-^{(k)} + rL_+^{(k)})]f, \tag{3.19}$$

where $L_{0,-,+}^{(k)}$ are the generators of the Lie algebra $s\ell(2,\mathbb{C})$, in differential operator form for the variable z_k, in the absence of randomness. Since we assume the stochastic terms to be independent Wiener processes, by applying the Feynman-Kac theorem, we can identify the correlation functions for the solutions of the stochastic Kuramoto model to expectation values of normal-ordered operator products of polynomials in (non-commutative) variables $\{z_k, \bar{z}_k\}, k = 1, 2, \ldots, n$. This happens to be entirely consistent with the identification $\bar{z} \to \frac{\partial}{\partial z}$ provided by (3.19), which means we have the following result:

THEOREM 3.4.1 *Let $\rho : s\ell(2, \mathbb{C}) \to L(H)$ be a Lie algebra representation into the space of linear operators on a Hilbert space with inner product $\langle , \rangle$. Denote by $t_k^\alpha = \rho(L_\alpha^{(k)})$, where $\alpha = 0, +, -$, such that*

$$[t_i^0, t_j^\pm] = \pm\delta_{ij} t_j^\pm, \quad [t_i^+, t_j^-] = 2\delta_{ij} t_j^0, \; i, j = 1, 2, \ldots, n. \tag{3.20}$$

Then the expectation value of the operator $\oplus_{k=1}^n t_k^-$, $J^- \equiv \langle v | \oplus_{k=1}^n t_k^- | v \rangle$, where v is a cyclic vector for the representation ρ, minimizing the energy functional

$$v = \arg\{\inf_{||\Psi||=1} \langle \Psi H \Psi \rangle\}, \quad H = \sum_k 2\omega_k t_k^0 - \lambda \sum_{k,j} t_k^+ t_j^-,$$

satisfies $J^-(t) = nr(t)$, where $r(t)$ is the Kuramoto order parameter defined in (3.2).

The averaged dynamical system is obtained by replacing the commutators by canonical Poisson brackets,

$$\{S_i^\alpha, S_j^\beta\} = 2\epsilon^{\alpha\beta\gamma} S_i^\gamma \delta_{ij}, \tag{3.21}$$

where $\mathbf{S}_l = 2\langle \mathbf{t}_l \rangle$ are smooth functions of time. In this limit, the problem can analyzed with tools of classical integrable systems, and the solution is known to be exact as $n \to \infty$. This problem was solved [39] by Sklyanin algebra techniques, and the solution provides us with the following dynamical phase transition picture:

Richardson showed [40] that the exact eigenvectors of his Hamiltonian is given by application of operators $b_k^\dagger = \sum_l \frac{t_l^\dagger}{2\omega_l - e_k}$ to the cyclic vector v. The unnormalized $n-$pair eigenvector reads $\Psi_R(\epsilon_i) = \prod_{k=1}^n b_k^\dagger |v\rangle$. The eigenvalues e_k satisfy the self-consistent (algebraic Bethe Ansatz) equations

$$\frac{1}{\lambda} = \sum_{p \neq k} \frac{2}{e_k - e_p} + \sum_l \frac{1}{2\omega_l - e_k}, \tag{3.22}$$

and $\Im(e_k) \neq 0$. Notice that the onset of synchronization corresponds to a single eigenvalue pair $e_0 = -\bar{e}_0 = 2iD$, and therefore we retrieve equation (1.44), where we have replaced summation over the frequencies ω_k by distribution average with distribution function $g(\omega)$.

The solutions found in [39] and later expanded upon include the uniform solution $|r(t)|$ =const. (which corresponds to the mean-field solution of Kuramoto), but also time-dependent solutions expressed through hyperelliptic theta functions, and which we identify to the “chimera” states observed in experiments.

References

[1] Yoshiki Kuramoto. *Self-entrainment of a population of coupled non-linear oscillators*, pages 420–422. Springer Berlin Heidelberg, Berlin, Heidelberg, 1975.

[2] S. Belan. Synchronization dynamics in diverse ensemble of noisy phase oscillators with asynchronous phase updates. *Phys. Rev. E*, 92:062910, Dec 2015.

[3] Alex Arenas, Albert Díaz-Guilera, Jurgen Kurths, Yamir Moreno, and Changsong Zhou. Synchronization in complex networks. *Physics reports*, 469(3):93–153, 2008.

[4] Ramin Hekmat. *Ad-hoc Networks: Fundamental Properties and Network Topologies*. Springer Netherlands, 2006.

[5] Martin Rohden, Andreas Sorge, Marc Timme, and Dirk Witthaut. Self-organized synchronization in decentralized power grids. *Phys. Rev. Lett.*, 109:064101, Aug 2012.

[6] Seth A.AU Anghel MarianAU Nishikawa TakashiTI RMotter, Adilson E.AU Myers. Spontaneous synchrony in power-grid networksja. *NATURE PHYSICS*, 9:191âĂŞ197, February 2013.

[7] Yoshiki Kuramoto and Ikuko Nishikawa. Statistical macrodynamics of large dynamical systems. case of a phase transition in oscillator communities. *Journal of Statistical Physics*, 49(3):569–605, 1987.

[8] Steven H. Strogatz. From kuramoto to crawford: exploring the onset of synchronization in populations of coupled oscillators. *Elsevier Science Physica D*, 143:1–20, 2000.

[9] Juan A. Acebrón, L. L. Bonilla, Conrad J. Pérez Vicente, Félix Ritort, and Renato Spigler. The kuramoto model: A simple paradigm for synchronization phenomena. *Rev. Mod. Phys.*, 77:137–185, Apr 2005.

[10] Steven H. Strogatz and Renato E. Mirollo. Stability of incoherence in a population of coupled oscillators. *Journal of Statistical Physics*, 63(3):613–635, 1991.

[11] James W. Swift, Steven H. Strogatz, and Kurt Wiesenfeld. Averaging of globally coupled oscillators. *Physica D: Nonlinear Phenomena*, 55(3):239 – 250, 1992.

[12] Steven H. Strogatz, Renato E. Mirollo, and Paul C. Matthews. Coupled nonlinear oscillators below the synchronization threshold: Relaxation by generalized landau damping. *Phys. Rev. Lett.*, 68:2730–2733, May 1992.

[13] Luis L. Bonilla, John C. Neu, and Renato Spigler. Nonlinear stability of incoherence and collective synchronization in a population of coupled oscillators. *Journal of Statistical Physics*, 67(1):313–330, 1992.

[14] J. L. van Hemmen and W. F. Wreszinski. Lyapunov function for the kuramoto model of nonlinearly coupled oscillators. *Journal of Statistical Physics*, 72(1):145–166, 1993.

[15] Y. Kuramoto and I. Nishikawa. *Onset of Collective Rhythms in Large Populations of Coupled Oscillators*. Springer Berlin Heidelberg, Berlin, Heidelberg, 1989.

[16] Hiroaki Daido. Intrinsic fluctuations and a phase transition in a class of large populations of interacting oscillators. *Journal of Statistical Physics*, 60(5):753–800, 1990.

[17] Alessandro Campa Shamik Gupta and Stefano Ruffo. Kuramoto model of synchronization: equilibrium and nonequilibrium aspects. *Journal of Statistical Mechanics: Theory and Experiment*, 2014(8):R08001, 2014.

[18] D. C. Roberts and R. Teodorescu. A linear path toward synchronization. *The European Physical Journal Special Topics*, 165(1):103–109, 2008.

[19] Hidetsugu Sakaguchi. Cooperative phenomena in coupled oscillator systems under external fields. *Progress of Theoretical Physics*, 79(1):39, 1988.

[20] Renato E. Mirollo and Steven H. Strogatz. Amplitude death in an array of limit-cycle oscillators. *Journal of Statistical Physics*, 60(1):245–262, 1990.

[21] Hiroaki Daido. Quasientrainment and slow relaxation in a population of oscillators with random and frustrated interactions. *Phys. Rev. Lett.*, 68:1073–1076, Feb 1992.

[22] Hiroaki Daido. Algebraic relaxation of an order parameter in randomly coupled limit-cycle oscillators. *Phys. Rev. E*, 61:2145–2147, Feb 2000.

[23] Scott Kirkpatrick and David Sherrington. Infinite-ranged models of spin-glasses. *Phys. Rev. B*, 17:4384–4403, Jun 1978.

[24] J. C. Stiller and G. Radons. Dynamics of nonlinear oscillators with random interactions. *Phys. Rev. E*, 58:1789–1799, Aug 1998.

[25] P. E. Kloeden and E. Platen. *Numerical Solution of Stochastic Differential Equations*. Springer, 1999.

[26] Brent Oksendal. *Stochastic Differential Equations, An Introduction with Applications, 6th Edition*. Springer, 2003.

[27] Zeev Schuss. *Theory and Applications of Stochastic Processes, An Analytic Approach*. Springer, 2010.

[28] Lawrence Evans. *An Introduction to Stochastic Differential Equations*. American Mathematical Society, 2010.

[29] Adrien Douady and Clifford J. Earle. Conformally natural extension of homeomorphisms of the circle. *Acta Math.*, 157:23–48, 1986.

[30] William Abikoff. Conformal barycenters and the douady-earle extension—a discrete dynamical approach. *Journal d'Analyse Mathématique*, 86(1):221–234, Dec 2002.

[31] William Abikoff and Taiping Ye. Computing the douady-earle extension. 211, 1997.

[32] Vladimir Jaćimović. Computing the douady–earle extension using kuramoto oscillators. *Analysis and Mathematical Physics*, Feb 2018.

[33] Jan R Engelbrecht Bolun Chen and Renato Mirollo. Hyperbolic geometry of kuramoto oscillator networks. *Journal of Physics A: Mathematical and Theoretical*, 50(35):355101, 2017.

[34] Razvan Teodorescu Bjorn Gustafsson and Alexander Vasil'ev. *Classical and Stochastic Laplacian Growth*. Birkhauser, 2014.

[35] Leandro Arosio. Abstract basins of attraction. In Filippo Bracci, Jisoo Byun, Hervé Gaussier, Kengo Hirachi, Kang-Tae Kim, and Nikolay Shcherbina, editors, *Complex Analysis and Geometry*, pages 57–66, Tokyo, 2015. Springer Japan.

[36] S. Díaz-Madrigal M. D. Contreras and P. Gumenyuk. *Iteration theory of holomorphic maps on taut manifolds*, volume 3. 1989.

[37] S. Díaz-Madrigal M. D. Contreras and P. Gumenyuk. Löwner chains in the unit disk. 3:975–1012., 01 2010.

[38] Georgy Ivanov and Alexander VasilâĂŹev. 01 2012.

[39] J. Dukelsky, S. Pittel, and G. Sierra. Colloquium: Exactly solvable richardson-gaudin models for many-body quantum systems. *Rev. Mod. Phys.*, 76:643–662, Aug 2004.

[40] R.W. Richardson and N. Sherman. Exact eigenstates of the pairing-force hamiltonian. *Nuclear Physics*, 52:221 – 238, 1964.

Appendix A

Kuramoto Model Simulation

```
//This program calculates the order parameter vs
//the coupling constant for Kuramoto Model
//Written by Wael Al-Sawai,  04-03-2017
    #include <cstdlib>
    #include <iostream>
    #include <math.h>
    #include <fstream>
    using namespace std;
    const int N=1500;
    double theta[N];
    double thetaInit[N];
    double omega[N];
    double Pi=3.14159265;
    int nstep=7000;
    double dt=0.01;//time step
    // Coupling constant
    double psi=0.0;
    double r=0.0;//coherence coefficient
    void Sum(double theta[],double & , double &);
    double mu=0.0;
    double sigma=2.0;
    double gSampler(double,double);
    //int main(int argc, char *argv[])
    int main()
    {
```

```
double K=0.0;
srand((unsigned) time(NULL));
ofstream myfile;
// generate thetaInit & omega

for (int i=0;i<N;i++){
//Initialize oscillators phases
        thetaInit[i]=(Pi)*double(i/(N-1));
        theta[i]=thetaInit[i];// intial condition
        //omega[i]=1+2*((double)rand()/(double)RAND_MAX);
        omega[i]=gSampler(mu,sigma);
}//end initializing for-loop

myfile.open ("Kuramoto.txt");

while (K<8.6){
        //Integrate the differentail equations



        for (int i=0;i<N;i++){
                for (int j=0;j<nstep;j++){
                theta[i]+=(omega[i]+(K*r)*sin(psi-theta[i]))*dt;
                } //nstep loop
        }//N loop

        Sum(theta, r, psi);
        myfile << K<<"    "<<r<<'\n';
        cout<<K<<'    '<<r<<endl;
        // Copy the initial arrays
        for (int i=0;i<N;i++){
                theta[i]=thetaInit[i];
        }//
```

```
            K+=0.1;
      }//While loop
      myfile.close();
      system("PAUSE");
      return EXIT_SUCCESS;

}

//
void Sum(double theta[], double & r, double & psi){

      double rr;
      double sumcos=0;
      double sumsin=0;
      //
      for(int i=0;i<N;i++){
            sumcos+=cos(theta[i]);
            sumsin+=sin(theta[i]);
      }//sum loop

      //
      rr=pow(sumcos,2.0)+pow(sumsin,2.0);
      r=pow(rr,0.5)/N;
      //
      psi=atan(sumcos/sumsin);
      if(sumcos<0){
            psi=psi+Pi;
      }//end if statement

}  //end function Sum
```

```
//Normally distributed random number
//generated using the Polar method

    double gSampler(double mu,double sigma){
        double U1, U2, W, mult;
//The variables X1 and X2 are made static so that it
//can hold the values from the previous call
        static double X1, X2;
        static int flag = 0;

        if (flag == 1)
        {
                flag = !flag;
                return (mu + sigma * (double) X2);
        }

        do
        {
                U1 = -1 + ((double) rand () / RAND_MAX) * 2;
                U2 = -1 + ((double) rand () / RAND_MAX) * 2;
                W = pow (U1, 2) + pow (U2, 2);
        }
        while (W >= 1 || W == 0);

        mult = sqrt ((-2 * log (W)) / W);
        X1 = U1 * mult;
        X2 = U2 * mult;

        flag = !flag;

        return (mu + sigma * (double) X1);
```

```
}// End of gsampler function
```