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Generalized D-Kaup-Newell integrable systems and their integrable couplings and Darboux transformations

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by

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Dedication

I would like to dedicate this dissertation to my beloved Carleigh, Cameron, Garry, Kathy, Meggin, Melanne, Bryan, Elizabeth, and Allison. Without all of your support, this undertaking would not be possible.

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Abstract

We present a new spectral problem, a generalization of the D-Kaup-Newell spectral problem, associated with the Lie algebra $\mathrm{sl}(2, \mathbb{R})$. Zero curvature equations furnish the soliton hierarchy. The trace identity produces the Hamiltonian structure for the hierarchy. Lastly, a reduction of the spectral problem is shown to have a different soliton hierarchy with a bi-Hamiltonian structure. The first major motivation of this dissertation is to present spectral problems that generate two soliton hierarchies with infinitely many commuting conservation laws and high-order symmetries, i.e., they are Liouville integrable.

We use the soliton hierarchies and a non-seimisimple matrix loop Lie algebra in order to construct integrable couplings. An enlarged spectral problem is presented starting from a generalization of the D-Kaup-Newell spectral problem. Then the enlarged zero curvature equations are solved from a series of Lax pairs producing the desired integrable couplings. A reduction is made of the original enlarged spectral problem generating a second integrable coupling system. Next, we discuss how to compute bilinear forms that are symmetric, ad-invariant, and non-degenerate on the given non-semisimple matrix Lie algebra to employ the variational identity. The variational identity is applied to the original integrable couplings of a generalized D-Kaup-Newell soliton hierarchy to furnish its Hamiltonian structures. Then we apply the variational identity to the reduced integrable couplings. The reduced coupling system has a bi-Hamiltonian structure. Both integrable coupling systems retain the properties of infinitely many commuting high-order symmetries and conserved densities of their original subsystems and, again, are Liouville integrable.

In order to find solutions to a generalized D-Kaup-Newell integrable coupling system, a theory of Darboux transformations on integrable couplings is formulated. The theory pertains to a spectral problem where the spectral matrix is a polynomial in λ of any order. An application to a generalized D-Kaup-Newell integrable couplings system is worked out, along with an explicit formula for the associated Bäcklund transformation. Precise one-soliton-like solutions are given for the m-th order generalized D-Kaup-Newell integrable coupling system.

1 Introduction

1.1 Background

The theory of integrable systems has vastly contributed to nonlinear science and modern mathematical physics. One major contributor is the branch of nonlinear science known as soliton theory. This theory describes various types of stable movements occurring in nature, such as a solitary water wave, solitary signals in optical fiber, etc., and has many applications in science and technology such as optical signal communication. Soliton research has furthered understanding of tidal bores, cyclones, and massive ocean waves like tsunamis. Mathematically speaking, it also gives many effective methods for getting explicit solutions for nonlinear partial differential equations. In general, this is not an easy task. However, for a special class of equations known as soliton equations, various methods have been found to get solutions.

Many important discoveries in the theory of integrable systems came from a simple observation of a soliton. In 1834, J. S. Russell reported his first observation of a soliton wave stating that it "rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed" [1, 2]. A half century later in 1895, D. Korteweg and G. de Vries [3] studied traveling shallow water waves created in a channel and found a $(1+1)$ dimensional, nonlinear equation describing the wave motion. This equation,

$$u_t - 6uu_x + u_{xxx} = 0, \tag{1}$$

famously became known as the KdV equation. They found a single-soliton solution [3] to the KdV (1) as

$$u(x,t) = -\frac{c}{2}\operatorname{sech}^2\left[\frac{1}{2}\sqrt{c}(x - ct - x_0)\right]. \tag{2}$$

More generally, the KdV equation is referred to as

$$u_t + \beta u u_x + \alpha u_{xxx} = 0, \tag{3}$$

where α, β are non-zero constants. Although the first soliton was observed in the 1830's, soliton theory was not given much attention until the 1960's.

In 1965, N. J. Zabusky and M. D. Kruskal [4] analyzed the nonlinear interacting process of the collision between solitary waves in plasma. Using computer simulations, they found that if two solitary waves traveling at different speeds collided, they would keep their original shape, energy, momentum, and direction. They named these waves "solitons" because this type of stability was usually observed in elastic particles. Zabusky and Kruskal also discovered numerical computations of solutions to the KdV equation (1). Formally, a solitary wave solution or *soliton* is a solution to any nonlinear equation that satisfies the following three properties [10]:

(i) represents a wave of permanent form;

(ii) is localized (asymptotically constant at $\pm\infty$ or obeys a periodicity conditions imposed on the original equation);

(iii) can interact strongly with other solitons and retain its size and shape.

Two years later, C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura determined a method for finding soliton solutions for the KdV equation famously known as the Inverse Scattering Transform (IST) [5]. The next year, P. D. Lax [6–8] generalized the IST and revealed more remarkable properties of the KdV. Lax discovered two linear operators, named a Lax pair, generate the KdV equation through a compatibility condition. In 1974, M. J. Ablowitz, D. J. Kaup, A. C.

Newel, and H. Segur [9] showed an infinite number of integrable equations can be found using the IST on nonlinear evolution equations. These contributions have provided the groundwork for the tools used for problems presented in this work.

This dissertation is organized by four chapters. The first chapter provides some preliminary material necessary for the chapters to follow. In the second chapter, the necessary information for integrable systems is presented to show the construction of two integrable soliton hiearchies associated with a generalized D-Kaup-Newell spectral problem. Chapter three is dedicated to the development and presentation of integrable couplings for the integrable soliton hierachies in chapter two. The fourth chapter formulates a theory for Darboux transformations of integrable couplings where the associated spectral matrix is a polynomial in λ of any degree. This theory is applied to the integrable couplings presented in chapter three where the spectral matrix is a polynomial in λ of degree two. We conclude the dissertation with a discussion of interesting problems for further research.

1.2 Preliminaries

1.2.1 Integrable systems

We now present some basic notation, assumptions, definitions, theorems, and corollaries for the integrable systems necessary for this dissertation. Let $x, t \in \mathbb{R}$ be independent variables in space and time, respectively.

DEFINITION 1.1. *The* Schwartz space, $S(\mathbb{R})$, *is the space of rapidly decreasing functions on* $\mathbb{R}$ *defined as*

$$S(\mathbb{R}) = \{f \in C^{\infty}(\mathbb{R}) : ||f||_{\alpha,\beta} < \infty, \forall \alpha, \beta \in \mathbb{Z}_+\}, \textit{ where} ||f||_{\alpha,\beta} = \sup_{x\in\mathbb{R}} |x^{\alpha}\partial^{\beta} f(x)|. \tag{4}$$

The operator $\partial = \frac{\partial}{\partial x}$ introduces an equivalence relation among elements of S,

$$f \sim g \leftrightarrow \exists h \in S \quad \text{such that} \quad f - g = \partial h.$$

The equivalence class that contains f is denoted by $\int f dx$,

$$\int f dx = \{f + \partial h | h \in S\}.$$

For our purposes, let $u_i = u_i(x,t), 1 \leq i \leq q$, be dependent variables in the Schwartz space on $\mathbb{R}$ for any fixed $t \in \mathbb{R}$ and $S^q(\mathbb{R},\mathbb{R})$ be the set of all vectors $u = (u_1, u_2, ..., u_q)^T$ [28].

DEFINITION 1.2. *An* evolution equation *is a partial differential equation of the form*

$$u_t = K(u) = K(u, u_x, u_{xx}, u_{xxx}, ...), \tag{5}$$

where $u(x,t)$ is a dependent variable, and $K(u)$ is a function on u and its derivatives with respect to x. If K is a nonlinear, it is called a nonlinear evolution equation.

Clearly, the KdV (1) is an example of a nonlinear evolution equation where $K(u) = 6uu_x - u_{xxx}$.

DEFINITION 1.3. *For any real-valued function $P(x,t,u)$, its* Gateaux derivative with respect to u in the direction of $v = (v_1, ..., v_N)^T \in S^q(\mathbb{R},\mathbb{R})$ *is defined by*

$$P'[v] = P'(u)[v] = \frac{\partial}{\partial \epsilon} P(u + \epsilon v)|_{\epsilon=0} = \frac{\partial}{\partial \epsilon} P(u_1 + \epsilon v_1, ..., u_N + \epsilon v_N)|_{\epsilon=0}.$$

We denote $\mathfrak{B}$ the space of all real-valued functions $P(x,t,u)$ which are C^∞-differentiable with respect to x and t and C^∞-Gateaux differentiable with respect to $u = u(x,t)$ as functions of x. Then set $\mathfrak{B}^q = \{(P_1, ..., P_q)^T | P_i \in \mathfrak{B}, 1 \leq i \leq q\}$ [45]. We consider evolution equations (5) with $K \in \mathfrak{B}^q$.

Definition 1.4. *For any two vector fields $K, S \in \mathfrak{B}^q$, define the product vector field to be*

$$[K, S] = K'[S] - S'[K], \tag{6}$$

which has been shown to be a commutator operation of $\mathfrak{B}^q$.

$(\mathfrak{B}^q, [\cdot, \cdot])$ forms a Lie algebra over the real field.

Definition 1.5. *Let u be a solution to an evolution equation (5). A vector field $S \in \mathfrak{B}^q$ is said to be a* symmetry *of (5) if the infinitesimal transformation*

$$u(t) \to u(t) + \epsilon S(u(t))$$

leaves (5) form invariant.

S is a symmetry of (5) if and only if S is a solution of the perturbation equation [13],

$$S_t = K'(u)[S],$$

where u solves (5). For a solution u of (5), and a vector field $S \in \mathfrak{B}^q$, we have

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + S'[u_t] = \frac{\partial S}{\partial t} + S'[K] = \frac{\partial S}{\partial t} + K'[S] - [K, S],$$

from the definition of $[\cdot, \cdot]$. Naturally, for symmetries of (5), we have the following theorem and corollary:

Theorem 1.1. *[11] $S \in \mathfrak{B}^q$ is a symmetry of (5) if and only if S satisfies*

$$\frac{\partial S}{\partial t} = [K, S]. \tag{7}$$

Corollary 1.1. *If a vector field S does not depend on t explicitly, i.e., $\frac{\partial S}{\partial t} = 0$, then S is a symmetry of (5) if and only if $[K, S] = 0$.*

A symmetry is a tool to map one solution to another. We can see this in the following example.

Example 1.1. *Let $K(u) = -u_{xxx} + 6uu_x$ and $S(u) = -u_x$. Now,*

$$\begin{aligned} K'(u)[S] =& \frac{\partial}{\partial \epsilon} K(u + \epsilon S)|_{\epsilon=0} = \frac{\partial}{\partial \epsilon}[-(u_{xxx} + \epsilon S_{xxx}) + 6(u + \epsilon S)(u_x + \epsilon S_x)]|_{\epsilon=0} \\ =& - S_{xxx} + 6uS_x + 6Su_x = u_{xxxx} - 6uu_{xx} - 6u_x^2 \\ S'(u)[K] =& \frac{\partial}{\partial \epsilon} S(u + \epsilon K)|_{\epsilon=0} = \frac{\partial}{\partial \epsilon}(-u_x - \epsilon K_x)|_{\epsilon=0} = -K_x \\ =& u_{xxxx} - 6uu_{xx} - 6u_x^2. \end{aligned} \tag{8}$$

We can clearly see that $[K, S] = 0$ and, by the previous Corollary 1.1, $S = -u_x$ is a symmetry of the KdV equation (1). We may also see that if we substitute $S(u) = -u_x$ into the perturbation equation $S_t = K'(u)[S] = -S_{xxx} + 6uS_x + 6Su_x$, a linear equation of S, we have

$$(-u_x)_t = -(-u_x)_{xxx} + 6u(-u_x)_x + 6u_x(-u_x), \tag{9}$$

which is the derivative of the KdV equation (1) with respect to x. It is widely known that the KdV equation has infinitely many of these symmetries.

Definition 1.6. *Let V denote the space of linear operators from $\mathfrak{B}^q$ to $\mathfrak{B}^q$. A linear operator $\Phi \in V$ is called a* recursion operator *for $u_t = K, K \in \mathfrak{B}^q$, if for any symmetry $S \in \mathfrak{B}^q$ of $u_t = K$, ΦS is again a symmetry of $u_t = K$.*

Example 1.2. *The recursion operator for the KdV equation (3), where $\alpha = 1$ and $\beta = 1$, is*

$$\Phi = \partial^2 + \frac{2}{3}u + \frac{1}{3}u_x\partial^{-1}, \quad \partial = \frac{\partial}{\partial x}, \quad \partial\partial^{-1} = \partial^{-1}\partial = 1. \tag{10}$$

If we apply Φ to a symmetry $S(u) = u_x$, we get

$$\begin{aligned}\Phi S =& \Phi u_x = (\partial^2 + \frac{2}{3}u + \frac{1}{3}u_x\partial^{-1})u_x = u_{xxx} + \frac{2}{3}uu_x + \frac{1}{3}u_x u \\ =& u_{xxx} + uu_x.\end{aligned} \tag{11}$$

$P(u) = \Phi S = u_{xxx} + uu_x$ is a new symmetry of the KdV equation. S corresponds to invariance under x-translation while P under t-translation.

Definition 1.7. *Let $\Phi \in V : \mathfrak{B}^q \to \mathfrak{B}^q$ be a linear operator and $K \in \mathfrak{B}^q$ be a vector field. The* Lie derivative $L_K\Phi \in V$ of the operator Φ with respect to K *is defined by*

$$(L_K\Phi)S = \Phi[K,S] - [K,\Phi S], \quad S \in \mathfrak{B}^q. \tag{12}$$

Definition 1.8. *For a linear operator $\Phi \in V$, its* Gateaux derivative operator $\Phi' : \mathfrak{B}^q \to \mathfrak{B}^q$ *is defined through*

$$\Phi'[K]S := \frac{\partial}{\partial\epsilon}\Phi(u + \epsilon K)S|_{\epsilon=0}, \quad K \in \mathfrak{B}^q, \quad S \in \mathfrak{B}^q.$$

We use the Gateaux derivative of Φ in the following theorem:

Theorem 1.2. *[12] A linear operator $\Phi \in V$ is a recursion operator for $K \in \mathfrak{B}^q$ if and only if*

$$L_K\Phi = \Phi'[K] - [K', \Phi] = 0, \tag{13}$$

i.e., Φ is invariant under K, when $\frac{\partial\Phi}{\partial t} = 0$.

We will show in the chapters that follow that each of the spectral problems presented generate infinitely many commuting symmetries.

1.2.2 Hamiltonian structures

We present next some definitions, propositions, theorems, corollaries, and a lemma on Hamiltonian structures. Let $\mathcal{F}$ denote the space of functionals $\mathcal{H} = \int f(u)dx$ where the function f is in the space $\mathfrak{B}$. Then we have that $\int \partial f(u)dx = 0$.

LEMMA 1.1. *For any $\mathcal{H} \in \mathcal{F}$, say $\mathcal{H} = \int f(u)dx$, $f \in \mathfrak{B}$, we have*

$$\frac{d}{dt}\mathcal{H} = \int \frac{d}{dt} f(u)dx. \tag{14}$$

DEFINITION 1.9. *[14] The* variational derivative $\frac{\delta \mathcal{P}}{\delta u}$ of a functional $\mathcal{P} \in \mathcal{F}$ with respect to u *is determined by*

$$\int \left(\frac{\delta \mathcal{P}}{\delta u}\right)^T \xi dx = \frac{\partial}{\partial \epsilon}\mathcal{P}(u + \epsilon\xi)|_{\epsilon=0}, \quad \xi \in \mathfrak{B}^q. \tag{15}$$

PROPOSITION 1.1. *For any $\mathcal{H} \in \mathcal{F}$, say $\mathcal{H} = \int f(u)dx$, $f \in \mathfrak{B}$, its time derivative can be represented as*

$$\frac{d}{dt}\mathcal{H} = \int \sum_{n=0}^{\infty}(-1)^n \frac{\partial^n}{\partial x^n}\left(\frac{\partial f}{\partial u^{(n)}}\right) dx, \quad \textit{where} \quad u^{(n)} = \frac{\partial^n u}{\partial x^n}, \tag{16}$$

and it hold that

$$\frac{\delta \mathcal{H}}{\delta u} = \sum_{n=0}^{\infty}(-1)^n \frac{\partial^n}{\partial x^n}\left(\frac{\partial f}{\partial u^{(n)}}\right) \tag{17}$$

where f is local and does not depend on t explicitly.

DEFINITION 1.10. *The* adjoint operator $J^\dagger : \mathfrak{B}^q \to \mathfrak{B}^q$ *of a linear operator $J : \mathfrak{B}^q \to \mathfrak{B}^q$ is determined by*

$$\int \xi^T J^\dagger \eta dx = \int \eta^T J \xi dx, \quad \xi, \eta \in \mathfrak{B}^q.$$

If $J^\dagger = -J$, then J is called *skew-adjoint*.

DEFINITION 1.11. *Let $J : \mathfrak{B}^q \to \mathfrak{B}^q$ be a linear differential operator. We define a bracket* $\{\cdot, \cdot\}$:

$\mathcal{F} \times \mathcal{F}$ as

$$\{\mathcal{P}, \mathcal{Q}\}_J := \int \left(\frac{\delta \mathcal{P}}{\delta u}\right)^T J \frac{\delta \mathcal{Q}}{\delta u} dx, \quad \mathcal{P}, \mathcal{Q} \in \mathcal{F}. \tag{18}$$

DEFINITION 1.12. *A linear operator $J : \mathfrak{B}^q \to \mathfrak{B}^q$ is called* Hamiltonian *if*

$$\{\mathcal{P}, \mathcal{Q}\} = \{\mathcal{P}, \mathcal{Q}\}_J = \int \left(\frac{\delta \mathcal{P}}{\delta u}\right)^T J \frac{\delta \mathcal{Q}}{\delta u} dx, \quad \mathcal{P}, \mathcal{Q} \in \mathcal{F}, \tag{19}$$

satisfies the skew-symmetry condition,

$$\{\mathcal{P}, \mathcal{Q}\} = -\{\mathcal{Q}, \mathcal{P}\}, \tag{20}$$

and the Jacobi identity,

$$\{\{\mathcal{P}, \mathcal{Q}\}, \mathcal{R}\} + \{\{\mathcal{Q}, \mathcal{R}\}, \mathcal{P}\} + \{\{\mathcal{R}, \mathcal{P}\}, \mathcal{Q}\} = 0. \tag{21}$$

If the operator J is Hamiltonian, then (18) is called a Poisson bracket*.*

LEMMA 1.2. *[46] Let $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ be functionals with variational derivatives $\frac{\delta \mathcal{P}}{\delta u} = P, \frac{\delta \mathcal{Q}}{\delta u} = Q, \frac{\delta \mathcal{R}}{\delta u} = R \in \mathfrak{B}^q$. Then the Jacobi identity (21) is equivalent to the expression*

$$\int \big(P \cdot J'[JR]Q + R \cdot J'[JQ]P + Q \cdot J'[JP]R\big)\, dx = 0, \tag{22}$$

or more commonly,

$$\begin{gathered} < P, J'[JR]Q > + < R, J'[JQ]P > + < Q, J'[JP]R >= \\ < P, J'[JR]Q > + \, cycle(P, Q, R) = 0. \end{gathered} \tag{23}$$

PROPOSITION 1.2. *[46] Let J be a $q \times q$ matrix differential operator with bracket (18) on the space of functionals. Then the bracket is skew-symmetric, i.e., (20) holds, if and only if J is skew-adjoint:*

$J^\dagger = -J$.

Proposition 1.3. *[46] Let J be a skew-adjoint $q \times q$ matrix differential operator. Then the bracket (21) satisfies the Jacobi identity if and only if (23) vanishes for all $P, Q, R \in \mathfrak{B}^q$.*

Corollary 1.2. *[46] If J is a skew-adjoint $q \times q$ matrix differential operator whose coefficients do not depend on u or its derivatives, then J is automatically a Hamiltonian operator.*

Definition 1.13. *[15] A pair of Hamiltonian operators $J, M : \mathfrak{B}^q \to \mathfrak{B}^q$ is called a* Hamiltonian pair *if $\alpha J + \beta M$, for all $\alpha, \beta \in \mathbb{R}$, is also a Hamiltonian operator.*

Definition 1.14. *A system of evolution equations $u_t = K, K \in \mathfrak{B}^q$ is called a* Hamiltonian system *if there is a Hamiltonian operator $J : \mathfrak{B}^q \to \mathfrak{B}^q$ and a functional $\mathcal{H} \in \mathcal{F}$ such that*

$$u_t = K(u) = J\frac{\delta \mathcal{H}}{\delta u}. \tag{24}$$

The functional $\mathcal{H}$ is called a Hamiltonian functional *of the system, and we say that the system possesses a* Hamiltonian structure *if it can be written in the form (24).*

To obtain Hamiltonian structure means to transform $u_t = K(u)$ into the form of (24).

Example 1.3. *The KdV equation (3), with $\alpha = \beta = -1$, can be written into Hamiltonian form in two different ways. We may write*

$$u_t = \partial(u_{xx} + \frac{1}{2}u^2) = J\frac{\delta \mathcal{H}_1}{\delta u}, \tag{25}$$

with the Hamiltonian operator and the Hamiltonian functional,

$$J = \partial, \quad \mathcal{H}_1 = \int [-\frac{1}{2}u_x^2 + \frac{1}{6}u^3]dx, \tag{26}$$

respectively. We can clearly see this using the definition of the variational derivative (17). We may also write

$$u_t = (\partial^3 + \frac{2}{3}u\partial + \frac{1}{3}u_x)u = M\frac{\delta \mathcal{H}_0}{\delta u}, \tag{27}$$

where the Hamiltonian operator and Hamiltonian functional are

$$M = \partial^3 + \frac{2}{3}u\partial + \frac{1}{3}u_x, \quad \mathcal{H}_0 = \int \frac{1}{2}u^2 dx, \tag{28}$$

respectively.

Definition 1.15. *[46] Any functional $\mathcal{P}(t,u) \in \mathcal{F}$ is a* conserved quantity *if along any solution u of (5) we have*

$$\frac{d}{dt}\mathcal{P}(t,u) = 0. \tag{29}$$

Example 1.4. *Assume the solution to the KdV equation (3), where $\alpha = 1$ and $\beta = 6$, is in S. Let the "mass" and "energy" be*

$$M := \int_{\mathbb{R}} u(x,t)dx, \quad \textit{and} \quad E := \int_{\mathbb{R}} \frac{1}{2}u^2(x,t)dx, \tag{30}$$

respectively. We may see that they are independent of time, i.e., conserved quantities. Differentiating both with respect to time variable t, gives

$$\begin{aligned} \frac{dM}{dt} &= \int (u_t)dx \\ &= \int (-6uu_x - u_{xxx})dx \\ &= \int \partial(-3u^2 - u_{xx})dx = 0, \end{aligned} \qquad \begin{aligned} \frac{dE}{dt} &= \int (uu_t)dx \\ &= \int (-6u^2u_x - uu_{xxx})dx \\ &= \int \partial(-2u^3 - uu_{xx} + \frac{1}{2}u_x^2)dx = 0. \end{aligned}$$

It is widely known that the KdV equation has infinitely many of these conservation laws.

For any solution u of (24), and for any functional $\mathcal{P} = \mathcal{P}(x,t,u) \in \mathcal{F}$, we have

$$\frac{d}{dt}\mathcal{P} = \frac{\partial \mathcal{P}}{\partial t} + \mathcal{P}'[u_t] = \frac{\partial \mathcal{P}}{\partial t} + \int \left(\frac{\delta \mathcal{P}}{\delta u}\right)^T u_t \, dx = \frac{\partial \mathcal{P}}{\partial t} + \int \left(\frac{\delta \mathcal{P}}{\delta u}\right)^T \frac{\delta \mathcal{H}}{\delta u} dx = \frac{\partial \mathcal{P}}{\partial t} + \{\mathcal{P}, \mathcal{H}\}_J.$$

THEOREM 1.3. *For any functional $\mathcal{P} \in \mathcal{F}, \mathcal{P}$ is a conserved quantity of (24) if and only if*

$$\frac{\partial \mathcal{P}}{\partial t} = \{\mathcal{H}, \mathcal{P}\}_J. \tag{31}$$

PROPOSITION 1.4. *A functional $\mathcal{P} \in \mathcal{F}$, which does not depend on t explicitly, is called a conserved quantity or conserved functional of the Hamiltonian system (24) if the Poisson bracket $\{\mathcal{H}, \mathcal{P}\}_J = 0$. We say $\mathcal{H}$ and $\mathcal{P}$ are in involution when their Poisson bracket is zero.*

DEFINITION 1.16. *[30] A Hamiltonian system is called to be* Liouville integrable*, if there exists a sequence of conserved functionals, $\{\mathcal{H}_n\}$, which are in involution with respect to their corresponding Poisson bracket:*

$$\{\mathcal{H}_n, \mathcal{H}_m\}_J = 0,$$

and the characteristics of whose associated Hamiltonian vector fields

$$K_n := J\frac{\delta \mathcal{H}_n}{\delta u}, \quad n \geq 0,$$

are independent.

PROPOSITION 1.5. *[28] Let J and Ψ be two linear operators mapping $\mathfrak{B}^q$ to itself. Suppose that*

(i) both J and $J\Psi$ are skew-adjoint, i.e.,

$$J^\dagger = -J, \quad J\Psi = \Psi^\dagger J;$$

(ii) there exists a series of functionals $\{\mathcal{H}_n\}$ for which it holds that

$$\Psi^n f(u) = \frac{\delta \mathcal{H}_n}{\delta u},$$

for some $f(u) \in \mathfrak{B}^q$. Then $\{\mathcal{H}_n\}$ is a common series of conserved densities for the whole hierarchy of equations,

$$u_t = J\Psi^n f(u)$$

and we have

$$\{\mathcal{H}_n, \mathcal{H}_m\}_J = 0.$$

DEFINITION 1.17. *[15] A system of evolution equations $u_t = K, K \in \mathfrak{B}^q$, is called a* bi-Hamiltonian system *if there exists a Hamiltonian pair $J, M : \mathfrak{B}^q \to \mathfrak{B}^q$ and functionals $\mathcal{H}, \mathcal{P} \in \mathcal{F}$, such that*

$$u_t = K(u) = J\frac{\delta \mathcal{H}}{\delta u} = M\frac{\delta \mathcal{P}}{\delta u}. \tag{32}$$

We say that the evolution system (5) possesses a bi-Hamiltonian structure *if it can be written in the form (32).*

To obtain bi-Hamiltonian structure means to transform $u_t = K(u)$ in (32).

THEOREM 1.4. *[12, 15] Let*

$$u_t = K(u) = J\frac{\delta \mathcal{H}}{\delta u} = M\frac{\delta \mathcal{P}}{\delta u}$$

be a bi-Hamiltonian system of evolution equations. Assume that the Hamiltonian operator J is invertible, and the linear operator Φ is defined by

$$\Phi := MJ^{-1}.$$

Next assume that for each $n = 0, 1, 2, ...,$ we can recursively define

$$K_n = \Phi K_{n-1}, \quad n \geq 1,$$

which implies for $n \geq 1, K_{n-1}$ lies in the image of J. Then there exists a sequence of Hamiltonian functionals, $\{\mathcal{H}_n\}$, such that

(i) for each $n \geq 1$, the evolution equation

$$u_{t_n} = K_n(u) = J\frac{\delta \mathcal{H}_n}{\delta u} = M\frac{\delta \mathcal{H}_{n-1}}{\delta u} \tag{33}$$

is a bi-Hamiltonian system;

(ii) the vector fields $\{K_n\}$ commute with each other, i.e. we have

$$[K_m, K_n] = 0, \quad n, m \geq 0; \tag{34}$$

(iii) the Hamiltonian functionals $\{\mathcal{H}_n\}$ are in involution with respect to both Poisson brackets

$$\{\mathcal{H}_n, \mathcal{H}_m\}_J = \{\mathcal{H}_n, \mathcal{H}_m\}_M = 0, \quad n, m \geq 0, \tag{35}$$

which implies there exists a sequence of infinitely many conserved quantities for each of the bi-Hamiltonian systems (33).

EXAMPLE 1.5. *In Example 1.3, we showed that the KdV has two Hamiltonian operators, i.e.,*

$$J = \partial, \quad \text{and} \quad M = \partial^3 + \frac{2}{3}u\partial + \frac{1}{3}u_x, \tag{36}$$

and it is bi-Hamiltonian

$$u_t = \partial(u_{xx} + \frac{1}{2}u^2) = (\partial^3 + \frac{2}{3}u\partial + \frac{1}{3}u_x)u = J\frac{\delta \mathcal{H}_1}{\delta u} = M\frac{\delta \mathcal{H}_0}{\delta u}. \quad (37)$$

It was mentioned in Example 1.2 that

$$\Phi = MJ^{-1} = (\partial^3 + \frac{2}{3}u\partial + \frac{1}{3}u_x)\partial^{-1} = \partial^2 + \frac{2}{3}u + \frac{1}{3}u_x\partial^{-1}, \quad (38)$$

is a recursion operator for the KdV. It is known the we can recursively define

$$K_n = \Phi K_{n-1}, \quad n \geq 1. \quad (39)$$

Therefore, there is a sequence of Hamiltonian functionals $\{\mathcal{H}_n\}$ *such that* $(i), (ii),$ *and* (iii) *from Theorem 1.4 hold.*

We will present two hierarchies in chapter two that possess infinitely many conserved quantities. One hierarchy will be shown to have bi-Hamiltonian structures. In chapter three, we enlarge the systems of evolution equations from chapter two to find two enlarged hierarchies again with infinitely many conserved densities. Each of the four hierarchies presented in this dissertation are shown to have infinitely many commuting high-order symmetries and conserved densities. Thus, the two soliton hierarchies from chapter two and their corresponding coupling systems in chapter three are Liouville integrable. The next chapter is entirely devoted to the development of two integrable systems that generalize the D-Kaup-Newell soliton hierarchy.

2 Infinite-dimensional integrable systems

2.1 Introduction

The study of soliton equations has been of considerable importance to the understanding of nonlinear phenomenon over the past few decades. In recent years, soliton theory has enriched the understanding of the nature of integrability in partial and ordinary differential equations (see [17]); one way is through the existence of infinitely many conservation laws and symmetries. Constructed from spectral problems associated with matrix Lie algebras, systems of solitons equations often give rise to soliton hierarchies [18]- [21], [28, 29]. Frequently, these hierarchies possess infinitely many symmetries and conserved functionals. Some of the most celebrated hierarchies of this particular type include the Ablowitz-Kaup-Newell-Segur [9], the Kaup-Newell [22], the D-Kaup-Newell [23], the KdV [6], and the Dirac hierarchies [24]. This chapter presents two spectral problems that generate different soliton hierarchies; both hierarchies have infinitely many high-order symmetries and conservation laws implying Liouville integrability [21, 30].

In this chapter, we will begin with a description of the tools used for a general method of soliton hierarchy construction. Next, we introduce a new spectral matrix and explain why it generalizes the D-Kaup-Newell spectral matrix. We then generate its soliton hierarchy and see its infinitely many commuting symmetries. We apply the trace identity to engender its Hamiltonian structures and discuss why it is Liouville integrable, i.e., the hierarchy has infinitely many commuting conserved functionals and high-order symmetries. We proceed with a presentation of a reduced spectral matrix which produces a completely different integrable soliton hierarchy with bi-Hamiltonian structures.

2.2 A method for soliton hierarchy construction

2.2.1 Lax pairs

As mentioned previously, P. D. Lax discovered what is known as a Lax pair in 1968 [6]. He found soliton solutions to nonlinear differential equations by relating the original differential equations to two linear operators through a compatibility condition. Lax discovered that some nonlinear partial differential equations

$$u_t = K(u) \tag{40}$$

have an equivalent formulation through linear partial differential equations

$$\mathcal{L}\psi = \lambda\psi, \quad \psi_t = \mathcal{A}\psi, \tag{41}$$

where $\mathcal{L}$ and $\mathcal{A}$ are linear differential operators and ψ is an eigenfunction of $\mathcal{L}$ corresponding to the eigenvalue λ. The linear differential operators $\mathcal{L}$ and $\mathcal{A}$ are said to be a Lax pair for (40). By simple calculation using (41), we have

$$\frac{d}{dt}(\mathcal{L}\psi) = \frac{d\mathcal{L}}{dt}\psi + \mathcal{L}\psi_t = \frac{d\mathcal{L}}{dt}\psi + \mathcal{L}\mathcal{A}\psi \tag{42}$$

and

$$\frac{d}{dt}(\mathcal{L}\psi) = \frac{d\mathcal{L}}{dt}(\lambda\psi) = \lambda\psi_t = \lambda(\mathcal{A}\psi) = \mathcal{A}(\lambda\psi) = \mathcal{A}\mathcal{L}\psi. \tag{43}$$

Thus, (40) is equivalent to

$$\frac{d\mathcal{L}}{dt} = [\mathcal{A}, \mathcal{L}], \tag{44}$$

where $[\mathcal{A}, \mathcal{L}] = \mathcal{A}\mathcal{L} - \mathcal{L}\mathcal{A}$ is the operator commutator.

We refer to an eigenvalue problem where the eigenvalue is independent of time as an *isospectral* eigenvalue problem. Throughout this dissertation, the eigenvalue problems presented will be isospectral eigenvalue problems. Let's look at an example of a Lax pair for a general KdV equation (3) seen in chapter one where $\alpha = \beta = 1$.

Example 2.1. *The KdV equation,*

$$u_t + uu_x + u_{xxx} = 0, \tag{45}$$

has a Lax pair of linear differential operators

$$\mathcal{L} = \partial^2 + \frac{1}{6}u, \quad \mathcal{A} = -4\partial^3 - u\partial - \frac{1}{2}u_x, \tag{46}$$

where the differential operator $\partial = \frac{\partial}{\partial x}$ *acting on a smooth function* $v(x)$ *is* $\partial v = v_x + v\partial$. *A brief calculation reveals to arbitrary* ψ *that*

$$(\mathcal{L}_t + [\mathcal{L}, \mathcal{A}])\psi = \frac{1}{6}(u_t + uu_x + u_{xxx})\psi = 0. \tag{47}$$

This shows that the linear Lax equation (44) is equivalent to the nonlinear KdV equation (45) given the Lax pair of linear differential operators (46).

The discovery of Lax pairs leads naturally into our next topic of zero curvature equations and soliton equations.

2.3 Zero curvature equations and soliton equations

To begin the discussion on zero curvature equations, let's introduce two linear first order spectral problems

$$\begin{cases} \phi_x = U(\lambda, u)\phi, \\ \phi_t = V(\lambda, u)\phi, \end{cases} \tag{48}$$

where $U(\lambda)$ and $V(\lambda)$ are matrix valued functions of x and t depending on the auxiliary variable λ, called the *spectral parameter*, ϕ is a column vector whose components depend on (x, t, λ), and u, a potential. After differentiating the spacial spectral problem with respect to t,

$$\phi_{xt} = U(\lambda, u)_t \phi + U(\lambda, u)\phi_t, \tag{49}$$

and the temporal problem with respect to x,

$$\phi_{tx} = V(\lambda, u)_x \phi + V(\lambda, u)\phi_x, \tag{50}$$

the consistency condition of $\phi_{xt} = \phi_{tx}$ shows, for all smooth enough ϕ,

$$\frac{\partial}{\partial t}(U(\lambda, u)\phi) - \frac{\partial}{\partial x}(V(\lambda, u)\phi) = \left(\frac{\partial}{\partial t}U(\lambda, u) - \frac{\partial}{\partial x}V(\lambda, u) + [U(\lambda, u), V(\lambda, u)]\right)\phi = 0, \tag{51}$$

or, equivalently,

$$U_t - V_x + [U, V] = 0, \tag{52}$$

known as the *zero curvature equation*.

THEOREM 2.1. *[26] The solution of*

$$\begin{cases} \phi_x = U(\lambda, u)\phi, \\ \phi_t = V(\lambda, u)\phi, \end{cases} \tag{53}$$

exists uniquely for any given initial data $\phi(x_0, t_0) = \phi_0$ *if and only if*

$$U_t - V_x + [U, V] = 0, \tag{54}$$

holds. (54) is known as the integrability condition, *or* compatibility condition, *of (53).*

Example 2.2. *[26, 35] The KdV equation (3) with $\alpha = 1$ and $\beta = 6$,*

$$u_t + 6uu_x + u_{xxx} = 0, \tag{55}$$

is the integrability condition of the linear system

$$\begin{cases} -\phi_{xx} - u\phi = \lambda\phi, \\ \phi_t = -4\phi_{xxx} - 6u\phi_x - 3u_x\phi, \end{cases} \tag{56}$$

which is called the Lax pair of the KdV. If we let $\phi_1 = \phi, \phi_2 = \phi_x, \Phi = (\phi_1, \phi_2)^T$, then the Lax pair (56) can be written in matrix form as

$$\begin{cases} \Phi_x = \begin{bmatrix} 0 & 1 \\ -\lambda - u & 0 \end{bmatrix} \Phi, \\ \Phi_t = \begin{bmatrix} u_x & 4\lambda - 2u \\ -4\lambda^2 - 2\lambda u + u_{xx} + 2u^2 & -u_x \end{bmatrix} \Phi. \end{cases} \tag{57}$$

Here is it clear to see that the Lax matrices are

$$U = \begin{bmatrix} 0 & 1 \\ -\lambda - u & 0 \end{bmatrix} \quad and \quad V = \begin{bmatrix} u_x & 4\lambda - 2u \\ -4\lambda^2 - 2\lambda u + u_{xx} + 2u^2 & -u_x \end{bmatrix}. \tag{58}$$

When we compute the zero curvature equation (54), we have

$$\begin{bmatrix} 0 & 0 \\ -u_t - u_{xxx} - 6uu_x & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{59}$$

Now, $U_t = U'[u_t] + \lambda_t U_\lambda$. We see that the zero curvature equation is true if and only if

$$U'[K] + \lambda_t U_\lambda - V_x + [U, V] = 0,$$

when u solves $u_t = K(u)$. As mentioned previously, we assume $\lambda_t = 0$. Under this assumption, we have the following theorem.

THEOREM 2.2. *[27] Assume that the spectral matrix U of (53) has an injective Gateaux derivative operator $U'[\cdot]$. For groups (V, K) and (W, S) satisfying the zero curvature equations,*

$$U'[K] - V_x + [U, V] = 0, \tag{60}$$

we have the equality

$$U'[[K, S]] - [[V, W]]_x + [U, [[V, W]]] = 0, \tag{61}$$

where

$$[[V, W]] := V'[S] - W'[K] + [V, W]. \tag{62}$$

2.3.1 Trace identity

The trace identity has proven to be a powerful tool for formulating Hamiltonian structures of hierarchies of differential equations and showing their integrability. G. Z. Tu discovered the well-known trace identity in 1986. A few years later, Tu showed that a hierarchy and its Hamiltonians can be constructed starting from what is known as the stationary zero curvature equation [31], i.e.,

$$W_x = [U, W], \tag{63}$$

via the trace identity. Later, he realized the Liouville integrability of zero curvature equations and the explicit formula using the trace identity.

We now introduce a formal definition of the trace identity. Let $\mathfrak{g}$ be a matrix semisimple Lie algebra. Let $U = U(\lambda, u)$ be an element of $\tilde{\mathfrak{g}}$, the corresponding loop algebra of $\mathfrak{g}$, i.e.,

$$\widetilde{\mathfrak{g}} = \left\{ \sum_{i \geq 0} A_i \lambda^{n-i} | A_i \in \mathfrak{g},\ n \in \mathbb{Z} \right\}. \tag{64}$$

Let $< x, y >$ be the Killing-Cartan form. Under the supposition that the solution W of (63), which is of given homogeneous rank, is unique up to a constant multiplier, it is proven [28] that for any solution W of (63) of homogeneous rank, there exists a constant γ such that for $\bar{W} = \lambda^\gamma W$, which is again a solution of (63), it holds that

$$\frac{\delta}{\delta u} \int \left\langle \bar{W}, \frac{\partial U}{\partial \lambda} \right\rangle dx = \frac{\partial}{\partial \lambda} \left\langle \bar{W}, \frac{\partial U}{\partial u} \right\rangle. \tag{65}$$

By a simple substitution $\bar{W} = \lambda^\gamma W$, we get

$$\frac{\delta}{\delta u} \int \left\langle W, \frac{\partial U}{\partial \lambda} \right\rangle dx = \left(\lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma \right) \left\langle W, \frac{\partial U}{\partial u} \right\rangle. \tag{66}$$

It is well known that the Killing form for a semisimple Lie algebra is proportional to the trace, i.e., $< x, y >= \alpha \mathrm{tr}(xy)$, for all $x, y \in \mathfrak{g}, \alpha \in \mathbb{R}$. Therefore, assuming $\alpha = 1$, we have

$$\frac{\delta}{\delta u} \int \mathrm{tr} \left(W \frac{\partial U}{\partial \lambda} \right) dx = \left(\lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma \right) \mathrm{tr} \left(W \frac{\partial U}{\partial u} \right). \tag{67}$$

The formula (67) is known as the *trace identity*.

2.3.2 A general scheme to construct a soliton hierarchy

A soliton hierarchy is an infinite sequence of soliton equations. For deeper understanding, we begin with a definition of a *soliton equation*.

Definition 2.1. *[25, 29, 31] If an evolution equation,*

$$u_t = K(u) = K(u, u_x, u_{xx}, ...), \tag{68}$$

can be presented by a zero curvature equation,

$$U_t - V_x + [U, V] = 0, \tag{69}$$

we call it a soliton equation*, and U and V its* Lax pair*.*

The following method may be used to construct an infinite sequence of soliton equations using a series of Lax pairs. It is commonly used as a general scheme for soliton hierarchy construction [29, 31]. A spectral problem is introduced

$$\phi_x = U\phi = U(u, \lambda)\phi \in \tilde{\mathfrak{g}}, \tag{70}$$

where $\tilde{\mathfrak{g}}$ is a matrix loop algebra based on a given matrix Lie algebra $\mathfrak{g}$, u, the dependent variable, and λ, the spectral parameter. We then assume the solution to the stationary zero curvature equation,

$$W_x = [U, W], \tag{71}$$

is of the form

$$W = W(u, \lambda) = \sum_{i \geq 0} W_i \lambda^{-i}, \quad W_i \in \mathfrak{g}, \quad i \geq 0, \tag{72}$$

and further introduce the Lax matrices,

$$V^{[m]} = V^{[m]}(u, \lambda) = (\lambda^m W)_+ + \Delta_m \in \tilde{\mathfrak{g}}, \quad m \geq 0, \tag{73}$$

where $(\lambda^m W)_+$ denotes the polynomial part of $\lambda^m W$ and $\Delta_m \in \tilde{\mathfrak{g}}$ is a modification term, to engender the temporal spectral problems,

$$\phi_{t_m} = V^{[m]}\phi = V^{[m]}(u,\lambda)\phi, \quad m \geq 0. \tag{74}$$

The temporal spectral problems (74) with the spacial spectral problem (70) formulate the zero curvature equations,

$$U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0, \quad m \geq 0. \tag{75}$$

For each m, the Lax pair U and $V^{[m]}$ are equivalent to the hierarchy of evolution equations,

$$u_{t_m} = K_m(u), \quad m \geq 0. \tag{76}$$

We call this a *soliton hierarchy* since each evolution equation (76) is a soliton equation. After the soliton hierarchy is found, the Hamiltonian structure of (76) is calculated by the trace identity [29, 31, 64] (in [64], the formal for γ is given),

$$\frac{\delta}{\delta u}\int \mathrm{tr}\left(\frac{\partial U}{\partial \lambda}W\right)dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\mathrm{tr}\left(\frac{\partial U}{\partial u}W\right), \gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\mathrm{tr}(W^2)|, \tag{77}$$

producing a hierarchy of Hamiltonian equations,

$$u_{t_m} = K_m(u) = J\frac{\delta \mathcal{H}_m}{\delta u}, \quad m \geq 0. \tag{78}$$

2.4 Two integrable soliton hierachies and their Hamiltonian structures

2.4.1 A simple matrix loop algebra

We must begin by introducting a simple matrix loop algebra. The two spectral matrices in this section are associated with $\mathrm{sl}(2,\mathbb{R})$, a three-dimensional special linear Lie algebra consisting of trace-free 2×2 matrices. The basis for this simple Lie algebra is

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, e_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \tag{79}$$

with commutator properties

$$[e_1, e_2] = 2e_2, \quad [e_1, e_3] = -2e_3, \quad [e_2, e_3] = e_1. \tag{80}$$

To generate the hierarchy, we use the following matrix loop algebra:

$$\widetilde{\mathrm{sl}}(2, \mathbb{R}) = \left\{ \sum_{i \geq 0} A_i \lambda^{n-i} | A_i \in \mathrm{sl}(2, \mathbb{R}), n \in \mathbb{Z} \right\}. \tag{81}$$

In particular, the matrix loop algebra $\widetilde{\mathrm{sl}}(2, \mathbb{R})$ contains elements of the form $\lambda^m e_1 + \lambda^l e_2 + \lambda^p e_3$ with arbitrary integers m, l, p. Many well-known soliton hierarchies are generated from the matrix loop algebra $\widetilde{\mathrm{sl}}(2, \mathbb{R})$ (see, e.g., [6], [9], [22]- [24]).

2.4.2 A generalized D-Kaup-Newell spectral problem

Let us introduce a spectral matrix [21]:

$$U = U(u, \lambda) = (\lambda^2 - r)e_1 + (\lambda p + s)e_2 + (\lambda q + v)e_3 = \begin{bmatrix} \lambda^2 - r & \lambda p + s \\ \lambda q + v & -\lambda^2 + r \end{bmatrix}, \tag{82}$$

and consider the following isospectral problem:

$$\phi_x = U\phi = \begin{bmatrix} \lambda^2 - r & \lambda p + s \\ \lambda q + v & -\lambda^2 + r \end{bmatrix} \phi,\ U \in \widetilde{\mathrm{sl}}(2,\mathbb{R}),\ u = \begin{bmatrix} p \\ q \\ r \\ s \\ v \end{bmatrix},\ \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}, \tag{83}$$

where $p, q, r, s,$ and v are potentials.

The D-Kaup-Newell spectral problem is known [23, 49] to be

$$\phi_x = U\phi = \begin{bmatrix} \lambda^2 + r & \lambda p \\ \lambda q & -\lambda^2 - r \end{bmatrix} \phi, \quad U \in \widetilde{\mathrm{sl}}(2,\mathbb{R}), \quad u = \begin{bmatrix} p \\ q \\ r \end{bmatrix}, \quad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}, \tag{84}$$

which depends on three potentials: $p, q,$ and r. The new spectral problem (83) is a generalization of the D-Kaup-Newell spectral problem adding two new potentials s and v. Previously, the cases $r = \alpha$ and $r = \alpha pq$, where α a constant, have been shown to generate integrable hierarchies [32, 33] for the D-Kaup-Newell spectral problem (84).

2.4.3 The soliton hierarchy

We assume a solution to the stationary zero curvature equation, $W_x = [U, W]$, to be of the form

$$W = ae_1 + be_2 + ce_3 = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \in \widetilde{\mathrm{sl}}(2,\mathbb{R}), \tag{85}$$

and we get the equations

$$\begin{cases} a_x = -qb\lambda + pc\lambda - vb + sc, \\ b_x = -2pa\lambda + 2b\lambda^2 - 2sa - 2rb, \\ c_x = 2qa\lambda - 2c\lambda^2 + 2va + 2rc. \end{cases} \tag{86}$$

Next, we assume that a, b, and c have Laurent expansions

$$a = \sum_{i\geq 0} a_i\lambda^{-i}, \quad b = \sum_{i\geq 0} b_i\lambda^{-i}, \quad c = \sum_{i\geq 0} c_i\lambda^{-i}, \tag{87}$$

which gives us the following recursive relations:

$$\begin{cases} b_{i+1} = \dfrac{b_{i-1,x}}{2} + pa_i + sa_{i-1} + rb_{i-1}, \\ c_{i+1} = -\dfrac{c_{i-1,x}}{2} + qa_i + va_{i-1} + rc_{i-1}, \qquad i \geq 1, \\ a_{i+1,x} = -q\dfrac{b_{i,x}}{2} - p\dfrac{c_{i,x}}{2} + (pv - qs)a_i - qrb_i + prc_i + sc_{i+1} - vb_{i+1}. \end{cases} \tag{88}$$

Now, we take the initial values of

$$a_0 = \alpha, \quad b_0 = c_0 = 0, \quad a_1 = 0, \quad b_1 = \alpha p, \quad c_1 = \alpha q, \tag{89}$$

and impose the conditions for integration

$$a_i|_{u=0} = b_i|_{u=0} = c_i|_{u=0} = 0, \quad i \geq 1, \tag{90}$$

to determine the sequence of $\{a_i, b_i, c_i | i \geq 1\}$ uniquely. For $i = 2, 3$, we have the following:

$b_2 = \alpha s, \quad c_2 = \alpha v, \quad a_2 = -\alpha\frac{1}{2}pq$

$b_3 = \alpha\frac{1}{2}(-p^2q + 2pr + p_x), \quad c_3 = -\alpha\frac{1}{2}(q^2p - 2qr + q_x),$

$a_3 = -\alpha\frac{1}{2}(pv + qs).$

We can see the localness of $\{a_i, b_i, c_i | 0 \le i \le 3\}$. This result can be proven for all $i \ge 0$ as follows:

PROPOSITION 2.1. *Let* $\{a_0, b_0, c_0, a_1, b_1, c_1\}$ *be given by equation (89). Then all functions* $\{a_i, b_i, c_i, i \ge 2\}$ *determined by equation (88) with the conditions in (90) are differential polynomials in* u *with respect to* x*, and thus, are local.*

Proof. We compute from the stationary zero curvature equation, $W_x = [U, W]$,

$$\frac{d}{dx}\mathrm{tr}(W^2) = 2\mathrm{tr}(WW_x) = 2\mathrm{tr}(W[U,W]) = 2(\mathrm{tr}(W^2U) - \mathrm{tr}(W^2U)) = 0, \tag{91}$$

and seeing that the $\mathrm{tr}(W^2) = 2(a^2 + bc)$, we have

$$a^2 + bc = (a^2 + bc)|_{u=0} = \alpha^2, \tag{92}$$

following from the initial data (89). Now, we use (87), the Laurent expansions of a, b, c, to give

$$a_i = \frac{\alpha}{2} - \frac{1}{2\alpha}\sum_{k+l=i,k,l\ge 1} a_k a_l - \frac{1}{2\alpha}\sum_{k+l=i,k,l\ge 0} b_k c_l, \quad i \ge 1. \tag{93}$$

Finally, based on the recursion relations (88) and (93), we use mathematical induction to see that all functions $\{a_i, b_i, c_i, i \ge 0\}$ are differential polynomials in u with respect to x, and therefore, are local. This completes the proof. □

Now, we need to solve the zero curvature equations,

$$U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0, \quad m \ge 0, \tag{94}$$

which is equivalent to (83) together with the temporal problems,

$$\phi_{t_m} = V^{[m]}\phi = V^{[m]}(u,\lambda)\phi, \quad m \geq 0. \tag{95}$$

In order to solve these, we introduce a series of Lax operators,

$$V^{[m]} = (\lambda^m W)_+ + \Delta_m \in \widetilde{\mathrm{sl}}(2,\mathbb{R}), \tag{96}$$

where $(\lambda^m W)_+$ denotes the polynomial part of $\lambda^m W$ and Δ_m is a modification term. Let

$$\Delta_m = \begin{bmatrix} \delta_{a,m} & \delta_{b,m} \\ \delta_{c,,m} & -\delta_{a,,m} \end{bmatrix}. \tag{97}$$

For ease of calculation, let's drop the m's in the matrix entries of Δ_m and simplify $V_x - [U, V^{[m]}]$:

$$\begin{aligned}
&V_x^{[m]} - [U, V^{[m]}] = \\
&\sum_{i=0}^{m} \lambda^{m-i} \begin{bmatrix} a_{i,x} & b_{i,x} \\ c_{i,x} & -a_{i,x} \end{bmatrix} + \begin{bmatrix} \delta_{a,x} & \delta_{b,x} \\ \delta_{c,x} & -\delta_{a,x} \end{bmatrix} \\
&- \left[\begin{bmatrix} \lambda^2 - r & \lambda p + s \\ \lambda q + v & -\lambda^2 + r \end{bmatrix}, \lambda^{m-i} \begin{bmatrix} a_i & b_i \\ c_i & -a_i \end{bmatrix} + \begin{bmatrix} \delta_a & \delta_b \\ \delta_c & -\delta_a \end{bmatrix} \right] = \\
&\sum_{i=0}^{m} \lambda^{m-i} \begin{bmatrix} a_{i,x} & b_{i,x} \\ c_{i,x} & -a_{i,x} \end{bmatrix} + \begin{bmatrix} \delta_{a,x} & \delta_{b,x} \\ \delta_{c,x} & -\delta_{a,x} \end{bmatrix} \\
&- \sum_{i=0}^{m} \lambda^{m-i} \begin{bmatrix} -qb_i\lambda + pc_i\lambda - vb_i + sc_i & -2pa_i\lambda + 2b_i\lambda^2 - 2sa_i - 2rb_i \\ 2qa_i\lambda - 2c_i\lambda^2 + 2va_i + 2rc_i & qb_i\lambda - pc_i\lambda + vb_i - sc_i \end{bmatrix} \\
&- \begin{bmatrix} -q\delta_b\lambda + p\delta_c\lambda - v\delta_b + s\delta_c & -2p\delta_a\lambda + 2\delta_b\lambda^2 - 2s\delta_a - 2r\delta_b \\ 2q\delta_a\lambda - 2\delta_c\lambda^2 + 2v\delta_a + 2r\delta_c & q\delta_b\lambda - p\delta_c\lambda + v\delta_b - s\delta_c \end{bmatrix}
\end{aligned} \tag{98}$$

Plugging in (94) in (98), we get three equations, namely,

$$
\begin{aligned}
r_{t_m} &= \sum_{i=0}^{m} \lambda^{m-i}(-a_{i,x}) - qb_i\lambda^{m-i+1} + pc_i\lambda^{m-i+1} - vb_i\lambda^{m-i} + sc_i\lambda^{m-i} \\
&- \delta_{a,x} - q\delta_b\lambda + p\delta_c\lambda - v\delta_b + s\delta_c, \\
\lambda p_{t_m} + s_{t_m} &= \sum_{i=0}^{m} \lambda^{m-i}b_{i,x} + 2pa_i\lambda^{m-i+1} - 2b_i\lambda^{m-i+2} + 2sa_i\lambda^{m-i} + 2rb_i\lambda^{m-i} \\
&+ \delta_{b,x} + 2p\delta_a\lambda - 2\delta_b\lambda^2 + 2s\delta_a + 2r\delta_b, \\
\lambda q_{t_m} + v_{t_m} &= \sum_{i=0}^{m} \lambda^{m-i}c_{i,x} - 2qa_i\lambda^{m-i+1} + 2c_i\lambda^{m-i+2} - 2va_i\lambda^{m-i} - 2rc_i\lambda^{m-i} \\
&+ \delta_{c,x} - 2q\delta_a\lambda + 2\delta_c\lambda^2 - 2v\delta_a - 2r\delta_c.
\end{aligned}
\tag{99}
$$

We plug the recursion relations (86) and (88) into (99) and get

$$
\begin{aligned}
r_{t_m} &= \sum_{i=0}^{m} \lambda^{m-i}(-a_{i,x}) - qb_i\lambda^{m-i+1} + pc_i\lambda^{m-i+1} - vb_i\lambda^{m-i} + sc_i\lambda^{m-i} \\
&- \delta_{a,x} - q\delta_b\lambda + p\delta_c\lambda - v\delta_b + s\delta_c \\
&= \sum_{i=0}^{m} \lambda^{m-i}(qb_{i+1} - pc_{i+1} + vb_i - sc_i) - qb_i\lambda^{m-i+1} + pc_i\lambda^{m-i+1} \\
&- vb_i\lambda^{m-i} + sc_i\lambda^{m-i} - \delta_{a,x} - q\delta_b\lambda + p\delta_c\lambda - v\delta_b + s\delta_c \\
&= \sum_{i=0}^{m} \lambda^{m-i}(qb_{i+1} - pc_{i+1}) - \sum_{i=0}^{m} \lambda^{m-i+1}(qb_i - pc_i) \\
&- \delta_{a,x} - q\delta_b\lambda + p\delta_c\lambda - v\delta_b + s\delta_c, \\
&= \sum_{i=1}^{m+1} \lambda^{m-i+1}(qb_i - pc_i) - \sum_{i=0}^{m} \lambda^{m-i+1}(qb_i - pc_i) - \delta_{a,x} - q\delta_b\lambda \\
&+ p\delta_c\lambda - v\delta_b + s\delta_c, \\
&= qb_{m+1} - pc_{m+1} - \lambda^{m+1}(qb_0 - pc_0) - \delta_{a,x} - q\delta_b\lambda + p\delta_c\lambda - v\delta_b + s\delta_c, \\
&= qb_{m+1} - pc_{m+1} - \delta_{a,x} - q\delta_b\lambda + p\delta_c\lambda - v\delta_b + s\delta_c, \\
\lambda p_{t_m} + s_{t_m} &= \sum_{i=0}^{m} \lambda^{m-i}b_{i,x} + 2pa_i\lambda^{m-i+1} - 2b_i\lambda^{m-i+2} + 2sa_i\lambda^{m-i} + 2rb_i\lambda^{m-i} \\
&+ \delta_{b,x} + 2p\delta_a\lambda - 2\delta_b\lambda^2 + 2s\delta_a + 2r\delta_b \\
&= \sum_{i=0}^{m} \lambda^{m-i}(-2pa_{i+1} + 2b_{i+2} - 2sa_i - 2rb_i) + 2pa_i\lambda^{m-i+1} - 2b_i\lambda^{m-i+2} \\
&+ 2sa_i\lambda^{m-i} + 2rb_i\lambda^{m-i} + \delta_{b,x} + 2p\delta_a\lambda - 2\delta_b\lambda^2 + 2s\delta_a + 2r\delta_b \\
&= -\sum_{i=0}^{m} \lambda^{m-i}2pa_{i+1} + \sum_{i=0}^{m} \lambda^{m-i}2b_{i+2} + \sum_{i=0}^{m} 2pa_i\lambda^{m-i+1} - \sum_{i=0}^{m} 2b_i\lambda^{m-i+2} \\
&+ \delta_{b,x} + 2p\delta_a\lambda - 2\delta_b\lambda^2 + 2s\delta_a + 2r\delta_b \\
&= -\sum_{i=1}^{m+1} \lambda^{m-i+1}2pa_i + \sum_{i=2}^{m+2} \lambda^{m-i+2}2b_i + \sum_{i=0}^{m} 2pa_i\lambda^{m-i+1} - \sum_{i=0}^{m} 2b_i\lambda^{m-i+2} \\
&+ \delta_{b,x} + 2p\delta_a\lambda - 2\delta_b\lambda^2 + 2s\delta_a + 2r\delta_b \\
&= \lambda^{m+1}2pa_0 - 2pa_{m+1} - \lambda^{m+2}2b_0 - \lambda^{m+1}2b_1 + \lambda 2b_{m+1} + 2b_{m+2} \\
&+ \delta_{b,x} + 2p\delta_a\lambda - 2\delta_b\lambda^2 + 2s\delta_a + 2r\delta_b \\
&= \lambda 2b_{m+1} - 2pa_{m+1} + 2b_{m+2} + \delta_{b,x} + 2p\delta_a\lambda - 2\delta_b\lambda^2 + 2s\delta_a + 2r\delta_b,
\end{aligned}
$$

$$\begin{aligned}
\lambda q_{t_m} + v_{t_m} &= \sum_{i=0}^{m} \lambda^{m-i} c_{i,x} - 2qa_i\lambda^{m-i+1} + 2c_i\lambda^{m-i+2} - 2va_i\lambda^{m-i} - 2rc_i\lambda^{m-i} \\
&+ \delta_{c,x} - 2q\delta_a\lambda + 2\delta_c\lambda^2 - 2v\delta_a - 2r\delta_c \\
&= \sum_{i=0}^{m} \lambda^{m-i}(2qa_{i+1} - 2c_{i+2} + 2va_i + 2rc_i) - 2qa_i\lambda^{m-i+1} + 2c_i\lambda^{m-i+2} \\
&- 2va_i\lambda^{m-i} - 2rc_i\lambda^{m-i} \\
&+ \delta_{c,x} - 2q\delta_a\lambda + 2\delta_c\lambda^2 - 2v\delta_a - 2r\delta_c \\
&= \sum_{i=0}^{m} \lambda^{m-i} 2qa_{i+1} - \sum_{i=0}^{m} \lambda^{m-i} 2c_{i+2} - \sum_{i=0}^{m} 2qa_i\lambda^{m-i+1} + \sum_{i=0}^{m} 2b_i\lambda^{m-i+2} \\
&+ \delta_{c,x} - 2q\delta_a\lambda + 2\delta_c\lambda^2 - 2v\delta_a - 2r\delta_c \\
&= \sum_{i=1}^{m+1} \lambda^{m-i+1} 2qa_i - \sum_{i=2}^{m+2} \lambda^{m-i+2} 2c_i - \sum_{i=0}^{m} 2qa_i\lambda^{m-i+1} + \sum_{i=0}^{m} 2c_i\lambda^{m-i+2} \\
&+ \delta_{c,x} - 2q\delta_a\lambda + 2\delta_c\lambda^2 - 2v\delta_a - 2r\delta_c \\
&= -2qa_0\lambda^{m+1} + 2qa_{m+1} - \lambda^{m+2}2c_0 + \lambda^{m+1}2c_1 - \lambda 2c_{m+1} - 2c_{m+2} \\
&+ \delta_{c,x} - 2q\delta_a\lambda + 2\delta_c\lambda^2 - 2v\delta_a - 2r\delta_c \\
&= -\lambda 2c_{m+1} + 2qa_{m+1} - 2c_{m+2} + \delta_{c,x} - 2q\delta_a\lambda + 2\delta_c\lambda^2 - 2v\delta_a - 2r\delta_c,
\end{aligned}$$

Comparing coefficients for powers of λ, we find $\delta_b = \delta_c = 0$ and the following five equations:

$$\begin{cases}
p_{t_m} = 2b_{m+1} + 2p\delta_a, \\
q_{t_m} = -2c_{m+1} - 2q\delta_a, \\
r_{t_m} = qb_{m+1} - pc_{m+1} - \delta_{a,x}, \\
s_{t_m} = -2pa_{m+1} + 2b_{m+2} + 2s\delta_a, \\
v_{t_m} = 2qa_{m+1} - 2c_{m+2} - 2v\delta_a.
\end{cases} \tag{100}$$

For simplicity, assume $\delta_a = 0$ to generate a hierarchy of soliton equations

$$u_{t_m} = \begin{bmatrix} p_{t_m} \\ q_{t_m} \\ r_{t_m} \\ s_{t_m} \\ v_{t_m} \end{bmatrix} = K_m = \begin{bmatrix} 2b_{m+1} \\ -2c_{m+1} \\ qb_{m+1} - pc_{m+1} \\ -2pa_{m+1} + 2b_{m+2} \\ 2qa_{m+1} - 2c_{m+2} \end{bmatrix} = \Phi \begin{bmatrix} 2b_m \\ -2c_m \\ qb_m - pc_m \\ -2pa_m + 2b_{m+1} \\ 2qa_m - 2c_{m+1} \end{bmatrix}, \quad m \geq 0. \tag{101}$$

Φ is a recursion operator determined by (88) and given by

$$\Phi = \begin{bmatrix} -p\partial^{-1}v - s\partial^{-1}q & -p\partial^{-1}s - s\partial^{-1}p & 2s\partial^{-1} & 1 - p\partial^{-1}q & -p\partial^{-1}p \\ q\partial^{-1}v + v\partial^{-1}q & q\partial^{-1}s + v\partial^{-1}p & -2v\partial^{-1} & q\partial^{-1}q & 1 + q\partial^{-1}p \\ (pv-qs)\partial^{-1}\frac{q}{2} & (pv-qs)\partial^{-1}\frac{p}{2} & -(pv-qs)\partial^{-1} & \frac{q}{2} & \frac{p}{2} \\ \begin{matrix}\frac{1}{2}\partial + r - s\partial^{-1}v \\ -pr\partial^{-1}q - \partial p\partial^{-1}\frac{q}{2}\end{matrix} & \begin{matrix}-s\partial^{-1}s - pr\partial^{-1}p \\ -\partial p\partial^{-1}\frac{p}{2}\end{matrix} & \begin{matrix}2pr\partial^{-1} \\ +\partial p\partial^{-1}\end{matrix} & -s\partial^{-1}q & -s\partial^{-1}p \\ \begin{matrix}v\partial^{-1}v + qr\partial^{-1}q \\ -\partial q\partial^{-1}\frac{q}{2}\end{matrix} & \begin{matrix}-\frac{1}{2}\partial + r + v\partial^{-1}s \\ +qr\partial^{-1}p - \partial q\partial^{-1}\frac{p}{2}\end{matrix} & \begin{matrix}-2qr\partial^{-1} \\ +\partial q\partial^{-1}\end{matrix} & v\partial^{-1}q & v\partial^{-1}p \end{bmatrix} \tag{102}$$

with $\partial = \dfrac{\partial}{\partial x}$ and ∂^{-1} as the inverse operator of ∂.

Proposition 2.2. *All $\{K_m\}$, as defined by (101), are commuting symmetries, i.e.,*

$$[K_k, K_l] = K_k'(u)[K_l] - K_l'(u)[K_k] = 0, \quad k, l \geq 0. \tag{103}$$

Proof. The Gateaux derivative of U defined by (82) along any direction $S = (S_1, S_2, S_3, S_4, S_5)^T$ is

$$U'[S] = \begin{bmatrix} -S_3 & \lambda S_1 + S_4 \\ \lambda S_2 + S_5 & S_3 \end{bmatrix}.$$

This shows $U'[S] = 0$ if and only if $S = 0$ and $U'[S]$ is injective. According to the results in [27], for the evolution equations defined by

$$u_{t_m} = K_m(u),$$

represented by a zero curvature formulation

$$U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0,$$

we have the following:

$$U'[[K_m, K_n]] - [[V^{[m]}, V^{[n]}]]_x + [U, [[V^{[m]}, V^{[n]}]]] = 0, \tag{104}$$

where $[[V^{[m]}, V^{[n]}]]$ is defined by

$$[[V^{[m]}, V^{[n]}]] := V^{[m]\prime}(u)[K_n] - V^{[n]\prime}(u)[K_m] + [V^{[m]}, V^{[n]}], \quad m, n \geq 0. \tag{105}$$

We will first show that

$$[[V^{[m]}, V^{[n]}]]|_{u=0} = 0. \tag{106}$$

By inspection, we see by (88) and (101) that

$$K_m|_{u=0} = 0, \quad \text{for} \quad m \geq 0,$$

and, therefore,

$$V^{[m]\prime}(u)[K_n] - V^{[n]\prime}(u)[K_m]|_{u=0} = 0, \quad \text{for} \quad m, n \geq 0.$$

Also, restricting the temporal spectral matrices to the case when $u = 0$ for all m, we have

$$V^{[m]}|_{u=0} = \begin{bmatrix} \alpha\lambda^m & 0 \\ 0 & -\alpha\lambda^m \end{bmatrix}. \tag{107}$$

Thus, it is easy to see that

$$[V^{[m]}, V^{[n]}]|_{u=0} = 0,$$

and (106) is proved.

We have the uniqueness property of the spectral problem presented: If a matrix solution $W \in \widetilde{\mathrm{sl}}(2, \mathbb{R})$ to the equation $U'[K] - W_x + [U, W] = 0$ with some vector-valued function K satisfies $W|_{u=0} = 0$, then $W = 0$ [43]. By the uniqueness property of the spectral problem, we have $[[V^{[m]}, V^{[n]}]] = 0$. This leaves

$$U'[[K_m, K_n]] = 0. \tag{108}$$

Recall, $U'[\cdot]$ is injective. Therefore, (108) implies

$$[K_m, K_n] = 0, \quad m, n \geq 0, \tag{109}$$

and the proposition is proved. □

Let's look at some of the reductions of this system. First, when $m = 6$, we may let $r, p, q =$

$0, v = -1, s = u$ and $\alpha = -4$ to find the KdV equation [3]

$$u_t = -6uu_x - u_{xxx}. \tag{110}$$

Also for $m = 6$, we let $r, p, q = 0, v = -u, s = u$ and $\alpha = -4$ to find the mKdV [34, 35]

$$u_t = -6u^2u_x - u_{xxx}. \tag{111}$$

We may see the NLSE [22] by letting $r, p, q = 0, v = -\bar{u}, s = u$ and $\alpha = -2i$ when $m = 4$

$$iu_t = 2|u|^2u + u_{xx}. \tag{112}$$

Lastly, when $m = 4$, we can see the Gerdjikov-Ivanov (G-I) equations [36, 37]

$$\begin{cases} q_t = \frac{1}{2}q^3r^2 - q^2r_x - q_{xx} \\ r_t = -\frac{1}{2}r^3q^2 - r^2q_x + r_{xx}, \end{cases} \tag{113}$$

if we let $r = -2qr, p = q, q = r, s = v = 0$ and $\alpha = -2$. The G-I equation is the third derivative nonlinear Schrödinger equation (DNLSIII) [37], or equivalently,

$$iq_t = -\frac{1}{2}q^3\bar{q}^2 + iq^2\bar{q}_x - q_{xx}. \tag{114}$$

2.4.4 Hamiltonian structure

The Hamiltonian structure for the hierarchy (101) is found using the trace identity [29, 31]. We compute

$$\begin{cases} \dfrac{\partial U}{\partial p} = \begin{bmatrix} 0 & \lambda \\ 0 & 0 \end{bmatrix}, \dfrac{\partial U}{\partial q} = \begin{bmatrix} 0 & 0 \\ \lambda & 0 \end{bmatrix}, \dfrac{\partial U}{\partial r} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \\ \dfrac{\partial U}{\partial s} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \dfrac{\partial U}{\partial v} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \dfrac{\partial U}{\partial \lambda} = \begin{bmatrix} 2\lambda & p \\ q & -2\lambda \end{bmatrix}. \end{cases} \tag{115}$$

The following traces are generated:

$$\begin{cases} \operatorname{tr}\left(W\frac{\partial U}{\partial \lambda}\right) = 4a\lambda + bq + cp, \operatorname{tr}\left(W\frac{\partial U}{\partial p}\right) = c\lambda, \operatorname{tr}\left(W\frac{\partial U}{\partial q}\right) = b\lambda, \\ \operatorname{tr}\left(W\frac{\partial U}{\partial r}\right) = -2a, \operatorname{tr}\left(W\frac{\partial U}{\partial s}\right) = c, \operatorname{tr}\left(W\frac{\partial U}{\partial v}\right) = b, \end{cases} \tag{116}$$

and applying the trace identity (77), we have

$$\frac{\delta}{\delta u}\int (4a\lambda + bq + cp)dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}[c\lambda, b\lambda, -2a, c, b]^T. \tag{117}$$

After balancing the coefficients of all powers of λ,

$$\frac{\delta}{\delta u}\int (4a_{m+2} + b_{m+1}q + c_{m+1}p)dx = (\gamma - m)[c_{m+1}, b_{m+1}, -2a_m, c_m, b_m]^T, \quad m \geq 0.$$

Considering the case where $m = 1$, we see $\gamma = 0$, and we have the following:

$$\frac{\delta}{\delta u}\int -\frac{4a_{m+2} + b_{m+1}q + c_{m+1}p}{m}dx = [c_{m+1}, b_{m+1}, -2a_m, c_m, b_m]^T, \quad m \geq 1. \tag{118}$$

A cumbersome calculation involving the recursion relations (88) show that

$$\frac{\delta \mathcal{H}_{m+1}}{\delta u} = \begin{bmatrix} c_{m+2} \\ b_{m+2} \\ -2a_{m+1} \\ c_{m+1} \\ b_{m+1} \end{bmatrix} = \Psi \begin{bmatrix} c_{m+1} \\ b_{m+1} \\ -2a_{m} \\ c_{m} \\ b_{m} \end{bmatrix} = \Psi \frac{\delta \mathcal{H}_{m}}{\delta u}, \tag{119}$$

where

$$\Psi = \Phi^{\dagger} = \begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} & \Psi_{15} \\ \Psi_{21} & \Psi_{22} & \Psi_{23} & \Psi_{24} & \Psi_{25} \\ \Psi_{31} & \Psi_{32} & \Psi_{33} & \Psi_{34} & \Psi_{35} \\ \Psi_{41} & \Psi_{42} & \Psi_{43} & \Psi_{44} & \Psi_{45} \\ \Psi_{51} & \Psi_{52} & \Psi_{53} & \Psi_{54} & \Psi_{55} \end{bmatrix}, \tag{120}$$

and

$$\begin{aligned}
&\Psi_{11} = v\partial^{-1}p + q\partial^{-1}s, \Psi_{12} = -v\partial^{-1}q - q\partial^{-1}v, \Psi_{13} = -\frac{q}{2}\partial^{-1}(pv - qs), \\
&\Psi_{14} = -\frac{1}{2}\partial + r + v\partial^{-1}s + q\partial^{-1}pr - \frac{q}{2}\partial^{-1}p\partial, \Psi_{15} = -v\partial^{-1}v - q\partial^{-1}qr - \frac{q}{2}\partial^{-1}q\partial, \\
&\Psi_{21} = s\partial^{-1}p + p\partial^{-1}s, \Psi_{22} = -s\partial^{-1}q - p\partial^{-1}v, \Psi_{23} = -\frac{p}{2}\partial^{-1}(pv - qs), \\
&\Psi_{24} = s\partial^{-1}s + p\partial^{-1}pr - \frac{p}{2}\partial^{-1}p\partial, \Psi_{25} = \frac{1}{2}\partial + r - s\partial^{-1}v - p\partial^{-1}qr - \frac{p}{2}\partial^{-1}q\partial, \\
&\Psi_{31} = -2\partial^{-1}s, \Psi_{32} = 2\partial^{-1}v, \Psi_{33} = \partial^{-1}(pv - qs), \\
&\Psi_{34} = -2\partial^{-1}pr + \partial^{-1}p\partial, \Psi_{35} = 2\partial^{-1}qr + \partial^{-1}q\partial, \\
&\Psi_{41} = 1 + q\partial^{-1}p, \Psi_{42} = -q\partial^{-1}q, \Psi_{43} = \frac{q}{2}, \Psi_{44} = q\partial^{-1}s, \Psi_{45} = -q\partial^{-1}v, \\
&\Psi_{51} = p\partial^{-1}p, \Psi_{52} = 1 - p\partial^{-1}q, \Psi_{53} = \frac{p}{2}, \Psi_{54} = p\partial^{-1}s, \Psi_{55} = -p\partial^{-1}v.
\end{aligned}$$

Note, Φ is from (102). Before we show the Hamiltonian structure for the soliton hierarchy (101), we must prove the following proposition.

PROPOSITION 2.3. *The operator J defined by*

$$J = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\partial & s & -v \\ 0 & 0 & -s & 0 & \partial + 2r \\ 0 & 0 & v & \partial - 2r & 0 \end{bmatrix} \tag{121}$$

is a Hamiltonian operator.

Proof. We can easily see by inspection that $J^\dagger = -J$, where $J^\dagger$ is the adjoint of J, and J is skew-adjoint. Therefore, we must only prove that J satisfies the equivalent formulation Jacobi condition

$$< Z, J'(u)[JX]Y > + \operatorname{cycle}(X, Y, Z) \equiv 0 \ (\operatorname{mod} \partial), \tag{122}$$

for all vector fields X, Y, and Z, where $J'(u)[X]$ denotes the Gateaux derivative of J with respect to u in the direction of X and $< \cdot, \cdot >$, the standard inner product. Assume

$$X = (X_1, X_2, X_3, X_4, X_5)^T, \quad Y = (Y_1, Y_2, Y_3, Y_4, Y_5)^T, \quad Z = (Z_1, Z_2, Z_3, Z_4, Z_5)^T,$$
$$W = (W_1, W_2, W_3, W_4, W_5)^T,$$

are five-dimensional vector functions. By (121), we immediately have

$$JX = \begin{bmatrix} 2X_2 \\ -2X_1 \\ \frac{1}{2}X_{3,x} + sX_4 - vX_5 \\ -sX_3 + X_{5,x} + 2rX_5 \\ vX_3 + X_{4,x} - 2rX_4 \end{bmatrix} := \begin{bmatrix} W_1(X) \\ W_1(X) \\ W_1(X) \\ W_1(X) \\ W_1(X) \end{bmatrix} = W.$$

Using the definition of the Gateaux derivative, we compute $J'[W]$ and, then, $J'[W]Y$ as:

$$J'[W] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & W_4 & -W_5 \\ 0 & 0 & -W_4 & 0 & 2W_3 \\ 0 & 0 & W_5 & -2W_3 & 0 \end{bmatrix},$$

and

$$J'[W]Y = \begin{bmatrix} 0 \\ 0 \\ W_4Y_4 - W_5Y_5 \\ -W_4Y_3 + 2W_3Y_5 \\ W_5Y_3 - 2W_3Y_4 \end{bmatrix},$$

Now, we see that

$$
\begin{aligned}
< Z, J'[W]Y > &= 2W_3(Z_4Y_5 - Z_5Y_4) + W_4(Z_3Y_4 - Z_4Y_3) + W_5(Z_5Y_3 - Z_3Y_5) \\
&= 2(\frac{1}{2}X_{3,x} + sX_4 - vX_5)(Z_4Y_5 - Z_5Y_4) \\
&+ (-sX_3 + X_{5,x} + 2rX_5)(Z_3Y_4 - Z_4Y_3) \\
&+ (vX_3 + X_{4,x} - 2rX_4)(Z_5Y_3 - Z_3Y_5) \\
&= [X_{3,x}(Z_4Y_5 - Z_5Y_4) + X_{4,x}(Z_5Y_3 - Z_3Y_5) + X_{5,x}(Z_3Y_4 - Z_4Y_3)] \\
&+ [2(sX_4 - vX_5)(Z_4Y_5 - Z_5Y_4) + (-sX_3 + 2rX_5)(Z_3Y_4 - Z_4Y_3) \\
&+ (vX_3 - 2rX_4)(Z_5Y_3 - Z_3Y_5)].
\end{aligned}
$$

We may make the following decomposition:

$$
< Z, J'[W]Y >= R(X, Y, Z) + S(X, Y, Z), \tag{123}
$$

where

$$
\begin{aligned}
R(X, Y, Z) &= X_{3,x}(Z_4Y_5 - Z_5Y_4) + X_{4,x}(Z_5Y_3 - Z_3Y_5) + X_{5,x}(Z_3Y_4 - Z_4Y_3), \\
S(X, Y, Z) &= 2(sX_4 - vX_5)(Z_4Y_5 - Z_5Y_4) + (-sX_3 + 2rX_5)(Z_3Y_4 - Z_4Y_3) \\
&+ (vX_3 - 2rX_4)(Z_5Y_3 - Z_3Y_5).
\end{aligned}
$$

For the functions R and S, we may make the following computation:

$$\begin{aligned} & R(X,Y,Z) + \mathrm{cycle}(X,Y,Z) \\ =& \partial(X_3Z_4Y_5 - X_3Y_4Z_5) + \mathrm{cycle}(X,Y,Z), \\ & S(X,Y,Z) + \mathrm{cycle}(X,Y,Z) \\ =& \partial\{\partial^{-1}[2(sX_4 - cX_5)(Z_4Y_5 - Z_5Y_4) + (-sX_3 + 2rX_5)(Z_3Y_4 - Z_4Y_3) \\ & + (vX_3 - 2rX_4)(Z_5Y_3 - Z_3Y_5)]\} + \mathrm{cycle}(X,Y,Z). \end{aligned} \tag{124}$$

Therefore, they are both total derivatives. Using (124), we see that J satisfies the Jacobi identity,

$$< Z, J'(u)[JX]Y > + \mathrm{cycle}(X,Y,Z) \equiv 0 \ (\mathrm{mod}\ \partial),$$

and J is a Hamiltonian operator. □

Thus, we arrive at the Hamiltonian structure for a generalized D-Kaup-Newell soliton hierarchy (101) [21]

$$u_{t_m} = K_m = \begin{bmatrix} 2b_{m+1} \\ -2c_{m+1} \\ -a_{m,x} - vb_m + sc_m \\ b_{m,x} + 2sa_m + 2rb_m \\ c_{m,x} - 2va_m - 2rc_m \end{bmatrix} = J \begin{bmatrix} c_{m+1} \\ b_{m+1} \\ -2a_m \\ c_m \\ b_m \end{bmatrix} = J\frac{\delta \mathcal{H}_m}{\delta u}, m \geq 0, \tag{125}$$

with the Hamiltonian operator

$$J = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\partial & s & -v \\ 0 & 0 & -s & 0 & \partial + 2r \\ 0 & 0 & v & \partial - 2r & 0 \end{bmatrix}, \tag{126}$$

and the Hamiltonian functionals

$$\mathcal{H}_0 = \int (-2\alpha r + \alpha pq) dx, \quad \mathcal{H}_m = \int -\frac{4a_{m+2} + b_{m+1}q + c_{m+1}p}{m} dx, \quad m \geq 1, \tag{127}$$

where $\mathcal{H}_0$ can be found directly from the vector $[c_1, b_1, -2a_0, c_0, b_0]^T$.

The above functionals (127) correspond to common conservation laws for each soliton system in the soliton hierarchy (101). Differential polynomial conservation laws can be generated either from some Riccati equation obtained from the underlying matrix spectral problem [39]- [41] or directly by a computer algebra system [42].

As a direct result of the Hamiltonian structures (125), the recursion structures (119) and (101), and the property $J\Psi = \Psi^\dagger J$, we can say that the soliton hierarchy (101) has infinitely many conserved functions in involution, i.e.,

$$\{\mathcal{H}_k, \mathcal{H}_l\}_J = \int \left(\frac{\delta \mathcal{H}_k}{\delta u}\right)^T J \frac{\delta \mathcal{H}_l}{\delta u} dx = 0, \quad k, l \geq 0. \tag{128}$$

and commutating symmetries,

$$[K_k, K_l] = K_k'(u)[K_l] - K_l'(u)[K_k] = 0, \quad k, l \geq 0. \tag{129}$$

Thus, the soliton hierarchy (101) is integrable in the Liouville sense. We would like to note that the commuting relations of the conserved functionals and symmetries are also consequences of the Virasoro algebra of Lax operators. For further reference on the algebraic structures of Lax operators and zero curvature equations, see [43]- [45].

We have an infinite set of functionals $\{\mathcal{H}_n\}$ which are involution in pairs, i.e., $\{\mathcal{H}_n, \mathcal{H}_m\}_J = 0$. Therefore, for the m-th system $u_{t_m} = J\frac{\delta \mathcal{H}_n}{\delta u}$, similar to an analysis in [64], we may compute

$$\begin{aligned}\frac{d}{dt_m}\mathcal{H}_n &= \int \frac{\delta \mathcal{H}_n}{\delta u} u_{t_m} dx \\ &= \int \frac{\delta \mathcal{H}_n}{\delta u} J \frac{\delta \mathcal{H}_m}{\delta u} \\ &= \{\mathcal{H}_n, \mathcal{H}_m\}_J = 0, \quad n \geq 0.\end{aligned}$$

Clearly, the soliton hierarchy (101) has infinitely many conserved quantities.

2.4.5 A bi-Hamiltonian reduced integrable hierarchy

Let's introduce a reduction of the spectral problem (82) [21]:

$$\phi_x = U(u,\lambda)\phi = \begin{bmatrix} \lambda^2 - \tilde{r} & \lambda p + s \\ \lambda q + v & -\lambda^2 + \tilde{r} \end{bmatrix} \phi,\ U \in \widetilde{\mathrm{sl}}(2,\mathbb{R}),\ u = \begin{bmatrix} p \\ q \\ s \\ v \end{bmatrix},\ \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}, \tag{130}$$

where $\tilde{r} = \frac{1}{2}pq$. We would like to construct its associated integrable soliton hierarchy possessing a bi-Hamiltonian structure. We will see that this spectral problem will generate a different soliton hierarchy than (101).

Assume a solution to stationary zero curvature equation to be of the same form,

$$W = ae_1 + be_2 + ce_3 = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \in \widetilde{\mathrm{sl}}(2, \mathbb{R}), \tag{131}$$

and we produce

$$\begin{cases} a_x = -qb\lambda + pc\lambda - vb + sc, \\ b_x = -2pa\lambda + 2b\lambda^2 - 2sa - pqb, \\ c_x = 2qa\lambda - 2c\lambda^2 + 2vb + pqc. \end{cases} \tag{132}$$

Taking the same Laurent expansions as before (87), the recursion relations are

$$\begin{cases} b_{i+1} = \dfrac{b_{i-1,x}}{2} + pa_i + sa_{i-1} + \frac{1}{2}pqb_{i-1}, \\ c_{i+1} = -\dfrac{c_{i-1,x}}{2} + qa_i + va_{i-1} + \frac{1}{2}pqc_{i-1}, \qquad i \geq 1, \\ a_{i+1,x} = -q\dfrac{b_{i,x}}{2} - p\dfrac{c_{i,x}}{2} + (pv - qs)a_i - \frac{1}{2}pq^2 b_i + \frac{1}{2}p^2 qc_i + sc_{i+1} - vb_{i+1}. \end{cases} \tag{133}$$

We note that we have a different set of recursion relations than those found in (88). In order to determine the sequence of $\{a_i, b_i, c_i | i \geq 1\}$ uniquely, we take the same initial values of

$$a_0 = \alpha, \quad b_0 = c_0 = 0, \quad a_1 = 0, \quad b_1 = \alpha p, \quad c_1 = \alpha q, \tag{134}$$

and conditions for integration

$$a_i|_{u=0} = b_i|_{u=0} = c_i|_{u=0} = 0, \quad i \geq 1. \tag{135}$$

Again, for all $i \geq 0$, $\{a_i, b_i, c_i\}$ are differential polynomials in u with respect to x.

PROPOSITION 2.4. *Let* $\{a_0, b_0, c_0, a_1, b_1, c_1\}$ *be given by equation (89). Then all functions* $\{a_i, b_i, c_i, i \geq$

2} *determined by equation (133) with the conditions in (90) are differential polynomials in* u *with respect to* x*, and thus, are local.*

Proof. We omit the proof as it is almost exactly the same as Proposition 2.1 but we use the recursion relations (133) instead of (88). □

After solving the zero curvature equation (94) and again taking the modification terms Δ_m to be zero, we generate a completely new soliton hierarchy

$$u_{t_m} = K_m = \begin{bmatrix} 2b_{m+1} \\ -2c_{m+1} \\ -2pa_{m+1} + 2b_{m+2} \\ 2qa_{m+1} - 2c_{m+2} \end{bmatrix} = \Phi \begin{bmatrix} 2b_m \\ -2c_m \\ -2pa_m + 2b_{m+1} \\ 2qa_m - 2c_{m+1} \end{bmatrix}, \quad m \geq 0, \tag{136}$$

with recursion operator

$$\Phi = \begin{bmatrix} -p\partial^{-1}v & -p\partial^{-1}s & 1 - p\partial^{-1}q & -p\partial^{-1}p \\ q\partial^{-1}v & q\partial^{-1}s & q\partial^{-1}q & 1 + q\partial^{-1}p \\ \frac{1}{2}\partial + \frac{1}{2}pq - s\partial^{-1}v & -s\partial^{-1}s & -s\partial^{-1}q & -s\partial^{-1}p \\ v\partial^{-1}v & -\frac{1}{2}\partial + \frac{1}{2}pq + v\partial^{-1}s & v\partial^{-1}q & v\partial^{-1}p \end{bmatrix}. \tag{137}$$

We may construct a Hamiltonian structure from the trace identity [29,31] for the above soliton hierarchy (136). We calculate

$$\frac{\partial U}{\partial p} = \begin{bmatrix} -\frac{1}{2}q & \lambda \\ 0 & \frac{1}{2}q \end{bmatrix}, \frac{\partial U}{\partial q} = \begin{bmatrix} -\frac{1}{2}p & 0 \\ \lambda & \frac{1}{2}p \end{bmatrix}, \tag{138}$$

and note $\dfrac{\partial U}{\partial s}, \dfrac{\partial U}{\partial v}$, and $\dfrac{\partial U}{\partial \lambda}$ are as the same as in (115). The new traces are found:

$$\mathrm{tr}\left(W\frac{\partial U}{\partial p}\right) = -aq + c\lambda, \quad \mathrm{tr}\left(W\frac{\partial U}{\partial q}\right) = -ap + b\lambda, \tag{139}$$

while $\mathrm{tr}\left(W\frac{\partial U}{\partial \lambda}\right), \mathrm{tr}\left(W\frac{\partial U}{\partial s}\right)$, and $\mathrm{tr}\left(W\frac{\partial U}{\partial v}\right)$ are as before in (116).

Plugging these into the trace identity (77), we have

$$\frac{\delta}{\delta u}\int (4a\lambda + bq + cp)dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}[-aq + c\lambda, -ap + b\lambda, c, b]^T. \tag{140}$$

After balancing the coefficients of all powers of λ and then considering the case where $m = 1$, we see $\gamma = 0$. We then have the following for $m \geq 1$:

$$\frac{\delta}{\delta u}\int -\frac{4a_{m+2} + b_{m+1}q + c_{m+1}p}{m}dx = [-a_m q + c_{m+1}, -a_m p + b_{m+1}, c_m, b_m]^T. \tag{141}$$

Therefore, we formulate the Hamiltonian structure for the reduced soliton hierarchy (136),

$$u_{t_m} = K_m = \begin{bmatrix} 2b_{m+1} \\ -2c_{m+1} \\ -2pa_{m+1} + 2b_{m+2} \\ 2qa_{m+1} - 2c_{m+2} \end{bmatrix} = J\begin{bmatrix} -a_{m+1}q + c_{m+2} \\ -a_{m+1}p + b_{m+2} \\ c_{m+1} \\ b_{m+1} \end{bmatrix} = J\frac{\delta \mathcal{H}_{m+1}}{\delta u}, m \geq 0, \tag{142}$$

with the Hamiltonian operator,

$$J = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{bmatrix}, \tag{143}$$

and the Hamiltonian functionals,

$$\mathcal{H}_m = \int -\frac{4a_{m+2} + b_{m+1}q + c_{m+1}p}{m} dx, \quad m \geq 1. \tag{144}$$

Obviously, J is skew-symmetric. The proof that J is Hamiltonian is simple as J doesn't depend on the potential u and the Gateaux derivative of J in any direction S with respect to u is $J'[S](u) = 0$.

A similar calculation as before but now involving the recursion relations (133) shows that

$$\frac{\delta \mathcal{H}_{m+1}}{\delta u} = \begin{bmatrix} -a_{m+1}q + c_{m+2} \\ -a_{m+1}p + b_{m+2} \\ c_{m+1} \\ b_{m+1} \end{bmatrix} = \Psi \begin{bmatrix} -a_m q + c_{m+1} \\ -a_m p + b_{m+1} \\ c_m \\ b_m \end{bmatrix} = \Psi \frac{\delta \mathcal{H}_m}{\delta u}, \tag{145}$$

where

$$\Psi = \Phi^\dagger = \begin{bmatrix} v\partial^{-1}p & -v\partial^{-1}q & -\frac{1}{2}\partial + \frac{1}{2}pq + v\partial^{-1}s & -v\partial^{-1}v \\ s\partial^{-1}p & -s\partial^{-1}q & s\partial^{-1}s & \frac{1}{2}\partial + \frac{1}{2}pq - s\partial^{-1}v \\ 1 + q\partial^{-1}p & -q\partial^{-1}q & q\partial^{-1}s & -q\partial^{-1}v \\ p\partial^{-1}p & 1 - p\partial^{-1}q & p\partial^{-1}s & -p\partial^{-1}v \end{bmatrix}, \tag{146}$$

where Φ is from (137).

It can be computed that all members in the soliton hierarchy (136) are bi-Hamiltonian (see [11, 13, 15, 46] for bi-Hamiltonian theory), i.e.,

$$u_{t_m} = K_m = J\frac{\delta \mathcal{H}_{m+1}}{\delta u} = M\frac{\delta \mathcal{H}_m}{\delta u}, \quad m \geq 1, \tag{147}$$

where the second Hamiltonian operator is

$$M = \Phi J = \begin{bmatrix} 2p\partial^{-1}p & 2-2p\partial^{-1}q & 2p\partial^{-1}s & -2p\partial^{-1}v \\ -2-2q\partial^{-1}p & 2q\partial^{-1}q & -2q\partial^{-1}s & 2q\partial^{-1}v \\ 2s\partial^{-1}p & -2s\partial^{-1}q & 2s\partial^{-1}s & \partial+pq-2s\partial^{-1}v \\ -2v\partial^{-1}p & 2v\partial^{-1}q & \partial-pq-2v\partial^{-1}s & 2v\partial^{-1}v \end{bmatrix}. \tag{148}$$

This means that J and M constitute a Hamiltonian pair, or, J, M, and $N = \alpha J + \beta M$, for any $\alpha, \beta \in \mathbb{R}$, are all Hamiltonian operators. As a direct result of the bi-Hamiltonian structure (147), we can say that the soliton hierarchy (136) is integrable in the Liouville sense:

$$\begin{cases} \{\mathcal{H}_k, \mathcal{H}_l\}_M = \int \left(\frac{\delta \mathcal{H}_k}{\delta u}\right)^T M \frac{\delta \mathcal{H}_l}{\delta u} dx = 0, \\ \{\mathcal{H}_k, \mathcal{H}_l\}_J = \int \left(\frac{\delta \mathcal{H}_k}{\delta u}\right)^T J \frac{\delta \mathcal{H}_l}{\delta u} dx = 0, \end{cases} \tag{149}$$

and

$$[K_k, K_l] = K_k'(u)[K_l] - K_l'(u)[K_k] = 0, \quad k, l \geq 0. \tag{150}$$

2.5 Summary

Starting with the matrix loop algebra $\widetilde{\mathrm{sl}}(2,\mathbb{R})$, we introduced a generalized D-Kaup-Newell spectral problem by adding two potentials s and v. Each system of equations in the soliton hierarchy has infinitely many commuting high-order symmetries and conserved functionals, and so, is Liouville integrable. Then we presented a reduction to the new spectral problem generating a different soliton hierarchy with a bi-Hamiltonian structure.

We would like to note that the soliton hierarchy (101) is different from the hierarchies presented in [9], [22], and [23]. The AKNS, Kaup-Newell, and D-Kaup-Newell hierarchies are found from the following spectral matrices, respectively:

$$U = \begin{bmatrix} \lambda & p \\ q & -\lambda \end{bmatrix}, U = \begin{bmatrix} \lambda^2 & \lambda p \\ \lambda q & -\lambda^2 \end{bmatrix}, U = \begin{bmatrix} \lambda^2 + r & \lambda p \\ \lambda q & -\lambda^2 - r \end{bmatrix}. \tag{151}$$

Recall, the spectral matrix for the soliton hierarchy (101) is

$$U = \begin{bmatrix} \lambda^2 - r & \lambda p + s \\ \lambda q + v & -\lambda^2 + r \end{bmatrix}. \tag{152}$$

It is clear that (152) is a generalization of the Kaup-Newell and D-Kaup-Newell spectral matrices. We will note that AKNS hierarchy [9] may be found from (152) by letting $p = q = r = 0$.

3 Integrable couplings

3.1 Introduction

Originally, integrable couplings were found in the study of centerless Virasoro symmetry algebras of integrable systems [64]. The problem of integrable couplings can be expressed as: "*For a given integrable system, how can we construct a non-trivial system of differential equations which is still integrable and includes the original system as a subsystem?* [59]" The quest for finding new integrable couplings has produced many ideas and become an important area of research in mathematical physics [14], [51]- [76]. Studying integrable couplings will facilitate with the complete classification of multiple component integrable systems.

Given an integrable system $u_t = K(u)$, an *integrable coupling* is a triangular system of the form

$$\begin{cases} u_t = K(u), \\ v_t = T(u, v), \end{cases} \tag{153}$$

where potentials u and v are scalar functions or vector functions with dependent variables $\bar{x} = (t, x_1, x_2, ...)$. The non-triviality condition is $\frac{\partial T}{\partial [u]} \neq 0$, where $[u]$ denotes a vector consisting of all derivatives of u with respect to the space variable. This condition guarantees that the new differential equations in the bigger system (153) involve the dependent variables of the original system. Integrable couplings were first constructed through perturbations [14, 51, 52] taking the form

$$\begin{cases} u_t = K(u), \\ v_t = K'(u)[v], \end{cases} \tag{154}$$

where $K'(u)[v] = \frac{\partial}{\partial \epsilon} K(u+\epsilon v, u_x+\epsilon v_x, ...)|_{\epsilon=0}$ is the Gateaux derivative. Then the spectral matrices were enlarged [53, 55]. In 2006, the connection between integrable couplings and semi-direct sums of Lie algebras was realized [59,60]. Since then, bi-integrable and tri-integrable couplings have been developed with examples associated with $\mathrm{sl}(2, \mathbb{R})$ and $\mathrm{so}(3, \mathbb{R})$ [61–63]. Very recently, a novel kind of AKNS integrable couplings were analyzed [75]; the enlarged spectral problem had an additional matrix block depending on the spectral parameter λ. The last section shows the construction of two examples which uses the same technique on generalized D-Kaup-Newell soliton hierarchies to enlarge the spectral problem producing integrable couplings for each.

This chapter has ends with two major sections: integrable couplings and Hamiltonian structures. In the integrable couplings section, we begin by enlarging the spectral problem (83) and solving the corresponding enlarged zero curvature equations. We prove its localness and show the structures of the integrable couplings. Next, we reduce the enlarged spectral matrix and follow the same procedure. This produces different integrable couplings for the reduced system. The section of Hamiltonian structures follows. Here, we find a non-degenerate, ad-invariant, symmetric bilinear form. The bilinear form is used in the variational identity to formulate Hamiltonian structures of a generalized D-Kaup-Newell integrable coupling system and then in the reduced integrable couplings. We find that the reduced integrable couplings have bi-Hamiltonian structures. Both hierarchies of integrable couplings retain the property of infinitely many high-order symmetries and conserved functionals from their subsystems. Before we begin the construction of integrable couplings, let's discuss: semi-direct sums and non-semisimple Lie algebras, the variational identity, and a general algorithm for formulating integrable couplings.

3.2 A method for constructing integrable couplings

3.2.1 Semi-direct sums and non-semisimple Lie algebras

Integrable couplings are directly related to semi-direct sums of Lie algebras [59, 60]. We construct integrable couplings for a soliton hierarchy by using a loop matrix Lie algebra. Define a triangular block matrix as follows:

$$M(A_1, A_2) = \begin{bmatrix} A_1 & A_2 \\ 0 & A_1 \end{bmatrix}. \tag{155}$$

It can easily be shown that matrices of this form are closed under matrix multiplication, i.e., constitute a Lie algebra. The associated matrix loop algebra $\tilde{\mathfrak{g}}(\lambda)$ is formed by all block matrices of the type:

$$\tilde{\mathfrak{g}}(\lambda) = \{M(A_1, A_2) | M \text{ defined by } (155), \text{entries of } A_i \text{ are Laurent series in } \lambda\}. \tag{156}$$

Since $\tilde{\mathfrak{g}}$ is a Lie algebra, it has a semi-direct sum decomposition [77] of the form:

$$\tilde{\mathfrak{g}} = \mathfrak{g} \Subset \mathfrak{g}_c, \tag{157}$$

where

$$\mathfrak{g} = \{M(A_1, 0) | M \text{ as in} (155), \text{entries of } A_1 \text{ are Laurent series in } \lambda \}$$

is semisimple and

$$\mathfrak{g}_c = \{M(0, A_2) | M \text{ as in} (155), \text{entries of } A_2 \text{ are Laurent series in } \lambda \}$$

is solvable. This means that $\mathfrak{g}, \mathfrak{g}_c \subseteq \tilde{\mathfrak{g}}$ have the property

$$[\mathfrak{g}, \mathfrak{g}_c] \subseteq \mathfrak{g}_c, \tag{158}$$

where $[\cdot, \cdot]$ is the Lie bracket of $\tilde{\mathfrak{g}}$. The subscript c indicates the connection with the construction of coupling systems.

We assume that (153) has a pair of Lax matrices $\bar{U}$ and $\bar{V}$ in a matrix loop Lie algebra $\tilde{\mathfrak{g}}$ associated with a spectral problems

$$\bar{\phi}_x = \bar{U}\bar{\phi}, \quad \bar{\phi}_t = \bar{V}\bar{\phi}. \tag{159}$$

Lax pairs of the enlarged system (153) must be of the form:

$$\bar{U}(\bar{u}, \lambda) = \begin{bmatrix} U(u, \lambda) & U_1(\bar{u}, \lambda) \\ 0 & U(u, \lambda) \end{bmatrix}, \quad \bar{V}(\bar{u}, \lambda) = \begin{bmatrix} V(u, \lambda) & V_1(\bar{u}, \lambda) \\ 0 & V(u, \lambda) \end{bmatrix}, \tag{160}$$

where $\bar{U}$ and $\bar{V}$ are block matrices in a non-semisimple Lie algebra, $\tilde{\mathfrak{g}}$. In summary, integrable couplings are integrable systems associated with non-semisimple Lie algebras, or semi-direct sums of Lie algebras [59, 60], which enlarge an original integrable system.

3.2.2 Variational identity

If a Lie algebra $\mathfrak{g}$ is semisimple, then all bilinear forms on $\mathfrak{g}$ that are non-degenerate, symmetric, and invariant under both the Lie bracket and Lie isomorphisms are equivalent to the Killing form up to a constant multiplier [68, 69]. The Killing form on a Lie algebra is non-degenerate if and only if $\mathfrak{g}$ is semisimple. When $\mathfrak{g}$ is non-semisimple, the trace identity no longer is useful for calculating Hamiltonian structures. As seen in the last section, integrable couplings are formed from semi-direct sums of Lie algebras. When $\mathfrak{g}_c$ from (157) is not zero, we have a non-semisimple Lie algebra.

To remedy this issue, Ma and Chen [64] developed the variational identity under a general bilinear form. The general bilinear form has to be non-degenerate, symmetric, and invariant under Lie product. A bilinear form $\langle \cdot, \cdot \rangle$ is said to be non-degenerate if $\langle A, B \rangle = 0$ for all vectors A, then $B = 0$, and if $\langle A, B \rangle = 0$ for all vectors B, then $A = 0$. The symmetric property and invariance under Lie product are as follows:

$$\langle A, B \rangle = \langle B, A \rangle \tag{161}$$

and

$$\langle A, [B, C] \rangle = \langle [A, B], C \rangle, \tag{162}$$

respectively. Ma and Chen found that removing the requirement of the bilinear form in the proof of the trace identity [28] that

$$\langle \rho(A), \rho(B) \rangle = \langle A, B \rangle \tag{163}$$

under an isomorphism ρ of the Lie algebra would allow for a more general bilinear form in the variational identity.

THEOREM 3.1. *[64] (the variational identity under general bilinear forms). Let $\tilde{\mathfrak{g}}$ be a matrix loop algebra, $\bar{U} = \bar{U}(\bar{u}, \lambda) \in \tilde{\mathfrak{g}}$ be homogeneous in rank and $\langle \cdot, \cdot \rangle$ denote a non-degenerate, symmetric, bilinear form invariant under the matrix Lie product. Assume that the stationary zero curvature equation $\bar{W}_x = [\bar{U}, \bar{W}]$ has a unique solution $\bar{W} \in \tilde{\mathfrak{g}}$ of a fixed rank up to a constant multiplier. Then for any solution $\bar{W} \in \tilde{\mathfrak{g}}$ of $\bar{W}_x = [\bar{U}, \bar{W}]$, being homogeneous in rank, we have the following variational identity:*

$$\frac{\delta}{\delta \bar{u}} \int \langle \bar{W}, \bar{U}_\lambda \rangle dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} \langle \bar{W}, \bar{U}_{\bar{u}} \rangle, \quad \gamma = -\frac{\lambda}{2} \frac{d}{d\lambda} ln |\langle \bar{W}, \bar{W} \rangle|. \tag{164}$$

3.2.3 A general algorithm for integrable couplings

A soliton hierarchy is usually associated with an enlarged spectral problem,

$$\bar{\phi}_x = \bar{U}\bar{\phi}, \qquad \bar{U} = \bar{U}(\bar{u}, \lambda) \in \tilde{\mathfrak{g}}, \tag{165}$$

where λ is a spectral parameter and $\tilde{\mathfrak{g}}$ is a matrix loop algebra associated with a matrix Lie agebra $\mathfrak{g}$, non-semisimple. Suppose that the corresponding stationary zero curvature equation,

$$\bar{W}_x = [\bar{U}, \bar{W}], \tag{166}$$

has a solution of the form

$$\bar{W} = \bar{W}(\bar{u}, \lambda) = \sum_{i \geq 0} \bar{W}_{0,i}\lambda^{-i}, \tag{167}$$

where $\bar{W}_{0,i} \in \tilde{\mathfrak{g}}$. We now introduce a sequence of enlarged temporal spectral problems,

$$\bar{\phi}_{t_m} = \bar{V}^{[m]}\bar{\phi} = \bar{V}^{[m]}(\bar{u}, \lambda)\bar{\phi}, \quad m \geq 0, \tag{168}$$

involving the Lax matrices defined by

$$\bar{V}^{[m]} = (\lambda^m \bar{W})_+ + \bar{\Delta}_m \in \tilde{\mathfrak{g}}, \quad m \geq 0, \tag{169}$$

where Q_+ denotes the polynomial part of Q in λ and the modification terms $\bar{\Delta}_m$ to ensure that the zero curvature equations,

$$\bar{U}_{t_m} - \bar{V}_x^{[m]} + [\bar{U}, \bar{V}^{[m]}] = 0, \qquad m \geq 0, \tag{170}$$

generate a soliton hierarchy with Hamiltonian structures,

$$\bar{u}_{t_m} = \bar{K}_m(\bar{u}) = \bar{J}\frac{\delta \bar{\mathcal{H}}_m}{\delta \bar{u}}, \;\; m \geq 0. \tag{171}$$

The Hamiltonian functionals are generally found by applying the variational identity (164) [64, 68, 69]. The soliton hierarchy (171) often has the communativity properties:

$$[\bar{K}_m, \bar{K}_n] = \bar{K}'_m(\bar{u})[\bar{K}_n] - \bar{K}'_n(\bar{u})[\bar{K}_m] = 0, \tag{172}$$

$$\{\bar{\mathcal{H}}_m, \bar{\mathcal{H}}_n\}_{\bar{J}} = \int \left(\frac{\delta \bar{\mathcal{H}}_m}{\delta \bar{u}}\right)^T \bar{J}\frac{\delta \bar{\mathcal{H}}_n}{\delta \bar{u}} dx = 0, \tag{173}$$

where $m, n \geq 0$. These properties imply that the hierarchy (171) possseses infinitely many commuting symmetries $\{\bar{K}_n\}_{n=0}^{\infty}$ and conserved functionals $\{\bar{\mathcal{H}}_n\}_{n=0}^{\infty}$.

4 Two integrable couplings and their Hamiltonian structures

4.1 An enlarged non-semisimple matrix loop algebra

We begin the construction of integrable couplings of a generalized D-Kaup-Newell soliton hierarchy by introducing an enlarged matrix loop algebra. Let $\bar{\mathrm{sl}}(2, \mathbb{R})$ be triangular block matrices such that

$$\bar{\mathrm{sl}}(2, \mathbb{R}) = \{M(A_1, A_2) | A_1, A_2 \in \mathrm{sl}(2, \mathbb{R}), \ \ M \text{ defined by } (155)\}. \tag{174}$$

The associated enlarged matrix loop algebra $\tilde{\bar{\mathrm{sl}}}(2, \mathbb{R})(\lambda)$ is formed by all block matrices of the type:

$$\tilde{\bar{\mathrm{sl}}}(2, \mathbb{R})(\lambda) = \left\{ \sum_{i \geq 0} M_i \lambda^{n-i} | \ \ M_i \in \bar{\mathrm{sl}}(2, \mathbb{R}), n \in \mathbb{Z} \right\}. \tag{175}$$

Throughout this section, we will conveniently refer to $\tilde{\bar{\mathrm{sl}}}(2, \mathbb{R})(\lambda)$ as $\tilde{\mathfrak{g}}(\lambda)$.

4.2 Generalized D-Kaup-Newell integrable couplings

A spectral matrix is chosen [76] from $\tilde{\mathfrak{g}}(\lambda)$:

$$\bar{U} = \bar{U}(\bar{u}, \lambda) = \begin{bmatrix} U & U_1 \\ 0 & U \end{bmatrix} = \left[\begin{array}{cc|cc} \lambda^2 - r_1 & \lambda p_1 + s_1 & \lambda^2 - r_2 & \lambda p_2 + s_2 \\ \lambda q_1 + v_1 & -\lambda^2 + r_1 & \lambda q_2 + v_2 & -\lambda^2 + r_2 \\ \hline 0 & 0 & \lambda^2 - r_1 & \lambda p_1 + s_1 \\ 0 & 0 & \lambda q_1 + v_1 & -\lambda^2 + r_1 \end{array}\right], \tag{176}$$

where $\{p_i, q_i, r_i, s_i, v_i, i = 1, 2\}$ are potentials and $\bar{u} = (u, v)^T, u = (p_1, q_1, r_1, s_1, v_1)^T, v = (p_2, q_2, r_2, s_2, v_2)^T$. The isospectral problem is

$$\bar{\phi}_x = \bar{U}\bar{\phi} = \left[\begin{array}{cc|cc} \lambda^2 - r_1 & \lambda p_1 + s_1 & \lambda^2 - r_2 & \lambda p_2 + s_2 \\ \lambda q_1 + v_1 & -\lambda^2 + r_1 & \lambda q_2 + v_2 & -\lambda^2 + r_2 \\ \hline 0 & 0 & \lambda^2 - r_1 & \lambda p_1 + s_1 \\ 0 & 0 & \lambda q_1 + v_1 & -\lambda^2 + r_1 \end{array}\right] \bar{\phi}, \bar{\phi} = \begin{bmatrix} \psi \\ \phi \end{bmatrix}, \tag{177}$$

where $\psi = (\psi_1, \psi_2)^T$ and $\phi = (\phi_1, \phi_2)^T$. Note that U is the same matrix as (83) with renamed variables $(p, q, r, s, v) \to (p_1, q_1, r_1, s_1, v_1)$.

Assume that the solution to the stationary zero curvature equation, $\bar{W}_x = [\bar{U}, \bar{W}]$, is of the form

$$\bar{W} = \begin{bmatrix} W & W_1 \\ 0 & W \end{bmatrix} = \left[\begin{array}{cc|cc} a & b & e & f \\ c & -a & g & -e \\ \hline 0 & 0 & a & b \\ 0 & 0 & c & -a \end{array}\right] \in \tilde{\mathfrak{g}}(\lambda), \tag{178}$$

then we get the following matrix formulas:

$$\begin{cases} W_x = UW - WU, \\ W_{1,x} = U_1 W - WU_1 + UW_1 - W_1 U. \end{cases} \tag{179}$$

Solving these two formulas, we get the differential equations:

$$\begin{cases} a_x = -q_1 b\lambda + p_1 c\lambda - v_1 b + s_1 c, \\ b_x = -2p_1 a\lambda + 2b\lambda^2 - 2s_1 a - 2r_1 b, \\ c_x = 2q_1 a\lambda - 2c\lambda^2 + 2v_1 a + 2r_1 c, \\ e_x = p_1 g\lambda + p_2 c\lambda - q_2 b\lambda - q_1 f\lambda + s_1 g + s_2 c - v_1 f - v_2 b, \\ f_x = 2b\lambda^2 + 2f\lambda^2 - 2p_1 e\lambda - 2p_2 a\lambda - 2r_1 f - 2r_2 b - 2s_1 e - 2s_2 a, \\ g_x = -2c\lambda^2 - 2g\lambda^2 + 2q_1 e\lambda + 2q_2 a\lambda + 2r_1 g + 2r_2 c + 2v_1 e + 2v_2 a. \end{cases} \tag{180}$$

By assuming a, b, c, e, f, g, have the following Laurent series expansions:

$$\begin{aligned} a &= \sum_{i=0}^{\infty} a_i \lambda^{-i}, & b &= \sum_{i=0}^{\infty} b_i \lambda^{-i}, & c &= \sum_{i=0}^{\infty} c_i \lambda^{-i}, \\ e &= \sum_{i=0}^{\infty} e_i \lambda^{-i}, & f &= \sum_{i=0}^{\infty} f_i \lambda^{-i}, & g &= \sum_{i=0}^{\infty} g_i \lambda^{-i}, \end{aligned} \tag{181}$$

and substituting (181) into (180), we have the recursion relations

$$\begin{cases} b_{i+1} &= \frac{b_{i-1,x}}{2} + p_1 a_i + s_1 a_{i-1} + r_1 b_{i-1}, \\ c_{i+1} &= -\frac{c_{i-1,x}}{2} + q_1 a_i + v_1 a_{i-1} + r_1 c_{i-1}, \\ a_{i+1,x} &= -q_1 \frac{b_{i,x}}{2} - p_1 \frac{c_{i,x}}{2} + (p_1 v_1 - q_1 s_1) a_i - q_1 r_1 b_i + p_1 r_1 c_i + s_1 c_{i+1} - v_1 b_{i+1}, \\ f_{i+1} &= \frac{f_{i-1,x}}{2} - b_{i+1} + p_2 a_i + p_1 e_i + s_2 a_{i-1} + s_1 e_{i-1} + r_2 b_{i-1} + r_1 f_{i-1}, \\ g_{i+1} &= -\frac{g_{i-1,x}}{2} - c_{i+1} + q_2 a_i + q_1 e_i + v_2 a_{i-1} + v_1 e_{i-1} + r_2 c_{i-1} + r_1 g_{i-1}, \\ e_{i+1,x} &= -\frac{g_{i,x}}{2} p_1 - \frac{f_{i,x}}{2} q_1 + (p_2 - p_1)[-\frac{c_{m,x}}{2} + v_1 a_m + r_1 c_m] \\ & +(q_1 - q_2)[\frac{b_{m,x}}{2} + s_1 a_m + r_1 b_m] + s_1 g_{i+1} + s_2 c_{i+1} - v_1 f_{i+1} - v_2 b_{i+1} \\ & +(p_1 v_2 - q_1 s_2) a_i + (p_1 v_1 - q_1 s_1) e_i + p_1 r_2 c_i - q_1 r_2 b_i \\ & +p_1 r_1 g_i - q_1 r_1 f_i, \end{cases} \tag{182}$$

for all $i \geq 1$ with initial values

$$\begin{aligned} &a_0 = \alpha, \quad b_0 = c_0 = 0, \quad a_1 = 0, \quad b_1 = \alpha p_1, \quad c_1 = \alpha q_1, \\ &e_0 = \beta, \quad f_0 = g_0 = 0, \quad e_1 = 0, \quad f_1 = (\beta - \alpha) p_1 + p_2 \alpha, \quad g_1 = (\beta - \alpha) q_1 + q_2 \alpha, \end{aligned} \tag{183}$$

and the conditions for integration

$$\begin{cases} a_i|_{u=0} = b_i|_{u=0} = c_i|_{u=0} = 0, \quad i \geq 1, \\ e_i|_{u=0} = f_i|_{u=0} = g_i|_{u=0} = 0, \quad i \geq 1, \end{cases} \tag{184}$$

which determine the sequence of $\{a_i, b_i, c_i, e_i, f_i, g_i | i \geq 0\}$ uniquely. For $i = 2, 3, 4$, we have the following:

$$
\begin{aligned}
&b_2 = \alpha s_1, \qquad c_2 = \alpha v_1, \qquad a_2 = -\alpha\frac{1}{2}p_1q_1,\\
&f_2 = (\beta-\alpha)s_1 + \alpha s_2, \qquad e_2 = (\alpha q_1 + \frac{1}{2}\beta q_1 - \frac{1}{2}q_2)p_1 - \frac{1}{2}\alpha p_2 q_1, \qquad g_2 = (\beta-\alpha)v_1 + \alpha v_2;\\
&b_3 = \alpha\frac{1}{2}(-p_1^2q_1 + 2p_1r_1 + p_{1,x}), \qquad c_3 = -\alpha\frac{1}{2}(q_1^2p_1 - 2q_1r_1 + q_{1,x}),\\
&a_3 = -\alpha\frac{1}{2}(p_1v_1 + q_1s_1),\\
&f_3 = \frac{1}{2}[(\beta-2\alpha)p_{1,x} + \alpha p_{2,x} + ((\beta+3\alpha)p_1^2q_1 + (2\beta-4\alpha)r_1p_1\\
&\quad - \alpha(p_2q_1p_1 - 2r_2p_1 + q_2p_1^2) + 2\alpha p_2r_1],\\
&g_3 = -\frac{1}{2}[(\beta-2\alpha)q_{1,x} + \alpha q_{2,x} - (-\beta+3\alpha)q_1^2p_1\alpha - (2\beta-4\alpha)r_1q_1\\
&\quad + \alpha(q_2p_1q_1 - 2r_2q_1 + p_2q_1^2) - 2\alpha q_2r_1],\\
&e_3 = (v_1\alpha - \frac{1}{2}v_1\beta - \frac{1}{2}v_2\alpha)p_1 + (s_1\alpha - \frac{1}{2}s_1\beta - \frac{1}{2}s_2\alpha)q_1 - \frac{1}{2}p_2v_1\alpha - \frac{1}{2}q_2s_1\alpha\\
&b_4 = \alpha\frac{1}{2}(-p_1^2v_1 - 2p_1q_1s_1 + 2r_1s_1 + s_{1,x}), \quad c_4 = -\alpha\frac{1}{2}(q_1^2s_1 + 2q_1p_1v_1 - 2r_1v_1 + v_{1,x}),\\
&a_4 = -\alpha\frac{1}{8}(-3q_1^2p_1^2 + 8q_1p_1r_1 + 4v_1s_1 + 2q_1p_{1,x} - 2p_1q_{1,x}),\\
&f_4 = \frac{1}{2}[(\beta-2\alpha)s_{1,x} + \alpha s_{2,x} - \alpha p_1^2v_2 + (-\beta+3\alpha)v_1p_1^2 + (((-2\beta+6\alpha)q_1 - 2\alpha q_2)s_1\\
&\quad - 2\alpha(q_1s_2 + p_2v_1))p_1 + (-2\alpha p_2q_1 + (-4\alpha+2\beta)r_1 + 2r_2\alpha)s_1 + 2\alpha r_1s_2],\\
&g_4 = -\frac{1}{2}[(\beta-2\alpha)v_{1,x} + \alpha v_{2,x} + \alpha q_1^2s_2 - (-\beta+3\alpha)s_1q_1^2 - (((-2\beta+6\alpha)p_1 - 2\alpha p_2)v_1\\
&\quad - 2\alpha(q_2s_1 + p_1v_2))q_1 - (-2\alpha q_2p_1 + (-4\alpha+2\beta)r_1 + 2r_2\alpha)v_1 - 2\alpha r_1v_2],\\
&e_4 = \frac{1}{8}[(-2\beta+6\alpha)q_1 - 2\alpha q_2)p_{1,x} + ((2\beta-6\alpha)p_1 + 2\alpha p_2)q_{1,x} + 2\alpha(-q_1p_{2,x} + p_1q_{2,x})\\
&\quad + 3q_1((\beta-4\alpha)q_1 + 2\alpha q_2)p_1^2 + (6\alpha p_2q_1^2 + ((-8\beta+24\alpha)r_1 - 8\alpha r_2)q_1\\
&\quad - 8\alpha q_2r_1)p_1 - 8\alpha q_1p_2r_1 + ((-4\beta+8\alpha)v_1 - 4\alpha v_2)s_1 - 4\alpha s_2v_1].
\end{aligned}
$$

All $\{a_i, b_i, c_i, e_i, f_i, g_i | i \geq 0\}$ can be proven as differential polynomials of $\bar{u}$ with respect to x.

Proposition 4.1. *Let $\{a_i, b_i, c_i, e_i, f_i, g_i | i = 0, 1\}$ be given by equations (183). Then all functions $\{a_i, b_i, c_i, e_i, f_i, g_i | i \geq 0\}$ determined by equation (182) with the conditions (184) are differential polynomials in $\bar{u}$ with respect to x, and thus, are local.*

Proof. We compute from the enlarged stationary zero curvature equation, $\bar{W}_x = [\bar{U}, \bar{W}]$,

$$\frac{d}{dx}\mathrm{tr}(\bar{W}^2) = 2\mathrm{tr}(\bar{W}\bar{W}_x) = 2\mathrm{tr}(\bar{W}[\bar{U}, \bar{W}]) = 2(\mathrm{tr}(\bar{W}^2\bar{U}) - \mathrm{tr}(\bar{W}^2\bar{U})) = 0, \tag{185}$$

and seeing that the $\mathrm{tr}(\bar{W}^2) = 4(a^2 + bc)$, we have

$$a^2 + bc = (a^2 + bc)|_{u=0} = \alpha^2, \tag{186}$$

following from the initial data (183). Now, we use (181), the Laurent expansions of a, b, c, to give

$$a_i = \frac{\alpha}{2} - \frac{1}{2\alpha} \sum_{k+l=i, k,l \geq 1} a_k a_l - \frac{1}{2\alpha} \sum_{k+l=i, k,l \geq 0} b_k c_l, i \geq 1. \tag{187}$$

Based on the recursion relation above (187) and the previous (182), we use mathematical induction to see that all functions $\{a_i, b_i, c_i, i \geq 0\}$ are differential polynomials in u with respect to x, and therefore, are local.

Now, we have

$$
\begin{aligned}
\frac{d}{dx}(2ae+fc+gb) =& 2a_xe+2ae_x+f_xc+fc_x+g_xb+gb_x \\
=& 2e(-q_1b\lambda+p_1c\lambda-v_1b+s_1c)+2a(p_1g\lambda+p_2c\lambda \\
& -q_2b\lambda-q_1f\lambda+s_1g+s_2c-v_1f-v_2b)+c(2b\lambda^2 \\
& +2f\lambda^2-2p_1e\lambda-p_2a\lambda-2r_1f-2r_2b-2s_1e-2s_2a) \\
& +f(2q_1a\lambda-2c\lambda^2+2v_1a+2r_1c)+b(-2c\lambda^2-2g\lambda^2 \\
& +2q_1e\lambda+q_2a\lambda+2r_1g+2r_2c+2v_1e+2v_2a) \\
& +g(-2p_1a\lambda+2b\lambda^2-2s_1a-2r_1b)=0.
\end{aligned}
$$

Similarly, we get

$$
2ae+fc+gb=(2ae+fc+gb)|_{\bar{u}=0}=\alpha\beta.
$$

Therefore, using the Laurent expansions of a, b, c, e, f, and g in (181), we have

$$
e_i=\beta-\frac{\beta}{\alpha}a_i-\frac{1}{2\alpha}\sum_{k+l=i,k,l\geq 0}f_kc_l-\frac{1}{2\alpha}\sum_{k+l=i,k,l\geq 0}g_kb_l-\frac{1}{\alpha}\sum_{k+l=i,k,l\geq 1}a_ke_l, \tag{188}
$$

for all $i\geq 1$. Using the localness of $\{a_i, b_i, c_i | i\geq 0\}$ and the recursive relations (182) and (188), we may see through mathematical induction that all functions $\{e_i, f_i, g_i | i\geq 0\}$ are differential polynomials in $\bar{u}$ with respect to x. This completes the proof. □

Now, we need to solve the zero curvature equations,

$$
\bar{U}_{t_m}-\bar{V}_x^{[m]}+[\bar{U},\bar{V}^{[m]}]=0, \quad m\geq 0, \tag{189}
$$

which are the compatibility conditions between (177) and the temporal problems,

$$\bar{\phi}_{t_m} = \bar{V}^{[m]}\bar{\phi} = \bar{V}^{[m]}(\bar{u},\lambda)\bar{\phi}, \quad m \geq 0. \tag{190}$$

In order to do this, we introduce a series of Lax operators,

$$\bar{V}^{[m]}(\bar{u},\lambda) = (\lambda^m \bar{W})_+. \tag{191}$$

Similar to the analysis in chapter two, we may show that we can take $\bar{\Delta}_m = 0$ from (169).

After solving (189), we generate a hierarchy of soliton equations, for all $m \geq 0$,

$$\bar{u}_{t_m} = \bar{K}_m = \begin{bmatrix} 2b_{m+1} \\ -2c_{m+1} \\ q_1 b_{m+1} - p_1 c_{m+1} \\ -2p_1 a_{m+1} + 2b_{m+2} \\ 2q_1 a_{m+1} - 2c_{m+2} \\ 2f_{m+1} + 2b_{m+1} \\ -2g_{m+1} - 2c_{m+1} \\ q_1 f_{m+1} + q_2 b_{m+1} - p_1 g_{m+1} - p_2 c_{m+1} \\ -2p_1 e_{m+1} - 2p_2 a_{m+1} + 2b_{m+2} + 2f_{m+2} \\ 2q_1 e_{m+1} + 2q_2 a_{m+1} - 2c_{m+2} - 2g_{m+2} \end{bmatrix}. \tag{192}$$

We have

$$\bar{K}_m = \bar{\Phi}\bar{K}_{m-1} = \bar{\Phi}^m \bar{K}_0, \quad m \geq 0, \tag{193}$$

where $\bar{\Phi}$ is a recursion operator determined from (182) and given by

$$\bar{\Phi} = \begin{bmatrix} \Phi & 0 \\ \Phi_1 - \Phi & \Phi \end{bmatrix}. \tag{194}$$

Φ is the same as (102) with renamed variables $(p, q, r, s, v) \to (p_1, q_1, r_1, s_1, v_1)$ and

$$\Phi_1 = \begin{bmatrix} [\Phi_1]_{11} & [\Phi_1]_{12} & [\Phi_1]_{13} & [\Phi_1]_{14} & [\Phi_1]_{15} \\ [\Phi_1]_{21} & [\Phi_1]_{22} & [\Phi_1]_{23} & [\Phi_1]_{24} & [\Phi_1]_{25} \\ [\Phi_1]_{31} & [\Phi_1]_{32} & [\Phi_1]_{33} & [\Phi_1]_{34} & [\Phi_1]_{35} \\ [\Phi_1]_{41} & [\Phi_1]_{42} & [\Phi_1]_{43} & [\Phi_1]_{44} & [\Phi_1]_{45} \\ [\Phi_1]_{51} & [\Phi_1]_{52} & [\Phi_1]_{53} & [\Phi_1]_{54} & [\Phi_1]_{55} \end{bmatrix}, \tag{195}$$

with

$$\begin{aligned}
[\Phi_1]_{11} =& - p_1\partial^{-1}v_2 - p_2\partial^{-1}v_1 - s_2\partial^{-1}q_1 - s_1\partial^{-1}q_2, \\
[\Phi_1]_{12} =& - p_1\partial^{-1}s_2 - p_2\partial^{-1}s_1 - s_2\partial^{-1}p_1 - s_1\partial^{-1}p_2, \\
[\Phi_1]_{13} =& 2s_2\partial^{-1} + 2s_1\partial^{-1}, [\Phi_1]_{14} = 1 - p_1\partial^{-1}q_2 - p_2\partial^{-1}q_1, [\Phi_1]_{15} = -p_1\partial^{-1}p_2 - p_2\partial^{-1}p_1, \\
[\Phi_1]_{21} =& q_1\partial^{-1}v_2 + q_2\partial^{-1}v_1 + v_1\partial^{-1}q_2 + v_2\partial^{-1}q_1, \\
[\Phi_1]_{22} =& q_1\partial^{-1}s_2 + q_2\partial^{-1}s_1 + v_1\partial^{-1}p_2 + v_2\partial^{-1}p_1, \\
[\Phi_1]_{23} =& - 2v_1\partial^{-1} - 2v_2\partial^{-1}, [\Phi_1]_{24} = q_1\partial^{-1}q_2 + q_2\partial^{-1}q_1, [\Phi_1]_{25} = 1 + q_1\partial^{-1}p_2 + q_2\partial^{-1}p_1,
\end{aligned}$$

$$
\begin{aligned}
[\Phi_1]_{31} =&\frac{(v_1p_2-q_2s_1)}{2}\partial^{-1}q_1+\frac{(p_1v_2-q_1s_2)}{2}\partial^{-1}q_1-\frac{(p_1v_1-q_1s_1)}{2}\partial^{-1}q_1\\
&+\frac{(p_1v_1-q_1s_1)}{2}\partial^{-1}q_2,\\
[\Phi_1]_{32} =&\frac{(v_1p_2-q_2s_1)}{2}\partial^{-1}p_1+\frac{(p_1v_2-q_1s_2)}{2}\partial^{-1}p_1-\frac{(p_1v_1-q_1s_1)}{2}\partial^{-1}p_1\\
&+\frac{(p_1v_1-q_1s_1)}{2}\partial^{-1}p_2,\\
[\Phi_1]_{33} =&-(p_1v_2-q_1s_2)\partial^{-1}-(p_2v_1-q_2s_1)\partial^{-1},[\Phi_1]_{34}=\frac{q_2}{2},[\Phi_1]_{35}=\frac{p_2}{2},\\
[\Phi_1]_{41} =&r_2-s_1\partial^{-1}v_2-s_2\partial^{-1}v_1-\partial\frac{(p_2-p_1)}{2}\partial^{-1}q_1-(r_1p_2+p_1r_2)\partial^{-1}q_1+p_1r_1\partial^{-1}q_1,\\
&-r_1p_1\partial^{-1}q_2-\partial\frac{p_1}{2}\partial^{-1}q_2,\\
[\Phi_1]_{42} =&-s_1\partial^{-1}s_2-s_2\partial^{-1}s_1-\partial\frac{(p_2-p_1)}{2}\partial^{-1}p_1-(r_1p_2+p_1r_2)\partial^{-1}p_1+p_1r_1\partial^{-1}p_1,\\
&-r_1q_1\partial^{-1}p_2-\partial\frac{p_1}{2}\partial^{-1}p_2,\\
[\Phi_1]_{43} =&\partial(p_2-p_1)\partial^{-1}+2(r_1p_2+p_1r_2)\partial^{-1}+\partial p_1\partial^{-1},\\
[\Phi_1]_{44} =&-s_1\partial^{-1}q_2-s_2\partial^{-1}q_1,[\Phi_1]_{45}=-s_1\partial^{-1}p_2-s_2\partial^{-1}p_1,\\
[\Phi_1]_{51} =&v_1\partial^{-1}v_2+v_2\partial^{-1}v_1+\partial\frac{(q_1-q_2)}{2}\partial^{-1}q_1+(r_1q_2+q_1r_2)\partial^{-1}q_1-q_1r_1\partial^{-1}q_1,\\
&+r_1q_1\partial^{-1}q_2-\partial\frac{q_1}{2}\partial^{-1}q_2,\\
[\Phi_1]_{52} =&r_2+v_1\partial^{-1}s_2+v_2\partial^{-1}s_1+\partial\frac{(q_1-q_2)}{2}\partial^{-1}p_1+(r_1q_2+q_1r_2)\partial^{-1}p_1-q_1r_1\partial^{-1}p_1\\
&+r_1q_1\partial^{-1}p_2-\partial\frac{q_1}{2}\partial^{-1}p_2,\\
[\Phi_1]_{53} =&-\partial(q_1-q_2)\partial^{-1}-2(r_1q_2+q_1r_2)\partial^{-1}+\partial q_1\partial^{-1},\\
[\Phi_1]_{54} =&v_1\partial^{-1}q_2+v_2\partial^{-1}q_1,[\Phi_1]_{55}=v_1\partial^{-1}p_2+v_2\partial^{-1}p_1,
\end{aligned}
$$

with $\partial=\dfrac{\partial}{\partial x}$ and ∂^{-1} as the inverse operator of ∂.

PROPOSITION 4.2. *All $\{\bar{K}_m\}$, as defined by (192), are commuting symmetries, i.e.,*

$$[\bar{K}_k,\bar{K}_l]=\bar{K}_k'(\bar{u})[\bar{K}_l]-\bar{K}_l'(\bar{u})[\bar{K}_k]=0,\quad k,l\geq 0. \tag{196}$$

Proof. The Gateaux derivative of $\bar{U}$ along any direction $\bar{S} = (S_1, S_2)^T$, where $S_i = (S_{1,i}, S_{2,i}, S_{3,i}, S_{4,i}, S_{5,i})$ for $i = 1, 2$, is

$$\bar{U}'[\bar{S}] = \begin{bmatrix} -S_{3,1} & \lambda S_{1,1} + S_{4,1} & -S_{3,2} & \lambda S_{1,2} + S_{4,2} \\ \lambda S_{2,1} + S_{5,1} & S_{3,1} & \lambda S_{2,2} + S_{5,2} & S_{3,2} \\ 0 & 0 & -S_{3,1} & \lambda S_{1,1} + S_{4,1} \\ 0 & 0 & \lambda S_{2,1} + S_{5,1} & S_{3,1} \end{bmatrix}.$$

This shows $\bar{U}'[\bar{S}] = 0$ if and only if $S = 0$ and $\bar{U}'[\bar{S}]$ is injective. Thus for the enlarged system, we still have the following [43]- [45]:

$$\bar{U}'[[\bar{K}_m, \bar{K}_n]] - [[\bar{V}^{[m]}, \bar{V}^{[n]}]]_x + [\bar{U}, [[\bar{V}^{[m]}, \bar{V}^{[n]}]]] = 0, \tag{197}$$

where $[[\bar{V}^{[m]}, \bar{V}^{[n]}]]$ is defined by

$$[[\bar{V}^{[m]}, \bar{V}^{[n]}]] := \bar{V}^{[m]'}(\bar{u})[\bar{K}_n] - \bar{V}^{[n]'}(\bar{u})[\bar{K}_m] + [\bar{V}^{[m]}, \bar{V}^{[n]}], \quad m, n \geq 0. \tag{198}$$

The method of proof is similar to Chapter 2. We begin proving

$$[[\bar{V}^{[m]}, \bar{V}^{[n]}]]|_{\bar{u}=0} = 0. \tag{199}$$

By inspection, we see by (192) and (182) that

$$\bar{K}_m|_{\bar{u}=0} = 0, \quad \text{for} \quad m \geq 0,$$

and, therefore,

$$\bar{V}^{[m]'}(\bar{u})[\bar{K}_n] - \bar{V}^{[n]'}(\bar{u})[\bar{K}_m]|_{\bar{u}=0} = 0, \quad \text{for} \quad m, n \geq 0.$$

We notice next that

$$\bar{V}^{[m]}|_{\bar{u}=0} = \begin{bmatrix} \alpha\lambda^m & 0 & \beta\lambda^m & 0 \\ 0 & -\alpha\lambda^m & 0 & -\beta\lambda^m \\ 0 & 0 & \alpha\lambda^m & 0 \\ 0 & 0 & 0 & -\alpha\lambda^m \end{bmatrix}. \tag{200}$$

Thus, it is easy to see that

$$[\bar{V}^{[m]}, \bar{V}^{[n]}]|_{\bar{u}=0} = 0,$$

and (199) is proved. By the uniqueness property of the enlarged spectral problem [43], we have $[\bar{V}^{[m]}, \bar{V}^{[n]}]] = 0$. This leaves

$$\bar{U}'[[\bar{K}_m, \bar{K}_n]] = 0. \tag{201}$$

$\bar{U}'[\cdot]$ is injective so (201) implies

$$[\bar{K}_m, \bar{K}_n] = 0, \quad m, n \geq 0, \tag{202}$$

and the proposition is proved. □

4.3 A specific reduction with two less potentials

A spectral matrix, $\bar{U}$, chosen from $\tilde{\mathfrak{g}}(\lambda)$, is of the form [76]:

$$\bar{U} = \left[\begin{array}{cc|cc} \lambda^2 - \widetilde{r_1} & \lambda p_1 + s_1 & \lambda^2 - \widetilde{r_2} & \lambda p_2 + s_2 \\ \lambda q_1 + v_1 & -\lambda^2 + \widetilde{r_1} & \lambda q_2 + v_2 & -\lambda^2 + \widetilde{r_2} \\ \hline 0 & 0 & \lambda^2 - \widetilde{r_1} & \lambda p_1 + s_1 \\ 0 & 0 & \lambda q_1 + v_1 & -\lambda^2 + \widetilde{r_1} \end{array}\right], \tag{203}$$

where $\widetilde{r_1} = \frac{1}{2}p_1q_1, \widetilde{r_2} = \frac{1}{2}(p_1q_2 + p_2q_1 - p_1q_1)$, $\{p_i, q_i, s_i, v_i, i = 1, 2\}$ are potentials, and $\bar{u} = (u, v)^T$, $u = (p_1, q_1, s_1, v_1)^T, v = (p_2, q_2, s_2, v_2)^T$. The corresponding spacial spectral problem is

$$\bar{\phi}_x = \bar{U}(\bar{u}, \lambda)\bar{\phi}, \quad \bar{\phi} = \begin{bmatrix} \psi \\ \phi \end{bmatrix}, \tag{204}$$

where $\psi = (\psi_1, \psi_2)^T$ and $\phi = (\phi_1, \phi_2)^T$.

Again, we assume that the solution to the stationary zero curvature equation, $\bar{W}_x = [\bar{U}, \bar{W}]$, is of the form

$$\bar{W} = \begin{bmatrix} W & W_1 \\ 0 & W \end{bmatrix} = \left[\begin{array}{cc|cc} a & b & e & f \\ c & -a & g & -e \\ \hline 0 & 0 & a & b \\ 0 & 0 & c & -a \end{array}\right] \in \tilde{\mathfrak{g}}(\lambda). \tag{205}$$

Solving the stationary zero curvature equation (189), we have the following differential equations:

$$\begin{cases} a_x &= -q_1 b\lambda + p_1 c\lambda - v_1 b + s_1 c, \\ b_x &= -2p_1 a\lambda + 2b\lambda^2 - 2s_1 a - p_1 q_1 b, \\ c_x &= 2q_1 a\lambda - 2c\lambda^2 + 2v_1 a + p_1 q_1 c, \\ e_x &= p_1 g\lambda + p_2 c\lambda - q_2 b\lambda - q_1 f\lambda + s_1 g + s_2 c - v_1 f - v_2 b, \\ f_x &= 2b\lambda^2 + 2f\lambda^2 - 2p_1 e\lambda - 2p_2 a\lambda - p_1 q_1 f - (p_1 q_2 + p_2 q_1 - p_1 q_1) b \\ & -2s_1 e - 2s_2 a, \\ g_x &= -2c\lambda^2 - 2g\lambda^2 + 2q_1 e\lambda + 2q_2 a\lambda + p_1 q_1 g + (p_1 q_2 + p_2 q_1 - p_1 q_1) c \\ & +2v_1 e + 2v_2 a. \end{cases} \tag{206}$$

By assuming a, b, c, e, f, g, have the Laurent expansions (181), we have the recursion relations

$$\begin{cases} b_{i+1} & = \frac{b_{i-1,x}}{2} + p_1 a_i + s_1 a_{i-1} + \frac{1}{2} p_1 q_1 b_{i-1}, \\ c_{i+1} & = -\frac{c_{i-1,x}}{2} + q_1 a_i + v_1 a_{i-1} + \frac{1}{2} p_1 q_1 c_{i-1}, \\ a_{i+1,x} & = -q_1 \frac{b_{i,x}}{2} - p_1 \frac{c_{i,x}}{2} + (p_1 v_1 - q_1 s_1) a_i - \frac{1}{2} p_1 q_1^2 b_i \\ & +\frac{1}{2} p_1^2 q_1 c_i + s_1 c_{i+1} - v_1 b_{i+1}, \\ f_{i+1} & = \frac{f_{i-1,x}}{2} - b_{i+1} + p_2 a_i + p_1 e_i + s_2 a_{i-1} + s_1 e_{i-1} \\ & +\frac{1}{2}(p_1 q_2 + p_2 q_1 - p_1 q_1) b_{i-1} + \frac{1}{2} p_1 q_1 f_{i-1}, \\ g_{i+1} & = -\frac{g_{i-1,x}}{2} - c_{i+1} + q_2 a_i + q_1 e_i + v_2 a_{i-1} + v_1 e_{i-1} \\ & +\frac{1}{2}(p_1 q_2 + p_2 q_1 - p_1 q_1) c_{i-1} + \frac{1}{2} p_1 q_1 g_{i-1}, \\ e_{i+1,x} & = -\frac{p_1 g_{i,x}}{2} - \frac{q_1 f_{i,x}}{2} + (p_2 - p_1)[-\frac{c_{m,x}}{2} + v_1 a_m + \frac{1}{2} p_1 q_1 c_m] \\ & +(q_1 - q_2))[\frac{b_{m,x}}{2} + s_1 a_m + \frac{1}{2} p_1 q_1 b_m] \\ & +s_1 g_{i+1} + s_2 c_{i+1} - v_1 f_{i+1} - v_2 b_{i+1} + (p_1 v_2 - q_1 s_2) a_i \\ & +(p_1 v_1 - q_1 s_1) e_i + \frac{1}{2} p_1 (p_1 q_2 + p_2 q_1 - p_1 q_1) c_i \\ & -\frac{1}{2} q_1 (p_1 q_2 + p_2 q_1 - p_1 q_1) b_i + \frac{1}{2} p_1^2 q_1 g_i - \frac{1}{2} p_1 q_1^2 f_i, \end{cases} \tag{207}$$

for all $i \geq 1$ with the same initial values (183) and conditions for integration (184) which determine the sequence of $\{a_i, b_i, c_i, e_i, f_i, g_i | i \geq 0\}$ uniquely. All $\{a_i, b_i, c_i, e_i, f_i, g_i\}$ can be proven as differential polynomials of $\bar{u}$ with respect to x.

Proposition 4.3. *Let $\{a_i, b_i, c_i, e_i, f_i, g_i | i = 0, 1\}$ be given by equation (183). Then all functions $\{a_i, b_i, c_i, e_i, f_i, g_i | i \geq 0\}$ determined by equations (207) with the conditions (184) are differential polynomials in $\bar{u}$ with respect to x, and thus, are local.*

Proof. For brevity, we leave the proof out. It is similar to Proposition 4.1. □

We solve the zero curvature equations (189) with the Lax matrices (191) to generate a hierarchy of soliton equations for all $m \geq 0$,

$$\bar{u}_{t_m} = \bar{K}_m = \begin{bmatrix} 2b_{m+1} \\ -2c_{m+1} \\ -2p_1 a_{m+1} + 2b_{m+2} \\ 2q_1 a_{m+1} - 2c_{m+2} \\ 2f_{m+1} + 2b_{m+1} \\ -2g_{m+1} - 2c_{m+1} \\ -2p_1 e_{m+1} - 2p_2 a_{m+1} + 2b_{m+2} + 2f_{m+2} \\ 2q_1 e_{m+1} + 2q_2 a_{m+1} - 2c_{m+2} - 2g_{m+2} \end{bmatrix}. \tag{208}$$

We have

$$\bar{K}_m = \bar{\Phi}\bar{K}_{m-1} = \bar{\Phi}^m \bar{K}_0, \quad m \geq 0, \tag{209}$$

where $\bar{\Phi}$ is a recursion operator determined from (207) and given by

$$\bar{\Phi} = \begin{bmatrix} \Phi & 0 \\ \Phi_1 - \Phi & \Phi \end{bmatrix}. \tag{210}$$

The matrix blocks of $\bar{\Phi}$ are defined by Φ from (137) with renamed variables $(p, q, r, s, v) \rightarrow$

$(p_1, q_1, r_1, s_1, v_1)$ and

$$\Phi_1 = \begin{bmatrix} -p_1\partial^{-1}v_2 & -p_1\partial^{-1}s_2 & 1-p_1\partial^{-1}q_2 & -p_1\partial^{-1}p_2 \\ -p_2\partial^{-1}v_1 & -p_2\partial^{-1}s_1 & -p_2\partial^{-1}q_1 & -p_2\partial^{-1}p_1 \\ & & & \\ q_1\partial^{-1}v_2 & q_1\partial^{-1}s_2 & q_1\partial^{-1}q_2 & 1+q_1\partial^{-1}p_2 \\ +q_2\partial^{-1}v_1 & +q_2\partial^{-1}s_1 & +q_2\partial^{-1}q_1 & +q_2\partial^{-1}p_1 \\ & & & \\ \widetilde{r_2} - s_1\partial^{-1}v_2 & -s_1\partial^{-1}s_2 & -s_1\partial^{-1}q_2 & -s_1\partial^{-1}p_2 \\ -s_2\partial^{-1}v_1 & -s_2\partial^{-1}s_1 & -s_2\partial^{-1}q_1 & -s_2\partial^{-1}p_1 \\ & & & \\ v_1\partial^{-1}v_2 & \widetilde{r_2} + v_2\partial^{-1}s_1 & v_1\partial^{-1}q_2 & v_1\partial^{-1}p_2 \\ +v_2\partial^{-1}v_1 & +v_1\partial^{-1}s_2 & +v_2\partial^{-1}q_1 & +v_2\partial^{-1}p_1 \end{bmatrix}, \tag{211}$$

where $\widetilde{r_2} = \frac{1}{2}(p_1q_2 + p_2q_1 - p_1q_1)$.

A specific example can be found from the reduced hierarchy of integrable couplings (208) when $m = 6$ by setting the eight potentials and α and β to be the following: $\{p_1 = q_1 = 0, s_1 = u, v_1 = -u, p_2 = q_2 = v, s_2 = w, v_2 = r, \alpha = -4, \beta = -8\}$. We find a coupled mKdV [34, 35] system of equations:

$$\begin{cases} u_t = -u_{xxx} - 6u^2u_x, \\ v_t = -v_{xxx} - 4uvu_x - 2u^2v_x + (4r - 4w)u^2, \\ w_t = -w_{xxx} + u_{xxx} + (6u^2 + (4v - 4w)u)u_x - 4u^2v_x - 2u^2w_x, \\ r_t = -r_{xxx} - u_{xxx} + (-6u^2 + (-4r + 4v)u)u_x - 2u^2r_x - 2u^2v_x. \end{cases} \tag{212}$$

4.3.1 Constructing bilinear forms over a non-semisimple Lie algebra

There is a systematic approach for generating Hamiltonian structures for the integrable couplings in (192) and (208) using the variational identity over the enlarged matrix loop algebra $\tilde{\mathfrak{g}}(\lambda)$ [64,68,69]. As seen in [64], there is a convenient method to constructing non-degenerate, symmetric, and ad-invariant bilinear forms on $\tilde{\mathfrak{g}}(\lambda)$ by rewriting $\tilde{\mathfrak{g}}(\lambda)$ into a vector form. The following four steps have been suggested in [64] to produce the required bilinear forms on $\tilde{\mathfrak{g}}(\lambda)$:

(1) Construct an isomorphism between the loop algebra $\tilde{\mathfrak{g}}(\lambda)$ and a vector Lie algebra;

(2) Derive the commutator on the vector Lie algebra;

(3) Compute the required non-degenerate, symmetric, and ad-invariant bilinear forms on the vector Lie algebra;

(4) Establish the corresponding bilinear forms on the original Lie algebra $\tilde{\mathfrak{g}}(\lambda)$.

The isomorphism

$$\sigma : \tilde{\mathfrak{g}}(\lambda) \to \mathbb{R}^6, A \mapsto (a_1, ..., a_6)^T, \tag{213}$$

where

$$A = M(A_1, A_2) \in \tilde{\mathfrak{g}}(\lambda), \quad A_i = \begin{bmatrix} a_{3i-2} & a_{3i-1} \\ a_{3i} & -a_{3i-2} \end{bmatrix}, i = 1, 2, \tag{214}$$

and a constant symmetric matrix,

$$F = \begin{bmatrix} 2\eta_1 & 0 & 0 & 2\eta_2 & 0 & 0 \\ 0 & 0 & \eta_1 & 0 & 0 & \eta_2 \\ 0 & \eta_1 & 0 & 0 & \eta_2 & 0 \\ 2\eta_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta_2 & 0 & 0 & 0 \\ 0 & \eta_2 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{215}$$

with arbitrary constants η_1 and η_2 furnish the bilinear forms on $\tilde{\mathfrak{g}}(\lambda)$ defined as

$$\begin{aligned}\langle A, B\rangle_{\tilde{\mathfrak{g}}(\lambda)} =&\langle\sigma(A), \sigma(B)\rangle_{\mathbb{R}^6} \\ =&(a_1, ..., a_6)F(b_1, ..., b_6)^T \\ =&(2a_1b_1 + a_2b_3 + a_3b_2)\eta_1 + (2a_1b_4 + a_2b_6 + a_3b_5 \\ &+ 2a_4b_1 + a_5b_3 + a_6b_2)\eta_2.\end{aligned} \tag{216}$$

The bilinear forms (216) are symmetric and ad-invariant due to the isomorphism σ. The bilinear forms, defined by (216), are non-degenerate iff the determinant of F is not zero, i.e.,

$$det(F) = -4\eta_2^6 \neq 0. \tag{217}$$

Therefore, we choose $\eta_2 \neq 0$ to obtain the required non-degenerate, symmetric, and ad-invariant bilinear forms over the enlarged matrix loop algebra $\tilde{\mathfrak{g}}(\lambda)$. For simplicity, we choose $\eta_1 = 0$ and $\eta_2 = 1$.

4.3.2 Hamiltonian structures of generalized D-Kaup-Newell integrable couplings

Now, we begin with the enlarged spectral matrix of the generalized D-Kaup-Newell hierarchy (177) and compute

$$\langle\bar{W}, \bar{U}_\lambda\rangle_{\tilde{\mathfrak{g}}(\lambda)} = (4a + 4e)\lambda + fq_1 + bq_2 + cp_2 + gp_1 \tag{218}$$

and

$$\langle\bar{W}, \bar{U}_{\bar{u}}\rangle_{\tilde{\mathfrak{g}}(\lambda)} = [g\lambda, f\lambda, -2e, g, f, c\lambda, b\lambda, -2a, c, b]^T. \tag{219}$$

Substituting the Laurent series and comparing powers of λ, and considering the case where $m = 1$ to see $\gamma = 0$, we have

$$\frac{\delta}{\delta \bar{u}} \int \frac{(4a_{m+2} + 4e_{m+2}) + f_{m+1}q_1 + b_{m+1}q_2 + c_{m+1}p_2 + g_{m+1}p_1}{m} dx = \\ [g_{m+1}, f_{m+1}, -2e_m, g_m, f_m, c_{m+1}, b_{m+1}, -2a_m, c_m, b_m]^T, \ \ m \geq 1. \tag{220}$$

Before we show the Hamiltonian structure for the soliton hierarchy (192), we must prove the following proposition.

PROPOSITION 4.4. *The operator $\bar{J}$ is a Hamiltonian operator defined by*

$$\bar{J} = \begin{bmatrix} 0 & J_1 \\ J_1 & J_2 \end{bmatrix} \tag{221}$$

where

$$J_1 = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\partial & s_1 & -v_1 \\ 0 & 0 & -s_1 & 0 & \partial + 2r_1 \\ 0 & 0 & v_1 & \partial - 2r_1 & 0 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & s_2 & -v_2 \\ 0 & 0 & -s_2 & 0 & 2r_2 \\ 0 & 0 & v_2 & -2r_2 & 0 \end{bmatrix}. \tag{222}$$

Proof. We can easily see by inspection that $\bar{J}^\dagger = -\bar{J}$, where $\bar{J}^\dagger$ is the adjoint of $\bar{J}$, and $\bar{J}$ is skew-adjoint. Note, we have proven that J_1 is a Hamiltonian operator in Chapter 2. We will use this fact to prove that $\bar{J}$ satisfies the Jacobi condition

$$< \bar{Z}, \bar{J}'(\bar{u})[\bar{J}\bar{X}]\bar{Y} > + \,\mathrm{cycle}(\bar{X}, \bar{Y}, \bar{Z}) \equiv 0 \ (\mathrm{mod}\ \partial), \tag{223}$$

for all vector fields $\bar{X}, \bar{Y}$, and $\bar{Z}$, where $\bar{J}'(\bar{u})[\bar{X}]$ denotes the Gateaux derivative of $\bar{J}$ with respect to $\bar{u}$ in the direction of $\bar{X}$ and $< \cdot, \cdot >$ denotes the standard inner product. Assume

$$\bar{X} = (X_1, X_2)^T, \quad \bar{Y} = (Y_1, Y_2)^T, \quad \bar{Z} = (Z_1, Z_2)^T, \quad \bar{W} = (W_1, W_2)^T,$$

and

$$X_i = (X_{1,i}, X_{2,i}, X_{3,i}, X_{4,i}, X_{5,i}), \quad Y_i = (Y_{1,i}, Y_{2,i}, Y_{3,i}, Y_{4,i}, Y_{5,i}),$$
$$Z_i = (Z_{1,i}, Z_{2,i}, Z_{3,i}, Z_{4,i}, Z_{5,i}), \quad W_i = (W_{1,i}, W_{2,i}, W_{3,i}, W_{4,i}, W_{5,i}), \quad i = 1, 2,$$

are ten-dimensional vector functions. By (221), we immediately have

$$\bar{J}\bar{X} = \begin{bmatrix} 0 & J_1 \\ J_1 & J_2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} J_1 X_2 \\ J_1 X_1 + J_2 X_2 \end{bmatrix} := \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}.$$

Using the definition of the Gateaux derivative, we compute $\bar{J}'[\bar{W}]$ and, then, $\bar{J}'[\bar{W}]\bar{Y}$ as:

$$\bar{J}'[\bar{W}] = \begin{bmatrix} 0 & J_1'[\bar{W}] \\ J_1'[\bar{W}] & J_2'[\bar{W}] \end{bmatrix} = \begin{bmatrix} 0 & J_1'[W_1] \\ J_1'[W_1] & J_2'[W_2] \end{bmatrix},$$

and

$$\bar{J}'[\bar{W}]\bar{Y} = \begin{bmatrix} 0 & J_1'[W_1] \\ J_1'[W_1] & J_2'[W_2] \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} J_1'[W_1] Y_2 \\ J_1'[W_1] Y_1 + J_2'[W_2] Y_2 \end{bmatrix},$$

Now, we see that

$$< \bar{Z}, \bar{J}'[\bar{W}]\bar{Y} > =< \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}, \begin{bmatrix} J_1'[W_1]Y_2 \\ J_1'[W_1]Y_1 + J_2'[W_2]Y_2 \end{bmatrix} >$$
$$=< Z_1, J_1'[W_1]Y_2 > + < Z_2, J_1'[W_1]Y_1 + J_2'[W_2]Y_2 >$$
$$=< Z_1, J_1'[W_1]Y_2 > + < Z_2, J_1'[W_1]Y_1 > + < Z_2, J_2'[W_2]Y_2 > .$$

Now, we may substitute $W_1 = J_1X_2$ and use the fact that J_1 is a Hamiltonian operator to see that

$$< Z_1, J_1'[J_1X_2]Y_2 > + \text{cycle}(X_2, Y_2, Z_1) \equiv 0 \text{ (mod } \partial),$$

and

$$< Z_2, J_1'[J_1X_2]Y_1 > + \text{cycle}(X_2, Y_1, Z_2) \equiv 0 \text{ (mod } \partial).$$

Therefore, we only have to show that

$$< Z_2, J_2'[W_2]Y_2 > + \text{cycle}(\bar{X}, \bar{Y}, \bar{Z}) \equiv 0 \text{ (mod } \partial). \tag{224}$$

Let's begin with computing $J_2'[W_2]$ and $J_2'[W_2]Y_2$. First,

$$J_2'[W_2] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & W_{4,2} & -W_{5,2} \\ 0 & 0 & -W_{4,2} & 0 & 2W_{3,2} \\ 0 & 0 & W_{5,2} & -2W_{3,2} & 0 \end{bmatrix},$$

and

$$
J_2'[W_2]Y_2 = \begin{bmatrix} 0 \\ 0 \\ W_{4,2}Y_{4,2} - W_{5,2}Y_{5,2} \\ -W_{4,2}Y_{3,2} + 2W_{3,2}Y_{5,2} \\ W_{5,2}Y_{3,2} - 2W_{3,2}Y_{4,2} \end{bmatrix}.
$$

Now, we see that

$$
\begin{aligned} < Z_2, J_2'[W_2]Y_2 > &= 2W_{3,2}(Z_{4,2}Y_{5,2} - Z_{5,2}Y_{4,2}) + W_{4,2}(Z_{3,2}Y_{4,2} - Z_{4,2}Y_{3,2}) \\ &+ W_{5,2}(Z_{5,2}Y_{3,2} - Z_{3,2}Y_{5,2}). \end{aligned} \tag{225}
$$

Recall,

$$
\bar{W} = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \begin{bmatrix} J_1X_2 \\ J_1X_1 + J_2X_2 \end{bmatrix},
$$

where

$$
W_2 = J_1X_1 + J_2X_2 = \begin{bmatrix} 2X_{2,1} + 2X_{2,2} \\ -2X_{1,1} - 2X_{1,2} \\ \frac{1}{2}X_{3,1,x} + s_1X_{4,1} - v_1X_{5,1} + s_2X_{4,2} - v_2X_{5,2} \\ -s_1X_{3,1} + X_{5,1,x} + 2r_1X_{5,1} - s_2X_{3,2} + 2r_2X_{5,2} \\ v_1X_{3,1} + X_{4,1,x} - 2r_1X_{4,1} + v_2X_{3,2} - 2r_2X_{4,2} \end{bmatrix}. \tag{226}
$$

Therefore, we see from (225) and (226) that

$$
\begin{aligned}
< Z_2, J_2'[W_2]Y_2 > &= 2(\frac{1}{2}X_{3,1,x} + s_1 X_{4,1} - v_1 X_{5,1} + s_2 X_{4,2} - v_2 X_{5,2})(Z_{4,2}Y_{5,2} - Z_{5,2}Y_{4,2}) \\
&+ (-s_1 X_{3,1} + X_{5,1,x} + 2r_1 X_{5,1} - s_2 X_{3,2} + 2r_2 X_{5,2})(Z_{3,2}Y_{4,2} - Z_{4,2}Y_{3,2}) \\
&+ (v_1 X_{3,1} + X_{4,1,x} - 2r_1 X_{4,1} + v_2 X_{3,2} - 2r_2 X_{4,2})(Z_{5,2}Y_{3,2} - Z_{3,2}Y_{5,2}) \\
&= [(X_{3,1,x})(Z_{4,2}Y_{5,2} - Z_{5,2}Y_{4,2}) + (X_{4,1,x})(Z_{5,2}Y_{3,2} - Z_{3,2}Y_{5,2}) \\
&+ (X_{5,1,x})(Z_{3,2}Y_{4,2} - Z_{4,2}Y_{3,2})] \\
&+ [(s_1 X_{4,1} - v_1 X_{5,1} + s_2 X_{4,2} - v_2 X_{5,2})(Z_{4,2}Y_{5,2} - Z_{5,2}Y_{4,2}) \\
&+ (-s_1 X_{3,1} + 2r_1 X_{5,1} - s_2 X_{3,2} + 2r_2 X_{5,2})(Z_{3,2}Y_{4,2} - Z_{4,2}Y_{3,2}) \\
&+ (v_1 X_{3,1} - 2r_1 X_{4,1} + v_2 X_{3,2} - 2r_2 X_{4,2})(Z_{5,2}Y_{3,2} - Z_{3,2}Y_{5,2})].
\end{aligned}
$$

We may make the following decomposition:

$$
< Z_2, J_2'[W_2]Y_2 > = R(\bar{X}, \bar{Y}, \bar{Z}) + S(\bar{X}, \bar{Y}, \bar{Z}), \tag{227}
$$

where

$$
\begin{aligned}
R(\bar{X}, \bar{Y}, \bar{Z}) &= (X_{3,1,x})(Z_{4,2}Y_{5,2} - Z_{5,2}Y_{4,2}) + (X_{4,1,x})(Z_{5,2}Y_{3,2} - Z_{3,2}Y_{5,2}) \\
&+ (X_{5,1,x})(Z_{3,2}Y_{4,2} - Z_{4,2}Y_{3,2}), \\
S(\bar{X}, \bar{Y}, \bar{Z}) &= (s_1 X_{4,1} - v_1 X_{5,1} + s_2 X_{4,2} - v_2 X_{5,2})(Z_{4,2}Y_{5,2} - Z_{5,2}Y_{4,2}) \\
&+ (-s_1 X_{3,1} + 2r_1 X_{5,1} - s_2 X_{3,2} + 2r_2 X_{5,2})(Z_{3,2}Y_{4,2} - Z_{4,2}Y_{3,2}) \\
&+ (v_1 X_{3,1} - 2r_1 X_{4,1} + v_2 X_{3,2} - 2r_2 X_{4,2})(Z_{5,2}Y_{3,2} - Z_{3,2}Y_{5,2}).
\end{aligned}
$$

For the functions R and S, we may make the following computation:

$$\begin{aligned}
R(\bar{X},\bar{Y},\bar{Z})+&\text{cycle}(\bar{X},\bar{Y},\bar{Z})\\
&= R(\bar{X},\bar{Y},\bar{Z}) + \text{cycle}(X_1, Y_2, Z_2)\\
&= \partial\{X_{3,1}Z_{4,2}Y_{5,2} - X_{3,1}Y_{4,2}Z_{5,2}\} + \text{cycle}(X_1, Y_2, Z_2),\\
S(\bar{X},\bar{Y},\bar{Z})+&\text{cycle}(\bar{X},\bar{Y},\bar{Z})\\
&= S(\bar{X},\bar{Y},\bar{Z}) + \text{cycle}(X_1, Y_2, Z_2) + \text{cycle}(X_2, Y_2, Z_2) \qquad (228)\\
&= \partial\{\partial^{-1}[(s_1X_{4,1} - v_1X_{5,1} + s_2X_{4,2} - v_2X_{5,2})(Z_{4,2}Y_{5,2} - Z_{5,2}Y_{4,2})\\
&+ (-s_1X_{3,1} + 2r_1X_{5,1} - s_2X_{3,2} + 2r_2X_{5,2})(Z_{3,2}Y_{4,2} - Z_{4,2}Y_{3,2})\\
&+ (v_1X_{3,1} - 2r_1X_{4,1} + v_2X_{3,2} - 2r_2X_{4,2})(Z_{5,2}Y_{3,2} - Z_{3,2}Y_{5,2})]\}\\
&+ \text{cycle}(\bar{X},\bar{Y},\bar{Z}).
\end{aligned}$$

Therefore, they are both total derivatives. Using (228), we see that $\bar{J}$ satisfies the Jacobi identity,

$$< \bar{Z}, \bar{J}'(\bar{u})[\bar{J}\bar{X}]\bar{Y} > + \text{cycle}(\bar{X},\bar{Y},\bar{Z}) \equiv 0 \ (\text{mod } \partial),$$

and $\bar{J}$ is a Hamiltonian operator. □

A long calculation involving the recursion relations (182) shows that

$$\frac{\delta \bar{\mathcal{H}}_{m+1}}{\delta \bar{u}} = \begin{bmatrix} g_{m+2} \\ f_{m+2} \\ -2e_{m+1} \\ g_{m+1} \\ f_{m+1} \\ c_{m+2} \\ b_{m+2} \\ -2a_{m+1} \\ c_{m+1} \\ b_{m+1} \end{bmatrix} = \bar{\Psi} \begin{bmatrix} g_{m+1} \\ f_{m+1} \\ -2e_{m} \\ g_{m} \\ f_{m} \\ c_{m+1} \\ b_{m+1} \\ -2a_{m} \\ c_{m} \\ b_{m} \end{bmatrix} = \bar{\Psi} \frac{\delta \bar{\mathcal{H}}_m}{\delta \bar{u}}, \tag{229}$$

where

$$\bar{\Psi} = \bar{\Phi}^{\dagger} = \begin{bmatrix} \Phi^{\dagger} & (\Phi_1 - \Phi)^{\dagger} \\ 0 & \Phi^{\dagger} \end{bmatrix}, \tag{230}$$

with Φ and Φ_1 from (102) and (195), respectively.

We consequently obtain Hamiltonian structures for the hierarchy of integrable couplings (192) [76], i.e.,

$$\bar{u}_{t_m} = \bar{J} \frac{\delta \bar{\mathcal{H}}_m}{\delta \bar{u}}, \;\; m \geq 0, \tag{231}$$

with the Hamiltonian functionals

$$\bar{\mathcal{H}}_m = \int \frac{(4a_{m+2} + 4e_{m+2}) + f_{m+1}q_1 + b_{m+1}q_2 + c_{m+1}p_2 + g_{m+1}p_1}{m} dx, \tag{232}$$

for $m \geq 1$, and

$$\bar{\mathcal{H}}_0 = \int [(\beta - \alpha)p_1 q_1 + \alpha(p_1 q_2 + p_2 q_1) - 2\beta r_1 - 2\alpha r_2] dx \tag{233}$$

calculated directly from $[g_1, f_1, -2e_0, g_0, f_0, c_1, b_1, -2a_0, c_0, b_0]^T$. The Hamiltonian operator in (231) is of the form:

$$\bar{J} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\partial & s_1 & -v_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -s_1 & 0 & \partial + 2r_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & v_1 & \partial - 2r_1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\partial & s_1 & -v_1 & 0 & 0 & 0 & s_2 & -v_2 \\ 0 & 0 & -s_1 & 0 & \partial + 2r_1 & 0 & 0 & -s_2 & 0 & 2r_2 \\ 0 & 0 & v_1 & \partial - 2r_1 & 0 & 0 & 0 & v_2 & -2r_2 & 0 \end{bmatrix}. \tag{234}$$

As a direct result of the Hamiltonian structures (231), the recursion structure (193) and (229), and the property $\bar{J}\bar{\Psi} = \bar{\Psi}^\dagger \bar{J}$, the hierarchy (192) has the following commutativity of flows:

$$\{\bar{\mathcal{H}}_k, \bar{\mathcal{H}}_l\}_{\bar{J}} = \int \left(\frac{\delta \bar{\mathcal{H}}_k}{\delta \bar{u}} \right)^T \bar{J} \frac{\delta \bar{\mathcal{H}}_l}{\delta \bar{u}} dx = 0, \tag{235}$$

and the commutativity of symmetries for $\{\bar{K}_n\}$, i.e.,

$$[\bar{K}_k, \bar{K}_l] = \bar{K}_k'(\bar{u})[\bar{K}_l] - \bar{K}_l'(\bar{u})[\bar{K}_k] = 0, \quad k, l \geq 0. \tag{236}$$

Therefore, the hierarchy (192) is Liouville integrable, as expected.

4.3.3 Bi-Hamiltonian structures of the reduced integrable couplings

Next, we focus on the reduced spectral matrix (204) and compute

$$\langle \bar{W}, \bar{U}_{\lambda} \rangle_{\tilde{\mathfrak{g}}(\lambda)} = (4a + 4e)\lambda + fq_1 + bq_2 + cp_2 + gp_1, \tag{237}$$

and

$$\langle \bar{W}, \bar{U}_{\bar{u}} \rangle_{\tilde{\mathfrak{g}}(\lambda)} = [(a-e)q_1 - aq_2 + g\lambda, (a-e)p_1 - ap_2 + f\lambda, g, f, -aq_1 + c\lambda, -ap_1 + b\lambda, c, b]^T. \tag{238}$$

Again, we compare powers of λ after substituting the Laurent series for a, b, c, e, f, g to get

$$\begin{aligned} &\frac{\delta}{\delta \bar{u}} \int \frac{(4a_{m+2} + 4e_{m+2}) + f_{m+1}q_1 + b_{m+1}q_2 + c_{m+1}p_2 + g_{m+1}p_1}{m} dx = \\ &[(a_m - e_m)q_1 - a_m q_2 + g_{m+1}, (a_m - e_m)p_1 - a_m p_2 + f_{m+1}, g_m, f_m, \\ &-a_m q_1 + c_{m+1}, -a_m p_1 + b_{m+1}, c_m, b_m]^T, \;\; m \geq 1. \end{aligned} \tag{239}$$

Now using the recursion relations (207), we have $\frac{\delta \bar{\mathcal{H}}_{m+1}}{\delta \bar{u}} = \bar{\Psi} \frac{\delta \bar{\mathcal{H}}_m}{\delta \bar{u}}$, i.e.,

$$\begin{bmatrix} (a_{m+1} - e_{m+1})q_1 - a_{m+1}q_2 + g_{m+2} \\ (a_{m+1} - e_{m+1})p_1 - a_{m+1}p_2 + f_{m+2} \\ g_{m+1} \\ f_{m+1} \\ -a_{m+1}q_1 + c_{m+2} \\ -a_{m+1}p_1 + b_{m+2} \\ c_{m+1} \\ b_{m+1} \end{bmatrix} = \bar{\Psi} \begin{bmatrix} (a_m - e_m)q_1 - a_m q_2 + g_{m+1} \\ (a_m - e_m)p_1 - a_m p_2 + f_m \\ g_m \\ f_m \\ -a_m q_1 + c_{m+1} \\ -a_m p_1 + b_{m+1} \\ c_m \\ b_m \end{bmatrix}, \tag{240}$$

where

$$\bar{\Psi} = \bar{\Phi}^{\dagger} = \begin{bmatrix} \Phi^{\dagger} & (\Phi_1 - \Phi)^{\dagger} \\ 0 & \Phi^{\dagger} \end{bmatrix}, \tag{241}$$

with Φ and Φ_1 from (137) and (211), respectively.

We finally obtain the bi-Hamiltonian structure for the hierarchy of integrable couplings (208),

$$\bar{u}_{t_m} = \bar{J}\frac{\delta \bar{\mathcal{H}}_{m+1}}{\delta \bar{u}} = \bar{M}\frac{\delta \bar{\mathcal{H}}_m}{\delta \bar{u}}, \ \ m \geq 0, \tag{242}$$

with the Hamiltonian functionals

$$\bar{\mathcal{H}}_m = \int \frac{(4a_{m+2} + 4e_{m+2}) + f_{m+1}q_1 + b_{m+1}q_2 + c_{m+1}p_2 + g_{m+1}p_1}{m} dx, \tag{243}$$

for $m \geq 1$, and the Hamiltonian operators,

$$\bar{J} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 & 0 & 0 & -2 & 0 \\ 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \end{bmatrix} \tag{244}$$

and $\bar{M} = \bar{\Phi}\bar{J}$ where $\bar{J}$ is above (244) and $\bar{\Phi}$ is the recursion operator (210) for the reduced integrable couplings (208). Recall, a bi-Hamiltonian property means that $\bar{J}$ and $\bar{M}$ constitute a Hamiltonian pair, or, $\bar{N} = \alpha\bar{J} + \beta\bar{M}$, for any $\alpha, \beta \in \mathbb{R}$, is a Hamiltonian operator. As a direct result of the bi-Hamiltonian structure (242), we can say that the soliton hierarchy (208) is integrable in the

Liouville sense:

$$\begin{cases} \{\bar{\mathcal{H}}_k, \bar{\mathcal{H}}_l\}_{\bar{J}} = & \int \left(\frac{\delta \bar{\mathcal{H}}_k}{\delta \bar{u}}\right)^T \bar{J} \frac{\delta \bar{\mathcal{H}}_l}{\delta \bar{u}} dx = 0, \\ \{\bar{\mathcal{H}}_k, \bar{\mathcal{H}}_l\}_{\bar{M}} = & \int \left(\frac{\delta \bar{\mathcal{H}}_k}{\delta \bar{u}}\right)^T \bar{M} \frac{\delta \bar{\mathcal{H}}_l}{\delta \bar{u}} dx = 0, \end{cases} \tag{245}$$

and

$$[\bar{K}_k, \bar{K}_l] = \bar{K}_k'(\bar{u})[\bar{K}_l] - \bar{K}_l'(\bar{u})[\bar{K}_k] = 0, \quad k, l \geq 0. \tag{246}$$

4.4 Summary

We enlarged the original generalization of a D-Kaup-Newell spectral problem from chapter two and generated its integrable couplings by solving the corresponding enlarged zero curvature equations. We then enlarged the previous chapters reduced spectral matrix and found its integrable couplings. Using the variational identity, the Hamiltonian structures for both a generalized D-Kaup-Newell integrable couplings and the reduced couplings were constructed. The reduced hierarchy of integrable couplings was found to have bi-Hamiltonian structures. Both hierarchies were shown to be Liouville integrable and possess infinitely many commuting symmetries and conserved densities.

5 Darboux Transformations

5.1 Introduction

A Darboux transformation is a powerful tool to generate new solutions from known solutions [26], [79]- [87]. Since the middle of the 1960's, Darboux transformations have been implemented in soliton theory [26], [79]- [82]. These transformations have mostly been used on integrable systems associated with semisimple matrix Lie algebras. Recently, the question of how to construct a Darboux transformation for integrable couplings associated with non-semisimple Lie algebras was addressed [86]. Darboux transformations were formulated for integrable couplings of an AKNS type, where the spacial spectral matrix is a degree one polynomial. In this chapter, we construct Darboux transformations for integrable couplings where the spectral matrix is any polynomial in λ.

We begin this chapter with a brief historical background of a Darboux transformation. Then the theory of Darboux transformations for integrable systems is reviewed. Built off of this theory, we develop the Darboux transformations for integrable couplings where the integrable system in the previous section is a subsystem. The chapter is finished with an application to a generalized D-Kaup-Newell integrable coupling system. An explicit solution set to the m-th order coupling system is constructed and presented.

5.2 Background

The original result, in 1882 by G. Darboux [88], was related to the theory of the one-dimensional Schrödinger equation,

$$-\phi_{xx} - u(x)\phi = \lambda\phi, \tag{247}$$

where $u(x)$, the potential, is given and λ is the spectral parameter. The discovery of Darboux can be summarized in the following theorem.

Theorem 5.1. *[88] If $u(x)$ and $\phi(x, \lambda)$ are two functions satisfying (247) and $f(x) = \phi(x, \lambda_0)$ is a solution of (247) for $\lambda = \lambda_0$, where λ_0 is a fixed constant, then the functions u' and ϕ' defined by*

$$u' = u + 2(\ln f)_{xx}, \quad \phi'(x, \lambda) = \phi_x(x, \lambda) - \frac{f_x}{f}\phi(x, \lambda), \tag{248}$$

satisfy

$$-\phi'_{xx} - u'\phi' = \lambda\phi', \tag{249}$$

which is of the same form as (247).

The transformation

$$(u, \phi) \to (u', \phi') \tag{250}$$

is the original Darboux transformation, which is valid for $f \neq 0$.

The result of Theorem 5.1 was applied to soliton theory in the middle of the 1960's when the KdV equation was found to be closely related to the Schrödinger equation [35].

Example 5.1. *[26] Recall, the KdV equation (3) with $\alpha = 1, \beta = 6$, is*

$$u_t + 6uu_x + u_{xxx} = 0. \tag{251}$$

We have seen that the KdV equation is the integrability condition (compatibility condition) of the *system of linear equations*

$$\begin{cases} -\phi(x,t)_{xx} - u\phi(x,t) = \lambda\phi(x,t), \\ \phi(x,t)_t = -4\phi(x,t)_{xxx} - 6u\phi(x,t)_x - 3u_x\phi(x,t) \end{cases} \tag{252}$$

which is called the Lax pair of the KdV equation. By integrability condition, we mean (251) is the necessary and sufficient for $(\phi_{xx})_t = (\phi_t)_{xx}$ *being true for all* λ.

Recognizing the first equation in (252) is just the Schrödinger equation (247), the Darboux transformation in Theorem 5.1 can be applied to the KdV equation. The transformation $(u,\phi) \rightarrow (u',\phi')$ *is invariant on the Lax pair of the KdV, i.e.,*

$$\begin{cases} -\phi'(x,t)_{xx} - u'\phi'(x,t) = \lambda\phi'(x,t), \\ \phi'(x,t)_t = -4\phi'(x,t)_{xxx} - 6u'\phi'(x,t)_x - 3u'_x\phi'(x,t). \end{cases} \tag{253}$$

Therefore, u' *is a solution to the KdV equation.*

If u *is a known solution to the KdV equation (251), the problem of finding another solution simplifies to finding a solution* ϕ *to the linear system (252). Once* ϕ *is found, take* $\lambda = \lambda_0$ *and let* $f(x,t) = \phi(x,t,\lambda_0)$. *Then* $u' = u + 2(\ln f)_{xx}$ *is a new solution to the KdV equation and* ϕ' *given by Theorem 5.1 is a solution of the Lax pair of the KdV equation corresponding to* u'. *This process may be repeated successively. Notice, given a solution* u*, we only need to solve a linear system* ϕ. *After that, each new solution in the chain*

$$(u,\phi) \rightarrow (u',\phi') \rightarrow (u'',\phi'') \rightarrow ... \tag{254}$$

is given algebraically by Theorem 5.1.

In the 1880's, Bäcklund transformations were developed for use in theories of differential equations and differential geometry. Later, there was found to be a close relationship between the inverse scattering transform (IST) mentioned in chapter one and a Bäcklund transformation, namely, every evolution equation solvable by IST has a corresponding Bäcklund transformation [10]. Let's define a Bäcklund transformation.

Definition 5.1. *[10] Suppose that we have two uncoupled partial differential equations, in two independent variables x and t, for two functions u and v; the two equations are expressed as*

$$P(u) = 0, \quad \textit{and} \quad Q(v) = 0, \tag{255}$$

where P and Q are two operators, which are in general nonlinear. Let $R_i = 0$ be a pair of relations

$$R_i(u, v, u_x, v_x, u_t, v_t, ...; x, t) = 0, \quad i = 1, 2, \tag{256}$$

between the two functions u and v. Then $R_i = 0$ is a Bäcklund transformation *if it is integrable for v when $P(u) = 0$ and if the resulting v is a solution of $Q(v) = 0$, and vice versa. If $P = Q$, so that u and v satisfy the same equation, then $R_i = 0$ is called an* auto-Bäcklund transformation.

Example 5.2. *[10] There is a transformation from the KdV to the mKdV (111) [34] known as the Miura transformation [35]. If we have the KdV from (251) and the mKdV as*

$$v_t - 6v^2 v_x + v_{xxx} = 0, \tag{257}$$

then the Miura transformation is

$$u = v^2 + v_x. \tag{258}$$

It is known that if v solves the (257), then u from (258) solves the KdV (1). We may eliminate higher derivatives from the mKdV and regard the Miura transformation (258) and the mKdV (257) as a Bäcklund transformation of the KdV (1).

Similar to the example of a Darboux transformation on the Lax pair of the KdV equation, there is a theory of Darboux transformations for integrable systems. We will see that these transformations generate Bäcklund transformations for solutions of certain integrable systems.

5.3 Darboux transformations of integrable systems

Let

$$u_t = K(u, u_x, u_{xx}, \ldots) \tag{259}$$

be an integrable system of partial differential equations where u is a function or a vector valued function. Suppose (259) is equivalent to the zero curvature equation

$$U_t - V_x + [U, V] = 0, \tag{260}$$

where two square matrices U and V, a Lax pair of (259), belong to a matrix loop algebra [20,59,60]. As stated before, the zero curvature equation is the compatibility condition of the spectral problems

$$\phi_x = U\phi = U(u, \lambda)\phi, \quad \phi_t = V\phi = V(u, \lambda)\phi, \tag{261}$$

where λ is the spectral parameter and ϕ is the vector eigenfunction. One way to find explicit solutions to the integrable system (259) is through Darboux transformations on the spectral problems (261).

We begin with a definition of a Darboux transformation for spectral problems (261).

Definition 5.2. *A* Darboux transformation *of the spectral problems (261) is a transformation $\phi' = D\phi$, with a square matrix $D = D(u, \lambda)$, where ϕ' and u' satisfy spectral problems of the same form as (261), i.e.,*

$$\phi'_x = U'\phi' = U'(u', \lambda)\phi', \quad \phi'_t = V'\phi' = V(u', \lambda)\phi', \tag{262}$$

and U' and V' are of the same form as U and V, respectively. The matrix D is called the Darboux matrix *of the spectral problem (261).*

For this section, we will be discussing spectral problems of this form:

$$\phi_x = U\phi = U(u,\lambda)\phi = \sum_{j=0}^{n} U_j \lambda^{n-j}\phi, \quad \phi_t = V^{[m]}\phi = V^{[m]}(u,\lambda)\phi = \sum_{j=0}^{m} V\lambda^{m-j}\phi. \tag{263}$$

Note, the spacial spectral matrix is a polynomial in λ of any order n. Any constant matrix K, independent of λ, is a Darboux matrix of degree 0. This case is considered trivial and not for the purposes of our discussion.

Assume that U and V are $N \times N$ matrices. Let a Darboux matrix, linear and of the first order in λ, be

$$D(\lambda) = \lambda I + S, \tag{264}$$

where I is an $N \times N$ identity matrix and S is an $N \times N$ matrix independent of λ. We may derive differential equations of S. Let's substitute D from (264) and $\phi' = D\phi$ into the first equation in (263). Then

$$\sum_{j=0}^{n} U_j' \lambda^{n-j}(\lambda I - S)\phi = ((\lambda I - S)\phi)_x = (\lambda I - S)\sum_{j=0}^{n} U_j \lambda^{n-j}\phi - S_x\phi. \tag{265}$$

Comparing the coefficients of powers of λ gives the following equalities:

$$U_0' = U_0, \quad U_{j+1}' = U_{j+1} + U_j' S - SU_j, \quad 0 \le j \le n-1, \tag{266}$$

and

$$S_x = U_n' S - SU_n. \tag{267}$$

These equalities produce

$$U_0' = U_0, \quad U_j' = U_j + \sum_{k=1}^{j} [U_{j-k}, S] S^{k-1}, \quad 1 \le j \le n, \tag{268}$$

and

$$S_x + [S, \sum_{j=0}^{n} U_j S^{n-j}] = 0. \tag{269}$$

Similarly, we substitute D and $\phi' = D\phi$ into the second equation in (263) giving

$$\sum_{j=0}^{m} V_j' \lambda^{m-j} (\lambda I - S)\phi = ((\lambda I - S)\phi)_{t_m} = (\lambda I - S) \sum_{j=0}^{m} V_j \lambda^{m-j} \phi - S_{t_m} \phi. \tag{270}$$

Again, we compare the coefficients of powers of λ gives the following equalities:

$$V_0' = V_0, \quad V_{j+1}' = V_{j+1} + V_j' S - S V_j, \quad 0 \le j \le m-1, \tag{271}$$

and

$$S_{t_m} = V_m' S - S V_m. \tag{272}$$

Then, we have the following:

$$V_0' = V_0, \quad V_j' = V_j + \sum_{k=1}^{j} [V_{j-k}, S] S^{k-1}, \quad 1 \le j \le m, \tag{273}$$

and

$$S_{t_m} + [S, \sum_{j=0}^{m} V_j S^{m-j}] = 0. \tag{274}$$

The analysis above may be summarized by the following theorem.

THEOREM 5.2. *[26] $D = \lambda I - S$ is a Darboux matrix for (263) if and only if S satisfies*

$$S_x + [S, \sum_{j=0}^{n} U_j S^{n-j}] = 0, \quad S_{t_m} + [S, \sum_{j=0}^{m} V_j S^{m-j}] = 0. \tag{275}$$

Additionally, the Darboux transformation generates a Bäcklund transformation of (263) as

$$U_0' = U_0, \quad U_j' = U_j + \sum_{k=1}^{j} [U_{j-k}, S] S^{k-1}, \quad 1 \leq j \leq n. \tag{276}$$

Let's introduce N distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_N$ with their corresponding eigenfunctions:

$$\phi_x^{(s)} = U(u, \lambda_s)\phi^{(s)}, \quad \phi_t^{(s)} = V(u, \lambda_s)\phi^{(s)}, \quad 1 \leq s \leq N, \tag{277}$$

where u is a given solution to (259). If $\det(H) \neq 0$, then suppose

$$S = H \Lambda H^{-1}, \tag{278}$$

where

$$H = (\phi^{(1)}, \phi^{(2)}, ..., \phi^{(N)}), \quad \Lambda = \mathrm{diag}(\lambda_1, \lambda_2, ..., \lambda_N). \tag{279}$$

THEOREM 5.3. *[26] The matrix $D = \lambda I - S$, where S is defined by (278), is a Darboux matrix for (263).*

A proof of the theorem shows that Theorem 5.2 is true for the given Darboux matrix $D = \lambda I - S$. The key strategy is to take $\phi^{(s)}, 1 \leq s \leq N$, to be eigenfunctions of (261) for $\lambda = \lambda_i$, then

$$\phi_x^{(s)} = \sum_{j=0}^{n} U_j \lambda^{n-j} h_i, \quad \phi_t^{(s)} = \sum_{j=0}^{m} V_j \lambda^{m-j} h_i. \tag{280}$$

This implies the following:

$$H_x = \sum_{j=0}^{n} U_j H \Lambda^{n-j}, \quad H_t = \sum_{j=0}^{n} V_j H \Lambda^{m-j}. \tag{281}$$

We expand on this idea in the next section when we formulate a Darboux and Bäcklund transformations for integrable couplings where the enlarged spacial spectral matrix is a polynomial of λ with degree n.

5.4 Darboux transformations of integrable couplings

Recall, an integrable coupling system is an extension of a given integrable system to a larger integrable system containing the original system as a subsystem of the form (153), where $u_t = K(u)$ is the original system. We may write (153) compactly as:

$$\bar{u}_t = \bar{K}(\bar{u}), \tag{282}$$

where the enlarged dependent variable is $\bar{u} = (u^T, v^T)^T$.

Let's start with a few assumptions. Assume a hierarchy of enlarged spectral problems,

$$\bar{\phi}_x = \bar{U}(\bar{u}, \lambda)\phi, \quad \bar{\phi}_{t_m} = \bar{V}^{[m]}(\bar{u}, \lambda)\phi, \tag{283}$$

is associated with integrable couplings,

$$\bar{u}_{t_m} = \bar{K}_m(\bar{u}) = (K_m^T(u), T_m^T(u, v))^T, \quad \bar{u} = (u^T, v^T)^T, \quad m \geq 0, \tag{284}$$

through the enlarged zero curvature equations,

$$\bar{U}_{t_m} - \bar{V}_x^{[m]} + [\bar{U}, \bar{V}^{[m]}] = 0, \tag{285}$$

where the enlarged Lax pairs are upper triangular block matrices,

$$\bar{U}(\bar{u},\lambda) = \begin{bmatrix} U(u,\lambda) & U_1(\bar{u},\lambda) \\ 0 & U(u,\lambda) \end{bmatrix}, \quad \bar{V}^{[m]}(\bar{u},\lambda) = \begin{bmatrix} V^{[m]}(u,\lambda) & V_1^{[m]}(\bar{u},\lambda) \\ 0 & V^{[m]}(u,\lambda) \end{bmatrix}. \tag{286}$$

In general, we may write

$$\begin{cases} \bar{U} = \sum_{j=0}^{n} \bar{U}_j \lambda^{n-j} = \sum_{j=0}^{n} \begin{bmatrix} U_j & U_{1j} \\ 0 & U_j \end{bmatrix} \lambda^{n-j}, \\ \bar{V}^{[m]} = \sum_{j=0}^{m} \bar{V}_j \lambda^{m-j} = \sum_{j=0}^{m} \begin{bmatrix} V_j & V_{1j} \\ 0 & V_j \end{bmatrix} \lambda^{m-j}, \end{cases} \tag{287}$$

where

$$\begin{cases} U(u,\lambda) = \sum_{j=0}^{n} U_j \lambda^{n-j}, \qquad U_1(\bar{u},\lambda) = \sum_{j=0}^{n} U_{1j} \lambda^{n-j}, \\ V^{[m]}(u,\lambda) = \sum_{j=0}^{m} V_j \lambda^{m-j}, \quad V_1^{[m]}(\bar{u},\lambda) = \sum_{j=0}^{m} V_{1j} \lambda^{m-j}, \quad m \geq 0, \end{cases} \tag{288}$$

are $N \times N$ matrices. Then the enlarged spectral problem may be viewed as:

$$\bar{\phi}_x = \bar{U}\bar{\phi} = \sum_{j=0}^{n} \bar{U}_j \lambda^{n-j} \bar{\phi}, \quad \bar{\phi}_{t_m} = \bar{V}^{[m]} \bar{\phi} = \sum_{j=0}^{m} \bar{V}_j \lambda^{m-j} \bar{\phi}. \tag{289}$$

We will discuss a class of Darboux matrices for the Lax pairs above without reductions, i.e., the entries of U_j, U_{1j}, V_j and V_{1j} are independent except for the partial differential equations (285).

Let $\bar{D} = \bar{D}(x,t,\lambda)$ be a $2N \times 2N$ matrix. If $\bar{\phi}' = \bar{D}\bar{\phi}$ satisfies a spectral problem of the same form as (289), i.e.,

$$\bar{\phi}'_x = \bar{U}'(\bar{u},\lambda)\bar{\phi}' = \sum_{j=0}^{n} \bar{U}'_j \lambda^{n-j} \bar{\phi}', \bar{\phi}'_{t_m} = \bar{V}'^{[m]}(\bar{u},\lambda)\bar{\phi}' = \sum_{j=0}^{m} \bar{V}'_j \lambda^{m-j} \bar{\phi}', m \geq 0, \tag{290}$$

then we have the Darboux transformation of the enlarged spectral problem (289) as:

$$(\bar{U}, \bar{V}^{[m]}, \bar{\phi}) \to (\bar{U}', \bar{V}'^{[m]}, \bar{\phi}') \tag{291}$$

where $\bar{D}$ is the Darboux matrix of (289).

We focus on a class of Darboux matrices of degree one that are linear in λ. Thus, the Darboux matrices are of the form $\bar{D} = \lambda\bar{I} - \bar{S}$ where $\bar{I} = \mathrm{diag}(I, I)$, I an $N \times N$ identity matrix, and $\bar{S}$ a $2N \times 2N$ matrix function. We have the same analysis of the first order Darboux matrix for the enlarged spectral problem as seen in the previous section 5.3 and may be summarized by the following theorem:

Theorem 5.4. *[86, 87] $\bar{D} = \lambda\bar{I} - \bar{S}$ is a Darboux matrix of degree one for (289) if and only if $\bar{S}$ satisfies*

$$\bar{S}_x + [\bar{S}, \sum_{j=0}^{n} \bar{U}_j \bar{S}^{n-j}] = 0, \quad \bar{S}_{t_m} + [\bar{S}, \sum_{j=0}^{m} \bar{V}_j \bar{S}^{m-j}] = 0. \tag{292}$$

Additionally, the Darboux transformation generates a Bäcklund transformation of (284) as

$$\bar{U}_0' = \bar{U}_0, \quad \bar{U}_j' = \bar{U}_j + \sum_{k=1}^{j} [\bar{U}_{j-k}, \bar{S}]\bar{S}^{k-1}, \quad 1 \le j \le n. \tag{293}$$

Let's introduce N enlarged eigenfunctions $\bar{\phi}_s$ with associated eigenvalues λ_s:

$$\bar{\phi}_x^{(s)} = \bar{U}(\bar{u}, \lambda_s)\bar{\phi}^{(s)}, \quad \bar{\phi}_{t_m}^{(s)} = \bar{V}^{[m]}(\bar{u}, \lambda_s)\bar{\phi}^{(s)}, \quad 1 \le s \le N, \tag{294}$$

where $\bar{u}$ is a given solution to (284). For the presentation of a Darboux transformation for integrable couplings, we further denote

$$\bar{\phi}^{(s)} = (\phi_1^{(s)T}, \phi^{(s)T})^T, \quad 1 \le s \le N, \tag{295}$$

where $\phi^{(s)}$ and $\phi_1^{(s)}$ are N dimensional column vector functions and

$$\bar{H} = \begin{bmatrix} H & H_1 \\ 0 & H \end{bmatrix}, \quad H = [\phi^{(1)}, \phi^{(2)}, ..., \phi^{(N)}], \quad H_1 = [\phi_1^{(1)}, \phi_1^{(2)}, ..., \phi_1^{(N)}]. \tag{296}$$

From (294), we clearly have

$$H_x = \sum_{j=0}^{n} U_j H \Lambda^{n-j}, \quad H_{1x} = \sum_{j=0}^{n} U_j H_1 \Lambda^{n-j} + U_{1j} H \Lambda^{n-j}, \tag{297}$$

and

$$H_t = \sum_{j=0}^{m} V_j H \Lambda^{m-j}, \quad H_{1t} = \sum_{j=0}^{m} V_j H_1 \Lambda^{m-j} + V_{1j} H \Lambda^{m-j}, \tag{298}$$

where Λ is defined (279) and, for convience, set $t_m = t$. The above notation provides a means for explicit calculation of a Darboux transformation for an integrable coupling (284).

Theorem 5.5. *[87] Let $\bar{H}$ be defined by (296) and $\bar{\Lambda} = diag(\Lambda, \Lambda)$, where Λ is defined by (279). Then $\bar{H}$ is invertible if and only if H is invertible. When H is invertible, $\bar{D} = \lambda \bar{I} - \bar{S}$ is a Darboux matrix of the unreduced enlarged spectral problem (289) where $\bar{S} = \bar{H}\bar{\Lambda}\bar{H}^{-1}$ and may be represented as*

$$\bar{S} = \begin{bmatrix} S & S_1 \\ 0 & S \end{bmatrix}, \quad S = H\Lambda H^{-1}, \quad S_1 = -SH_1H^{-1} + H_1\Lambda H^{-1}. \tag{299}$$

Moreover, the Bäcklund transformation for the integrable coupling (284) is

$$\begin{cases} U_0' = U_0, \quad U_j' = U_j + \sum_{k=1}^{j} [U_{j-k}, S] S^{k-1}, \quad U_{10}' = U_{10}, \quad 1 \leq j \leq n, \\ U_{1j}' = U_{1j} + \sum_{k=1}^{j} [U_{1j-k}, S] S^{k-1} + (U_{j-k}\hat{S} - SU_{j-k})\hat{S}^{k-1} - \hat{S}U_{j-k}S^{k-1}, \end{cases} \tag{300}$$

where $\hat{S}^k = -S^k H_1 H^{-1} + H_1 \Lambda^k H^{-1}$ is the $(1,2)$-th matrix block of $\bar{S}$ raised to the k-th power.

Proof. $\bar{H}$ is an upper triangular matrix with diagonal blocks H. Therefore, $\bar{H}$ is invertible if and only if H is invertible.

Assume H is invertible. By direct calculation, we easily see $\bar{H}^{-1}$ as:

$$\bar{H}^{-1} = \begin{bmatrix} H^{-1} & -H^{-1}H_1H^{-1} \\ 0 & H^{-1} \end{bmatrix},$$

where $\bar{H}^{-1}$ depends on H^{-1} and H_1. Then, we work out $\bar{S}$:

$$\begin{aligned}
\bar{S} &= \bar{H}\bar{\Lambda}\bar{H}^{-1} \\
&= \begin{bmatrix} H & H_1 \\ 0 & H \end{bmatrix} \begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda \end{bmatrix} \begin{bmatrix} H^{-1} & -H^{-1}H_1H^{-1} \\ 0 & H^{-1} \end{bmatrix} \\
&= \begin{bmatrix} H\Lambda & H_1\Lambda \\ 0 & H\Lambda \end{bmatrix} \begin{bmatrix} H^{-1} & -H^{-1}H_1H^{-1} \\ 0 & H^{-1} \end{bmatrix} \\
&= \begin{bmatrix} H\Lambda H^{-1} & -H\Lambda H^{-1}H_1H^{-1} + H_1\Lambda H^{-1} \\ 0 & H\Lambda H^{-1} \end{bmatrix} \\
&= \begin{bmatrix} H\Lambda H^{-1} & -SH_1H^{-1} + H_1\Lambda H^{-1} \\ 0 & H\Lambda H^{-1} \end{bmatrix} \\
&= \begin{bmatrix} S & S_1 \\ 0 & S \end{bmatrix}.
\end{aligned}$$

This shows the representation of S and S_1 in (299).

In order to show that $\bar{D} = \lambda\bar{I} - \bar{S}$ is a Darboux matrix of the enlarged spectral problem (289) where $\bar{S} = \bar{H}\bar{\Lambda}\bar{H}^{-1}$, we need to show

$$\bar{S}_x + [\bar{S}, \sum_{j=0}^{n} \bar{U}_j \bar{S}^{n-j}] = 0, \quad \bar{S}_{t_m} + [\bar{S}, \sum_{j=0}^{m} \bar{V}_j \bar{S}^{m-j}] = 0, \tag{301}$$

from Theorem 5.4. Before we begin analyzing the first equation, let's observe that

$$\begin{aligned}
\bar{S}^k &= (\bar{H}\bar{\Lambda}\bar{H}^{-1})^k \\
&= \underbrace{(\bar{H}\bar{\Lambda}\bar{H}^{-1})(\bar{H}\bar{\Lambda}\bar{H}^{-1})...(\bar{H}\bar{\Lambda}\bar{H}^{-1})}_{k\text{-times}} \\
&= \bar{H}\bar{\Lambda}^k\bar{H}^{-1} \\
&= \begin{bmatrix} H & H_1 \\ 0 & H \end{bmatrix} \begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda \end{bmatrix}^k \begin{bmatrix} H^{-1} & -H^{-1}H_1H^{-1} \\ 0 & H^{-1} \end{bmatrix} \\
&= \begin{bmatrix} H & H_1 \\ 0 & H \end{bmatrix} \begin{bmatrix} \Lambda^k & 0 \\ 0 & \Lambda^k \end{bmatrix} \begin{bmatrix} H^{-1} & -H^{-1}H_1H^{-1} \\ 0 & H^{-1} \end{bmatrix} \\
&= \begin{bmatrix} H\Lambda^kH^{-1} & -H\Lambda^kH^{-1}H_1H^{-1} + H_1\Lambda^kH^{-1} \\ 0 & H\Lambda^kH^{-1} \end{bmatrix} \\
&= \begin{bmatrix} S^k & -S^kH_1H^{-1} + H_1\Lambda^kH^{-1} \\ 0 & S^k \end{bmatrix}.
\end{aligned}$$

For simplicity, we name the $(1,2)$-th block of $\bar{S}^k$ to be $\hat{S}^k$ and see that $\hat{S}^k = -S^kH_1H^{-1} + H_1\Lambda^kH^{-1}$.

The commutator in the first equation of (301) may be expanded to:

$$[\bar{S}, \sum_{j=0}^{n} \bar{U}_j \bar{S}^{n-j}] = \left[\begin{bmatrix} S & S_1 \\ 0 & S \end{bmatrix}, \sum_{j=0}^{n} \begin{bmatrix} U_j & U_{1j} \\ 0 & U_j \end{bmatrix} \begin{bmatrix} S & S_1 \\ 0 & S \end{bmatrix}^{n-j} \right]$$

$$= \left[\begin{bmatrix} S & S_1 \\ 0 & S \end{bmatrix}, \textstyle\sum_{j=0}^{n} \begin{bmatrix} U_j & U_{1j} \\ 0 & U_j \end{bmatrix} \begin{bmatrix} S^{n-j} & \hat{S}^{n-j} \\ 0 & S^{n-j} \end{bmatrix}\right]$$

$$= \sum_{j=0}^{n} \begin{bmatrix} S & S_1 \\ 0 & S \end{bmatrix} \begin{bmatrix} U_j & U_{1j} \\ 0 & U_j \end{bmatrix} \begin{bmatrix} S^{n-j} & \hat{S}^{n-j} \\ 0 & S^{n-j} \end{bmatrix}$$

$$- \sum_{j=0}^{n} \begin{bmatrix} U_j & U_{1j} \\ 0 & U_j \end{bmatrix} \begin{bmatrix} S^{n-j+1} & \hat{S}^{n-j+1} \\ 0 & S^{n-j+1} \end{bmatrix}$$

$$= \sum_{j=0}^{n} \begin{bmatrix} SU_j & SU_{1j} + S_1 U_j \\ 0 & SU_j \end{bmatrix} \begin{bmatrix} S^{n-j} & \hat{S}^{n-j} \\ 0 & S^{n-j} \end{bmatrix}$$

$$- \sum_{j=0}^{n} \begin{bmatrix} U_j S^{n-j+1} & U_j \hat{S}^{n-j+1} + U_{1j} S^{n-j+1} \\ 0 & U_j S^{n-j+1} \end{bmatrix}$$

$$= \sum_{j=0}^{n} \begin{bmatrix} SU_j S^{n-j} & SU_{1j} S^{n-j} + S_1 U_j S^{n-j} + SU_j \hat{S}^{n-j} \\ 0 & SU_j S^{n-j} \end{bmatrix}$$

$$- \sum_{j=0}^{n} \begin{bmatrix} U_j S^{n-j+1} & U_j \hat{S}^{n-j+1} + U_{1j} S^{n-j+1} \\ 0 & U_j S^{n-j+1} \end{bmatrix}$$

$$= \sum_{j=0}^{n} \begin{bmatrix} [S, U_j S^{n-j}] & [S, U_{1j} S^{n-j}] + S_1 U_j S^{n-j} + (SU_j - U_j \hat{S}) \hat{S}^{n-j} \\ 0 & [S, U_j] S^{n-j} \end{bmatrix}$$

$$= \sum_{j=0}^{n} \begin{bmatrix} [S, U_j S^{n-j}] & [S, U_{1j} S^{n-j}] + \hat{S} U_j S^{n-j} + (SU_j - U_j \hat{S}) \hat{S}^{n-j} \\ 0 & [S, U_j] S^{n-j} \end{bmatrix}$$

as $S_1 = \hat{S} = [\bar{S}]_{12}$.

Therefore, in order to prove the first equation of (301), we need to show

$$\sum_{j=0}^{n}\begin{bmatrix} S_x+[S,U_jS^{n-j}] & S_{1,x}+[S,U_{1j}S^{n-j}]+\hat{S}U_jS^{n-j}+(SU_j-U_j\hat{S})\hat{S}^{n-j} \\ 0 & S_x+[S,U_j]S^{n-j}\end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{302}$$

First, using (297), we have

$$\begin{aligned} S_x =&(H\Lambda H^{-1})_x \\ =&H_x\Lambda H^{-1} - H\Lambda H^{-1}H_xH^{-1} \\ =&(\sum_{j=0}^{n} U_jH\Lambda^{n-j})\Lambda H^{-1} - H\Lambda H^{-1}(\sum_{j=0}^{n} U_jH\Lambda^{n-j})H^{-1} \\ =&\sum_{j=0}^{n} U_jH\Lambda^{n-j+1}H^{-1} - H\Lambda H^{-1}U_jH\Lambda^{n-j}H^{-1} \\ =&\sum_{j=0}^{n} U_jS^{n-j+1} - SU_jS^{n-j} \\ =&\sum_{j=0}^{n} -[S,U_jS^{n-j}]. \end{aligned}$$

This shows that each block of the diagonal of the matrix on the left side of equation (302) is 0.

We will now show that the $(1,2)$-th block on the left side of equation (302) is 0. Again, by (297) and $S_x = \sum_{j=0}^{n} -[S, U_j S^{n-j}]$, we have

$$\begin{aligned}
S_{1x} =& (-SH_1H^{-1} + H_1\Lambda H^{-1})_x \\
=& -S_xH_1H^{-1} - SH_{1x}H^{-1} + SH_1H^{-1}H_xH^{-1} \\
&+ H_{1x}\Lambda H^{-1} - H_1\Lambda H^{-1}H_xH^{-1} \\
=& \sum_{j=0}^{n}([S, U_jS^{n-j}])H_1H^{-1} - S(U_jH_1\Lambda^{n-j} + U_{1j}H\Lambda^{n-j})H^{-1} \\
&+ SH_1H^{-1}(U_jH\Lambda^{n-j})H^{-1} + (U_jH_1\Lambda^{n-j} + U_{1j}H\Lambda^{n-j})\Lambda H^{-1} \\
&- H_1\Lambda H^{-1}(U_jH\Lambda^{n-j})H^{-1} \\
=& \sum_{j=0}^{n}[S, U_jS^{n-j}]H_1H^{-1} - SU_jH_1\Lambda^{n-j}H^{-1} - \underline{SU_{1j}S^{n-j}} \\
&+ SH_1H^{-1}U_jS^{n-j} + U_jH_1\Lambda^{n-j+1}H^{-1} + \underline{U_{1j}S^{n-j+1}} \\
&- H_1\Lambda H^{-1}U_jS^{n-j} \\
=& \sum_{j=0}^{n}[U_{1j}S^{n-j}, S] + SU_jS^{n-j}H_1H^{-1} - U_jS^{n-j+1}H_1H^{-1} \\
&- SU_jH_1\Lambda^{n-j}H^{-1} + SH_1H^{-1}U_jS^{n-j} + U_jH_1\Lambda^{n-j+1}H^{-1} \\
&- H_1\Lambda H^{-1}U_jS^{n-j} \\
=& \sum_{j=0}^{n}[U_{1j}S^{n-j}, S] - SU_j\underbrace{(-S^{n-j}H_1H^{-1} + H_1\Lambda^{n-j}H^{-1})}_{\hat{S}^{n-j}} \\
&+ U_j\underbrace{(-S^{n-j+1}H_1H^{-1} + H_1\Lambda^{n-j+1}H^{-1})}_{\hat{S}^{n-j+1}} \\
&- \underbrace{(-SH_1H^{-1} + H_1\Lambda H^{-1})}_{\hat{S}}U_jS^{n-j} \\
=& \sum_{j=0}^{n} -[S, U_{1j}S^{n-j}] - \hat{S}U_jS^{n-j} - (SU_j - U_j\hat{S})\hat{S}^{n-j}.
\end{aligned}$$

Thus, (302) is proved.

Now, let's have a look at the second equation in (301),

$$\bar{S}_{t_m} + [\bar{S}, \sum_{j=0}^{m} \bar{V}_j \bar{S}^{m-j}] = 0.$$

Similar to the reasoning before, we have

$$[\bar{S}, \sum_{j=0}^{m} \bar{V}_j \bar{S}^{m-j}] = \sum_{j=0}^{m} \begin{bmatrix} [S, V_j S^{m-j}] & [S, V_{1j} S^{m-j}] + \hat{S} V_j S^{m-j} + (S V_j - V_j \hat{S}) \hat{S}^{m-j} \\ 0 & [S, V_j] S^{m-j} \end{bmatrix}.$$

Therefore, we need to show

$$\sum_{j=0}^{m} \begin{bmatrix} S_t + [S, V_j S^{m-j}] & S_{1t} + [S, V_{1j} S^{m-j}] + \hat{S} V_j S^{m-j} + (S V_j - V_j \hat{S}) \hat{S}^{m-j} \\ 0 & S_t + [S, V_j] S^{m-j} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{303}$$

Note, we use $t_m = t$ for ease in notation. Again, we can prove this equality block by block using (298). We omit the proof as it is almost identical to the proof of (302). In conclusion of the above results, the enlarged matrix $\bar{S} = \bar{H} \bar{\Lambda} \bar{H}^{-1}$ satisfies the two conditions in Theorem 5.4, namely the conditions (301), and thus, $\bar{D} = \lambda \bar{I} - \bar{S}$ is a Darboux matrix of the enlarged spectral problem (289).

Lastly, we recall equation (293). A brief calculation of the equation $\bar{U}_j' = \bar{U}_j$

$+ \sum_{k=1}^{j} [\bar{U}_{j-k}, \bar{S}]\bar{S}^{k-1}$ gives:

$$\begin{bmatrix} U_j' & U_{1j}' \\ 0 & U_j' \end{bmatrix} = \begin{bmatrix} U_j & U_{1j} \\ 0 & U_j \end{bmatrix}$$
$$+ \sum_{k=1}^{j} \begin{bmatrix} [U_{j-k}, S]S^{k-1} & [U_{1,j-k}, S]S^{k-1} + (U_{j-k}\hat{S} - SU_{j-k})\hat{S}^{k-1} - \hat{S}U_{j-k}S^{k-1} \\ 0 & [U_{j-k}, S]S^{k-1} \end{bmatrix}.$$

Looking at each block, we see clearly the Bäcklund tranformations (300) as stated in the theorem. This completes the proof. □

In the next section, we provide an example of the Darboux tranformations on integrable coupling where the spacial spectral matrix $\bar{U}$ is a polynomial in λ of degree 2. The integrable couplings are of a generalized D-Kaup-Newell soliton hierarchy presented in the previous chapter.

5.5 An application to integrable couplings of a generalized D-Kaup-Newell hierarchy

5.5.1 Integrable couplings of a generalized D-Kaup-Newell hierarchy

In Chapter 3, we constructed integrable couplings for a generalized D-Kaup-Newell soliton hierarchy (192) [76]. The first four integrable couplings in (192), i.e., $\bar{u}_{t_i} = \bar{K}_i(\bar{u}), i = 0, 1, 2, 3,$ are

$$\bar{u}_{t_0} = \begin{bmatrix} p_1 \\ q_1 \\ r_1 \\ s_1 \\ v_1 \\ p_2 \\ q_2 \\ r_2 \\ s_2 \\ v_2 \end{bmatrix}_{t_0} = \bar{K}_0(\bar{u}) = \begin{bmatrix} 2\alpha p_1 \\ -2\alpha q_1 \\ 0 \\ 2\alpha s_1 \\ -2\alpha v_1 \\ 2\beta p_1 + 2\alpha p_2 \\ -2\beta q_1 - 2\alpha q_2 \\ 0 \\ 2\beta s_1 + 2\alpha s_2 \\ -2\beta v_1 - 2\alpha v_2 \end{bmatrix}, \tag{304}$$

$$\bar{u}_{t_1} = \begin{bmatrix} p_1 \\ q_1 \\ r_1 \\ s_1 \\ v_1 \\ p_2 \\ q_2 \\ r_2 \\ s_2 \\ v_2 \end{bmatrix}_{t_1} = \begin{bmatrix} 2\alpha s_1 \\ -2\alpha v_1 \\ \alpha(s_1 q_1 - v_1 p_1) \\ \alpha(p_{1,x} + 2r_1 p_1) \\ -\alpha(-q_{1,x} + 2r_1 q_1) \\ 2\beta s_1 + 2\alpha s_2 \\ -2\beta v_1 - 2\alpha v_2 \\ ((\alpha-\beta)v_1 - \alpha v_2)p_1 + q_1((\beta-\alpha)s_1 + \alpha s_2) - \alpha(v_1 p_2 - s_1 q_2) \\ (\beta-\alpha)p_{1,x} + \alpha p_{2,x} + p_1((2\beta - 2\alpha)r_1 + 2\alpha r_2) + 2\alpha p_2 r_1 \\ (\beta-\alpha)q_{1,x} + \alpha q_{2,x} - q_1((2\beta-2\alpha)r_1 + 2\alpha r_2) - 2\alpha q_2 r_1 \end{bmatrix}, \qquad (305)$$

$$\bar{u}_{t_2} = \begin{bmatrix} p_1 \\ q_1 \\ r_1 \\ s_1 \\ v_1 \\ p_2 \\ \\ q_2 \\ \\ r_2 \\ s_2 \\ \\ v_2 \end{bmatrix}_{t_2} = \begin{bmatrix} \alpha(p_{1,x} - p_1^2 q_1 + 2r_1 p_1) \\ \alpha(q_{1,x} + p_1 q_1^2 - 2r_1 q_1) \\ \frac{1}{2}\alpha(p_1 q_1)_x \\ \alpha(s_{1,x} + 2r_1 s_1 - s_1 p_1 q_1) \\ \alpha(v_{1,x} + v_1 q_1 p_1 - 2r_1 v_1) \\ (\beta-\alpha)p_{1,x} + \alpha p_{2,x} + ((-\beta+2\alpha)q_1 - \alpha q_2)p_1^2 \\ +p_1((2\beta-2\alpha)r_1 - \alpha(2p_2 q_1 - 2r_2)) + 2\alpha p_2 r_1) \\ (\beta-\alpha)q_{1,x} + \alpha q_{2,x} - ((-\beta+2\alpha)p_1 - \alpha p_2)q_1^2 \\ +q_1((-2\beta+2\alpha)r_1 + \alpha(2q_2 p_1 - 2r_2)) - 2\alpha q_2 r_1) \\ \frac{1}{2}[(\beta-2\alpha)(p_1 q_1)_x + \alpha(q_2 p_{1,x} + p_2 q_{1,x}) + \alpha(p_1 q_{2,x} + q_1 p_{2,x})] \\ (\beta-\alpha)s_{1,x} + \alpha s_{2,x} + r_1((2\beta-2\alpha)s_1 + 2\alpha s_2) - \alpha s_2 q_1 p_1 + 2\alpha s_1 r_2 \\ +s_1((-\beta+2\alpha)q_1 - \alpha q_2)p_1 - \alpha p_2 q_1) \\ (\beta-\alpha)v_{1,x} + \alpha v_{2,x} + r_1((-2\beta+2\alpha)v_1 - 2\alpha v_2) + \alpha v_2 q_1 p_1 - 2\alpha v_1 r_2 \\ +v_1((-\beta+2\alpha)q_1 - \alpha q_2)p_1 - \alpha p_2 q_1) \end{bmatrix}, \qquad (306)$$

and

$$
\begin{bmatrix} p_1 \\ q_1 \\ r_1 \\ s_1 \\ v_1 \\ p_2 \\ \\ q_2 \\ \\ r_2 \\ \\ \\ \\ s_2 \\ \\ \\ \\ \\ v_2 \\ \\ \\ \\ \\ \end{bmatrix}_{t_3} = \begin{bmatrix}
\alpha(-v_1p_1^2 - 2s_1p_1q_1 + 2r_1s_1 + s_{1,x}) \\
\alpha(2v_1p_1q_1 + s_1q_1^2 - 2r_1v_1 + v_{1,x}) \\
\frac{1}{2}\alpha(s_{1,x}q_1 + v_{1,x}p_1 + (p_1v_1 - q_1s_1)(q_1p_1 - 2r_1)) \\
\frac{1}{2}\alpha(-2s_1v_1p_1 - 2q_1p_1p_{1,x} - 2s_1^2q_1 - 2p_1^2q_1r_1 - p_1^2q_{1,x} + 4p_1r_1^2 + 2p_1r_{1,x} + 4r_1p_{1,x} + p_{1,xx}) \\
\frac{1}{2}\alpha(2s_1v_1q_1 - 2q_1p_1q_{1,x} + 2v_1^2p_1 + 2q_1^2p_1r_1 - q_1^2p_{1,x} - 4q_1r_1^2 + 2q_1r_{1,x} + 4r_1q_{1,x} + q_{1,xx}) \\
(\beta-\alpha)s_{1,x} + \alpha s_{2,x} - p_1^2((\beta-2\alpha)v_1 + \alpha v_2) + p_1(((-2\beta+4\alpha)q_1 - 2q_2\alpha)s_1 \\
-2\alpha(s_2q_1 + p_2v_1)) + ((2\beta-2\alpha)r_1 + 2\alpha r_2 - 2p_2q_1\alpha)s_1 + 2\alpha s_2r_1 \\
(\beta-\alpha)v_{1,x} + \alpha v_{2,x} + q_1^2((\beta-2\alpha)s_1 + \alpha s_2) + (((2\beta-4\alpha)p_1 + 2\alpha p_2)v_1 \\
+2\alpha(s_1q_2 + v_2p_1))q_1 + ((-2\beta+2\alpha)r_1 - 2\alpha r_2 + 2\alpha q_2p_1)v_1 - 2\alpha v_2r_1 \\
\frac{1}{2}((\beta-2\alpha)q_1 + \alpha q_2)s_{1,x} + \frac{1}{2}((\beta-2\alpha)p_1 + \alpha p_2))v_{1,x} + \frac{1}{2}\alpha q_1s_{2,x} + \frac{1}{2}\alpha v_{2,x}p_1 \\
+\frac{1}{2}(((\beta-3\alpha)v_1 + \alpha v_2)q_1 + \alpha q_2v_1)p_1^2 + \frac{1}{2}(((-\beta+3\alpha)s_1 - \alpha s_2)q_1^2 - 2\alpha(-p_2v_1 \\
+q_2s_1)q_1 + ((-2\beta+4\alpha)v_1 - 2v_2\alpha)r_1 - 2r_2v_1\alpha)p_1 - \frac{1}{2}\alpha s_1p_2q_1^2 \\
+\frac{1}{2}(((2\beta-4\alpha)s_1 + 2s_2\alpha)r_1 + 2r_2s_1\alpha)q_1 + r_1\alpha(-p_2v_1 + q_2s_1) \\
\frac{1}{2}(\beta-2\alpha)p_{1,xx} + \frac{1}{2}\alpha p_{2,xx} + \frac{1}{2}(((-2\beta+6\alpha)q_1 - 2\alpha q_2)p_1 - 2p_2q_1\alpha + (4\beta-8\alpha)r_1 \\
+4r_2\alpha))p_{1,x} + \frac{1}{2}((2\beta-4\alpha)p_1 + 2p_2\alpha))r_{1,x} - \alpha(p_1q_1 - 2r_1)p_{2,x} - \frac{1}{2}p_1((\beta-3\alpha)p_1 \\
+2\alpha p_2)q_{1,x} + p_1r_{2,x}\alpha + \frac{1}{2}((-2\beta+6\alpha)r_1 - 2\alpha r_2)q_1 - 2r_1q_2\alpha)p_1^2 + \frac{1}{2}(-4\alpha p_2q_1r_1 \\
+(4\beta-8\alpha)r_1^2 + 8\alpha r_2r_1 + ((-2\beta+4\alpha)v_1 - \frac{1}{2}\alpha p_1^2q_{2,x} - 2\alpha v_2)s_1 - 2\alpha v_1s_2)p_1 \\
-s_1((\beta-2\alpha)s_1 + 2\alpha s_2)q_1 - \alpha(-2p_2r_1^2 + s_1(s_1q_2 + p_2v_1)) \\
-\frac{1}{2}(\beta-2\alpha)q_{1,xx} - \frac{1}{2}\alpha q_{2,xx} + \frac{1}{2}(((-2\beta+6\alpha)p_1 - 2p_2\alpha)q_1 - 2q_2p_1\alpha + (4\beta-8\alpha)r_1 \\
+4r_2\alpha))q_{1,x} - \frac{1}{2}q_1((\beta-3\alpha)q_1 + 2q_2\alpha)p_{1,x} + \frac{1}{2}((2\beta-4\alpha)q_1 + 2q_2\alpha)r_{1,x} \\
-\alpha(p_1q_1 - 2r_1)q_{2,x} - \frac{1}{2}\alpha q_1^2p_{2,x} + q_1r_{2,x}\alpha - \frac{1}{2}(((2\beta-6\alpha)r_1 + 2\alpha r_2)p_1 + 2r_1p_2\alpha)q_1^2 \\
+\frac{1}{2}(4\alpha p_1r_1q_2 + (-4\beta+8\alpha)r_1^2 - 8\alpha r_2r_1 + ((2\beta-4\alpha)s_1 + 2s_2\alpha)v_1 \\
+2\alpha s_1v_2)q_1 + ((\beta-2\alpha)v_1 + 2\alpha v_2)v_1p_1 + \alpha(-2q_2r_1^2 + v_1(s_1q_2 + p_2v_1))
\end{bmatrix}. \tag{307}
$$

where $\bar{u}_{t_3} = \bar{K}_3$ is nonlinear in four potentials of u, namely, p_1, q_1, s_1, v_1.

5.5.2 Darboux transformations of generalized D-Kaup-Newell integrable couplings

We will apply Theorem 5.5 to get Darboux transformations for the integrable couplings (192) [87]. We may write the spectral matrix (176) as the following:

$$\begin{cases} U = U(u,\lambda) = \lambda^2 U_0 + \lambda U_1 + U_2, \quad U_1 = U_1(\bar{u},\lambda) = \lambda^2 U_{10} + \lambda U_{11} + U_{12} \\ U_0 = U_{10} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad U_1 = \begin{bmatrix} 0 & p_1 \\ q_1 & 0 \end{bmatrix}, \quad U_2 = \begin{bmatrix} r_1 & s_1 \\ v_1 & -r_1 \end{bmatrix}, \\ U_{11} = \begin{bmatrix} 0 & p_2 \\ q_2 & 0 \end{bmatrix}, \quad U_{12} = \begin{bmatrix} r_2 & s_2 \\ v_2 & -r_2 \end{bmatrix}. \end{cases} \tag{308}$$

Assume two different eigenvalues λ_1 and λ_2 and let

$$\phi_{jk} = \phi_j(\lambda_k), \quad \psi_{jk} = \psi_j(\lambda_k), \quad j,k = 1,2, \tag{309}$$

where ϕ_j, ψ_k are eigenfunctions. In order to compute

$$S = H\Lambda H^{-1}, \quad S_1 = -SH_1H^{-1} + H_1\Lambda H^{-1}, \tag{310}$$

we identify

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad H = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}, \quad H_1 = \begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{bmatrix}. \tag{311}$$

Theorem 5.5 presents a Darboux matrix of the first order in λ as

$$\bar{D} = \lambda \bar{I} - \bar{S}, \quad \bar{I} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \quad \bar{S} = \begin{bmatrix} S & S_1 \\ 0 & S_1 \end{bmatrix}, \tag{312}$$

where I is a 2×2 identity matrix. The corresponding Darboux transformation is

$$\bar{\phi}' = \bar{D}\bar{\phi}, \quad \bar{U}_0' = \bar{U}_0, \quad \bar{U}_1' = \bar{U}_1 + [\bar{U}_0, \bar{S}], \quad \bar{U}_2' = \bar{U}_2 + [\bar{U}_1, \bar{S}] + [\bar{U}_0, \bar{S}]\bar{S}, \tag{313}$$

which provides the Bäcklund transformations:

$$\begin{cases} U_0' = U_0, \quad U_1' = U_1 + [U_0, S], \quad U_2' = U_2 + [U_1, S] + [U_0, S]S, \\ U_{10}' = U_{10}, \quad U_{11}' = U_{11} + [U_{10}, S] + [U_0, S_1] - SU_0, \\ U_{12}' = U_{12} + [U_{11}, S] + [U_1, S_1] - SU_1 + [U_{10}, S]S + (U_0 S_1 - SU_0)S_1 - S_1 U_0 S, \end{cases} \tag{314}$$

where we note $S_1 = \hat{S}$ as seen in Theorem 5.5. Let's label an initial solution $(U_1, U_2, U_{11}, U_{12})$ and eigenfunction $\bar{\phi}$ as

$$\begin{cases} U_1 = U_1^{[0]} = \begin{bmatrix} 0 & p_1^{[0]} \\ q_1^{[0]} & 0 \end{bmatrix}, \quad U_2 = U_2^{[0]} = \begin{bmatrix} r_1^{[0]} & s_1^{[0]} \\ v_1^{[0]} & -r_1^{[0]} \end{bmatrix}, \\ U_{11} = U_{11}^{[0]} = \begin{bmatrix} 0 & p_2^{[0]} \\ q_2^{[0]} & 0 \end{bmatrix}, \quad U_{12} = U_{12}^{[0]} = \begin{bmatrix} r_2^{[0]} & s_2^{[0]} \\ v_2^{[0]} & -r_2^{[0]} \end{bmatrix}, \quad \bar{\phi} = \bar{\phi}^{[0]} = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \phi_1 \\ \phi_2 \end{bmatrix}, \end{cases} \tag{315}$$

and the new solution $(U_1', U_2', U_{11}', U_{12}')$ and eigenfunction $\bar{\phi}'$ obtained from the Darboux transformation (313) as

$$\begin{cases} U_1 = U_1^{[1]} = \begin{bmatrix} 0 & p_1^{[1]} \\ q_1^{[1]} & 0 \end{bmatrix}, \quad U_2 = U_2^{[1]} = \begin{bmatrix} r_1^{[1]} & s_1^{[1]} \\ v_1^{[1]} & -r_1^{[1]} \end{bmatrix}, \\ U_{11} = U_{11}^{[1]} = \begin{bmatrix} 0 & p_2^{[1]} \\ q_2^{[1]} & 0 \end{bmatrix}, \quad U_{12} = U_{12}^{[1]} = \begin{bmatrix} r_2^{[1]} & s_2^{[1]} \\ v_2^{[1]} & -r_2^{[1]} \end{bmatrix}, \quad \bar{\phi} = \bar{\phi}^{[1]} = \begin{bmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \\ \tilde{\phi}_1 \\ \tilde{\phi}_2 \end{bmatrix}. \end{cases} \tag{316}$$

Therefore, we state the Bäcklund transformation taking the initial solution to the new solution:

$$\begin{cases} U_1^{[1]} = U_1^{[0]} + [U_0, S], \quad U_2^{[1]} = U_2^{[0]} + [U_1^{[0]}, S] + [U_0, S]S, \\ U_{11}^{[1]} = U_{11}^{[0]} + [U_{10}, S] + [U_0, S_1] - SU_0, \\ U_{12}^{[1]} = U_{12}^{[0]} + [U_{11}^{[0]}, S] + [U_1^{[0]}, S_1] - SU_1^{[0]} + [U_{10}, S]S + (U_0S_1 - SU_0)S_1 - S_1U_0S, \end{cases} \tag{317}$$

which defines each new solution from the initial solution as

$$
\begin{cases}
p_1^{[1]} = p_1^{[0]} + \frac{2(\lambda_2-\lambda_1)\phi_{11}\phi_{12}}{\phi_{11}\phi_{22}-\phi_{12}\phi_{21}}, \\
q_1^{[1]} = q_1^{[0]} + \frac{2(\lambda_2-\lambda_1)\phi_{21}\phi_{22}}{\phi_{11}\phi_{22}-\phi_{12}\phi_{21}}, \\
r_1^{[1]} = r_1^{[0]} + \frac{2(\lambda_1-\lambda_2)^2\phi_{11}\phi_{12}\phi_{21}\phi_{22}}{(\phi_{11}\phi_{22}-\phi_{12}\phi_{21})^2}, \\
s_1^{[1]} = s_1^{[0]} + \frac{2(\lambda_1-\lambda_2)\phi_{11}\phi_{12}(\lambda_1\phi_{12}\phi_{21}-\lambda_2\phi_{11}\phi_{12})}{(\phi_{11}\phi_{22}-\phi_{12}\phi_{21})^2}, \\
v_1^{[1]} = v_1^{[0]} + \frac{2(\lambda_2-\lambda_1)\phi_{21}\phi_{22}(\lambda_1\phi_{11}\phi_{22}-\lambda_2\phi_{12}\phi_{21})}{(\phi_{11}\phi_{22}-\phi_{12}\phi_{21})^2}, \\
p_2^{[1]} = p_2^{[0]} + \frac{2(\lambda_2-\lambda_1)((\phi_{12}\phi_{22}-\phi_{12}\psi_{22}+\psi_{12}\phi_{22})\phi_{11}^2-(\phi_{21}\phi_{11}-\psi_{21}\phi_{11}+\psi_{11}\phi_{21})\phi_{12}^2)}{(\phi_{11}\phi_{22}-\phi_{12}\phi_{21})^2}, \\
q_2^{[1]} = q_2^{[0]} + \frac{2(\lambda_2-\lambda_1)((-\phi_{21}\phi_{22}+\psi_{12}\phi_{22}-\phi_{12}\psi_{22})\phi_{21}^2+(\phi_{21}\phi_{11}-\phi_{21}\psi_{11}+\phi_{11}\psi_{21})\phi_{22}^2)}{(\phi_{11}\phi_{22}-\phi_{12}\phi_{21})^2}, \\
r_2^{[1]} = r_2^{[0]} + \frac{2(\lambda_2-\lambda_1)^2\phi_{22}(((\phi_{22}-\psi_{22})\phi_{21}+\psi_{21}\phi_{22})\phi_{12}+\phi_{21}\phi_{22}\psi_{12})\phi_{1,1}{}^2}{(\phi_{11}\phi_{22}-\phi_{12}\phi_{21})^3} - \frac{2(\lambda_2-\lambda_1)^2\phi_{12}{}^2\phi_{21}{}^2\phi_{22}\psi_{11}}{(\phi_{11}\phi_{22}-\phi_{12}\phi_{21})^3} \\
-\frac{2(\lambda_2-\lambda_1)^2\left(((\phi_{22}+\psi_{22})\phi_{21}-\psi_{21}\phi_{22})\phi_{12}-\phi_{2,1}\phi_{22}\psi_{12}+\phi_{22}{}^2\psi_{11}\right)\phi_{21}\phi_{12}\phi_{11}}{(\phi_{11}\phi_{22}-\phi_{12}\phi_{21})^3}, \\
s_2^{[1]} = s_2^{[0]} - \frac{2(\lambda_2-\lambda_1)\phi_{22}\lambda_2((\phi_{22}-\psi_{22})\phi_{12}+\phi_{22}\psi_{12})\phi_{11}{}^3+\lambda_1\phi_{12}{}^3\phi_{21}{}^2\psi_{11}}{(\phi_{11}\phi_{22}-\phi_{12}\phi_{21})^3} \\
-\frac{2(\lambda_2-\lambda_1)\left(\left(((\lambda_1+\lambda_2)\phi_{22}+\psi_{22}(\lambda_2-2\,\lambda_1))\phi_{21}-2\,\phi_{22}\psi_{21}\left(\lambda_2-\frac{1}{2}\,\lambda_1\right)\right)\phi_{12}-\phi_{2,1}\phi_{22}\psi_{12}(\lambda_2-2\,\lambda_1)\right)\phi_{12}\phi_{11}{}^2}{(\phi_{11}\phi_{22}-\phi_{12}\phi_{21})^3} \\
-\frac{2(\lambda_2-\lambda_1)2\,\phi_{21}\left(-\frac{1}{2}\,\lambda_1(\phi_{21}-\psi_{2,1})\phi_{12}+\phi_{22}\psi_{11}\left(\lambda_2-\frac{1}{2}\,\lambda_1\right)\right)\phi_{12}{}^2\phi_{11}}{(\phi_{11}\phi_{22}-\phi_{12}\phi_{21})^3}, \\
v_2^{[1]} = v_2^{[0]} + \frac{2(\lambda_2-\lambda_1)((\phi_{12}-\psi_{12})\phi_{22}+\phi_{12}\psi_{22})\lambda_2\phi_{12}\phi_{21}{}^3+\lambda_1\phi_{11}{}^2\phi_{22}{}^3\psi_{21}}{(\phi_{11}\phi_{22}-\phi_{12}\phi_{21})^3} \\
-\frac{2(\lambda_2-\lambda_1)\phi_{22}\left(\left(((\lambda_1+\lambda_2)\phi_{12}+\psi_{12}(\lambda_2-2\,\lambda_1))\phi_{11}-2\,\psi_{11}\left(\lambda_2-\frac{1}{2}\,\lambda_1\right)\phi_{12}\right)\phi_{22}-\phi_{11}\phi_{12}\psi_{22}(\lambda_2-2\,\lambda_1)\right)\phi_{21}{}^2}{(\phi_{11}\phi_{22}-\phi_{12}\phi_{21})^3} \\
+\frac{2(\lambda_2-\lambda_1)\phi_{22}{}^2\left(\lambda_1(\phi_{11}-\psi_{11})\phi_{22}-2\,\psi_{21}\left(\lambda_2-\frac{1}{2}\,\lambda_1\right)\phi_{12}\right)\phi_{11}\phi_{21}}{(\phi_{11}\phi_{22}-\phi_{12}\phi_{21})^3}.
\end{cases}
\tag{318}
$$

This above process can be repeated to get a Darboux transformations of higher order in λ. For instance, we may find a new solution $(p_1^{[2]}, q_1^{[2]}, r_1^{[2]}, s_1^{[2]}, v_1^{[2]}, p_2^{[2]}, q_2^{[2]}, r_2^{[2]}, s_2^{[2]}, v_2^{[2]})$ associated with two new eigenvalues $\tilde{\lambda_1}$ and $\tilde{\lambda_2}$ from the solution $(p_1^{[1]}, q_1^{[1]}, r_1^{[1]}, s_1^{[1]}, v_1^{[1]}, p_2^{[1]}, q_2^{[1]}, r_2^{[1]}, s_2^{[1]}, v_2^{[1]})$. We compute

$$\begin{cases} U_1^{[2]} = U_1^{[1]} + [U_0, \tilde{S}], \quad U_2^{[2]} = U_2^{[1]} + [U_1^{[1]}, \tilde{S}] + [U_0, \tilde{S}]\tilde{S}, \\ U_{11}^{[2]} = U_{11}^{[1]} + [U_{10}, \tilde{S}] + [U_0, \tilde{S}_1] - \tilde{S}U_0, \\ U_{12}^{[2]} = U_{12}^{[1]} + [U_{11}^{[1]}, \tilde{S}] + [U_1^{[1]}, \tilde{S}_1] - \tilde{S}U_1^{[1]} + [U_{10}, \tilde{S}]\tilde{S} + (U_0\tilde{S}_1 - \tilde{S}U_0)\tilde{S}_1 - \tilde{S}_1U_0\tilde{S}, \end{cases} \tag{319}$$

where $\tilde{S}$ and $\tilde{S}_1$ are defined as

$$\tilde{S} = \tilde{H}\tilde{\Lambda}\tilde{H}^{-1}, \quad \tilde{S}_1 = -\tilde{S}\tilde{H}_1\tilde{H} + \tilde{H}_1\tilde{\Lambda}\tilde{H}^{-1}, \tag{320}$$

through the matrices

$$\tilde{\Lambda} = \begin{bmatrix} \tilde{\lambda_1} & 0 \\ 0 & \tilde{\lambda_2} \end{bmatrix}, \quad \tilde{H} = \begin{bmatrix} \tilde{\phi_{11}} & \tilde{\phi_{12}} \\ \tilde{\phi_{21}} & \tilde{\phi_{22}} \end{bmatrix}, \quad \tilde{H}_1 = \begin{bmatrix} \tilde{\psi_{11}} & \tilde{\psi_{12}} \\ \tilde{\psi_{21}} & \tilde{\psi_{22}} \end{bmatrix}, \tag{321}$$

where $\tilde{\phi_{jk}} = \tilde{\phi_j}(\tilde{\lambda_k})$ and $\tilde{\psi_{jk}} = \tilde{\psi_j}(\tilde{\lambda_k}), j, k = 1, 2.$

For the example following, we will denote the time variable t_m by only t for simplicity.

Example - A solution to the m-th order system

We will be considering the m-th order integrable coupling system $\bar{u}_t = \bar{K}_m$ defined by (192). From (178) and (191), we have

$$\bar{V}^{[m]} = \begin{bmatrix} V^{[m]} & V_1^{[m]} \\ 0 & V^{[m]} \end{bmatrix} = \left[\begin{array}{cc|cc} a^{[m]} & b^{[m]} & e^{[m]} & f^{[m]} \\ c^{[m]} & -a^{[m]} & g^{[m]} & -e^{[m]} \\ \hline 0 & 0 & a^{[m]} & b^{[m]} \\ 0 & 0 & c^{[m]} & -a^{[m]} \end{array}\right] = \sum_{i=0}^{m} \left[\begin{array}{cc|cc} a_i & b_i & e_i & f_i \\ c_i & -a_i & g_i & -e_i \\ \hline 0 & 0 & a_i & b_i \\ 0 & 0 & c_i & -a_i \end{array}\right] \lambda^{m-i}, \tag{322}$$

where $\{a_i, b_i, c_i, e_i, f_i, g_i | 1 \leq i \leq m\}$ are given by (182).

Let's look at the spectral problems (177) and (190) when $\bar{u}=0$, i.e.,

$$\begin{cases} \bar{\phi}_x = \begin{bmatrix} \psi_{1,x} \\ \psi_{2,x} \\ \phi_{1,x} \\ \phi_{2,x} \end{bmatrix} = \begin{bmatrix} \lambda^2 & 0 & \lambda^2 & 0 \\ 0 & -\lambda^2 & 0 & -\lambda^2 \\ 0 & 0 & \lambda^2 & 0 \\ 0 & 0 & 0 & -\lambda^2 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \phi_1 \\ \phi_2 \end{bmatrix} = \bar{U}|_{\bar{u}=0}\bar{\phi}, \\ \bar{\phi}_t = \begin{bmatrix} \psi_{1,t} \\ \psi_{2,t} \\ \phi_{1,t} \\ \phi_{2,t} \end{bmatrix} \begin{bmatrix} \alpha\lambda^m & 0 & \beta\lambda^m & 0 \\ 0 & -\alpha\lambda^m & 0 & -\beta\lambda^m \\ 0 & 0 & \alpha\lambda^m & 0 \\ 0 & 0 & 0 & -\alpha\lambda^m \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \phi_1 \\ \phi_2 \end{bmatrix} = \bar{V}|_{\bar{u}=0}\bar{\phi}. \end{cases} \tag{323}$$

Solving (323), we get the zero seed solution of (177) and (190) and generate the following eigenfunctions associated with the eigenvalue λ:

$$\begin{cases} \psi_1 = \chi_1(\lambda) = (\beta\lambda^m\mu_1 t + \lambda^2\mu_1 + \mu_3)e^{\alpha\lambda^m t + \lambda^2 x}, \\ \psi_2 = \chi_2(\lambda) = -(\beta\lambda^m\mu_2 t + \lambda^2\mu_2 - \mu_4)e^{-\alpha\lambda^m t - \lambda^2 x}, \\ \phi_1 = \chi_3(\lambda) = \mu_1 e^{\alpha\lambda^m t + \lambda^2 x}, \\ \phi_2 = \chi_4(\lambda) = \mu_2 e^{-\alpha\lambda^m t - \lambda^2 x}, \end{cases} \tag{324}$$

where $\mu_i, 1 \leq i \leq 4$ are arbitrary constants. In order to obtain analytic solutions by the Darboux and Bäcklund transformations in Theorem 5.5, we choose the following vectors of eigenfunctions to be linearly independent:

$$\psi_1(\lambda_1) = \chi_1(\lambda_1), \quad \psi_2(\lambda_1) = \chi_2(\lambda_1), \quad \phi_1(\lambda_1) = \chi_3(\lambda_1), \quad \phi_2(\lambda_1) = \chi_4(\lambda_1), \tag{325}$$

and

$$\psi_1(\lambda_2) = -\chi_1(\lambda_2), \quad \psi_2(\lambda_2) = \chi_2(\lambda_2), \quad \phi_1(\lambda_2) = -\chi_3(\lambda_2), \quad \phi_2(\lambda_2) = \chi_4(\lambda_2), \tag{326}$$

associated with two different eigenvalues λ_1 and λ_2, respectively. Then we use the Bäcklund transformation (317) to obtain a one-soliton-like solution to the m-th order generalized D-Kaup-Newell integrable coupling system [87]:

$$\begin{cases} p_1 = \frac{\mu_1}{\mu_2}(\lambda_1 - \lambda_2)e^{\tau_1+\tau_2} \operatorname{sech}\xi, \\ q_1 = -\frac{\mu_2}{\mu_1}(\lambda_1 - \lambda_2)e^{-\tau_1-\tau_2} \operatorname{sech}\xi, \\ r_1 = -\frac{1}{2}(\lambda_1 - \lambda_2)^2 \operatorname{sech}^2\xi, \\ s_1 = \frac{\mu_1}{2\mu_2}(\lambda_1 - \lambda_2)(e^{2\tau_1}\lambda_2 + e^{2\tau_2}\lambda_1) \operatorname{sech}^2\xi, \\ v_1 = -\frac{\mu_2}{2\mu_1}(\lambda_1 - \lambda_2)(e^{-2\tau_2}\lambda_1 + e^{-2\tau_1}\lambda_2) \operatorname{sech}^2\xi, \\ p_2 = \frac{\eta_1}{\mu_2^2} \operatorname{sech}^2\xi, \\ q_2 = \frac{\eta_2}{\mu_1^2} \operatorname{sech}^2\xi, \\ r_2 = -\frac{\eta_3}{2}(\lambda_1 - \lambda_2)^2 \operatorname{sech}^3\xi, \\ s_2 = -\frac{\eta_4}{2\mu_2^2}(\lambda_1 - \lambda_2) \operatorname{sech}^3\xi, \\ v_2 = \frac{\eta_5}{2\mu_1^2}(\lambda_1 - \lambda_2) \operatorname{sech}^3\xi, \end{cases} \tag{327}$$

where

$$
\begin{cases}
\xi = \left(-{\lambda_2}^{1+n}\beta\, t\mu_1\mu_2 + {\lambda_2}^{n}\beta\, t\lambda_1\mu_1\mu_2 + \left(\left({\lambda_2}^{2}x\mu_2 + \frac{1}{2}\,\mu_2 - \frac{1}{2}\,\mu_4\right)\mu_1 + \frac{1}{2}\,\mu_2\mu_3\right)(\lambda_1-\lambda_2)\right) e^{2\tau_1} \\
+e^{2\tau_2}\left({\lambda_1}^{1+n}\beta\, t\mu_1\mu_2 - {\lambda_1}^{n}\beta\, t\lambda_2\mu_1\mu_2 + (\lambda_1-\lambda_2)\left(\left({\lambda_1}^{2}x\mu_2 + \frac{1}{2}\,\mu_2 - \frac{1}{2}\,\mu_4\right)\mu_1 + \frac{1}{2}\,\mu_2\mu_3\right)\right), \\
\eta_2 = \left(-{\lambda_2}^{1+n}\beta\, t\mu_1\mu_2 + {\lambda_2}^{n}\beta\, t\lambda_1\mu_1\mu_2 + \left(\left({\lambda_2}^{2}x\mu_1 - \frac{1}{2}\,\mu_1 + \frac{1}{2}\,\mu_3\right)\mu_2 - \frac{1}{2}\,\mu_1\mu_4\right)(\lambda_1-\lambda_2)\right) \mathrm{e}^{-2\tau_1} \\
+\left({\lambda_1}^{1+n}\beta\, t\mu_1\mu_2 - {\lambda_1}^{n}\beta\, t\lambda_2\mu_1\mu_2 + \left(\left({\lambda_1}^{2}x\mu_1 - \frac{1}{2}\,\mu_1 + \frac{1}{2}\,\mu_3\right)\mu_2 - \frac{1}{2}\,\mu_1\mu_4\right)(\lambda_1-\lambda_2)\right) \mathrm{e}^{-2\tau_2}, \\
\eta_3 = \left({\lambda_1}^{m}\beta\, t - {\lambda_2}^{m}\beta\, t + \frac{1}{2} + x\left({\lambda_1}^{2} - {\lambda_2}^{2}\right)\right)\mathrm{e}^{-\xi} \\
-\left({\lambda_1}^{m}\beta\, t - {\lambda_2}^{m}\beta\, t - \frac{1}{2} + x\left({\lambda_1}^{2} - {\lambda_2}^{2}\right)\right)\mathrm{e}^{\xi}, \\
\eta_4 = (-{\lambda_1}^{1+m}t\beta\,\mu_1\mu_2 - \lambda_1(({\lambda_1}^{2}x\mu_2 + \frac{1}{2}\,\mu_2 - \frac{1}{2}\,\mu_4)\mu_1 + \frac{1}{2}\,\mu_2\mu_3))e^{-\tau_1+3\tau_2} \\
+(-{\lambda_2}^{1+m}t\beta\,\mu_1\mu_2 - \lambda_2(({\lambda_2}^{2}x\mu_2 + \frac{1}{2}\,\mu_2 - \frac{1}{2}\,\mu_4)\mu_1 + \frac{1}{2}\,\mu_2\mu_3))e^{-\tau_2+3\tau_1} \\
+({\lambda_1}^{1+m}t\beta\,\mu_1\mu_2 + {\lambda_2}^{1+m}t\beta\,\mu_1\mu_2 - 2\,\lambda_2{\lambda_1}^{m}t\beta\,\mu_1\mu_2 - 2\,\lambda_1{\lambda_2}^{m}t\beta\,\mu_1\mu_2 \\
+((({\lambda_1}^{2}x - 3\,\lambda_1\lambda_2x + {\lambda_2}^{2}x - \frac{1}{2})\mu_2 + \frac{1}{2}\,\mu_4)\mu_1 - \frac{1}{2}\,\mu_2\mu_3)(\lambda_1+\lambda_2))e^{\tau_1+\tau_2}, \\
\eta_5 = ({\lambda_1}^{1+m}t\beta\,\mu_1\mu_2 + \lambda_1(({\lambda_1}^{2}x\mu_2 - \frac{1}{2}\,\mu_2 - \frac{1}{2}\,\mu_4)\mu_1 + \frac{1}{2}\,\mu_2\mu_3))e^{\tau_1-3\tau_2} \\
+({\lambda_2}^{1+m}t\beta\,\mu_1\mu_2 + \lambda_2(({\lambda_2}^{2}x\mu_2 - \frac{1}{2}\,\mu_2 - \frac{1}{2}\,\mu_4)\mu_1 + \frac{1}{2}\,\mu_2\mu_3))e^{-3\tau_1+\tau_2} \\
-({\lambda_1}^{1+m}t\beta\,\mu_1\mu_2 + {\lambda_2}^{1+m}t\beta\,\mu_1\mu_2 - 2\,\lambda_2{\lambda_1}^{m}t\beta\,\mu_1\mu_2 - 2\,\lambda_1{\lambda_2}^{m}t\beta\,\mu_1\mu_2 \\
+((({\lambda_1}^{2}x - 3\,\lambda_1\lambda_2x + {\lambda_2}^{2}x + \frac{1}{2})\mu_2 + \frac{1}{2}\,\mu_4)\mu_1 - \frac{1}{2}\,\mu_2\mu_3)(\lambda_1+\lambda_2))e^{-\tau_1-\tau_2}.
\end{cases}
\tag{328}
$$

Note, r_1 is a one-soliton solution. Specifically, we may let $m = 0, 1, 2, 3$, in order to solve the corresponding integrable coupling systems (304), (305), (**??**), and (**??**), respectively.

5.6 Summary

Darboux transformations for integrable couplings were formulated from the algebraic structure of non-semisimple matrix Lie algebras. The spectral matrix of the integrable couplings may be a polynomial of arbitrary order in λ. The resulting transformations were applied to generalized D-Kaup-Newell integrable couplings and solutions to the m-th coupling system were found. Explicit formulas of soliton-like solutions were presented. The example of solutions to the m-th coupling system shows that a set of solutions to each coupling system has been worked out. It is expected that these transformations may be used to find explicit solutions for a number of integrable couplings.

6 Concluding Remarks

We constructed two soliton hierarchies in chapter two. Each hierarchy has infinitely many high-order symmetries and conserved densities. Both hierarchies were formulated from spectral matrices based in $\widetilde{\mathrm{sl}}(2,\mathbb{R})$. Recently, the matrix Lie loop algebra $\widetilde{\mathrm{so}}(3,\mathbb{R})$ defined by

$$\widetilde{\mathrm{so}}(3,\mathbb{R}) = \left\{ \sum_{i\geq 0} A_i \lambda^{n-i} | A_i \in \mathrm{so}(3,\mathbb{R}), n \in \mathbb{Z} \right\} \tag{329}$$

with basis elements

$$e_1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{330}$$

and commutator relations

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2, \tag{331}$$

have been studied [32], [47]- [50]. Mostly, spectral matrices with two dependent variables have been discussed:

$$U(u,\lambda) = \lambda e_1 + pe_2 + qe_3,$$
$$U(u,\lambda) = \lambda^2 e_1 + \lambda pe_2 + \lambda qe_3,$$
$$U(u,\lambda) = \lambda e_1 + \lambda pe_2 + \lambda qe_3,$$
$$U(u,\lambda) = \lambda qe_1 + (\lambda^2 + \lambda p)e_2 + (-\lambda^2 + \lambda p)e_3.$$

A spectral matrix with three dependent variables has been analyzed [49] of the following form:

$$U(u,\lambda) = (\lambda^2 + r)e_1 + \lambda pe_2 + \lambda qe_3.$$

We hope to see more soliton hierarchies using three or more dependent variables. For instance, what kind of beautiful structure is possible for a spectral matrix similar to (152) associated with $\mathrm{so}(3,\mathbb{R})$?

The third chapter uses a relatively new idea of having $\frac{\partial U_1}{\partial \lambda} \neq 0$ in the enlarged spectral matrix $\bar{U}$. This idea helped generate two integrable couplings for a generalized D-Kaup-Newell soliton hierarchy and for a special reduction. Although the calculations are more difficult, many new applications may arise from integrable couplings starting from enlarged spectral matrices of this form. One application is the Darboux transformation method for the construction of soliton solutions to integrable couplings. Such a new process of construction creates new integrable systems associated with non-semisimple Lie algebras and brings us new insightful thoughts to classify integrable systems from an algebraic point of view.

The last chapter formulated Darboux transformations for integrable couplings from a spectral matrix which is a polynomial of order n in λ, i.e., $\bar{U}(\bar{u},\lambda) = \sum_{j=0}^{n} \bar{U}_j \lambda^{j-n}$. The Darboux transformations were applied to the integrable couplings found in chapter three. It is expected that many other integrable couplings can be solved using this method. In particular, based on the structure of the spectral matrix, we predict the transformations may be applied to an integrable coupling system where the subsystem is of AKNS type and we find integrable couplings through perturbations, i.e., a spectral matrix given as

$$\bar{U} = \bar{U}(\bar{u}, \lambda) = \begin{bmatrix} U & U_1 \\ 0 & U \end{bmatrix} = \left[\begin{array}{cc|cc} \lambda & p & 0 & s \\ q & -\lambda & v & 0 \\ \hline 0 & 0 & \lambda & p \\ 0 & 0 & q & -\lambda \end{array}\right]. \tag{332}$$

We also imagine that solutions may be found for a spectral matrix that generates integrable couplings of the Kaup-Newell type,

$$\bar{U} = \bar{U}(\bar{u}, \lambda) = \begin{bmatrix} U & U_1 \\ 0 & U \end{bmatrix} = \left[\begin{array}{cc|cc} \lambda^2 & \lambda p & 0 & \lambda s \\ \lambda q & -\lambda^2 & \lambda v & 0 \\ \hline 0 & 0 & \lambda^2 & \lambda p \\ 0 & 0 & \lambda q & -\lambda^2 \end{array}\right], \tag{333}$$

through the Darboux and Bäcklund transformations formulated in chapter four.

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