May 2018

Orthogonal Polynomials With Respect to the Measure Supported Over the Whole Complex Plane

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by

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A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
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Date of Approval:
May 4, 2018

Keywords: Orthogonal polynomials, Riemann-Hilbert problem, Random Matrices, Skeleton, Discontinuity, Multiple orthogonal polynomials

Dedication

This paper is dedicated to my father, my family and my friends.

Acknowledgments

First of all, I would like to express my deep gratitude to my supervisor, Dr. Seung-Yeop Lee, for his patient guidance, great support and encouragement throughout my Ph.D research studies. I have been very lucky to have a supervisor who cared about my research and my life, and who responded to my questions promptly. I benefited greatly from his fruitful suggestions, especially when we were exploring new methods. Without his incredible support, I would have never been able to complete this thesis.

I also would like to thank Dr. Dmitry Khavinson and Dr. Catherine Bénéteau for providing me the opportunity to present my research in Analysis Seminar and the travel support for the summer school in Quebec City, Canada. I would like to thank Dr. Razvan Teodorescu and Dr. Evguenii Rakhmanov for the helpful discussions and constructive suggestions on my current research during Analysis Seminar. I would like to express my sincere gratitude to my committee members, Dr. Abey López-García (the chairperson), Dr. Dmitry Khavinson, Dr. Evguenii Rakhmanov, Dr. Razvan Teodorescu and Dr. Wen-Xiu Ma for their valuable advice, comments and suggestions. I am also indebted to the faculty and staff at the Department of Mathematics and Statistics at University of South Florida. A special thank goes to Dr. Yun-Cheng You, who took care of both my study and life in the first two years.

In addition, I would like to thank my best friend Bin Shi, my qualifying exams study group Morgan McAnally and Kristina Hilton, my graduate classmates Matthew Fleeman, Sayed Zoalroshd, Yuan Zhou, Xiang Gu, Wael Al-Sawai, Solomon Manukure, Emanuele Zappala, Jun-Yi Tu and Fu-Dong Wang.

Finally, I want to take the opportunity to thank my parents Zhi-Gang Yang and Gang-Zhen Liu and sister Xue Yang, brother-in-law Zhong-Zhi Dong and two cute nieces Xing-Jie Dong and Ruo-Lin Dong for their unconditional support, my friends Peng-Fei Liang, Lin-Yu Yu, Han-Ze Zhang, Meng-Ying He, Ying-Wei Yang, Hong-Zhan Li and Lan Xu and other people whose names are not mentioned here.

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Abstract

In chapter 1, we present some background knowledge about random matrices, Coulomb gas, orthogonal polynomials, asymptotics of planar orthogonal polynomials and the Riemann-Hilbert problem. In chapter 2, we consider the monic orthogonal polynomials, $\{P_{n,N}(z)\}_{n=0,1,\cdots}$, that satisfy the orthogonality condition,

$$\int_{\mathbb{C}} P_{n,N}(z)\overline{P_{m,N}(z)}e^{-NQ(z)}dA(z) = h_{n,N}\delta_{nm} \quad (n,m = 0,1,2,\cdots),$$

where $h_{n,N}$ is a (positive) norming constant and the external potential is given by

$$Q(z) = |z|^2 + \frac{2c}{N}\log\frac{1}{|z-a|}, \quad c > -1, \quad a > 0.$$

The orthogonal polynomial is related to the interacting Coulomb particles with charge $+1$ for each, in the presence of an extra particle with charge $+c$ at a. For N large and a fixed "c" this can be a small perturbation of the Gaussian weight. The polynomial $P_{n,N}(z)$ can be characterized by a matrix Riemann–Hilbert problem [2]. We then apply the standard nonlinear steepest descent method [10, 11] to derive the strong asymptotics of $P_{n,N}(z)$ when n and N go to ∞. From the asymptotic behavior of $P_{n,N}(z)$, we find that, as we vary c, the limiting distribution behaves discontinuously at $c = 0$. We observe that the mother body (a kind of potential theoretic skeleton) also behaves discontinuously at $c = 0$. The smooth interpolation of the discontinuity is obtained by further scaling of $c = e^{-\eta N}$ in terms of the parameter $\eta \in [0,\infty)$. To obtain the results for arbitrary values of c, we used the "partial Schlesinger transform" method developed in [5] to derive an arbitrary order correction in the Riemann–Hilbert analysis.

In chapter 3, we consider the case of multiple logarithmic singularities. The planar orthogonal polynomials $\{p_n(z)\}_{n=0,1,\cdots}$ with respect to the external potential that is given by

$$Q(z) = |z|^2 + 2\sum_{j=1}^{l} c_j \log\frac{1}{|z-a_j|},$$

where $\{a_1, a_2, \cdots, a_l\}$ is a set of nonzero complex numbers and $\{c_1, c_2, \cdots, c_l\}$ is a set of positive real numbers. We show that the planar orthogonal polynomials $p_n(z)$ with l logarithmic singularities in the potential are the multiple orthogonal polynomials $p_{\mathbf{n}}(z)$ (Hermite-Padé polynomials) of Type II with l measures of degree $|\mathbf{n}| = n = \kappa l + r$, $\mathbf{n} = (n_1, \cdots, n_l)$ satisfying the orthogonality condition,

$$\frac{1}{2\mathrm{i}} \int_{\Gamma} p_{\mathbf{n}}(z) z^k \chi_{\mathbf{n}-\mathbf{e}_j}(z) \mathrm{d}z = 0, \quad 0 \le k \le n_j - 1, \quad 1 \le j \le l,$$

where Γ is a certain simple closed curve with counterclockwise orientation and

$$\chi_{\mathbf{n}-\mathbf{e}_j}(z) := \prod_{i=1}^{l} (z - a_i)^{c_i} \int_0^{\bar{z}\times\infty} \frac{\prod_{i=1}^{l} (s - \bar{a}_i)^{n_i + c_i}}{(s - \bar{a}_j) \mathrm{e}^{zs}} \, \mathrm{d}s.$$

Such equivalence allows us to formulate the $(l+1) \times (l+1)$ Riemann–Hilbert problem for $p_n(z)$. We also find the ratio between the determinant of the moment matrix corresponding to the multiple orthogonal polynomials and the determinant of the moment matrix from the original planar measure.

Chapter 1

Introduction

1.1 Random Matrices

In 1930s, random matrices first appeared in mathematical statistics, however they did not draw much attention at that time. In 1950s, random matrix theory was introduced to the theoretical physics community as a subject of intensive study by Wigner in his work on nuclear physics [39]. Since that time, the random matrix theory has been developed by many authors, particularly, Dyson, Gaudin, and Mehta [12, 32]. A random matrix is a matrix whose entries are random variables corresponding to given probability distribution. As the entries are random, its eigenvalues and its eigenvectors are also random. Understanding statistical properties of the random matrix will help us to understand the probability distributions of its eigenvalues and its eigenvectors. As we know, random matrix theory has reached an important place in many areas of physics and mathematics. For example, number theory, integrable systems, asymptotics of orthogonal polynomials, infinite-dimensional diffusions, communication technology, financial mathematics and so on. In the physical models, the systems are characterized by their Hamiltonian, which are represented by Hermitian matrices. For the simplest example, let us consider the particular Hermitian ensemble, the Gaussian Unitary Ensemble (GUE) (see [10, 32]).

Theorem 1.1.1 *Every $n \times n$ Hermitian matrix M can be diagonalized by a Unitary matrix $U \in \mathcal{U}(n)$ (i.e. $U^*U = UU^* = I_n$) and its eigenvalues are real. (see [29])*

Let $\mathcal{M} = \{M, M_{ij} = \overline{M_{ji}}\}$ denote the space of $n \times n$ *Hermitian matrices.* Let the probability distributions $P^{(n)}$ on $\mathcal{M}$ be given by

$$P^{(n)}(M)\,\mathrm{d}M = c\,\mathrm{e}^{-F(M)}\,\mathrm{d}M = c\,\mathrm{e}^{-F(M)}\prod_{i=1}^{n}\mathrm{d}M_{ii}\prod_{i<j}^{n}(\mathrm{d}X_{ij}\mathrm{d}Y_{ij})\,,$$

where $\mathrm{d}M$ stands for the natural Lebesgue measure which is invariant under translations, $M_{ij} =$

$X_{ij} + \mathrm{i}Y_{ij}$ denotes the entry M_{ij} of M with $M_{ij} = \overline{M_{ji}}$ and c is a norming constant such that

$$c \int_{\mathcal{M}} \mathrm{e}^{-F(M)} \mathrm{d}M = 1.$$

Moreover, we require that

$$\mathrm{e}^{-F(\widetilde{M})} \mathrm{d}\widetilde{M} = \mathrm{e}^{-F(M)} \mathrm{d}M,$$

where $\widetilde{M} = UMU^{-1}$ for any unitary matrix U. This formula means $P^{(n)}(M)\,\mathrm{d}M$ is invariant under every automorphism $M \to UMU^{-1}$ from $\mathcal{M}$ into itself. By the claim in [10], we have

$$\mathrm{d}\widetilde{M} = \mathrm{d}M. \tag{1.1}$$

Therefore,

$$\mathrm{e}^{-F(UMU^{-1})} = \mathrm{e}^{-F(M)}$$

for all unitary matrices U and Hermitian matrices M. Choosing U to diagonalize M, it follows that $F(M)$ depends only on the eigenvalues of M and also that $F(M)$ must depend symmetrically on the eigenvalues. Particularly, our interest is in the case of

$$F(M) = \operatorname{tr} M^2 = \sum_{j=1}^{n} \lambda_j^2,$$

which gives the probability distribution for the Gaussian Unitary Ensemble (GUE).

After integrating out the unitary conjugation, we expect measure on the matrices can be written as the measure on the eigenvalues as follows,

$$c\,\mathrm{e}^{-\mathrm{tr}M^2} \mathrm{d}M \to \frac{1}{\mathcal{Z}_n} \mathrm{e}^{-\sum_{i=1}^{n} \lambda_i^2} \prod_{i<j} (\lambda_i - \lambda_j)^2 \prod_{i=1}^{n} \mathrm{d}\lambda_i,$$

where

$$\mathcal{Z}_n = \int_{\mathbb{R}^n} \mathrm{e}^{-\sum_{i=1}^{n} \lambda_i^2} \prod_{i<j} (\lambda_i - \lambda_j)^2 \prod_{i=1}^{n} \mathrm{d}\lambda_i.$$

As we will see in the next subsection, from the measure on the eigenvalues, the eigenvalues can be considered Coulomb particles confined into a real axis.

Let $\mathcal{M} = \{M, M^*M = MM^*\}$ denote the space of n by n *Normal matrices* which is also called *Normal matrix ensemble*, where M^* is the conjugate transpose of M. Eigenvalues of normal matrices are complex. Similarly, the probability distributions on $\mathcal{M}$ is given by

$$\frac{1}{\mathcal{Z}_n}\prod_{i<j}|\lambda_i-\lambda_j|^2\cdot\exp\left(-\sum_{j=1}^{n}Q(\lambda_j)\right)\cdot\prod_{j=1}^{n}\mathrm{d}A(\lambda_j),$$

where

$$\mathcal{Z}_n=\int_{\mathbb{C}^n}\prod_{i<j}|\lambda_i-\lambda_j|^2\cdot\exp\left(-\sum_{j=1}^{n}Q(\lambda_j)\right)\cdot\prod_{j=1}^{n}\mathrm{d}A(\lambda_j)$$

and $\mathrm{d}A$ denotes the standard Lebesgue measure on the plane. This represents Coulomb gas on the plane with respect to the external potential Q.

1.2 Coulomb Gas

In the 1950s, Wigner's works presented the basic idea of the Coulomb gas model. And then, in 1960s, a series of papers by Dyson [12] showed the exact correspondence between the eigenvalue distributions of some random matrix models and the statistical mechanics of classical two-dimensional Coulomb gas, which attracted the attention of physicists and mathematicians.

In the two-dimensional Coulomb gas model (or the one-component plasma model), we consider n particles as a system of point charges with the same sign located at points $\{z_j\}_{j=1}^n$ in the complex plane, influenced by an external potential. The potential of interaction between z_j and z_k (logarithmic repulsion) is

$$\log\frac{1}{|z_j-z_k|^2},\quad j\neq k,\quad j,k\in\{1,\cdots,n\},$$

while the external potential is denoted by $Q(z)$. The function

$$Q:\mathbb{C}\to\mathbb{R}\cup\{+\infty\}$$

is lower semi-continuous and sufficiently large to force the particles to condensate in a scaling limit on a certain finite portion of the plane, called the "droplet", which is the support of the equilibrium measure. The details will be described in the main Chapters. For the external potential $Q(z)$, we have the following theorem [36] to define the equilibrium measure:

Theorem 1.2.1 *There is a unique probability measure* $\mathrm{d}\mu^*$ *in the plane that minimizes the func-*

tional $\mathcal{L}(\mu)$,

$$\mathcal{L}(\mu) = \int_{\mathbb{C}} Q(z)\mathrm{d}\mu(z) + \int_{\mathbb{C}^2} \log \frac{1}{|z-w|}\mathrm{d}\mu(z)\mathrm{d}\mu(w).$$

The minimizer $\mathrm{d}\mu^*$ *can be characterized by*

$$Q(z) - 2\int_{\mathbb{C}} \log|z-w|\mathrm{d}\mu^*(w) + l \geq 0$$

for all $z \in \mathbb{C}$ *with equality on the support of the measure* μ^**. The constant* l *is called modified Robin's constant and the measure* μ^* *is called the equilibrium measure.*

The combined potential energy resulting from particle interaction and the external potential is the function $\mathcal{E}_Q : \mathbb{C}^n \to \mathbb{R} \cup \{\infty\}$ given by

$$\mathcal{E}_Q(z) = \frac{1}{2}\sum_{j\neq k}^{n} \log \frac{1}{|z_j - z_k|^2} + N\sum_{j=1}^{n} Q(z_j), \quad z = (z_1, \cdots, z_n) \in \mathbb{C}^n,$$

where the summation indices j, k are assumed confined to the set $\{1, \cdots, n\}$. We are interested in the scaling limit where n and N tend to infinity while n/N is a fixed positive number. The particles are then distributed by Gibbs distribution,

$$\frac{1}{\mathcal{Z}_n}\mathrm{e}^{-\frac{\beta}{2}\mathcal{E}_Q(z)}\prod_{j=1}^{n}\mathrm{d}A(z_j),$$

where

$$\mathcal{Z}_n = \int_{\mathbb{C}^n} \mathrm{e}^{-\frac{\beta}{2}\mathcal{E}_Q}\prod_{j=1}^{n}\mathrm{d}A(z_j).$$

Here β is a positive parameter called inverse temperature and $0 < \mathcal{Z}_n < \infty$. In terms of the usual Vandermonde expression

$$V_n(z_1, \cdots, z_n) = \prod_{j<k}(z_k - z_j),$$

we may write the Gibbs distribution in the form of

$$\frac{1}{\mathcal{Z}_n}|V_n(z_1, \cdots, z_n)|^{\beta}\,\mathrm{e}^{-\frac{\beta}{2}N\sum_j Q(z_j)}\prod_{j=1}^{n}\mathrm{d}A(z_j).$$

When we consider the case of $\beta = 2$, the probability measure on $\mathcal{M}$ matches the one for the eigenvalues of normal matrices, which is given by

$$\frac{1}{\mathcal{Z}_n}\prod_{i<j}|z_i - z_j|^2 \cdot \exp\left(-N\sum_{j=1}^{n} Q(z_j)\right)\cdot\prod_{j=1}^{n}\mathrm{d}A(z_j), \tag{1.2}$$

where

$$\mathcal{Z}_n = \int_{\mathbb{C}^n}\prod_{i<j}|z_i - z_j|^2 \cdot \exp\left(-N\sum_{j=1}^{n} Q(z_j)\right)\cdot\prod_{j=1}^{n}\mathrm{d}A(z_j).$$

1.3 Orthogonal Polynomials

For the probability measure in (1.2), a connection to orthogonal polynomials can be provided by Heine's formula. It says that the averaged characteristic polynomial of the n particles is the (monic) orthogonal polynomial of degree n, i.e., $p_n(z) = \mathbb{E}\prod_{j=1}^{n}(z - z_j)$,

$$p_n(z) = \frac{1}{\widehat{D}_{n-1}}\det\begin{bmatrix} M_{00} & M_{10} & \cdots & M_{n0} \\ M_{01} & M_{11} & \cdots & M_{n1} \\ \vdots & \vdots & \vdots & \vdots \\ M_{0,n-1} & M_{1,n-1} & \cdots & M_{n,n-1} \\ 1 & z & \cdots & z^n \end{bmatrix}, \quad \widehat{D}_{n-1} = \det\begin{bmatrix} M_{00} & M_{10} & \cdots & M_{n-1,0} \\ M_{01} & M_{11} & \cdots & M_{n-1,1} \\ \vdots & \vdots & \vdots & \vdots \\ M_{0,n-1} & M_{1,n-1} & \cdots & M_{n-1,n-1} \end{bmatrix},$$

satisfies the orthogonality condition,

$$\int_{\mathbb{C}} p_n(z)\overline{p_m(z)}\mathrm{e}^{-NQ(z)}\mathrm{d}A(z) = h_n\delta_{nm} \quad (n, m = 0, 1, 2, \ldots), \tag{1.3}$$

where M_{ij} is defined by

$$M_{ij} = \int_{\mathbb{C}} z^i \bar{z}^j \mathrm{e}^{-NQ(z)}\mathrm{d}A(z)$$

and h_n is a (positive) norming constant. Note the expectation (in Heine's formula) is taken with respect to the measure in (1.2).

For $n \geq 1$, let us set $\mathbb{M}_n = [M_{ij}]_{0\leq i,j\leq n-1}$ to be the matrix of moments in terms of the measure $\mathrm{e}^{-NQ(z)}\mathrm{d}A(z)$. We define

$$\widehat{D}_n = \det \mathbb{M}_n.$$

For $Q < \infty$ almost everywhere, one can show that $\widehat{D}_n > 0$ (or, equivalently that $\mathbb{M}_n$ is positive definite). For an arbitrary nonzero vector $(s_1, \cdots, s_n) \in \mathbb{C}^n$, we have

$$0 < \left\| \sum_{i=0}^{n} s_i x^i \right\|_{L_Q^2}^2 = \int_{\mathbb{C}} \left(\sum_{i=0}^{n} s_i z^i \right) \left(\sum_{j=0}^{n} \overline{s_j} \bar{z}^j \right) \mathrm{e}^{-NQ(z)} \mathrm{d}A(z) = \sum_{i=0}^{n} \sum_{j=0}^{n} s_i \overline{s_j} M_{ij}.$$

1.4 Asymptotics of Planar Orthogonal Polynomial

The orthogonal polynomials with respect to a measure supported on the plane are called *planar orthogonal polynomials*. Such polynomials have been of interest due to its connection to two–dimensional Coulomb gas [1]. Moreover these polynomials appear [37] in the quantized version of Hele-Shaw flow, a type of growth model in the two–dimensional plane. These connections to physical system, Coulomb gas and Hele-Shaw flow, motivate one to study the large degree behavior of the polynomials. We recommend the recent paper [23] for an important progress in this regard and for the related history. Still lacking, until now, is the understanding of the limiting zero distribution when the degree of the polynomial goes to infinity. Several studies [2, 3, 7, 25, 27, 28] have shown that the zeros tend to certain one–dimensional set. In all of these cases the planar orthogonal polynomials in question turn out to be either classical orthogonal polynomials or multiple orthogonal polynomials [14, 24], whose asymptotic behavior is possible to study [35] due to rich algebraic structure such as finite term recurrence relation.

The statistical behavior of the particles has been studied [1] for a large class of potentials in various contexts including random normal matrices and two-dimensional Coulomb gas. For example, in the scaling limit where n and N tend to infinity while n/N is fixed, it is known [22] that the counting measure of the particles converges weakly,

$$\mathbb{E}\frac{1}{N}\sum_{j=1}^{n}\delta(z-z_j) \to \frac{\Delta Q}{4\pi}\chi_K$$

where $\Delta Q = (\partial_x^2 + \partial_y^2)Q$, χ_K is the characteristic function of the compact set $K \subset \mathbb{C}$ that we will call a *droplet* following [22], and the expectation is taken with respect to the measure in (1.2).

As a connection between orthogonal polynomials and Coulomb gas can be provided by Heine's formula, one might wonder if the zero distribution of P_n would tend to the averaged distribution of the particles. Though this is the case with the orthogonal polynomials on the real line (that corresponds to the particles confined on the line), in the cases of two-dimensional orthogonal polynomials so far studied [2, 3, 7, 25, 27, 28], the limiting zero distribution is observed to be concentrated on a small subset of the droplet, on some kind of potential-theoretic *skeleton* of K.[1]

A *skeleton of* K will refer to a subset of (the polynomial hull of) K with zero area, such that there exists a measure that is supported exactly on the skeleton and that generates the same logarithmic potential in the exterior of (the polynomial hull of) K as the Lebesgue measure supported on K. One characteristic of such skeleton is that it can be discontinuous under the continuous variation of the droplet K. A simple example [19, 20] comes from the sequence of polygons converging to a disk. The skeleton of the polygon, which is the set of rays connecting each vertex to the center, does not converge to the skeleton of the disk, the single point at the center. Such discontinuity can also occur, as we will see, when the perturbed droplets have real analytic boundary.

1.5 Riemann-Hilbert problem

We consider the following Riemann-Hilbert problem on the oriented contour (piecewise smooth) Γ, which has a positive side and a negative side. Fix an integer $n \geq 0$ and seek a 2×2 matrix function $Y = Y_n(z)$ such that it satisfies the following conditions,

$$\begin{cases} Y(z) \quad \text{is analytic in } \mathbb{C}\setminus\Gamma, \\ Y_+(z) = Y_-(z)\begin{bmatrix} 1 & w(z) \\ 0 & 1 \end{bmatrix}, \quad z\in\Gamma, \\ Y(z) = \left(I + \mathcal{O}\left(z^{-1}\right)\right)\begin{bmatrix} z^n & 0 \\ 0 & z^{-n} \end{bmatrix}, \quad z\to\infty. \end{cases}$$

Here $Y_\pm(z)$ describe the limits of $Y(z')$ as $z'\to z\in\Gamma$ from the $+$ (respectively, negative) side of Γ. If there exists Y such that it solves the above R-H problem, then we can prove Y is unique. Indeed, if Y solves the R-H problem

$$\det Y_+(z) = \det Y_-(z)\det\begin{bmatrix} 1 & w(z) \\ 0 & 1 \end{bmatrix} = \det Y_-(z).$$

Hence, $\det Y(z)$ is analytic in $\mathbb{C}$. Moreover, $\det Y(z) = 1 + \mathcal{O}\left(z^{-1}\right)$ as $z\to\infty$. Therefore, $\det Y(z)\equiv 1$, and so $Y^{-1}(z)$ is analytic in $\mathbb{C}\setminus\Gamma$. Suppose $\widetilde{Y}(z)$ is another solution to the R-H problem, then

[1] In some cases, the skeleton is also called "mother body" [19, 20].

for any $z \in \Gamma$,

$$\left[\widetilde{Y}Y^{-1}\right]_+(z) = \widetilde{Y}_+(z)Y_+^{-1}(z) = \widetilde{Y}_-(z)\begin{bmatrix}1 & w(z)\\ 0 & 1\end{bmatrix}\left(Y_-(z)\begin{bmatrix}1 & w(z)\\ 0 & 1\end{bmatrix}\right)^{-1} = \left[\widetilde{Y}Y^{-1}\right]_-(z).$$

Hence, $\widetilde{Y}Y^{-1}$ is analytic in $\mathbb{C}$ and $\widetilde{Y}Y^{-1} \to I$ as $z \to \infty$. Thus, by Liouville's Theorem,

$$Y = \widetilde{Y}.$$

We will show that $Y_{11}(z)$ is the orthogonal polynomial corresponding to the measure $w(z)\,\mathrm{d}z$ on Γ. By the jump condition of the Riemann-Hilbert problem, we have $[Y_{11}]_+ = [Y_{11}]_-$, therefore $Y_{11}(z)$ is analytic in $\mathbb{C}$. Moreover, by the asympototic behavior of $Y(z)$,

$$Y(z) = \begin{bmatrix}Y_{11}(z) & Y_{12}(z)\\ Y_{21}(z) & Y_{22}(z)\end{bmatrix} = \begin{bmatrix}z^n + \mathcal{O}\left(z^{n-1}\right) & \mathcal{O}\left(z^{-n-1}\right)\\ \mathcal{O}\left(z^{n-1}\right) & z^{-n} + \mathcal{O}\left(z^{-n-1}\right)\end{bmatrix},$$

i.e., $Y_{11}(z)$ is a monic polynomial. Moreover,

$$[Y_{12}(z)]_+ = [Y_{12}(z)]_- + Y_{11}(z)w(z)$$

with $Y_{12}(z) \to 0$ as $z \to \infty$. Hence, by applying the Plemelj-Sokhotsky formula (see page 23 of [18]),

$$Y_{12}(z) = \frac{1}{2\pi \mathrm{i}}\int_\Gamma \frac{Y_{11}(s)w(s)}{s-z}\mathrm{d}s.$$

Since

$$Y_{12}(z) = -\frac{1}{2\pi \mathrm{i}}\int_\Gamma Y_{11}(s)w(s)\left(\frac{1}{z} + \frac{s}{z^2} + \cdots + \frac{s^n}{z^{n+1}} + \cdots\right)\mathrm{d}s$$

and

$$Y_{12}(z) = \mathcal{O}\left(z^{-n-1}\right),$$

we obtain that

$$\int_\Gamma Y_{11}(s)s^j w(s)\,\mathrm{d}s = 0, \quad 0 \le j \le n-1.$$

Thus, $Y_{11}(s)$ is orthogonal to s^j for $0 \le j \le n-1$ with respect to the measure $w(s)\,\mathrm{d}s$.

Similarly, for the Riemann-Hilbert problem for Type II multiple orthogonal polynomials, we have an analogous result. Let Γ be a simple closed oriented curve. The Riemann-Hilbert problem:

$$\begin{cases} Y : \text{is holomorphic matrix function in } \mathbb{C} \setminus \Gamma, \\ Y_+(z) = Y_-(z) \begin{bmatrix} 1 & w_1(z) & \cdots & w_l(z) \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \text{ on } \Gamma, \\ Y(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right) \begin{bmatrix} z^n & 0 & \cdots & 0 \\ 0 & z^{-n_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z^{-n_l} \end{bmatrix}, \quad \text{as } z \to \infty, \end{cases}$$

where $\sum_{j=1}^{l} n_j = n$ and the subscript $\pm$ in $Y_\pm$ represents the limiting value when approaching Γ from the corresponding sides of the contour. We have $Y_{11}(z)$ is a Type II multiple orthogonal polynomial satisfying the orthogonality condition:

$$\int_\Gamma p_{\mathbf{n}}(z)\, z^k w_j(z)\, \mathrm{d}z = 0, \quad 0 \le k \le n_j - 1, \quad 1 \le j \le l.$$

Lastly, we will introduce the Small Norm Theorem[4, 21], which plays an important technical role in the asymptotics analysis of the solutions to the Riemann-Hilbert problem.

Theorem 1.5.1 *Suppose a Riemann-Hilbert problem is posed on the oriented contour* Γ *(piecewise smooth) for a matrix function* $\mathcal{E}(z)$*,*

$$\begin{cases} \mathcal{E}_+(z) &= \mathcal{E}_-(z)\left(I + \delta G(z)\right), \quad z \in \Gamma, \\ \mathcal{E}(z) &= I + \mathcal{O}\left(z^{-1}\right), \quad z \to \infty, \end{cases} \tag{1.4}$$

where $\det\left(I + \delta G(z)\right) = 1$*, the subscript* $\pm$ *in* $\mathcal{E}_\pm(z)$ *represents the limiting value when approaching* Γ *from the corresponding sides of the contour. Let* N_p *be the norm in* $L^p(\Gamma, |dz|)$ *of the matrix function* $\delta G(z)$*. Then there exists a constant* C_Γ *such that if* $N_\infty < \frac{1}{C_\Gamma}$ *the solution of the R-H* *problem exists and*

$$\|\mathcal{E}(z)-I\| \leq \frac{C_\Gamma N_2}{1-C_\Gamma N_\infty}, \quad \textit{for } z \in \Gamma.$$

$$\|\mathcal{E}(z)-I\| \leq \frac{1}{2\pi\, dist(z,\Gamma)}\left(N_1+\frac{C_\Gamma N_2^2}{1-C_\Gamma N_\infty}\right), \quad \textit{for } z \in \mathbb{C}\setminus\Gamma.$$

In the following proof, we will use the fact about the L^2- boundedness of the Cauchy operators (cf. to [31]). For any $f \in L^p(\Gamma, |dz|)$ with $1 < p < \infty$,

$$\|C_\pm f\|_{L^p(\Gamma)} = \left\| \lim_{z'\to z_\pm} \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{f(s)}{s-z'}\,\mathrm{d}s \right\|_{L^p(\Gamma)} \leq C_\Gamma \|f\|_{L^p(\Gamma)}$$

for some constant C_Γ. In other words, the Cauchy operators C_+ and C_- are bounded in the space $L^p(\Gamma)$ for all $1 < p < \infty$.

Proof. First of all, we will show the Riemann-Hilbert problem (1.4) is equivalent to the following singular integral equation,

$$\mathcal{E}(z) = I + \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{\mathcal{E}_-(s)\delta G(s)}{s-z}\,\mathrm{d}s. \tag{1.5}$$

By taking the boundary conditions from the $+$ and $-$ sides of Γ in the equation (1.5), we have

$$\begin{aligned} \mathcal{E}_+(z) &= I + \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{\mathcal{E}_-(s)\delta G(s)}{s-z_+}\,\mathrm{d}s, \\ \mathcal{E}_-(z) &= I + \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{\mathcal{E}_-(s)\delta G(s)}{s-z_-}\,\mathrm{d}s. \end{aligned}$$

Therefore,

$$\mathcal{E}_+(z)-\mathcal{E}_-(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{\mathcal{E}_-(s)\delta G(s)}{s-z_+}\,\mathrm{d}s - \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{\mathcal{E}_-(s)\delta G(s)}{s-z_-}\,\mathrm{d}s = \mathcal{E}_-(z)\delta G(z),$$

which shows the identity (1.5) has the same jump condition of the Riemann-Hilbert problem (1.4), thus the equivalence holds. Let us rewrite the euqation (1.5) as

$$\mathcal{E}(z)-I = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{\delta G(s)}{s-z}\mathrm{d}s + \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{(\mathcal{E}_-(s)-I)\,\delta G(s)}{s-z}\,\mathrm{d}s.$$

By taking the boundary condition from the $(-)$ side of Γ in the above identity, we have,

$$\mathcal{E}_-(z)-I = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{\delta G(s)}{s-z_-}\mathrm{d}s + \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{(\mathcal{E}_-(s)-I)\,\delta G(s)}{s-z_-}\,\mathrm{d}s.$$

For convenience, let us denote $\mathcal{E}_-(z) - I$ by $f(z)$, denote $\frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{\delta G(s)}{s - z_-} \mathrm{d}s$ by δh and denote $\frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{f(s)\delta G(s)}{s - z_-} \mathrm{d}s$ by $\mathcal{L}(f)$, we have

$$(Id - \mathcal{L})(f) = \delta h, \tag{1.6}$$

this can be considered as an equation in $L^2(\Gamma)$, which implies

$$\begin{aligned} f &= (Id - \mathcal{L})^{-1}\delta h = \textstyle\sum_{j=0}^{\infty} \mathcal{L}^j(\delta h), \\ \|f\|_{L^2} &\leq \frac{\|\delta h\|_{L^2}}{1 - \|\mathcal{L}\|_{L^2}}. \end{aligned} \tag{1.7}$$

The solution exists if the operator norm of $\mathcal{L}$ is less than 1. This is because: for any $f_1, f_2 \in L^2(\Gamma)$,

$$\|\mathcal{L}(f_1) - \mathcal{L}(f_2)\|_{L^2} \leq \|\mathcal{L}\|_{L^2} \|f_1 - f_2\|_{L^2}.$$

If $\|\mathcal{L}\|_{L^2} < 1$, then the mapping $\mathcal{L}$ is a contraction mapping. Hence, the solution to (1.6) exists. Moreover, since $\mathcal{L}$ is multiplication on the right by δG, then

$$\|\mathcal{L}\|_{L^2} \leq C_\Gamma \|\delta G\|_\infty,$$

where C_Γ is the norm of the Cauchy operator on Γ. Therefore, the solution of the Riemann-Hilbert Problem exists when $N_\infty < \frac{1}{C_\Gamma}$. Since

$$\delta h = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{\delta G(s)}{s - z} \mathrm{d}s,$$

we obtain

$$\|\delta h\|_{L^2} \leq C_\Gamma \|\delta G\|_{L^2}$$

such that

$$\|\mathcal{E}_-(z) - I\|_{L^2} = \|f\|_{L^2} \leq \frac{\|\delta h\|_{L^2}}{1 - \|\mathcal{L}\|_{L^2}} \leq \frac{C_\Gamma N_2}{1 - C_\Gamma N_\infty}.$$

Last, we will estimate $\mathcal{E}(z)$ for $z \notin \Gamma$,

$$\begin{aligned}
\|\mathcal{E}(z) - I\| \quad &\le \quad \left\| \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{\delta G(s)}{s - z} \, \mathrm{d}s \right\|_{L^2} + \left\| \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{(\mathcal{E}_-(s) - I) \, \delta G(s)}{s - z} \, \mathrm{d}s \right\|_{L^2} \\
&\le \quad \frac{1}{2\pi} \frac{N_1}{\mathrm{dist}(z, \Gamma)} + \frac{1}{2\pi} \frac{\|\mathcal{E}_-(z) - I\|_{L^2} N_2}{\mathrm{dist}(z, \Gamma)} = \frac{1}{2\pi \, \mathrm{dist}(z, \Gamma)} \left(N_1 + \frac{C_\Gamma N_2^2}{1 - C_\Gamma N_\infty} \right).
\end{aligned} \tag{1.8}$$

$\square$

Chapter 2

Discontinuity in the asymptotic behavior of planar orthogonal polynomials under a perturbation of the Gaussian weight

2.1 Introduction

In this chapter we consider the external potential given by

$$Q(z) = |z|^2 + \frac{2c}{N} \log \frac{1}{|z-a|}, \quad c > -1, \quad a > 0. \tag{2.1}$$

When N is large and $c \ll N$, this represents a small perturbation of the Gaussian weight. It corresponds to the interacting Coulomb particles with charge $+1$ for each, in the presence of an extra particle with charge $+c$ at a. By a simple rotation of the plane, the above Q covers the case with any nonzero $a \in \mathbb{C}$. Since one characteristic of the skeleton is that it can be discontinuous under the continuous variation of the droplet K, we ask whether the zero distribution of the corresponding orthogonal polynomial P_n also exhibits the similar discontinuity under the variation of the underlying droplet or, equivalently, under the variation of the external potential.

We are interested in the scaling limit where N and n go to infinity while the ratio, n/N, is a fixed positive number. Below we will set $N = n$ without losing generality since the orthogonality (1.3) gives the relation

$$P_{n,N}(z;a) = \left(\frac{n}{N}\right)^{n/2} P_{n,n}\left(\sqrt{\frac{N}{n}}z; \sqrt{\frac{N}{n}}a\right),$$

where $P_{n,N}(z;a) = P_{n,N}(z)$ stands for orthogonal polynomials with respect to the external potential given by (2.1). Though we will mostly use N, we will keep n whenever the expression holds true for general $n \neq N$.

2.1.1 Limiting skeleton

The potential (2.1) has been studied in [2] with the slightly different notation. Let us define γ by c/N (c in [2] is c/N in our notation). Then $Q(z)$ can be written as

$$Q(z) = |z|^2 + 2\gamma \log \frac{1}{|z-a|}.$$

To state Theorem 2.1.1 let us introduce K_γ, μ_γ and $\mathcal{S}_\gamma$, and define μ and $\mathcal{S}$.

Let $K_\gamma \subset \mathbb{C}$ be the compact set, called a *droplet*, so that

$$\mu_\gamma^{(2\mathrm{D})} = \frac{1}{4\pi} \chi_{K_\gamma}$$

is the unique probability measure that minimizes the energy functional,

$$I[\mu] = \int Q \,\mathrm{d}\mu + \frac{1}{2} \iint \log \frac{1}{|z-w|} \mathrm{d}\mu(z) \mathrm{d}\mu(w).$$

Let $\mathcal{S}_\gamma = \operatorname{supp} \mu_\gamma$ be the *skeleton of* K_γ, that is, the compact subset of $\mathbb{C}$ with zero area such that the probability measure μ_γ generates the same logarithmic potential as $\mu_\gamma^{(2\mathrm{D})}$:

$$U^{\mu_\gamma}(z) = U^{\mu_\gamma^{(2\mathrm{D})}}(z), \qquad z \notin \text{(polynomial convex hull of } K_\gamma). \tag{2.2}$$

Here we denote $U^m(z) = -\int \log |z-w| \, dm(w)$ for a positive Borel measure m. We note that this definition of skeleton is not conventional; $\mathbb{C} \setminus \mathcal{S}_\gamma$ does not have to be connected. Such a skeleton may not be unique in general. We give explicit definitions of $\mathcal{S}_\gamma$ and μ_γ in Section 2.2.

We define the *limiting skeleton* $\mathcal{S}$ by

$$\mathcal{S} = \{ z \in \mathbb{C} : \ \operatorname{Re}(\log z - az) = \log \beta - a\beta, \ \ \operatorname{Re} z \le \beta \}, \tag{2.3}$$

where

$$\beta = \min\{a, 1/a\}.$$

From the equivalent representation of $\mathcal{S}$ in the real coordinates by

$$\mathcal{S} = \left\{ (x,y) \in \mathbb{R}^2 : x^2 + y^2 = \beta^2 \mathrm{e}^{2a(x-\beta)}, \ x \le \beta \right\}.$$

It is a simple exercise to show that, $\mathcal{S} \subset \operatorname{clos}\mathbb{D}$ is a simple closed curve that encloses the origin and intersects β, where $\mathbb{D}$ is the unit disk. We will denote the interior and the exterior of $\mathcal{S}$ by $\operatorname{Int}\mathcal{S}$ and $\operatorname{Ext}\mathcal{S}$ respectively. See Figure 1 for some illustration of $\mathcal{S}$.

We define μ to be the probability measure supported on $\mathcal{S}$ given by

$$\mathrm{d}\mu(z) = \rho(z)\mathrm{d}\ell(z) = \frac{1}{2\pi}\left|a - \frac{1}{z}\right|\mathrm{d}\ell(z), \quad z \in \mathcal{S}, \tag{2.4}$$

where $d\ell$ is the arclength measure on $\mathcal{S}$. Alternatively, the same measure can be written in terms of holomorphic differential by

$$\mathrm{d}\mu(z) = \frac{1}{2\pi\mathrm{i}}\left(\frac{1}{z} - a\right)\mathrm{d}z.$$

This is because $\left(\frac{1}{z} - a\right)\mathrm{d}z = \pm\mathrm{i}\left|a - \frac{1}{z}\right|\mathrm{d}\ell(z)$, the sign is determined at the intersection of $\mathcal{S}$ with the real axis.

Theorem 2.1.1 *As $\gamma \to 0$ we have the convergences:*

$$K_\gamma \to \operatorname{clos}\mathbb{D}, \quad \mu_\gamma \to \mu, \quad \mathcal{S}_\gamma \to \mathcal{S},$$

in the appropriate senses (i.e., respectively in Hausdorff metric, in weak-$*$, and in Hausdorff metric).

Remark 1. In Theorem 2.1.1, we define the Hausdorff metric $d_H(X, Y)$ by

$$d_H(X, Y) = \max\{\sup_{x\in X}\inf_{y\in Y} d(x, y), \sup_{y\in Y}\inf_{x\in X} d(x, y)\}$$

for X and Y are two non-empty subsets of metric space (M, d), we choose the metric to be the Euclidean metric. The proof is in Section 2.2.

Remark 2. In both examples, the one by Gustafsson [19] and the one from the above theorem – the discontinuity occurs when the droplet becomes a disk. It is an interesting question whether the discontinuity occurs with other shapes than disk.

2.1.2 Strong asymptotics of P_N and the location of zeros

Let us define

$$\begin{aligned}\phi_A(z) &= a(z-\beta) - \log\frac{z}{\beta},\\ \phi(z) &= \begin{cases}\phi_A(z), & z\in \operatorname{Ext}\mathcal{S},\\ -\phi_A(z), & z\in\operatorname{Int}\mathcal{S}.\end{cases}\end{aligned}\tag{2.5}$$

Note that $\operatorname{Re}\phi\equiv 0$ on $\mathcal{S}$.

Let U be a certain neighborhood of $\mathcal{S}\setminus\{\beta\}$ where $\operatorname{Re}\phi\le 0$. See Figure 7 and the paragraph below Lemma 2.3.1 for more details. Let D_β be a disk neighborhood of β with a fixed radius such that the map $\zeta : D_\beta\to\mathbb{C}$ given below is univalent.

$$\zeta(z) = \begin{cases}\sqrt{2N\phi_A(z)} = a\sqrt{N}(z-\beta)(1+\mathcal{O}(z-\beta)) & \text{for}\quad a>1,\\ -N\phi_A(z) = \dfrac{1-a^2}{a}N(z-\beta)\left(1+\mathcal{O}(z-\beta)\right) & \text{for}\quad a<1.\end{cases}\tag{2.6}$$

Theorem 2.1.2 *For $a>1$ and for any fixed nonzero $c>-1$, we have*

$$P_N(z) = \begin{cases} z^N\left(\dfrac{z}{z-\beta}\right)^c\left(1+\mathcal{O}\left(\dfrac{1}{N}\right)\right), & z\in\operatorname{Ext}\mathcal{S}\setminus(U\cup D_\beta),\\[2ex] -\dfrac{\beta^N\sqrt{2\pi}(a^2-1)^c}{N^{1/2-c}a\Gamma(c)}\dfrac{\mathrm{e}^{Na(z-\beta)}}{z-\beta}\left(\dfrac{z-\beta}{z-a}\right)^c\left(1+\mathcal{O}\left(\dfrac{1}{\sqrt{N}}\right)\right), & z\in\operatorname{Int}\mathcal{S}\setminus(U\cup D_\beta),\\[2ex] z^N\left(\dfrac{z}{z-\beta}\right)^c\left(1+\mathcal{O}\left(\dfrac{1}{N}\right)\right) & \\ \qquad -\dfrac{\beta^N\sqrt{2\pi}(a^2-1)^c}{N^{1/2-c}a\Gamma(c)}\dfrac{\mathrm{e}^{Na(z-\beta)}}{z-\beta}\left(\dfrac{z-\beta}{z-a}\right)^c\left(1+\mathcal{O}\left(\dfrac{1}{\sqrt{N}}\right)\right), & z\in U\setminus D_\beta\\[2ex] z^N\left(\left(\dfrac{z\zeta(z)}{z-\beta}\right)^c\mathrm{e}^{\frac{\zeta^2(z)}{4}}D_{-c}(\zeta(z))+\mathcal{O}\left(\dfrac{1}{\sqrt{N}}\right)\right), & z\in D_\beta.\end{cases}$$

Here D_{-c} be the parabolic cylinder function or Weber function and is defined by (see the identity (12.5.6) in [34])

$$D_{-c}(\zeta) := \frac{\mathrm{e}^{\frac{\zeta^2}{4}}}{\mathrm{i}\sqrt{2\pi}}\int_{\epsilon-\mathrm{i}\infty}^{\epsilon+\mathrm{i}\infty}\mathrm{e}^{-\zeta s+\frac{s^2}{2}}s^{-c}\mathrm{d}s,\quad \epsilon>0.\tag{2.7}$$

Theorem 2.1.3 *For $a < 1$ and for any fixed nonzero $c > -1$, we have*

$$P_N(z) = \begin{cases} z^N \left(\frac{z}{z-a}\right)^c \left(1 + \mathcal{O}\left(\frac{1}{N^\infty}\right)\right), & z \in \operatorname{Ext} \mathcal{S} \setminus (U \cup D_\beta), \\ -\frac{a^{1+N}(1-a^2)^{c-1}}{N^{1-c}\Gamma(c)} \frac{\mathrm{e}^{Na(z-a)}}{z-a} \left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right), & z \in \operatorname{Int} \mathcal{S} \setminus (U \cup D_\beta), \\ z^N \left(\frac{z}{z-a}\right)^c \left(1 + \mathcal{O}\left(\frac{1}{N^\infty}\right)\right) & \\ \qquad - \frac{a^{1+N}(1-a^2)^{c-1}}{N^{1-c}\Gamma(c)} \frac{\mathrm{e}^{Na(z-a)}}{z-a} \left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right), & z \in U \setminus D_\beta, \\ z^N \left(\left(\frac{z}{z-a}\right)^c \left(1 + \mathcal{O}\left(\frac{1}{N^\infty}\right)\right) \right. & \\ \qquad \left. - \left(\frac{z\zeta(z)}{z-a}\right)^c \frac{1}{\mathrm{e}^{\zeta(z)}} \left(\hat{f}(\zeta(z)) + \mathcal{O}\left(\frac{1}{N}\right)\right) \right), & z \in D_\beta. \end{cases} \tag{2.8}$$

Here,

$$\hat{f}(\zeta) = \frac{-1}{2\pi \mathrm{i}} \int_{\mathcal{L}} \frac{\mathrm{e}^s}{s^c(s-\zeta)} \mathrm{d}s,$$

where the contour $\mathcal{L}$ begins at $-\infty$, circles the origin once in the counterclockwise direction, and returns to $-\infty$. The error bound $\mathcal{O}(1/N^\infty)$ means $o(1/N^k)$ for an arbitrary integer k.

One can check that the branch cut discontinuity of $(z/(z-a))^c$ in the last equation of (2.8) is canceled by the discontinuity of $\hat{f}$ so that the asymptotic expression of P_N in D_β is analytic.

From Theorem 2.1.2 and 2.1.3, one can notice that the zeros of P_N can appear when the two terms in the asymptotic expressions of P_N in $U \setminus D_\beta$ cancel each other and hence must have the same order with respect to N. Such cancellation may be expressed in terms of ϕ_A as we presently explain below.

$$\begin{aligned} \left(\frac{z}{z-\beta}\right)^c &= \mathrm{e}^{N\phi_A(z)} \left(\frac{z-\beta}{z-a}\right)^c \frac{\sqrt{2\pi}(a^2-1)^c}{a\Gamma(c) N^{\frac{1}{2}-c}(z-\beta)}, && \text{for} \quad a > 1, \\ \left(\frac{z}{z-a}\right)^c &= \mathrm{e}^{N\phi_A(z)} \frac{a(1-a^2)^{c-1}}{N^{1-c}\Gamma(c)(z-a)}, && \text{for} \quad a < 1. \end{aligned}$$

Taking the logarithm of the absolute values on both sides and after simple calculations, we get

$$-\operatorname{Re} \phi_A(z) = \left(c - \frac{1}{2}\right) \frac{\log N}{N} - \frac{\log \Gamma(c)}{N} + \frac{1}{N} \log \left| \left(\frac{z-\beta}{z-a}\right)^c \frac{\sqrt{2\pi}(a^2-1)^c}{a(z-\beta)^{1-c} z^c} \right|, \qquad a > 1, \tag{2.9}$$

$$-\operatorname{Re} \phi_A(z) = \frac{(c-1)\log N}{N} - \frac{\log \Gamma(c)}{N} + \frac{1}{N} \log \left| \frac{a(1-a^2)^{c-1}}{(z-a)^{1-c} z^c} \right|, \qquad a < 1. \tag{2.10}$$

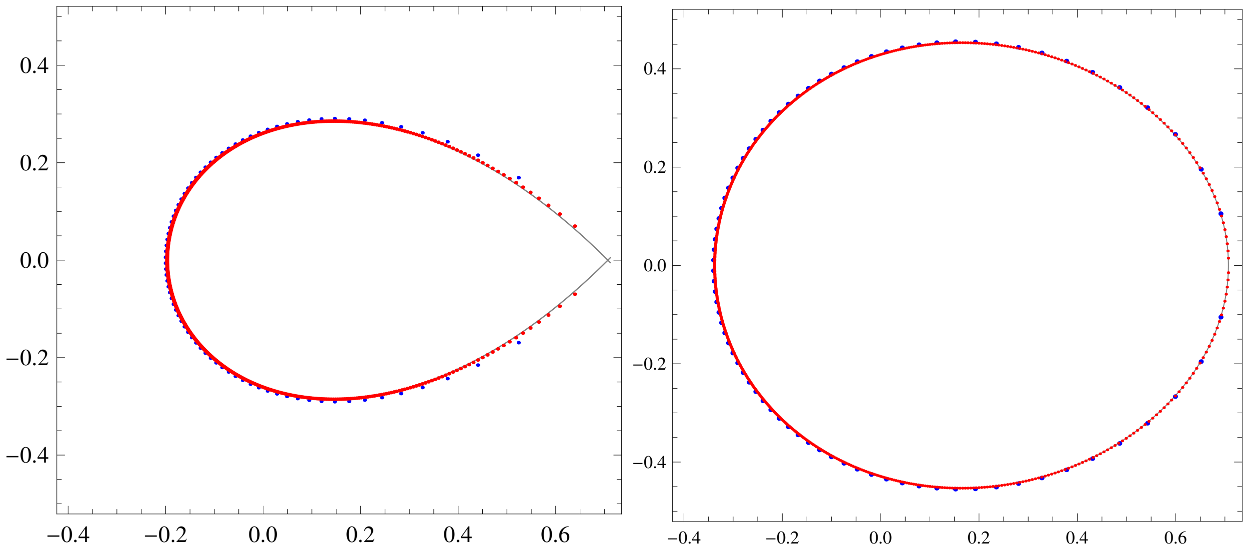

Figure 1.: The zeros of orthogonal polynomials with degrees 80 (blue) and 600 (red) for $c = 1$. The left is for $a = \sqrt{2}$ and the right is for $a = 1/\sqrt{2}$. In both cases, zeros are close to the curves representing $\mathcal{S}$.

As we will show in Lemma 2.3.1, $\operatorname{Re}\phi_A$ is positive (resp. negative) in $U \cap \operatorname{Int}\mathcal{S}$ (cf. to Fig. 8) (resp. in $U \cap \operatorname{Ext}\mathcal{S}$). For $a > 1$, since the dominant term in the right hand side of (2.9) is $\left(c - \frac{1}{2}\right)\frac{\log N}{N}$, the zeros will approach $\mathcal{S}$ from $\operatorname{Ext}\mathcal{S}$ for $c > \frac{1}{2}$ and from $\operatorname{Int}\mathcal{S}$ for $c < \frac{1}{2}$. For $a < 1$, since the dominant term in the right hand side of (2.10) is $(c-1)\frac{\log N}{N}$, the zeros will approach $\mathcal{S}$ from $\operatorname{Ext}\mathcal{S}$ for $c > 1$ and from $\operatorname{Int}\mathcal{S}$ for $c < 1$. See Figure 1. We also remark, without proof, that the limiting distribution of the zeros is given by μ which is explicitly given in (2.4). This can be proven, for example, using the method in [36] (Chapter III) and [33] (Theorem 2.3).

We remark that the case $-1 < c < 0$ is essentially treated in [3]. We note that the limiting locus of zeros remains the same for both the positive and negative c (which seems unexpected according to Remark 1.2 in [3]). It turns out that, as the value of c gets bigger, we need higher order corrections in the Riemann-Hilbert nonlinear steepest descent analysis [11]. To obtain the result that works for an arbitrary value of c, therefore, we need an arbitrary order correction in the nonlinear steepest descent analysis shown later. This is done in Section 2.5 using the method developed in [5].

We found that the limiting support of the zeros does not depend on c. Even for c decaying as a power of N (e.g., $c = N^{-1000}$) the limiting support of the zeros converges to $\mathcal{S}$. However, when c

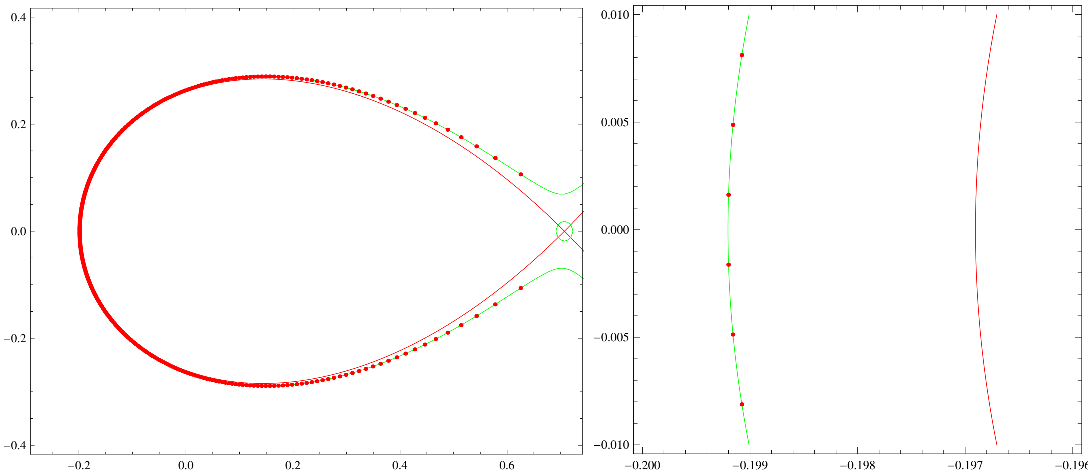

Figure 2.: Zeros of orthogonal polynomials when $a = \sqrt{2}$, $c = 1$ and $N = 300$. The red line is $\mathcal{S}$ and the green line is the solution set of (2.9). The right figure is the enlarged view of the left figure.

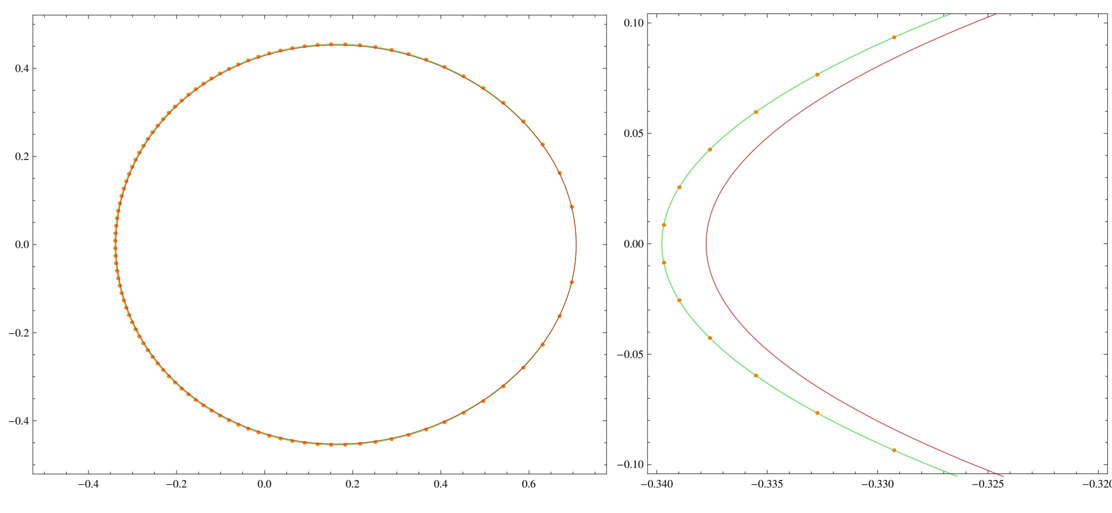

Figure 3.: When $a = 1/\sqrt{2}$, $c = 1$ and $N = 100$. The red line is $\mathcal{S}$ and the green line is the solution set of (2.10). The right figure is the enlarged view of the left figure.

decays exponentially in N, say $c = \mathrm{e}^{-\eta N}$, the right hand sides of both (2.9) and (2.10) converge to

$$-\eta = -\lim_{N\to\infty} \frac{\log \Gamma(\mathrm{e}^{-\eta N})}{N}, \quad \eta > 0$$

and the zeros approach the curve in $\operatorname{Int} \mathcal{S}$ given by the equation

$$\operatorname{Re} \phi_A(z) = \eta. \tag{2.11}$$

A similar "sensitive behavior of zeros with respect to a parameter" has been observed in [26].

It is simple to observe that the family of curves given by (2.11) for $0 \le \eta < \infty$ continuously interpolates between the curve $\mathcal{S}$ and the origin. In Figure 11, we show the curves satisfying (2.11) for $\eta = 0.2$ and $\eta = 0.4$, with the corresponding zeros.

To establish the behavior of zeros for *scaling* c, however, Theorem 2.1.2 and 2.1.3 are not enough as the error bounds in the theorems are for *fixed* c. For *c that scales to zero with N* we will prove Theorem 2.4.5 and 2.6.3 where the error bounds are *uniform in* c.

Remark 3. A simple way to understand the phenomenon is to recall the well–known instability of roots of polynomials, for example, the zeros of $P_n(z) = z^n + a/n^k$ still tend to the uniform distribution on the unit circle as $n \to \infty$ (for any fixed positive k) although the polynomial is a $\mathcal{O}(n^{-k})$ perturbation of the monomial. This simple model example already shows that a perturbation that interpolates between the two behaviors would require to have $a = \mathrm{e}^{-n\eta}$. From this perspective it can be expected to see that the exponentially small perturbations of the potential Q may interpolate the too different behaviors.

Remark 4. The main message of the paper is that the asymptotic zero locus can be quite sensitive to the small perturbation of the underlying measure. In Figure 5 we give another numerical plot that supports such statement. The example considers the orthogonal polynomials with the orthogonality measure supported on the restricted domain (cutoff) as described by E (cf. to Fig. 5). Though the cutoff may be considered as a "small perturbation" to the underlying Coulomb particle system, it seems to affect the polynomials significantly.

In the next section we prove Theorem 2.1.1 about the limiting skeleton. In section 2.3 we prove the asympototic result for $a > 1$ and c near 0. In section 2.4 we prove the similar result for an arbitrary c. In section 2.5 we prove the asympototic result mostly following the arguments from the previous two sections. In the section 2.7, we argue that the similar method will give the result

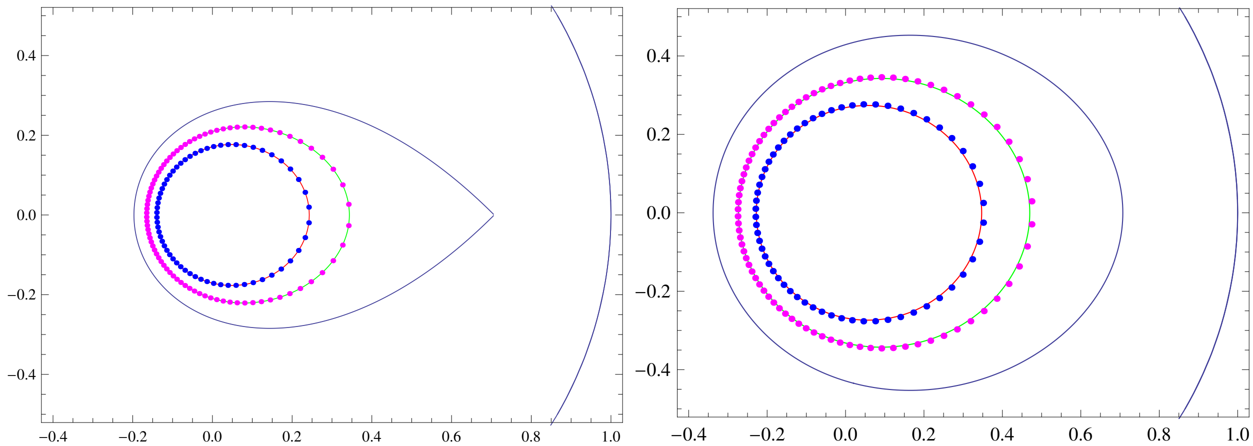

Figure 4.: The zeros of orthogonal polynomials with degrees 60 (blue) and 80 (magenta) for $c = e^{-\eta n}$, where $\eta = 0.4$ (blue) and $\eta = 0.2$ (magenta). The left is for $a = \sqrt{2}$ and the right is for $a = 1/\sqrt{2}$. In both cases, zeros seem to converge to the curves given by (2.11) for the corresponding values of a and c.

for the critical case of $a \approx 1$, by showing that the local parametrix satisfies the Riemann-Hilbert problem for Painlevé IV equation.

2.2 The proof of Theorem 2.1.1

For the convenience of the readers we reproduce the useful definitions from [2].

For $a < 1$ and a sufficiently small γ we define

$$K_\gamma = \overline{D(0, \sqrt{1+\gamma})} \setminus D(a, \sqrt{\gamma}), \tag{2.12}$$

where $D(a, r)$ stands for the disc with radius r centered at a.

For $a < 1$ we define $\mathcal{S}_\gamma$ to be the simple closed curve enclosing $[0, a]$ and intersecting

$$\beta_\gamma = \frac{a^2 + 1 - \sqrt{(1-a^2)^2 - 4a^2\gamma}}{2a} > a,$$

such that the quadratic differential $y_\gamma(z)^2 \mathrm{d}z^2$ is real and negative on $\mathcal{S}_\gamma$ where

$$y_\gamma(z) := (-1)^{\chi_{\mathrm{Int}\,\mathcal{S}_\gamma}} \left[a + \frac{\gamma}{z-a} - \frac{1+\gamma}{z} \right].$$

Here, we denote the interior of a simple closed curve $\mathcal{S}_\gamma$ by $\mathrm{Int}\,\mathcal{S}_\gamma$. (We recall that χ is the characteristic function.)

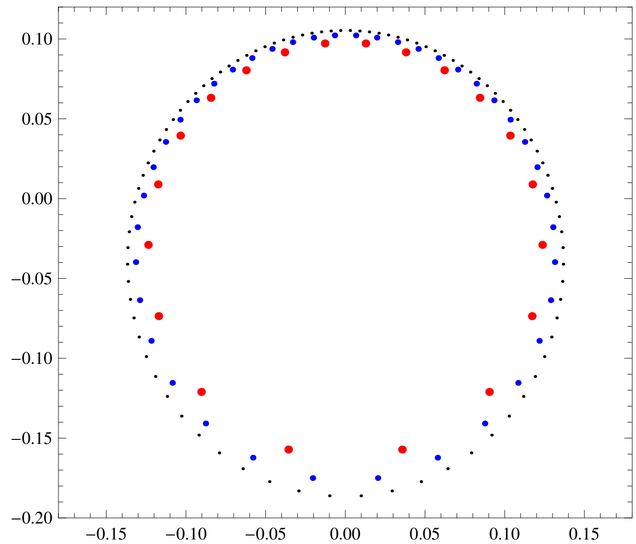

Figure 5.: The zeros of orthogonal polynomials with degrees $\{20, 40, 90\}$ and with the orthogonality measure given by $\chi_E e^{-n|z|^2} dA(z)$ where $E = (-\infty, +\infty) \times [-3i/2, +i\infty) \subset \mathbb{C}$. The plot suggests that the limiting support of zeros is not the origin.

For $a \geq 1$, the set K_γ is defined to be the closure of the interior of the real analytic Jordan curve given by the image of the unit circle under f_γ, where

$$f_\gamma(\nu) = \rho\nu - \frac{\kappa}{\nu - \alpha} - \frac{\kappa}{\alpha},$$

and parameters $\rho > 0, \kappa \geq 0$, and $0 < \alpha \leq 1/a$ are given in terms of a and γ below. First,

$$\rho = \frac{1 + a^2\alpha^2}{2a\alpha}, \quad \kappa = \frac{(1-\alpha^2)(1-a^2\alpha^2)}{2a\alpha}.$$

The parameter α is given by the unique solution of $P_\gamma(\alpha^2) = 0$ such that $0 < \alpha \leq 1/a$ where

$$P_\gamma(X) := X^3 - \left(\frac{a^2 + 4\gamma + 2}{2a^2}\right) X^2 + \frac{1}{2a^4}.$$

The existence is easily seen since $P_\gamma(0) > 0$ and $P_\gamma(1/a^2) = -2\gamma/a^6 < 0$. Moreover, $P_\gamma(X)$ is monotonically decreasing on $(0, 1/a]$, we can see the uniqueness of α. We note that, as γ goes to zero, α goes to $1/a$, κ goes to zero and ρ goes to 1.

For $a \geq 1$ we define $\mathcal{S}_\gamma$ to be the smooth arc with the endpoints at

$$\beta_\gamma := \alpha\rho - \frac{\kappa}{\alpha} + 2i\sqrt{\kappa\rho} \quad \text{and} \quad \overline{\beta_\gamma}$$

such that the quadratic differential $y_\gamma(z)^2dz^2$ is real and negative on $\mathcal{S}_\gamma$ where

$$y_\gamma(z) := \frac{a(z-b_\gamma)\sqrt{(z-\beta_\gamma)(z-\overline{\beta_\gamma})}}{z(z-a)}, \quad b_\gamma = \frac{\rho}{\alpha}.$$ [1]

For all values of a, we define the probability measure μ_γ supported on $\mathcal{S}_\gamma$ by

$$d\mu_\gamma = \frac{1}{2\pi}|y_\gamma(z)|\mathrm{d}\ell_\gamma,$$

where $\mathrm{d}\ell_\gamma$ is the arclength measure of $\mathcal{S}_\gamma$.

For all values of a, we define ϕ_γ by

$$\phi_\gamma(z) = \int_{\beta_\gamma}^{z} y_\gamma(s)\,\mathrm{d}s,$$

where the integration contour lies in the simply connected domain $\mathbb{C}\setminus([0,\infty)\cup[\beta_\gamma,\overline{\beta_\gamma}])$, where $[\beta_\gamma,\overline{\beta_\gamma}]$ stands for the vertical line segment connecting β_γ and $\overline{\beta_\gamma}$ (for $a\geq 1$, $[\beta_\gamma,\overline{\beta_\gamma}]$ is a point on $\mathbb{R}^+$). One can consider ϕ_γ to be defined *over the whole complex plane* by analytic continuation over $[0,\infty)\cup[\beta_\gamma,\overline{\beta_\gamma}]$ *consistently* for all γ.

Lemma 2.2.1 *As γ goes to 0, ϕ_γ converges to $\phi_0 := \phi_{\gamma=0}$ uniformly over compact subsets in $\mathbb{C}\setminus\{0,a\}$.*

Proof. It is simple to check that, as γ goes to zero, β_γ converges to β and b_γ converges to a. Therefore $y_\gamma(z)$ converges to $y_{\gamma=0}(z)$, by choosing the branch cut of y_γ at $[\beta_\gamma,\overline{\beta_\gamma}]$ that converges to β. This convergence is uniform away from the singularities of y_γ at 0 and a. $\square$

Lemma 2.2.2 *Let $I=\{\mathrm{i}t : -2\pi\leq t\leq 0\}$. The mapping $\phi_\gamma : \mathcal{S}_\gamma\setminus\{\beta_\gamma,\overline{\beta_\gamma}\}\to I\setminus\{0,-2\pi\mathrm{i}\}$ is invertible.*

Proof. We prove this for $a>1$ as the other case is similiar. We get $\phi_\gamma(\beta_\gamma)=0$ by definition. We have

$$\phi_\gamma(\overline{\beta_\gamma}) = \int_{\beta_\gamma}^{\overline{\beta_\gamma}} y_\gamma(s)\,\mathrm{d}s = \frac{1}{2}\oint y_\gamma(s)\,\mathrm{d}s,$$

where, in the first integral, the integration contour can be taken along $\mathcal{S}_\gamma$ and, in the second integral, the integration contour goes *around* $\mathcal{S}_\gamma$ counterclockwise while the branch cut of y_γ is placed at

[1]In [2] $b_\gamma = \alpha/\rho$, a typo.

$\mathcal{S}_\gamma$ (instead of at $[\beta_\gamma, \overline{\beta_\gamma}]$). The latter integration contour can be deformed into three clockwise contours around $\infty, 0$ and a, which leads to

$$\phi_\gamma(\overline{\beta_\gamma}) = -\frac{2\pi \mathrm{i}}{2}\left(\operatorname*{Res}_{z=\infty} y_\gamma(z) + \operatorname*{Res}_{z=0} y_\gamma(z) + \operatorname*{Res}_{z=a} y_\gamma(z)\right).$$

By Lemma 2.19 in [2], we have $\mathrm{Res}_{z=\infty} y_\gamma(z) = 1, \mathrm{Res}_{z=0} y_\gamma(z) = 1+\gamma$, and $\mathrm{Res}_{z=a} y_\gamma(z) = -\gamma$ and, therefore, we have $\phi_\gamma(\overline{\beta_\gamma}) = -2\pi\mathrm{i}$. Since ϕ_γ is continuous on S_γ (here we again place the branch cut of y_γ at $[\beta_\gamma, \overline{\beta_\gamma}]$) we have $I \subset \phi_\gamma(\mathcal{S}_\gamma)$. Since ϕ_γ has no critical point in $\mathcal{S}_\gamma$ except at the endpoints, ϕ_γ is 1-to-1 and $I = \phi_\gamma(\mathcal{S}_\gamma)$. □

Lemma 2.2.3 *Let $\{K_j \subset \mathbb{C}\}_{j=1}^\infty$ be a bounded sequence of compact sets such that K_∞, the set of limit points of $\{K_j\}_{j=1}^\infty$, is also compact. If K_j are connected, $b_j \in K_j$ and $\lim_{j\to\infty} b_j = b_\infty$, then $b_\infty \in K_\infty$.*

Proof. If not, there exist open sets O_1 and O_2 such that K_∞ is the disjoint union of $K_\infty \bigcap O_1$ and $K_\infty \bigcap O_2$. Since K_∞ is compact and since both $K_\infty \bigcap O_1$ and $K_\infty \bigcap O_2$ are closed in the relative topology of K_∞, both $K_\infty \bigcap O_1$ and $K_\infty \bigcap O_2$ are compact and, therefore, there are disjoint open neighborhoods of the two disjoint compact sets (a property of a Hausdorff space). Without loss of generality, we can call the disjoint neighborhoods by O_1 and O_2. Suppose $b_\infty \in O_2$. For j large enough we have $K_j \subset O_1 \bigcup O_2$ and $b_j \in O_2$ and, therefore, $K_j \subset O_2$ because K_j is connected. This is a contradiction. □

Proof of Theorem 2.1.1. Assume $\mathcal{S}_\gamma$ does not converge to $\mathcal{S}$ in Hausdorff metric. Then there exist a sequence $\{p_j\} \subset \mathcal{S}$ and $\{\gamma_j\} \to 0$ such that $dist(p_j, \mathcal{S}_{\gamma_j}) > 2\epsilon$ for some $\epsilon > 0$. Taking a limit point $z \in \mathcal{S}$ of $\{p_j\}$ and choosing a subsequence if necessary we can assume $dist(z, \mathcal{S}_{\gamma_j}) > \epsilon$ for all j's. Such z cannot be $\beta \in \mathcal{S}$ because $\{\beta_{\gamma_j} \in \mathcal{S}_{\gamma_j}\}$ converges to β as j goes to ∞. Since $\phi_{\gamma_j} : \mathcal{S}_{\gamma_j} \setminus \{\beta_{\gamma_j}, \overline{\beta_{\gamma_j}}\} \to I \setminus \{0, -2\pi\mathrm{i}\}$ is invertible by Lemma 2.2.2, we can define

$$z_j := \phi_{\gamma_j}^{-1} \circ \phi_0(z) \in \mathcal{S}_{\gamma_j}.$$

Let z_∞ be a limit point of $\{z_j\}$, then $z_\infty \notin \{0, a\}$ because $\mathcal{S}_{\gamma_j}$ is uniformly away from 0 and a for sufficiently small γ_j. We also have $z_\infty \neq \beta$ (and similarly, $z_\infty \neq \overline{\beta}$) because, if not, $|z_j - \beta_{\gamma_j}|$ would go to zero while $|\phi_{\gamma_j}(z_j) - \phi_{\gamma_j}(\beta_{\gamma_j})| = |\phi_0(z)| > 0$.

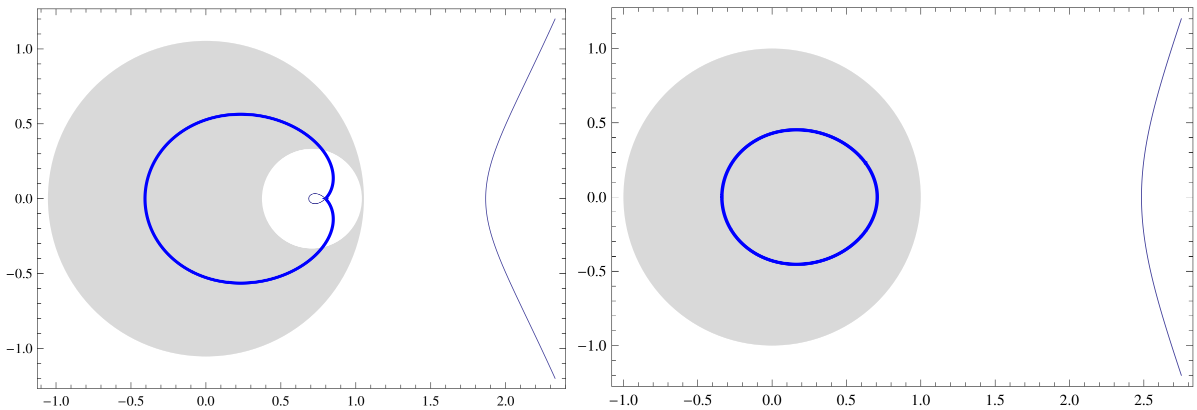

Figure 6.: Illustration of the convergence, $\mathcal{S}_\gamma \to \mathcal{S}$ and $K_\gamma \to K$, when $a = 1/\sqrt{2}$. For $\gamma = 1/9$ (left), $\mathcal{S}_\gamma$ is drawn with thick line and the rest of the set $\{z : \operatorname{Re}\phi_\gamma(z) = 0\}$ is drawn with the thin line; K is the shaded region. Same for $\gamma = 0$ (right).

Since $(\operatorname{clos}\{z_j\}) \cap \{0, a\} = \emptyset$, Lemma 2.2.1 says that

$$|\phi_0(z) - \phi_0(z_j)| = |\phi_{\gamma_j}(z_j) - \phi_0(z_j)| \overset{j\to\infty}{\longrightarrow} 0.$$

Since a subsequence of $\{\phi_0(z_j)\}$ converges to $\phi_0(z_\infty)$ by the continuity of ϕ_0, we have

$$\phi_0(z) = \phi_0(z_\infty). \tag{2.13}$$

Let $\mathcal{S}_\infty$ be the set of limit points of $\{\mathcal{S}_{\gamma_j}\}$. By Lemma 2.2.3, $\beta \in \mathcal{S}_\infty$. Since $\mathcal{S}$ is the only component of $\phi_0^{-1}(I)$ containing β, we have $\mathcal{S}_\infty \subset \mathcal{S}$. From (2.13) and $z_\infty \in \mathcal{S} \setminus \{\beta, \overline{\beta}\}$, we get $z = z_\infty$ by Lemma 2.2.2. This is a contradiction because z_∞ is a limit point of $\{\mathcal{S}_{\gamma_j}\}$ and, therefore, $dist(z, z_\infty) \geq \epsilon$. This concludes the proof of $\mathcal{S}_\gamma \to \mathcal{S}$.

For $a < 1$, the convergence of K_γ to $\operatorname{clos}\mathbb{D}$ follows from (2.12).

For $a \geq 1$, we need to show that $\partial K_\gamma = f_\gamma(\partial\mathbb{D})$ converges to $\partial\mathbb{D}$. Recall that, as γ goes to zero, α goes to $1/a$, κ goes to zero and ρ goes to 1. It follows that $\lim_{\gamma\to 0} f_\gamma(v) = v$, which means $K_\gamma \to \operatorname{clos}\mathbb{D}$.

For all a, the convergence of μ_γ to μ follows from the facts $\mathcal{S}_\gamma \to \mathcal{S}$ and $\lim_{\gamma\to 0} |y_\gamma(z)| = 2\pi\rho(z)$, where $\rho(z)$ is defined in (2.4).

2.3 Matrix Riemann-Hilbert Problem

The following fact is from [2]: Let Γ be a simple closed curve enclosing the line segment $[0,a] \subset \mathbb{C}$ and oriented counterclockwise. Let the analytic function $\omega_{n,N}$ on $\mathbb{C} \setminus [0,a]$ be defined by

$$\omega_{n,N}(z) := \left(\frac{z-a}{z}\right)^c \frac{\mathrm{e}^{-Naz}}{z^n},$$

where we choose the principal branch, i.e. $\left(\frac{z-a}{z}\right)^c$ goes to 1 as $z \to \infty$. Then the Riemann-Hilbert problem,

$$\begin{cases} Y(z) \text{ is holomorphic in } \mathbb{C} \setminus \Gamma, \\ Y_+(z) = Y_-(z) \begin{bmatrix} 1 & \omega_{n,N}(z) \\ 0 & 1 \end{bmatrix}, & z \in \Gamma, \\ Y(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right) \begin{bmatrix} z^n & 0 \\ 0 & z^{-n} \end{bmatrix}, & z \to \infty, \end{cases}$$

has the unique solution given by

$$Y(z) = \begin{bmatrix} P_n(z) & \dfrac{1}{2\pi \mathrm{i}} \displaystyle\int_\Gamma \frac{P_n(w)\omega_{n,N}(w)}{w-z} \mathrm{d}w \\ Q_{n-1}(z) & \dfrac{1}{2\pi \mathrm{i}} \displaystyle\int_\Gamma \frac{Q_{n-1}(w)\omega_{n,N}(w)}{w-z} \mathrm{d}w \end{bmatrix},$$

where $Q_{n-1}(z)$ is the unique polynomial of degree $n-1$ such that

$$\frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{Q_{n-1}(w)\omega_{n,N}(w)}{w-z} \mathrm{d}w = \frac{1}{z^n}\left(1 + \mathcal{O}\left(\frac{1}{z}\right)\right).$$

Lemma 2.3.1 *For $a<1$, there exists a neighborhood V of $\overline{\operatorname{Int}\mathcal{S}}$ such that $\operatorname{Re}\phi(z) < 0$ on $V \setminus \mathcal{S}$ and the boundary of V is a smooth Jordan curve. For $a \geq 1$, there exists a domain V such that it contains $\overline{\operatorname{Int}\mathcal{S}} \setminus \{\beta\}$ and its boundary, ∂V, is a smooth Jordan curve that intersects β. Also $\mathcal{S}$ is smooth except at β, where it makes a corner with the inner angle $\pi/2$ (i.e. towards $\operatorname{Int}\mathcal{S}$). Lastly, $\operatorname{Re}\phi > 0$ on $(\beta, a]$.*

Proof. From the definition (2.5) of ϕ, $\operatorname{Re}\phi$ is a harmonic function away from $\mathcal{S}$ and the origin. Since $\operatorname{Re}\phi(z)$ diverges to $-\infty$ as z goes to 0, $\operatorname{Re}\phi(z)$ has to be negative everywhere in $\operatorname{Int}\mathcal{S}$ –

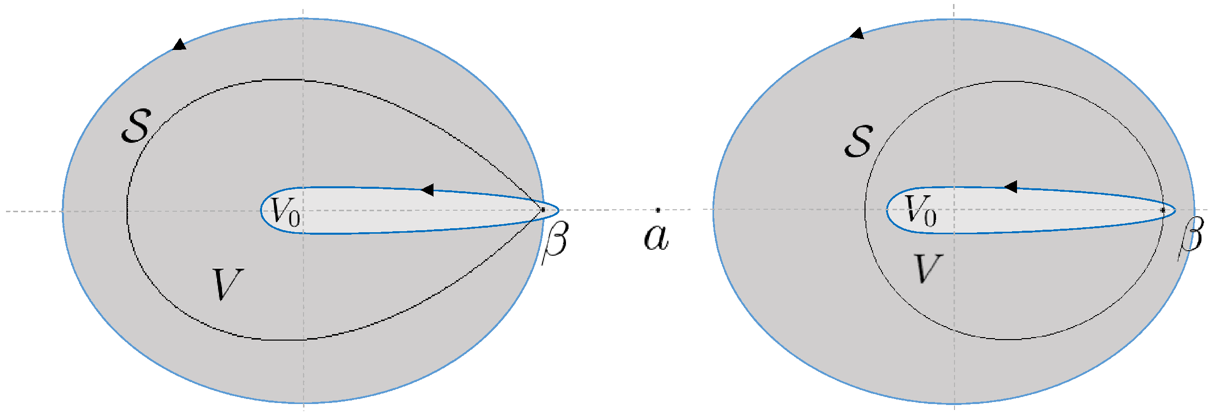

Figure 7.: V and V_0 for $a > 1$ (left) and $a < 1$ (right), $\mathcal{S}$ is the black curve, V is the interior of the contour enclosing the shaded region, V_0 is the interior of the contour enclosing the non–shaded region. These domains are used to define the domain U at (2.14).

otherwise $\operatorname{Re}\phi(z)$ has a local maximum in $\operatorname{Int}\mathcal{S}$, which is impossible. For $a < 1$, since the only critical point, $1/a$, of ϕ is away from $\mathcal{S}$ and since $\operatorname{Re}\phi_A$ is harmonic in a neighborhood of $\mathcal{S}$, $\operatorname{Re}\phi$ is negative in the vicinity of $\mathcal{S}$. For $a \geq 1$, since β is the only critical point of ϕ_A, the claim in the lemma about the local shape of $\mathcal{S}$ near β and about ∂V being intersecting β follows by the local analysis of the harmonic function $\operatorname{Re}\phi_A(z)$. (By (2.6), we have $\phi_A(z) \sim \frac{a^2}{2}(z-\beta)^2$. Moreover, by (2.5), we choose different sign of $\phi_A(z)$ to $\phi(z)$ depending z is inside or outside of $\mathcal{S}$.) Specifically, $\operatorname{Re}\phi_A(z)$ is positive along the real axis on $(0,\infty)\setminus\{\beta\}$, and is negative near β in the vertical direction (i.e. imaginary direction) from β. $\square$

Using V from the above lemma, we define the domain U as

$$U = V \setminus \overline{V_0}. \tag{2.14}$$

Here V_0 is a small open neighborhood of $[0,\beta]$ such that its boundary, ∂V_0, is a smooth Jordan curve that is arbitrarily close to $[0,\beta]$, see Figure 7. The region U is simply connected (when $a \geq 1$) or doubly connected (when $a < 1$) open neighborhood of $\mathcal{S}\setminus\overline{V_0}$, disjoint from $[0,a]$ and with a (piecewise) smooth boundary. We assign the counterclockwise orientation on $\partial U \cap \operatorname{Ext}\mathcal{S}$ with respect to the domain U and the counterclockwise orientation on $\partial U \cap \operatorname{Int}\mathcal{S}$ with respect to V_0.

From now on we let Γ exactly match $\mathcal{S}$ inside U and away from a small neighborhood of β. When $a > 1$, a part of the contour Γ goes outside U around the line segment $[\beta,a]$, see Figure 8. Near β the reader should not be concerned too much about the exact arrangement of Γ and U as it

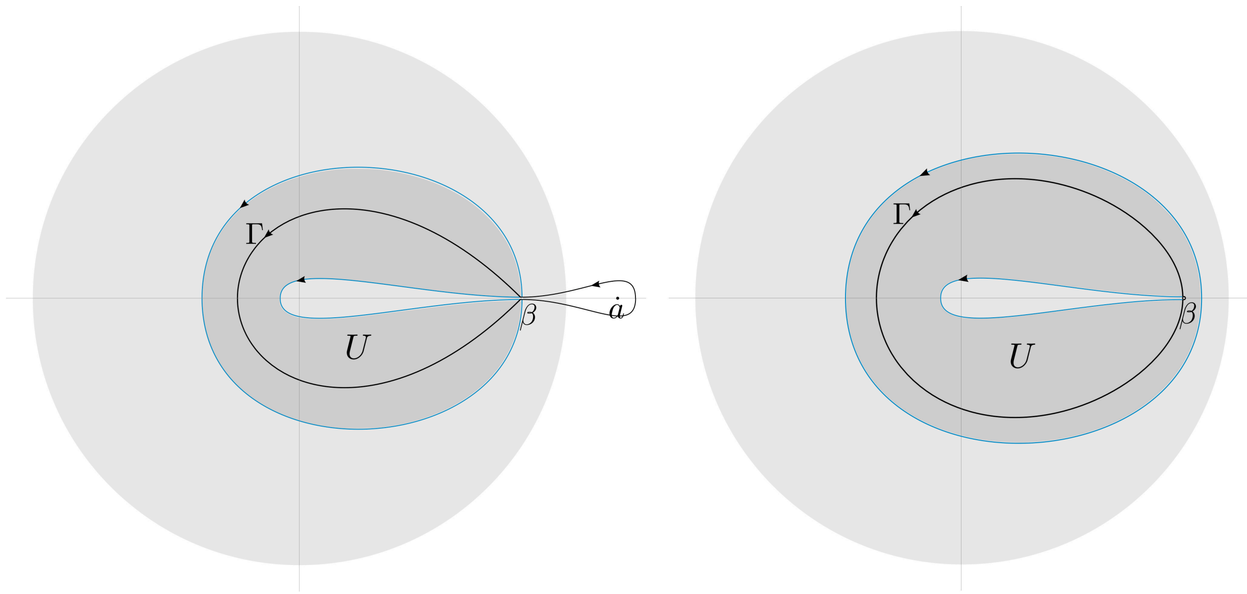

Figure 8.: Contours for the Riemann-Hilbert problem of Φ when $a > 1$ (left) and $a < 1$ (right). Γ is the black curves and U is the shaded region bounded by the blue curves.

will become clear when we define the local parametrix, which is a series of transformations to the Riemann-Hilbert problem.

We define the complex logarithmic potential of μ in (2.4) by

$$g(z) = \int \log(z - w)\, \mathrm{d}\mu(w),$$

where the specific branch of the log is chosen below. As a function of z, this equals $\log z$ (modulo $2\pi\mathrm{i}$) when $z \in \operatorname{Ext}\mathcal{S}$ by (2.2) and Theorem 2.1.1, and has continuous real part, since the jump of g on $\mathcal{S}$ is purely imaginary. These properties and (2.3) determine the explicit expression of this function as follows,

$$g(z) = \begin{cases} \log z, & z \in \overline{\operatorname{Ext}\mathcal{S}}, \\ az + \log\beta - a\beta, & z \in \operatorname{Int}\mathcal{S}. \end{cases}$$

From the g-function above, we can write

$$\phi(z) = az + \log z - 2g(z) + \ell, \quad \ell = \log\beta - a\beta,$$

so that $\operatorname{Re}\phi(z) = 0$ when $z \in \mathcal{S}$.

Following the standard nonlinear steepest descent method [10, 11] applied to the matrix Riemann-

Hilbert problem for Y, we define Z, as the final object after the multiple transforms of Y, by

$$Z(z) = \mathrm{e}^{\frac{-N\ell}{2}\sigma_3} Y(z)\, \mathrm{e}^{-Ng(z)\sigma_3} \mathrm{e}^{\frac{N\ell}{2}\sigma_3} \begin{bmatrix} 1 & 0 \\ \star \left(\dfrac{z}{z-a}\right)^c \mathrm{e}^{N\phi(z)} & 1 \end{bmatrix}, \tag{2.15}$$

where

$$\star = \begin{cases} 1, & \text{when } z \in U \cap \operatorname{Ext}\Gamma, \\ -1, & \text{when } z \in U \cap \operatorname{Int}\Gamma, \\ 0, & \text{when } z \notin U. \end{cases}$$

Then, Z solves the following Riemann-Hilbert problem:

$$\begin{cases} Z_+(z) = Z_-(z) \begin{bmatrix} 1 & 0 \\ \left(\frac{z}{z-a}\right)^c \mathrm{e}^{N\phi(z)} & 1 \end{bmatrix}, & z \in \partial U, \\ Z_+(z) = Z_-(z) \begin{bmatrix} 0 & \left(\frac{z-a}{z}\right)^c \\ -\left(\frac{z}{z-a}\right)^c & 0 \end{bmatrix}, & z \in \Gamma \cap U, \\ Z_+(z) = Z_-(z) \begin{bmatrix} 1 & \left(\frac{z-a}{z}\right)^c \mathrm{e}^{-N\phi(z)} \\ 0 & 1 \end{bmatrix}, & z \in \Gamma \setminus U. \\ Z(z) = I + \mathcal{O}(z^{-1}), & z \to \infty. \end{cases} \tag{2.16}$$

We define

$$\Phi(z) = \begin{cases} \begin{bmatrix} \left(\dfrac{z}{z-\beta}\right)^c & 0 \\ 0 & \left(\dfrac{z-\beta}{z}\right)^c \end{bmatrix}, & z \in \operatorname{Ext}\Gamma, \\ \begin{bmatrix} 0 & \left(\dfrac{z-a}{z-\beta}\right)^c \\ -\left(\dfrac{z-\beta}{z-a}\right)^c & 0 \end{bmatrix}, & z \in \operatorname{Int}\Gamma, \end{cases}$$

that satisfies the Riemann-Hilbert problem,

$$\begin{cases} \Phi_+(z) = \Phi_-(z) \begin{bmatrix} 0 & \left(\dfrac{z-a}{z}\right)^c \\ -\left(\dfrac{z}{z-a}\right)^c & 0 \end{bmatrix}, & z \in \mathcal{S}, \\ \Phi(z) = I + \mathcal{O}\left(\dfrac{1}{z}\right), & z \to \infty. \end{cases}$$

Note that, when $a \le 1$ and $z \in \operatorname{Int}\mathcal{S}$ we have $\Phi(z) = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$. Also note that Φ is not the only solution to the above Riemann-Hilbert problem – for any rational matrix function $\mathcal{R}(z)$ with a pole at β such that $\mathcal{R}(\infty) = I$, $\mathcal{R}(z)\Phi(z)$ is a solution. We will use this fact in the next section.

2.4 $a > 1$: when c near 0

From the definition of ϕ_A in (2.5), we obtain

$$\phi_A(z) = \frac{a^2}{2}(z-\beta)^2\left(1 + \mathcal{O}(z-\beta)\right).$$

Let D_β be a disk centered at β such that there exists a univalent map $\zeta : D_\beta \to \mathbb{C}$ as defined in (2.6). Under the mapping ζ the contour $\mathcal{S}$ maps into $[0, \mathrm{e}^{3\pi\mathrm{i}/4}t] \cup [0, \mathrm{e}^{-3\pi\mathrm{i}/4}t]_{t\in[0,\infty)}$.

In this section we intend to find $\mathcal{P} : D_\beta \to \mathbb{C}^{2\times 2}$ such that

$$Z^\infty(z) = \Phi(z)\left(\frac{z-a}{z}\right)^{\frac{c}{2}\sigma_3}\mathcal{P}(z)\left(\frac{z-a}{z}\right)^{-\frac{c}{2}\sigma_3}, \quad z \in D_\beta \tag{2.17}$$

satisfies the jump condition of Z at (2.16), i.e., we require $\mathcal{P}$ to satisfy in D_β:

$$\begin{cases} \mathcal{P}_+(z) = \mathcal{P}_-(z)\begin{bmatrix} 1 & \mathrm{e}^{-\zeta(z)^2/2} \\ 0 & 1 \end{bmatrix}, & z \in \Gamma \setminus U, \\ \mathcal{P}_+(z) = \mathcal{P}_-(z)\begin{bmatrix} 1 & 0 \\ \mathrm{e}^{\zeta(z)^2/2} & 1 \end{bmatrix}, & z \in \partial U \cap \operatorname{Ext}\Gamma, \\ \mathcal{P}_+(z) = \mathcal{P}_-(z)\begin{bmatrix} 1 & 0 \\ \mathrm{e}^{-\zeta(z)^2/2} & 1 \end{bmatrix}, & z \in \partial U \cap \operatorname{Int}\Gamma, \\ \mathcal{P}_+(z) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\mathcal{P}_-(z)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, & z \in \Gamma \cap U, \\ \mathcal{P}_+(z) = \mathrm{e}^{-c\pi\mathrm{i}\sigma_3}\mathcal{P}_-(z)\mathrm{e}^{c\pi\mathrm{i}\sigma_3}, & z \in \mathbb{R}, \end{cases} \tag{2.18}$$

and the boundary condition, $\mathcal{P}(z) \sim I$ on ∂D_β. The fourth equation of (2.18) comes from Φ in (2.17) and the last equation comes from the (conjugating) factors $\left((z-a)/z\right)^{\pm(c/2)\sigma_3}$ in (2.17). The jump contours $\Gamma \setminus U$ and $\partial U \cap \operatorname{Int}\Gamma$ can be pushed arbitrarily close to the real axis, so that the jump contours of $\mathcal{P}$ consists of $\mathbb{R}$, $i\mathbb{R}$ and $\{t\,\mathrm{e}^{\pm\mathrm{i}3\pi/4}\}_{0<t<\infty}$. See Figure 9 for the illustration of the jump contours in D_β.

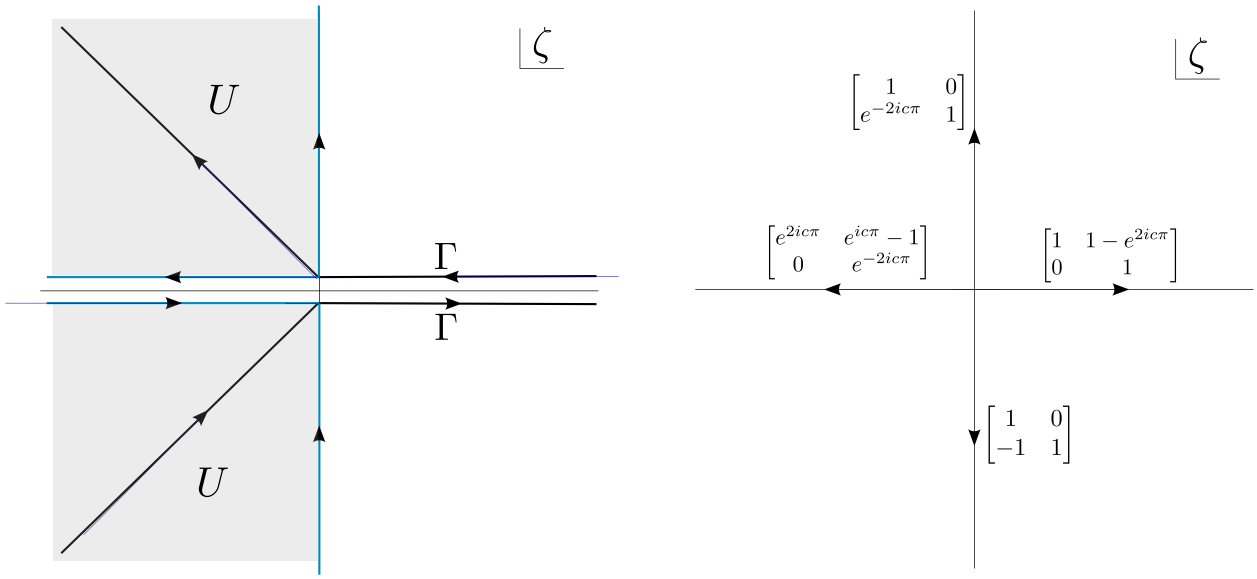

Figure 9.: Jump contours of $\mathcal{P}$ (2.18) in D_β (left) and the jump matrices of W (right)

We want to transform $\mathcal{P}$ into a new matrix function W that has only *constant jump matrices from the right*. Such transform may be given by

$$W(z) := \zeta(z)^{-c\sigma_3} S \cdot \mathcal{P}(z) \cdot T(\zeta(z))^{-1} S^{-1}, \tag{2.19}$$

using a diagonal matrix T and a piecewise constant matrix S defined below:

$$T(\zeta) = \begin{cases} \exp\left(\dfrac{\zeta^2}{4}\sigma_3\right), & |\arg\zeta| < 3\pi/4, \\ \exp\left(-\dfrac{\zeta^2}{4}\sigma_3\right), & \text{otherwise}, \end{cases} \tag{2.20}$$

and

$$S = \begin{cases} I, & \operatorname{Im}\zeta < 0 \cap |\arg\zeta| < 3\pi/4, \\ \mathrm{e}^{c\pi\mathrm{i}\sigma_3}, & \operatorname{Im}\zeta > 0 \cap |\arg\zeta| < 3\pi/4, \\ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, & \operatorname{Im}\zeta < 0 \cap |\arg\zeta| \geq 3\pi/4, \\ \mathrm{e}^{c\pi\mathrm{i}\sigma_3}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, & \operatorname{Im}\zeta > 0 \cap |\arg\zeta| \geq 3\pi/4. \end{cases} \tag{2.21}$$

Here we choose S such that $S^{-1}\zeta(z)^{c\sigma_3}$ satisfies all the *left* jumps of $\mathcal{P}$, i.e.,

$$\left(S^{-1}\zeta^{c\sigma_3}\right)_+ = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \left(S^{-1}\zeta^{c\sigma_3}\right)_-, \qquad z \in \Gamma \cap U,$$

$$\left(S^{-1}\zeta^{c\sigma_3}\right)_+ = e^{-c\pi \mathrm{i}\sigma_3} \left(S^{-1}\zeta^{c\sigma_3}\right)_-, \qquad z \in \mathbb{R},$$

so that W has the jump matrices only from the *right*. Furthermore, the jump matrices of W are constant matrices because of the right multipliction of T^{-1} in (2.19). The jump on $\{t\,\mathrm{e}^{\pm \mathrm{i}3\pi/4}\}_{0<t<\infty}$ disappears after the right multiplication by S^{-1}. We summarize the jump matrices of W below,

$$W_+(z) = W_-(z) \begin{cases} \begin{bmatrix} 1 & 1-\mathrm{e}^{2\mathrm{i}c\pi} \\ 0 & 1 \end{bmatrix}, & \zeta(z) \in \mathbb{R}^+, \\ \begin{bmatrix} 1 & 0 \\ \mathrm{e}^{-2\mathrm{i}c\pi} & 1 \end{bmatrix}, & \zeta(z) \in \mathrm{i}\mathbb{R}^+, \\ \begin{bmatrix} \mathrm{e}^{2\mathrm{i}c\pi} & e^{2\mathrm{i}c\pi}-1 \\ 0 & \mathrm{e}^{-2\mathrm{i}c\pi} \end{bmatrix}, & \zeta(z) \in \mathbb{R}^-, \\ \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, & \zeta(z) \in \mathrm{i}\mathbb{R}^-. \end{cases} \tag{2.22}$$

The following fact can be checked by a direct calculation.

Lemma 2.4.1 *For $z \in D_\beta$ we have*

$$\Phi(z) \left(\frac{z-a}{z}\right)^{\frac{c}{2}\sigma_3} S^{-1}\zeta^{c\sigma_3} = \left(N^{c/2}\eta(z)\right)^{\sigma_3},$$

where $\eta : D_\beta \to \mathbb{C}$,

$$\eta(z) := \frac{\mathrm{e}^{-ic\pi/2}}{N^{c/2}} \left(\frac{a-z}{z}\right)^{\frac{c}{2}} \left(\frac{z\,\zeta(z)}{z-\beta}\right)^{c}$$

is a nonvanishing analytic function in D_β independent of N.

Using the parabolic cylinder function (2.7) we define $\mathcal{W} : \mathbb{C} \setminus (\mathbb{R} \cup \mathrm{i}\mathbb{R}) \to \mathbb{C}^{2\times 2}$ as

$$\mathcal{W}(\zeta) = \begin{cases} \begin{bmatrix} D_{-c}(\zeta) & \frac{\mathrm{i}\sqrt{2\pi}\mathrm{e}^{\frac{c\pi\mathrm{i}}{2}}}{\Gamma(c)} D_{-1+c}(\mathrm{i}\zeta) \\ -\frac{\Gamma(c+1)}{\sqrt{2\pi}\mathrm{e}^{c\pi\mathrm{i}}} D_{-1-c}(\zeta) & \mathrm{e}^{-\frac{c\pi\mathrm{i}}{2}} D_c(\mathrm{i}\zeta) \end{bmatrix}, & -\frac{\pi}{2} < \arg(\zeta) < 0, \\ \begin{bmatrix} D_{-c}(\zeta) & -\frac{\mathrm{i}\sqrt{2\pi}\mathrm{e}^{\frac{3c\pi\mathrm{i}}{2}}}{\Gamma(c)} D_{-1+c}(-\mathrm{i}\zeta) \\ -\frac{\Gamma(c+1)}{\sqrt{2\pi}\mathrm{e}^{c\pi\mathrm{i}}} D_{-1-c}(\zeta) & \mathrm{e}^{\frac{c\pi\mathrm{i}}{2}} D_c(-\mathrm{i}\zeta) \end{bmatrix}, & 0 < \arg(\zeta) < \frac{\pi}{2}, \\ \begin{bmatrix} \mathrm{e}^{-c\pi\mathrm{i}} D_{-c}(-\zeta) & -\frac{\mathrm{i}\sqrt{2\pi}\mathrm{e}^{\frac{3c\pi\mathrm{i}}{2}}}{\Gamma(c)} D_{-1+c}(-\mathrm{i}\zeta) \\ \frac{\Gamma(1+c)}{\sqrt{2\pi}\mathrm{e}^{2c\pi\mathrm{i}}} D_{-1-c}(-\zeta) & \mathrm{e}^{\frac{c\pi\mathrm{i}}{2}} D_c(-\mathrm{i}\zeta) \end{bmatrix}, & \frac{\pi}{2} < \arg(\zeta) < \pi, \\ \begin{bmatrix} \mathrm{e}^{c\pi\mathrm{i}} D_{-c}(-\zeta) & \frac{\mathrm{i}\sqrt{2\pi}\mathrm{e}^{\frac{c\pi\mathrm{i}}{2}}}{\Gamma(c)} D_{-1+c}(\mathrm{i}\zeta) \\ \frac{\Gamma(1+c)}{\sqrt{2\pi}} D_{-1-c}(-\zeta) & \mathrm{e}^{-\frac{c\pi\mathrm{i}}{2}} D_c(\mathrm{i}\zeta) \end{bmatrix}, & \pi < \arg(\zeta) < \frac{3\pi}{2}. \end{cases} \tag{2.23}$$

Lemma 2.4.2 *There exists the asymptotic expansion of $D_{-c}(\zeta)$ given by*

$$D_{-c}(\zeta) = \mathrm{e}^{-\frac{\zeta^2}{4}} \zeta^{-c} \left(\sum_{s=0}^{n-1} (-1)^s \frac{(c)_{2s}}{s!(2\zeta^2)^s} + \varepsilon_n(\zeta) \right), \quad |\arg \zeta| < \frac{\pi}{2}. \tag{2.24}$$

There exists a constant $C > 0$ independent of c so that

$$|\varepsilon_n(\zeta)| \le C \left| \frac{\left(\frac{c}{2}\right)_n \left(\frac{c+1}{2}\right)_n}{n!(\zeta^2)^n} \right|, \quad |\arg \zeta| < \frac{\pi}{2}.$$

Here, $(\cdot)_n$ is Pochhammer's Symbol defined by $(x)_n = \dfrac{\Gamma(x+n)}{\Gamma(x)}$.

Proof. By the identities (12.7.14) and (13.7.4) in [34] we can write

$$D_{-c}(\zeta) = 2^{-c/2} \mathrm{e}^{-\zeta^2/4} U\left(\frac{c}{2}, \frac{1}{2}, \frac{\zeta^2}{2} \right),$$

where U has the following asymptotic expansion as $|\zeta| \to \infty$.

$$U\left(\frac{c}{2}, \frac{1}{2}, \frac{\zeta^2}{2} \right) = \left(\frac{\zeta^2}{2} \right)^{-\frac{c}{2}} \sum_{s=0}^{n-1} \left(-\frac{\zeta^2}{2} \right)^{-s} \frac{\left(\frac{c}{2}\right)_s \left(\frac{c+1}{2}\right)_s}{s!(2\zeta^2)^s} + \widehat{\varepsilon_n}\left(\frac{\zeta^2}{2} \right), \quad |\arg \zeta| < \frac{\pi}{2}.$$

The error term $\widehat{\varepsilon_n}$ is bounded by

$$\left| \widehat{\varepsilon_n}\left(\frac{\zeta^2}{2} \right) \right| \le 2^{\frac{c}{2}+n+1} \alpha \left| \frac{(\frac{c}{2})_n (\frac{c+1}{2})_n}{n!(\zeta^2)^{n+\frac{c}{2}}} \right| \exp\left(\frac{4\alpha\rho}{|\zeta^2|} \right),$$

where

$$\alpha = \frac{1}{1-\sigma}, \quad \sigma = \left|\frac{1-2c}{\zeta^2}\right|, \quad \rho = \left|\frac{c^2-c+1}{4}\right| + \frac{\sigma(1+\frac{\sigma}{4})}{(1-\sigma)^2}.$$

We have

$$|\varepsilon_n(\zeta)| = 2^{-\frac{c}{2}}|\zeta|^c \left|\widehat{\varepsilon_n}\left(\frac{\zeta^2}{2}\right)\right| \leq C\left|\frac{(\frac{c}{2})_n(\frac{c+1}{2})_n}{n!(\zeta^2)^n}\right|.$$

where

$$C = \frac{2^{n+1}|\zeta^2|}{(|\zeta^2|-|1-2c|)} \exp\left(\left|\frac{c^2-c+1}{4(|\zeta^2|-|1-2c|)}\right| + \frac{|1-2c|(|\zeta^2|+\frac{|1-2c|}{4})}{(|\zeta^2|-|1-2c|)^3}\right).$$

For $\dfrac{|\zeta^2|}{|1-2c|}$ big enough, we have $C \leq 2^{n+2}$. $\square$

Though the lemma only concerns $|\arg\zeta| < \pi/2$, this turns out to cover every term that appears in $\mathcal{W}(\zeta)$ of (2.23) and leads to the following lemma.

Lemma 2.4.3 *$\mathcal{W}(\zeta(z))$ satisfies (2.22) and the asymptotic behavior*

$$\mathcal{F}(\zeta) := \mathcal{W}(\zeta)\,\zeta^{c\sigma_3}\mathrm{e}^{\frac{\zeta^2}{4}\sigma_3} = I + \frac{C_1}{\zeta} + \frac{C_2}{\zeta^2} + \mathcal{O}\left(\frac{1}{\zeta^3}\right) \tag{2.25}$$

as $|\zeta|$ goes to ∞, where

$$C_1 = \begin{bmatrix} 0 & \dfrac{\sqrt{2\pi}\mathrm{e}^{\mathrm{i}\pi c}}{\Gamma(c)} \\ -\dfrac{\Gamma(c+1)}{\sqrt{2\pi}\mathrm{e}^{\mathrm{i}\pi c}} & 0 \end{bmatrix} \quad \textit{and} \quad C_2 = \begin{bmatrix} -\dfrac{c(c+1)}{2} & 0 \\ 0 & \dfrac{c(c-1)}{2} \end{bmatrix}.$$

Moreover, as $c \to 0$ and $|\zeta| \to \infty$, we get

$$\mathcal{F}(\zeta)F_1(\zeta)^{-1} = I + \begin{bmatrix} \mathcal{O}\left(c\,\zeta^{-2}\right) & \mathcal{O}\left(c\,\zeta^{-3}\right) \\ \mathcal{O}\left(\zeta^{-1}\right) & \mathcal{O}\left(c\,\zeta^{-2}\right) \end{bmatrix}, \tag{2.26}$$

$$\mathcal{F}(\zeta)F_1(\zeta)^{-1}F_2(\zeta)^{-1} = I + \mathcal{O}\left(\zeta^{-3}\right), \tag{2.27}$$

where

$$F_1(\zeta) = I + \frac{1}{\zeta}\begin{bmatrix} 0 & \dfrac{\sqrt{2\pi}\mathrm{e}^{\mathrm{i}\pi c}}{\Gamma(c)} \\ 0 & 0 \end{bmatrix}, \tag{2.28}$$

$$F_2(\zeta) = I + \begin{bmatrix} -\dfrac{c(c+1)}{2}\dfrac{1}{\zeta^2} & \dfrac{\sqrt{2\pi}\mathrm{e}^{\mathrm{i}\pi c}c^2(c+1)^2}{4\Gamma(c+1)}\dfrac{1}{\zeta^3} \\ -\dfrac{\Gamma(c+1)}{\sqrt{2\pi}\mathrm{e}^{\mathrm{i}\pi c}}\dfrac{1}{\zeta} & \dfrac{c(c+1)}{2}\dfrac{1}{\zeta^2} \end{bmatrix}. \tag{2.29}$$

The error bound in (2.26) *is uniform over* $c \in [-1/2, 1/2]$ *as* ζ *tends to infinity, and the error bound in* (2.27) *is for a fixed* c.

Proof. The proof of that $\mathcal{W}$ satisfies (2.22) is straightforward if one uses the following identities [9, 38]:

$$\begin{aligned} D_{-c}(\zeta) &= \frac{\Gamma(1-c)}{\sqrt{2\pi}}\left[\mathrm{e}^{\frac{-c\pi\mathrm{i}}{2}}D_{c-1}(\mathrm{i}\zeta) + \mathrm{e}^{\frac{c\pi\mathrm{i}}{2}}D_{c-1}(-\mathrm{i}\zeta)\right], \\ D_{-c}(\zeta) &= \mathrm{e}^{-c\pi\mathrm{i}}D_{-c}(-\zeta) + \frac{\sqrt{2\pi}}{\Gamma(c)}\mathrm{e}^{\frac{(1-c)\pi\mathrm{i}}{2}}D_{c-1}(-\mathrm{i}\zeta), \\ D_{-c}(\zeta) &= \mathrm{e}^{c\pi\mathrm{i}}D_{-c}(-\zeta) + \frac{\sqrt{2\pi}}{\Gamma(c)}\mathrm{e}^{\frac{(c-1)\pi\mathrm{i}}{2}}D_{c-1}(\mathrm{i}\zeta). \end{aligned}$$

The proof of the asymptotic behavior is based on Lemma 2.4.2 regarding the asymptotic behavior of the parabolic cylinder function. By Lemma 2.4.2, letting $n = 1$, we have

$$|\varepsilon_1(\zeta)| \le C\left|\frac{c(c+1)}{\zeta^2}\right|, \quad |\arg\zeta| < \frac{\pi}{2}.$$

This leads to $D_{-c}(\zeta) = \mathrm{e}^{-\zeta^2/4}\zeta^{-c}\left(1 + \mathcal{O}\left(c(c+1)/\zeta^2\right)\right)$. Similarly, we can obtain the asymptotic expression for $D_{-1+c}(\mathrm{i}\zeta)$, $D_{-1-c}(\zeta)$, and $D_c(\mathrm{i}\zeta)$ and we get

$$\mathcal{F}(\zeta) = F_1(\zeta) + \begin{bmatrix} \mathcal{O}\left(\dfrac{c(c+1)}{\zeta^2}\right) & \mathcal{O}\left(\dfrac{(c-1)(c-2)}{\zeta^3\Gamma(c)}\right) \\ \mathcal{O}\left(\dfrac{\Gamma(c+1)}{\zeta}\right) & \mathcal{O}\left(\dfrac{c(c-1)}{\zeta^2}\right) \end{bmatrix}.$$

This leads to (2.26) using $\Gamma(c) = c^{-1}(1 + \mathcal{O}(c))$. Similarly, the equations (2.27) and (2.25) follow from Lemma 2.4.2. □

Let H be a holomorphic matrix function on D_β with determinant 1. We define W by

$$W(z) = H(z)\mathcal{W}(\zeta(z)), \quad z \in D_\beta. \tag{2.30}$$

Combining (2.19), (2.25) and (2.30), the expression in (2.17) can be written as

$$
\begin{aligned}
&\Phi(z)\left(\tfrac{z-a}{z}\right)^{\frac{c}{2}\sigma_3}\mathcal{P}(z)\left(\tfrac{z-a}{z}\right)^{-\frac{c}{2}\sigma_3}\\
&=\Phi(z)\left(\frac{z-a}{z}\right)^{\frac{c}{2}\sigma_3}S^{-1}\zeta^{c\sigma_3}H(z)\mathcal{W}(z)\,ST(\zeta(z))\left(\frac{z-a}{z}\right)^{-\frac{c}{2}\sigma_3}\\
&=\Phi(z)\left(\frac{z-a}{z}\right)^{\frac{c}{2}\sigma_3}S^{-1}\zeta^{c\sigma_3}H(z)\,\mathcal{F}(\zeta(z))\,\zeta(z)^{-c\sigma_3}\mathrm{e}^{-\frac{\zeta(z)^2}{4}\sigma_3}ST(\zeta(z))\left(\frac{z-a}{z}\right)^{-\frac{c}{2}\sigma_3}.
\end{aligned}
$$

By (2.20), (2.21) and Lemma 2.4.1, we obtain

$$
\zeta^{-c\sigma_3}\mathrm{e}^{\frac{-\zeta^2}{4}\sigma_3}ST(\zeta(z))\left(\frac{z-a}{z}\right)^{-\frac{c}{2}\sigma_3}=\zeta^{-c\sigma_3}S\left(\frac{z-a}{z}\right)^{-\frac{c}{2}\sigma_3}=\left(N^{c/2}\eta(z)\right)^{-\sigma_3}\Phi(z).
$$

The above equations lead to the following Lemma.

Lemma 2.4.4 *When $z\in D_\beta$, we have*

$$
\Phi(z)\left(\frac{z-a}{z}\right)^{\frac{c}{2}\sigma_3}\mathcal{P}(z)\left(\frac{z-a}{z}\right)^{-\frac{c}{2}\sigma_3}=\left(N^{c/2}\eta(z)\right)^{\sigma_3}H(z)\,\mathcal{F}(\zeta(z))\left(N^{c/2}\eta(z)\right)^{-\sigma_3}\Phi(z). \tag{2.31}
$$

Theorem 2.4.5 *For $a>1$ and $-1/2\le c\le 1/2$, we get*

$$
P_N(z)=\begin{cases}
z^N\left(\dfrac{z}{z-\beta}\right)^c\left(1+\mathcal{O}\left(\dfrac{1}{N^{c+1/2}}\right)\right), & z\in\operatorname{Ext}\mathcal{S}\setminus(U\cup D_\beta),\\[2ex]
z^N\Bigg(\left(\dfrac{z}{z-\beta}\right)^c-\dfrac{\sqrt{2\pi}(a^2-1)^c}{N^{1/2-c}a\Gamma(c)}\dfrac{e^{N\phi_A(z)}}{(z-\beta)}\left(\dfrac{z-\beta}{z-a}\right)^c & \\
\qquad+\mathcal{O}\left(\dfrac{1}{N^{c+1/2}},\dfrac{\mathrm{e}^{N\phi_A}}{N^{c+1/2}}\right)\Bigg), & z\in U\setminus D_\beta,\\[2ex]
z^N\left(\left(\dfrac{z\zeta}{z-\beta}\right)^c\mathrm{e}^{\frac{\zeta^2(z)}{4}}D_{-c}(\zeta(z))+\mathcal{O}\left(\dfrac{1}{N^{1/2}},\dfrac{1}{N^{2c+1/2}}\right)\right), & z\in D_\beta.
\end{cases}
$$

The error bounds are uniform in $c\in[-1/2,1/2]$. The big $\mathcal{O}$ notation with multiple arguments is defined by $\mathcal{O}(A,B)=\mathcal{O}(A)+\mathcal{O}(B)$.

This theorem is similar to Theorem 2.1.2 except that the range of c is restricted to $[-1/2,1/2]$ and the error bounds are uniform in the range.

Proof. Using F_1 in (2.28) we can define a meromorphic matrix function with determinant 1 and a simple pole at β by

$$\mathcal{R}(z) = I + \frac{\sqrt{2\pi}\,(a^2-1)^c}{N^{1/2-c}a\,\Gamma(c)}\frac{1}{z-\beta}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} \tag{2.32}$$

such that we can set

$$H(z) = \left(N^{c/2}\eta(z)\right)^{-\sigma_3}\mathcal{R}(z)\left(N^{c/2}\eta(z)\right)^{\sigma_3}F_1(\zeta(z))^{-1}, \tag{2.33}$$

i.e., the above matrix has determinant 1 and is holomorphic at β.

Now we define the strong asymptotics of Z that we will denote by

$$Z^\infty(z) := \begin{cases} \mathcal{R}(z)\Phi(z), & z \notin D_\beta,\\ \Phi(z)\left(\dfrac{z-a}{z}\right)^{\frac{c}{2}\sigma_3}\mathcal{P}(z)\left(\dfrac{z-a}{z}\right)^{-\frac{c}{2}\sigma_3}, & z \in D_\beta, \end{cases} \tag{2.34}$$

where the second line is given in Lemma 2.4.4. We get

$$\begin{aligned} Z_+^\infty(z)\left(Z_-^\infty(z)\right)^{-1} &= \Phi(z)\left(\frac{z-a}{z}\right)^{\frac{c}{2}\sigma_3}\mathcal{P}(z)\left(\frac{z-a}{z}\right)^{-\frac{c}{2}\sigma_3}\Phi^{-1}(z)\mathcal{R}^{-1}(z)\\ &= \left(N^{c/2}\eta(z)\right)^{\sigma_3}H(z)\mathcal{F}(z)\left(N^{c/2}\eta(z)\right)^{-\sigma_3}\mathcal{R}^{-1}(z)\\ &= \left(N^{c/2}\eta(z)\right)^{\sigma_3}H(z)\widehat{\mathcal{F}}(\zeta)H^{-1}(z)\left(N^{c/2}\eta(z)\right)^{-\sigma_3}, \end{aligned} \tag{2.35}$$

where in the last line we set

$$\widehat{\mathcal{F}}(\zeta) = \mathcal{F}(\zeta)F_1(\zeta)^{-1}.$$

Defining the error matrix by

$$\mathcal{E}(z) := Z^\infty(z)\,Z^{-1}(z),$$

we have

$$\begin{aligned} \mathcal{E}_+(z)\mathcal{E}_-^{-1}(z) &= Z^\infty(z)_+\left(Z_-^\infty(z)\right)^{-1}\\ &= \left(N^{c/2}\eta(z)\right)^{\sigma_3}H(z)\widehat{\mathcal{F}}(\zeta)H^{-1}(z)\left(N^{c/2}\eta(z)\right)^{-\sigma_3}\\ &= I + \begin{bmatrix} \mathcal{O}\left(\dfrac{c}{N}\right) & \mathcal{O}\left(\dfrac{c}{N^{3/2-c}}\right)\\ \mathcal{O}\left(\dfrac{1}{N^{1/2+c}}\right) & \mathcal{O}\left(\dfrac{c}{N}\right)\end{bmatrix} = I + \mathcal{O}\left(\frac{1}{N^{1/2+c}}\right), \quad z \in \partial D_\beta, \end{aligned} \tag{2.36}$$

where in the last equality we used the asymptotic behavior (2.26) for $\widehat{\mathcal{F}}(\zeta) = \mathcal{F}(\zeta)F_1(\zeta)^{-1}$, and the asymptotic behavior of H given below:

$$H=\begin{bmatrix}1 & h(z)\\ 0 & 1\end{bmatrix},\qquad h(z)=\frac{\sqrt{2\pi}\,(a^2-1)^c}{\sqrt{N}\eta^2(z)a\,\Gamma(c)}\frac{1}{z-\beta}-\frac{1}{\zeta(z)}\frac{\sqrt{2\pi}\mathrm{e}^{\mathrm{i}\pi c}}{\Gamma(c)}=\mathcal{O}\left(\frac{c}{\sqrt{N}}\right). \tag{2.37}$$

One can check that the jump of $\mathcal{E}$ is exponentially small in N away from ∂D_β using Lemma 2.3.1 and (2.16). By the Small Norm Theorem (Theorem 1.5.1) we obtain $\mathcal{E}(z)=I+\mathcal{O}\left(1/N^{c+1/2}\right)$ and, therefore, $Z^\infty(z)Z^{-1}(z)=I+\mathcal{O}\left(1/N^{c+1/2}\right)$. Note that the error bound is uniform over $c\in[-1/2,1/2]$.

Using (2.15) we have (see (2.15) for the definition of $(\star)$):

$$\begin{aligned}
Y(z)&=\mathrm{e}^{\frac{N\ell}{2}\sigma_3}Z(z)\begin{bmatrix}1 & 0\\ (-\star)\left(\dfrac{z}{z-a}\right)^c\mathrm{e}^{N\phi(z)} & 1\end{bmatrix}\mathrm{e}^{\frac{-N\ell}{2}\sigma_3}\mathrm{e}^{Ng(z)\sigma_3}\\
&=\mathrm{e}^{\frac{N\ell}{2}\sigma_3}\left(I+\mathcal{O}\left(\frac{1}{N^{1/2+c}}\right)\right)Z^\infty(z)\begin{bmatrix}1 & 0\\ (-\star)\left(\dfrac{z}{z-a}\right)^c\mathrm{e}^{N\phi(z)} & 1\end{bmatrix}\mathrm{e}^{\frac{-N\ell}{2}\sigma_3}\mathrm{e}^{Ng(z)\sigma_3}.
\end{aligned}$$

Using (2.34), we calculate the strong asymptotics for $z\in(\operatorname{Ext}\mathcal{S}\cap U)\setminus D_\beta$ as an example.

$$\begin{aligned}
P_N(z)=[Y(z)]_{11}&=\left[\left(I+\mathcal{O}\left(\frac{1}{N^{1/2+c}}\right)\right)Z^\infty(z)\begin{bmatrix}1 & 0\\ (-\star)\left(\frac{z}{z-a}\right)^c\mathrm{e}^{N\phi(z)} & 1\end{bmatrix}\right]_{11}\mathrm{e}^{Ng(z)}\\
&=\left[\left(I+\mathcal{O}\left(\frac{1}{N^{1/2+c}}\right)\right)\mathcal{R}(z)\Phi(z)\begin{bmatrix}1 & 0\\ -\left(\frac{z}{z-a}\right)^c\mathrm{e}^{N\phi(z)} & 1\end{bmatrix}\right]_{11}\mathrm{e}^{Ng(z)}\\
&=\left[\left(I+\mathcal{O}\left(\frac{1}{N^{1/2+c}}\right)\right)\begin{bmatrix}1 & \frac{\sqrt{2\pi}\left(a^2-1\right)^c}{N^{1/2-c}a\,\Gamma(c)}\frac{1}{z-\beta}\\ 0 & 1\end{bmatrix}\begin{bmatrix}\left(\frac{z}{z-\beta}\right)^c & 0\\ 0 & \left(\frac{z-\beta}{z}\right)^c\end{bmatrix}\begin{bmatrix}1 & 0\\ -\left(\frac{z}{z-a}\right)^c\mathrm{e}^{N\phi(z)} & 1\end{bmatrix}\right]_{11}z^N\\
&=\left[\left(1+\mathcal{O}\left(\frac{1}{N^{1/2+c}}\right)\right)\left(\left(\frac{z}{z-\beta}\right)^c-\left(\frac{z-\beta}{z-a}\right)^c\frac{\sqrt{2\pi}(a^2-1)^c}{a\Gamma(c)N^{1/2-c}(z-\beta)}\mathrm{e}^{N\phi(z)}\right)\right.\\
&\qquad\left.-\,\mathcal{O}\left(\frac{1}{N^{1/2+c}}\right)\left(\frac{z-\beta}{z-a}\right)^c\mathrm{e}^{N\phi(z)}\right]z^N\\
&=z^N\left(\left(\frac{z}{z-\beta}\right)^c-\left(\frac{z-\beta}{z-a}\right)^c\frac{\sqrt{2\pi}(a^2-1)^c}{a\Gamma(c)N^{1/2-c}(z-\beta)}\mathrm{e}^{N\phi(z)}+\mathcal{O}\left(\frac{1}{N^{1/2+c}}\right)\right).
\end{aligned} \tag{2.38}$$

A similar calculation will give the following for $z \in (\operatorname{Int}\mathcal{S} \cap U) \setminus D_\beta$:

$$P_N(z) = \mathrm{e}^{Ng(z)}\Big(\Big(\frac{z}{z-\beta}\Big)^c \mathrm{e}^{N\phi(z)} - \Big(\frac{z-\beta}{z-a}\Big)^c \frac{\sqrt{2\pi}(a^2-1)^c}{a\Gamma(c)N^{1/2-c}(z-\beta)} + \mathcal{O}\Big(\frac{1}{N^{1/2+c}}\Big)\Big).$$

For $z \in (\operatorname{Ext}\mathcal{S} \setminus U) \cap D_\beta$ we calculate the strong asymptotics using (2.34), (2.25) and Lemma 2.4.4 to represent $\mathcal{P}$ in terms of $\mathcal{W}$ in (2.23) and $H(z)$ in (2.33):

$$\begin{aligned}
P_N(z) = [Y(z)]_{11} &= \left[\Big(I + \mathcal{O}\Big(\frac{1}{N^{1/2+c}}\Big)\Big) Z^\infty \begin{bmatrix} 1 & 0 \\ -\star\,\left(\frac{z}{z-a}\right)^c \mathrm{e}^{N\phi(z)} & 1 \end{bmatrix}\right]_{11} \mathrm{e}^{Ng(z)} \\
&= \left[\Big(I + \mathcal{O}\Big(\frac{1}{N^{1/2+c}}\Big)\Big) \Phi(z) \Big(\frac{z-a}{z}\Big)^{\frac{c}{2}\sigma_3} \mathcal{P}(z) \Big(\frac{z-a}{z}\Big)^{-\frac{c}{2}\sigma_3}\right]_{11} z^N \\
&= \left[\Big(I + \mathcal{O}\Big(\frac{1}{N^{1/2+c}}\Big)\Big) \Big(N^{c/2}\eta(z)\Big)^{\sigma_3} H(z)\,\mathcal{F}(\zeta(z)) \Big(N^{c/2}\eta(z)\Big)^{-\sigma_3} \Phi(z)\right]_{11} z^N \\
&= \Bigg[\Big(I + \mathcal{O}\Big(\frac{1}{N^{1/2+c}}\Big)\Big) \Big(N^{c/2}\eta(z)\Big)^{\sigma_3} \begin{bmatrix} 1 & h(z) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} D_{-c}(\zeta) & \frac{\mathrm{i}\sqrt{2\pi}\mathrm{e}^{\frac{c\pi\mathrm{i}}{2}}}{\Gamma(c)} D_{-1+c}(\mathrm{i}\zeta) \\ -\frac{\Gamma(c+1)}{\sqrt{2\pi}\mathrm{e}^{c\pi\mathrm{i}}} D_{-1-c}(\zeta) & \mathrm{e}^{-\frac{c\pi\mathrm{i}}{2}} D_c(\mathrm{i}\zeta) \end{bmatrix} \\
&\qquad \cdot \zeta^{c\sigma_3} \mathrm{e}^{\frac{\zeta^2}{4}\sigma_3} \Big(N^{c/2}\eta(z)\Big)^{-\sigma_3} \begin{bmatrix} \left(\frac{z}{z-\beta}\right)^c & 0 \\ 0 & \left(\frac{z-\beta}{z}\right)^c \end{bmatrix}\Bigg]_{11} z^N \\
&= \Bigg[\Big(\frac{z}{z-\beta}\Big)^c \zeta(z)^c \mathrm{e}^{\frac{\zeta(z)^2}{4}} \Big(D_{-c}(\zeta) - h(z)\frac{\Gamma(c+1)}{\sqrt{2\pi}\mathrm{e}^{c\pi\mathrm{i}}} D_{-1-c}(\zeta)\Big)\Big(1 + \mathcal{O}\Big(\frac{1}{N^{1/2+c}}\Big)\Big) \\
&\qquad + \mathcal{O}\Big(\frac{1}{N^{1/2+2c}}\Big)\Bigg] z^N \\
&= \Big[\Big(\frac{z}{z-\beta}\Big)^c \zeta(z)^c \mathrm{e}^{\frac{\zeta(z)^2}{4}} D_{-c}(\zeta) + \mathcal{O}\Big(\frac{1}{\sqrt{N}}, \frac{1}{N^{1/2+2c}}\Big)\Big] z^N.
\end{aligned} \tag{2.39}$$

We used (2.37) in the last equality. Note that the above error bounds are uniform for $c \in [-1/2, 1/2]$.

We skip the calculations for other regins since they are similar. □

2.5 $a > 1$: Proof of Theorem 2.1.2

The proof of Theorem 2.1.2 is identical to the above proof of Theorem 2.4.5 except that we use different $\mathcal{R}$ and H (hence different $\mathcal{P}$). The construction of $\mathcal{R}$ and H will be more involved and will be useful for the next case of $a < 1$ and, therefore, we will describe the construction in a more general setting.

Here we describe how to construct $\mathcal{R}$ and $\mathcal{P}$ inductively so that the jump, $Z^\infty_+(Z^\infty_-)^{-1}$, of Z^∞ is close to the identity up to $\mathcal{O}(N^{-L})$ for any given $L > 0$. The inductive method that we describe here involves only algebraic manipulations, e.g., taking the inverses of relatively small matrices.

We introduce several notations that we will use in this section.

Let us recall that ζ is a univalent function in D_β such that $\zeta(\beta) = 0$ and $N^{-\tau_a}\zeta(z)/(z-\beta)$ is an N-independent and non-vanishing holomorphic function, where (we include the case $a < 1$ later):

$$\tau_a = \begin{cases} 1/2 & \text{for } a > 1, \\ 1 & \text{for } a < 1. \end{cases}$$

The lemma below generalizes the definition of $\widehat{\mathcal{F}}$ that we used in the previous section.

Lemma 2.5.1 *Let $\mathcal{F}$ be a piecewise analytic matrix function with determinant* 1 *and its asymptotic expansion around ∞ given by*

$$\mathcal{F} = I + \frac{C_1}{\zeta} + \frac{C_2}{\zeta^2} + \cdots,$$

where C_j's are constant 2×2 matrices. For any positive integer L, there exists a positive number of k and a decomposition

$$\mathcal{F}(\zeta) = \widehat{\mathcal{F}}(\zeta) F_k(\zeta) \cdots F_1(\zeta), \tag{2.40}$$

such that, for all $1 \le j \le k$, F_j is a rational function with its only singularity at the origin, $F_j(\infty) = I$, $F_j(\zeta) - I$ is nilpotent and

$$\widehat{\mathcal{F}}(\zeta) = I + \mathcal{O}\left(\zeta^{-L}\right).$$

Proof. Assume

$$\mathcal{F}(\zeta) = I + \frac{C_0}{\zeta^m} + \mathcal{O}\left(\frac{1}{\zeta^{m+1}}\right), \quad C_0 = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}.$$

Since $\det \mathcal{F} = 1$, we have $c_{11} + c_{22} = 0$. One can write C_0 as the sum of three nilpotent matrices, $C_0 = N_1 + N_2 + N_3$, where

$$N_1 = \begin{bmatrix} c_{11} & -c_{11}^2 \\ 1 & -c_{11} \end{bmatrix}, \quad N_2 = \begin{bmatrix} 0 & c_{12} - c_{11}^2 \\ 0 & 0 \end{bmatrix}, \quad N_3 = \begin{bmatrix} 0 & 0 \\ c_{21} - 1 & 0 \end{bmatrix}.$$

We get

$$\mathcal{F}(\zeta)\left(I + \frac{N_1}{\zeta^m}\right)^{-1}\left(I + \frac{N_2}{\zeta^m}\right)^{-1}\left(I + \frac{N_3}{\zeta^m}\right)^{-1} = I + \mathcal{O}\left(\frac{1}{\zeta^{m+1}}\right).$$

The straightforward induction finishes the proof of the lemma. $\square$

Given $\{F_k\}_{k=1,2,\cdots}$, we will define $\{H_k\}$ and $\{R_k\}$ inductively. Let $H_0 = I$. Assume that H_{k-1} is holomorphic and non-vanishing at β and $H_{k-1}(z) = I + \mathcal{O}(1/N^{\tau_a})$. We define

$$\widetilde{F}_k(z) := \left(N^{\frac{c}{2}}\eta(z)\right)^{\sigma_3} H_{k-1}(z)F_k(\zeta(z))H_{k-1}^{-1}(z)\left(N^{\frac{c}{2}}\eta(z)\right)^{-\sigma_3}. \tag{2.41}$$

If F_k satisfies the property described in Lemma 2.5.1, we have the following truncated Laurent series expansion near β,

$$\widetilde{F}_k^{-1}(z) = N^{\frac{c}{2}\sigma_3}\left(I + \sum_{j=-\infty}^{m_k} \frac{A_j}{(z-\beta)^j}\right)N^{-\frac{c}{2}\sigma_3},$$

for some positive integer m_k and some constant matrices $\{A_j\}$. Given $\{A_j\}$, the lemma below constructs $\{R_k\}$ inductively.

Lemma 2.5.2 *Given $\widetilde{F}_k(z)$ as above, the unique rational matrix function R_k such that its only singularity is at β, $R_k(\infty) = I$ and $R_k(z)\widetilde{F}_k^{-1}(z)$ is holomorphic at β, is given by*

$$R_k(z) = N^{\frac{c}{2}\sigma_3}\left(I + \sum_{j=1}^{m_k} \frac{B_j}{(z-\beta)^j}\right)N^{-\frac{c}{2}\sigma_3},$$

where for a sufficiently large N B_j's are given by

$$[B_{m_k}, B_{m_{k-1}}, \cdots, B_1] = -[A_{m_k}, A_{m_{k-1}}, \cdots, A_1]\left(I + \widetilde{M}\right)^{-1}.$$

The $2m_k \times 2m_k$ matrix $\widetilde{M}$ is given in block form by

$$\widetilde{M} = \begin{bmatrix} A_0 & A_{-1} & \cdots & A_{1-m_k} \\ A_1 & A_0 & \cdots & A_{2-m_k} \\ \vdots & \ddots & \ddots & \vdots \\ A_{m_{k-1}} & \cdots & A_1 & A_0 \end{bmatrix}$$

and for a sufficiently large N, $I + \widetilde{M}$ is invertible. Moreover, $\det R_k \equiv 1$.

Proof. Let

$$M = \begin{bmatrix} A_{m_k} & A_{m_{k-1}} & \cdots & A_1 \\ & A_{m_k} & \cdots & A_2 \\ & & \ddots & \vdots \\ & & & A_{m_k} \end{bmatrix},$$

In order to make $R_k(z)\widetilde{F}_k^{-1}(z)$ holomorphic at β, we require all the pole terms of $R_k(z)\widetilde{F}_k^{-1}(z)$ to vanish. We obtain

$$[B_{m_k}, B_{m_{k-1}}, \cdots, B_1] \cdot M = 0, \tag{2.42}$$

$$[B_{m_k}, B_{m_{k-1}}, \cdots, B_1](I + \widetilde{M}) + [A_{m_k}, A_{m_{k-1}}, \cdots, A_1] = 0, \tag{2.43}$$

where the first equation comes from the the poles of the orders $2m_k, 2m_k - 1, \cdots, m_k + 1$, and the second equation comes from the poles orders $m_k, m_k - 1, \cdots, 1$.

Let's explain a useful bound on A_j's. If $F_k(\zeta) = I + \mathcal{O}(\zeta^{-m_k})$, then $F_k(\zeta(z)) = I + \mathcal{O}(N^{-m_k \tau_a})$ on ∂D_β. Therefore, we have $A_j = \mathcal{O}(N^{-m_k\tau_a})$ and $\|\widetilde{M}\| = \mathcal{O}(N^{-m_k\tau_a})$. Hence $I + \widetilde{M}$ is invertible for a sufficiently large N so that, from (2.43), we can obtain

$$[B_{m_k}, B_{m_{k-1}}, \cdots, B_1] = -[A_{m_k}, A_{m_{k-1}}, \cdots, A_1] \left(I + \widetilde{M}\right)^{-1}.$$

Let us show that (2.42) is satisfied. Since $F_k(\zeta) - I$ is nilpotent, $\widetilde{F}_k^{-1}(z) - I$ is nilpotent and therefore,

$$\left(\sum_{j=-\infty}^{m_k} \frac{A_j}{(z-\beta)^j} \right)^2 = 0.$$

This implies $M^2 = 0$ and $M\widetilde{M} = -\widetilde{M}M$. Then,

$$\begin{aligned} [B_{m_k}, B_{m_{k-1}}, \cdots, B_1] \cdot M &= -[A_{m_k}, A_{m_{k-1}}, \cdots, A_1] \left(I + \widetilde{M}\right)^{-1} \cdot M \\ &= -[M]_{\text{1st row}} \left(I - \widetilde{M} + \widetilde{M}^2 + \cdots\right) \cdot M \\ &= -\left[M \cdot \left(I - \widetilde{M} + \widetilde{M}^2 + \cdots\right) \cdot M\right]_{\text{1st row}} \\ &= -\left[MM - M\widetilde{M}M + M\widetilde{M}^2 M + \cdots\right]_{\text{1st row}} \\ &= -\left[MM + M^2\widetilde{M} + M^2\widetilde{M}^2 + \cdots\right]_{\text{1st row}} = 0 \end{aligned}$$

The "1st row" means the 1st row in the 2×2 block matrix. Since $R_k(z)\widetilde{F}_k^{-1}(z)$ is holomorphic at

β and $\det \widetilde{F}_k^{-1}(z) \equiv 1$, $\det R_k(z)$ is holomorphic at β. Since $\det R_k(\infty) = 1$, we have $\det R_k \equiv 1$.

Now we show that R_k is unique. Assume $\widetilde{R}_k$ also satisfies all the conditions satisfied by R_k in the lemma. Then, $R_k \widetilde{R}_k^{-1}$ is holomorphic away from β, $R_k(z)\widetilde{R}_k(z)^{-1} \to I$ as $z \to \infty$, and $R_k \widetilde{R}_k^{-1} = R_k \widetilde{F}_k^{-1} (\widetilde{R}_k \widetilde{F}_k^{-1})^{-1}$ is holomorphic at β. Thus, $R_k = \widetilde{R}_k$. ☐

Corollary 2.5.3 *If $F_k(\zeta) = I + \mathcal{O}(\zeta^{-m})$, then $N^{-\frac{c}{2}\sigma_3} R_k(z) N^{\frac{c}{2}\sigma_3} = I + \mathcal{O}(N^{-\tau_a m})$ when $z \in \partial D_\beta$.*

Proof. From $A_j = \mathcal{O}(N^{-m\tau_a})$, it follows that $B_j = \mathcal{O}(N^{-m\tau_a})$. By Lemma 2.5.2, this ends the proof. ☐

Using $R_k(z)$ from the above lemma, we define $H_k(z)$ by

$$H_k(z) = \left(N^{\frac{c}{2}}\eta(z)\right)^{-\sigma_3} R_k(z)\widetilde{F}_k^{-1}(z) \left(N^{\frac{c}{2}}\eta(z)\right)^{\sigma_3} H_{k-1}(z). \tag{2.44}$$

Since $H_0 = I$, by induction, $H_k(z)$ is holomorphic at β and has determinant 1. By Corollary 2.5.3 we get

$$H_k(z) = I + \mathcal{O}(N^{-\tau_a}), \quad z \in \overline{D_\beta}. \tag{2.45}$$

Lemma 2.5.4 *For $z \in \partial D_\beta$, we have*

$$Z_+^\infty(z) \left(Z_-^\infty(z)\right)^{-1} = \left(N^{c/2}\eta(z)\right)^{\sigma_3} H(z)\widehat{\mathcal{F}}(\zeta)H^{-1}(z) \left(N^{c/2}\eta(z)\right)^{-\sigma_3}.$$

Proof. We have

$$\begin{aligned} Z_+^\infty(z) \left(Z_-^\infty(z)\right)^{-1} &= \Phi(z) \left(\frac{z-a}{z}\right)^{\frac{c}{2}\sigma_3} \mathcal{P}(z) \left(\frac{z-a}{z}\right)^{-\frac{c}{2}\sigma_3} \Phi^{-1}(z)\mathcal{R}^{-1}(z) \\ &= \left(N^{c/2}\eta(z)\right)^{\sigma_3} H(z)\mathcal{F}(z) \left(N^{c/2}\eta(z)\right)^{-\sigma_3} \mathcal{R}^{-1}(z) \\ &= \left(N^{c/2}\eta(z)\right)^{\sigma_3} H(z)\widehat{\mathcal{F}}(\zeta)H^{-1}(z) \left(N^{c/2}\eta(z)\right)^{-\sigma_3}. \end{aligned} \tag{2.46}$$

The first equality is from (2.34), the second equality comes from Lemma 2.4.4, and the last equality follows from (2.40) and

$$H = H_k = \left(N^{c/2}\eta\right)^{-\sigma_3} R_k \cdots R_1 \left(N^{c/2}\eta\right)^{\sigma_3} F_1^{-1} \cdots F_k^{-1}, \tag{2.47}$$

which follows from the inductive definition of H_k in (2.44) with $H_0 = I$. The theorem is proved using Lemma 2.5.1 and (2.45). ☐

Proof of Theorem 2.1.2. Contrary to the proof of Theorem 2.4.5, all the error bounds will be for a *fixed c*.

Here, we construct $\{R_j\}$ and $\{H_j\}$ inductively from the initial data $R_1 = \mathcal{R}$ and $H_1 = H$ where $\mathcal{R}$ and H given by (2.32) and (2.33).

By (2.29) with (2.41) a calculation of $\widetilde{F}_2(z)$ leads to,

$$\begin{aligned}\widetilde{F}_2(z) &= \left(N^{\frac{c}{2}}\eta(z)\right)^{\sigma_3} H_1(z)F_2(\zeta(z))H_1^{-1}(z)\left(N^{\frac{c}{2}}\eta(z)\right)^{-\sigma_3} \\ &= N^{\frac{c}{2}\sigma_3}\left(I + \begin{bmatrix} \mathcal{O}\left(\frac{1}{N}\right) & \mathcal{O}\left(\frac{1}{N^{3/2}}\right) \\ \mathcal{O}\left(\frac{1}{\sqrt{N}}\right) & \mathcal{O}\left(\frac{1}{N}\right)\end{bmatrix}\right) N^{-\frac{c}{2}\sigma_3}. \qquad z \in \partial D_\beta.\end{aligned}$$

Estimating $\widetilde{F}_2(z)$ by using $H_1 = I + \mathcal{O}(N^{-1/2})$ in (2.45) gives the same result except the bound at (12)-entry above may be relaxed to $\mathcal{O}(N^{-1})$. Then, by Lemma 2.5.2, we have

$$R_2(z) = N^{\frac{c}{2}\sigma_3}\left(I + \begin{bmatrix} \mathcal{O}\left(\frac{1}{N}\right) & \mathcal{O}\left(\frac{1}{N}\right) \\ \mathcal{O}\left(\frac{1}{\sqrt{N}}\right) & \mathcal{O}\left(\frac{1}{\sqrt{N}}\right)\end{bmatrix}\right) N^{-\frac{c}{2}\sigma_3}.$$

Using $R_1 = \mathcal{R}$ with (2.32) we get

$$R_2R_1 = N^{\frac{c}{2}\sigma_3}\left(I + \begin{bmatrix} \mathcal{O}\left(\frac{1}{N}\right) & \frac{\sqrt{2\pi}\left(a^2-1\right)^c}{\sqrt{N}a\,\Gamma(c)}\frac{1}{z-\beta} + \mathcal{O}\left(\frac{1}{N}\right) \\ \mathcal{O}\left(\frac{1}{\sqrt{N}}\right) & \mathcal{O}\left(\frac{1}{\sqrt{N}}\right)\end{bmatrix}\right) N^{-\frac{c}{2}\sigma_3}.$$

From (2.27), a further decompositions of $\mathcal{F}$ gives $F_k = I + \mathcal{O}\left(\zeta^{-3}\right)$ for $k \geq 3$. Then, by Corollary 2.5.3, we get

$$R_k \cdots R_3 = N^{\frac{c}{2}\sigma_3}(I + \mathcal{O}(N^{-3/2}))N^{-\frac{c}{2}\sigma_3}$$

and

$$R_k \cdots R_1 = N^{\frac{c}{2}\sigma_3}\left(I + \begin{bmatrix} \mathcal{O}\left(\frac{1}{N}\right) & \frac{\sqrt{2\pi}\left(a^2-1\right)^c}{\sqrt{N}a\,\Gamma(c)}\frac{1}{z-\beta} + \mathcal{O}\left(\frac{1}{N}\right) \\ \mathcal{O}\left(\frac{1}{\sqrt{N}}\right) & \mathcal{O}\left(\frac{1}{\sqrt{N}}\right)\end{bmatrix}\right) N^{-\frac{c}{2}\sigma_3}, \quad z \in \partial D_\beta.$$

Using Lemma 2.5.1, we can have $\widehat{\mathcal{F}}(\zeta) = I + \mathcal{O}\left(\zeta^{-L}\right)$ for an arbitrary L. Using Lemma 2.5.4 with

$$\mathcal{R} = R_k \cdots R_1 \quad \text{and} \quad H = H_k = I + \mathcal{O}(N^{-1/2}),$$

we get $Z_+^\infty\left(Z_-^\infty\right)^{-1} = I + \mathcal{O}(N^{-L})$ on ∂D_β. From the argument similar to one used in the proof of

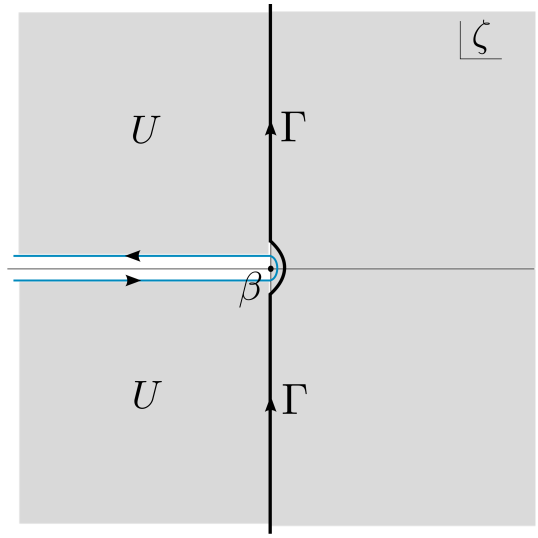

Figure 10.: Jump contours of $\mathcal{P}$ (2.48) in D_β (left); the shaded region (everywhere except the negative real axis) is U.

Theorem 2.4.5, we obtain

$$Y(z) = \mathrm{e}^{\frac{N\ell}{2}\sigma_3}\left(I + \mathcal{O}\left(\frac{1}{N^L}\right)\right) Z^\infty(z) \begin{bmatrix} 1 & 0 \\ -\star\left(\dfrac{z}{z-a}\right)^c \mathrm{e}^{N\phi(z)} & 1 \end{bmatrix} \mathrm{e}^{\frac{-N\ell}{2}\sigma_3}\mathrm{e}^{Ng(z)\sigma_3}$$

uniformly on any compact set for an arbitrary positive integer L. The proof is finished by calculations similar to (2.38) and (2.39).

2.6 $a < 1$: c near 0 and Proof of Theorem 2.1.3

In this section, we consider the case $a < 1$ following closely the analysis of previous two sections for the case $a > 1$.

From (2.5), we obtain

$$\phi_A(z) = \frac{a^2-1}{a}(z-\beta)\left(1 + \mathcal{O}(z-\beta)\right).$$

We define $\zeta : D_\beta \to \mathbb{C}$ by (2.6) where D_β is a sufficiently small fixed disc centered at $z = \beta$ such that ζ is one-to-one. Under the mapping ζ the contour $\mathcal{S}$ maps to the imaginary axis.

Inside D_β we want to find $\mathcal{P}$ such that

$$Z^\infty(z) = \Phi(z)\left(\frac{z-a}{z}\right)^{\frac{c}{2}\sigma_3}\mathcal{P}(z)\left(\frac{z-a}{z}\right)^{-\frac{c}{2}\sigma_3}$$

satisfies the jump conditions of Z in (2.16), i.e.,

$$\begin{cases} \mathcal{P}_+(z) = \mathcal{P}_-(z) \begin{bmatrix} 1 & 0 \\ \mathrm{e}^{\zeta(z)} & 1 \end{bmatrix}, & z \in \partial U \cap D_\beta, \\ \mathcal{P}_+(z) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathcal{P}_-(z) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, & z \in \Gamma \cap D_\beta, \\ \mathcal{P}_+(z) = \mathrm{e}^{-c\pi\mathrm{i}\sigma_3} \mathcal{P}_-(z) \mathrm{e}^{c\pi\mathrm{i}\sigma_3}, & z \in (-\infty, a] \cap D_\beta. \end{cases} \tag{2.48}$$

Let us define S by

$$S = S(\zeta) = \begin{cases} I, & |\arg\zeta| < \pi/2, \\ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, & \text{otherwise.} \end{cases}$$

Here we choose S so that $S^{-1}\zeta(z)^{\frac{c}{2}\sigma_3}$ satisfies *the left jump of* $\mathcal{P}(z)$ from the second and the third equations of (2.48). Then, the matrix function

$$W(z) = \zeta(z)^{-\frac{c}{2}\sigma_3} S \mathcal{P}(z) S^{-1} \zeta(z)^{\frac{c}{2}\sigma_3} \tag{2.49}$$

satisfies

$$W_+(z) = W_-(z) \begin{bmatrix} 1 & -\zeta(z)^{-c}\mathrm{e}^{\zeta(z)} \\ 0 & 1 \end{bmatrix}, \quad z \in \partial U \cap D_\beta.$$

Let H be a holomorphic matrix (that will be specified later). A solution to the above jump condition can be written as $W(z) = H(z)\mathcal{F}(\zeta(z))$, where

$$\mathcal{F}(\zeta) := \begin{bmatrix} 1 & \dfrac{-1}{2\mathrm{i}\pi} \displaystyle\int_{\mathcal{L}} \frac{\mathrm{e}^s}{s^c(s-\zeta)} \mathrm{d}s \\ 0 & 1 \end{bmatrix}. \tag{2.50}$$

Here the contour $\mathcal{L}$ is the image of ∂U under ζ. It begins at $-\infty$, circles the origin once in the counterclockwise direction, and returns to $-\infty$.

Lemma 2.6.1 *For $z \in D_\beta$ we have*

$$\Phi(z) \left(\frac{z-a}{z} \right)^{\frac{c}{2}\sigma_3} S^{-1} \zeta(z)^{\frac{c}{2}\sigma_3} = \left(N^{c/2} \eta(z) \right)^{\sigma_3},$$

where $\eta : D_\beta \to \mathbb{C}$ *and*

$$\eta(z) := \frac{1}{N^{c/2}} \left(\frac{z\,\zeta(z)}{z-\beta} \right)^{c/2},$$

is a nonvanishing N*-independent analytic function in* D_β*.*

By Lemma 2.6.1, (2.49) and if $W = H\mathcal{F}$ we get

$$\begin{aligned}
&\Phi(z) \left(\frac{z-a}{z} \right)^{\frac{c}{2}\sigma_3} \mathcal{P}(z) \left(\frac{z-a}{z} \right)^{-\frac{c}{2}\sigma_3} \\
&= \Phi(z) \left(\frac{z-a}{z} \right)^{\frac{c}{2}\sigma_3} S^{-1} \zeta^{(c/2)\sigma_3} W(z)\, \zeta^{-(c/2)\sigma_3} S \left(\frac{z-a}{z} \right)^{-\frac{c}{2}\sigma_3} \\
&= \Phi(z) \left(\frac{z-a}{z} \right)^{\frac{c}{2}\sigma_3} S^{-1} \zeta^{(c/2)\sigma_3} H(z)\, \mathcal{F}(\zeta(z))\, \zeta^{-(c/2)\sigma_3} S \left(\frac{z-a}{z} \right)^{-\frac{c}{2}\sigma_3} \\
&= \left(N^{c/2} \eta(z) \right)^{\sigma_3} H(z)\, \mathcal{F}(\zeta(z)) \left(N^{c/2} \eta(z) \right)^{-\sigma_3} \Phi(z).
\end{aligned} \tag{2.51}$$

This proves the same statement as in Lemma 2.4.4 for $a < 1$.

Lemma 2.6.2 *When* $|\zeta|$ *goes to* ∞*,* $\mathcal{F}$ *in* (2.50) *satisfies*

$$\mathcal{F}(\zeta) F_1(\zeta)^{-1} = I + \mathcal{O}\left(\frac{1}{|\zeta^2|} \right) \tag{2.52}$$

uniformly over $c \in (-1, 2)$ *and*

$$\mathcal{F}(\zeta) F_1(\zeta)^{-1} \cdots F_k(\zeta)^{-1} = I + \mathcal{O}\left(\frac{1}{|\zeta^{k+1}|} \right), \tag{2.53}$$

where

$$F_k(\zeta) = I + \frac{c_k}{\zeta^k} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad c_k = \frac{1}{2\mathrm{i}\pi} \int_{\mathcal{L}} \frac{s^{k-1} \mathrm{e}^s}{s^c} \mathrm{d}s = \frac{\sin(c\pi)\,\Gamma(k-c)}{\pi(-1)^{k-1}}. \tag{2.54}$$

Proof. We only give the proof of (2.52) as the proof of (2.53) is similar. The only nonzero entry of $\left(\mathcal{F}F_1^{-1}-I\right)$ is the (12)−entry. For $\arg|\zeta|<\pi/2$, we have

$$\begin{aligned}
\left|\left(\mathcal{F}(\zeta)F_1(\zeta)^{-1}\right)_{12}\right| &= \frac{1}{2\pi}\left|\int_{\mathcal{L}}\frac{\mathrm{e}^s}{s^c(s-\zeta(z))}\mathrm{d}s+\int_{\mathcal{L}}\frac{\mathrm{e}^s}{s^c\zeta(z)}\mathrm{d}s\right| \\
&\le \frac{1}{2\pi}\int_{\mathcal{L}}\left|\frac{\mathrm{e}^s s}{s^c(\zeta(z)-s)\zeta(z)}\right||\mathrm{d}s| \\
&\le \frac{1}{2\pi}\int_{\mathcal{L}}\left|\frac{\mathrm{e}^s s}{s^c\zeta^2}\right||\mathrm{d}s| \\
&= \frac{1}{2\pi|\zeta^2|}\int_{\mathcal{L}}\left|\frac{\mathrm{e}^s s}{s^c}\right||\mathrm{d}s|.
\end{aligned}$$

In the second inequality, we use $|\zeta-s|\ge|\zeta|$ for $\operatorname{Re}\zeta>0$ and $s\in(-\infty,0]$. One can prove that the last integral is finite by deforming the contour away from the origin so that the integrant is bounded from above.

When $|\arg\zeta|\ge\pi/2$ a similar argument using the deformation of integration contour leads to the proof of the lemma. Note that the branch cut $(-\infty,0)$ of s^c and the integration contour $\mathcal{L}$ can be deformed, respectively, into $\{te^{\mathrm{i}\theta_0}\}_{0<t<\infty}$ for $\pi/2\le|\theta_0|\le\pi$ and the corresponding contour around the new branch cut. We shall omit the further details. □

Theorem 2.6.3 *For $a<1$ we get*

$$P_N(z)=\begin{cases} z^N\left(\dfrac{z}{z-a}\right)^c\left(1+\mathcal{O}\left(\dfrac{1}{N^{2-c}}\right)\right), & z\in\operatorname{Ext}\mathcal{S}\setminus(U\cup D_\beta),\\[2ex] z^N\left(\left(\dfrac{z}{z-a}\right)^c-\dfrac{a(1-a^2)^{c-1}}{N^{1-c}\Gamma(c)}\dfrac{e^{N\phi_A(z)}}{(z-a)}+\mathcal{O}\left(\dfrac{1}{N^{2-c}},\dfrac{\mathrm{e}^{N\phi_A}}{N^{2-c}}\right)\right), & z\in U\setminus D_\beta,\\[2ex] z^N\left(\left(\dfrac{z}{z-a}\right)^c-\left(\dfrac{z\zeta(z)}{z-a}\right)^c\dfrac{1}{\mathrm{e}^{\zeta(z)}}\left(\hat{f}(\zeta(z))+\mathcal{O}\left(\dfrac{c}{N}\right)\right)+\mathcal{O}\left(\dfrac{1}{N^{2-c}}\right)\right), & z\in D_\beta,\end{cases}$$

where

$$\hat{f}(\zeta)=\frac{-1}{2\mathrm{i}\pi}\int_{\mathcal{L}}\frac{\mathrm{e}^s}{s^c(s-\zeta)}\mathrm{d}s.$$

Here the contour $\mathcal{L}$ is the image of ∂U under ζ, and it begins at $-\infty$, circles the origin once in the counterclockwise direction, and returns to $-\infty$. The error bounds are uniform over $-1<c<2$.

Proof. From F_1 in (2.54) one can obtain R_1 using Lemma 2.5.2 and obtain H_1 by (2.47):

$$\begin{aligned} R_1(z) &= I + \frac{a(1-a^2)^{c-1}}{N^{1-c}\Gamma(c)} \frac{1}{z-a} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \\ H_1(z) &= \left(N^{c/2}\eta(z)\right)^{-\sigma_3} R_1(z) \left(N^{c/2}\eta(z)\right)^{\sigma_3} F_1(\zeta(z))^{-1} = \begin{bmatrix} 1 & h(z) \\ 0 & 1 \end{bmatrix}, \end{aligned} \tag{2.55}$$

where (using $c_1 = 1/\Gamma(c)$ that appears in F_1)

$$h(z) = \left(\frac{z-a}{z\zeta(z)}\right)^c \left(\frac{a(1-a^2)^{c-1}}{N^{1-c}\Gamma(c)} \frac{1}{z-a}\right) - \frac{1}{\zeta(z)\Gamma(c)} = \mathcal{O}\left(\frac{c}{N}\right). \tag{2.56}$$

Setting $\mathcal{R} = R_1$ and $H = H_1$, we can define Z^∞ by (2.31) and (2.34). Defining the error matrix by $\mathcal{E} = Z^\infty Z^{-1}$, by the similar calculation as (2.36) with $\widehat{\mathcal{F}} = \mathcal{F}F_1^{-1}$ and (2.52), we get

$$\mathcal{E}_+(z)\mathcal{E}_-^{-1}(z) = I + \mathcal{O}\left(\frac{1}{N^{2-c}}\right), \quad z \in \partial D_\beta,$$

uniformly over $c \in (-1, 2)$. By the same argument as in the proof of Theorem 2.4.5 we obtain

$$Z(z) = \left(I + \mathcal{O}\left(\frac{1}{N^{2-c}}\right)\right) Z^\infty(z).$$

The proof is finished by the calculations identical to those in (2.38) and (2.39). To add a little more detail, inside D_β we need to use (2.56) to obtain the final result. Below we write the strong asymptotics *before* using (2.56) as an example.

$$\left(\left(\frac{z}{z-a}\right)^c - \left(\frac{z\zeta(z)}{z-a}\right)^c \left(\hat{f}(\zeta(z)) + h(z)\right) \mathrm{e}^{N\phi(z)} + \mathcal{O}\left(\frac{1}{N^{2-c}}\right)\right) \mathrm{e}^{Ng(z)}, \qquad z \in \operatorname{Ext}\mathcal{S} \cap D_\beta.$$

We omit the computation. ☐

Proof of Theorem 2.1.3. The proof will be similar to the above proof and the proof of Theorem 2.1.2.

By (2.54), (2.55) and (2.41), we obtain

$$\begin{aligned} \widetilde{F}_2(z) &= \left(N^{\frac{c}{2}}\eta(z)\right)^{\sigma_3} H_1(z) F_2(\zeta(z)) H_1^{-1}(z) \left(N^{\frac{c}{2}}\eta(z)\right)^{-\sigma_3} \\ &= N^{\frac{c}{2}\sigma_3} \left(I + \begin{bmatrix} 0 & \mathcal{O}\left(N^{-2}\right) \\ 0 & 0 \end{bmatrix}\right) N^{-\frac{c}{2}\sigma_3}, \qquad z \in \partial D_\beta. \end{aligned} \tag{2.57}$$

From Lemma 2.5.2 and (2.57), we have

$$R_2(z) = N^{\frac{c}{2}\sigma_3}\left(I + \begin{bmatrix} 0 & \mathcal{O}\left(N^{-2}\right) \\ 0 & 0 \end{bmatrix}\right) N^{-\frac{c}{2}\sigma_3}.$$

Combined with R_1 in (2.55), we derive

$$R_2 R_1 = N^{\frac{c}{2}\sigma_3}\left(I + \begin{bmatrix} 0 & \dfrac{a(1-a^2)^{c-1}}{N\Gamma(c)}\dfrac{1}{z-a} + \mathcal{O}\left(\dfrac{1}{N^2}\right) \\ 0 & 0 \end{bmatrix}\right) N^{-\frac{c}{2}\sigma_3}.$$

From (2.54) in Lemma 2.6.2, we have $F_k = I + \mathcal{O}\left(\zeta^{-3}\right)$ for $k \geq 3$. By Corollary 2.5.3, we obtain

$$R_k \cdots R_3 = N^{\frac{c}{2}\sigma_3}(I + \mathcal{O}(N^{-3}))N^{-\frac{c}{2}\sigma_3}.$$

In fact, following the inductive construction of R_k and H_k in Section 2.5, one can find that R_k's are all upper triangular matrices. Therefore, we get

$$R_k \cdots R_1 = N^{\frac{c}{2}\sigma_3}\left(I + \begin{bmatrix} 0 & \dfrac{a(1-a^2)^{c-1}}{N\Gamma(c)}\dfrac{1}{z-a} + \mathcal{O}\left(\dfrac{1}{N^2}\right) \\ 0 & 0 \end{bmatrix}\right) N^{-\frac{c}{2}\sigma_3}, \quad z \in \partial D_\beta.$$

Using Lemma 2.5.1, we can have $\widehat{\mathcal{F}}(\zeta) = I + \mathcal{O}\left(\zeta^{-L}\right)$ for an arbitrary L. Using Lemma 2.5.4 with

$$\mathcal{R} = R_k \cdots R_1 \quad \text{and} \quad H = H_k = I + \mathcal{O}(N^{-1}),$$

we get $Z_+^\infty \left(Z_-^\infty\right)^{-1} = I + \mathcal{O}(N^{-L})$ on ∂D_β. From the argument similar to that in the proof of Theorem 2.4.5, we obtain

$$Y(z) = \mathrm{e}^{\frac{N\ell}{2}\sigma_3}\left(I + \mathcal{O}\left(\frac{1}{N^L}\right)\right) Z^\infty(z) \begin{bmatrix} 1 & 0 \\ -\star\left(\dfrac{z}{z-a}\right)^c \mathrm{e}^{N\phi(z)} & 1 \end{bmatrix} \mathrm{e}^{\frac{-N\ell}{2}\sigma_3}\mathrm{e}^{Ng(z)\sigma_3}$$

for an arbitrary positive integer L. The proof is finished by calculations similar to those in (2.38) and (2.39).

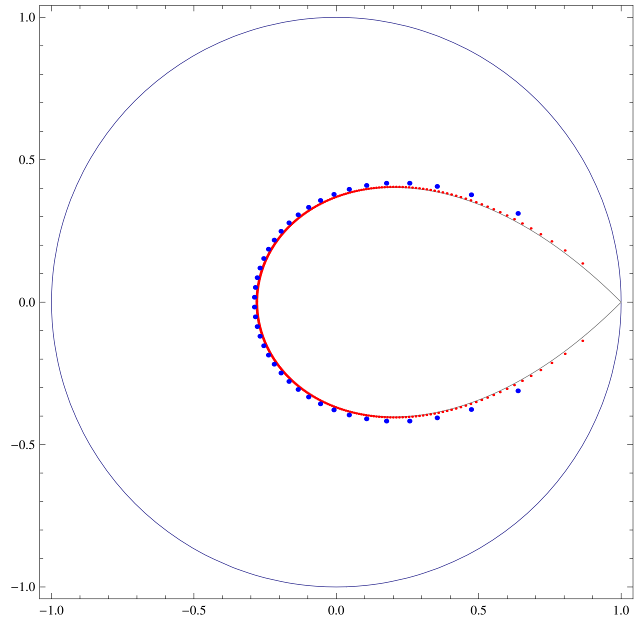

Figure 11.: The zeros of orthogonal polynomials with degrees 40 (blue) and 300 (red), $c = 1$ and $a = 1$. The solid line inside the disk is $\mathcal{S}$.

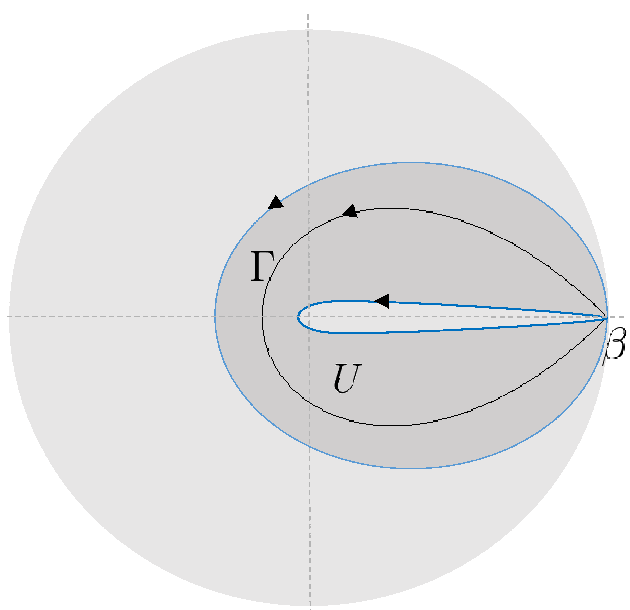

Figure 12.: Contours for the Riemann-Hilbert problem of Φ when $a \approx 1$. Γ is the black curves and U is the shaded region bounded by the blue curves.

2.7 Critical case: $a = 1$

In this section we consider $a = 1 + \mathcal{O}(1/\sqrt{N})$. Here we only argue that the strong asymptotics *can* be obtained through the parametrix of Painlevé IV equation (as suggested in [3]) following the similar steps described previously.

There is a disk D_1 centered at 1 such that there exists a univalent map $\zeta : D_1 \to \mathbb{C}$ that satisfies

$$(\zeta(z) + x)^2 = N\phi_A(z) - N\phi_A\,(1/a)\,,$$

where

$$x := \sqrt{N\phi_A(a) - N\phi_A(1/a)} = \sqrt{2N}(a-1)(1 + \mathcal{O}(a-1)).$$

Under the mapping ζ, we have $\zeta(a) = 0$ and the critical point of ϕ_A is mapped to $-x$; note that $\phi(1/a)$ is the *critical value* of ϕ_A.

Inside D_1 we require that $\Phi(z)\left(\frac{z-a}{z}\right)^{\frac{c}{2}\sigma_3}\mathcal{P}(z)\left(\frac{z-a}{z}\right)^{-\frac{c}{2}\sigma_3}$ satisfies the jump conditions (2.16) for Z. With the boundary condition of $\mathcal{P}$ on ∂D_1 this leads to the following jumps of $\mathcal{P}$ *inside* D_1:

$$\begin{cases} \mathcal{P}_+(z) = \mathcal{P}_-(z)\begin{bmatrix} 1 & 0 \\ \mathrm{e}^{-N\phi_A(z)} & 1 \end{bmatrix}, & z \in \partial U \cap \operatorname{Int}\Gamma, \\ \mathcal{P}_+(z) = \mathcal{P}_-(z)\begin{bmatrix} 1 & 0 \\ \mathrm{e}^{N\phi_A(z)} & 1 \end{bmatrix}, & z \in \partial U \cap \operatorname{Ext}\Gamma, \\ \mathcal{P}_+(z) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\mathcal{P}_-(z)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, & z \in \Gamma \cap U, \\ \mathcal{P}_+(z) = \mathrm{e}^{-c\pi \mathrm{i}\sigma_3}\mathcal{P}_-(z)\mathrm{e}^{c\pi\mathrm{i}\sigma_3}, & z \in (0, a], \\ \mathcal{P}(z) = I + o\,(1)\,, & z \in \partial D_1. \end{cases}$$

Here U and Γ are defined similarly to those for $a > 1$ except the segment $[\beta, a]$ becomes a point at 1, see Figure 12. We will show that such $\mathcal{P}$ can be written in terms of the solution of the Painlevé IV equation. To achieve this, we want to transform $\mathcal{P}$ into a new matrix function, W, with only *constant jump matrices from the right.* Such transform is given by

$$W(z) = \mathrm{e}^{-\frac{\ell x}{2}\sigma_3}\zeta(z)^{\frac{c}{2}\sigma_3}S \cdot \mathcal{P}(z)\cdot T(z)^{-1}\,S^{-1}, \quad z \in D_1, \tag{2.58}$$

using a diagonal matrix T, a piecewise constant matrix S and a constant ℓ_x defined by

$$T(z) = \exp\left(\frac{N}{2}(-1)^\nu \phi_A(z)\sigma_3\right) = \exp\left[\frac{(-1)^\nu}{2}\left(\zeta(z)^2 + 2x\zeta(z) + \ell_x\right)\sigma_3\right],$$

$$\ell_x = x^2 + N\phi_A(1/a), \quad S = S(z) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^\nu,$$

where

$$\nu = \begin{cases} 0, & z \in \operatorname{Ext}\Gamma, \\ 1, & z \in \operatorname{Int}\Gamma. \end{cases}$$

Here, we chose S such that $S^{-1}\zeta(z)^{-\frac{c}{2}\sigma_3}$ satisfies all the *left* jumps of $\mathcal{P}$, i.e.,

$$\left(S^{-1}\zeta(z)^{-\frac{c}{2}\sigma_3}\right)_+ = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\left(S^{-1}\zeta(z)^{-\frac{c}{2}\sigma_3}\right)_-, \quad z \in \Gamma \cap U,$$

$$\left(S^{-1}\zeta^{-\frac{c}{2}\sigma_3}\right)_+ = \mathrm{e}^{-c\pi i\sigma_3}\left(S^{-1}\zeta^{-\frac{c}{2}\sigma_3}\right)_-, \qquad z \in [-\infty, 0].$$

Consequently, W has the jump matrices only from the *right.* Furthermore, the jump matrices of W are constant matrices because of the right multipliction of T in (2.58), and the jump on Γ disappears by the right multiplication by S^{-1}. We obtain the jump condition of W by

$$W_+(z) = W_-(z)\begin{cases} \begin{bmatrix} 1 & 0 \\ s_1 & 1 \end{bmatrix}, & \zeta(z) \in \mathbb{R}^+, \\ \begin{bmatrix} 1 & s_2 \\ 0 & 1 \end{bmatrix}, & \zeta(z) \in \mathrm{i}\mathbb{R}^+, \\ \begin{bmatrix} 1 & 0 \\ s_3 & 1 \end{bmatrix}, & \zeta(z) \in \mathbb{R}^-, \\ \begin{bmatrix} 1 & s_4 \\ 0 & 1 \end{bmatrix}, & \zeta(z) \in \mathrm{i}\mathbb{R}^-, \end{cases}$$

where $s_1 = 0$, $s_2 = 1$, $s_3 = \mathrm{e}^{2\mathrm{i}c\pi} - 1$ and $s_4 = -\mathrm{e}^{-2\mathrm{i}c\pi}$. The boundary condition on ∂D_1 gives

$$W(z) = \zeta(z)^{\frac{c}{2}\sigma_3}\left(I + o\left(1\right)\right)\mathrm{e}^{\left(\frac{\zeta(z)^2}{2} + x\zeta(z)\right)\sigma_3}, \quad z \in \partial D_\beta.$$

Here we used that $\ell_x = \mathcal{O}(1)$ for $a = 1 + \mathcal{O}(1/\sqrt{N})$. According to page 34 of [9] (or page 182 of

[15]) the Riemann-Hilbert problem for the Painlevé IV parametrix Ψ, following the notation in [9], satisfies exactly the jump condition above and the boundary condition:

$$\Psi(\zeta, x) = \left(I + \frac{\Psi_{-1}(x)}{\zeta} + \frac{\Psi_{-2}(x)}{\zeta^2} + \mathcal{O}\left(\frac{1}{\zeta^3}\right)\right) \mathrm{e}^{\left(\frac{\zeta^2}{2}+x\zeta\right)} \zeta^{-\Theta_\infty \sigma_3}, \quad z \to \infty,$$

when

$$(1 + s_2 s_3)\mathrm{e}^{2\mathrm{i}\pi\Theta_\infty} + [s_1 s_4 + (1 + s_3 s_4)(1 + s_1 s_2)]\mathrm{e}^{-2\mathrm{i}\pi\Theta_\infty} = 2\cos 2\pi\Theta.$$

In our case we get $\Theta = c/2$, $\Theta_\infty = -c/2$. It means that, using the same strategy as in Sections 2.4 and 2.6, we could get a similar result regarding the asymptotics of orthogonal polynomials in terms of Painlevé IV equation:

$$\frac{d^2u}{dx^2} = \frac{1}{2u}\left(\frac{du}{dx}\right)^2 + \frac{3}{2}u^3 + 4xu^2 + (2 + 2x^2 - 4\Theta_\infty)u - \frac{8\Theta^2}{u},$$

where the solution u is related to the Riemann-Hilbert problem by

$$u(x) = -2x - \frac{d}{dx}\log\left((\Psi_{-1})(x)_{12}\right).$$

2.8 Lax pair: how the numerical calculation is done

Define $\widetilde{Y}(z)$ by $\widetilde{Y}(z) = \widetilde{Y}_n(z) = Y(z)\begin{bmatrix} \left(\frac{z-a}{z}\right)^c \frac{1}{\mathrm{e}^{Naz}} & 0 \\ 0 & z^n \end{bmatrix}$, then the Riemann-Hilbert problem for $\widetilde{Y}(z)$ is

$$\begin{cases} \widetilde{Y}(z) \text{ is holomorphic in } \mathbb{C}\setminus\Gamma, & \\ \widetilde{Y}_+(z) = \widetilde{Y}_-(z)\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, & z \in \Gamma, \\ \widetilde{Y}_+(z) = \widetilde{Y}_-(z)\begin{bmatrix} \mathrm{e}^{2c\pi\mathrm{i}} & 0 \\ 0 & 1 \end{bmatrix}, & z \in (0, a), \\ \widetilde{Y}(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right)\begin{bmatrix} \left(\frac{z-a}{z}\right)^c \frac{z^n}{\mathrm{e}^{Naz}} & 0 \\ 0 & 1 \end{bmatrix}, & z \to \infty. \end{cases}$$

We observe that $\widetilde{Y}_n(z)$ and $\widetilde{Y}_{n+1}(z)$ have the same jump matrices. Since $\det Y(z) \equiv 1$, the inverse of $\widetilde{Y}(z)$ exists in $\mathbb{C} \setminus (\Gamma \cup (0,a))$, and we can define

$$A_n(z) = \frac{\mathrm{d}\widetilde{Y}_n(z)}{\mathrm{d}z} \widetilde{Y}_n(z)^{-1}.$$

The matrix function $A_n(z)$ is meromorphic and can be determined by identifying the singularities. For $z \to \infty$, writing (we know that c_n below is not related to the charge "c" in the potential)

$$\widetilde{Y}_n(z) = \left(I + \frac{1}{z} \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix} + \cdots \right) \begin{bmatrix} \left(\frac{z-a}{z}\right)^c \frac{z^n}{\mathrm{e}^{Naz}} & 0 \\ 0 & 1 \end{bmatrix},$$

we get

$$A_n(z) = \begin{bmatrix} -Na & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{z} \begin{bmatrix} n & Nab_n \\ -Nac_n & 0 \end{bmatrix} + \mathcal{O}\left(z^{-2}\right).$$

Similarly we obtain the following for $z \to 0$:

$$\widetilde{Y}_n(z) = \begin{bmatrix} \alpha_n & \beta_n \\ \gamma_n & \eta_n \end{bmatrix} (I + \mathcal{O}(z)) \begin{bmatrix} \left(\frac{z-a}{z}\right)^c \frac{1}{\mathrm{e}^{Naz}} & 0 \\ 0 & z^n \end{bmatrix}$$

$$A_n(z) = \frac{1}{z} \begin{bmatrix} -c - (c+n)\beta_n\gamma_n & (c+n)\alpha_n\beta_n \\ -(c+n)\gamma_n\eta_n & n + (c+n)\beta_n\gamma_n \end{bmatrix}.$$

Therefore,

$$\begin{aligned} A_n(z) = {} & \begin{bmatrix} -Na & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{z} \begin{bmatrix} -c - (c+n)\beta_n\gamma_n & (c+n)\alpha_n\beta_n \\ -(c+n)\gamma_n\eta_n & n + (c+n)\beta_n\gamma_n \end{bmatrix} \\ & + \frac{1}{z-a} \begin{bmatrix} (c+n)\left(1 + \beta_n\gamma_n\right) & Nab_n - (c+n)\alpha_n\beta_n \\ -Nac_n + (c+n)\gamma_n\eta_n & -n - (c+n)\beta_n\gamma_n \end{bmatrix}. \end{aligned}$$

Defining $M_n(z) = \widetilde{Y}_{n+1}(z)\widetilde{Y}_n(z)^{-1}$ we obtain, by a similar procedure as above, that

$$M_n(z) = \begin{bmatrix} z + a_{n+1} - a_n & -b_n \\ c_{n+1} & 1 \end{bmatrix}.$$

The compatibility of the Lax pair,

$$\frac{\mathrm{d}\widetilde{Y}_n(z)}{\mathrm{d}z} = A_n(z)\widetilde{Y}_n(z),$$
$$\widetilde{Y}_{n+1}(z) = M_n(z)\widetilde{Y}_n(z),$$

gives

$$A_{n+1}(z)M_n(z) = \frac{\mathrm{d}M_n(z)}{\mathrm{d}z} + M_n(z)A_n(z).$$

This yields the following recurrence relation:

$$\begin{aligned}
a_{n+1} &= a_n + \frac{b_n\left(1+\beta_n\gamma_n\right)}{\alpha_n\beta_n}, \quad \alpha_{n+1} = \frac{b_n}{\beta_n}, \quad \gamma_{n+1} = -\frac{1}{\beta_n},\\
b_{n+1} &= \frac{(1+n+a^2N)b_n}{aN} + \frac{(c+n)\alpha_n\beta_n}{N} + \frac{b_n^2\left(1+\beta_n\gamma_n\right)}{\alpha_n\beta_n},\\
\beta_{n+1} &= \frac{\tilde{c}}{(1+c+n)\left((c+n)\alpha_n\beta_n - aNb_n\right)\alpha_n^2\beta_n},
\end{aligned}$$

where

$$\begin{aligned}
\tilde{c} &= a^2N - c - a(1+2(c+n))\alpha_n\beta_n + \Big(a^2N - c - a(c+n)\alpha_n\beta_n\Big)\beta_n\gamma_n\\
&\quad + (c+n)(c+n+1)\alpha_n^3\beta_n^3 + aN^2b_n^3\left(1+\beta_n\gamma_n\right)^2,\\
a_0 &= 0, \quad b_0 = a, \quad \alpha_0 = 1, \quad \beta_0 = 1+a^2N, \quad \gamma_0 = 0.
\end{aligned}$$

The last line contains the initial conditions for the recurrence relation. We used the above relations to generate the orthogonal polynomials numerically.

Chapter 3

Planar Orthogonal Polynomials As Type II Multiple Orthogonal Polynomials

3.1 Introduction

In this chapter, we consider the external potential $Q(z)$ given by

$$Q(z) = |z|^2 + 2\sum_{j=1}^{l} c_j \log \frac{1}{|z-a_j|},$$

where $\{a_1, a_2, \cdots, a_l\}$ is a set of nonzero complex numbers and $\{c_1, c_2, \cdots, c_l\}$ is a set of positive real numbers. Let $p_n(z)$ be the monic polynomial of degree n satisfying the orthogonality:

$$\int_{\mathbb{C}} p_n(z)\, \overline{p_m(z)}\, \mathrm{e}^{-|z|^2} |W(z)|^2 \,\mathrm{d}A(z) = h_n\, \delta_{nm}, \quad n, m \geq 0, \tag{3.1}$$

where $\mathrm{d}A$ is the Lebesgue area measure of the complex plane and h_n is the positive norming constant. We define, for $l \geq 1$, the multi-valued function W by

$$W(z) = \prod_{j=1}^{l} (z-a_j)^{c_j}, \quad z \in \mathbb{C}, \tag{3.2}$$

where $\{c_1, \cdots, c_l\}$ are positive real numbers and $\{a_1, \cdots, a_l\}$ are distinct points in $\mathbb{C}$.

The main result of this chapter is that our polynomials $\{p_n\}$ are multiple orthogonal polynomials of Type II. To introduce the main theorem, let us prepare several notations. To remove the unnecessary complication, we assume that a_j's are all nonzero and the arguments of a_j's are all different. Without loss of generality, we may assume:

$$0 \leq \arg a_1 < \cdots < \arg a_l < 2\pi. \tag{3.3}$$

To determine the branch of the multi-valued function W, we define the union of contours,

$$\mathbf{B} = \bigcup_{j=1}^{l} \mathbf{B}_j, \quad \mathbf{B}_j = \{a_j t : t \geq 1\}, \tag{3.4}$$

where the contours are directed towards the infinity. In the rest of the paper, we define $W : \mathbb{C} \setminus \mathbf{B} \to \mathbb{C}$ to be an analytic branch of (3.2). Let $\mathbf{B}^*$ and $\mathbf{B}_j^*$ be the complex-conjugate images of $\mathbf{B}$ and $\mathbf{B}_j$. Let $\overline{W} : \mathbb{C} \setminus \mathbf{B}^* \to \mathbb{C}$ be defined by

$$\overline{W}(z) = \overline{W(\bar{z})} = \prod_{j=1}^{l} (z - \bar{a}_j)^{c_j}. \tag{3.5}$$

Let $\mathbf{k} = (k_1, \cdots, k_l)$ with non–negative integers k_j's. When $\arg z \notin \{\arg a_1, \cdots, \arg a_l\}$, we define

$$\chi_{\mathbf{k}}(z) = W(z) \int_0^{\bar{z} \times \infty} \prod_{j=1}^{l} (s - \bar{a}_j)^{k_j} \overline{W}(s) \mathrm{e}^{-zs} \, \mathrm{d}s, \tag{3.6}$$

where the represented integration contour is $\{\bar{z}t | \, t \geq 0\}$.

Definition 3.1.1 *Let Γ be a simple closed curve with counterclockwise orientation, that connects $\{a_1, \cdots, a_l\}$, encloses the origin, and does not intersect $\mathbf{B} \setminus \{a_1, \cdots, a_l\}$. Explicitly, we may choose $\Gamma = \overline{a_1 a_2} \cup \cdots \cup \overline{a_{l-1} a_l} \cup \overline{a_l a_1}$ to be the union of l line segments.*

Definition 3.1.2 *Let $\mathbf{n} = (n_1, \cdots, n_l)$, where n_j's are non–negative integers. We define $p_{\mathbf{n}}(z)$ to be the monic polynomial of degree $|\mathbf{n}| := \sum_{j=1}^{l} n_j = n$ satisfying the orthogonality condition:*

$$\int_\Gamma p_{\mathbf{n}}(z) \, z^k \chi_{\mathbf{n}-\mathbf{e}_j}(z) \, \mathrm{d}z = 0, \quad 0 \leq k \leq n_j - 1, \quad 1 \leq j \leq l. \tag{3.7}$$

Here, $\mathbf{e}_j$ is the unit vector with one for the jth entry and zeros for all the other entries.

Definition 3.1.3 *We define $q_{\mathbf{n}}^{(i)}(z)$ to be the monic polynomial of degree $|\mathbf{n}| - 1$ satisfying the orthogonality condition:*

$$\int_\Gamma q_{\mathbf{n}}^{(i)}(z) \, z^k \chi_{\mathbf{n}-\mathbf{e}_j}(z) \, \mathrm{d}z = 0, \quad 0 \leq k \leq n_j - 1 - \delta_{ij}, \quad 1 \leq i, \, j \leq l.$$

Multiple orthogonal polynomials are related to Hermite–Padé approximation for a system of Markov functions. For type II Hermite–Padé approximation, we look for rational functions approx-

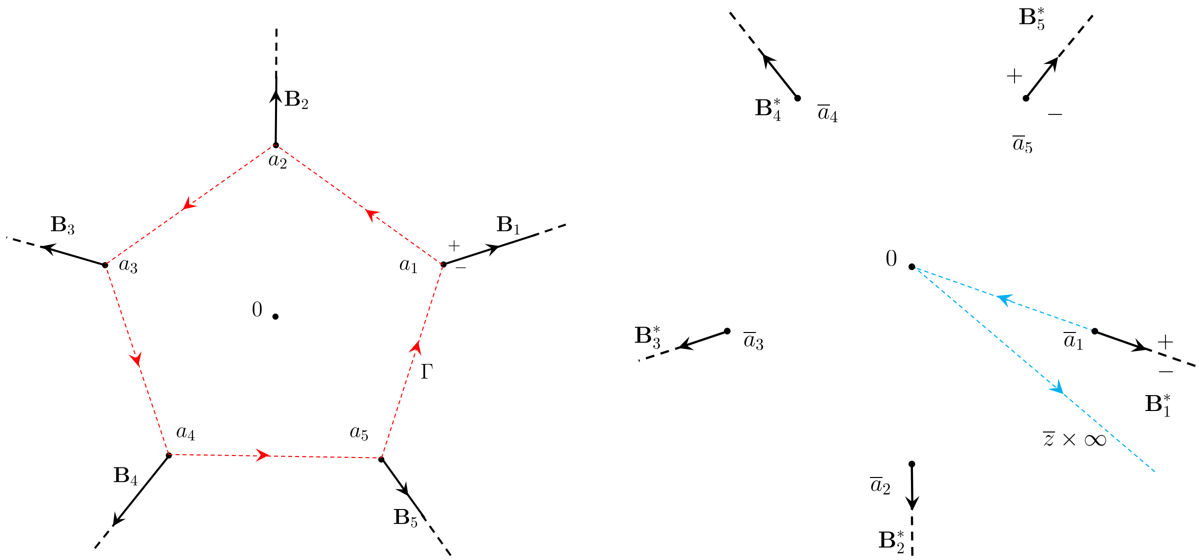

Figure 13.: Contours when $l = 5$. In the left are contours for $\mathbf{B}$ (black) and Γ (dotted red); In the right are the complex conjugate image of the right, and the integration contour for $\widetilde{\chi}_{\mathbf{k}}$ (dotted blue).

imating Markov functions near infinity, which consists of finding a polynomial $P_{\mathbf{n}}$ of degree $|\mathbf{n}|$ and polynomials $Q_{\mathbf{n},j}$ $(j = 1, \cdots, l)$ such that

$$P_{\mathbf{n}}(z)f_j(z) - Q_{\mathbf{n},j}(z) = \mathcal{O}\left(\frac{1}{z^{n_j+1}}\right), \quad z \to \infty,\ j = 1, \cdots, l,$$

where $f_1, \cdots, f_l$ are l Markov functions defined in our context by

$$f_j(z) = \int_\Gamma \frac{\chi_{\mathbf{n}-\mathbf{e}_j}(s)}{z - s}\,\mathrm{d}s, \quad z \notin \Gamma,\ j = 1 \cdots, l$$

while $Q_{\mathbf{n},j}(z)$ are defined by

$$Q_{\mathbf{n},j}(z) = \int_\Gamma \frac{(P_{\mathbf{n}}(z) - P_{\mathbf{n}}(s))\,\chi_{\mathbf{n}-\mathbf{e}_j}(s)}{z - s}\,\mathrm{d}s.$$

In our context, $P_{\mathbf{n}} = p_{\mathbf{n}}$. We now state the main results:

Theorem 3.1.4 *Given positive integers n and l, we define a non–negative integer κ and a non–negative integer $0 \le r < l$ such that $n = \kappa l + r$. Then,*

$$p_n(z) = p_{\mathbf{n}}(z),$$

where $\mathbf{n} = \mathbf{n}(n,l) = (\underbrace{\kappa+1,\cdots,\kappa+1}_{r},\underbrace{\kappa,\cdots,\kappa}_{l-r})$.

The next theorem is an immediate consequence. (A more general version is proved in [24].)

Theorem 3.1.5 *Let the* $(l+1)$ *by* $(l+1)$ *matrix function be given by*

$$Y(z) = \begin{array}{cc} \begin{bmatrix} p_{\mathbf{n}}(z) & \frac{1}{2\pi \mathrm{i}}\int_\Gamma \frac{p_{\mathbf{n}}(w)\chi_{\mathbf{n}-\mathbf{e}_1}(w)}{w-z}\mathrm{d}w & \cdots & \frac{1}{2\pi \mathrm{i}}\int_\Gamma \frac{p_{\mathbf{n}}(w)\chi_{\mathbf{n}-\mathbf{e}_l}(w)}{w-z}\mathrm{d}w \\ \vdots & \vdots & \vdots & \vdots \\ \gamma_j\, q_{\mathbf{n}}^{(j)}(z) & \frac{\gamma_j}{2\pi \mathrm{i}}\int_\Gamma \frac{q_{\mathbf{n}}^{(j)}(w)\chi_{\mathbf{n}-\mathbf{e}_1}(w)}{w-z}\mathrm{d}w & \cdots & \frac{\gamma_j}{2\pi \mathrm{i}}\int_\Gamma \frac{q_{\mathbf{n}}^{(j)}(w)\chi_{\mathbf{n}-\mathbf{e}_l}(w)}{w-z}\mathrm{d}w \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} & \begin{array}{l} \\ \\ \leftarrow (j+1)th\ row, \\ \\ \end{array} \end{array}$$

where the constant γ_j *in the* $(j+1)th$ *row is given by*

$$\gamma_j = -\left(\frac{1}{2\pi \mathrm{i}}\int_\Gamma q_{\mathbf{n}}^{(j)}(w)w^m\chi_{\mathbf{n}-\mathbf{e}_j}(w)\mathrm{d}w\right)^{-1}, \qquad m = \begin{cases} \kappa & \text{for} \quad 1 \le j \le r; \\ \kappa - 1 & \text{for} \quad r+1 \le j \le l, \end{cases}$$

is the unique solution of the Riemann-Hilbert problem given below.

$$\begin{cases} Y : \mathbb{C}\setminus\Gamma \to \mathbb{C}^{(l+1)\times(l+1)} \ \text{is a holomorphic matrix function,} \\ \\ Y_+(z) = Y_-(z)J(z) \ \text{on } \Gamma, \\ \\ Y(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right)\begin{bmatrix} z^n & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & z^{-(\kappa+1)}I_{r\times r} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & z^{-\kappa}I_{(l-r)\times(l-r)} \end{bmatrix}, \quad \text{as } z\to\infty, \end{cases}$$

the subscript $\pm$ *in* $Y_\pm$ *above represents the limiting value when approaching* Γ *from the corresponding sides of the directed contour, and*

$$J(z) = \begin{bmatrix} 1 & \chi_{\mathbf{n}-\mathbf{e}_1}(z) & \cdots & \chi_{\mathbf{n}-\mathbf{e}_l}(z) \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Remark. For $l=1$, the contour Γ is a closed curve around the origin passing through a_1. Since $\omega_{n,N}$ is the analytic on $\mathbb{C}\setminus[0,a]$, one can see that the jump contour Γ can be deformed to enclose the line segment $[0,a_1]$, to match the one in [2].

Let us define the moments

$$\begin{aligned}
\nu^{(i)}_{jk} &:= \frac{1}{2\mathrm{i}}\int_\Gamma z^{j+k}\,\chi_{\mathbf{n}-\mathbf{e}_i}(z)\,\mathrm{d}z = \frac{1}{2\mathrm{i}}\int_\Gamma z^{j+k}\,\widetilde{\chi}_{\mathbf{n}-\mathbf{e}_i}(z)\,\mathrm{d}z,\\
\mu_{jk} &:= \frac{1}{2\mathrm{i}}\int_\Gamma z^{j}\,\chi^\infty_k(z)\,\mathrm{d}z = \int_{\mathbb{C}} z^j\,\bar{z}^k\,\mathrm{e}^{-|z|^2}|W(z)|^2\,\mathrm{d}A(z).
\end{aligned} \tag{3.8}$$

Theorem 3.1.6 *Let n, l, κ, r and $\mathbf{n}=(\underbrace{\kappa+1,\cdots,\kappa+1}_{r},\underbrace{\kappa,\cdots,\kappa}_{l-r})$ be given as in Theorem 2.1.1. For $\nu^{(i)}_{jk}$ and μ_{jk} given above, set the $n\times n$ matrices of moments d_n and D_n to be*

$$d_n = \begin{bmatrix} & & \vdots & \\ \hdashline \nu^{(i)}_{0,0} & \nu^{(i)}_{1,0} & \cdots & \nu^{(i)}_{n-1,0}\\ \vdots & \vdots & \vdots & \vdots\\ \nu^{(i)}_{0,n_j-1} & \nu^{(i)}_{1,n_j-1} & \cdots & \nu^{(i)}_{n-1,n_j-1}\\ \hdashline & & \vdots & \end{bmatrix},\quad D_n = \begin{bmatrix} \mu_{0,0} & \mu_{1,0} & \cdots & \mu_{n-1,0}\\ \mu_{0,1} & \mu_{1,1} & \cdots & \mu_{n-1,1}\\ \vdots & \vdots & \ddots & \vdots\\ \mu_{0,n-1} & \mu_{1,n-1} & \cdots & \mu_{n-1,n-1}\end{bmatrix},$$

where

$$n_j = \begin{cases} \kappa+1 & \text{for} \quad 1\le j\le r;\\ \kappa & \text{for} \quad r+1\le j\le l.\end{cases}$$

Then there exists a unique constant matrix A_n such that $d_n = A_nD_n$. Moreover it satisfies

$$\begin{aligned}
\det A_n &= (-1)^{n(n-1)/2}\left(\prod_{i=1}^{l}\prod_{j=1}^{n_i-1}(c_i+j)^j\right)\prod_{i<j}(\bar{a}_j-\bar{a}_i)^{n_in_j}\\
&= (-1)^{n(n-1)/2}\left(\prod_{i=1}^{l}\prod_{j=1}^{\kappa-1}(c_i+j)^j\right)\left(\prod_{i=1}^{r}(c_i+\kappa)^\kappa\right)\prod_{1\le i<j\le l}(\bar{a}_j-\bar{a}_i)^{\kappa^2}\\
&\times \prod_{1\le i<j\le r}(\bar{a}_j-\bar{a}_i)^{2\kappa+1}\prod_{j=r+1}^{l}\prod_{i=1}^{r}(\bar{a}_j-\bar{a}_i)^{\kappa}.
\end{aligned} \tag{3.9}$$

Theorem 3.1.5 provides a way to study such planar orthogonal polynomials by the nonlinear steepest descent analysis of matrix Riemann–Hilbert problem, see [2, 7, 25, 28]. Theorem 3.1.6 suggests that the partition function of the corresponding Coulomb Gas system (see [30] and the reference therein) can be calculated using the tau–function from the Riemann-Hilbert problem [6]. This is currently work in progress.

3.2 Proof of Theorem 3.1.4

3.2.1 Area Integral via Contour Integral

The following definitions will be useful.

$$\begin{aligned} \chi_m(z) &:= W(z) \int_0^{\bar z} s^m \overline{W}(s)\, \mathrm{e}^{-zs} \mathrm{d}s, \\ \chi_m^\infty(z) &:= W(z) \int_0^{\bar z \times \infty} s^m \overline{W}(s)\, \mathrm{e}^{-zs} \mathrm{d}s. \end{aligned} \tag{3.10}$$

Both are well defined if $\arg z \neq \arg a_j$ for all j. The following lemma holds.

Lemma 3.2.1 *Let* $\mathbf{S} = \bigcup_{j=1}^{l} \mathbf{S}_j$ *where* $\mathbf{S}_j = \{a_j t : 0 \le t \le 1\}$. $\chi_m^\infty(z) - \chi_m(z)$ *has continuous extension to* $\mathbb{C} \setminus \mathbf{S}$ *and, given* $k > 0$, *there exists* $C > 0$ *such that*

$$|z^k|\, |\chi_m^\infty(z) - \chi_m(z)| \le C\, \mathrm{e}^{-(|z|-1)^2} \tag{3.11}$$

for all z *such that* $|z| > 2$.

Proof. It is enough to check the continuity on $\mathbf{B}_1 \setminus \{a_1\}$. The piecewise analytic functions, W and $\overline{W}$, satisfy the following jump conditions,

$$\begin{aligned} W_+(z) &= \mathrm{e}^{-2\pi \mathrm{i} c_j} W_-(z), \quad z \in \mathbf{B}_j, \\ \overline{W}_+(z) &= \mathrm{e}^{-2\pi \mathrm{i} c_j} \overline{W}_-(z), \quad z \in \mathbf{B}_j^*. \end{aligned} \tag{3.12}$$

Here, the subscripts $\pm$ stand for the boundary values taken on $\pm$ sides of $\mathbf{B}$; we assign $\pm$ sides at each point of $\mathbf{B} \setminus \{a_1, a_2, \cdots, a_l\}$ and $\mathbf{B}^* \setminus \{\bar a_1, \bar a_2, \cdots, \bar a_l\}$ in a standard way, see Figure 13.

Let $p \in \mathbf{B}_1 \setminus \{a_1\}$. Note that when z approaches p from " + " side of $\mathbf{B}_1$, $\bar z$ approaches $\mathbf{B}_1^*$ from " − " side. Then, we have

$$\begin{aligned} [\chi_m^\infty(p) - \chi_m(p)]_+ &= [W(p)]_+ \int_{\bar p}^{\bar p \times \infty} s^m \left[\overline{W}(s)\right]_- \mathrm{e}^{-ps} \mathrm{d}s \\ &= [W(p)]_- \textstyle\int_{\bar p}^{\bar p \times \infty} s^m \left[\overline{W}(s)\right]_+ \mathrm{e}^{-ps} \mathrm{d}s \\ &= [\chi_m^\infty(p) - \chi_m(p)]_-, \end{aligned} \tag{3.13}$$

where we used (3.12) in the second equality. This proves the continuity statement. To prove the statement about the bound, we use the elementary estimate that, given $k>0$, there exists $C>0$ such that

$$|z^k|\,|W(z)| \le C\,\mathrm{e}^{|z|}$$

for all $z\in\mathbb{C}$. Then, for some $C>0$ and $|z|>2$, we get

$$\begin{aligned}\left|z^k\left(\chi_m^\infty(z)-\chi_m(z)\right)\right| &= \left|z^k W(z)\int_{\bar z}^{\bar z\times\infty} s^m\overline{W}(s)\,\mathrm{e}^{-zs}\mathrm{d}s\right| \le C\,\mathrm{e}^{|z|}\int_{\bar z}^{\bar z\times\infty}\mathrm{e}^{|s|}\,\mathrm{e}^{-zs}|\mathrm{d}s| \\ &\le C\,\mathrm{e}^{|z|}\left|\int_{|\bar z|}^{\infty}\mathrm{e}^{x}\,\mathrm{e}^{-|z|x}\mathrm{d}x\right| = C\,\frac{\mathrm{e}^{-|z|^2+2|z|}}{|z|-1}\le \widetilde{C}\,\mathrm{e}^{-(|z|-1)^2}.\end{aligned}\tag{3.14}$$

□

Proposition 3.2.2 *For an arbitrary polynomial $p(z)$ we have the following identity:*

$$\int_{\mathbb{C}} p(z)\,\bar z^m\,\mathrm{e}^{-|z|^2}|W(z)|^2\,\mathrm{d}A(z) = \frac{1}{2\mathrm{i}}\int_\Gamma p(z)\,\chi_m^\infty(z)\,\mathrm{d}z. \tag{3.15}$$

Proof. We apply Green's theorem to change the integral over $\mathbb{C}$ to the integral over a contour. First we observe that

$$\bar z^m\,|W(z)|^2\mathrm{e}^{-|z|^2} = \frac{\partial\chi_m(z)}{\partial\bar z},\qquad z\in\mathbb{C}\setminus\mathbf{B}. \tag{3.16}$$

Therefore, defining $D_R := \{z\,|\,|z|<R\}$, we get

$$\begin{aligned}\int_{\mathbb{C}} p(z)\,\bar z^m\,|W(z)|^2\mathrm{e}^{-|z|^2}\,\mathrm{d}A(z) &= \lim_{R\to\infty}\int_{D_R} p(z)\,\bar z^m\,|W(z)|^2\mathrm{e}^{-|z|^2}\,\mathrm{d}A(z)\\ &= \lim_{R\to\infty}\int_{D_R\setminus\mathbf{B}} p(z)\,\frac{\partial\chi_m(z)}{\partial\bar z}\,\mathrm{d}A(z)\\ &= \lim_{R\to\infty}\frac{1}{2\mathrm{i}}\left(\int_{\partial D_R} p(z)\,\chi_m(z)\,\mathrm{d}z + \sum_{j=1}^m\int_{\mathbf{B}_j\cap D_R} p(z)\,[\chi_m(z)]_-^+\mathrm{d}z\right),\end{aligned}\tag{3.17}$$

where we used Green's theorem in the last equality.

Since $\chi_m^{(\infty)}(z)$ is analytic in $\mathbb{C}\setminus(\mathbf{S}\cup\mathbf{B})$, by a deformation of the contour, we get the identity

$$\int_\Gamma p(z)\,\chi_m^\infty(z)\,\mathrm{d}z = \int_{\partial D_R} p(z)\,\chi_m^\infty(z)\,\mathrm{d}z + \sum_{j=1}^m\int_{\mathbf{B}_j\cap D_R} p(z)\,[\chi_m(z)]_-^+\mathrm{d}z \tag{3.18}$$

Using this identity, the right hand side of (3.17) becomes

$$\lim_{R\to\infty} \frac{1}{2\mathrm{i}} \int_{\partial D_R} p(z)\, (\chi_m(z) - \chi_m^{\infty}(z))\, \mathrm{d}z + \frac{1}{2\mathrm{i}} \int_{\Gamma} p(z)\, \chi_m^{\infty}(z)\, \mathrm{d}z = \frac{1}{2\mathrm{i}} \int_{\Gamma} p(z)\, \chi_m^{\infty}(z)\, \mathrm{d}z, \tag{3.19}$$

where the last equality holds because of (3.11) in Lemma 3.2.1. This proves Proposition 3.2.2. □

3.2.2 Several Lemmas

Definition 3.2.3 *All the vectors in this section have only non-negative entries. For two vectors,* $\mathbf{k}$ *and* $\mathbf{s}$*, we say* $\mathbf{k} \geq \mathbf{s}$ *if* $\mathbf{k} - \mathbf{s}$ *has only non-negative entries. If, in addition,* $\mathbf{k} \neq \mathbf{s}$ *then we say* $\mathbf{k} > \mathbf{s}$*. The* j*th entry of* $\mathbf{k}$ *is denoted by* $[\mathbf{k}]_j$*. As before, the length of a vector* $|\mathbf{k}| = [\mathbf{k}]_1 + \cdots + [\mathbf{k}]_l$*.*

Lemma 3.2.4 *For any* $n \geq 1$*, we have*

$$\textit{span}\{\chi_j^{\infty} :\, 0 \leq j < n\} = \textit{span}\{\chi_{\mathbf{k}} :\, |\mathbf{k}| < n\}. \tag{3.20}$$

Proof. For $n = 0$, the lemma holds because $\chi_0^{\infty}(z) = \chi_{\mathbf{0}}(z)$. Assume that the lemma holds for $n = n_0$. If $|\mathbf{k}| = n_0 + 1$ we have

$$\begin{aligned} \chi_{\mathbf{k}}(z) - \chi_{n_0+1}^{\infty}(z) &= W(z) \int_0^{\bar{z}\times\infty} \prod_{j=1}^{l} (s - \bar{a}_j)^{k_j}\, \overline{W}(s) \mathrm{e}^{-zs} \mathrm{d}s - W(z) \int_0^{\bar{z}\times\infty} s^{n_0+1}\, \overline{W}(s) \mathrm{e}^{-zs} \mathrm{d}s \\ &= W(z) \int_0^{\bar{z}\times\infty} \{\text{polynomial in s of degree } \leq n_0\} \times \overline{W}(s) \mathrm{e}^{-zs} \mathrm{d}s. \end{aligned} \tag{3.21}$$

Since the last term belongs to both spans in (3.20) for $n = n_0$, $\chi_{\mathbf{k}}$ belongs to the left span in (3.20) with $n = n_0 + 1$ and $\chi_{n_0+1}^{\infty}$ belongs to the right span in (3.20) with $n = n_0 + 1$. □

To prove $p_n = p_{\mathbf{n}}$, one may try to show that

$$\text{span}\,\{\chi_j^{\infty}(z) |\, 0 \leq j < n\} = \text{span}\,\{z^k \chi_{\mathbf{n}-\mathbf{e}_j}\, |\, 0 \leq k < [\mathbf{n}]_j, 1 \leq j \leq l\}. \tag{3.22}$$

In fact, it is enough to show that the above equality up to functions ψ that satisfies $\langle p, \psi \rangle = 0$ for all polynomial p. For example, we have $\langle p, \psi \rangle = 0$ for

$$\psi(z) = W(z) \int_0^{\bar{a}_1} \prod_{j=1}^{l} (s - \bar{a}_j)^{k_j}\, \overline{W}(s) \mathrm{e}^{-zs} \mathrm{d}s. \tag{3.23}$$

Since ψ is analytic in $\mathbb{C} \setminus \mathbf{B}$ and, therefore, the integration contour in $\int_{\Gamma} p(z)\psi(z) dz$ is contractible to a point. This allows us to consider, instead of $\chi_{\mathbf{k}}$ in (3.22),

$$\widetilde{\chi}_{\mathbf{k}} := \chi_{\mathbf{k}} - W(z)\int_0^{\bar{a}_1}\prod_{j=1}^{l}(s-\bar{a}_j)^{k_j}\overline{W}(s)\mathrm{e}^{-zs}\mathrm{d}s.$$

As a result, using Lemma 3.2.4, the proof of Theorem 2.1.1 is reduced to proving the following Proposition.

Proposition 3.2.5 *For any $n \geq 1$ and $l \geq 1$, let $\mathbf{n}$ be as in Theorem 2.1.1. Then the following holds.*

$$\mathit{span}\{\widetilde{\chi}_{\mathbf{k}}(z) :\, |\mathbf{k}| < n\} = \mathit{span}\{z^k\widetilde{\chi}_{\mathbf{n}-\mathbf{e}_j}(z)\,|\,0 \leq k < [\mathbf{n}]_j, 1 \leq j \leq l\}. \tag{3.24}$$

The proof of this proposition will be in the next subsection. The following Lemma justifies why it is useful to use $\widetilde{\chi}_{\mathbf{k}}$ instead of $\chi_{\mathbf{k}}$.

Lemma 3.2.6

$$z\widetilde{\chi}_{\mathbf{k}}(z) = \sum_{j=1}^{l}(c_j+k_j)\widetilde{\chi}_{\mathbf{k}-\mathbf{e}_j}(z). \tag{3.25}$$

Proof. We have

$$\begin{aligned}
0 &= W(z)\left[\prod_{j=1}^{l}(s-\bar{a}_j)^{c_j+k_j}\mathrm{e}^{-zs}\right]\Bigg|_{\bar{a}_1}^{\bar{z}\times\infty} \\
&= W(z)\int_{\bar{a}_1}^{\bar{z}\times\infty}\partial_s\left[\prod_{j=1}^{l}(s-\bar{a}_j)^{c_j+k_j}\mathrm{e}^{-zs}\right]\mathrm{d}s \\
&= W(z)\int_{\bar{a}_1}^{\bar{z}\times\infty}\left(\sum_{j=1}^{l}\frac{c_j+k_j}{s-\bar{a}_j}-z\right)\prod_{j=1}^{l}(s-\bar{a}_j)^{c_j+k_j}\mathrm{e}^{-zs}\,\mathrm{d}s \\
&= \sum_{j=1}^{l}(c_j+k_j)\widetilde{\chi}_{\mathbf{k}-\mathbf{e}_j}(z) - z\widetilde{\chi}_{\mathbf{k}}(z).
\end{aligned}$$

□

Corollary 3.2.7 *Let $\mathbf{k} = (k_1, k_2, \cdots, k_l)$ and $s \leq \min\{k_j\}_{j=1}^{l}$ be a positive integer. Then $z^s\widetilde{\chi}_{\mathbf{k}}(z)$ can be represented as a linear combination of $\{\widetilde{\chi}_{\mathbf{k}-\mathbf{s}}(z)|\,|\mathbf{s}| = s\}$. Furthermore, the coefficient of $\widetilde{\chi}_{\mathbf{k}-s\mathbf{e}_m}(z)$ is nonzero for all $1 \leq m \leq l$.*

Proof. The corollary is true when $s = 1$ by Lemma 3.2.6. Assume, for some $1 \leq s < \min\{k_j\}_{j=1}^{l}$, that $z^s\widetilde{\chi}_{\mathbf{k}}(z)$ is a linear combination of $\widetilde{\chi}_{\mathbf{k}-\mathbf{s}}(z)$ for $|\mathbf{s}| = s$ and the coefficient of $\{\widetilde{\chi}_{\mathbf{k}-s\mathbf{e}_m}(z)\}_{m=1}^{l}$ are all non-vanishing.

Then $z^{s+1}\widetilde{\chi}_{\mathbf{k}}(z)$ is a linear combination of $z\widetilde{\chi}_{\mathbf{k}-\mathbf{s}}(z)$ and, therefore, of $\widetilde{\chi}_{\mathbf{k}-\mathbf{s}-\mathbf{e}_m}(z)$ with $|\mathbf{s}| = s$ and $1 \leq m \leq l$. Since the term $\widetilde{\chi}_{\mathbf{k}-(s+1)\mathbf{e}_m}(z)$ comes only from $z\widetilde{\chi}_{\mathbf{k}-s\mathbf{e}_m}(z)$ and since the coefficient of $\widetilde{\chi}_{\mathbf{k}-s\mathbf{e}_m}(z)$ is non-zero, the coefficient at $\widetilde{\chi}_{\mathbf{k}-(s+1)\mathbf{e}_m}(z)$ is non-zero. Note that all the coefficients in the right hand side of (3.25) are non-zero. By induction, this ends the proof. □

Lemma 3.2.8 *For $n \neq m$, we have*

$$\widetilde{\chi}_{\mathbf{k}+\mathbf{e}_n}(z) - \widetilde{\chi}_{\mathbf{k}+\mathbf{e}_m}(z) + (\bar{a}_n - \bar{a}_m)\,\widetilde{\chi}_{\mathbf{k}}(z) = 0. \tag{3.26}$$

Proof. Since

$$(s - \bar{a}_n) - (s - \bar{a}_m) + (\bar{a}_n - \bar{a}_m) = 0,$$

we obtain,

$$0 = W(z)\int_{\bar{a}_1}^{\bar{z}\times\infty} [(s - \bar{a}_n) - (s - \bar{a}_m) + (\bar{a}_n - \bar{a}_m)] \prod_{j=1}^{l} (s - \bar{a}_j)^{c_j + k_j} \mathrm{e}^{-zs} \mathrm{d}s.$$

By the definition of $\widetilde{\chi}_{\mathbf{k}}(z)$, (3.26) holds. □

3.2.3 Proof of Proposition 3.2.5

By Corollary 3.2.7, we have $\supset$. To prove the opposite inclusion, we note that any vector $\mathbf{k}$ can be uniquely represented as

$$\mathbf{k} = \mathbf{n} + \mathbf{m} - \mathbf{s},$$

where $[\mathbf{m}]_j[\mathbf{s}]_j = 0$, i.e., $\mathbf{m}$ and $\mathbf{s}$ cannot be both non-vanishing in any of the entries. It is then enough to show the following claim.

Claim: For all $\mathbf{s} \leq \mathbf{n}$ and $\mathbf{m}$ satisfying $|\mathbf{n} + \mathbf{m} - \mathbf{s}| < n$,

$$\widetilde{\chi}_{\mathbf{n}+\mathbf{m}-\mathbf{s}} \in \operatorname{span}\{z^k \widetilde{\chi}_{\mathbf{n}-\mathbf{e}_j}(z) \,|\, 0 \leq k < [\mathbf{n}]_j, 1 \leq j \leq l\}.$$

We prove this claim in two steps.

Step 1: For all $\mathbf{0} < \mathbf{s} \leq \mathbf{n}$, $\widetilde{\chi}_{\mathbf{n}-\mathbf{s}} \in \operatorname{span}\{z^k \widetilde{\chi}_{\mathbf{n}-\mathbf{e}_j}(z) \,|\, 0 \leq k < [\mathbf{n}]_j, 1 \leq j \leq l\}$. If $|\mathbf{s}| = 1$ then the inclusion is immediate. Let the inclusion hold for $|\mathbf{s}| \leq m - 1$, for some $m < n$. (If $m \geq n$ then there is nothing to prove.) Below we claim that the inclusion holds for $|\mathbf{s}| = m$, which proves Step 1 by induction.

1. If $\mathbf{s}$ has more than one non-zero entries, i.e., $[\mathbf{s}]_i \neq 0$ and $[\mathbf{s}]_j \neq 0$,

$$\widetilde{\chi}_{\mathbf{n}-\mathbf{s}}(z) = \frac{1}{\bar{a}_i - \bar{a}_j}\left(\widetilde{\chi}_{\mathbf{n}-\mathbf{s}+\mathbf{e}_j}(z) - \widetilde{\chi}_{\mathbf{n}-\mathbf{s}+\mathbf{e}_i}(z)\right).$$

   The left hand side belongs to the span in Claim since the right hand side does so by assumption.

2. Assume $\mathbf{s}$ has exactly one non-zero entry, i.e., $\mathbf{s} = m\mathbf{e}_j$ for some j. From $\mathbf{s} < \mathbf{n}$ we have $m \leq [\mathbf{n}]_j$. Since $z^{m-1}\widetilde{\chi}_{\mathbf{n}-\mathbf{e}_j}(z)$ is a linear combination of $\{\widetilde{\chi}_{\mathbf{n}-\widetilde{\mathbf{s}}} : |\widetilde{\mathbf{s}}| = m\}$ where the term $\widetilde{\chi}_{\mathbf{n}-m\mathbf{e}_j}$ appears with non-zero coefficient (see Corollary 3.2.7), and since all the other terms in the linear combination belongs to the span in item 1, $\widetilde{\chi}_{\mathbf{n}-m\mathbf{e}_j}$ also belongs to the span in Claim.

Step 2: Step 1 showed Claim for $|\mathbf{m}| = 0$. Assume that Claim is true when $|\mathbf{m}| \leq k-1$. We will show that Claim holds when $|\mathbf{m}| \leq k$, i.e. $\widetilde{\chi}_{\mathbf{n}+\mathbf{m}-\mathbf{s}}$ belongs to the span in Claim for $|\mathbf{m}| = k$. Let $\mathbf{m}$ satisfy $|\mathbf{m}| = k \geq 1$. There exists j such that $[\mathbf{m}]_j > 0$. Then $\widetilde{\chi}_{\mathbf{n}+(\mathbf{m}-\mathbf{e}_j)-\mathbf{s}}$ belongs to the span in the claim by the assumption. Since $|\mathbf{n} + (\mathbf{m} - \mathbf{e}_j) - \mathbf{s}| < n - 1$ we have $|\mathbf{s}| > 0$ and there exists $i \neq j$ such that $[\mathbf{s}]_i > 0$. Then $\widetilde{\chi}_{\mathbf{n}+(\mathbf{m}-\mathbf{e}_j)-(\mathbf{s}-\mathbf{e}_i)}$ also belongs to the span by the assumption. Since, by Lemma 3.2.8, we have

$$\widetilde{\chi}_{\mathbf{n}+\mathbf{m}-\mathbf{s}} = \widetilde{\chi}_{\mathbf{n}+(\mathbf{m}-\mathbf{e}_j)-(\mathbf{s}-\mathbf{e}_i)} + (\bar{a}_i - \bar{a}_j)\widetilde{\chi}_{\mathbf{n}+(\mathbf{m}-\mathbf{e}_j)-\mathbf{s}},$$

the left hand side belongs to the span. This ends the proof of Proposition 3.2.5 and Theorem 2.1.1.

3.3 Proof of Theorem 3.1.6

Proof. Since $\det D_n = \prod_{j=0}^{n-1} h_j > 0$ where h_j is defined in (3.1), D_n is an invertible matrix and this proves the existence and the uniqueness of A_n. In the remainder of the proof, we will construct

A_n using induction. Let us consider the jth column of d_n,

$$\begin{bmatrix} \nu_{j,0}^{(1)} \\ \nu_{j,1}^{(1)} \\ \vdots \\ \nu_{j,n_1-1}^{(1)} \\ \vdots \\ \nu_{j,0}^{(l)} \\ \nu_{j,1}^{(l)} \\ \vdots \\ \nu_{j,n_l-1}^{(l)} \end{bmatrix} = \frac{1}{2\mathrm{i}} \int_\Gamma z^j \, V_{\mathbf{n}}(z) \, \mathrm{d}z, \quad \text{where } V_{\mathbf{n}} = V_{\mathbf{n}}(z) = \begin{bmatrix} \chi_{\mathbf{n}-\mathbf{e}_1} \\ z\chi_{\mathbf{n}-\mathbf{e}_1} \\ \vdots \\ z^{n_1-1}\chi_{\mathbf{n}-\mathbf{e}_1} \\ \vdots \\ \chi_{\mathbf{n}-\mathbf{e}_l} \\ z\chi_{\mathbf{n}-\mathbf{e}_l} \\ \vdots \\ z^{n_l-1}\chi_{\mathbf{n}-\mathbf{e}_l} \end{bmatrix}.$$

We will find a constant $(n+1)\times(n+1)$ matrix B_n such that, for all z,

$$B_n V_{\mathbf{n}+\mathbf{e}_{r+1}}(z) = \left[\begin{array}{c} \chi_{\mathbf{n}}(z) \\ \hdashline V_{\mathbf{n}}(z) \end{array}\right].$$

This means that

$$B_n d_{n+1} = \left[\begin{array}{cccc:c} \nu_{0,0} & \nu_{1,0} & \cdots & \nu_{n-1,0} & \nu_{n,0} \\ \hdashline & & & & \nu_{n,0}^{(1)} \\ & & & & \vdots \\ & & & & \nu_{n,n_1-1}^{(1)} \\ & & d_n = A_n D_n & & \vdots \\ & & & & \nu_{n,0}^{(l)} \\ & & & & \vdots \\ & & & & \nu_{n,n_l-1}^{(l)} \end{array}\right],$$

where $\nu_{j,0}$ is given by $\nu_{j,0} = \dfrac{1}{2\mathrm{i}} \displaystyle\int_\Gamma z^j \, \chi_{\mathbf{n}}(z) \, \mathrm{d}z$. The matrix B_n can be obtained by three successive linear transformations on $V_{\mathbf{n}+\mathbf{e}_{r+1}}$ that we describe below.

$$\begin{bmatrix} \chi_{\mathbf{n}+\mathbf{e}_{r+1}-\mathbf{e}_1} \\ z\chi_{\mathbf{n}+\mathbf{e}_{r+1}-\mathbf{e}_1} \\ \vdots \\ z^{\kappa}\chi_{\mathbf{n}+\mathbf{e}_{r+1}-\mathbf{e}_1} \\ \hdashline \vdots \\ \hdashline \chi_{\mathbf{n}+\mathbf{e}_{r+1}-\mathbf{e}_r} \\ z\chi_{\mathbf{n}+\mathbf{e}_{r+1}-\mathbf{e}_r} \\ \vdots \\ z^{\kappa}\chi_{\mathbf{n}+\mathbf{e}_{r+1}-\mathbf{e}_r} \\ \hdashline \chi_{\mathbf{n}} \\ z\chi_{\mathbf{n}} \\ \vdots \\ z^{\kappa}\chi_{\mathbf{n}} \\ \hdashline \chi_{\mathbf{n}+\mathbf{e}_{r+1}-\mathbf{e}_{r+2}} \\ z\chi_{\mathbf{n}+\mathbf{e}_{r+1}-\mathbf{e}_{r+2}} \\ \vdots \\ z^{\kappa-1}\chi_{\mathbf{n}+\mathbf{e}_{r+1}-\mathbf{e}_{r+2}} \\ \hdashline \vdots \\ \hdashline \chi_{\mathbf{n}+\mathbf{e}_{r+1}-\mathbf{e}_l} \\ z\chi_{\mathbf{n}+\mathbf{e}_{r+1}-\mathbf{e}_l} \\ \vdots \\ z^{\kappa-1}\chi_{\mathbf{n}+\mathbf{e}_{r+1}-\mathbf{e}_l} \end{bmatrix} \xrightarrow{(A)} \begin{bmatrix} \chi_{\mathbf{n}-\mathbf{e}_1} \\ z\chi_{\mathbf{n}-\mathbf{e}_1} \\ \vdots \\ z^{\kappa}\chi_{\mathbf{n}-\mathbf{e}_1} \\ \hdashline \vdots \\ \hdashline \chi_{\mathbf{n}-\mathbf{e}_r} \\ z\chi_{\mathbf{n}-\mathbf{e}_r} \\ \vdots \\ z^{\kappa}\chi_{\mathbf{n}-\mathbf{e}_r} \\ \hdashline \chi_{\mathbf{n}} \\ z\chi_{\mathbf{n}} \\ \vdots \\ z^{\kappa}\chi_{\mathbf{n}} \\ \hdashline \chi_{\mathbf{n}-\mathbf{e}_{r+2}} \\ z\chi_{\mathbf{n}-\mathbf{e}_{r+2}} \\ \vdots \\ z^{\kappa-1}\chi_{\mathbf{n}-\mathbf{e}_{r+2}} \\ \hdashline \vdots \\ \hdashline \chi_{\mathbf{n}-\mathbf{e}_l} \\ z\chi_{\mathbf{n}-\mathbf{e}_l} \\ \vdots \\ z^{\kappa-1}\chi_{\mathbf{n}-\mathbf{e}_l} \end{bmatrix} \xrightarrow{(B)} \begin{bmatrix} \chi_{\mathbf{n}-\mathbf{e}_1} \\ z\chi_{\mathbf{n}-\mathbf{e}_1} \\ \vdots \\ z^{\kappa}\chi_{\mathbf{n}-\mathbf{e}_1} \\ \hdashline \vdots \\ \hdashline \chi_{\mathbf{n}-\mathbf{e}_r} \\ z\chi_{\mathbf{n}-\mathbf{e}_r} \\ \vdots \\ z^{\kappa}\chi_{\mathbf{n}-\mathbf{e}_r} \\ \hdashline \chi_{\mathbf{n}} \\ \chi_{\mathbf{n}-\mathbf{e}_{r+1}} \\ \vdots \\ z^{\kappa-1}\chi_{\mathbf{n}-\mathbf{e}_{r+1}} \\ \hdashline \chi_{\mathbf{n}-\mathbf{e}_{r+2}} \\ z\chi_{\mathbf{n}-\mathbf{e}_{r+2}} \\ \vdots \\ z^{\kappa-1}\chi_{\mathbf{n}-\mathbf{e}_{r+2}} \\ \hdashline \vdots \\ \hdashline \chi_{\mathbf{n}-\mathbf{e}_l} \\ z\chi_{\mathbf{n}-\mathbf{e}_l} \\ \vdots \\ z^{\kappa-1}\chi_{\mathbf{n}-\mathbf{e}_l} \end{bmatrix} \xrightarrow{(C)} \begin{bmatrix} \chi_{\mathbf{n}} \\ \chi_{\mathbf{n}-\mathbf{e}_1} \\ z\chi_{\mathbf{n}-\mathbf{e}_1} \\ \vdots \\ z^{\kappa}\chi_{\mathbf{n}-\mathbf{e}_1} \\ \hdashline \vdots \\ \hdashline \chi_{\mathbf{n}-\mathbf{e}_r} \\ z\chi_{\mathbf{n}-\mathbf{e}_r} \\ \vdots \\ z^{\kappa}\chi_{\mathbf{n}-\mathbf{e}_r} \\ \hdashline \chi_{\mathbf{n}-\mathbf{e}_{r+1}} \\ \vdots \\ z^{\kappa-1}\chi_{\mathbf{n}-\mathbf{e}_{r+1}} \\ \hdashline \chi_{\mathbf{n}-\mathbf{e}_{r+2}} \\ z\chi_{\mathbf{n}-\mathbf{e}_{r+2}} \\ \vdots \\ z^{\kappa-1}\chi_{\mathbf{n}-\mathbf{e}_{r+2}} \\ \hdashline \vdots \\ \hdashline \chi_{\mathbf{n}-\mathbf{e}_l} \\ z\chi_{\mathbf{n}-\mathbf{e}_l} \\ \vdots \\ z^{\kappa-1}\chi_{\mathbf{n}-\mathbf{e}_l} \end{bmatrix},$$

Above, each arrow means the linear transformation given by

$$\begin{aligned} B_n^{(1)} \text{ LHS of (A)} &= \text{RHS of (A)}, \\ B_n^{(2)} \text{ LHS of (B)} &= \text{RHS of (B)}, \\ B_n^{(3)} \text{ LHS of (C)} &= \text{RHS of (C)}, \end{aligned}$$

$$B_n^{(1)} = \left[\begin{array}{ccc|c|ccc}
\frac{I_{\kappa+1}}{\bar{a}_1-\bar{a}_{r+1}} & \cdots & \mathbf{0} & -\frac{I_{\kappa+1}}{\bar{a}_1-\bar{a}_{r+1}} & & & \\
\vdots & \ddots & \vdots & \vdots & & \mathbf{0} & \\
\mathbf{0} & \cdots & \frac{I_{\kappa+1}}{\bar{a}_r-\bar{a}_{r+1}} & -\frac{I_{\kappa+1}}{\bar{a}_r-\bar{a}_{r+1}} & & & \\
\hline
 & \mathbf{0} & & I_{\kappa+1} & & \mathbf{0} & \\
\hline
 & & & -\frac{I_\kappa}{\bar{a}_{r+2}-\bar{a}_{r+1}} & \frac{I_\kappa}{\bar{a}_{r+2}-\bar{a}_{r+1}} & \cdots & \mathbf{0} \\
 & \mathbf{0} & & \vdots & \vdots & \ddots & \vdots \\
 & & & -\frac{I_\kappa}{\bar{a}_l-\bar{a}_{r+1}} & \mathbf{0} & \cdots & \frac{I_\kappa}{\bar{a}_l-\bar{a}_{r+1}}
\end{array}\right],$$

$$B_n^{(2)} = \left[\begin{array}{ccccc|c|ccc}
 & & I_{(\kappa+1)r+1} & & & \mathbf{0} & & \mathbf{0} & \\
\hline
-\frac{c_1+\kappa+1}{c_{r+1}+\kappa}I_\kappa & \mathbf{0}_{\kappa\times 1} & \cdots & -\frac{c_r+\kappa+1}{c_{r+1}+\kappa}I_\kappa & \mathbf{0}_{\kappa\times 2} & \frac{I_\kappa}{c_{r+1}+\kappa} & -\frac{c_{r+2}+\kappa}{c_{r+1}+\kappa}I_\kappa & \cdots & -\frac{c_l+\kappa}{c_{r+1}+\kappa}I_\kappa \\
\hline
 & & \mathbf{0} & & & \mathbf{0} & & I_{\kappa(l-r-1)} &
\end{array}\right],$$

$$B_n^{(3)} = \left[\begin{array}{c|c|c}
\mathbf{0} & 1 & \mathbf{0} \\
\hline
I_{(\kappa+1)r} & \mathbf{0}_{(\kappa+1)\times 1} & \mathbf{0} \\
\hline
\mathbf{0} & \mathbf{0} & I_{\kappa(l-r)}
\end{array}\right],$$

where I_m is the m by m identity matrix and $\mathbf{0}_{j\times k}$ is the zero matrix of size j by k. We used Lemma 3.2.8 in the transformation (A) and Lemma 3.2.6 in (B). This gives $B_n = B_n^{(3)}B_n^{(2)}B_n^{(1)}$.

Using $d_n = A_nD_n$ we obtain that

$$B_nd_{n+1} = B_nA_{n+1}D_{n+1} = \left[\begin{array}{ccc|c}
C_0 & \cdots & C_{n-1} & 1 \\
\hline
 & A_n & & \mathbf{0}
\end{array}\right] D_{n+1}. \tag{3.27}$$

The identity at the first row follows from

$$\nu_{j,0} = \frac{1}{2\mathrm{i}}\int_\Gamma z^j\,\chi_{\mathbf{n}}(z)\,\mathrm{d}z = \frac{1}{2\mathrm{i}}\int_\Gamma z^j\,\sum_{k=0}^{n} C_k\chi_k^\infty(z)\,\mathrm{d}z = \sum_{k=0}^{n} C_k\mu_{jk},$$

where C_k is given by $\prod_{i=1}^{l}(z-\bar{a}_i)^{n_j} = \sum_{k=0}^{n} C_k z^k$. We also used that the upper $n \times n$ diagonal submatrix of D_{n+1} is D_n.

Calculating the determinant of (3.27) and using $B_n = B_n^{(3)} B_n^{(2)} B_n^{(1)}$, we arrive at

$$\begin{aligned} \det A_{n+1} &= (-1)^{(n+2)} \left(\det B_n^{(1)} \det B_n^{(2)} \det B_n^{(3)}\right)^{-1} \det A_n \\ = \quad & (-1)^{(n+2)+\sum_{i\leq r} n_i} \left(\prod_{i<r+1} (\bar{a}_i - \bar{a}_{r+1})^{n_i} \right) \left(\prod_{j>r+1} (\bar{a}_{r+1} - \bar{a}_j)^{n_j} \right) (c_{r+1} + \kappa)^{\kappa} \det A_n. \end{aligned} \tag{3.28}$$

Now we can prove (3.9) by induction. When $\mathbf{n} = (1, 0 \cdots, 0)$ (i.e. $\kappa = 0$ and $r = 1$), by the definition of $\nu_{jk}^{(i)}$ and μ_{jk}, we have $\nu_{0,0}^{(1)} = \mu_{0,0}$. This proves $d_1 = D_1$ with $\det A_1 = 1$. If (3.9) holds up to $n \leq N$ then (3.9) holds for $n = N+1$ by (3.28). Recall that if $\mathbf{n}(N, l) = (n_1, \cdots, n_l)$ and $N = \kappa l + r$ then $\mathbf{n}(N+1, l) = (n_1, \cdots, n_{r+1} + 1, \cdots, n_l)$, increasing only the $(r+1)$th entry by one. This ends the proof of Theorem 3.1.6. $\square$

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