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Generalizations of Quandles and their cohomologies

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by

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Dedication

This dissertation is dedicated to my father and late mother.

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Abstract

Quandles are distributive algebraic structures originally introduced independently by David Joyce [33] and Sergei Matveev [43] in 1979, motivated by the study of knots. In this dissertation, we discuss a number of generalizations of the notion of quandles. In the first part of this dissertation we discuss biquandles, in the context of augmented biquandles, a representation of biquandles in terms of actions of a set by an augmentation group. Using this representation we are able to develop a homology and cohomology theory for these structures.

We then introduce an n-ary generalization of the notion of quandles. We discuss a number of properties of these structures and provide a number of examples. Also discussed are methods of obtaining n-ary quandles through iteration of binary quandles, and obtaining binary quandles from n-ary quandles, along with a classification of low order ternary quandles.

We build upon this generalization, introducing n-ary f-quandles, and similarly discuss examples, properties, and relations between the n-ary structures and their binary counter parts, as well as low order classification of ternary f-quandles. Finally we present cohomology theory for general n-ary f-quandles.

Chapter 1
Introduction

In this chapter, we give a brief overview of quandles, as well as f-quandles, a generalization of quandles, their history and connections to knot theory. Following a brief discussion of the history of quandles, we will discuss their structure, and their accompanying cohomology theory. Our introduction will conclude with a discussion of the motivation behind the n-ary generalizations of these structures, as well as an overview of the organization of this dissertation. Throughout this section we will state many definitions that form the basis of the work presented in this dissertation.

1.1 History

The term quandle first appeared in the 1979 Ph.D. thesis of David Joyce, published in 1982, [33], followed by a separate work by Sergey V. Matveev [43]. Joyce and Matveev introduced the *knot quandle*, as a classifying invariant of the knot [33, 43]. The notion of the quandle however can be traced back further.

In 1942 Mituhisa Takasaki introduced the *kei* an algebraic structure as an abstraction of symmetric transformations [52], which Joyce's work refers to as ”involutory quandles”. The notion was later rediscovered and generalized by John Conway and Gavin Wraith under the name ”wracks” [21]. Roger Fenn and Colin Rourke would later use the term ”rack” to refer to a generalization of the quandle [27]. Interest in quandles grew as their relations to knot theory were investigated by Joyce [33] and Matveev [43] in the early 1980s. Since that time extensive research has been published regarding quandles, including classification [35,46,55], use as a knot invariant [20,23,34,51], cohomology [12], and more [19, 36]. The study of quandles and racks has also lead to a number of related structures and generalizations. Generalizations of knots, such as virtual knots, singular knots, and pseudoknots, have lead to related generalizations of the quandles, including biracks [15], singquandles [18], and psyquandles [45] respectively. Additionally the algebraic interest in quandles and distributive structures has lead to algebraically motivated generalizations such as f-quandles [16], a generalization in the same family as Hom-algebras.

The concept of Hom-algebras has its origins in the work of Hartwig, Larsson, and Silvestrov [31] to provide a general framework for q-deformations of Lie algebras over vector fields. Such deformations arise in theoretical physics in the context of string theory, lattice models, quantum scattering, and other related areas. The initial motivation for their work was the creation of a general approach for dealing with examples of q-deformations of Witt and Virasoro algebras constructed in the early nineties which were observed to fail to satisfy the Jacobi identity, while other q-deformations of ordinary Lie algebras preserved the identity. In their work Hartwig, Larsson, and Silvestrov introduced Hom-Lie algebras as well as related concepts of quasi-Lie algebras and quasi-Hom-Lie algebras. In the years since the introduction of this form of generalization, it has garnered significant interest, leading to a growing body of work on related structures [38, 49], and work applying similar generalization to a number of other algebraic structures and ideas, including Hom-associative algebras [42], the previously noted f-quandles [16], the Hom-Yang-Baxter equation [56], and others [40].

While binary operations are the standard in the study of both quandles and hom-structures, generalizations of binary operations can be found as early as the nineteenth century when Cayley introduced cubic generalizations of matrices. Ternary operations, and more general n-ary operations, has also appeared naturally in various fields of theoretical and mathematical physics. Ternary Lie algebras first appeared in the context of Nambu mechanics [44], a generalization of hamiltonian mechanics, whose algebraic formulation was achieved by Takhtajan [53]. Ternary Lie algebras also find application in String and Superstring theory in the context of Nahm equations [7, 10]. Since their introduction construction and classification of ternary and more general n-ary Lie algebras have been studied [9, 29, 38], as well as work on deformations and cohomologies [30, 50, 54]. This work has included considering n-ary generalizations of Hom-Lie algebras [6], Hom-Leibniz algebras [41], as well as other Hom-associative and non-associative hom-algebras [3, 42].

1.2 Quandles and Knot Theory

While we will be considering quandles in a purely algebraic sense, due to their deep connection with knot theory we will introduce them in that context. To do so we will first give a few definitions, beginning with the definition of a knot, and of knot equivalence.

Definition 1.2.1. *A knot is an embedding of the circle S^1 into $\mathbb{R}^3$ or S^3, that is, a simple closed curve. A link is a finite, ordered collection of disjoint knots.*

Two knots are equivalent if one can be continuously deformed into the other. More precisely, knots K and K' are equivalent if they are ambient isotopic.

Definition 1.2.2. [24] *K is ambient isotopic to K' if there is a continuous map $H : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ such that $H(K,0) = K$, $H(K,1) = K'$, and $H(x,t)$ is injective for all $t \in [0,1]$. Such a map is called an "ambient isotopy."*

Commonly, knots are represented using knot diagrams, a projection of the knot onto the plane with finitely many double points, called crossings. In standard knot diagrams over and under crossings are indicated by drawing the under strand broken at the crossing.

The utility of knot diagrams in studying knots is clear, especially considering the impact of Kurt Reidemeister's theorem regarding the equivalence of knot diagrams, and thus the knots they represent.

Theorem 1.2.3. [1] *Two knot diagrams, K_1 and K_2, are equivalent if, and only if, one can be changed into the other by a finite sequence of planar isotopies and Reidemeister moves.*

Due to this powerful theorem, it is enough to show that a quantity from a knot diagram is invariant under the Reidemister moves to prove it is a knot invariant. One of the simpler of such invariants is tricolorability, and its generalization Fox n-colorability, where a set of colors, or labels, are used to label each arc in a knot diagram, such that at least two distinct labels are used, and at any crossing all three arcs are either uniformly or uniquely labeled.

In order to consider labeling from an algebraic perspective, let X be a set of labels, let $\triangleright$ be a binary operation representing the action of the change in labeling at a crossing,

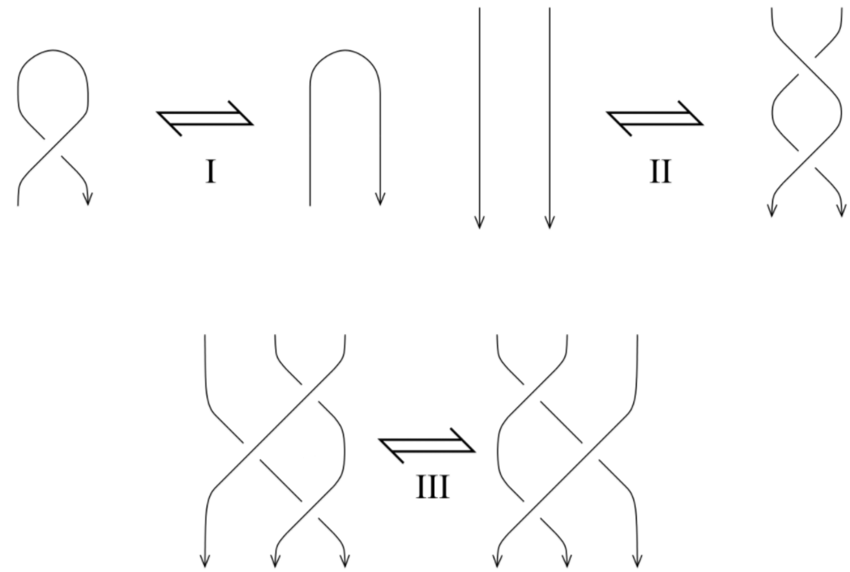


Figure 1.: Reidemeister moves

as shown in the diagram below. Thus, applying this labeling scheme to the Reidemeister diagrams, as above, the diagram equivalencies provide us relations on a free magma over the set X. By taking these relations as axioms we can form algebraic structures that encapsulate the labeling invariant.

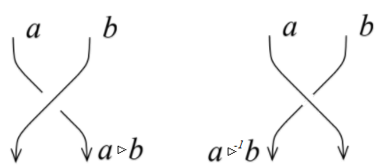


Figure 2.: Quandle action at positive and negative crossings.

First we will consider the relation given by Reidemeister type III move, $(a \triangleright b) \triangleright c = (a \triangleright c) \triangleright (b \triangleright c)$ as shown in figure 3. This leads to the following definition.

Definition 1.2.4. *A shelf is a pair $(X, \triangleright)$, where X is a non-empty set with a binary operation $\triangleright$ satisfying the following identity for all $x, y, z \in X$:*

$$(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z) \tag{1.1}$$

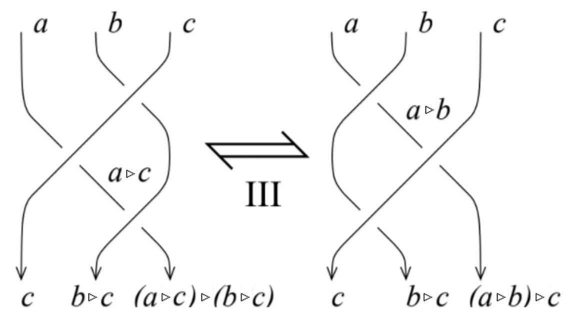


Figure 3.: Quandle labeling of Reidemeister type III move

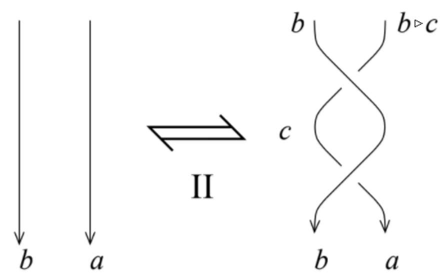


Figure 4.: Quandle labeling of Reidemeister type II move

Now we consider the relation given by Reidemeister type II move, as shown in figure 4. Thus right multiplication by a fixed element must be invertible. This gives us the notion of a rack.

Definition 1.2.5. *A rack is a shelf such that for any $b, c \in X$ there exists a unique $a \in X$ such that*

$$a \triangleright b = c \tag{1.2}$$

Remark 1.2.6. *As right multiplication by a fixed element is invertible, it is often convenient to denote this action $\triangleright^{-1}$. That is, if $a \triangleright b = c$, then $c \triangleright^{-1} b = a$.*

Finally, from the Reidemeister type I move we obtain the relation $x \triangleright x = x$ as shown in figure 5. Adding this final axiom gives us the notion of a quandle.

Definition 1.2.7. *A quandle is a rack such that*

$$a \triangleright a = a, \forall a \in X \tag{1.3}$$

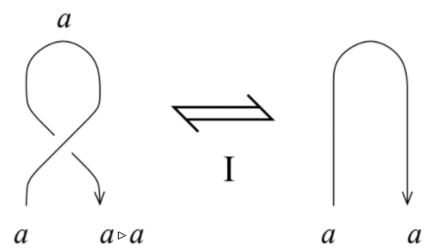


Figure 5.: Quandle labeling of Reidemeister type I move

Remark 1.2.8. *While we have presented the quandle with right distributivity, and invertibility of right multiplication, formulation using left distributivity and invertibility of left multiplication instead is equivalent.*

Having defined the notion of a quandle, we will now define the notion of quandle morphisms.

Definition 1.2.9. *Let* $(X, \triangleright_1)$ *and* $(Y, \triangleright_2)$ *be quandles (resp. racks). Let* $\phi : X \to Y$ *be a map. Then* ϕ *is a morphism of quandles (resp. racks) if* $\phi(x \triangleright_1 y) = \phi(x) \triangleright_2 \phi(y)$ *for all* $x, y \in X$.

To illustrate these ideas we provide a few common examples of quandles and types of quandles.

Example 1.2.10. *Given a set* X *and an operation* $\triangleright$ *defined* $x \triangleright y = x$ *for all* $x, y \in X$, $(X, \triangleright)$ *is a quandle called the* trivial quandle *on* X.

Example 1.2.11. *For a group* G*, define an operation as* n*-fold conjugation, that is* $a \triangleright b = b^{-n}ab^n$. *Then* $(G, \triangleright)$ *is a quandle.*

Example 1.2.12. *The* dihedral quandle *on* $\mathbb{Z}_n$ *is defined* $a \triangleright b \equiv 2b - a \mod n$*, and can be identified with the set of reflections of a regular* n*-gon with conjugation as the operation. Note that the dihedral quandle is simultaneously right and left distributive.*

Example 1.2.13. *Given a quandle* $(X, \triangleright)$*, we say* $(X, \triangleright)$ *is a connected quandle if for all* $x, y \in X$*, there exists some* $z_1, z_2, \ldots, z_n \in X$ *such that* $(\cdots((x \triangleright^{\pm 1} z_1) \triangleright^{\pm 1} z_2) \cdots \triangleright^{\pm 1} z_n = y$. *That is, the inner automorphism group of* $(X, \triangleright)$ *act transitively on* X.

Note that definition 1.2.5 may be reformulated as follows.

Definition 1.2.14. *A rack is a shelf, $(Q, \triangleright)$, such that for all $x \in Q$, the map $\beta_x : Q \to Q$ defined $\beta_x(y) = y \triangleright x$ is invertible.*

This formulation where $\beta : Q \to Inn(Q)$ leads us to a reformulation of quandles, known as *augmented quandles*, in a manner similar to modules with the automorphism group of the quandle playing the roles of scalers.

Definition 1.2.15. *Let $(X, \triangleright)$ be a quandle and G be a group acting on X, and $\epsilon : X \to G$, an augmentation map such that for all $x \in X$ and $g \in G$*

- $\epsilon(x)x = x$
- $\epsilon(gx) = g\epsilon(x)g^{-1}$

The natural example of such a structure being the case where $G = Inn(Q)$ and ϵ mapping x to β_x. This formulation of quandles was introduced by Joyce [33] and for racks by Fenn and Rourke [27], and has been useful in providing insight into quandles.

1.3 Cohomology Theory of Quandles

In this section we will review the notion of quandle cohomology, an invariant introduced by Carter, Jelsovsky, Kamada, Langford, and Saito [14], and further generalized by Clark, Graña, and Saito [12].

Let $(X, \triangleright)$ be a rack. Consider the free abelian group generated by $X^{\times n}$, which we denote $C_n^R(X)$, and define the family of homomorphisms $\partial_n : C_n^R(X) \to C_{n+1}^R(X)$ such that $\partial_n = 0$ for $n \leq 1$ and for $n \geq 2$,

$$\begin{aligned}\partial_n \phi(x_1, \dots, x_{n+1}) \quad &= \\ &(-1)^{n+1} \sum_{i=2}^{n+1} (-1)^i \Big(\eta_{[x_1, \dots, \hat{x}_i, \dots, x_{n+1}], [x_i, \dots, x_{n+1}]} \phi(x_1, \dots, \hat{x}_i, \dots, x_{n+1}) \\ &- \phi(x_1 \triangleright x_i, x_2 \triangleright x_i, \dots, x_{i-1} \triangleright x_i, x_{i+1}, \dots x_{n+1}) \Big) \\ &+ (-1)^{n+1} \tau_{[x_1, x_3, \dots, x_{n+1}], [x_2, \dots, x_{n+1}]} \phi(x_2, \dots, x_{n+1}),\end{aligned}$$

where $\hat{x}_i$ denotes the removal of x_i from the sequence, $[x_1, x_2, \ldots, x_n] = (\cdots((x_1 \triangleright x_2) \triangleright x_3) \cdots) \triangleright x_n$, and η and τ represent families of automorphisms and endomorphisms respectively. Then $C^R_\bullet(X) = \{C^n_R(X), \partial^n\}$ is a chain complex, and by defining $C^D_n(X)$ as the subset of $C^R_n(X)$ generated by the $(n+1)$-tuples $(x_1, x_2, .., x_n, x_{n+1})$ such that $x_i = x_{i+1}$ for some $i \in 1, 2, \ldots, n$, we obtain the *degenerate subcomplex*. If $(X, \triangleright)$ is a quandle, $\partial_n(C^D_n(X)) \subset C^D_{n-1}(X)$, and thus the degenerate subcomplex is indeed a subcomplex of $C^R_\bullet(X)$. From this one obtains the quandle chain complex $C^Q_\bullet = \{C^Q_n(X), \delta^n_Q\}$ where $C^Q_n(X) = C^R_n(X)/C^D_n(X)$ and ∂^n_Q is the induced homomorphism. Note, R, D, and Q, simply denote rack, degenerate, and quandle respectively.

From this we may define a cochain complex, letting A be an abelian group define $C^n_R = \text{Hom}(C_n, A)$ and the coboundary operator $\delta^n : C^n \to C^{n+1}$, and defining C^n_D and C^n_Q in a similar manner.

Definition 1.3.1. *Define* $Z^n_Q(X, A) = ker(\delta^n) \subset C^n_Q(X)$*, and* $B^n_Q(X, A) = Im(\delta^n) \subset C^n_Q(X)$*. Then the* n*-th quandle cohomology group of* $(X, \triangleright)$ *with coefficients in* A *is defined*

$$H^n_Q(X, A) = H^n(C^\bullet_n(X, A) = Z^n_Q(X, a)/B^n_Q(X, A).$$

We will take this opportunity to review low dimensional cocycles. For an abelian group A, a quandle 2-cocycle with coefficients in A is a functions $\phi : X \times X \to A$ such that $\phi(x, x) = 0$ and

$$\phi(x, y) + \phi(x \triangleright y, z) = \phi(x, z) + \phi(x \triangleright z, y \triangleright z).$$

These conditions can be obtained from the Reidemeister moves, by first labeling the arcs via a quandle, and then using ϕ to represent the weight at the crossing as shown below.

As the weights in the equivalent diagrams must be equal we obtain the two conditions from Reidemister moves I and III, while Reidemeister move II provides no additional condition as the signs of the crossings cancel out.

Similar methods show a quandle 3-cocyle is a function $\theta : X \times X \times X \to A$

$$\theta(w, x, y) + \theta(w \triangleright y, x \triangleright y, z) + \theta(w, y, z) = \theta(w \triangleright x, y, z) + \theta(w, x, z) + \theta(w \triangleright z, x \triangleright z, y \triangleright z)$$

with $\theta(x, x, y) = 0$ and $\theta(x, y, y) = 0$.

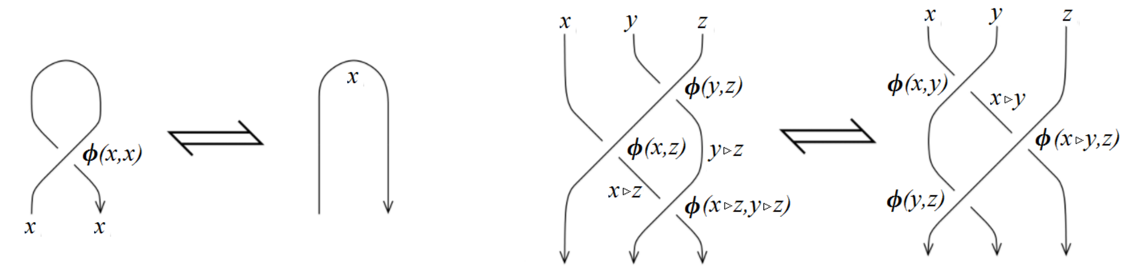


Figure 6.: Reidemeister move I and III cocycle coloring

1.4 f-quandles

In this section we will review the hom generalization of binary quandles, referred to as f-quandles, originally presented in [16]. We will begin with their axiomatic definition.

Definition 1.4.1. *A f-shelf is a triple $(X, *, f)$ in which X is a set, $*$ is a binary operation on X, and $f\colon X \to X$ is a map such that, for any $x, y, z \in X$, the equality*

$$(x * y) * f(z) = (x * z) * (y * z) \tag{1.4}$$

holds. A f-rack is a twisted-shelf such that, for any $x, y \in X$, there exists a unique $z \in X$ such that

$$z * y = f(x). \tag{1.5}$$

A f-quandle is a twisted-rack such that, for each $x \in X$, the equality

$$x * x = f(x) \tag{1.6}$$

holds.

Having defined the f-quandle structure, we will also introduce f-quandle morphisms.

Definition 1.4.2. *Let $(X, \triangleright_1, f_1)$ and $(Y, \triangleright_2, f_2)$ be f-racks (resp. f-quandles). Let $\phi : X \to Y$ be a map. Then ϕ is a morphism of f-racks (resp. f-quandles) if $\phi(x \triangleright_1 y) = \phi(x) \triangleright_2 \phi(y)$ for all $x, y \in X$, and $\phi \circ f_1 = f_2 \circ \phi$.*

We note that for f-quandles, the first condition implies the second. We include a few examples to illustrate these notions.

Example 1.4.3. *Let $(X, \triangleright)$ be a quandle. Then $(X, \triangleright, id_X)$ is an f-quandle.*

Example 1.4.4. *Let $(A, +)$ be an abelian group. Then for $f : A \to A$ defined $f(a) = 2a$, $(A, +, f)$ is an f-quandle.*

1.5 Biquandles

This section will review the definitions and motivations of biracks, originally presented in [28] and biquandles, as found in [26, 37, 47].

Definition 1.5.1. [24] *A birack is a set X equipped with two binary operations $*_1, *_2$, and a bijective map $\pi : X \to X$ such that for all $a, b, c \in X$,*

- $\pi(a *_2 a) = a *_1 a$ *and* $\pi(a) *_2 a = a *_1 \pi(a)$,
- *The map $H(a, b) = (b *_2 a, a *_1 b)$ is invertible,*
- $(a *_2 b) *_2 (c *_1 b) = (a *_2 c) *_2 (b *_2 c)$,
- $(a *_1 b) *_1 (c *_1 b) = (a *_1 c) *_1 (b *_2 c)$,
- $(a *_1 b) *_2 (c *_1 b) = (a *_2 c) *_1 (b *_2 c)$.

A birack such that π is the identity map is called a biquandle.

Note that every quandle $(X, \triangleright)$ is a birack with π equal to the identity map, $a *_1 b = a \triangleright b$ and $a *_2 b = a$.

The geometric motivation for biracks come from labeling semiarcs (the portion of an arc between two consecutive crossings) in an oriented framed link diagram with elements of X. Framed knots and links can be thought of as 'thickened' knots, and as such a standard Reidemeister type I move introduces a *kink* in the knot, much like it would in a ribbon. As such the standard type I move does not apply and is replaced by the move show in

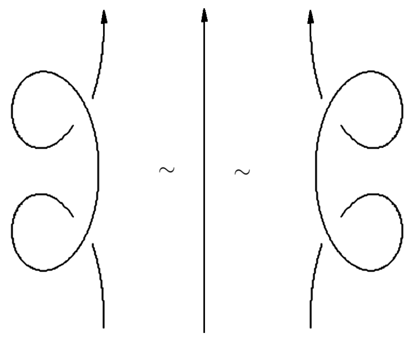


Figure 7.: Framed Reidemeister Type 1 move

figure 1.5. Labeling this and the remaining Reidemeister moves yields a set of necessary and sufficient conditions for labelings before and after the move to correspond bijectively leading to the axioms above, just as in the case for standard knot. In this case the map π is the kink map, representing the effect of kinks on labeling.

In this context the birack map $H : X \times X \to X \times X$ represents the change in labeling occurring at a crossing as show in figure 1.5.

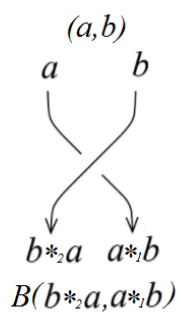


Figure 8.: Birack map

Chapter 2

Augmented Biquandles

In this chapter we discuss a generalization of the quandle, the biquandle and birack, first introduced by [28], reformulated in terms of actions of a set by an augmentation group, in the same manner as augmented quandles. The bulk of this chapter originally appeared in [15].

2.1 Definitions and Properties

Definition 2.1.1. Let X be a set and G be a subgroup of the group of bijections $g : X \to X$. An *augmented birack structure* on (X, G) consists of maps $\alpha, \beta, \bar{\alpha}, \bar{\beta} : X \to G$ (i.e., for each $x \in X$ we have bijections $\alpha_x : X \to X$, $\beta_x : X \to X$, $\bar{\alpha}_x : X \to X$ and $\bar{\beta}_x : X \to X$) and a distinguished element $\pi \in G$ satisfying

(i) For all $x \in X$, we have

$$\alpha_{\pi(x)}(x) = \beta_x \pi(x) \quad \text{and} \quad \bar{\beta}_{\pi(x)}(x) = \bar{\alpha}_x \pi(x),$$

(ii) For all $x, y \in X$ we have

$$\bar{\alpha}_{\beta_x(y)} \alpha_y(x) = x, \quad \bar{\beta}_{\alpha_x(y)} \beta_y(x) = x, \quad \alpha_{\bar{\beta}_x(y)} \bar{\alpha}_y(x) = x,$$
$$\text{and} \quad \beta_{\bar{\alpha}_x(y)} \bar{\beta}_y(x) = x,$$

and

(iii) For all $x, y \in X$, we have

$$\alpha_{\alpha_x(y)} \alpha_x = \alpha_{\beta_y(x)} \alpha_y, \quad \beta_{\alpha_x(y)} \alpha_x = \alpha_{\beta_y(x)} \beta_y, \quad \text{and} \quad \beta_{\alpha_x(y)} \beta_x = \beta_{\beta_y(x)} \beta_y.$$

Definition 2.1.2. *Given an augmented birack (G, X), if X is a finite set, then G is a subgroup of the symmetric group $S_{|X|}$, and there exist some smallest $N \in \mathbb{Z}^+$ such that $\pi^N = 1 \in G$. This N is called the characteristic of the augmented birack (G, X).*

Example 2.1.3. Let $\tilde{\Lambda} = \mathbb{Z}[t^{\pm 1}, s, r^{\pm 1}]/(s^2 - (1 - t^{-1}r)s)$, let X be a $\tilde{\Lambda}$-module and let G be the group of invertible linear transformations of X. Then (G, X) is an augmented biquandle with

$$\alpha_x(y) = ry, \quad \beta_y(x) = tx - tsy, \quad \bar{\alpha_y}(x) = r^{-1}x, \quad \bar{\beta}_x(y) = sr^{-1}x + t^{-1}y,$$

$$\text{and} \quad \pi(x) = (t^{-1}r + s)x.$$

For example, we have

$$\beta_{\alpha_x(z)}\alpha_x(y) = \beta_{rz}(ry) = try - tsrz = r(ty - tsz) = \alpha_{\beta_z(x)}\beta_z(y).$$

An augmented birack of this type is known as a (t, s, r)-*birack*.

Example 2.1.4. We can define an augmented birack structure symbolically on the finite set $X = \{1, 2, 3, \ldots, n\}$ by explicitly listing the maps $\alpha_x, \beta_x : X \to X$ for each $x \in X$. This is conveniently done by giving a $2n \times n$ matrix whose upper block has (i, j) entry $\alpha_j(i)$ and whose lower block has (i, j) entry $\beta_j(i)$, which we might denote by $M_{(G,X)} = \left[\begin{array}{c} \alpha_j(i) \\ \hline \beta_j(i) \end{array}\right]$. Such a matrix defines an augmented birack with G being the symmetric group S_n provided the maps thus defined satisfy the augmented birack axioms; note that if the axioms are satisfied, then the maps $\pi, \bar{\alpha_x}$ and $\bar{\beta}_x$ are determined by the maps α_x, β_x. For example, the matrix

$$M_{(G,X)} = \left[\begin{array}{ccc} 2 & 2 & 2 \\ 1 & 1 & 1 \\ 3 & 3 & 3 \\ \hline 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{array}\right]$$

encodes the (t, s, r)-birack structure on $X = \{1, 2, 3\} = \mathbb{Z}_3$ with $t = 1, s = 2, r = 2$.

An augmented birack defines a birack map $B : X \times X \to X \times X$ as defined in the introduction by setting

$$B(x, y) = (\beta_x^{-1}(y), \alpha_{\beta_x^{-1}(y)}(x)).$$

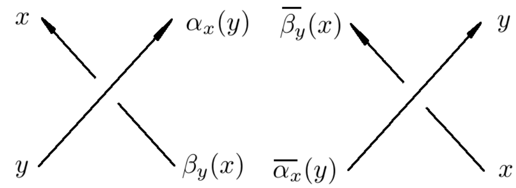


Figure 9.: Birack labeling for positive crossings

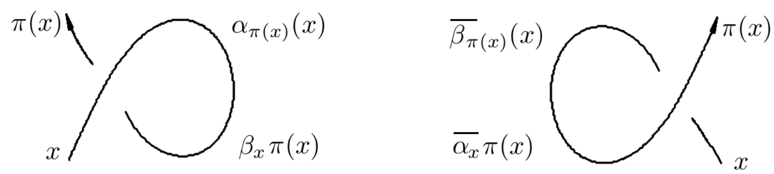


Figure 10.: Birack kink map diagram

The names are chosen so that if we orient a crossing, positive or negative, with the strands oriented upward, then the unbarred actions go left-to-right and the barred actions go right-to-left, with α and β standing for "above" and "below"[1]. Thus, $\alpha_x(y)$ is the result of y going above x left-to-right and $\bar{\beta}_y(x)$ is the result of x going below y from right-to-left.

The element $\pi \in G$ is the *kink map* which encodes the change of semiarc labels when going through a positive kink. In particular, each (G, X)-labeling of a framed oriented knot or link diagram before a framed type I move corresponds to a unique (G, X)-labeling after the move. If $\pi = 1$ is the identity element in G, our augmented birack is an *augmented biquandle*; labelings of an oriented link by an augmented biquandle are independent of framing.

Axiom (ii) is equivalent to the condition that the map $S : X \times X \to X \times X$ defined by

$$S(x, y) = (\alpha_x(y), \beta_y(x))$$

is a bijection with inverse

$$S^{-1}(y, x) = (\bar{\beta}_y(x), \bar{\alpha_x}(y)).$$

Note that the condition that the components of S are bijective is not sufficient to make

[1]Thanks to Scott Carter for this observation.

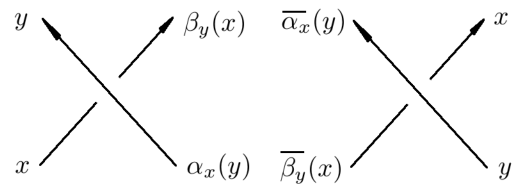
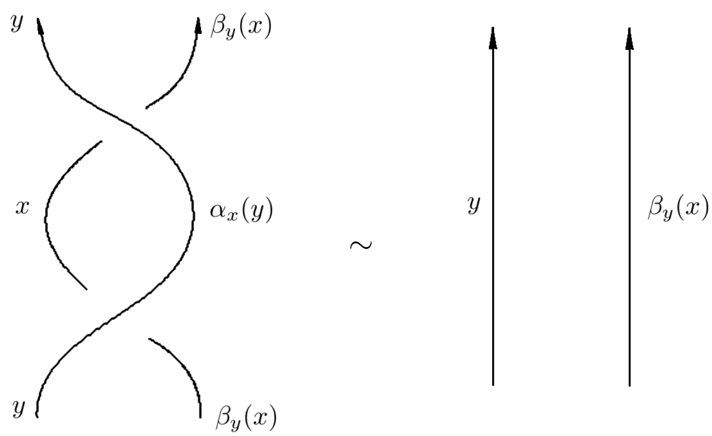
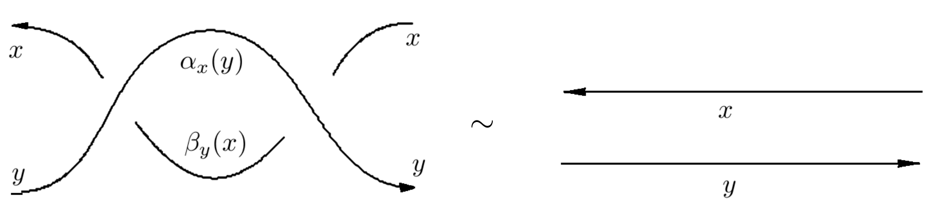


Figure 11.: Birack labeling for positive crossings

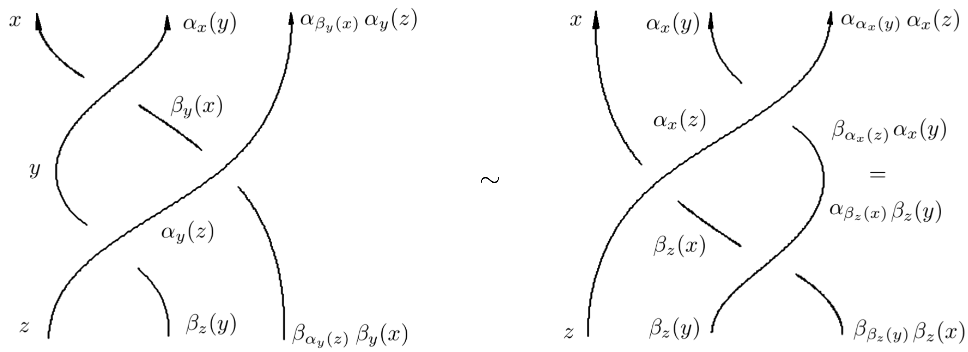
S bijective; for instance, if X is any abelian group, the map $S(x,y) = (\alpha_x(y), \beta_y(x)) = (x+y, x+y)$ has bijective component maps but is not bijective as a map of pairs. The maps $\bar{\alpha_x}, \bar{\beta_x}$ are the components of the inverse of the sideways map; we can interpret them as labeling rules going right to left. At negatively oriented crossings, the top and bottom labels are switched. Note that (G,X)-labelings of a framed knot or link correpond bijectively before and after both forms of type II moves: *direct type II* moves where both strands are oriented in the same direction



and *reverse type II* moves in which the strands are oriented in opposite directions.



Axiom (iii) encodes the conditions arising from the Reidemeister III move:



Thus by construction we have

Theorem 2.1.5. *If L and L' are oriented framed links related by oriented framed Reidemeister moves and (G, X) is an augmented birack, then there is a bijection between the set of labelings of L by (G, X), denoted $\mathcal{L}(L, (G, X))$, and the set of labelings and the set of labelings of L' by (G, X), denoted $\mathcal{L}(L', (G, X))$.*

2.2 Homology

Let (X, G) be an augmented birack. Let $C_n = \mathbb{Z}[X^n]$ be the free abelian group generated by ordered n-tuples of elements of X and let $C^n(X) = \{f : C_n \to \mathbb{Z} \mid f \in Hom(C_n, \mathbb{Z})\}$. For $k = 1, 2, \ldots, n$, define maps $\partial'_k, \partial''_k : C_n(X) \to C_{n-1}(X)$ by

$$\partial'_k(x_1 \ldots, x_n) = (x_1, \ldots, \widehat{x_k}, \ldots, x_n)$$

and

$$\partial''_k(x_1, \ldots, x_n) = (\beta_{x_k}(x_1), \ldots, \beta_{x_k}(x_{k-1}), \widehat{x_k}, \alpha_{x_k}(x_{k+1}), \ldots, \alpha_{x_k}(x_n))$$

where the $\widehat{\ }$ indicates that the entry is deleted, i.e.

$$(x_1, \ldots, \widehat{x_k}, \ldots, x_n) = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n).$$

Theorem 2.2.1. *The map $\partial_n : C_n(X) \to C_{n-1}(X)$ given by*

$$\partial_n(\vec{x}) = \sum_{k=1}^{n} (-1)^k (\partial'_k(\vec{x}) - \partial''_k(\vec{x}))$$

is a boundary map; the map $\delta^n : C^n(X) \to C^{n+1}(X)$ given by $\delta^n(f) = f\partial_{n+1}$ is the corresponding coboundary map. The quotient group $H_n(X) = \text{Ker } \partial_n/\text{Im } \partial_{n+1}$ is the nth augmented birack homology of (X, G), and the quotient group $H^n(X) = \text{Ker } \delta^n/\text{Im } \delta^{n-1}$ is the nth augmented birack cohomology of (X, G).

To prove theorem 2.2.1, we will find it convenient to first prove a few key lemmas.

Lemma 2.2.2. *Let $j < k$. Then $\partial'_j \partial'_k(\vec{x}) = \partial'_{k-1}\partial'_j$.*

Proof. We compute

$$\begin{aligned} \partial'_j \partial'_k(\vec{x}) &= \partial'_j(x_1, \dots, \widehat{x_k}, \dots, x_n) \\ &= \partial'_j(x_1, \dots, \widehat{x_j} \dots, \widehat{x_k}, \dots, x_n) \end{aligned}$$

obtaining the input vector with the entries in the jth and kth positions deleted. On the other hand, if we first delete the jth entry, each entry with subscript greater than j is now shifted into one lower position; in particular, x_k is now in the $(k-1)$st position and we have

$$\begin{aligned} \partial'_{k-1}\partial'_j(\vec{x}) &= \partial'_j(x_1, \dots, \widehat{x_j}, \dots, x_n) \\ &= \partial'_j(x_1, \dots, \widehat{x_j}, \dots, \widehat{x_k}, \dots, x_n) \end{aligned}$$

as required. □

Corollary 2.2.3. *The map $\partial' : C_n \to C_{n-1}$ defined by $\sum_{k=1}^{n}(-1)^k \partial'(\vec{x})$ is a boundary map.*

Proof. If we apply ∂' twice, each term with first summation index less than the second summation index is matched by an equal term with first summation index greater than the second summation index but of opposite sign:

$$\begin{aligned} \partial'(\partial'(\vec{x})) &= \sum_{j<k}(-1)^{j+k}\partial'_j\partial'_k(\vec{x}) + \sum_{j>k}(-1)^{j+k}\partial'_j(\partial'_k(\vec{x})) \\ &= \sum_{j>k}(-1)^{j+k+1}\partial'_j\partial'_k(\vec{x}) + \sum_{j>k}(-1)^{j+k}\partial'_j(\partial'_k(\vec{x})) \\ &= 0. \end{aligned}$$

□

Lemma 2.2.4. *If $j < k$ we have $\partial'_j \partial''_k(\vec{x}) = \partial''_{k-1}\partial'_j(\vec{x})$.*

Proof. On the one hand,

$$\begin{aligned}
\partial'_j \partial''_k(\vec{x}) &= \partial'_j(\beta_{x_k}(x_1), \ldots, \beta_{x_k}(x_{k-1}), \widehat{x_k}, \alpha_{x_k}(x_{k+1}), \ldots, \alpha_{x_k}(x_n)) \\
&= (\beta_{x_k}(x_1), \ldots, \beta_{x_k}(x_{j-1}), \widehat{\beta_{x_k}(x_j)}, \beta_{x_k}(x_{j+1}), \ldots, \beta_{x_k}(x_{k-1}), \\
&\quad \alpha_{x_k}(x_{k+1}), \ldots, \alpha_{x_k}(x_n))).
\end{aligned}$$

On the other hand, applying ∂'_j first shifts x_k into the $(k-1)$ position and we have

$$\begin{aligned}
\partial''_{k+1}\partial'_j(\vec{x}) &= \partial''_{k+1}(x_1, \ldots, \widehat{x_j}, \ldots, x_n) \\
&= (\beta_{x_k}(x_1), \ldots, \beta_{x_k}(x_{j-1}), \widehat{x_j}, \beta_{x_k}(x_{j+1}), \ldots, \beta_{x_k}(x_{k-1}), \widehat{x_k}, \\
&\quad \alpha_{x_k}(x_{k+1}), \ldots, \alpha_{x_k}(x_n)))
\end{aligned}$$

as required.

□

Lemma 2.2.5. *If $j < k$ we have $\partial''_j \partial'_k(\vec{x}) = \partial'_{k-1}\partial''_j(\vec{x})$.*

Proof. On the one hand,

$$\begin{aligned}
\partial''_j \partial'_k(\vec{x}) &= \partial''_j(x_1, \ldots, \widehat{x_k}, \ldots, x_n) \\
&= (\beta_{x_j}(x_1), \ldots, \beta_{x_j}(x_{j-1}), \\
&\quad \widehat{x_j}, \alpha_{x_j}(x_{j+1}), \ldots, \alpha_{x_j}(x_{k-1}), \widehat{x_k}, \alpha_{x_j}(x_{k+1}), \ldots, \alpha_{x_j}(x_n)))
\end{aligned}$$

As above, applying ∂''_j shifts the positions of the entries with indices greater than j, and we have

$$\begin{aligned}\partial'_{k-1}\partial''_j(\vec{x}) &= \partial'_{k-1}(\beta_{x_j}(x_1),\dots,\beta_{x_j}(x_{j-1}),\widehat{x_j},\alpha_{x_j}(x_{j+1}),\dots,\alpha_{x_j}(x_n))\\ &= (\beta_{x_j}(x_1),\dots,\beta_{x_j}(x_{j-1}),\widehat{x_j},\alpha_{x_j}(x_{j+1}),\dots,\alpha_{x_j}(x_{k-1}),\widehat{\alpha_{x_j}(x_k)},\\ &\quad \alpha_{x_j}(x_{k+1}),\dots\alpha_{x_j}(x_n))\end{aligned}$$

as required. □

The final lemma depends on the augmented birack axioms.

Lemma 2.2.6. *If $j < k$ we have $\partial''_j\partial''_k(\vec{x}) = \partial''_{k-1}\partial''_j(\vec{x})$.*

Proof. We have

$$\begin{aligned}\partial''_j\partial''_k(\vec{x}) &= \partial''_j(\beta_{x_k}(x_1),\dots,\beta_{x_k}(x_{k-1}),\widehat{x_k},\alpha_{x_k}(x_{k+1}),\dots,\alpha_{x_k}(x_n))\\ &= (\beta_{\beta_{x_k}(x_j)}\beta_{x_k}(x_1),\dots,\beta_{\beta_{x_k}(x_j)}\beta_{x_k}(x_{j-1}),\widehat{\beta_{x_k}(x_j)},\alpha_{\beta_{x_k}(x_j)}\beta_{x_k}(x_{j+1}),\dots,\\ &\qquad \alpha_{\beta_{x_k}(x_j)}\beta_{x_k}(x_{k-1}),\widehat{x_k},\alpha_{\beta_{x_k}(x_j)}\alpha_{x_k}(x_{k+1}),\dots,\alpha_{\beta_{x_k}(x_j)}\alpha_{x_k}(x_n))\end{aligned}$$

while again applying ∂''_j first shifts the positions of the entries with indices greater than j, and we have

$$\begin{aligned}\partial''_{k-1}\partial''_j(\vec{x}) &= \partial''_{k-1}(\beta_{x_j}(x_1),\dots,\beta_{x_j}(x_{j-1}),\widehat{x_j},\alpha_{x_j}(x_{j+1}),\dots,\alpha_{x_j}(x_n))\\ &= (\beta_{\alpha_{x_j(x_k)}}\beta_{x_j}(x_1),\dots,\beta_{\alpha_{x_j(x_k)}}\beta_{x_j}(x_{j-1}),\widehat{x_j},\beta_{\alpha_{x_j(x_k)}}\alpha_{x_j}(x_{j+1}),\dots,\\ &\qquad \beta_{\alpha_{x_j(x_k)}}\alpha_{x_j}(x_{k-1}),\widehat{\alpha_{x_j}(x_k)},\alpha_{\alpha_{x_k}(x_j)}\alpha_{x_j}(x_{k+1}),\dots,\alpha_{\alpha_{x_k}(x_j)}\alpha_{x_j}(x_n))\end{aligned}$$

and the two are equal after application of the augmented birack axioms. □

Corollary 2.2.7. *The map $\partial'' : C_n \to C_{n-1}$ defined by $\partial''(\vec{x}) = \sum_{k=1}^{n}(-1)^k\partial''_k(\vec{x})$ is a boundary map.*

Proof. As with ∂', we observe that every term in $\partial''_{n-1}\partial''_n(\vec{x})$ with $j < k$ is matched by an equal term with $j > k$ but with opposite sign. □

Remark 2.2.8. Corollary 2.2.7 shows that the conditions in augmented birack axiom (iii) are precisely the conditions required to make ∂'' a boundary map. This provides a non-knot theoretic alternative motivation for the augmented birack structure.

Proof. (of theorem 2.2.1) We must check that $\partial_{n-1}\partial_n = 0$. Our lemmas show that each term in the sum with $j < k$ is matched by an equal term with opposite sign with $j > k$. We have

$$\begin{aligned}
\partial_{n-1}(\partial_n(x_1,\ldots,x_n)) &= \partial_{n-1}\left(\sum_{k=0}^{n}(-1)^k\left(\partial'_k(\vec{x})-\partial''_k(\vec{x})\right)\right)\\
&= \sum_{j=0}^{n-1}\left(\sum_{k=0}^{n}(-1)^{k+j}\left(\partial'_j\partial'_k(\vec{x})-\partial''_j\partial'_k(\vec{x})-\partial'_j\partial''_k(\vec{x})+\partial''_j\partial''_k(\vec{x})\right)\right)\\
&= \sum_{j<k}(-1)^{k+j}\left(\partial'_j\partial'_k(\vec{x})-\partial''_j\partial'_k(\vec{x})-\partial'_j\partial''_k(\vec{x})+\partial''_j\partial''_k(\vec{x})\right)\\
&\quad+\sum_{j>k}(-1)^{k+j}\left(\partial'_j\partial'_k(\vec{x})-\partial''_j\partial'_k(\vec{x})-\partial'_j\partial''_k(\vec{x})+\partial''_j\partial''_k(\vec{x})\right)\\
&= \sum_{j<k}(-1)^{k+j}\left(\partial'_j\partial'_k(\vec{x})-\partial''_j\partial'_k(\vec{x})-\partial'_j\partial''_k(\vec{x})+\partial''_j\partial''_k(\vec{x})\right)\\
&\quad+\sum_{j<k}(-1)^{k+j-1}\left(\partial'_j\partial'_k(\vec{x})-\partial''_j\partial'_k(\vec{x})-\partial'_j\partial''_k(\vec{x})+\partial''_j\partial''_k(\vec{x})\right)\\
&= 0.
\end{aligned}$$

□

Definition 2.2.9. Let (G, X) be an agumented birack of characteristic N. Say that an element $\vec{v}$ of $C_n(X)$ is N-*degenerate* if $\vec{v}$ is a linear combination of elements of the form

$$\sum_{k=1}^{N}(x_1,\ldots,x_{j-1},\pi^k(x_j),\pi^{k-1}(x_j),x_{j+2},\ldots,x_n).$$

Denote the set of N-degenerate n-chains and n-cochains as $C_n^D(X)$ and $C_D^n(X)$ and the homology and cohomology groups, H_n^D and H_D^n.

Theorem 2.2.10. *The sets of N-degenerate chains form a subcomplex of (C_n, ∂).*

Proof. We must show that $\vec{v} \in C_n^D(X)$ implies $\partial(\vec{v}) \in C_{n-1}^D(X)$. Using linearity it is enough to prove that

$$\partial\left(\sum_{k=1}^{N}(x_1,\ldots,x_{j-1},\pi^k(x_j),\pi^{k-1}(x_j),x_{j+2},\ldots,x_n)\right)$$

is N-degenerate. Let $\vec{u} = \sum_{k=1}^{N}(x_1,\ldots,x_{j-1},\pi^k(x_j),\pi^{k-1}(x_j),x_{j+2},\ldots,x_n)$, we have:

$$\begin{aligned}
\partial(\vec{u}) &= \partial\left[\sum_{k=1}^{N}(x_1,\ldots,x_{j-1},\pi^k(x_j),\pi^{k-1}(x_j),x_{j+2},\ldots,x_n)\right] \\
&= \sum_{k=1}^{N}\partial(x_1,\ldots,x_{j-1},\pi^k(x_j),\pi^{k-1}(x_j),x_{j+2},\ldots,x_n) \\
&= \sum_{k=1}^{N}\left\{\sum_{i=1}^{j-1}(-1)^i[(x_1,\ldots,\widehat{x_i},\ldots,\pi^k(x_j),\pi^{k-1}(x_j),x_{j+2},\ldots,x_n)\right. \\
&\quad -(\beta_{x_i}(x_1),\ldots,\beta_{x_i}(x_{i-1}),\widehat{x_i},\alpha_{x_i}(x_{i+1}),\ldots,\alpha_{x_i}(\pi^k(x_j)),\alpha_{x_i}(\pi^{k-1}(x_j)), \\
&\quad \alpha_{x_i}(x_{j+2}),\ldots,\alpha_{x_i}(x_n))]\} \\
&\quad +\left\{\sum_{k=1}^{N}\{(-1)^j[(x_1,\ldots,x_{j-1},\pi^{k-1}(x_j),x_{j+2},\ldots,x_n)\right. \\
&\quad -(\beta_{\pi^k(x_j)}(x_1),\ldots,\beta_{\pi^k(x_j)}(x_{j-1}),\alpha_{\pi^k(x_j)}(\pi^{k-1}(x_j)),\alpha_{\pi^k(x_j)}(x_{i+2}),\ldots, \\
&\quad \alpha_{\pi^k(x_j)}(x_n))] + (-1)^{j+1}\left[(x_1,\ldots,x_{j-1},\pi^k(x_j),x_{j+2},\ldots,x_n)\right. \\
&\quad -(\beta_{\pi^{k-1}(x_j)}(x_1),\ldots,\beta_{\pi^{k-1}(x_j)}(x_{j-1}),\beta_{\pi^{k-1}(x_j)}(x_j),\alpha_{\pi^{k-1}(x_j)}(x_{j+2}), \\
&\quad \left.\ldots,\alpha_{\pi^{k-1}(x_j)}(x_n))\right]\} \\
&\quad +\sum_{k=1}^{N}\left\{\sum_{i=j+2}^{n}(-1)^i\left[(x_1,\ldots,\pi^k(x_j),\pi^{k-1}(x_j),\ldots,\widehat{x_i},\ldots,x_n)\right.\right. \\
&\quad -(\beta_{x_i}(x_1),\ldots,\beta_{x_i}(x_{j-1}),\beta_{x_i}(\pi^k(x_j)),\beta_{x_i}(\pi^{k-1}(x_j)),\ldots,\widehat{x_i}, \\
&\quad -\alpha_{x_i}(x_{i+1}),\ldots,\alpha_{x_i}(x_n))]\} \qquad (1)
\end{aligned}$$

where as usual $(x_1,\ldots,\widehat{x_i},\ldots,x_n)$ means $(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)$. Now the rest of the proof is based on the following two facts: (1) $\pi^N = 1$ and (2) $\alpha_{\pi^k(x)}(\pi^{k-1}(x)) =$

$\beta_{\pi^{k-1}(x)}(\pi^k(x))$ which is obtained by induction from axiom (i) in the definition 2.1.1 (of augmented birack).

The following sum vanishes:

$$\sum_{k=1}^{N}\big\{\big[(x_1,\ldots,x_{j-1},\pi^{k-1}(x_j),x_{j+2},\ldots,x_n)$$
$$-\ (\beta_{\pi^k(x_j)}(x_1),\ldots,\beta_{\pi^k(x_j)}(x_{j-1}),\alpha_{\pi^k(x_j)}(\pi^{k-1}(x_j)),\alpha_{\pi^k(x_j)}(x_{i+2}),$$
$$\ldots,\alpha_{\pi^k(x_j)}(x_n))$$
$$-\big[(x_1,\ldots,x_{j-1},\pi^k(x_j),x_{j+2},\ldots,x_n)\big]$$
$$-\ (\beta_{\pi^{k-1}(x_j)}(x_1),\ldots,\beta_{\pi^{k-1}(x_j)}(x_{j-1}),\beta_{\pi^{k-1}(x_j)}(x_j),\alpha_{\pi^{k-1}(x_j)}(x_{j+2}),\ldots,$$
$$\alpha_{\pi^{k-1}(x_j)}(x_n))\big]\big\}$$

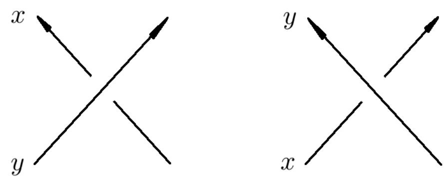
because $\alpha_{\pi^k(x)}(\pi^{k-1}(x)) = \beta_{\pi^{k-1}(x)}(\pi^k(x))$ and $\pi^N = 1$. The rest of the sums can be written as combination of degenerate elements as in the proof of theorem 2 in [25]. □

Definition 2.2.11. The quotient groups $H_n^{NR}(X) = H_n(C_n(X)/C_n^D(X))$ and $H^n_{NR}(X) = H^n(C_n(X)/C^n_D(X)$ are the N*-Reduced Birack Homology* and N*-Reduced Birack Cohomology* groups.

2.3 Cocycles

In this section we will use augmented birack cocycles to enhance the augmented birack counting invariant analogously to previous work.

Let $L_{\vec{w}}$ be an oriented framed link diagram with framing vector $\vec{w}$ and a labeling $f \in \mathcal{L}(L_{\vec{w}},(G,X))$ by an augmented birack (G,X) of characteristic N. For a choice of $\phi \in Z^2_{NR}$, we define an integer-valued signature of the labeling called a *Boltzmann weight* by adding contributions from each crossing as pictured below. Orienting the crossing so that both strands are oriented upward, each crossing contributes ϕ evaluated on the pair of labels on the left side of the crossing with the understrand label listed first.



$+\phi(x,y)$ $\qquad$ $-\phi(x,y)$

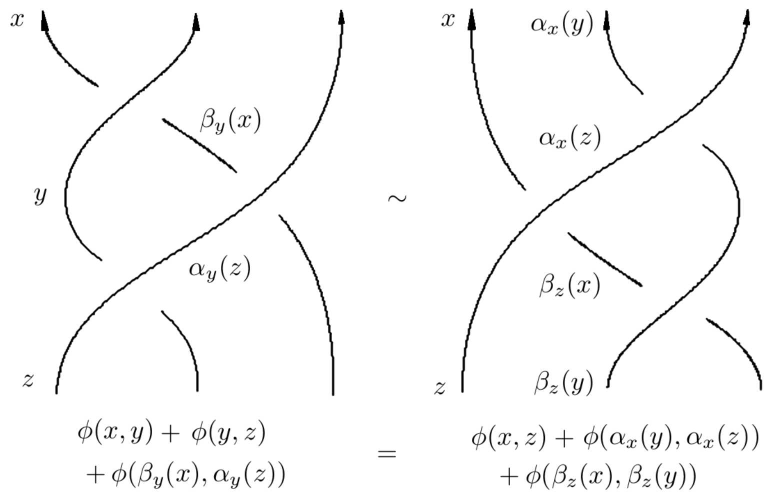
Then as we can easily verify, the Boltzmann weight $BW(f) = \sum_{\text{crossings}} \pm\phi(x,y)$ is unchanged by framed oriented Reidemeister moves and N-phone cord moves, that is the composition of N kinks. Starting with move III, note that $\phi \in Z^2(x)$ implies that

$$\begin{aligned}
(\delta^2\phi)(x,y,z) &= \phi(\partial_2(x,y,z)) \\
&= \phi((y,z) - (\alpha_x(y),\alpha_x(z)) - (x,z) + (\beta_y(x),\alpha_y(z)) + (x,y) - \\
&\quad (\beta_z(x),\beta_z(y))) \\
&= \phi(y,z) - \phi(\alpha_x(y),\alpha_x(z)) - \phi(x,z) + \phi(\beta_y(x),\alpha_y(z)) + \phi(x,y) - \\
&\quad \phi(\beta_z(x),\beta_z(y)) \\
&= 0
\end{aligned}$$

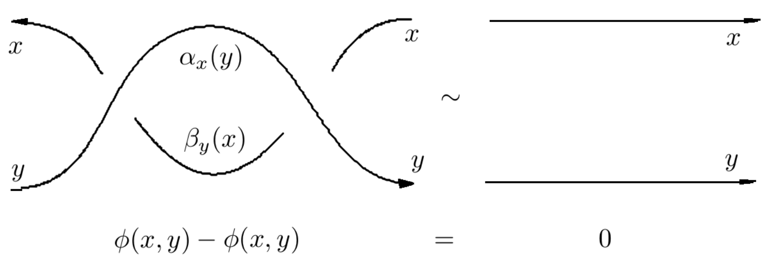
and in particular we have

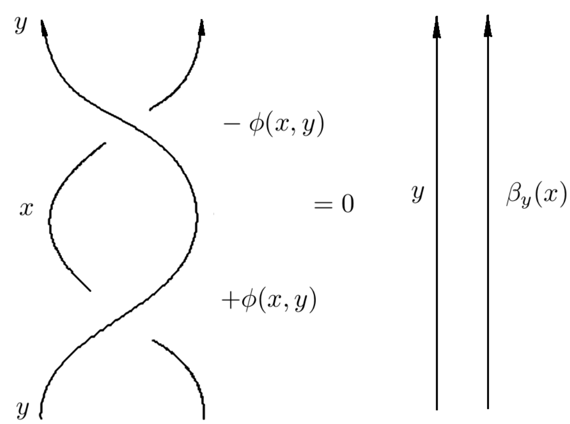
$$\phi(y,z) + \phi(\beta_y(x),\alpha_y(z)) + \phi(x,y) = \phi(\alpha_x(y),\alpha_x(z)) + \phi(x,z) + \phi(\beta_z(x),\beta_z(y)).$$

Then both sides of the Reidemeister III move contribute the same amount to the Boltzmann weight:

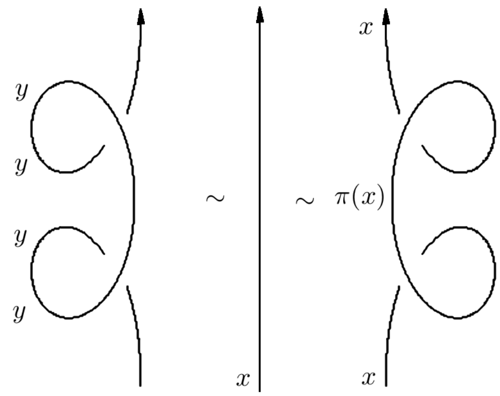


Both sides of both type II moves contribute zero to the Bolztmann weight:



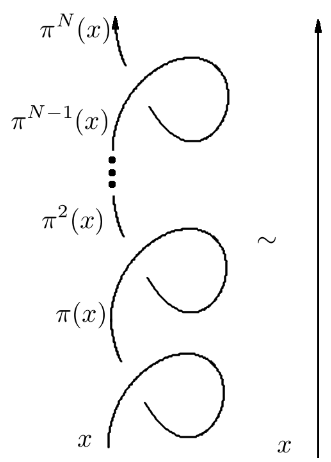


Similarly, both sides of the framed type I moves contribute zero; here we use the alternate form of the framed type I move for clarity, with $y = \bar{\alpha}_x \pi(x) = \bar{\beta}_{\pi(x)}(x)$:



$$\phi(y,y) - \phi(y,y) \quad = 0 = \ \phi(x,\pi(x)) - \phi(x,\pi(x))$$

Finally, the N-phone cord move contributes a degenerate N-chain:



$$\sum_{k=1}^{N} \phi(\pi^k(x), \pi^{k-1}(x)) \quad = 0$$

Putting it all together, we have our main result:

Theorem 2.3.1. *Let L be an oriented unframed link of c components and (G, X) be a finite augmented birack of characteristic N. For each $\phi \in Z^2_{NR}(X)$, the multiset $\Phi^M_\phi(L)$ and polynomial $\Phi_\phi(L)$ defined by*

$$\Phi^M_\phi(L) = \{BW(f) \mid f \in \mathcal{L}(L_{\vec{w}}, (G, X)), \vec{w} \in (\mathbb{Z}_N)^c\}$$

and

$$\Phi_\phi(L) = \sum_{\vec{w}\in(\mathbb{Z}_N)^c} \left(\sum_{f\in\mathcal{L}(L_{\vec{w}},(G,X))} u^{BW(f)} \right)$$

are invariants of L known as the augmented birack 2-cocycle invariants of L.

Remark 2.3.2. We note that if $\phi \in Z^2(X)$ then the corresponding quantities,

$$\Phi_\phi^M(L_{\vec{w}}) = \{BW(f) \mid f \in \mathcal{L}(L_{\vec{w}},(G,X))\} \quad \text{and} \quad \Phi_\phi(L_{\vec{w}}) = \sum_{f\in\mathcal{L}(L_{\vec{w}},(G,X))} u^{BW(f)},$$

are invariants of $L_{\vec{w}}$ as a framed link.

Remark 2.3.3. If L is a virtual link, $\Phi_\phi^M(L)$ and $\Phi_\phi(L)$ are invariants of L under virtual isotopy via the usual convention of ignoring the virtual crossings.

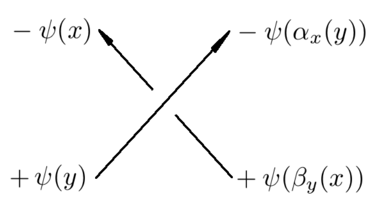
As in quandle homology, we have

Theorem 2.3.4. *Let (G,X) be an augmented birack. If $\phi \in Z^2(X)$ is a coboundary, then for any (G,X)-labeling f of a framed link $L_{\vec{w}}$ the Boltzmann weight $BW(f) = 0$.*

Proof. If $\phi \in H^2(X)$ is a coboundary, then there is a map $\psi \in H^1$ such that $\psi = \delta^2\phi = (\phi\delta_2)$. Then for any (x,y) we have

$$\phi(x,y) = \psi(\delta_2(x,y)) = \psi(y) - \psi(\alpha_x(y)) - \psi(x) + \psi(\beta_y(x))$$

and the Boltzmann weight can be pictured at a crossing as below.



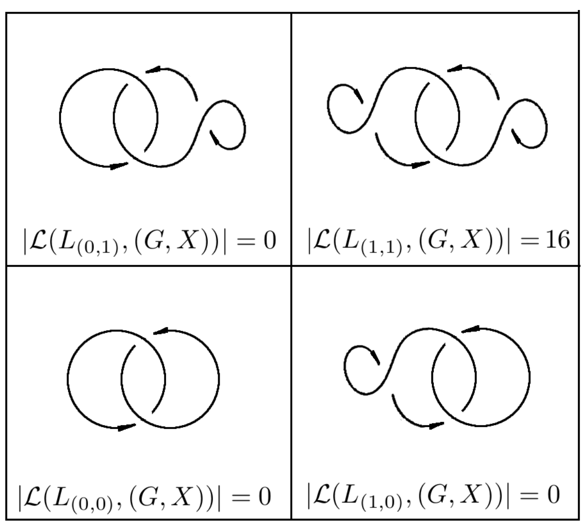
In particular, every semiarc labeled x contributes a $+\psi(x)$ at its tail and a $-\psi(x)$ at its head, so each semiarc contributes zero to the Boltzmann weight. □

Corollary 2.3.5. *Cohomologous cocycles define the same $\Phi_\phi(L)$ and $\Phi_\phi^M(L)$ invariants.*

Example 2.3.6. Let $X = \{1, 2, 3, 4\}$ be the set of four elements and $G = S_4$ the group of permutations of X. The pair (G, X) has augmented birack structures including

$$M_{(G,X)} = \left[\begin{array}{cccc} 2 & 3 & 3 & 2 \\ 4 & 1 & 1 & 4 \\ 1 & 4 & 4 & 1 \\ 3 & 2 & 2 & 3 \\ \hline 3 & 2 & 2 & 3 \\ 1 & 4 & 4 & 1 \\ 4 & 1 & 1 & 4 \\ 2 & 3 & 3 & 2 \end{array}\right].$$

This augmented birack has kink map $\pi = (14)(23)$ and hence characteristic $N = 2$. Thus, to find a complete tile of labelings of a link L, we'll need to consider diagrams of L with framing vectors $\vec{w} \in (\mathbb{Z}_2)^2 = \{(0,0), (0,1), (1,0), (1,1)\}$. Then for instance the Hopf link $L = L2a1$ has no labelings in framings $(0,0)$, $(1,0)$ and $(0,1)$ and sixteen labelings in framing $(1,1)$, for a counting invariant value of $\Phi^{\mathbb{Z}}_{(G,X)}(L2a1) = 16 + 0 + 0 + 0 = 16$.



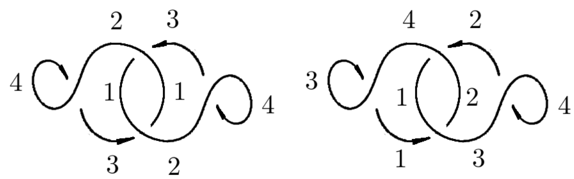
Note that $C^2(X)$ has $\mathbb{Z}$–basis $\{\chi_{ij} \mid i, j = 1, 2, 3, 4, 5\}$ where

$$\chi_{(i,j)}((i', j')) = \begin{cases} 1 & (i, j) = (i', j') \\ 0 & (i, j) \neq (i', j'). \end{cases}$$

The function $\phi : X \times X \to \mathbb{Z}$ defined by

$$\phi = \chi_{(2,1)} + \chi_{(2,4)} + \chi_{(3,1)} + \chi_{(3,4)}$$

is an N-reduced 2-cocycle in $H^2_{NR}(G, X)$. We then compute $\Phi_\phi(L)$ by finding the Boltzmann weight for each labeling.



$BW(f) = 0$ $\qquad$ $BW(f) = 1$

In the labeling on the left, we have

$$BW(f) = \phi(1,1) + \phi(1,1) + \phi(3,2) + \phi(3,2) = 0 + 0 + 0 + 0 = 0$$

and in the labeling on the right we have

$$BW(f) = \phi(1,2) + \phi(2,1) + \phi(2,3) + \phi(1,4) = 0 + 1 + 0 + 0 = 1.$$

Repeating for all 14 other labelings, we get $\Phi_\phi(L2a1) = 8 + 8u$. Similarly the unlink $L0a1$ and $(4,2)$-torus link $L4a1$ have counting invariant value $\Phi^{\mathbb{Z}}_{(G,X)}(L0a1) = \Phi^{\mathbb{Z}}_{(G,X)}(L4a1) = 16$ with respect to (G, X) but augmented birack cocycles invariant values $\Phi_\phi(L0a1) = 16$ and $\Phi_\phi(L4a1) = 8 + 8u^2$ respectively.

Chapter 3

n-ary Quandles

In this chapter we introduce the notion of n-ary distributive sets, specifically n-ary quandles, and discuss some notable properties.

Ternary and n-ary operations are natural generalizations of binary operations and appear in numerous areas of mathematics and physics. Ternary associative structures, a set X with operation μ satisfying the condition $\mu(\mu(x,y,z),u,v) = \mu(x,\mu(y,z,u),v) = \mu(x,y,\mu(z,u,v))$, have been considered, such as in papers by H. Ataguema and A. Makhlouf [4, 5]. Such structures are called totally associative algebras, differentiating them from partially associative Lie-type algebras. Early axiomatic treatment of non-associative structures appeared in 1949 in work by N. Jacobson [32]. He discussed Lie triple systems, a subspace of a Lie algebra that is closed with respect to the iterated Lie bracket, $[[x,y],z]$ in connection with problems in Jordan theory and quantum mechanics. Further progress in theoretical quantum mechanics and the generalization of Hamiltonian mechanics known as Nambu mechanics after its discovery by Y. Nambu [44] along with work by S. Okubo [48] fueled further work on n-ary algebras. Even more recently work in string theory and M-brane theory has lead to extensive work on Bagger-Lambert algebras, an algebra with a ternary operation [7]. More recent work in this area has led to the study of Lie n-racks by G. Biyogmam [11].

In this chapter we present the definition of n-ary distributive structures along with relevant morphisms and a number of examples. We will use the ternary case as an illustrative introduction to these concepts. We also present a number of constructions for ternary and general n-ary distributive structures, as well as a classification of ternary distributive structures of orders two and three.

The bulk of this chapter originally appeared in [22].

3.1 Definitions and Properties of n-ary Quandles

Definition 3.1.1. *A ternary quandle is a pair (Q,T) where Q is a set and $T : Q^{\times 3} \to Q$ is a ternary operation satisfying the following conditions:*

1. *For all $x, y, z, u, v \in Q$*

$$T(T(x,y,z),u,v) = T(T(x,u,v),T(y,u,v),T(z,u,v)). \quad \textit{(right distributivity)} \tag{3.1}$$

2. *For all $y, z \in Q$, the map $R_{y,z} : Q \to Q$ given by*

$$R_{y,z}(x) = T(x,y,z)$$

is invertible.

3. *For all $x \in Q$,*

$$T(x,x,x) = x. \tag{3.2}$$

If T satisfies only condition (1), then (Q,T) is said to be a ternary shelf. If both conditions (1) and (2) are satisfied then (Q,T) is said to be a ternary rack.

This ternary generalization is easily extended to the n-ary case as follows:

Definition 3.1.2. *An n-ary quandle is a pair (Q,T) where Q is a set and $T : Q^{\times n} \to Q$ is an n-ary operation satisfying the following conditions:*

1.

$$T(T(x_1,\dots,x_n),u_1,\dots,u_{n-1}) =$$
$$T(T(x_1,u_1,\dots,u_{n-1}),T(x_2,u_1,\dots,u_{n-1}),\dots,T(x_n,u_1,\dots,u_{n-1})),$$

$\forall x_i, u_i \in Q$ *(n-ary distributivity).*

2. *For all $a_1,\dots,a_{n-1} \in Q$, the map $R_{a_1,\dots,a_{n-1}} : Q \to Q$ given by*

$$R_{a_1,\dots,a_{n-1}}(x) = T(x,a_1,\dots,a_{n-1})$$

is invertible.

3. For all $x \in Q$,

$$T(x, \cdots, x) = x.$$

If T satisfies only condition (1), then (Q, T) is said to be an n-ary shelf. If both conditions (1) and (2) are satisfied then (Q, T) is said to be an n-ary rack.

This leads us to natural definitions for n-ary rack and n-ary quandle morphisms.

Definition 3.1.3. *Let (Q_1, T_1) and (Q_2, T_2) be two n-ary racks (resp. quandles). A map $\phi : Q_1 \to Q_2$ is said to be a n-ary rack (resp. quandle) morphism if $\phi(T_1(x_1, \ldots, x_n)) = T_2(\phi(x_1), \ldots, \phi(x_n))$ for all $x \in Q_1$.*

$$\begin{array}{ccc} Q_1^{\times n} & \xrightarrow{T_1} & Q_1 \\ \downarrow{\scriptstyle \phi^{\times n}} & & \downarrow{\scriptstyle \phi} \\ Q_2^{\times n} & \xrightarrow{T_2} & Q_2 \end{array}$$

To illustrate these definitions we present a number of examples of n-ary quandles.

Example 3.1.4. *For any set Q let T be an n-ary operation over Q defined:*

$$T(x_1, x_2, \ldots, x_n) = x_1.$$

Then (Q, T) is a n-ary quandle, called the trivial n-ary quandle over Q.

Example 3.1.5. *Define the n-ary operation T over $\mathbb{Z}_m$ by*

$$T(x_1, \ldots, x_n) = a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

where $\sum_{i=1}^{n} a_i = 1 \mod m$.

Then $(\mathbb{Z}_m, T)$ is a n-ary quandle. We call this an affine n-ary quandle.

Example 3.1.6. *For any quandle $(Q, \triangleright)$, the operation $T(x_1, x_2, \ldots, x_n) = (\cdots(x_1 \triangleright x_2) \triangleright x_3) \cdots \triangleright x_n)$ defines an n-ary quandle structure on Q.*

For the sake of readability we will make use of a more succinct notation, where $x_1x_2 = x_1 \triangleright x_2$ and for longer strings, we iterate the operation from left to right, i.e. $x_1x_2x_3 = (x_1 \triangleright x_2) \triangleright x_3$ and $x_1x_2 \ldots x_n = (\cdots(x_1 \triangleright x_2) \triangleright x_3) \cdots) \triangleright x_n$.

Thus the distributivity condition for standard quandles (1.1) can be written as $xyz = (xz)(yz)$*. Now we note applying (1.1) to* $x_1x_2\dots x_{i-1}x_ix_{i+1}$ *shows*

$$x_1x_2\dots x_{i-1}x_ix_{i+1} = (x_1x_2\dots x_{i-1}x_{i+1})(x_ix_{i+1})$$

$$T(T(x_1,x_2,\dots,x_n),x_{n+1},\dots,x_{2n-1}) = x_1x_2x_3\dots x_nx_{n+1}\dots x_{2n-1}$$

$$= ((x_1\dots x_{n-1}x_{n+1})(x_nx_{n+1}))x_{n+2}x_{n+3}\dots x_{2n-1}.$$

Iterating this process of distributing x_i *for* $n+2 \le i \le 2n-1$ *gives us:*

$$= (x_1\dots x_{n-1}x_{n+1}\dots x_{2n-1})(x_nx_{n+1}\dots x_{2n-1}).$$

Repeating the same process of distributing $x_{n+1}\dots x_{2n-1}$ *to the first* $i-1$ *terms and the* i*-th term in succession for* $i = n-1$ *to* $i = 2$ *gives us:*

$$= (x_1x_{n+1}\dots x_{2n-1})(x_2x_{n+1}\dots x_{2n-1})\dots(x_nx_{n+1}\dots x_{2n-1})$$

$$= T(x_1,x_{n+1},\dots,x_{2n-1})T(x_2,x_{n+1},\dots,x_{2n-1})\dots T(x_1,x_{n+1},\dots,x_{2n-1})$$

$$= T(T(x_1,x_{n+1},\dots,x_{2n-1}),T(x_2,x_{n+1},\dots,x_{2n-1}),\dots,T(x_1,x_{n+1},\dots,x_{2n-1})).$$

Since R_{x_i} *is a bijection,* $R_{x_2x_3\dots x_n} = R_{x_n}\circ R_{x_{n-1}}\circ\cdots\circ R_{x_2}$ *is as well. The idempotency of the resulting operator is clear.*

We call (Q,T) *the* n*-ary quandle induced by* $(Q,\triangleright)$*.*

Note that the above example also shows that iterating a rack or shelf would similarly result in an n-ary rack or n-ary shelf respectively.

Remark 3.1.7. *We note at this time that just as we may derive an* n*-ary quandle from a binary quandle, we may also derive a binary quandle from an* n*-ary quandle by defining* $a \triangleright b = T(a,b,b,\dots,b)$*. We further note that this will not in general return the original binary quandle from the induced* n*-ary quandle.*

Example 3.1.8. *Consider an affine* n*-ary quandle* $(\mathbb{Z}_m,T)$*. Then* $(\mathbb{Z}_m,\triangleright)$*, where* $\triangleright$ *is defined*

$$s \triangleright t = T(s,t,t,\dots,t) = a_1s + (a_2+\dots+a_n)t.$$

As $a_1 + (a_2 + \cdots + a_n) = 1 \mod m$*, by substituting* $a_2 + a_3 + \cdots + a_n = 1 - a_1$ *we see* $(\mathbb{Z}_m, \triangleright)$ *is a standard affine quandle.*

Example 3.1.9. *Let* G *be a group, and let* $n = 2k+1$ *for some* $k \in \mathbb{Z}$*. Then* (G, T) *where* T *is an* n*-ary operation defined* $T(x_1, x_2, \dots, x_n) = x_1 x_2^{-1} x_3 \dots x_{n-1}^{-1} x_n$ *is an* n*-ary quandle.*

It is obvious that this structure fulfills the second and third axioms. Starting with the right hand side of the first axiom, we see:

$$\begin{aligned}
&T(T(x_1, y_2, \dots, y_n), T(x_2, y_2, \dots, y_n), \dots, T(x_n, y_2, \dots, y_n)) \\
&\qquad = x_1 y_2^{-1} \dots y_n (x_2 y_2^{-1} \dots y_n)^{-1} \dots x_n y_2^{-1} \dots y_n \\
&\qquad = x_1 y_2^{-1} \dots y_n y_n^{-1} y_{n-1} \dots y_2 x_2^{-1} \dots x_n y_2^{-1} \dots y_n \\
&\qquad = x_1 x_2^{-1} \dots x_n y_2^{-1} \dots y_n = T(T(x_1, x_2, \dots, x_n), y_2, \dots y_n).
\end{aligned}$$

Thus the first axiom holds.

3.2 Construction and Classification of Ternary Quandles

Having provided a number of methods for constructing n-ary quandles in the previous section, we give in this section the classification of ternary quandles up to isomorphisms. We provide all ternary quandles of order 2 and 3. Moreover, we describe an additional method of constructing ternary quandles coming from groups.

3.2.1 Ternary quandles of order two

We have the following lemma which states that there are two non-isomorphic ternary quandle structures on a set of two elements.

Lemma 3.2.1. *In size two, all ternary quandles are affine, and are divided into two isomorphism classes, represented by the trivial ternary quandle, and the one with* $T(x, y, z) = x + y + z$ (mod 2).

Proof. Let $Q = \{1,2\}$ and T be a ternary quandle operation on Q. Then we have $T(1,1,1) = 1$ and $T(2,1,1) = 2$. Similarly, we have $T(2,2,2) = 2$ and $T(1,2,2) = 1$. Now we need to choose a value for $T(1,1,2)$. We distinguish two cases:

Case 1: Assume $T(1,1,2) = 1$, this implies $T(2,1,2) = 2$ (by second axiom). We claim that in this case $T(1,2,1)$ can not equal 2, otherwise $T(2,2,1) = 1$ (again axiom (II)). Now use axiom (I) of right-self-distributivity to get

$$T(T(2,1,2),2,1) = T(T(2,2,1), T(1,2,1), T(2,2,1))$$

implying that $T(2,2,1) = T(1,2,1)$ but this contradicts the bijectivity of axiom (II). Then $T(1,2,1) = 1$ and $T(2,2,1) = 2$. This ends the proof for case 1.

Case 2: Assume $T(1,1,2) = 2$, this implies $T(2,1,2) = 1$ (by second axiom). As in case 1, we prove similarly that $T(1,2,1)$ can not equal 1, thus $T(1,2,1) = 2$ and $T(2,2,1) = 1$. Now, the only non-trivial bijection of the set $\{1,2\}$ is the transposition sending 1 to 2. It's easy to see that this transposition is not a homomorphism between the two ternary quandles given in case 1 and case 2. □

3.2.2 Ternary quandles of order three

To help classify the ternary quandles two observations are useful. First we note that every ternary quandle is related to some (binary) quandle.

Remark 3.2.2. *If (Q,τ) is a ternary quandle, then $(Q,*)$, where $x * y = \tau(x,y,y)$ is a (binary) quandle.*

We shall refer to this related quandle as the *associated quandle*. We now consider how the relation between associated quandles extends to the ternary quandles.

Lemma 3.2.3. *Let (Q,τ) be a ternary quandle, and $(Q,*)$, be the associated quandle defined by $x * y = \tau(x,y,y)$. If $(R,*')$ is a quandle such that $(Q,*) \cong (R,*')$, then there exists a ternary quandle $(R,\tau') \cong (Q,T)$ such that $x *' y = \tau'(x,y,y)$.*

Proof. This is easily shown by setting $\tau'(x,y,z) = \phi(\tau(\phi^{-1}(x),\phi^{-1}(y),\phi^{-1}(z)))$ where $\phi : Q \to R$ is an isomorphism from $(Q,*)$ to $(R,*')$. □

Remark 3.2.4. *While we are limiting our discussion to ternary quandle in this section, we note at this time that just as remark 3.2.2 is a special case of remark 3.1.7, lemma 3.2.3 is easily extended to the general case of n-ary quandles.*

With these facts we now see that we may limit the task of generating isomorphically distinct ternary quandles by generating them based on isomorphically distinct quandles. We developed a simple program using the conditions defining a ternary quandle to compute all ternary quandles of order 3. The output of which we used to obtain the following result.

Lemma 3.2.5. *There are 31 isomophically distinct ternary quandles of order 3. Six of these are affine: the trivial ternary quandle τ_0, as well as two more with trivial associated quandle, τ_{14} defined $\tau(x,y,z) = x+y+2z \ (mod\ 3)$, and τ_{15} defined $\tau(x,y,z) = x+2y+z \ (mod\ 3)$, as wells as three with the connected associated quandle, τ_1 defined $\tau(x,y,z) = 2x+2z \ (mod\ 3)$, τ_2 defined $\tau(x,y,z) = 2x+y+z \ (mod\ 3)$, and τ_5 defined $\tau = 2x+2y \ (mod\ 3)$.*

Additionally we found that 14 were connected (that is, the group generated by the maps $R_{a,b}$ acts transitively on the set), including the non-trivial affine structures, as well as the remaining structures with the connected quandle as their associated quandle and 6 with the trivial associated quandle $\tau_6, \tau_7, \tau_{10}, \tau_{12}, and \tau_{13}$ through τ_{16}.

Since for each fixed a, b, the map $x \mapsto \tau(x, a, b)$ is a permutation, then in the following table we describe all ternary quandles of order three in terms of the columns of the Cayley table. Each column is a permutation of the elements and is described in standard notation that is by explicitly writing it in terms of products of disjoint cycles. Thus for a given z we give the permutations resulting from fixing $y = 1, 2, 3$. For example, the ternary set $\tau_{12}(x, y, z)$ with the Cayley Table 1 will be represented with the permutations $(1), (12), (13); (12), (1), (23); (13), (23), (1)$. This will appear on Table 3 as shown in Table 2.

Additionally we organize the table based on the associated quandle, given in similar permutation notation.

Table 1: Cayley representation of ternary quandle τ_{12}

z=1			z=2			z=3		
1	2	3	2	1	1	3	1	1
2	1	2	1	2	3	2	3	2
3	3	1	3	3	2	1	2	3

Table 2: Permutation representation of ternary quandle τ_{12}

τ	z=1	z=2	z=3
τ_{12}	(1),(12),(13)	(12),(1),(23)	(13),(23),(1)

Table 3: Isomorphism classes of ternary quandles of order 3

Ternary Distributive Sets With Associated Quandle (1),(1),(1)							
τ	z=1	z=2	z=3	τ	z=1	z=2	z=3
τ_0	(1),(1),(1)	(1),(1),(1)	(1),(1),(1)	τ_1	(1),(1),(1)	(1),(1),(1)	(12),(12),(1)
τ_2	(1),(1),(1)	(1),(1),(23)	(1),(23),(1)	τ_3	(1),(1),(1)	(23),(1),(1)	(23),(1),(1)
τ_4	(1),(1),(1)	(23),(1),(23)	(23),(23),(1)	τ_5	(1),(1),(12)	(1),(1),(12)	(12),(12),(1)
τ_6	(1),(1),(123)	(123),(1),(1)	(1),(123),(1)	τ_7	(1),(1),(132)	(132),(1),(1)	(1),(132),(1)
τ_8	(1),(1),(13)	(13),(1),(13)	(13),(1),(1)	τ_9	(1),(23),(23)	(23),(1),(23)	(23),(23),(1)
τ_{10}	(1),(23),(23)	(13),(1),(13)	(12),(12),(1)	τ_{11}	(1),(12),(12)	(12),(1),(12)	(1),(1),(1)
τ_{12}	(1),(12),(13)	(12),(1),(23)	(13),(23),(1)	τ_{13}	(1),(123),(123)	(123),(1),(123)	(123),(123),(1)
τ_{14}	(1),(123),(132)	(132),(1),(123)	(123),(132),(1)	τ_{15}	(1),(132),(123)	(123),(1),(132)	(132),(123),(1)
τ_{16}	(1),(13),(12)	(23),(1),(12)	(23),(13),(1)				
Ternary Distributive Sets With Associated Quandle (1),(1),(12)							
τ	z=1	z=2	z=3	τ	z=1	z=2	z=3
τ_0	(1),(1),(1)	(1),(1),(1)	(1),(1),(12)	τ_1	(1),(1),(1)	(1),(1),(1)	(12),(12),(12)
τ_2	(1),(1),(12)	(1),(1),(12)	(1),(1),(12)	τ_3	(1),(1),(12)	(1),(1),(12)	(12),(12),(12)
τ_4	(1),(12),(1)	(12),(1),(1)	(1),(1),(12)	τ_5	(1),(12),(1)	(12),(1),(1)	(12),(12),(12)
τ_6	(1),(12),(12)	(12),(1),(12)	(1),(1),(12)	τ_7	(1),(12),(12)	(12),(1),(12)	(12),(12),(12)
Ternary Distributive Sets With Associated Quandle (23),(13),(12)							
τ	z=1	z=2	z=3	τ	z=1	z=2	z=3
τ_0	(23),(1),(1)	(1),(13),(1)	(1),(1),(12)	τ_1	(23),(23),(23)	(13),(13),(13)	(12),(12),(12)
τ_2	(23),(12),(13)	(12),(13),(23)	(13),(23),(12)	τ_3	(23),(123),(132)	(132),(13),(123)	(123),(132),(12)
τ_4	(23),(132),(123)	(123),(13),(132)	(132),(123),(12)	τ_5	(23),(13),(12)	(23),(13),(12)	(23),(13),(12)

3.2.3 Ternary distributive structures from groups

We will briefly discuss ternary distributive structures coming from groups. We have the following sufficient conditions.

Lemma 3.2.6. *Let x, y, z be three fixed elements in a group G. Let $w(x, y, z) = a_1^{e_1} a_2^{e_2} \dots a_n^{e_n}$ such that $a_i \in \{x, y, z\}$ and $e_i = \pm 1$. If w is defined such that (I) $\sum_{i=1}^{n} e_i = 1$, (II) there exists a unique i such that $a_i = x$, and (III) w(x,y,z) satisfies equation (3.1) of Definition 3.1.1, then w defines a ternary quandle over the group G.*

The condition $\sum_{i=1}^{n} e_i = 1$ ensures that $w(x, x, x) = x$ and the condition of a unique x ensures $R_{y,z}$ is invertible.

Example 3.2.7. *Using the sufficient conditions, we found three families of group words defining ternary quandles over G. Words of the form $x(a^{-1}b)^n$, $(ab^{-1})^n x$ and wxw^{-1}, where $a, b \in \{y, z\}$, and w is any word over $\{y, z\}$.*

Proof. The first and second conditions are immediately clear, we need only show

$$w(w(x, y, z), u, v) = w(w(x, u, v), w(y, u, v), w(z, u, v)).$$

First consider words of the form $w(x, y, z) = x(a^{-1}b)^n$ where $a, b \in \{y, z\}$. Then we have $w(w(x, y, z), u, v) = x(a^{-1}b)^n(c^{-1}d)^n$ where c, d represent the corresponding u, v. The right hand side become:

$$\begin{aligned}
w(w(x, u, v), w(y, u, v), w(z, u, v)) &= w(x(c^{-1}d)^n, y(c^{-1}d)^n, z(c^{-1}d)^n) \\
&= x(c^{-1}d)^n((c^{-1}d)^{-n}a^{-1}b(c^{-1}d)^n)^n \\
&= x(c^{-1}d)^n(c^{-1}d)^{-n}a^{-1}b(c^{-1}d)^n((c^{-1}d)^{-n}a^{-1}b(c^{-1}d)^n)^{n-1} \\
&= xa^{-1}b(c^{-1}d)^n((c^{-1}d)^{-n}a^{-1}b(c^{-1}d)^n)^{n-1} \\
&= x(a^{-1}b)^2(c^{-1}d)^n((c^{-1}d)^{-n}a^{-1}b(c^{-1}d)^n)^{n-2} \\
&= x(a^{-1}b)^n(c^{-1}d)^n.
\end{aligned}$$

The proof for words of the form $(a^{-1}b)^n x$ is a mirror of the above.

Now we consider words of the form $w(x,y,z) = \bar{w}(y,z)x\bar{w}^{-1}(y,z)$. Considering the left hand side, we see $w(w(x,y,z),u,v) = \bar{w}(u,v)\bar{w}(y,z)x\bar{w}^{-1}(y,z)\bar{w}^{-1}(u,v)$. The right hand side reduces as follows:

$$\begin{aligned}
&w(w(x,u,v),w(y,u,v),w(z,u,v)) \\
&\quad = w(\bar{w}(u,v)x\bar{w}^{-1}(u,v),\bar{w}(u,v)y\bar{w}^{-1}(u,v),\bar{w}(u,v)z\bar{w}^{-1}(u,v)) \\
&\quad = \bar{w}(\bar{w}(u,v)y\bar{w}^{-1}(u,v),\bar{w}(u,v)z\bar{w}^{-1}(u,v))\bar{w}(u,v)x\bar{w}^{-1}(u,v) \\
&\quad \bar{w}^{-1}(\bar{w}(u,v)y\bar{w}^{-1}(u,v),\bar{w}(u,v)z\bar{w}^{-1}(u,v)).
\end{aligned}$$

Now note that for $\bar{w}(\bar{w}(u,v)y\bar{w}^{-1}(u,v),\bar{w}(u,v)z\bar{w}^{-1}(u,v))$ each 'character' in the word is prefixed by $\bar{w}(u,v)$ and suffixed by $\bar{w}^{-1}(u,v)$, thus these terms will cancel inside the word and we have

$$\bar{w}(\bar{w}(u,v)y\bar{w}^{-1}(u,v),\bar{w}(u,v)z\bar{w}^{-1}(u,v)) = \bar{w}(u,v)\bar{w}(y,z)\bar{w}^{-1}(u,v).$$

Similarly $\bar{w}^{-1}(\bar{w}(u,v)y\bar{w}^{-1}(u,v),\bar{w}(u,v)z\bar{w}^{-1}(u,v)) = \bar{w}(y,z)\bar{w}^{-1}(u,v)$. Thus,

$$\begin{aligned}
&w(w(x,u,v),w(y,u,v),w(z,u,v)) \\
&\quad = \bar{w}(u,v)\bar{w}(y,z)\bar{w}^{-1}(u,v)\bar{w}(u,v)x\bar{w}^{-1}(u,v)\bar{w}(y,z)\bar{w}^{-1}(u,v) \\
&\quad = \bar{w}(u,v)\bar{w}(y,z)x\bar{w}^{-1}(y,z)\bar{w}^{-1}(u,v).
\end{aligned}$$

And all three groups are shown to fulfill the third condition. □

Chapter 4

n-ary f-quandles

In this chapter we introduce the notion of n-ary f-distributive sets, specifically n-ary f-quandles, a generalization of the n-ary distributive sets, and discuss a number of their properties and relations to both n-ary quandles and binary f-quandles.

This type of generalization, which introduces a map that deforms the standard operation, was first introduced in the context of quantum deformations of algebras over vector fields. Generally known as Hom-algebras, a study of Lie type algebras was done by Hartwig, Larsson and Silvestrov in [31], and n-ary Hom-type algebras were addressed by Ataguema, Makhlouf, and Silvestrov in [6]. The unifying feature of these generalizations is the introduction of a homomophism which 'twists' the usual identities. Initial work applying this type of generalization to the binary quandle structure was introduced in [16].

The bulk of the material in the following three chapters originally appeared in [17].

4.1 Definitions and Properties of n-ary f-quandles

In this section we present a further generalization of the quandle, applying our n-ary generalization of the quandle to the f-quandles discussed in the introduction and originally presented in [16]. Again we will begin by presenting the ternary case, before discussing the general n-ary case.

Definition 4.1.1. *A ternary f-distributive set is a triple (Q, T, f) where Q is a set, f is a map, and $T : Q^{\times 3} \to Q$ is a ternary operation satisfying the following conditions:*

1. *For all $x, y, z, u, v \in Q$*

$$T(T(x,y,z), f(u), f(v)) = T(T(x,u,v), T(y,u,v), T(z,u,v)). \quad \textit{(right distributivity)} \tag{4.1}$$

2. *For all $y, z, w \in Q$, there exists a unique x such that*

$$T(x, y, z) = f(w)$$

is invertible.

3. *For all $x \in Q$,*

$$T(x, x, x) = f(x). \tag{4.2}$$

If T satisfies only condition (1), then (Q, T, f) is said to be a ternary f-shelf. If both conditions (1) and (2) are satisfied then (Q, T, f) is said to be a ternary f-rack. If all three conditions (1), (2) and (3) are satisfied then (Q, T, f) is said to be a ternary f-quandle.

Extending this definition to the general n-ary case leads us to the following definition.

Definition 4.1.2. *An n-ary f-distributive set is a triple (Q, T, f) where Q is a set, f is a map, and $T : Q^{\times n} \to Q$ is an n-ary operation satisfying the following conditions:*

1.

$$T(T(x_1, \ldots, x_n), f(u_1), \ldots, f(u_{n-1})) =$$
$$T(T(x_1, u_1, \ldots, u_{n-1}), T(x_2, u_1, \ldots, u_{n-1}), \ldots, T(x_n, u_1, \ldots, u_{n-1})),$$

$\forall x_i, u_i \in Q$ *(distributivity).*

2. *For all $a_1, \ldots, a_{n-1} \in Q$, the map $R_{a_1,\ldots,a_{n-1}} : Q \to Q$ given by*

$$R_{a_1,\ldots,a_{n-1}}(x) = T(x, a_1, \ldots, a_{n-1})$$

is invertible.

3. *For all $x \in Q$,*

$$T(x, \ldots, x) = f(x).$$

If T satisfies only condition (1), then (Q, T, f) is said to be an n-ary f-shelf. If both conditions (1) and (2) are satisfied then (Q, T, f) is said to be an n-ary f-rack. If all three conditions (1), (2) and (3) are satisfied then (Q, T, f) is said to be an n-ary f-quandle.

It is useful to note the following.

Proposition 4.1.3. *If (X, T, f) is an n-ary f-quandle of finite order, then right multiplication $R_{x_2,x_3,\ldots,x_n}(y) = T(y, x_2, x_3, \ldots, x_n)$ is a bijection.*

Proof. Assume $T(y, x_2, x_3, \ldots, x_n) = T(z, x_2, x_3, \ldots, x_n)$. Then, $f(T(y, x_2, x_3, \ldots, x_n)) = f(T(z, x_2, x_3, \ldots, x_n)) = f(T(z, x_2, x_3, \ldots, x_n)$ and by the second condition of definition, $f(y) = f(z)$. Then

$$\begin{aligned} & T(T(y, z, z, \ldots, z), f(x_2), f(x_3), \ldots, (x_n)) \\ &= T(T(y, x_2, \ldots, x_n), T(z, x_2, \ldots, x_n), \ldots, T(z, x_2, \ldots, x_n)) \\ &= T(T(z, x_2, \ldots, x_n), T(z, x_2, \ldots, x_n), \ldots, T(z, x_2, \ldots, x_n)) \\ &= f(T(z, x_2, \ldots, x_n)) \\ &= T(f(z), f(x_2), \ldots, f(x_n)) \end{aligned}$$

Again, by the second condition of 4.1, we have $T(y, z, z, \ldots, z) = f(z) = T(z, z, \ldots, z)$. Thus $y = z$. $\square$

The notion of morphism of n-ary f-quandles is given in the following definition, requiring a preservation of the n-ary operation, T, as well as the action of the morphism f.

Definition 4.1.4. *Let (Q_1, T_1, f_1) and (Q_2, T_2, f_2) be two n-ary f-racks (resp. f-quandles, f-shelves). A map $\phi : Q_1 \to Q_2$ is an n-ary f-rack (resp. f-quandle, f-shelve) morphism if it satisfies the conditions:*

$$\phi(T_1(x_1, x_2, \ldots, x_n)) = T_2(\phi(x_1), \phi(x_2), \ldots, \phi(x_n))$$

and

$$\phi \circ f_1 = f_2 \circ \phi$$

$$\begin{array}{ccc} Q_1^{\times n} & \xrightarrow{T_1} & Q_1 \\ \downarrow{\scriptstyle \phi^{\times n}} & & \downarrow{\scriptstyle \phi} \\ Q_2^{\times n} & \xrightarrow{T_2} & Q_2 \end{array}$$

Note that in the case of n-ary quandles, condition 3 forces the morphism f to be dependent upon the operation T. This leads us to the following remarks.

Remark 4.1.5. *In the case of n-ary f-quandles, the first condition implies the second condition as condition 3 gives*

$$\phi \circ f_1(x) = \phi \circ T_1(x, x, \dots, x).$$

Remark 4.1.6. *Let (Q, T, f) be an n-ary f-quandle, then f is an n-ary f-quandle homomorphism.*

$$\begin{aligned} T(f(x_1), f(x_2), \dots, f(x_n)) &= T(T(x_1, x_1, \dots, x_1), f(x_2), \dots, f(x_n)) \\ = T(T(x_1, x_2, \dots, x_n, &T(x_1, x_2, \dots, x_n), \dots, T(x_1, x_2, \dots, x_n)) \\ &= f(T(x_1, x_2, \dots, x_n)). \end{aligned}$$

Now we give an example of an n-ary f-quandle.

Example 4.1.7. *Define the n-ary operation T over $\mathbb{Z}_m$ by*

$$T(x_1, x_2, \dots, x_n) = a_1 x_1 + a_2 x_2 + \dots + a_n x_n.$$

Then $(\mathbb{Z}_m, T)$ is a n-ary f-quandle. We call this an affine n-ary f-quandle.

Example 4.1.8. *We may obtain a ternary f-quandle (X, T, F) from a binary f-quandle $(X, *, f)$ by defining $T(a, b, c) = (a * b) * f(c)$ and $F(x) = f^2(x)$. Starting with the left-hand side of condition 4.3 we see,*

$$\begin{aligned} &T(T(x, y, z), F(u), F(v)) \\ &\quad = (((x * y) * f(z)) * f^2(u)) * f^3(v) \\ &\quad = (((x * y) * f(u)) * (f(z) * f(u))) * f^3(v) \\ &\quad = (((x * y) * f(u)) * f^2(v)) * ((f(z) * f(u)) * f^2(v)) \\ &\quad = (((x * u) * (y * u)) * f^2(v)) * f((z * u) * f(v)) \\ &\quad = (((x * u) * f(v)) * ((y * u) * f(v))) * f(T(z, u, v)) \\ &\quad = (T(x, u, v) * T(y, u, v)) * f(T(z, u, v)) \\ &\quad = T(T(x, u, v), T(y, u, v), T(z, u, v)). \end{aligned}$$

Thus distributivity holds. As $R_{b,c} = r_{f(c)} \circ r_b$, where r_a is right muliplication by a in the binary f-quandle, the bijectivity of right multiplication of binary f-quandles extends to T. Finally $T(x,x,x) = (x * x) * f(x) = f(x) * f(x) = f^2(x) = F(x)$, thus all three conditions hold.

Example 4.1.9. *In the following example we extend the iteration to give a procedure for obtaining a n-ary f-quandle from a binary one. Let $(X, *, f)$ be a binary f-quandle and define $T(x_1, x_2, x_3, \ldots, x_n) = ((\cdots (x_1 * x_2) * f(x_3)) * \cdots) * f^{n-2}(x_n)$ and $F(x) = f^{n-1}(x)$.*

First we see that,

$$
\begin{aligned}
&T(x, x, \ldots, x) \\
&\qquad = (\cdots((x * x) * f(x)) * f^2(x)) * \cdots) * f^{n-2}(x) \\
&\qquad = (\cdots(f(x) * f(x)) * f^2(x)) * \cdots) * f^{n-2}(x) \\
&\qquad = (\cdots(f^2(x) * f^2(x)) * f^3(x)) * \cdots) * f^{n-2}(x) \\
&\qquad = \cdots = f^{n-2}(x) * f^{n-2} = f^{n-1}(x) = F(x).
\end{aligned}
$$

As in the ternary case, $R_{x_2,x_3,\ldots,x_n} = r_{f^{n-2}(x_n)} \circ \ldots \circ r_b$ again the bijectivity of right multiplication extends from the binary to the n-ary case.

Finally, we can see that distributivity holds as follows:

$$
\begin{aligned}
&T(T(x_1, x_2, \ldots, x_n), F(u_2), F(u_3), \ldots, F(u_n)) \\
=&(\cdots(x_1 * x_2) * f(x_3)) * \cdots) * f^{n-2}(x_n)) * f^{n-1}(u_2)) * f^n(u_3)) * \cdots) * f^{2n-3}(u_n) \\
=&(\cdots([..(x_1 * x_2) * f(x_3)) * \cdots) * f^{n-3}(x_{n-1})) * f^{n-2}(u_2)] * [f^{n-2}(x_n) * f^{n-2}(u_2)]) \\
&* f^n(u_3)) * \cdots) * f^{2n-3}(u_n) \\
=&[(\cdots(x_1 * x_2) * f(x_3)) * \cdots) * f^{n-3}(x_{n-1})) * f^{n-2}(u_2)) * f^{n-1}(u_3)) * \cdots) * f^{2n-4}(u_n)] \\
&* [(\cdots(f^{n-2}(x_n) * fn-2(u_2)) * f^{n-1}(u_3)) * \cdots) * f^{2n-4}(u_n)] \\
=&[(\cdots(x_1 * x_2) * f(x_3)) * \cdots) * f^{n-3}(x_{n-1})) * f^{n-2}(u_2)) * f^{n-1}(u_3)) * \cdots) * f^{2n-4}(u_n)] \\
&* f^{n-2}(T(x_n, u_2, \ldots, u_n)) \\
=&([(\cdots(x_1 * x_2) * f(x_3)) * \cdots) * f^{n-4}(x_{n-2})) * f^{n-3}(u_2)) * f^{n-2}(u_3)) * \cdots) * f^{2n-5}(u_n)] \\
&* [(\cdots(f^{n-3}(x_{n-1}) * f^{n-3}(u_2)) * f^{n-2}(u_3)) * \cdots) * f^{2n-5}(u_n)]) * f^{n-2}(T(x_n, u_2, \ldots, u_n)) \\
=&([(\cdots(x_1 * x_2) * f(x_3)) * \cdots) * f^{n-4}(x_{n-2})) * f^{n-3}(u_2)) * f^{n-2}(u_3)) * \cdots) * f^{2n-5}(u_n)] \\
&* f^{n-3}(T(x_{n-1}, u_2, \ldots, u_n))) * f^{n-2}(T(x_n, u_2, \ldots, u_n)) \\
=&(\cdots(T(x_1, u_2, \ldots, u_n) * T(x_2, u_2, \ldots, u_n)) * f(T(x_3, u_2, \ldots, u_n))) * \cdots) \\
&* f^{n-2}(T(x_n, u_2, \ldots, u_n)) \\
=&T(T(x_1, u_2, \ldots, u_n), T(x_2, u_2, \ldots, u_n)), \ldots, T(x_n, u_2, \ldots, u_n)).
\end{aligned}
$$

4.2 Construction and Classification of n-ary f-quandles

Proposition 4.2.1. *Let (X, T, f) be a finite f-quandle and ϕ be an n-ary f-quandle morphism. Then (X, T_ϕ, f_ϕ) is an f-quandle with $T_\phi(x_1, x_2, \ldots, x_n) = \phi(T(x_1, x_2, \ldots, x_n))$ and $f_\phi(x) = \phi(f(x))$ if and only if ϕ is an automorphism.*

Proof. Assume ϕ is an automorphism, and let $x_1, x_2, \ldots, x_n, u_2, u_3, \ldots, u_n \in X$.

$$\begin{aligned}
&T_\phi(T_\phi(x_1, \ldots, x_n), f_\phi(u_2), f_\phi(u_3), \ldots, f_\phi(u_n)) \\
&\qquad = \phi(T(\phi(T(x_1, \ldots, x_n)), \phi(f(u_2)), \ldots, \phi(f(u_n)))) \\
&\qquad = \phi^2(T(T(x_1, \ldots, x_n), f(u_2), \ldots, f(u_n))) \\
&\qquad = \phi^2(T(T(x_1, u_2, \ldots, u_n), T(x_2, u_2, \ldots, u_n), \ldots, T(x_n, u_2, \ldots, u_n)) \\
&\qquad = \phi(T(T_\phi(x_1, u_2, \ldots, u_n), \ldots, T_\phi(x_n, u_2, \ldots, u_n))) \\
&\qquad = T_\phi(T_\phi(x_1, u_2, \ldots, u_n), \ldots, T_\phi(x_n, u_2, \ldots, u_n)).
\end{aligned}$$

The remaining conditions are easier to see. As $R_{a_1,\ldots,a_{n-1}}$ is a bijection for all $a_1, \ldots, a_n$ so is $R_{\phi(a_1),\ldots,\phi(a_{n-1})}(x) = T(\phi(x), \phi(a_1), \ldots, \phi(a_{n-1})) = \phi(T(x, a_1, \ldots, a_{n-1}))$, while the final condition is immediately guaranteed since ϕ is an automorphism. $\square$

Remark 4.2.2. *Note that an n-ary quandle (rsp. rack, shelf), (Q, T) may be viewed as an n-ary f-quandle (resp. rack, shelf) (Q, T, id_Q), with the identity map as the morphism.*

Corollary 4.2.3. *In the case where f is the identity map, Proposition 4.2.1 shows that any usual quandle along with any automorphism gives rise to an f-quandle.*

Example 4.2.4. *Let $(\mathbb{Z}_m, T)$ be an n-ary quandle such that $T(x_1, x_2, \ldots, x_n) = a_1x_1 + a_2x_2 + \cdots + a_nx_n$ where $\sum a_n = 1$. Let $f(x) = x + b$. As $T(f(x_1), f(x_2), \ldots, f(x_n)) = \sum a_i(x_i+b) = \sum a_ix_i + \sum a_ib = \sum a_ix_i + b = f(T(x_1, x_2, \ldots x_n))$, f is an automorphism of (Z_m, T)*

Then (Z_m, T, f) is an n-ary f-quandle.

We note at this time that the classification of binary f-quandles given in [16].

Below we present a complete classification of connected ternary f-quandles of orders 2 and 3, giving a representative of each isomorphism class.

For order 2, we found 6 distinct isomorphism classes, each of which can be defined over $\mathbb{Z}_2$ by one of the following maps: $\tau(x, y, z) = x$, $\tau(x, y, z) = x + 1$, $\tau(x, y, z) = x + y$, $\tau(x, y, z) = x + z$, $\tau(x, y, z) = x + y + z$, or $\tau(x, y, z) = x + y + z + 1$.

Table 4: Isomorphism classes of ternary f-quandles of order 3

z=1	z=2	z=3	z=1	z=2	z=3
(1),(1),(1)	(1),(2 3),(1)	(1),(1),(2 3)	(1),(1),(1)	(1),(2 3),(2 3)	(1),(2 3),(2 3)
(1),(1),(1)	(2 3),(2 3),(1)	(2 3),(1),(2 3)	(1),(1),(1)	(2 3),(2 3),(2 3)	(2 3),(2 3),(2 3)
(1),(1),(1)	(1 2 3),(1 2 3),(1 2 3)	(1 3 2),(1 3 2),(1 3 2)	(1),(1),(1)	(1 3 2),(1 3 2),(1 3 2)	(1 2 3),(1 2 3),(1 2 3)
(1),(1),(1 2 3)	(1),(1 3 2),(1 3 2)	(1 2 3),(1 3 2),(1 2 3)	(1),(1),(1 3 2)	(1 2 3),(1 3 2),(1 3 2)	(1 2 3),(1),(1 2 3)
(1),(2 3),(2 3)	(1),(2 3),(1)	(1),(1),(2 3)	(1),(2 3),(2 3)	(1),(2 3),(2 3)	(1),(2 3),(2 3)
(1),(2 3),(2 3)	(2 3),(2 3),(1)	(2 3),(1),(2 3)	(1),(2 3),(2 3)	(2 3),(2 3),(2 3)	(2 3),(2 3),(2 3)
(1),(2 3),(2 3)	(2 3),(1 2 3),(2 3)	(2 3),(2 3),(1 3 2)	(1),(1 2),(1 3)	(1 3),(1 2 3),(1 2)	(1 2),(1 3),(1 3 2)
(1),(1 2 3),(1 2 3)	(1),(1 3 2),(1)	(1 3 2),(1 3 2),(1 2 3)	(1),(1 2 3),(1 3 2)	(1),(1 2 3),(1 3 2)	(1),(1 2 3),(1 3 2)
(1),(1 2 3),(1 3 2)	(1 2 3),(1 3 2),(1)	(1 3 2),(1),(1 2 3)	(1),(1 3 2),(1 2 3)	(1),(1 3 2),(1 2 3)	(1),(1 3 2),(1 2 3)
(1),(1 3 2),(1 2 3)	(1 3 2),(1 2 3),(1)	(1 2 3),(1),(1 3 2)	(1),(1 3),(1 2)	(1 2),(1 2 3),(1 3)	(1 3),(1 2),(1 3 2)
(2 3),(1),(1)	(1),(2 3),(1)	(1),(1),(2 3)	(2 3),(1),(1)	(1),(2 3),(2 3)	(1),(2 3),(2 3)
(2 3),(1),(1)	(2 3),(2 3),(1)	(2 3),(1),(2 3)	(2 3),(1),(1)	(2 3),(2 3),(2 3)	(2 3),(2 3),(2 3)
(2 3),(1),(1)	(1 2 3),(2 3),(1 2 3)	(1 3 2),(1 3 2),(2 3)	(2 3),(2 3),(2 3)	(1),(2 3),(1)	(1),(1),(2 3)
(2 3),(2 3),(2 3)	(1),(2 3),(2 3)	(1),(2 3),(2 3)	(2 3),(2 3),(2 3)	(2 3),(2 3),(1)	(2 3),(1),(2 3)
(2 3),(2 3),(2 3)	(2 3),(2 3),(2 3)	(2 3),(2 3),(2 3)	(2 3),(2 3),(2 3)	(1 2),(1 2),(1 2)	(1 3),(1 3),(1 3)
(2 3),(1 2),(1 3)	(2 3),(1 2),(1 3)	(2 3),(1 2),(1 3)	(2 3),(1 2),(1 3)	(1 3),(2 3),(1 2)	(1 2),(1 3),(2 3)
(2 3),(1 2 3),(1 3 2)	(1),(2 3),(1 3 2)	(1),(1 2 3),(2 3)	(2 3),(1 3 2),(1 2 3)	(1 3 2),(2 3),(1)	(1 2 3),(1),(2 3)
(2 3),(1 3),(1 2)	(1 2),(2 3),(1 3)	(1 3),(1 2),(2 3)	(2 3),(1 3),(1 2)	(1 3),(1 2),(2 3)	(1 2),(2 3),(1 3)
(1 2),(1),(1 3 2)	(1 3 2),(2 3),(1)	(1),(1 3 2),(1 3)	(1 2),(2 3),(1 3)	(1 2),(2 3),(1 3)	(1 2),(2 3),(1 3)
(1 2),(1 2),(1 2)	(2 3),(2 3),(2 3)	(1 3),(1 3),(1 3)	(1 2),(1 2 3),(1 2 3)	(1 2 3),(2 3),(1 2 3)	(1 2 3),(1 2 3),(1 3)
(1 2),(1 3 2),(1)	(1),(2 3),(1 3 2)	(1 3 2),(1),(1 3)	(1 2),(1 3),(2 3)	(1 3),(2 3),(1 2)	(2 3),(1 2),(1 3)
(1 2 3),(1),(1)	(1),(1 2 3),(1)	(1),(1),(1 2 3)	(1 2 3),(1),(1 2 3)	(1 2 3),(1 2 3),(1)	(1),(1 2 3),(1 2 3)
(1 2 3),(1),(1 3 2)	(1 3 2),(1 2 3),(1)	(1),(1 3 2),(1 2 3)	(1 2 3),(2 3),(1 3)	(1 2),(1 2 3),(1 3)	(1 2),(2 3),(1 2 3)
(1 2 3),(1 2),(1 2)	(2 3),(1 2 3),(2 3)	(1 3),(1 3),(1 2 3)	(1 2 3),(1 2 3),(1)	(1),(1 2 3),(1 2 3)	(1 2 3),(1),(1 2 3)
(1 2 3),(1 2 3),(1 2 3)	(1 2 3),(1 2 3),(1 2 3)	(1 2 3),(1 2 3),(1 2 3)	(1 2 3),(1 2 3),(1 3 2)	(1 3 2),(1 2 3),(1 2 3)	(1 2 3),(1 3 2),(1 2 3)
(1 2 3),(1 3 2),(1)	(1),(1 2 3),(1 3 2)	(1 3 2),(1),(1 2 3)	(1 2 3),(1 3 2),(1 2 3)	(1 2 3),(1 2 3),(1 3 2)	(1 3 2),(1 2 3),(1 2 3)
(1 2 3),(1 3 2),(1 3 2)	(1 3 2),(1 2 3),(1 3 2)	(1 3 2),(1 3 2),(1 2 3)	(1 2 3),(1 3),(2 3)	(1 3),(1 2 3),(1 2)	(2 3),(1 2),(1 2 3)

As mentioned in remark 4.2.2, the class of n-ary quandles can be considered a subclass of n-ary f-quandles. Thus we present in the following table only those isomorphism classes which contain no elements found in table 3.

Chapter 5

Extensions of n-ary quandles and n-ary f-quandles

In this chapter we investigate the notion of extension for ternary f-quandles. We define generalized cohomology theory for ternary f-quandle 3-cocycles and give examples. We give an explicit formula relating group 3-cocycles to Ternary f-quandle 3-cocycles, when the ternary f-quandle is constructed from a group. As in any standard algebraic structures we show that the second cohomology group classifies extensions [2, 13].

5.1 Extensions with dynamical cocycles and Extensions with constant cocycles

In this section we construct extensions following the ideas presented in [2].

Proposition 5.1.1. *Let (X, T, F) be a Ternary f-quandle and A be a non-empty set. Let $\alpha : X\times X\times X \to$ Fun$(A\times A\times A, A)$ be a function and $f, g : A \to A$ are maps. Then, $X\times A$ is a Ternary f-quandle by the operation $T((x, a), (y, b), (z, c)) = (T(x, y, z), \alpha_{x,y,z}(a, b, c))$, where $T(x, y, z)$ denotes the Ternary f-quandle product in X, if and only if α and g satisfies the following conditions:*

1. $\alpha_{x,x,x}(a, a, a) = g(a)$ *for all $x \in X$ and $a \in A$;*
2. $\alpha_{x,y,z}(-, b, c) : A \to A$ *is a bijection for all $x, y, z \in X$ and for all $b, c \in A$;*
3. $\alpha_{T(x,y,z),f(u),f(v)}(\alpha_{x,y,z}(a, b, c), g(d), g(e)) =$ $\alpha_{T(x,u,v),T(y,u,v),T(z,u,v)}(\alpha_{x,u,v}(a, d, e), \alpha_{y,u,v}(b, d, e), \alpha_{z,u,v}(c, d, e))$ *for all $x, y, z, u, v \in X$ and $a, b, c, d, e \in A$.*

Such function α is called a dynamical Ternary f-quandle cocycle or dynamical Ternary f-rack cocycle (when it satisfies above conditions).

The Ternary f-quandle constructed above is denoted by $X \times_\alpha A$, and it is called *extension* of X by a dynamical cocycle α. The construction is general, as Andruskiewitch and Graña showed in [2].

Assume (X, T, F) is a Ternary f-quandle and α be a dynamical f-cocycle with assomap g. For $x \in X$, define $T_x(a, b, c) := \alpha_{x,x,x}(a, b, c)$. Then it is easy to see that (A, T_x, F) is a Ternary f-quandle for all $x \in X$.

Remark 5.1.2. *When $x = y = z$ on (3) above, we get*
$\alpha_{f(x),f(u),f(v)}(\alpha_{x,x,x}(a, b, c), g(d), g(e)) =$
$\alpha_{T(x,u,v),T(x,u,v),T(x,u,v)}(\alpha_{x,u,v}(a, d, e), \alpha_{x,u,v}(b, d, e), \alpha_{x,u,v}(c, d, e))$
for all $a, b, c, d, e \in A$.

Now, we discuss Extensions with constant cocycles. Let (X, T, F) be a ternary f-rack and $\lambda : X \times X \times X \to S_A$ where S_A is group of permutations of A.
If $\lambda_{T(x,y,z),F(u),F(v)}\lambda_{x,y,z} = \lambda_{T(x,u,v),T(y,u,v),T(z,u,v)}\lambda_{x,u,v}$ we say λ is a *constant ternary f-rack cocycle*.
If (X, T, F) is a ternary f-quandle and further satisfies $\lambda_{x,x,x} = id$ for all $x \in X$, then we say λ is a *constant ternary f-quandle cocycle*.

5.2 Modules over Ternary f-rack

Here we introduce the notion of modules over ternary f-racks in order to define a generalized cohomology theory.

Definition 5.2.1. *Let (X, T, f) be a Ternary f-rack and A be an abelian group. A structure of X-module on A consists of an endomorphism g, a family of automorphisms $(\eta_{ijk})_{i,j,k \in X}$, and two families of endomorphisms $(\tau_{ijk})_{i,j,k \in X}$ and $(\mu_{ijk})_{i,j,k \in X}$ of A satisfying the fol-*

lowing conditions:

$$\eta_{T(x,y,z),f(u),f(v)}\eta_{x,y,z} = \eta_{T(x,u,v),T(y,u,v),T(z,u,v)}\eta_{x,u,v} \tag{5.1}$$

$$\eta_{T(x,y,z),f(u),f(v)}\tau_{x,y,z} = \tau_{T(x,u,v),T(y,u,v),T(z,u,v)}\eta_{y,u,v} \tag{5.2}$$

$$\eta_{T(x,y,z),f(u),f(v)}\mu_{x,y,z} = \mu_{T(x,u,v),T(y,u,v),T(z,u,v)}\eta_{z,u,v} \tag{5.3}$$

$$\tau_{T(x,y,z),f(u),f(v)}g = \eta_{T(x,u,v),T(y,u,v),T(z,u,v)}\tau_{x,u,v} + \tau_{T(x,u,v),T(y,u,v),T(z,u,v)}\tau_{y,u,v}$$
$$+\mu_{T(x,u,v),T(y,u,v),T(z,u,v)}\tau_{z,u,v} \tag{5.4}$$

$$\mu_{T(x,y,z),f(u),f(v)}g = \eta_{T(x,u,v),T(y,u,v),T(z,u,v)}\mu_{x,u,v} + \tau_{T(x,u,v),T(y,u,v),T(z,u,v)}\mu_{y,u,v}$$
$$+\mu_{T(x,u,v),T(y,u,v),T(z,u,v)}\mu_{z,u,v} \tag{5.5}$$

In the n-ary case, we generalized the above definition as follows.

Definition 5.2.2. *Let (X, T, f) be a n-ary f-rack, A be an abelian group. A structure of X-module on A consists of an endomorphism g, a family of automorphisms $(\eta_{i_1 i_2 \dots i_n})_{i_1,i_2,\dots,i_n \in X}$, and a family of endomorphisms $(\tau^i_{i_1 i_2 \dots i_n})_{i_1,i_2,\dots,i_n \in X}$ of A, satisfying the following conditions:*

$$\eta_{T(x_1,x_2,\dots,x_n),f(y_2),f(y_3),\dots,f(y_n)}\eta_{x_1,x_2,\dots,x_n} =$$
$$\eta_{T(x_1,y2,\dots,y_n),T(x_2,y_2,\dots,y_n),\dots,T(x_n,y_2,\dots,y_n)}\eta_{x_1,y_2,\dots,y_n} \tag{5.6}$$

$$\eta_{T(x_1,x_2,\dots,x_n),f(y_2),f(y_3),\dots,f(y_n)}\tau^i_{x_1,x_2,\dots,x_n} =$$
$$\tau^i_{T(x_1,y2,\dots,y_n),T(x_2,y_2,\dots,y_n),\dots,T(x_n,y_2,\dots,y_n)}\eta_{x_i,y_2,\dots,y_n} \tag{5.7}$$

$$\tau^i_{T(x_1,x_2,\dots,x_n),f(y_2),f(y_3),\dots,f(y_n)}g =$$
$$\eta_{T(x_1,y2,\dots,y_n),T(x_2,y_2,\dots,y_n),\dots,T(x_n,y_2,\dots,y_n)}\tau^i_{x_1,y_2,\dots,y_n}$$
$$+\sum_{j=1}^{n-1}\tau^j_{T(x_1,y2,\dots,y_n),T(x_2,y_2,\dots,y_n),\dots,T(x_n,y_2,\dots,y_n)}\tau^i_{x_j,y_2,\dots,y_n} \tag{5.8}$$

Remark 5.2.3. *If X is a Ternary f-quandle, a Ternary f-quandle structure of X-module on A is a structure of an X-module further satisfies*

$$\tau_{f(x),f(u),f(v)}g = (\eta_{T(x,u,v),T(x,u,v),T(x,u,v)} + \tau_{T(x,u,v),T(x,u,v),T(x,u,v)} +$$
$$\mu_{T(x,u,v),T(x,u,v),T(x,u,v)})\tau_{x,u,v} \tag{5.9}$$

and

$$\mu_{f(x),f(u),f(v)}g = (\eta_{T(x,u,v),T(x,u,v),T(x,u,v)} + \tau_{T(x,u,v),T(x,u,v),T(x,u,v)} + \mu_{T(x,u,v),T(x,u,v),T(x,u,v)})\mu_{x,u,v} \quad (5.10)$$

Furthermore, if $f, g = id$ maps, then it satisfies

$\eta_{T(x,u,v),T(x,u,v),T(x,u,v)} + \tau_{T(x,u,v),T(x,u,v),T(x,u,v)} + \mu_{T(x,u,v),T(x,u,v),T(x,u,v)} = id$ *and*

$\eta_{T(x,u,v),T(x,u,v),T(x,u,v)} + \tau_{T(x,u,v),T(x,u,v),T(x,u,v)} + \mu_{T(x,u,v),T(x,u,v),T(x,u,v)} = id.$

Remark 5.2.4. *When $x = y = z$ in* (5.1)*, we get*

$$\eta_{f(x),f(u),f(v)}\eta_{x,x,x} = \eta_{T(x,u,v),T(x,u,v),T(x,u,v)}\eta_{x,u,v}$$

Example 5.2.5. *Let A be a non-empty set, (X, T, F) be a Ternary f-quandle, and κ be a generalized 3-cocycle. For $a, b, c \in A$, let*

$$\alpha_{x,y,z}(a, b, c) = \eta_{x,y,z}(a) + \tau_{x,y,z}(b) + \mu_{x,y,z}(c) + \kappa_{x,y,z}.$$

Then, it can be verified directly that α is a dynamical cocycle and the following relations hold:

$$\eta_{T(x,y,z),f(u),f(v)}\eta_{x,y,z} = \eta_{T(x,u,v),T(y,u,v),T(z,u,v)}\eta_{x,u,v} \quad (5.11)$$

$$\eta_{T(x,y,z),f(u),f(v)}\tau_{x,y,z} = \tau_{T(x,u,v),T(y,u,v),T(z,u,v)}\eta_{y,u,v} \quad (5.12)$$

$$\eta_{T(x,y,z),f(u),f(v)}\mu_{x,y,z} = \mu_{T(x,u,v),T(y,u,v),T(z,u,v)}\eta_{z,u,v} \quad (5.13)$$

$$\tau_{T(x,y,z),f(u),f(v)}g = \eta_{T(x,u,v),T(y,u,v),T(z,u,v)}\tau_{x,u,v} + \tau_{T(x,u,v),T(y,u,v),T(z,u,v)}\tau_{y,u,v} + \mu_{T(x,u,v),T(y,u,v),T(z,u,v)}\tau_{z,u,v} \quad (5.14)$$

$$\mu_{T(x,y,z),f(u),f(v)}g = \eta_{T(x,u,v),T(y,u,v),T(z,u,v)}\mu_{x,u,v} + \tau_{T(x,u,v),T(y,u,v),T(z,u,v)}\mu_{y,u,v} + \mu_{T(x,u,v),T(y,u,v),T(z,u,v)}\mu_{z,u,v} \quad (5.15)$$

$$\eta_{T(x,y,z),f(u),f(v)}\kappa_{x,y,z} + \kappa_{T(x,y,z),f(u),f(v)} = \eta_{T(x,u,v),T(y,u,v),T(z,u,v)}\kappa_{x,u,v} + \tau_{T(x,u,v),T(y,u,v),T(z,u,v)}\kappa_{y,u,v} + \mu_{T(x,u,v),T(y,u,v),T(z,u,v)}\kappa_{z,u,v} + \kappa_{T(x,u,v),T(y,u,v),T(z,u,v)} \quad (5.16)$$

Definition 5.2.6. *When κ further satisfies $\kappa_{x,x,x} = 0$ in (5.16) for any $x \in X$, we call it a generalized Ternary f-quandle 3-cocycle.*

Example 5.2.7. *Let (X, T, F) be a Ternary f-quandle and A be an abelian group. Set $\eta_{x,y,z} = \tau_{x,y,z}, \mu_{x,y,z} = 0, \kappa_{x,y,z} = \phi(x, y, z)$.*
Then ϕ is a 3-cocycle. That is,

$$\phi(x, y, z) + \phi(T(x, y, z), f(u), f(v)) = \phi(x, u, v) + \phi(y, u, v)+ \\ \phi(T(x, u, v), T(y, u, v), T(z, u, v))$$

Example 5.2.8. *Let $\Gamma = \mathbb{Z}[P, Q, R]$ denote the ring of Laurent polynomials. Then any Γ-module M is a $\mathbb{Z}(X)$-module for any Ternary f-quandle (X, T, F) by defining $\eta_{x,y,z}(a) = Pa$, $\tau_{x,y,z}(b) = Qb$ and $\mu_{x,y,z}(c) = Rc$ for any $x, y, z \in X$ and $a, b, c \in \mathbb{Z}$.*

Chapter 6

Cohomology of n-ary quandles and n-ary f-quandles

In this chapter, we will begin by reviewing cohomology theory of standard n-ary quandles before presenting a cohomology theory for higher-ary f-quandles.

6.1 Cohomology of n-ary f-quandles

In this section we generalize the cohomology theory of the previous section to the context of n-ary f-quandles.

Let $(X, *, f)$ be a ternary f-rack where $f : X \to X$ is a ternary f-rack morphism. We will define the most generalized cohomology theories of f-racks as follows (the reader is advised to review examples 6.2.1, 6.2.2, 6.2.3 which may help illuminate the dense technical notation that follows).

For a sequence of elements $(x_1, x_2, x_3, x_4, \ldots, x_{2p+1}) \in X^{2p+1}$ define

$$[x_1, x_2, x_3, x_4, \ldots, x_{2p+1}] =$$
$$T(\ldots T(x_1, x_2, x_3), f(x_4), f(x_5)), f^2(x_6), f^2(x_7))) \ldots) f^{p-1}(x_{2p}), f^{p-1}(x_{2p+1})).$$

More generally, if we are considering an n-ary f-rack, we define the bracket as follows:

$$[x_1, x_2, x_3, x_4, \ldots, x_{(n-1)p+1}] = T(\ldots T(x_1, \ldots, x_n), f(x_{n+1}), \ldots, f(x_{2n-1})) \ldots$$
$$\ldots, f^{p-1}(x_{p(n-2)+1}), \ldots, f^{p-1}(x_{(n-1)p+1}))$$

Notice that for $i = (p-1)j + 1 < n$ we have the following equality.

$$[x_1, x_2, x_3, x_4, \ldots, x_n] =$$
$$T([x_1, \ldots, x_{i-1}, x_{i+p}, \ldots, x_n], f^{i-2}[x_i, x_{i+p}, \ldots, x_n], f^{i-2}[x_{i+1}, x_{i+p}, \ldots, x_m], \ldots$$
$$\ldots, f^{i-2}[x_{i+p-1}, x_{i+p}, \ldots, x_n]$$

This relation is obtained by applying the first axiom of f-quandles $(p-i)$ times, first grouping the first $(i-1)$ terms together, then iterating this process, again grouping and iterating each.

We provide cohomology theory for the f-rack by defining a co-chain complex.

Theorem 6.1.1. *Consider the free left $\mathbb{Z}(X)$-module $C_p(X) = \mathbb{Z}(X)X^p$ with basis X^p. For an abelian group A, denote $C^p(X, A) := Hom_{\mathbb{Z}(X)}(C_p(X), A)$. The operators $\partial = \partial_p : C_p(X) \to C_{p-1}(X)$ defined:*

$$\partial_p\phi(x_1, \ldots, x_{(n-1)p+1})$$
$$= (-1)^{p+1}\sum_{i=2}^{p+1}(-1)^i\{\eta_{[A(i)],F^{i-2}([B_2(i)]),F^{i-2}([B_3(i)]),\ldots,F^{i-2}([B_n(i)])}\phi(A(i))$$
$$-\phi(T(C_1(i)), T(C_2(i)), \ldots, T(C_{(n-1)i-1}(i)), F(x_{(n-1)i}), F(x_{(n-1)i+1}), \ldots, F(x_{(n-1)p+1}))$$
$$+ (-1)^{p+1}\sum_{j=1}^{n-1}\tau^i_{[B_1(0)],[B_2(0)],\ldots,[B_{n-1}(0)]}\phi(B_j(0)),$$

where $A(i) = x_1, x_2, \ldots, \hat{x}_{(n-1)i}, \hat{x}_{(n-1)i+1}, \ldots, \hat{x}_{(n-1)(i+1)}, x_{n(i+1)-i}, \ldots, x_{(n-1)p+1}$,
$B_k(i) = x_{(n-1)i+k}, x_{(n-1)i+n+1}, x_{(n-1)i+2}, \ldots, x_{(n-1)p+1}$,
$C_k(i) = x_k, x_{(n-1)i}, x_{(n-1)i+1}, \ldots, x_{(n-1)i+n-2}$,

defines a boundary map for the chain complex, and the map $\delta = \delta^p : C^{p+1}(X) \to C^p(X)$ defined $\delta^p(f) = f\partial^{p+1}$ defines a coboundary map for the cochain complex.

Proof. To prove that $\partial^{p+1}\partial^p = 0$, and thus ∂ is a coboundary map we will break the composition into pieces, using the linearity of η and τ^i.

First we will show that the composition of the i^{th} term of the first summand of ∂^p with the j^{th} term of the first summand of ∂^{p+1} cancels with the $(j+1)^{th}$ term of the first summand of ∂^p with the i^{th} term of the first summand of ∂^{p+1} for $i \leq j$. As the sign of these terms are opposite, we need only to show that the compositions are equal up to their sign. For the sake of readability we will introduce the following, based on A and B above:

$$\begin{aligned} A(i,j) \quad &= x_1, \ldots, \hat{x}_{(n-1)i}, \hat{x}_{(n-1)i+1}, \ldots, \hat{x}_{ni}, x_{ni+1}, \ldots, x_{(n-1)j-1}, \\ &\hat{x}_{(n-1)j}, \hat{x}_{(n-1)j+1}, \ldots, \hat{x}_{nj}, \ldots, x_{(n-1)p+1}, \end{aligned}$$

$$B(i,j) = x_{(n-1)i+k}, x_{ni+1}, x_{ni+2}, \ldots, \hat{x}_{(n-1)j}, \hat{x}_{(n-1)j+1}, \ldots, \hat{x}_{nj}, \ldots, x_{(n-1)p+1}.$$

Now we can see that the composition of the i^{th} term of the first summand of ∂^p with the j^{th} term of the first summand of ∂^{p+1} can be rewritten as follows:

$$\begin{aligned} &\eta_{[A(i)],F^{i-2}[B_0(i)],F^{i-2}[B_1(i)],\ldots,F^{i-2}[B_{n-2}(i)]}\eta_{[A(i,j+1)],F^{j-1}[B_0(j+1)],F^{j-1}[B_1(j+1)],\ldots,F^{j-1}[B_{n-2}(j+1)]} \\ &= \eta_{T([A(i,j)],F^{j-1}[B_0(j+1)],\ldots,F^{j-1}[B_{n-2}(j+1)]),T(F^{i-2}[B_0(i,j)],F^{j-1}[B_0(j+1)],\ldots,F^{j-1}[B_{n-2}(j+1)])} \\ &\quad {}_{T(F^{i-2}[B_1(i,j)],F^{j-1}[B_0(j+1)],\ldots,F^{j-1}[B_{n-2}(j+1)]),\ldots,T(F^{i-2}[B_{n-2}(i,j)],F^{j-1}[B_0(j+1)],\ldots,F^{j-1}[B_{n-2}(j+1)])} \\ &\quad \eta_{[A(i,j+1)],F^{j-1}[B_0(j+1)],F^{j-1}[B_1(j+1)],\ldots,F^{j-1}[B_{n-2}(j+1)]} \\ &= \eta_{T([A(i,j)],F^{i-2}[B_0(i,j)],\ldots,F^{i-2}[B_{n-2}(i,j)]),F^{j-1}[B_0(j+1)],\ldots,F^{j-1}[B_{n-2}(j+1)]} \\ &\quad \eta_{[A(i,j)],F^{i-2}[B_0(i,j)],\ldots,F^{i-2}[B_{n-2}(i,j)]} \\ &= \eta_{[A(j+1)],F^{j-1}[B_0(j+1)],F^{j-1}[B_1(j+1)],\ldots,F^{j-1}[B_{n-2}(j+1)]}\eta_{[A(i,j)],F^{i-2}[B_0(i,j)],\ldots,F^{i-2}[B_{n-2}(i,j)]} \end{aligned}$$

Which is precisely the $(j+1)^{th}$ term of the first summand of ∂^p with the i^{th} term of the first summand of ∂^{p+1}.

Similar manipulations show that the composition of τ^i from ∂^p with the i^{th} term of the first sum of ∂^{p+1} cancels with the composition of the $(i+1)^{th}$ term of the first sum of ∂^p with τ^i from ∂^{p+1}. For the sake of brevity we will omit showing these manipulations, but the table below presents all relations which are canceled by similar manipulations.

In the table η_i represents the i^{th} summand of the first sum, $\circ_i$ represents the i^{th} summand of the second sum, with order of composition determining its origin in δ^p or δ^{p+1}.

$$\begin{aligned}
\eta_i\eta_j &= \eta_{j+1}\eta_i \\
\eta_i\circ_j &= \circ_{j+1}\eta_i \\
\eta_i\tau^i &= \tau^i\eta_{i+1} \\
\tau^i\circ_i &= \circ_{i+1}\tau^i \\
\circ_i\circ_j &= \circ_{j+1}\circ_i
\end{aligned}$$

All these relations leave $m+1$ remaining terms, which cancel via the third axiom in Definition. □

6.2 Examples

Example 6.2.1. *By specializing $n = 2$ in the theorem 6.1.1, the coboundary operator simplifies to:*

$$\begin{aligned}
&\partial\phi(x_1, \ldots, x_{2p+1}) \\
&\quad = \sum_{i=1}^{p}(-1)^i \eta_{\{A,B,C\}}\phi(x_1, \ldots, \hat{x}_{2i}, \hat{x}_{2i+1}, \ldots, x_{2p+1}) \\
&\quad - \sum_{i=1}^{p}(-1)^i \phi(T(x_1, x_{2i}, x_{2i+1}), \ldots, T(x_{2i-1}, x_{2i}, x_{2i+1}), F(x_{2i+2}), \ldots, F(x_{2p+1})) \\
&\quad + (-1)^{2p+1}\tau_{[x_1,x_4,\ldots,x_{2n+1}],[x_2,x_4,\ldots,x_{2p+1}],[x_3,\ldots,x_{2p+1}]}\phi(x_2, x_4, \ldots, x_{2p+1}) \\
&\quad + (-1)^{2p+1}\mu_{[x_1,x_4,\ldots,x_{2n+1}],[x_2,x_4,\ldots,x_{2p+1}],[x_3,\ldots,x_{2p+1}]}\phi(x_3, x_4, \ldots, x_{2p+1}).
\end{aligned}$$

where $A = [x_1, \ldots, \hat{x}_{2i}, \hat{x}_{2i+1}, \ldots, x_{2p+1}]$, $B = F^{\{i-1\}}[x_{2i}, x_{2i+2}, x_{2i+3}, \ldots, x_{2p+1}]$, $C = F^{\{i-1\}}[x_{2i+1}, x_{2i+2}, \ldots, x_{2p+1}]$.

Further specializing example 6.2.1, we have the following result.

Example 6.2.2. *In this example, we compute the first and second cohomology groups of the ternary Alexander f-quandle $X = \mathbb{Z}_3$ with coefficients in the abelian group $\mathbb{Z}_3$. In the ternary f-quandle under consideration, we have $P = 2$, $Q = R = 1$, that is*

$T(x_1, x_2, x_3) = Px_1 + Qx_2 + Rx_3$ and $f(x) = (P + Q + R)x$ as in example 4.1.7. Now Setting η to be multiplication by P, τ to be multiplication by Q, and μ to be multiplication by R we have the 1-cocycle condition for $\phi : X \to A$ as

$$P\phi(x) + Q\phi(y) + R\phi(z) - \phi(T(x, y, z)) = 0$$

and the 2-cocycle condition as

$$\begin{aligned} &P\psi(x_1, x_2, x_3) + \psi(T(x_1, x_2, x_3), f(x_4), f(x_5)) \\ = \;& P\psi(x_1, x_4, x_5) + Q\psi(x_2, x_4, x_5) + R\psi(x_3, x_4, x_5) \\ + \;& \psi(T(x_1, x_4, x_5), T(x_2, x_4, x_5), T(x_3, x_4, x_5)). \end{aligned}$$

A direct computation gives $H^1(X = \mathbb{Z}_3, A = \mathbb{Z}_3)$ is 2-dimensional, with basis $\{2\chi_0 + \chi_1, 2\chi_0 + \chi_2\}$. As such the dim$(Im(\delta_1)) = 1$, and additional calculation gives dim$(ker(\delta_2)) = 3$, thus H^2 is also 2-dimensional.

Lastly we consider a binary case, obtaining, as expected, a familiar result.

Example 6.2.3. *Let η be the multiplication by T and τ be the multiplication by S in Example 4.1.7. The 1-cocycle condition is written for a function $\phi : X \to A$ as*

$$T\phi(x) + S\phi(y) - \phi(x * y) = 0.$$

Note that this means that $\phi : X \to A$ is a quandle homomorphism.

For $\psi : X \times X \to A$, the 2-cocycle condition can be written as

$$\begin{aligned} &T\psi(x_1, x_2) + \psi(x_1 * x_2, f(x_3)) \\ = \;& T\psi(x_1, x_3) + S\psi(x_2, x_3) + \psi(x_1 * x_3, x_2 * x_3). \end{aligned}$$

In [16], the groups H^1 and H^2 with coefficients in the abelian group $\mathbb{Z}_3$ of the f-quandle $X = \mathbb{Z}_3$, $T = 1, S = 2$ and $f(x) = 0$ were computed. More precisely, $H^1(\mathbb{Z}_3, \mathbb{Z}_3)$ is 1-dimensional with a basis $\chi_1 + 2\chi_2$ and H^2 is 1-dimension with a basis

$$\phi = \chi_{(0,1)} + 2\chi_{(0,2)} + 2\chi_{(1,0)} + \chi_{(1,2)} + 2\chi_{(2,1)}.$$

Appendix A

Ternary f-quandle code

Here we present the C++ code that we used to generate ternary f-quandles.

```
#include <iostream>
#include <fstream>
#include <string>
#include <sstream>
#include <algorithm>
using namespace std;

int main ()
{
int size, Hom[9]={0}, sizeFact=1, j, k, Valid, IndxSize;
int permu[10]={1,2,3,4,5,6,7,8,9}, Indx[256]={0}, placE;
//Name and location of file to output to
string outFileName="C:\\Cpp\\TQuandWMaps\\QuandXXTables.txt";
//Initialize to first posible structure under lexicographic order,
int Quand[4][4][4]={{{1,2,3,4},{1,2,3,4},{1,2,3,4},{1,2,3,4}}
,{{1,2,3,4},{1,2,3,4},{1,2,3,4},{1,2,3,4}},{{1,2,3,4},{1,2,3,4},
{1,2,3,4},{1,2,3,4}},{{1,2,3,4},{1,2,3,4},{1,2,3,4},{1,2,3,4}}};

//Order of structure to generate chosen when the program is run
cout << "Size:";
cin >> size;

//initialize the output to file, setting the second X in name to size
ofstream outFile;
```

```
outFileName[25]='0'+size;
outFile.open(outFileName);

//sizeFact is set to size!
for (int a=1; a<=size; a++)
sizeFact *= a;

IndxSize = size*size;

//Runs through validation check, valid table dump,
// and incrementing structure procedure
while (Indx[IndxSize]<1)
{

//Used for debugging
/*
for (int c=0; c<size; c++)
{
for (int d=0; d<size; d++)
cout << Quand[c][d];
cout << endl;
}
*/


//Define f(a)=T(a,a,a)
for (int k=0; k<size; k++)
Hom[k]=Quand[k][k][k];
```

```
//Check condition T(T(l,m,n),p,q) = T(T(l,p,q),T(m,p,q),T(n,p,q))
//for all l,m,n,p,q
Valid=1;
for (int l=0; l<size; l++)
{
for (int m=0; m<size; m++)
{
for (int n=0; n<size; n++)
{
for (int q=0; q<size; q++)
{
for (int p=0; p<size; p++)
{
if (Quand[Quand[l][m][q]-1][Hom[n]-1][Hom[p]-1]
!= Quand[Quand[l][n][p]-1][Quand[m][n][p]-1][Quand[q][n][p]-1])
{
//if the condition fails for any sets 'Valid' to 0 and escapes loops
Valid=0;
n=size;
m=size;
l=size;
p=size;
q=size;
}
}
}
}
}
```

```
}
}

//Dump Valid tables to file
if (Valid)
{
for (int c=0; c<size; c++)
{
for (int d=0; d<size; d++)
{
for (int e=0; e<size; e++)
outFile << Quand[e][c][d];
outFile << endl;
}
outFile << endl;
}
outFile << endl;

//Used for debugging
/*
cout << "Valid" << Quand[4][0] << Quand[4][1] << Quand[4][2]
<< Quand[4][3] << Quand[4][4] << endl;
*/
}

//Sets Quand to next table in lexicalgraphic order, by changing
```

```
//the permutation in Quand[size-1][size-1] first
k=0;
j=1;
do
{


if (k==size)
{
k=0;
j++;
}

k++;

//Uses next_permutation to increment to next structure in
//lexicographic order
} while( !(next_permutation(Quand[size-j][size-k]
,Quand[size-j][size-k]+size)));;


//increments the counter for the while loop
Indx[0]++;
placE=0;

while (Indx[placE] == sizeFact)
{
Indx[placE]=0;
```

```
placE++;
Indx[placE]++;
}



}



outFile.close();

system("pause");


return 0;
}
```

Here we present the C++ code used to reduce the set of generated ternary structures to an isomorphically distinct set.

```
#include <iostream>
#include <fstream>
#include <string>
#include <sstream>
#include <algorithm>

using namespace std;

int main ()
{
int CurrStruct[9][9][9], GeningHom[9], InverseHom[9];
int StructSize, PositFileName, FileNameLength;
int CheckMatch;

string StructLine, GenedIsoLine;

string InFileName, OutFileName;
string IsoFile="C:\\Cpp\\Iso\\IsoFile.txt";

//initialize file input streams
ifstream InputFile; //file to reduce
ifstream GenedIso;  //reduced file

//initialize file output streams
ofstream ReducedFile;
ofstream GeningIso;
```

```
//Get InFileName and Structure size
InFileName = "C:\\Cpp\\TQuandWMaps\\QuandX3Tables.txt";
StructSize = 3;


//Generate ReducedFileName
PositFileName = InFileName.find_last_of("\\");
FileNameLength = InFileName.size();

OutFileName = InFileName.substr(0, PositFileName+1) + "Reduced\\"
 + InFileName.substr(PositFileName+1, FileNameLength);

//Debug
cout << OutFileName << endl;


//Open Input and Output Files
InputFile.open(InFileName);
ReducedFile.open(OutFileName);

//Clear Previous Temp File
remove("C:\\Cpp\\Iso\\IsoFile.txt");

//OuterMostLoop
while ( getline(InputFile,StructLine) )
    {
```

```
//Read in next structure
for (int a=0; a<StructSize; a++)
{
for (int b=0; b<StructSize; b++)
{
for (int c=0; c<StructSize; c++)
{
CurrStruct[a][b][c]=StructLine.at(c)-49;
cout << CurrStruct[a][b][c];
}
getline (InputFile,StructLine);
cout << endl;
}
getline (InputFile,StructLine);
cout << endl;
}

//Compare to list of generated structures
GenedIso.open(IsoFile);
CheckMatch = 1;

if (GenedIso.is_open())
{
while ( getline(GenedIso, GenedIsoLine) && CheckMatch != 0 )
{
CheckMatch=0;
//Read in iso gen structures and compare with current structure
for (int aa=0; aa<StructSize; aa++)
```

```
{
for (int bb=0; bb<StructSize; bb++)
{
for (int cc=0; cc<StructSize; cc++)
{
cout << GenedIsoLine.at(bb)-48; //debug output


if (CurrStruct[aa][bb][cc] != GenedIsoLine.at(cc)-48)
  CheckMatch++;
}

getline (GenedIso, GenedIsoLine);
}
cout << endl; //debug output

getline (GenedIso, GenedIsoLine);
}

//if CheckMatch=0 CurStruct is isomorphic to a
//previous structure and loop exits
}
}


GenedIso.close();
GenedIso.clear();
```

```
//If not on list (loop exits with checkmatch not 0)
if (CheckMatch != 0)
{
//Append to reducedFile
for (int dd=0; dd<StructSize; dd++)
{
for (int ee=0; ee<StructSize; ee++)
{
for (int ff=0; ff<StructSize; ff++)
{
ReducedFile << CurrStruct[dd][ee][ff]+1;
}
ReducedFile << endl;
}
ReducedFile << endl;
}
ReducedFile << endl;



//Generate isomorphic structures and Append to geningIso
for (int ee=0; ee<StructSize; ee++)
GeningHom[ee]=ee;

GeningIso.open(IsoFile ,ios::out | ios::app);
do{

for (int z=0; z<StructSize; z++)
{
```

```
for (int y=0; y<StructSize; y++)
{
if (GeningHom[y]==z)
InverseHom[z]=y;
}
}


for (int ff=0; ff<StructSize; ff++)
{
for (int gg=0; gg<StructSize; gg++)
{
for (int hh=0; hh<StructSize; hh++)
{
GeningIso <<
InverseHom[CurrStruct[GeningHom[ff]][GeningHom[gg]][GeningHom[hh]]];
}
GeningIso << endl;
}
GeningIso << endl;
}
GeningIso << endl;
}while(next_permutation(GeningHom,GeningHom+StructSize));
GeningIso.close();
GeningIso.clear();
}

}//OuterMostLoop End
```

```
system("pause");

remove("C:\\Cpp\\Iso\\IsoFile.txt");

return 0;
}
```

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