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Lump, complexiton and algebro-geometric solutions to soliton equations

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by

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Dedication

This paper is dedicated to my family.

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Table of Contents

Abstract

In chapter 2, we study two Kaup-Newell-type matrix spectral problems, derive their soliton hierarchies within the zero curvature formulation, and furnish their bi-Hamiltonian structures by the trace identity to show that they are integrable in the Liouville sense. In chapter 5, we obtain the Riemann theta function representation of solutions for the first hierarchy of generalized Kaup-Newell systems.

In chapter 3, using Hirota bilinear forms, we discuss positive quadratic polynomial solutions to generalized bilinear equations, which generate lump or lump-type solutions to nonlinear evolution equations, and propose an algorithm for computing higher-order lump or lump-type solutions. In chapter 4, we study mixed exponential and trigonometric wave solutions (called complexitons) to general bilinear equations, and propose two methods to find complexitons to generalized bilinear equations. We also succeed in proving that by choosing suitable complex coefficients in soliton solutions, multi-complexitons are actually real wave solutions from complex soliton solutions and establish the linear superposition principle for complexion solutions.

In each chapter, we present computational examples.

Chapter 1

Intruoduction

1.1 Background

It was Scott Russell in 1834 who observed the solitary waves at a canal. Sixty yearys later, the Korteweg-de Vries (KdV) equation [37] was derived by D. Korteweg and G. de Vries in 1895 to describe surface water in long, narrow and shallow rivers. In 1965 N. J. Zabusky and M. D. Kruskal [80] found "solitary-wave pulses", for which they named them "solitons", in their numerical analysis for the KdV equation. Thereafter in 1967, the inverse scattering method (IST)[16] was proposed by C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura to solve initial value problems for the KdV equation with fast-decaying initial data. In 1968, P. Lax [38] introduced a whole hierarchy of KdV equations by introducing the concept of the Lax pairs. There exist many interesting soliton hierarchies, which consist of commuting evolution equations, and the celebrated examples include the Korteweg-de Vries hierarchy [38], the Ablowitz-Kaup-Newell-Segur hierarchy [2] and the Kaup-Newell hierarchy [36], the Wadati-Konno-Ichikawa hierarchy [77], the Boiti-Pempinelli-Tu hierarchy [8], the Dirac hierarchy [26] and the coupled AKNS-Kaup-Newell hierarchy [58]. Spectral problems associated with matrix Lie algebras are a starting point in generating soliton hierarchies (see, e.g., [1, 2, 3, 9, 12, 19, 36, 58, 74]). Not long ago, the three-dimensional Lie algebra so(3,$\mathbb{R}$) has been introduced in producing soliton hierarchies. The first few examples of soliton hierarchies associated with so(3,$\mathbb{R}$) are the AKNS, the Kaup-Newell, the WKI, and the Heisenberg type soliton hieiarchy [45, 46, 54, 49].

1.2 Integrable systems

Integrable systems are nonlinear differential or difference equations which in some way can be solved analytically. This means that we can solve these equations by finite steps of algebraic operations and integrations. In this section, we sketch some of the concepts of integrability, important theorems that we will use in this dissertation. In the finite dimensional situation, for the classical theory of Hamiltonian dynamical systems, the Liouville integrability is widely accepted and the key result is the Liouville-Arnold theorem [4]. A finite-dimensional Hamiltonian system is completely integrable in the Liouville sense, if there exist canonical coordinates on the phase space known as action-angle variables, such that the transformed Hamiltonian systems depend only on the action variables. As a result, the action variables are conserved. Meanwhile, the angle variables evolve linearly in the evolution parameters, therefore one can explicitly solve (or integrate) Hamiltons equations.

However, in the infinite-dimensional systems, there is no unify accepted integrable concept. We will concentrate our discussion on the integrability in the Liouville sense. The main references for this section are [4, 42, 67, 68].

1.2.1 Lax pair

The Lax Pair and the zero curvature equation

We consider a spectral problem first. Let L be a function or an operator of time t, $\lambda \in \mathbb{C}$ be the spectral parameter, and φ be the spectral function (eigenfunction). Suppose

$$L\varphi = \lambda\varphi, \qquad \varphi_t = A\varphi. \tag{1.2.1}$$

Then

$$\begin{aligned} \frac{\partial}{\partial t} L\varphi &= L_t\varphi + L\varphi_t = L_t\varphi + LA\varphi \\ &= \frac{\partial}{\partial t}\lambda\varphi = \lambda_t\varphi + \lambda\varphi_t = \lambda_t\varphi + AL\varphi. \end{aligned}$$

Since φ is general, we obtain

$$\frac{\partial L}{\partial t} + [L, A] = \lambda_t, \tag{1.2.2}$$

which means that the eigenvalues of L are independent of t if $\lambda_t \equiv 0$, we call the spectral problem (1.2.1) isospectral.

Definition 1.2.1 (Lax pairs)**.** *A Lax pair is a pair of matrix $L(t)$ and $A(t)$ of functions or operators satisfying Lax equation:*

$$\frac{dL}{dt} + [L, A] = 0, \tag{1.2.3}$$

where $[L, A] = LA - AL = -[A, L]$ is the matrix commutator.

Lax found that when $L = \partial^2 + u$ (a Sturm-Liouville operator), $A = -4\partial^3 - 6u\partial - 3u_x$ where $\partial := \frac{d}{dx}$, the Lax equation is equivalent to the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0. \tag{1.2.4}$$

The crucial importance of Lax's observation is that any equation that can be cast into the Lax equation for operators L and A has shared many of the features of the KdV equation, including an infinite number of local conservation laws.

In general, the spectral problem with potential u can be written as

$$\phi_x = U(x, t, u, \lambda)\phi, \tag{1.2.5}$$

$$\phi_t = V(x, t, u, \lambda)\phi, \tag{1.2.6}$$

where $U, V \in \mathbb{C}^{n\times n}$ are matrix functions, $u(x, t)$ is a given vector function, ϕ is an n-dimensional spectral function.

We arrive at the **zero curvature equation** by the compatibility condition $\phi_{tx} = \phi_{xt}$

$$U_t - V_x + [U, V] = 0. \tag{1.2.7}$$

A pair of matrices (U, V) is also called a Lax pair if they satisfy the zero-curvature equation (1.2.7).

Example 1.2.2. *If we introduce a vector $\phi = [\varphi\,,\ \varphi_x]^T$, and matrices*

$$U = \begin{bmatrix} 0 & 1 \\ \lambda - u & 0 \end{bmatrix}, \quad V = \begin{bmatrix} u_x & -(4\lambda + 2u) \\ u_{xx} - (\lambda - u)(4\lambda + 2u) & -u_x \end{bmatrix}.$$

Then the zero curvature equation of U and V is equivalent to the KdV equation.

1.2.2 Generalized vector field and symmetry

Let p and q be two positive integers. Suppose independent variables $x = (x_1, \cdots, x_p)^T \in X = \mathbb{R}^p$ and dependent variables $u = (u_1, \cdots, u_q)^T \in U = \mathbb{R}^q$. The open set $M \subset X \times U$. $\mathcal{A}$ is the algebra of differential functions $P[u] = P(x, u^{(n)})$ over M, where n is a positive integer and P depends only on spatial variables x and spatial derivatives of u up to n-th order. We define $\mathcal{F}$ to be the quotient space of $\mathcal{A}$ by the subspace of total divergence $\{f : f = \text{div } P,\ \exists P \in \mathcal{A}\}$ and denote its element by $\mathcal{P} = \int P dx$.

Definition 1.2.3. *A generalized vector field*

$$\mathbf{v} := \sum_{j=1}^{p} \xi_j[u]\frac{\partial}{\partial x_j} + \sum_{k=1}^{q} \varphi_k[u]\frac{\partial}{\partial u_k},$$

in which $(\xi_1, \cdots, \xi_p)^T \in \mathcal{A}^p$ *and* $(\varphi_1, \cdots, \varphi_q)^T \in \mathcal{A}^q$ *are all smooth differential functions.*

When $\xi_1 = \cdots = \xi_p = 0$*, the vector field (denoted by* $\mathbf{v}_Q$*) is called an evolutionary vector field, where* $Q = [\varphi_1[u], \cdots, \varphi_q[u]]^T$ *is called its characteristic.*

Definition 1.2.4 (Total derivative)**.** *Let* $P(x, u^{[n]}) \in \mathcal{A}$*. For any given* $j : 1 \leq j \leq p$*, the total derivative of a function* P *with respect to* x_j *is defined by*

$$\mathbf{D}_j P = \frac{\partial P}{\partial x_j} + \sum_{k=1}^{q} \sum_{\#J \leq n} \frac{\partial u_{k,J}}{\partial x_j} \cdot \frac{\partial P}{\partial u_{k,J}}, \tag{1.2.8}$$

where the multi-indices $J = (j_1, \cdots, j_m)$ *with* $1 \leq j_i \leq p$ *for* $1 \leq i \leq k$ *and* $\#J := m$*. And*

$$u_{k,J} := \frac{\partial^m u_k}{\partial x_{j_i} \cdots \partial x_{j_m}}. \tag{1.2.9}$$

Definition 1.2.5 (Prolongation)**.** *The prolonged generalized vector field is defined by*

$$pr^{(n)}\mathbf{v} = \mathbf{v} + \sum_{k=1}^{q} \sum_{\#J \leq n} \varphi_k^J[u]\frac{\partial}{\partial u_{k,J}},$$

where the coefficients φ_k^J *are determined by*

$$\varphi_k^J[u] := \mathbf{D}_J(\varphi_k - \sum_{j=1}^{p} \xi_j u_{k,j}) + \sum_{j=1}^{p} \xi_j u_{k,(J,j)}. \tag{1.2.10}$$

Definition 1.2.6 (Symmetry)**.** *Let a generalized vector field* $\mathbf{v}$ *de called a generalized infinitesimal symmetry of a system of differential equations*

$$\Delta_j[u] = \Delta_j(x, u^{(n)}) = 0, \quad j \in \{1, \cdots, r\}, \tag{1.2.11}$$

if and only if the prolonged generalized vector field satisfies

$$pr\mathbf{v}[\Delta_j] = 0, \quad j \in \{1, \cdots, r\}, \tag{1.2.12}$$

for every smooth function $u = f(x)$ *satisfying* (1.2.11).

Definition 1.2.7. *A system of evolution equations reads*

$$u_t = K[u], \quad K \in \mathcal{A}^q, \tag{1.2.13}$$

where $K[u] = K(x, u^{(n)})$ *does not depend on* t.

Definition 1.2.8 (Recursive operator)**.** *Let* $\Delta = 0$ *be a system of differential equations. A linear operator* $\mathcal{R} : \mathcal{A}^q \to \mathcal{A}^q$ *is called a recursive operator for* Δ *if whenever* $\mathbf{v}_Q$ *is an evolutionary symmetry of* $\Delta = 0$*, so is* $\mathbf{v}_{\mathcal{R}Q}$.

Definition 1.2.9 (Pseudo-differential operator)**.** *A differential operator is given by*

$$\mathcal{D} := \sum_{j=1}^{n} P_i[u]\big(\frac{\partial}{\partial x}\big)^j.$$

If $P_n \not\equiv 0$*, then* n *is the order of* $\mathcal{D}$.

A (formal) pseudo-differential operator is defined by

$$\mathcal{D} := \sum_{j=-\infty}^{n} P_i[u]\big(\frac{\partial}{\partial x}\big)^j.$$

The calculus of variations

Let $\Omega \subset \mathbb{R}^p$ be an open, connected subset with smooth boundary $\partial\Omega$. Finding the extrema of a functional

$$\mathcal{L}[u] = \int_\Omega L(x, u^{(n)})dx$$

in functions $u = f(x) \in R^q$ defined on Ω is named a variational problem and the integrand $L(x, u^{(n)}) \in \mathcal{A}$ is termed the Lagrangian of $\mathcal{L}$.

Definition 1.2.10 (The variational derivative)**.** *Let function* $u = f(x)$ *be a smooth function defined on* Ω*. The variational derivative of* $\mathcal{L}$ *is tagged as* $\delta\mathcal{L}[u]$ *with the property that*

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\mathcal{L}[f+\varepsilon\eta] = \int_\Omega \delta\mathcal{L}[f(x)]\cdot\eta(x)dx = \int_\Omega \sum_{j=1}^{q}\delta_j\mathcal{L}[f(x)]\eta_j(x)dx \tag{1.2.14}$$

whenever $\eta = (\eta_1, \cdots, \eta_q)$ *is a smooth function with compact support in* Ω*. The component* $\delta_j\mathcal{L} = \dfrac{\delta\mathcal{L}}{\delta u_j}$ *is the variational derivative of* $\mathcal{L}$ *with respect to* $u_j, j = 1, \cdots, q$.

Definition 1.2.11 (The Euler operator)**.** *Let $1 \leq n \leq q$. The n-th Euler operator is defined by*

$$E_n = \sum_J (-1)^k \mathbf{D}_J \frac{\partial}{\partial u_{n,J}}, \tag{1.2.15}$$

where the sum over all $J = (j_1, \cdots, j_k)$ with $1 \leq j_k \leq p$, and $\mathbf{D}_J := \mathbf{D}_{j_1} \cdots \mathbf{D}_{j_k}$ with $\mathbf{D}_j$ being the jth total derivative.

It is shown in [68] that for $1 \leq n \leq q$

$$\delta \mathcal{L}[u] = E(L) = [E_1[L], \cdots, E_q[L]]^T. \tag{1.2.16}$$

Note: There are only finitely many terms in the summation for any given L.

1.2.3 Hamiltonian structure and integrability

Hamiltonian operator

Definition 1.2.12 (The Hamiltonian operator)**.** *Suppose that $\mathcal{D} : \mathcal{A}^q \to \mathcal{A}^q$ is a linear operator such that the Poisson bracket*

$$\{\mathcal{P}, \mathcal{Q}\} := \int \delta\mathcal{P} \cdot \mathcal{D} \delta \mathcal{Q} dx, \quad \forall \mathcal{P}, \mathcal{Q} \in \mathcal{F} \tag{1.2.17}$$

satisfies

a) the conditions of skew-symmetry

$$\{\mathcal{P}, \mathcal{Q}\} = -\{\mathcal{Q}, \mathcal{P}\}, \quad \forall \mathcal{P}, \mathcal{Q} \in \mathcal{F}, \tag{1.2.18}$$

b) the Jacobi identity

$$\{\{\mathcal{P}, \mathcal{Q}\}, \mathcal{R}\} + \{\{\mathcal{R}, \mathcal{P}\}, \mathcal{Q}\} + \{\{\mathcal{Q}, \mathcal{R}\}, \mathcal{P}\} = 0, \quad \forall \mathcal{P}, \mathcal{Q}, \mathcal{R} \in \mathcal{F}. \tag{1.2.19}$$

Then $\mathcal{D}$ is called a Hamiltonian operator.

Proposition 1.2.13 ([68])**.** *If $\mathcal{D}$ is a Hamiltonian operator, then it is skew-adjont: $\mathcal{D}^* = -\mathcal{D}$, where the adjoint operator $\mathcal{D}^*$ is determined by*

$$\int_\Omega P \cdot \mathcal{D} Q dx = \int_\Omega Q \cdot \mathcal{D}^* P dx, \qquad \forall P, Q \in \mathcal{A}. \tag{1.2.20}$$

Proposition 1.2.14 ([68])**.** *A skew-adjont differential operator $\mathcal{D}$ is Hamiltonian if and only if*

$$\int [P \cdot pr\mathbf{v}_{\mathcal{D}R}(\mathcal{D})Q + R \cdot pr\mathbf{v}_{\mathcal{D}Q}(\mathcal{D})P + Q \cdot pr\mathbf{v}_{\mathcal{D}P}(\mathcal{D})R]dx = 0 \tag{1.2.21}$$

for all q-tuples $P, Q, R \in \mathcal{A}^q$.

A Hamiltonian system of evolution equations reads

$$\frac{\partial u}{\partial t} = \mathcal{D} \cdot \delta\mathcal{H}, \tag{1.2.22}$$

where $\mathcal{D}$ is Hamiltonian operator and $\mathcal{H} = \int H dx \in \mathcal{F}$ is a Hamiltonian functional.

Definition 1.2.15. *A conservation law is of the form*

$$\frac{\partial T}{\partial t} + \mathrm{Div} X = 0, \tag{1.2.23}$$

in which Div is the spatial divergence of X. T is called conserved density and X the associated flux.

$$I = \int T dx \tag{1.2.24}$$

is said a constant of motion, or an integral of the differential equation, since $I_t = 0$.

Definition 1.2.16 (Liouville integrability[49])**.** *The evolution equation* (1.2.22) *is call to be Liouville integrable, if it admits infinitely many conserved functionals $\{\mathcal{H} : n = 1, 2, \cdots\}$ which are in involution in pairs*

$$\{\mathcal{H}_n, \mathcal{H}_m\} = 0, \qquad n, m \geq 0, \tag{1.2.25}$$

and the characteristic of whose associated Hamiltonian vector fields

$$K_n := \mathcal{D}\frac{\delta\mathcal{H}_n}{\delta u}, \quad n \geq 0, \tag{1.2.26}$$

are independent.

Definition 1.2.17 (Bi-Hamiltonian pair)**.** *A pair of skew-adjoint q dimensional differential operators $\mathcal{D}$ and $\mathcal{E}$ is called to form a Hamiltonian pair if $a\mathcal{D}+b\mathcal{E}$ is a Hamiltonian operator for any $a, b \in \mathbb{R}$.*

If a system

$$u_t = K_1[u] = \mathcal{D}\delta\mathcal{H}_1 = \mathcal{E}\delta\mathcal{H}_0. \tag{1.2.27}$$

with $\mathcal{D}$ and $\mathcal{E}$ being a Hamiltonian pair, and $\mathcal{H}_0$ and $\mathcal{H}_1$ are Hamiltonian functionals. Then it is said a bi-Hamiltonian system.

The concept of the bi-Hamiltonian systems was introduced by F. Magri [63] and the following theorem was extracted from the book of P. Olver.

Theorem 1.2.18 ([68]). *Let a bi-Hamiltonian system of evolution equation be*

$$u_t = K_1[u] = \mathcal{D}\delta\mathcal{H}_1 = \mathcal{E}\delta\mathcal{H}_0. \tag{1.2.28}$$

Assume that $\mathcal{D}$ is nondegenerate (i.e.,if $\tilde{\mathcal{D}} \cdot \mathcal{D} \equiv 0$ then $\tilde{\mathcal{D}} \equiv 0$). Then $\mathcal{R} = \mathcal{E} \cdot \mathcal{D}^{-1}$ is a recursive operator of (1.2.28)*, and let $K_0 = \mathcal{D}\delta\mathcal{H}_0$ and for each $n = 1, 2, \cdots$ we can define*

$$K_n = \mathcal{R}K_{n-1}. \tag{1.2.29}$$

Then there exists a sequence of functionals $\{\mathcal{H}_n : n = 1, 2, \cdots\}$ such that

a) for each $n \geq 0$,

$$u_t = K_n[u] = \mathcal{D}\delta\mathcal{H}_n = \mathcal{E}\delta\mathcal{H}_{n-1} \tag{1.2.30}$$

is a bi-Hamiltonian system;

b) the corresponding evolutionary vector fields $\mathbf{v}_n = \mathbf{v}_{K_n}, n \geq 0$, all mutually commute:

$$[\mathbf{v}_n, \mathbf{v}_m] = 0, \qquad n, m \geq 0; \tag{1.2.31}$$

c) the Hamiltonian functionals $\mathcal{H}_n, n \geq 0$, are all in involution with respect to either Poisson bracket

$$\{\mathcal{H}_n, \mathcal{H}_m\}_{\mathcal{D}} = \{\mathcal{H}_n, \mathcal{H}_m\}_{\mathcal{E}} = 0, \qquad n, m \geq 0, \tag{1.2.32}$$

and hence provide an infinite collection of conservation laws for each of the bi-Hamiltonian systems in a).

1.3 The Hirota method

We give a short introduction to the Hirota direct method. We want to solve a nonlinear partial differential or difference equation

$$F[u] = F(u, u_x, u_t, \cdots) = 0.$$

One of the central ideas of the Hirota method is bilinearization of above nonlinear equations. There are many types of dependent variable transformations, but the most common examples are rational, logarithmic and bi-logarithmic transformations.

As the first step we transform $F[u]$ to a quadratic form in the dependent variables by using a transformation $u = T(f, g)$. We call this new form as a bilinear form of $F[u]$. We should note that for some equations we may not find such a transformation. Another remark is that some integrable equations like the Korteweg-de Vries (KdV), Kadomtsev-Pethviashvili (KP) and Toda lattice (Tl) equations can be transformed to a single bilinear equation but many of them like the modified Korteweg-de Vries (mKdV), sine-Gordon (sG), and nonlinear Schrödinger (nlS) equations can only be written as combination of bilinear equations. Now we introduce the Hirota D-operator which is the key object in Hirota method.

1.3.1 Hirota derivatives

Let f, g be differentiable functions of x. The D-operator is a bilinear binary differential operator, acting on f and g, which was introduced by R. Hirota in 1971:

$$(D_x f \cdot g)(x) := \Big(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\Big) f(x) g(x')\Big|_{x'=x} = f'(x)g(x) - f(x)g'(x). \tag{1.3.1}$$

The D-operator is now called the Hirota derivative.

It is easy to see that the D-operator has an equivalent definition:

$$(D_x f \cdot g)(x) := \frac{\partial}{\partial a} f(x+a)g(x-a)\Big|_{a=0}. \tag{1.3.2}$$

The D-operator possesses the following properties.

1. The D-operator is bilinear: for any constants c_1 and c_2 it is true that

$$D_x(c_1 f_1 + c_2 f_2) \cdot g = c_1 D_x f_1 \cdot g + c_2 D_x f_2 \cdot g,$$
$$D_x f \cdot (c_1 g_1 + c_2 g_2) = c_1 D_x f \cdot g_1 + c_2 D_x f \cdot g_2.$$

2. The D-operator is asymmetric:

$$D_x g \cdot f = -D_x f \cdot g,$$

especially

$$D_x f \cdot f = 0.$$

3. When $g \equiv 1$, we have

$$D_x f \cdot 1 = \partial_x f,$$

where $\partial_x := \dfrac{\partial}{\partial x}$.

4. The exponential identity: if a is a constant, then

$$\exp(aD_x) f \cdot g = f(x+a)g(x-a).$$

In general, we can extend the definition of D-operators to high dimensional spaces. Suppose the integer $M \geq 1, x \in \mathbb{R}^M$. Let $\partial = (\partial_1, \partial_2, \cdots, \partial_M)^T$ with $\partial_j := \dfrac{\partial}{\partial x_j}$ and let $D = (D_1, D_2, \cdots, D_M)^T$, where $D_j, 1 \leq j \leq M$, are D-operators with respect to $x_j, 1 \leq j \leq M$, which are defined as follows.

Let f, g be infinitely differentiable functions in $\mathbb{R}^M$, $x = (x_1, x_2, \cdots, x_M)^T$ and $x' = (x'_1, x'_2, \cdots, x'_M)^T$. Then

$$\begin{aligned}(D_j f \cdot g)(x) &= (D_{x_j} f \cdot g)(x) \\ &:= \Big(\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x'_j}\Big) f(x) g(x')\Big|_{x'=x} = f_{x_j}(x)g(x) - f(x)g_{x_j}(x). \end{aligned} \tag{1.3.3}$$

The higher order D-operator is defined by

$$\begin{aligned} & (D_1^{n_1} D_2^{n_2} \cdots D_M^{n_M} f \cdot g)(x) \\ = & \prod_{j=1}^{M} (\partial_{x_j} - \partial_{x'_j})^{n_j} f(x) g(x')\Big|_{x'=x} \\ = & \sum_{k_1=0}^{n_1} \cdots \sum_{k_M=0}^{n_M} (-1)^{\sum_{j=1}^{M}(n_j - k_j)} \prod_{j=1}^{M} \binom{n_j}{k_j} \partial_j^{k_j} f(x) \partial_j^{n_j - k_j} g(x), \end{aligned} \tag{1.3.4}$$

where $n_1, n_2, \cdots, n_M$ are nonnegative integers, $\dbinom{n}{k} := \dfrac{n!}{k!(n-k)!}$ is the binomial coefficient.

One of the important properties of the Hirota derivatives is that when interchanging the functions f and g, we have

$$D_1^{n_1} D_2^{n_2} \cdots D_M^{n_M} f \cdot g = (-1)^{n_1 + n_2 + \cdots + n_M} D_1^{n_1} D_2^{n_2} \cdots D_M^{n_M} g \cdot f, \tag{1.3.5}$$

which implies that if $n = n_1 + n_2 + \cdots + n_M$ is an odd number, then

$$D_1^{n_1} D_2^{n_2} \cdots D_M^{n_M} f \cdot f = 0. \tag{1.3.6}$$

Example 1.3.1. *Examples of the Hirota derivatives:*

$$
\begin{aligned}
D_x^2 f \cdot g &= f_{xx}g - 2f_x g_x + f g_{xx}, \\
D_x^2 f \cdot f &= 2(f_{xx}f - f_x^2), \\
D_x D_t f \cdot g &= D_t D_x f \cdot g = f_{xt}g - f_t g_x - f_x g_t + f g_{xt}, \\
D_x^3 f \cdot g &= f_{xxx}g - 3f_{xx}g_x + 3f_x g_{xx} - f g_{xxx}, \\
D_x^2 D_t f \cdot g &= f_{xxt}g - 2f_{xt}g_x - f_{xx}g_t + 2f_x g_{xt} + f_t g_{xx} - f g_{xxt}, \\
D_x^4 f \cdot f &= 2(f_{xxxx}f - 4f_{xxx}f_x + 3f_{xx}^2), \\
D_x D_y D_z D_t f \cdot f &= 2(f_{xyzt}f - f_{xyt}f_z - f_{xyz}f_t - f_{xzt}f_y - f_{yzt}f_x + f_{xy}f_{zt} \\
&\quad + f_{xz}f_{yt} + f_{xt}f_{yz}).
\end{aligned} \tag{1.3.7}
$$

1.3.2 The Hirota method

In the following discussion, we will consider the KPII equation for $\sigma = 1$.

$$(u_t + 6uu_x + u_{xxx})_x + \sigma u_{yy} = 0. \tag{1.3.8}$$

In order to solve the KP equation, we apply the transformation $u = 2(\ln f)_{xx}$ from equation (1.3.8) to get the bilinear KP equation

$$(D_x D_t + D_x^4 + D_y^2) f \cdot f = 0, \tag{1.3.9}$$

and we define the polynomial corresponding to the bilinear KP equation by

$$P_1(x, y, t) := x^4 + y^2 + xt. \tag{1.3.10}$$

Now we consider travelling wave solutions to the KP equation (1.3.9). For any positive integer j, define the function

$$\eta_j(x, y, t) := k_j x + w_j t + l_j y + \eta_j^0, \quad k_j, w_j, l_j, \eta_j^0 \in \mathbb{C}. \tag{1.3.11}$$

Consider

$$f = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \cdots. \tag{1.3.12}$$

It is easy to check, for any integers $j, j' > 0$ and $m, n, r \geq 0$, we have

$$D_x^m D_y^n D_t^r \exp(\eta_j) \cdot \exp(\eta_{j'}) = (k_j - k_{j'})^m (l_j - l_{j'})^n (w_j - w_{j'})^r \exp(\eta_j + \eta_{j'}), \tag{1.3.13}$$

and thus

$$D_x^m D_y^n D_t^r \exp(\eta_j) \cdot \exp(\eta_j) = 0, \tag{1.3.14}$$

if $m + n + r > 0$.

In order to get the one soliton solution, suppose $f_0 = 1$ and $f_1 = \exp(\eta_j)$ for some $j \in \mathbb{N}$. Substituting the above expansion into (1.3.9) and collecting terms of each order of ε, we have

$$\begin{aligned}
\varepsilon \quad &: (D_x D_t + D_x^4 + D_y^2)\{f_1 \cdot 1 + 1 \cdot f_1\} = 0,\\
\varepsilon^2 \quad &: (D_x D_t + D_x^4 + D_y^2)\{f_2 \cdot 1 + f_1 \cdot f_1 + 1 \cdot f_2\} = 0,\\
\varepsilon^3 \quad &: (D_x D_t + D_x^4 + D_y^2)\{f_3 \cdot 1 + f_2 \cdot f_1 + f_1 \cdot f_2 + 1 \cdot f_3\} = 0,\\
&\cdots .
\end{aligned}$$

We denote $\partial = [\partial_x, \partial_y, \partial_t]^T$. The coefficient of ε is equivalent to

$$P_1(\partial) f_1 = P_1(k_j, l_j, w_j) f_1 = 0. \tag{1.3.15}$$

Therefore, we have the **dispersion relation**

$$0 = P_1(k_j, l_j, w_j) = k_j^4 + l_j^2 + w_j k_j, \tag{1.3.16}$$

or (if $k_j \neq 0$)

$$w_j = -\frac{k_j^4 + l_j^2}{k_j}. \tag{1.3.17}$$

From the coefficient of ε^2, we have

$$2P_1(\partial) f_2 = -P_1(D) f_1 \cdot f_1 = 0. \tag{1.3.18}$$

We can choose $f_2 = 0$ and f can be truncated as $f = 1 + \varepsilon f_1$. Taking $\varepsilon = 1$, we get an exact solution of (1.3.8)

$$u = 2(\ln f)_{xx} = \frac{k_j^2}{2} \operatorname{sech}^2 \frac{\eta_j}{2}. \tag{1.3.19}$$

For the two soliton solution we set $f_1 = \exp(\eta_1) + \exp(\eta_2)$ with the coefficients of η_1 and η_2 satisfing (1.3.16). Setting the coefficients of ε^2 to 0, we get

$$\begin{aligned}
2P_1(\partial) f_2 \quad &= -P_1(D) f_1 \cdot f_1 = -2P_1(D) \exp(\eta_1) \cdot \exp(\eta_2)\\
&= -2P_1(k_1 - k_2, l_1 - l_2, w_1 - w_2) \exp(\eta_1 + \eta_2),
\end{aligned} \tag{1.3.20}$$

and then we can take $f_2 = a_{12}\exp(\eta_1+\eta_2)$ where

$$\begin{aligned} a_{12} &= -\frac{P_1(k_1-k_2, l_1-l_2, w_1-w_2)}{P_1(k_1+k_2, l_1+l_2, w_1+w_2)} \\ &= \frac{4(k_1^3k_2+k_1k_2^3)-6k_1^2k_2^2+2l_1l_2+k_1w_2+k_2w_1}{4(k_1^3k_2+k_1k_2^3)+6k_1^2k_2^2+2l_1l_2+k_1w_2+k_2w_1}. \end{aligned} \tag{1.3.21}$$

Calculating the coefficient of ε^3 tells

$$\begin{aligned} 2P_1(\partial)f_3 &= -2P_1(D)f_1\cdot f_2 \\ &= -2P_1(k_2,l_2,w_2)\exp(2\eta_1+\eta_2) - 2P_1(k_1,l_1,w_1)\exp(\eta_1+2\eta_2) \\ &= 0. \end{aligned} \tag{1.3.22}$$

We take $f_3 = 0$ to have a finitely truncated function $f = 1+\varepsilon f_1+\varepsilon^2 f_2$. Let $\varepsilon = 1$. Then

$$f = 1+\exp(\eta_1)+\exp(\eta_2)+a_{12}\exp(\eta_1+\eta_2). \tag{1.3.23}$$

The two soliton solution reads

$$\begin{aligned} u &= 2(\ln f)_{xx} \\ &= -2\frac{k_1^2e^{\eta_1}+k_2^2e^{\eta_2}+\{(k_1-k_2)^2+a_{12}[(k_1+k_2)^2+k_1^2e^{\eta_2}+k_2^2e^{\eta_1}]\}e^{\eta_1+\eta_2}}{(1+e^{\eta_1}+e^{\eta_2}+a_{12}e^{\eta_1+\eta_2})^2}. \end{aligned} \tag{1.3.24}$$

In general, according to [32], the N-soliton solution of the KP equation can be written as

$$f = \sum \exp\left[\sum_{j=1}^N \mu_j\eta_j + \sum_{j<k} A_{jk}\mu_j\mu_k\right] = 1+\sum_{n=1}^N \sum_{\sum_j \mu_j = n} \exp\left[\sum_{j=1}^N \mu_j\eta_j + \sum_{j<k} A_{jk}\mu_j\mu_k\right], \tag{1.3.25}$$

where the first sum means a summation over all possible combinations of $\mu_j = 0, 1$, the functions η_j are defined by (1.3.11) and the coefficients of η_j satisfy the dispersion relation (1.3.16) for $j = 1, 2, \cdots, N$, and for $1 \le j, j' \le N, j \ne j'$

$$e^{A_{jj'}} = a_{jj'} = \frac{4(k_j^3k_{j'}+k_jk_{j'}^3)-6k_j^2k_{j'}^2+2l_jl_{j'}+k_jw_{j'}+k_{j'}w_j}{4(k_j^3k_{j'}+k_jk_{j'}^3)+6k_j^2k_{j'}^2+2l_jl_{j'}+k_jw_{j'}+k_{j'}w_j}. \tag{1.3.26}$$

For example, when $N = 3$,

$$\begin{aligned} f = \;& 1+e^{\eta_1}+e^{\eta_2}+e^{\eta_3}+a_{12}e^{\eta_1+\eta_2}+a_{13}e^{\eta_1+\eta_3} \\ & +a_{23}e^{\eta_2+\eta_3}+a_{12}a_{13}a_{23}e^{\eta_1+\eta_2+\eta_3} \end{aligned} \tag{1.3.27}$$

is a three-soliton solution if $\eta_j, j = 1, 2, 3$, are all real.

In general, formula (1.3.25) presents the N-soliton solution of the bilinear KP equation for real $\eta_j, j = 1, \cdots, N$.

1.3.3 Generalized bilinear derivatives

Let $p \in \mathbb{N}$ be given. For $f, g \in C^\infty(\mathbb{R}^M)$, let $D_p = (D_{p,1}, D_{p,2}, \cdots, D_{p,1})^T$, the generalized bilinear derivative or the D_p-operator is defined as follows [44]:

$$\begin{aligned}
&(D_{p,1}^{n_1} D_{p,2}^{n_2} \cdots D_{p,M}^{n_M} f \cdot g)(x) \\
= &\prod_{j=1}^{M} (\partial_{x_j} + \alpha_p \partial_{x'_j})^{n_j} f(x) g(x') \Big|_{x'=x} \\
= &\sum_{k_1=0}^{n_1} \cdots \sum_{k_M=0}^{n_M} \alpha_p^{\sum_{j=1}^{M}(n_j - k_j)} \prod_{j=1}^{M} \binom{n_j}{k_j} \partial_j^{k_j} f(x) \partial_j^{n_j - k_j} g(x), \qquad (1.3.28)
\end{aligned}$$

where $n_1, n_2, \cdots, n_M$ are nonnegative integers , and for any integer m, the mth power of α_p is defined by

$$\alpha_p^m := (-1)^{r(m)}, \qquad (1.3.29)$$

where $r(m) \equiv m \bmod p$ with $0 \leq r(m) < p$.

When $p = 2k$ ($k \in \mathbb{N}$), all the general bilinear derivatives become the Hirota derivatives, since $m - r(m)$ is even, we have $\alpha_p^m = (-1)^{r(m)} = (-1)^m$. Therefore $D_{2k,j} = D_j$ for $j = 1, \cdots, M$.

Now we consider $p = 3$, we have

$$\alpha_3^1 = -1, \alpha_3^2 = \alpha_3^3 = 1, \alpha_3^4 = -1, \alpha_3^5 = \alpha_3^6 = 1, \cdots \qquad (1.3.30)$$

It is easy to get

Example 1.3.2. *Examples of the general bilinear derivatives*

$$\begin{aligned}
D_{3,x} f \cdot g &= D_x f \cdot g = f_x g - f g_x, \\
D_{3,x} D_{3,y} f \cdot g &= D_x D_y f \cdot g = f_{xy} g - f_y g_x - f_x g_y + f g_{xy}, \\
D_{3,x} D_{3,y} D_{3,z} f \cdot g &= f_{xyz} g - f_{xy} g_z - f_{xz} g_y + f_x g_{yz} - f_{yz} g_x + f_y g_{xz} + f_z g_{xy} + f g_{xyz}, \\
D_{3,x} D_{3,y} D_{3,z} D_{3,t} f \cdot g &= f_{xyzt} g - f_{xyz} g_t - f_{xyt} g_z + f_{xy} g_{zt} - f_{xzt} g_y + f_{xz} g_{yt} \\
&\quad + f_{xt} g_{yz} + f_x g_{yzt} - f_{yzt} g_x + f_{yz} g_{xt} + f_{yt} g_{xz} + f_y g_{xzt} \\
&\quad + f_{zt} g_{xy} + f_z g_{xyt} + f_t g_{xyz} - f g_{xyzt},
\end{aligned}$$

Remark 1.3.3. *When* $p = 3$, (1.3.6) *is not true.*

Let M be a natural number and we will discuss the following general Hirota bilinear equation

$$P(D)f \cdot f = P(D_1, D_2, \cdots, D_M)f \cdot f = 0, \tag{1.3.31}$$

where P is a polynomial of M variables. Since the terms of odd powers are all zeros, we assume that P is an even polynomial, i.e., $P(-x) = P(x)$, and to generate non-zero polynomial solutions, we require that P has no constant term, i.e., $P(0) = 0$.

Moreover, we set

$$P(x) = \sum_{i,j=1}^{M} p_{ij}x_i x_j + \sum_{i,j,k,l=1}^{M} p_{ijkl}x_i x_j x_k x_l + \text{other terms}, \tag{1.3.32}$$

where p_{ij} and p_{ijkl} are coefficients of terms of second- and fourth-degree, to determine quadratic function solutions.

For convenience's sake, we adopt the index notation for partial derivatives of f:

$$f_{i_1 i_2 \cdots i_k} = \frac{\partial^k f}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_k}},\ 1 \le i_1, i_2, \cdots, i_k \le M. \tag{1.3.33}$$

Using this notation, we have the compact expressions for the second- and fourth-order Hirota bilinear derivatives:

$$D_i D_j f \cdot f = 2(f_{ij}f - f_i f_j),\ 1 \le i, j \le M, \tag{1.3.34}$$

and

$$\begin{aligned} & D_i D_j D_k D_l (f \cdot f) \\ = {} & 2\big[f_{ijkl}f - f_{ijk}f_l - f_{ijl}f_k - f_{ikl}f_j \\ & -f_{jkl}f_i + f_{ij}f_{kl} + f_{ik}f_{jl} + f_{il}f_{jk}\big],\ 1 \le i, j, k, l \le M. \end{aligned} \tag{1.3.35}$$

Motivated by Bell polynomial theories on soliton [47], we take the dependent variable transformations:

$$u = 2(\ln f)_{x_1},\ u = 2(\ln f)_{x_1 x_1}, \tag{1.3.36}$$

to formulate nonlinear differential equations from Hirota bilinear equations. Many integrable nonlinear equations can be generated this way [9, 32].

We will also consider a generalized bilinear equation

$$P(D_p)f \cdot f = P(D_{p,1}, D_{p,2}, \cdots, D_{p,M})f \cdot f = 0. \tag{1.3.37}$$

In this situation, we do not assume that P is even.

Example 1.3.4 (The KdV equation [32]). *Let us consider the KdV equation*

$$u_t + 6uu_x + u_{xxx} = 0. \tag{1.3.38}$$

We can use the transformation $u = 2(\ln f)_{xx}$ *to get the bilinear KdV equation*

$$(D_x D_t + D_x^4) f \cdot f = 0. \tag{1.3.39}$$

On the other hand, the partial differential equation

$$8w_t + 3w^4 + 12w^2 w + 12w_x^2 = 0, \tag{1.3.40}$$

can be changed to the generalized bilinear KdV equation

$$(D_{3,x} D_{3,t} + D_{3,x}^4) f \cdot f = 0 \tag{1.3.41}$$

under the transformation $w = 2(\ln f)_x$.

Example 1.3.5 (The modified KdV equation [32]).

$$u_t + 6u^2 u_x + u_{xxx} = 0. \tag{1.3.42}$$

We can use the bi-logarithmic transformation $u = i(\ln f/g)_x$ *to get the bilinear form of m-KdV equation*

$$\left\{ \begin{array}{rcl} (D_t + D_x^3) f \cdot g & = & 3\lambda D_x f \cdot g, \\ D_x^2 f \cdot g & = & \lambda f \cdot g, \end{array} \right. \tag{1.3.43}$$

where λ *is an arbitrary function.*

Chapter 2

Soliton hierarchies with bi-Hamiltonian structures

2.1 Introduction

In this chapter, we would like to consider two generalized Kaup-Newell spectral problems possessing two unknown potential functions associated with sl(2,$\mathbb{R}$) and so(3,$\mathbb{R}$) respectively. They are reduced spectral problems of the two D-Kaup-Newell spectral problems associated with sl(2,$\mathbb{R}$) and so(3,$\mathbb{R}$), which are presented respectively in [19, 74] and possess three dependent variables.

A standard procedure for generating soliton hierarchies [40, 76] is stated as follows. Let $\tilde{\mathfrak{g}}$ be the matrix loop algebra associated with a given matrix Lie algebra $\mathfrak{g}$. We first introduce a spatial spectral problem

$$\phi_x = U\phi,\ U = U(u, \lambda) \in \tilde{\mathfrak{g}}, \tag{2.1.1}$$

where u denotes a column vector of potential functions and λ is a spectral parameter. Then we take a solution of the form

$$W = W(u, \lambda) = \sum_{i\geq 0} W_i \lambda^{-i},\ W_i \in \mathfrak{g},\ i \geq 0, \tag{2.1.2}$$

to the stationary zero curvature equation

$$W_x = [U, W]. \tag{2.1.3}$$

Then, we try to determine Lax matrices

$$V^{[m]} = V^{[m]}(u, \lambda) = (\lambda^{f(m)} W)_+ + \Delta_m \in \tilde{\mathfrak{g}},\ m \geq 0, \tag{2.1.4}$$

$(P)_+$ denoting the polynomial part of P in λ and f being an appropriate function from $\mathbb{N}$ to $\mathbb{N}$, to formulate the temporal spectral problems

$$\phi_{t_m} = V^{[m]}\phi,\ m \geq 0. \tag{2.1.5}$$

The modification terms $\Delta_m \in \tilde{\mathfrak{g}}$ should be selected such that the corresponding zero curvature equations

$$U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0,\ m \geq 0, \tag{2.1.6}$$

will produce a hierarchy of soliton equations:

$$u_{t_m} = K_m(u),\ m \geq 0. \tag{2.1.7}$$

A soliton hierarchy usually possesses Hamiltonian structures

$$u_{t_m} = K_m(u) = J\frac{\delta \mathcal{H}_m}{\delta u},\ m \geq 0, \tag{2.1.8}$$

where $\frac{\delta}{\delta u}$ is the variational derivative, J is a Hamiltonian operator, and $\mathcal{H}_m,\ m \geq 0$, are common conserved functionals. Such Hamiltonian structures can often be generated by applying the trace identity where $\mathfrak{g}$ is semisimple [40, 76]:

$$\frac{\delta}{\delta u}\int \mathrm{tr}(\frac{\partial U}{\partial \lambda}W)dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\mathrm{tr}(\frac{\partial U}{\partial u}W),\ \gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\mathrm{tr}(W^2)|, \tag{2.1.9}$$

and the variational identity where $\mathfrak{g}$ is non-semisimple [43, 52]:

$$\frac{\delta}{\delta u}\int \langle\frac{\partial U}{\partial \lambda}, W\rangle dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\langle\frac{\partial U}{\partial u}, W\rangle,\ \gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\langle W, W\rangle|, \tag{2.1.10}$$

where $\langle\cdot,\cdot\rangle$ is a non-degenerate, symmetric, and ad-invariant bilinear form on the underlying matrix loop algebra $\tilde{\mathfrak{g}}$.

The three-dimensional real special linear Lie algebra $\mathrm{sl}(2,\mathbb{R})$ has the basis

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ e_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ e_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \tag{2.1.11}$$

whose commutators are

$$[e_1, e_2] = 2e_2,\ [e_2, e_3] = e_1,\ [e_3, e_1] = 2e_3; \tag{2.1.12}$$

whereas the special orthogonal Lie algebra $\mathrm{so}(3,\mathbb{R})$ has the basis

$$e_1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},\ e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix},\ e_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.1.13}$$

whose commutators are

$$[e_1, e_2] = e_3,\ [e_2, e_3] = e_1,\ [e_3, e_1] = e_2. \tag{2.1.14}$$

We will adopt the following two matrix loop algebras associated with $\mathrm{sl}(2,\mathbb{R})$ and $\mathrm{so}(3,\mathbb{R})$:

$$\widetilde{\mathrm{sl}}(2,\mathbb{R}) = \Big\{ \sum_{i\geq 0} M_i \lambda^{n-i} \mid M_i \in \mathrm{sl}(2,\mathbb{R}),\ i \geq 0 \text{ and } n \in \mathbb{Z} \Big\}, \tag{2.1.15}$$

and

$$\widetilde{\mathrm{so}}(3,\mathbb{R}) = \Big\{ \sum_{i\geq 0} M_i \lambda^{n-i} \mid M_i \in \mathrm{so}(3,\mathbb{R}),\ i \geq 0 \text{ and } n \in \mathbb{Z} \Big\}. \tag{2.1.16}$$

Those are spaces of all Laurent series in λ with coefficients in $\mathrm{sl}(2,\mathbb{R})$ or $\mathrm{so}(3,\mathbb{R})$ and with a finite polynomial part.

The rest of the chapter is structured as follows. In sections 2.2 and 2.3, we will introduce two general Kaup-Newell spectral problem associated with $\mathrm{sl}(2,\mathbb{R})$ and $\mathrm{so}(3,\mathbb{R})$ and then generate the corresponding soliton hierarchies of bi-Hamiltonian equations, respectively. In section 2.4, we introduce the gauge transformation to transform above two spectral problems to their equivalent spectral problems. In last section, we will make a conclusion and a few remarks.

2.2 A generalized Kaup-Newell soliton hierarchy associate with $\widetilde{\mathrm{sl}}(2,\mathbb{R})$

In this section we will derive a soliton hierarchy associate with the matrix loop algebra $\widetilde{\mathrm{sl}}(2,\mathbb{R})$. We begin with a spectral problem and let e_1, e_2, e_3 be defined by (2.1.11) with commutators introduced in (2.1.12).

2.2.1 Matrix spectral problem I

Let α be an arbitrary real constant. Let us introduce a spectral matrix

$$U = U(u,\lambda) = (\lambda^2+\alpha)e_1 + \lambda p e_2 + \lambda q e_3 = \begin{bmatrix} \lambda^2+\alpha & \lambda p \\ \lambda q & -\lambda^2-\alpha \end{bmatrix}, \tag{2.2.1}$$

and consider the following isospectral problem

$$\phi_x = U\phi = \begin{bmatrix} \lambda^2+\alpha & \lambda p \\ \lambda q & -\lambda^2-\alpha \end{bmatrix} \phi,\ u = \begin{bmatrix} p \\ q \end{bmatrix},\ \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}, \tag{2.2.2}$$

associated with $\widetilde{\mathrm{sl}}(2,\mathbb{R})$.

It is known [19] that the D-Kaup-Newell spectral problem associated with $\widetilde{\mathrm{sl}}(2,\mathbb{R})$ refers to

$$\phi_x = U\phi = \begin{bmatrix} \lambda^2 + r & \lambda p \\ \lambda q & -\lambda^2 - r \end{bmatrix} \phi,\ u = \begin{bmatrix} p \\ q \\ r \end{bmatrix},\ \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix},$$

which possesses three potentials: p, q and r. The new spectral problem (2.2.2) is just a reduced case of the above D-Kaup-Newell spectral problem under $r = \alpha$. It is, actually, also a generalization of the standard Kaup-Newell spectral problem [36], which corresponds to the case of $\alpha = 0$. There is another interesting reduction $r = \alpha pq$ of the D-Kaup-Newell spectral problem associated with $\widetilde{\mathrm{sl}}(2,\mathbb{R})$, which generates an integrable hierarchy (see [19, 55] for details).

2.2.2 Soliton hierarchy

We define a matrix

$$W = ae_1 + be_2 + ce_3 = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \in \widetilde{\mathrm{sl}}(2,\mathbb{R}), \tag{2.2.3}$$

and then, the stationary zero curvature equation $W_x = [U, W]$ turns out

$$\begin{cases} a_x = \lambda(pc - qb), \\ b_x = 2(\lambda^2 + \alpha)b - 2\lambda pa, \\ c_x = 2\lambda qa - 2(\lambda^2 + \alpha)c. \end{cases} \tag{2.2.4}$$

We further assume that

$$a = \sum_{i\geq 0} a_i\lambda^{-2i},\ b = \sum_{i\geq 0} b_i\lambda^{-2i-1},\ c = \sum_{i\geq 0} c_i\lambda^{-2i-1}, \tag{2.2.5}$$

and take the initial values

$$a_0 = 1,\ b_0 = p,\ c_0 = q, \tag{2.2.6}$$

which are required by

$$a_{0,x} = pc_0 - qb_0,\ b_0 - pa_0 = 0,\ qa_0 - c_0 = 0. \tag{2.2.7}$$

Now based on (2.2.4), we have

$$\begin{cases} a_{i,x} = pc_i - qb_i, \\ b_{i,x} = 2\alpha b_i + 2b_{i+1} - 2pa_{i+1}, \quad i \geq 0. \\ c_{i,x} = 2qa_{i+1} - 2\alpha c_i - 2c_{i+1}, \end{cases} \tag{2.2.8}$$

From this, we can derive the recursion relations for $i \geq 0$,

$$\begin{cases} a_{i+1,x} = \alpha q b_i - \alpha p c_i - \frac{q}{2} b_{i,x} - \frac{p}{2} c_{i,x}, \\ b_{i+1} = \frac{1}{2} b_{i,x} - \alpha b_i + p a_{i+1}, \\ c_{i+1} = q a_{i+1} - \frac{1}{2} c_{i,x} - \alpha c_i, \end{cases} \tag{2.2.9}$$

since (2.2.8) tells

$$\begin{aligned} a_{i+1,x} &= p c_{i+1} - q b_{i+1} \\ &= p(q a_{i+1} - \tfrac{1}{2} c_{i,x} - \alpha c_i) - q(\tfrac{1}{2} b_{i,x} - \alpha b_i + p a_{i+1}) \\ &= \alpha q b_i - \alpha p c_i - \tfrac{q}{2} b_{i,x} - \tfrac{p}{2} c_{i,x},\ i \geq 0. \end{aligned}$$

We impose the conditions for integration:

$$a_i|_{u=0} = b_i|_{u=0} = c_i|_{u=0} = 0,\ i \geq 1, \tag{2.2.10}$$

to determine the sequence of $\{a_i, b_i, c_i\ :\ i \geq 1\}$ uniquely. We list the first two sets as follows:

$$\begin{aligned} a_1 &= -\tfrac{1}{2} pq, \\ b_1 &= \tfrac{1}{2} p_x - \alpha p - \tfrac{1}{2} p^2 q, \\ c_1 &= -\tfrac{1}{2} q_x - \alpha q - \tfrac{1}{2} p q^2; \\ a_2 &= \alpha pq + \tfrac{1}{4}(p q_x - p_x q) + \tfrac{3}{8} p^2 q^2, \\ b_2 &= \alpha^2 p - \alpha p_x + \tfrac{1}{4} p_{xx} - \tfrac{3}{4} p p_x q + \tfrac{3}{2} \alpha p^2 q + \tfrac{3}{8} p^3 q^2, \\ c_2 &= \alpha^2 q + \alpha q_x + \tfrac{1}{4} q_{xx} + \tfrac{3}{4} p q q_x + \tfrac{3}{2} \alpha p q^2 + \tfrac{3}{8} p^2 q^3. \end{aligned}$$

Based on the recursion relations (2.2.9), we obtain

$$\begin{bmatrix} c_{i+1} \\ b_{i+1} \end{bmatrix} = \Psi \begin{bmatrix} c_i \\ b_i \end{bmatrix},\ i \geq 0, \tag{2.2.11}$$

where the recursion matrix

$$\Psi = \begin{bmatrix} -\alpha - \frac{1}{2}\partial - \alpha q \partial^{-1} p - \frac{1}{2} q \partial^{-1} p \partial & \alpha q \partial^{-1} q - \frac{1}{2} q \partial^{-1} q \partial \\ -\alpha p \partial^{-1} p - \frac{1}{2} p \partial^{-1} p \partial & -\alpha + \frac{1}{2}\partial + \alpha p \partial^{-1} q - \frac{1}{2} p \partial^{-1} q \partial \end{bmatrix}, \tag{2.2.12}$$

in which $\partial = \frac{\partial}{\partial x}$. We will see that all vectors $[c_i, b_i]^T,\ i \geq 0$, are gradient, and will generate conserved functionals.

From the first three sets of $\{a_j, b_j, c_j\}$, we know they are all local. Now we show all the a_j's, b_j's and c_j's are local for any $j \geq 0$.

Proposition 2.2.1. *The functions a_j, b_j and c_j determined by the recursive relations* (2.2.8) *are all local for* $j \geq 0$.

Proof. Since $W_x = [U, W]$, we have

$$\frac{d}{dx} tr(W^2) = 2tr(WW_x) = 2tr(X[U, W]) = 0,$$

and so, due to $tr(W^2) = 2(a^2 + bc)$ and the initial data $(a_0, b_0, c_0) = (1, p, q)$ we have

$$2(a^2 + bc)\big|_{u=0} = 2.$$

Using the Laurent series of λ gives for $k > 0$

$$\sum_{j=0}^{k} a_j a_{k-j} + \sum_{j=0}^{k-1} b_j c_{k-1-j} = 0. \tag{2.2.13}$$

Then

$$a_k = -\frac{1}{2}\Big(\sum_{j=i}^{k-1} a_j a_{k-j} + \sum_{j=0}^{k-1} b_j c_{k-1-j}\Big), \quad k \geq 0.$$

Our conclusion follows from (2.2.9) by the induction. □

Now for each $m \geq 0$, we introduce

$$\begin{aligned} V^{[m]} &= \lambda(\lambda^{2m+1} W)_+ \\ &= (\lambda^{2m+2} W)_+ - a_{m+1} e_1 \\ &= \sum_{i=0}^{m} [a_i \lambda^{2(m-i)+2} e_1 + b_i \lambda^{2(m-i)+1} e_2 + c_i \lambda^{2(m-i)+1} e_3], \end{aligned} \tag{2.2.14}$$

and the corresponding zero curvature equations

$$U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0, \ m \geq 0, \tag{2.2.15}$$

engender a hierarchy of solution equations

$$u_{t_m} = K_m = \begin{bmatrix} b_{m,x} - 2\alpha b_m \\ c_{m,x} + 2\alpha c_m \end{bmatrix} = J \begin{bmatrix} c_m \\ b_m \end{bmatrix}, \ m \geq 0, \tag{2.2.16}$$

where

$$J = \begin{bmatrix} 0 & \partial - 2\alpha \\ \partial + 2\alpha & 0 \end{bmatrix}. \tag{2.2.17}$$

It is obvious that J is a Hamiltonian operator by Proposition since it is skew-adjoint and does not depend on the potentials. We can get the adjoint operator of Ψ using integration by parts.

$$\Psi^* = \begin{bmatrix} -\alpha + \frac{1}{2}\partial + \alpha p\partial^{-1}q - \frac{1}{2}\partial p\partial^{-1}q & \alpha p\partial^{-1}p - \frac{1}{2}\partial p\partial^{-1}p \\ -\alpha q\partial^{-1}q - \frac{1}{2}\partial q\partial^{-1}q & -\alpha - \frac{1}{2}\partial - \alpha q\partial^{-1}p - \frac{1}{2}\partial q\partial^{-1}p. \end{bmatrix}$$

It is easy to compute that

$$\frac{\partial U}{\partial \lambda} = \begin{bmatrix} 2\lambda & p \\ q & -2\lambda \end{bmatrix}, \ \frac{\partial U}{\partial p} = \begin{bmatrix} 0 & \lambda \\ 0 & 0 \end{bmatrix}, \ \frac{\partial U}{\partial q} = \begin{bmatrix} 0 & 0 \\ \lambda & 0 \end{bmatrix},$$

and so, we have

$$\mathrm{tr}(W\frac{\partial U}{\partial \lambda}) = 4\lambda a + qb + pc, \ \mathrm{tr}(W\frac{\partial U}{\partial p}) = \lambda c, \ \mathrm{tr}(W\frac{\partial U}{\partial q}) = \lambda b.$$

By the trace identity (2.1.9), we get

$$\frac{\delta}{\delta u}\int (4\lambda a + qb + pc)dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\begin{bmatrix} \lambda c \\ \lambda b \end{bmatrix}.$$

Balancing coefficients of all power of λ in above equality tells

$$\frac{\delta}{\delta u}\int (4a_{m+1} + qb_m + pc_m)dx = (\gamma - 2m)\begin{bmatrix} c_m \\ b_m \end{bmatrix}, \ m \geq 0.$$

Taking $m = 1$, we obtain $\gamma = 0$, and further, we arrive at

$$\frac{\delta}{\delta u}\int (-\frac{4a_{m+1} + qb_m + pc_m}{2m})dx = \begin{bmatrix} c_m \\ b_m \end{bmatrix}, \ m \geq 1. \tag{2.2.18}$$

Therefore, we get Hamiltonian structures for the generalized Kaup-Newell soliton hierarchy (2.2.16) associated with $\widetilde{\mathrm{sl}}(2, \mathbb{R})$:

$$u_{t_m} = K_m = J\begin{bmatrix} c_m \\ b_m \end{bmatrix} = J\frac{\delta \mathcal{H}_m}{\delta u}, \ m \geq 0, \tag{2.2.19}$$

where the Hamiltonian functionals are given by

$$\begin{aligned} \mathcal{H}_0 &= \int pq dx, \\ \mathcal{H}_m &= \int (-\frac{4a_{m+1} + qb_m + pc_m}{2m})dx, \ m \geq 1. \end{aligned} \tag{2.2.20}$$

Now we introduce an operator

$$M = J\Psi = \Psi^* J = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \tag{2.2.21}$$

where all elements can be explicitly worked out:

$$\begin{cases} M_{11} = 2\alpha^2 p\partial^{-1}p + \alpha p\partial^{-1}p\partial - \alpha\partial p\partial^{-1}p - \frac{1}{2}\partial p\partial^{-1}p\partial, \\ M_{12} = \frac{1}{2}\partial^2 - 2\alpha\partial + 2\alpha^2 - 2\alpha^2 p\partial^{-1}q + \alpha\partial p\partial^{-1}q + \alpha p\partial^{-1}q\partial - \frac{1}{2}\partial p\partial^{-1}q\partial, \\ M_{21} = -\frac{1}{2}\partial^2 - 2\alpha\partial - 2\alpha^2 - 2\alpha^2 q\partial^{-1}p - \alpha\partial q\partial^{-1}p - \alpha q\partial^{-1}p\partial - \frac{1}{2}\partial q\partial^{-1}p\partial, \\ M_{22} = 2\alpha^2 q\partial^{-1}q + \alpha\partial q\partial^{-1}q - \alpha q\partial^{-1}q\partial - \frac{1}{2}\partial q\partial^{-1}q\partial. \end{cases} \tag{2.2.22}$$

It is easy to verify that M is skew-adjoint. By a long and tedious computation with Maple, we can verify that J and M constitute a Hamiltonian pair [15, 63] , i.e., any linear combination N of J and M is skew-symmetry and satisfies the Jacobi identity:

$$\int K^T N'(u)[NS]T dx + \text{cycle}(K, S, T) = 0$$

for all vector fields K, S and T. Thus, the operator

$$\Phi = \Psi^* \tag{2.2.23}$$

is hereditary, that is, it satisfies that

$$\Phi'(u)[\Phi K]S - \Phi\Phi'(u)[K]S = \Phi'(u)[\Phi S]K - \Phi\Phi'(u)[S]K$$

for all vector fields K and S (see [14] for definition of hereditary operators). Here Φ' denotes the Gateaux derivative of Φ as usual.

The above condition for the hereditary operators is equivalent to

$$L_{\Phi K}\Phi = \Phi L_K \Phi \tag{2.2.24}$$

for any vector field K. Here $L_K\Phi$ is the Lie derivative defined by

$$(L_K\Phi)S := \Phi[K, S] - [K, \Phi S],$$

where $[\cdot, \cdot]$ is the Lie bracket of vector fields. Note that for any autonomous operator $\Psi = \Psi(u, u_x, \cdots)$ is a recursion operator of a given evolution equation $u_t = K(u)$ if and only if Φ satisfies

$$L_K\Phi = \Phi'[K] - [K', \Phi] = 0,$$

S' denoting the Gateaux derivative operator of a vector field S (see [68] for more details).

For the hierarchy (2.2.16), it is direct to show that

$$L_{K_0}\Phi = \Phi'[K_0] - [K_0', \Phi] = 0. \tag{2.2.25}$$

Thus, it follows now that the hierarchy (2.2.16) is bi-Hamiltonian:

$$u_{t_m} = K_m = J\frac{\delta \mathcal{H}_m}{\delta u} = M\frac{\delta \mathcal{H}_{m-1}}{\delta u},\ m \geq 1, \tag{2.2.26}$$

and Φ is a common hereditary recursion operator for the whole hierarchy (2.2.16). All this implies that the hierarchy (2.2.16) is Liouville integrable [49]. We point out that no bi-Hamiltonian structure was presented for the D-Kaup-Newell soliton hierarchy in [19], though the hierarchy was shown to have infinitely many symmetries.

When $m = 0$, we get a linear system:

$$u_{t_0} = \begin{bmatrix} p \\ q \end{bmatrix}_{t_0} = K_0 = \begin{bmatrix} p_x - 2\alpha p \\ q_x + 2\alpha q \end{bmatrix} = J\frac{\delta \mathcal{H}_0}{\delta u}. \tag{2.2.27}$$

When $m = 1$, we have a nonlinear system of bi-Hamiltonian equations:

$$\begin{aligned} u_{t_1} &= \begin{bmatrix} p \\ q \end{bmatrix}_{t_1} = K_1 \\ &= \begin{bmatrix} 2\alpha^2 p + \alpha p^2 q + \frac{1}{2}p_{xx} - 2\alpha p_x - \frac{1}{2}p^2 q_x - pqp_x \\ -2\alpha^2 q - \alpha pq^2 - 2\alpha q_x - \frac{1}{2}q_{xx} - pqq_x - \frac{1}{2}p_x q^2 \end{bmatrix} \\ &= J\frac{\delta \mathcal{H}_1}{\delta u} = M\frac{\delta \mathcal{H}_0}{\delta u}, \end{aligned} \tag{2.2.28}$$

where $\mathcal{H}_1$ can also be explicitly given by

$$\mathcal{H}_1 = \int \big[-\alpha pq - \frac{1}{4}p^2q^2 - \frac{1}{4}(pq_x - p_x q)\big]\,dx. \tag{2.2.29}$$

2.3 A generalized Kaup-Newell soliton hierarchy associate with $\widetilde{\mathrm{so}}(3,\mathbb{R})$

In this section, we will construct the second soliton hierarchy based on the 3-dimensional orthogonal Lie algebra $\widetilde{\mathrm{so}}(3,\mathbb{R})$. The basis of $\widetilde{\mathrm{so}}(3,\mathbb{R})$ consisting of e_1, e_2, e_3 is defined by (2.1.13).

2.3.1 Matrix spectral problem II

Let us introduce the spectral matrix with a real constant α:

$$\begin{aligned} U &= U(u,\lambda) = (\lambda^2+\alpha)e_1 + \lambda p e_2 + \lambda q e_3 \\ &= \begin{bmatrix} 0 & -\lambda q & -\lambda^2-\alpha \\ \lambda q & 0 & -\lambda p \\ \lambda^2+\alpha & \lambda p & 0 \end{bmatrix}, \end{aligned} \tag{2.3.1}$$

and consider the following isospectral problem

$$\begin{aligned} &\phi_x = U\phi = \begin{bmatrix} 0 & -\lambda q & -\lambda^2-\alpha \\ \lambda q & 0 & -\lambda p \\ \lambda^2+\alpha & \lambda p & 0 \end{bmatrix}\phi, \\ &u = \begin{bmatrix} p \\ q \end{bmatrix},\ \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}, \end{aligned} \tag{2.3.2}$$

associated with $\widetilde{\mathrm{so}}(3,\mathbb{R})$.

The D-Kaup-Newell spectral problem associated with $\widetilde{\mathrm{so}}(3,\mathbb{R})$ was studied recently in [74]:

$$\phi_x = U\phi = \begin{bmatrix} 0 & -\lambda q & -\lambda^2-r \\ \lambda q & 0 & -\lambda p \\ \lambda^2+r & \lambda p & 0 \end{bmatrix}\phi, u = \begin{bmatrix} p \\ q \\ r \end{bmatrix},\ \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix},$$

which possesses three potentials: p, q and r. The new spectral problem (2.3.2) is a reduced case of the above D-Kaup-Newell spectral problem under $r = \alpha$. In fact, it is also a generalization of the standard Kaup-Newell spectral problem in [46], which corresponds to the case of $\alpha = 0$. Another interesting reduction $r = \alpha(p^2+q^2)$ of the D-Kaup-Newell type spectral problem associated with $\widetilde{\mathrm{so}}(3,\mathbb{R})$, generating an integrable hierarchy, has been proposed and studied in [39].

2.3.2 Soliton hierarchy

Define

$$W = ae_1 + be_2 + ce_3 = \begin{bmatrix} 0 & -c & -a \\ c & 0 & -b \\ a & b & 0 \end{bmatrix} \in \widetilde{\mathrm{so}}(3,\mathbb{R}), \tag{2.3.3}$$

and then, the stationary zero curvature equation $W_x = [U, W]$ becomes

$$\begin{cases} a_x = \lambda(pc - qb), \\ b_x = \lambda qa - (\lambda^2 + \alpha)c, \\ c_x = (\lambda^2 + \alpha)b - \lambda pa. \end{cases} \tag{2.3.4}$$

We further assume that

$$a = \sum_{i \geq 0} a_i \lambda^{-2i},\ b = \sum_{i \geq 0} b_i \lambda^{-2i-1},\ c = \sum_{i \geq 0} c_i \lambda^{-2i-1}, \tag{2.3.5}$$

and take the initial values

$$a_0 = 1,\ b_0 = p,\ c_0 = q, \tag{2.3.6}$$

which are required by

$$a_{0,x} = pc_0 - qb_0,\ -c_0 + qa_0 = 0,\ -pa_0 + b_0 = 0. \tag{2.3.7}$$

Now based on (2.3.4), we have

$$\begin{cases} a_{i,x} = pc_i - qb_i, \\ b_{i,x} = qa_{i+1} - c_{i+1} - \alpha c_i, \quad i \geq 0. \\ c_{i,x} = b_{i+1} + \alpha b_i - pa_{i+1}, \end{cases} \tag{2.3.8}$$

From this, we can derive the recursion relations

$$\begin{cases} a_{i+1,x} = -pb_{i,x} + \alpha qb_i - qc_{i,x} - \alpha pc_i, \\ b_{i+1} = c_{i,x} - \alpha b_i + pa_{i+1}, \qquad i \geq 0, \\ c_{i+1} = -b_{i,x} - \alpha c_i + qa_{i+1}, \end{cases} \tag{2.3.9}$$

since (2.3.8) tells

$$\begin{aligned} a_{i+1,x} &= pc_{i+1} - qb_{i+1} \\ &= p(-b_{i,x} - \alpha c_i + qa_{i+1}) - q(c_{i,x} - \alpha b_i + pa_{i+1}) \\ &= -pb_{i,x} + \alpha qb_i - qc_{i,x} - \alpha pc_i,\ i \geq 0. \end{aligned}$$

We impose the conditions for integration:

$$a_i|_{u=0} = b_i|_{u=0} = c_i|_{u=0} = 0,\ i \geq 1, \tag{2.3.10}$$

to determine the sequence of $\{a_i, b_i, c_i\ :\ i \geq 1\}$ uniquely.

It follows directly from the recursion relations (2.3.9) that we have

$$\begin{bmatrix} b_{i+1} \\ c_{i+1} \end{bmatrix} = \Psi \begin{bmatrix} b_i \\ c_i \end{bmatrix},\ i \geq 0, \tag{2.3.11}$$

where

$$\Psi = \begin{bmatrix} -\alpha - p\partial^{-1}p\partial + \alpha p\partial^{-1}q & \partial - \alpha p\partial^{-1}p - p\partial^{-1}q\partial \\ -\partial - q\partial^{-1}p\partial + \alpha q\partial^{-1}q & -\alpha - \alpha q\partial^{-1}p - q\partial^{-1}q\partial \end{bmatrix}. \tag{2.3.12}$$

We will see that all vectors $(b_i, c_i)^T$, $i \geq 0$, above are gradient, which will generate conserved functionals.

Similar to the previous section, $\{a_j, b_j, c_j \,:\, j \geq 0\}$, are all local.

Proposition 2.3.1. *The functions a_j, b_j and c_j defiend by* (2.3.9) *are all local for $j \geq 0$.*

Now for each $m \geq 0$, we introduce

$$\begin{aligned} V^{[m]} &= \lambda(\lambda^{2m+1}W)_+ \\ &= (\lambda^{2m+2}W)_+ - a_{m+1}e_1 \\ &= \sum_{i=0}^{m}[a_i\lambda^{2(m-i)+2}e_1 + b_i\lambda^{2(m-i)+1}e_2 + c_i\lambda^{2(m-i)+1}e_3], \end{aligned} \tag{2.3.13}$$

and the corresponding zero curvature equations

$$U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0,\ m \geq 0, \tag{2.3.14}$$

engender a hierarchy of solution equations

$$u_{t_m} = K_m = \begin{bmatrix} b_{m,x} + \alpha c_m \\ c_{m,x} - \alpha b_m \end{bmatrix} = J \begin{bmatrix} b_m \\ c_m \end{bmatrix},\ m \geq 0, \tag{2.3.15}$$

with J being defined by

$$J = \begin{bmatrix} \partial & \alpha \\ -\alpha & \partial \end{bmatrix}. \tag{2.3.16}$$

It is direct to check that J is Hamiltonian.

It is easy to see that

$$\frac{\partial U}{\partial \lambda} = \begin{bmatrix} 0 & -q & -2\lambda \\ q & 0 & -p \\ 2\lambda & p & 0 \end{bmatrix},\ \frac{\partial U}{\partial p} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix},\ \frac{\partial U}{\partial q} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and so, we obtain

$$\mathrm{tr}(W\frac{\partial U}{\partial \lambda}) = -4\lambda a - 2pb - 2qc,\ \mathrm{tr}(W\frac{\partial U}{\partial p}) = -2\lambda b,\ \mathrm{tr}(W\frac{\partial U}{\partial q}) = -2\lambda c.$$

By the trace identity we obtain Hamiltonian structures for the reduced D-Kaup-Newell hierarchy (2.3.15) associated with $\widetilde{\mathrm{so}}(3,\mathbb{R})$:

$$u_{t_m} = K_m = J\begin{bmatrix} b_m \\ c_m \end{bmatrix} = J\frac{\delta \mathcal{H}_m}{\delta u},\ m \geq 0, \tag{2.3.17}$$

where the Hamiltonian functionals are given by

$$\begin{aligned} \mathcal{H}_0 &= \int \frac{1}{2}(p^2+q^2)dx, \\ \mathcal{H}_m &= \int (-\frac{2a_{m+1}+pb_m+qc_m}{2m})dx,\ m \geq 1. \end{aligned} \tag{2.3.18}$$

Introduce a second Hamiltonian operator

$$M = J\Psi = \Psi^* J = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \tag{2.3.19}$$

where all elements can be explicitly worked out:

$$\begin{cases} M_{11} = -2\alpha\partial + \alpha^2 q\partial^{-1}q + \alpha\partial p\partial^{-1}q - \alpha q\partial^{-1}p\partial - \partial p\partial^{-1}p\partial, \\ M_{12} = -\alpha^2 + \partial^2 - \alpha^2 q\partial^{-1}p - \alpha q\partial^{-1}q\partial - \alpha\partial p\partial^{-1}p - \partial p\partial^{-1}q\partial, \\ M_{21} = \alpha^2 - \partial^2 - \alpha^2 p\partial^{-1}q + \alpha\partial q\partial^{-1}q + \alpha p\partial^{-1}p\partial - \partial q\partial^{-1}p\partial, \\ M_{22} = -2\alpha\partial + \alpha^2 p\partial^{-1}p - \alpha\partial q\partial^{-1}p + \alpha p\partial^{-1}q\partial - \partial q\partial^{-1}q\partial. \end{cases} \tag{2.3.20}$$

A direct computation by Maple can show that J and M constitute a Hamiltonian pair. Thus once proving $L_{K_0}\Phi = 0$, we can show by the same argument as in the previous section that the operator

$$\Phi = \Psi^* = \begin{bmatrix} -\alpha - \partial p\partial^{-1}p - \alpha q\partial^{-1}p & \partial - \partial p\partial^{-1}q - \alpha q\partial^{-1}q \\ -\partial + \alpha p\partial^{-1}p - \partial q\partial^{-1}p & -\alpha + \alpha p\partial^{-1}q - \partial q\partial^{-1}q \end{bmatrix} \tag{2.3.21}$$

is a common hereditary recursion operator for the whole hierarchy (2.3.15), and the hierarchy (2.3.15) is bi-Hamiltonian:

$$u_{t_m} = K_m = J\frac{\delta \mathcal{H}_m}{\delta u} = M\frac{\delta \mathcal{H}_{m-1}}{\delta u},\ m \geq 1. \tag{2.3.22}$$

All this shows that the A generalized Kaup-Newell hierarchy (2.3.15) associated with $\widetilde{\mathrm{so}}(3,\mathbb{R})$ is Liouville integrable [49].

When $m = 0$, we get a linear system

$$u_{t_0} = \begin{bmatrix} p \\ q \end{bmatrix}_{t_0} = K_0 = \begin{bmatrix} p_x + \alpha q \\ -\alpha p + q_x \end{bmatrix} = J\frac{\delta \mathcal{H}_0}{\delta u}. \tag{2.3.23}$$

When $m = 1$, we obtain a nonlinear system of bi-Hamiltonian equations:

$$\begin{aligned} u_{t_1} &= \begin{bmatrix} p \\ q \end{bmatrix}_{t_1} = K_1 \\ &= \begin{bmatrix} -\alpha^2 q - 2\alpha p_x + q_{xx} - \frac{1}{2}(\alpha q + p_x)(p^2+q^2) - p(pp_x + qq_x) \\ \alpha^2 p - 2\alpha q_x - p_{xx} + \frac{1}{2}(\alpha p + q_x)(p^2+q^2) - q(pp_x + qq_x) \end{bmatrix} \\ &= J\frac{\delta \mathcal{H}_1}{\delta u} = M\frac{\delta \mathcal{H}_0}{\delta u}, \end{aligned} \tag{2.3.24}$$

where $\mathcal{H}_1$ can also be explicitly given by

$$\mathcal{H}_1 = \int \Big[\frac{1}{2}(-\alpha p^2 - \alpha q^2 - p_x q + p q_x) - \frac{1}{8}(p^2+q^2)^2\Big] dx. \tag{2.3.25}$$

We point out that no bi-Hamiltonian structure was presented for the two D-Kaup-Newell soliton hierarchies associated with $\widetilde{\mathrm{so}}(3,\mathbb{R})$ [74]. Only quasi-Hamiltonian strcutures were established for the second D-Kaup-Newell soliton hierarchy in [74].

2.4 Concluding remarks

We have studied the two generalized Kaup-Newell spectral problems associated with $\widetilde{\mathrm{sl}}(2,\mathbb{R})$ and $\widetilde{\mathrm{so}}(3,\mathbb{R})$ and generated their hierarchies of bi-Hamiltonian equations via the zero curvature formulation. The Liouville integrability of the resulting soliton hierarchies has been shown upon establishing the bi-Hamiltonian structures through the trace identity.

Unlike the D-Kaup-Newell soliton hierarchies [19, 74], the two soliton hierarchies we obtained present bi-Hamiltonian structures, though our spectral problems involve less potentials. Moreover, the newly presented Hamiltonian pairs display extended recursion operator structures from the known Kaup-Newell recursion operators associated with $\widetilde{\mathrm{sl}}(2,\mathbb{R})$ and $\widetilde{\mathrm{so}}(3,\mathbb{R})$ [36, 46]. Our spectral problem differs from that discussed in [78] and also soliton hierarchies are dissimilar.

Chapter 3

A class of lump and lump-type solutions of evolution equations

3.1 Introduction

The Hirota direct method [32] is one of the most powerful approaches for constructing multisoliton solutions to integrable equations. Its successful idea is to use a transformation of dependent variables to convert nonlinear differential equations into bilinear forms defined in terms of Hirota bilinear derivatives.

In recent years, there has been a growing interest in rationally localized solutions in the space [7, 25, 33, 65], particularly lump solutions, localized in all directions in the space (see, e.g., references [1, 24, 35, 73] for typical examples). The KPI equation

$$(u_t + 6uu_x + u_{xxx})_x - u_{yy} = 0 \tag{3.1.1}$$

has the following lump solutions [64]

$$u = 4\frac{-[x + ay + (a^2 - b^2)t]^2 + b^2(y + 2at)^2 + 3/b^2}{\{[x + ay + (a^2 - b^2)t]^2 + b^2(y + 2at)^2 + 3/b^2\}^2}, \tag{3.1.2}$$

where a and $b \neq 0$ are two free real constants. More generally, the KPI equation (3.1.1) admits the lump solution [50] with the six free parameters.

In this chapter, we would like to consider Hirota bilinear equations and generalized bilinear equations, and focus on some classes of generalized bilinear differential equations involving the generalized bilinear derivatives with the prime number $p = 2, 3$ presented in Ref. [44]. We will search for their positive quadratic and quartic function solutions by symbolic computation with Maple. Further, we will present lump-type solutions to the corresponding nonlinear differential equations generated by $u = 2(\ln f)_x$. The nonlinear differential equations can be achieved by rational transformation $u = f/g$, the bi-logarithmic transformation $u = 2(\ln f/g)_x$, etc [32]. We also have a short discussion on higher-degree polynomial solutions. It is hoped that the study will help us recognize characteristics of integrability more concretely. A few concluding remarks will be given at the end of the chapter.

3.2 Non-negative quadratic functions and solutions to bilinear forms

We consider real polynomials on $\mathbb{R}^M$ in this chapter.

3.2.1 Non-negative quadratic polynomials

Definition 3.2.1. *Assume that f is a polynomial on $\mathbb{R}^M$. Then f is positive if $f(x) > 0,\ \forall x \in \mathbb{R}^M$; f is non-negative or positive semidefinite (PSD) if $f(x) \geq 0,\ \forall x \in \mathbb{R}^M$.*

A quadratic polynomial $f : \mathbb{R}^M \to \mathbb{R}$ can always be expressed as

$$f(x) = x^T Ax - 2b^T x + c, \tag{3.2.1}$$

where $A \in \mathbb{R}^{M\times M}$ is a symmetric matrix, $b \in \mathbb{R}^M$ denotes a column vector, $c \in \mathbb{R}$ is a constant and T denotes matrix transpose. A, b, c are uniquely determined by f under the condition of $A = A^T$.

We introduce the the Moore-Penrose pseudoinverse [70] of a matrix to describe the non-negativity (or positivity) of a quadratic function.

Definition 3.2.2. *Let a matrix $A \in \mathbb{R}^{N\times M}$. We call a matrix $A^+ \in \mathbb{R}^{M\times N}$ the Moore-Penrose pseudoinverse of A if it satisfies*

$$AA^+A = A,\ A^+AA^+ = A^+,\ (AA^+)^T = AA^+,\ (A^+A)^T = A^+A. \tag{3.2.2}$$

We denote that a positive-semidefinite matrix $A \in \mathbb{R}^{M\times M}$ by $A \geq 0$ and positive-definite by $A > 0$. Namely, $A \geq 0$ means that $x^T Ax \geq 0$ for all $x \in \mathbb{R}^M$ and $A > 0$ iff $x^T Ax > 0$ for all non-zero $x \in \mathbb{R}^M$. The following theorem gives a description of non-negative and positive quadratic functions.

Proposition 3.2.3. *Let a quadratic function f be defined by (3.2.1). Then f is non-negative if and only if $A \geq 0$, $b \in$ range(A) and*

$$d = c - b^T A^+ b \geq 0. \tag{3.2.3}$$

Moreover, f is positive if and only if $A \geq 0$, $b \in$ range(A) and $d = c - b^T A^+ b > 0$.

Proof. If $b \in \text{range}(A)$, then there exists a vector $\alpha \in \mathbb{R}^M$ such that $A\alpha = b$. We get

$$\begin{aligned} f(x) &= x^T A A x - b^T x - x^T b + c \\ &= (x-\alpha)^T A (x-\alpha) + c - \alpha^T A \alpha \\ &= (x-\alpha)^T A (x-\alpha) + c - b^T A^+ b. \end{aligned} \tag{3.2.4}$$

The sufficiency is immediately implied by (3.2.4).

The necessity. Suppose that $A \geq 0$ is false. Then there exists a vector $\beta \in \mathbb{R}^M$ such that $\beta^T A \beta < 0$, and further for $r \in \mathbb{R}$, we have

$$f(r\beta) = r^2 \beta^T A \beta - 2 r b^T \beta + c \to -\infty, \text{ as } r \to \infty.$$

This is a contradiction to the assumption that f is non-negative. Therefore, we have $A \geq 0$.

Now let $b = b^{(1)} + b^{(2)}$ with $b^{(1)} \in \text{range}(A)$ and $b^{(2)} \in \text{range}(A)^\perp$. Assume that $\alpha \in \mathbb{R}^M$ satisfies $A\alpha = b^{(1)}$. Consider $x = \alpha + r b^{(2)}$, with r being a positive number. Then we can have

$$\begin{aligned} f(x) &= x^T A x - 2\alpha^T A x - 2 b^{(2)T} x + c \\ &= (x-\alpha)^T A (x-\alpha) - 2 b^{(2)T} x + c - \alpha^T A \alpha \\ &= r^2 b^{(2)T} A b^{(2)} - 2 b^{(2)T} \alpha - 2 r b^{(2)T} b^{(2)} + c - \alpha^T A \alpha \\ &= -2 r b^{(2)T} b^{(2)} - 2 b^{(2)T} \alpha + c - \alpha^T A \alpha \to -\infty, \text{ as } r \to \infty, \end{aligned}$$

if $b^{(2)} \neq 0$. Therefore $b^{(2)} = 0$, since f is non-negative. This implies $b \in \text{range}(A)$. Further, $d = f(\alpha) \geq 0$. The last conclusion is obvious. □

Corollary 3.2.4. *If function $f(x) = x^T A x - 2 b^T x + c$ is bounded below. Then $d = c - b^T A^+ b$ is the largest low bound.*

Next, we will explore relations between quadratic function solutions and sums of squares of linear functions, and discuss quadratic function solutions which can be written as sums of squares of linear functions.

Proposition 3.2.5. *Let a non-negative quadratic function f be defined by* (3.2.1). *Suppose r = rank(A). Then there exist $b^{(j)} \in \mathbb{R}^M$, $c_j \in \mathbb{R}$, $1 \leq j \leq r$, such that*

$$f(x) = \sum_{j=1}^{r} (b^{(j)T} x + c_j)^2 + d \tag{3.2.5}$$

with $d \geq 0$.

Moreover, if $f(x) = \sum_{j=1}^{s} (\hat{b}^{(j)T} x + \hat{c}_j)^2 + \hat{h}$, *where* $\hat{b}^{(j)} \in \mathbb{R}^M$, $\hat{c}_j \in \mathbb{R}$, $1 \leq j \leq s$, $\hat{h} \in \mathbb{R}$, *then* $s \geq r$.

Proof. By proposition 3.2.3, we know $A \geq 0$. We consider the singular value decomposition of the matrix A:

$$A = V \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T, \tag{3.2.6}$$

where $V \in \mathbb{R}^{M \times M}$ is orthogonal and

$$\Sigma = \text{diag}(d_1, \cdots, d_r),\ d_1 \geq \cdots \geq d_r > 0.$$

Upon denoting $V = (v^{(1)}, v^{(2)}, \cdots, v^{(M)})$ and setting

$$b^{(j)} = \sqrt{d_j}\, v^{(j)},\ c_j = -\alpha^T b^{(j)},\ 1 \leq j \leq r, \tag{3.2.7}$$

we have

$$A = \sum_{j=1}^{r} d_j v^{(j)} v^{(j)T} = \sum_{j=1}^{r} (\sqrt{d_j}\, v^{(j)})(\sqrt{d_j}\, v^{(j)})^T = \sum_{j=1}^{r} b^{(j)} b^{(j)T},$$

and thus

$$\begin{aligned} f(x) &= \sum_{j=1}^{r} (x-\alpha)^T b^{(j)} b^{(j)T} (x-\alpha) + d \\ &= \sum_{j=1}^{r} [(x-\alpha)^T b^{(j)}][(x-\alpha)^T b^{(j)}]^T + d \\ &= \sum_{j=1}^{r} (b^{(j)T} x + c_j)^2 + d. \end{aligned}$$

Note that we can rewrite

$$f(x) = \sum_{j=1}^{s} (\hat{b}^{(j)T} x + \hat{c}_j)^2 + \hat{h} = x^T A x - 2b^T x + c,$$

Therefore $A = \sum_{j=1}^{s} \hat{b}^{(j)} \hat{b}^{(j)T}$. Set $\hat{B} = (\hat{b}^{(1)}, \hat{b}^{(2)}, \cdots, \hat{b}^{(s)})$. Then $A = \hat{B}\hat{B}^T$ and so

$$r = \text{rank}(A) = \text{rank}\hat{B} \leq s.$$

This completes the proof. □

Based on Proposition 3.2.3, and noting that constant functions are particular linear functions, the following result is a direct consequence of Proposition 3.2.5.

Corollary 3.2.6. *Any non-negative quadratic function can be written as a sum of squares of linear functions.*

This corollary guarantees that completing squares can transform non-negative quadratic functions into sums of squares of linear functions.

3.2.2 Positive quadratic function solutions to bilinear forms

Let $\alpha = (\alpha_1, \cdots, \alpha_M)^T \in \mathbb{R}^M$ be a fixed vector. Consider following quadratic function:

$$f(x) = (x-\alpha)^T A(x-\alpha) + d, \tag{3.2.8}$$

where the real matrix $A = (a_{ij})_{M\times M}$ is symmetric and $d \in \mathbb{R}$ is a constant. Proposition 3.2.3 implies that when $A \geq 0$ and $d > 0$, this presents the class of positive quadratic functions.

We will discuss the following general Hirota bilinear equation

$$P(D)f\cdot f = P(D_1, D_2, \cdots, D_M)f\cdot f = 0, \tag{3.2.9}$$

where P is a polynomial of M variables $x = (x_1, \cdots, x_M)$ and $D = (D_1, D_2, \cdots, D_M)$ is a vector of Hirota operators. Since the terms of odd powers are all zeros, we assume that P is an even polynomial, i.e., $P(-x) = P(x)$, and to generate non-zero polynomial solutions, we also require that P has no constant term, i.e., $P(0) = 0$. Moreover, we set

$$P(x) = \sum_{i,j=1}^{M} p_{ij}x_ix_j + \sum_{i,j,k,l=1}^{M} p_{ijkl}x_ix_jx_kx_l + \text{higher order terms}, \tag{3.2.10}$$

for p_{ij} and p_{ijkl} are coefficients of terms of second- and fourth-degree, to determine quadratic function solutions. Without loss of the generality, we require $p_{ij} = p_{ji}, 1 \leq, i, j \leq M$. We denote the coefficient matrix of the second order Hirota bilinear derivative terms by $P^{(2)} = (p_{ij})_{M\times M} \in \mathbb{R}^{M\times M}$ then it is symmetric: $P^{(2)} = (P^{(2)})^T$.

Obviously, we have

$$D_{i_1}D_{i_2}\cdots D_{i_k}f\cdot f = 0,\ 1 \leq i_j \leq M,\ 1 \leq j \leq k,\ k > 4,$$

for any quadratic function f. Moreover, because all odd-order Hirota bilinear derivative terms in the Hirota bilinear equation (1.3.31) are zero, the bilinear equation (1.3.31) is reduced to

$$Q(D)f\cdot f = 0, \tag{3.2.11}$$

where

$$Q(x)=\sum_{i,j=1}^{M}p_{ij}x_ix_j+\sum_{i,j,k,l=1}^{M}p_{ijkl}x_ix_jx_kx_l, \tag{3.2.12}$$

since $Q(D)f\cdot f=P(D)f\cdot f$, where f is quadratic.

Now we compute the second- and fourth-order Hirota bilinear derivatives of a positive quadratic function defined by (3.2.8). Note that

$$f_i=2\sum_{k=1}^{M}a_{ik}(x_k-\alpha_k)=2A_i^T(x-\alpha),\ f_{ij}=2a_{ij},\ 1\le i,j\le M,$$

where A_i is the ith column vector of A for $1\le i\le M$. We denote $y=x-\alpha$. Then using (1.3.34), we have

$$\begin{aligned}\sum_{i,j=1}^{M}p_{ij}D_iD_jf\cdot f&=4\sum_{i,j=1}^{M}p_{ij}a_{ij}f-8\sum_{i,j=1}^{M}p_{ij}y^TA_iA_j^Ty\\&=4d\sum_{i,j=1}^{M}p_{ij}a_{ij}+4y^T\Big[\sum_{i,j=1}^{M}p_{ij}(a_{ij}A-A_iA_j^T-A_jA_i^T)\Big]y.\end{aligned} \tag{3.2.13}$$

By (1.3.35), the fourth-order Hirota bilinear derivatives of f in (3.2.8) read

$$D_iD_jD_kD_lf\cdot f=2(f_{ij}f_{kl}+f_{ik}f_{jl}+f_{il}f_{jk})=8(a_{ij}a_{kl}+a_{ik}a_{jl}+a_{il}a_{jk}), \tag{3.2.14}$$

wgere $1\le i,j,k,l\le M$. Thus, if (3.2.8) solves the Hirota bilinear equation (1.3.31), i.e., the reduced Hirota bilinear equation (3.2.11), then we have

$$\begin{aligned}&8\sum_{i,j,k,l=1}^{M}p_{ijkl}(a_{ij}a_{kl}+a_{ik}a_{jl}+a_{il}a_{jk})+4d\sum_{i,j=1}^{M}p_{ij}a_{ij}\\&+y^T\Big[\sum_{i,j=1}^{M}p_{ij}(a_{ij}A-A_iA_j^T-A_jA_i^T)\Big]y=0.\end{aligned} \tag{3.2.15}$$

Note $x\in\mathbb{R}^M$ is arbitrary, and so is $y=x-\alpha$. Therefore, we obtain the following result.

Theorem 3.2.7. *Let $A=(a_{ij})_{M\times M}\in\mathbb{R}^{M\times M}$ be symmetric and $d\in\mathbb{R}$ be arbitrary. A quadratic function f defined by (3.2.8) solves the Hirota bilinear equation (3.2.11) if and only if*

$$2\sum_{i,j,k,l=1}^{M}p_{ijkl}(a_{ij}a_{kl}+a_{ik}a_{jl}+a_{il}a_{jk})+d\sum_{i,j=1}^{M}p_{ij}a_{ij}=0 \tag{3.2.16}$$

and

$$\sum_{i,j=1}^{M}p_{ij}(a_{ij}A-A_iA_j^T-A_jA_i^T)=\sum_{i,j=1}^{M}p_{ij}a_{ij}A-2AP^{(2)}A=0, \tag{3.2.17}$$

where A_i denotes the ith column vector of the symmetric matrix A for $1\le i\le M$.

Corollary 3.2.8. *If $f(x) = x^T Ax + d$ solves the Hirota bilinear equation (1.3.31), then for any $\alpha \in \mathbb{R}^M$, $f(x-\alpha)$ solves the Hirota bilinear equation (1.3.31), too.*

Proof. This is because (3.2.16) and (3.2.17) only depend on the matrix A and the constant d, but do not depend on the shift vector α. □

When $P^{(2)} = 0$, the matrix equation (3.2.17) is automatically satisfied and the scalar equation (3.2.16) reduces to

$$\sum_{i,j,k,l=1}^{M} p_{ijkl}(a_{ij}a_{kl} + a_{ik}a_{jl} + a_{il}a_{jk}) = 0. \tag{3.2.18}$$

If $M \geq 2$, for a fixed matrix A, obviously there exists infinitely many non-zero solutions of $p_{ijkl},\ 1 \leq i,j,k,l \leq M$, to the equation (3.2.18).

Let us now consider quadratic function solutions with $|A| \neq 0$.

If $M = 1$, then $a_{11} \neq 0$. Therefore, (3.2.16) and (3.2.17) equivalently yield

$$p_{11} = p_{1111} = 0.$$

This means that a bilinear ordinary differential equation defined by (1.3.31) has a quadractic function solution if and only if the least degree of a polynomial P must be greater than 5.

If $M = 2$, we have following example in (1+1)-dimensions. Consider the function $f(x,t) = 2x^2 - 2xt + 2t^2 + 4$, then $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} > 0$ with $|A| = 3 > 0$. This quadratic polynomial is positive, and solves the following (1+1)-dimensional Hirota bilinear equation:

$$(D_x^4 - 2D_x^2 - 2D_tD_x - 2D_t^2) f \cdot f = 0,$$

where the symmetric coefficient matrix $P^{(2)} = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix}$ is not zero. This function f leads to lump solutions to the corresponding nonlinear equations under $u = 2(\ln f)_x$ or $u = 2(\ln f)_{xx}$.

When $M \geq 3$, there is a totally different situation. What kind of Hirota bilinear equations (1.3.31) can possess a quadratic function solution defined by (3.2.8) with $|A| \neq 0$? The following theorem provides a complete answer to this question.

Theorem 3.2.9. *Let $M \geq 3$. Assume that a quadratic function f defined by* (3.2.8) *solves the Hirota bilinear equation* (3.2.9) *with P defined by* (3.2.10)*. If $|A| \neq 0$, i.e., A is non-singular, then*

$$p_{ij} + p_{ji} = 0,\ 1 \leq i,j \leq M, \tag{3.2.19}$$

which means that the Hirota bilinear equation (3.2.9) *doesn't contain any second-order Hirota bilinear derivative term.*

Before we prove the theorem, we introduce a lemma as follows.

Lemma 3.2.10. *Suppose that* $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{M\times M}$*. For any orthogonal matrix* $U = (u_{ij}) \in \mathbb{R}^{M\times M}$*, let* $\tilde{A} = (\tilde{a}_{ij}) = UAU^T, \tilde{B} = (\tilde{b}_{ij}) = UBU^T$*. Then*

$$\sum_{i,j=1}^{M} \tilde{a}_{ij}\tilde{b}_{ij} = \sum_{i,j=1}^{M} a_{ij}b_{ij}. \tag{3.2.20}$$

Proof. It is easy to see

$$\tilde{a}_{ij} = \sum_{k=1}^{M} u_{ik}(\sum_{l=1}^{M} a_{kl}u_{jl}) = \sum_{k,l=1}^{M} a_{kl}u_{ik}u_{jl}.$$

Similarly we have

$$\tilde{b}_{ij} = \sum_{k,l=1}^{M} b_{kl}u_{ik}u_{jl}.$$

Since matrix U is orthogonal, we have for any $i:\ 1 \le i \le M$,

$$\sum_{i=1}^{M} u_{ik}u_{ij} = \begin{cases} 1, & k = j, \\ 0, & k \neq j. \end{cases}$$

Then

$$\begin{aligned}
\sum_{i,j=1}^{M} \tilde{a}_{ij}\tilde{b}_{ij} &= \sum_{k,l=1}^{M} a_{kl}u_{ik}u_{jl} \sum_{k,l=1}^{M} a_{kl}u_{ik}u_{jl} \\
&= \sum_{i,j=1}^{M}\sum_{k,l=1}^{M} a_{kl}b_{kl}u_{ik}^2u_{jl}^2 + \sum_{i,j=1}^{M} \sum_{\substack{k,l,k'l'=1 \\ (k,l)\neq(k',l')}}^{M} a_{kl}b_{k'l'}u_{ik}u_{jl}u_{ik'}u_{jl'} \\
&= \sum_{k,l=1}^{M} a_{kl}b_{kl}\Big(\sum_{i=1}^{M} u_{ik}^2\Big) \times \Big(\sum_{j=1}^{M} u_{jl}^2\Big) \\
&\quad + \sum_{\substack{k,l,k'l'=1 \\ (k,l)\neq(k',l')}}^{M} a_{kl}b_{k'l'}\Big(\sum_{i=1}^{M} u_{ik}u_{ik'}\Big) \times \Big(\sum_{j=1}^{M} u_{jl}u_{jl'}\Big) \\
&= \sum_{k,l=1}^{M} a_{kl}b_{kl}.
\end{aligned}$$

□

Proof of Theorem 3.2.9. First, assume that $P^{(2)T} = P^{(2)}$. Then, (3.2.17) becomes

$$\tilde{a}A - 2AP^{(2)}A = 0, \text{ where } \tilde{a} = \sum_{i,j=1}^{M} p_{ij}a_{ij}. \tag{3.2.21}$$

Since A is symmetric, there exists an orthogonal matrix $U \in \mathbb{R}^{M\times M}$ such that

$$\hat{A} = U^T AU = \mathrm{diag}(\hat{a}_1, \cdots, \hat{a}_M).$$

Set $\hat{P}^{(2)} = U^T P^{(2)}U$, and by (3.2.21), we have

$$\tilde{a}\hat{A} - 2\hat{A}\hat{P}^{(2)}\hat{A} = 0. \tag{3.2.22}$$

Since $|A| \neq 0$, we have $|\hat{A}| \neq 0$. Thus, (3.2.22) tells that $\hat{P}^{(2)} = \frac{\tilde{a}}{2}\hat{A}^{-1}$ and further $\hat{P}^{(2)}$ is diagonal. Therefore, we can express

$$\hat{P}^{(2)} = \mathrm{diag}(\hat{p}_1, \cdots, \hat{p}_M).$$

Plugging the two diagonal matrices $\hat{A}$ and $\hat{P}^{(2)}$ into (3.2.22) engenders

$$\tilde{a} = 2\hat{a}_k\hat{p}_k, \ 1 \leq k \leq M. \tag{3.2.23}$$

On the other hand, lemma 3.2.10 shows that $\tilde{a} = \sum_{i,j=1}^{M} a_{ij}p_{ij}$ is an invariant under an orthogonal similarity transformation, and thus, from $\hat{A} = U^T AU$ and $\hat{P}^{(2)} = U^T P^{(2)}U$, we have

$$\tilde{a} = \sum_{k=1}^{M} \hat{a}_k\hat{p}_k. \tag{3.2.24}$$

Now a combination of (3.2.23) and (3.2.24) tells that $M\tilde{a} = 2\tilde{a}$. Since $M \geq 3$, we see $\tilde{a} = 0$, and so, $\hat{P}^{(2)} = 0$, which implies that $P^{(2)} = 0$.

Second, if $P^{(2)}$ is not symmetric, noting that

$$\sum_{i,j=1}^{N} p_{ij}x_ix_j = \sum_{i,j=1}^{N} \bar{p}_{ij}x_ix_j, \ \bar{p}_{ij} = \frac{p_{ij}+p_{ji}}{2}, \ 1 \leq i,j \leq M.$$

we can begin with a symmetric coefficient matrix of second order Hirota bilinear derivative terms, $\bar{P}^{(2)} = (\bar{p}_{ij})_{M\times M}$, to analyze quadratic function solutions. Thus, as we just showed, $\bar{P}^{(2)} = 0$. This is exactly what we need to get. The proof is finished. □

Theorem 3.2.9 tells us about the case of $|A| \neq 0$, which says that if a Hirota bilinear equation admits a quadratic function solution determined by (3.2.8) with $|A| \neq 0$, then it cannot contain any second-order Hirota bilinear derivative term.

For the KPI and KPII equations, since the corresponding symmetric coefficient matrix $P^{(2)}$ is not zero, Theorem 3.2.9 tells that any quadratic function solution f cannot be expressed as a sum of squares of three linear functions and a constant: $f = g_1^2 + g_2^2 + g_3^2 + d$, where

$$g_i = c_{i1}x + c_{i2}y + c_{i3}t + c_{i4},\ 1 \le i \le 3,$$

with $(c_{ij})_{3\times 3}$ being non-singular, which will also be showed clearly later.

The other case is $|A| = 0$, for which there is no requirement on inclusion of second-order Hirota bilinear derivative terms. Obviously, when $A = \mathrm{diag}(a_1, \cdots, a_{M-1}, 0) \neq 0$, (3.2.21) has a non-zero symmetric matrix solution $P^{(2)} = \mathrm{diag}(\underbrace{0, \cdots, 0}_{M-1}, 1) \neq 0$ with $\tilde{a} = 0$, and (3.2.16) has infinitely many non-zero solutions for $\{p_{ijkl} | \, 1 \le i, j, k, l, \le M\}$. Therefore, we can have both second- and fourth-order Hirota bilinear derivative terms in the Hirota bilinear equation (1.3.31).

3.3 Applications to generalized KP equations

Let us first consider the generalized Kadomtsev-Petviashvili (gKP) equations in $(N+1)$-dimensions:

$$(u_t + 6uu_{x_1} + u_{x_1x_1x_1})_{x_1} + \sigma(u_{x_2x_2} + u_{x_3x_3} + \cdots + u_{x_Nx_N}) = 0, \tag{3.3.1}$$

where $\sigma = \mp 1$ and $N \ge 2$. When $\sigma = -1$, it is called the gKPI equation, and when $\sigma = 1$, the gKPII equation.

Denote $x = (x_1, x_2, \cdots, x_N, t)^T \in \mathbb{R}^{N+1}$. Take a positive quadratic function:

$$f(x) = x^T A x + d \tag{3.3.2}$$

with $A = A^T \in \mathbb{R}^{(N+1)\times(N+1)}$, $A \ge 0$ and $d > 0$. For any $x \in \mathbb{R}^{N+1}$, the rational function

$$u = 2(\ln f)_{x_1x_1} = \frac{2(ff_{11} - f_1^2)}{f^2}$$

is analytical in $\mathbb{R}^{N+1}$. Substituting it into (3.3.1), we have

$$\begin{aligned}
&(u_t + 6uu_{x_1} + u_{x_1x_1x_1})_{x_1} + \sigma \sum_{j=2}^{N} u_{x_jx_j} \\
&= \frac{\partial^2}{\partial x_1^2}\Big[f^{-2}(D_1^4 + D_1 D_{N+1} + \sigma \sum_{j=2}^{N} D_j^2) f \cdot f\Big] = 0,\ \sigma = \mp 1,
\end{aligned}$$

where D_{N+1} is the Hirota bilinear derivative with respect to time t. Therefore, if f solves the bilinear gKPI or gKPII equation:

then $u = 2(\ln f)_{x_1x_1}$ solves the gKPI or gKPII equation in (3.3.1). Such a solution process provides us with lump or lump-type solutions to the gKPI or gKPII equation.

Theorem 3.3.1. *A positive quadratic function f defined by (3.3.2) solves the bilinear gKPI or gKPII equation by (3.3.1) if and only if*

$$6a_{11}^2 + d\tilde{a} = 0, \tag{3.3.3}$$

and

$$\tilde{a}A - (A_1A_{N+1}^T + A_{N+1}A_1^T) - 2\sigma \sum_{i=2}^{N} A_iA_i^T = 0, \tag{3.3.4}$$

where

$$\tilde{a} := a_{1N+1} + \sigma \sum_{i=2}^{N} a_{ii} \leq 0. \tag{3.3.5}$$

Proof. An application of Theorem 3.2.7 to the bilinear gKPI and gKPII equations in (3.3.1) tells (3.3.3) and (3.3.4). The property $\tilde{a} \leq 0$ in (3.3.5) follows from (3.3.3) and $d > 0$. The proof is finished. □

If $\tilde{a} = 0$, then we have $a_{11} = 0$ by (3.3.3). Since $A \geq 0$, we have $a_{1,N+1} = 0$. Further

$$\sigma \sum_{i=2}^{N} a_{ii} = \tilde{a} - a_{1N+1} = 0.$$

However, $\sigma \neq 0$ and $a_{ii} \geq 0$ for $i = 1, \cdots, N+1$. Thus, $a_{22} = \cdots = a_{NN} = 0$, and there exists only a non-zero solution $A = (a_{ij})_{(N+1)\times(N+1)}$ with all $a_{ij} = 0$ except $a_{N+1,N+1}$. The corresponding solution is $u = 2(\ln f)_{x_1x_1} \equiv 0$, a trivial solution.

Now let us introduce

$$B = 2\bar{P}^{(2)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2\sigma I_{N-1} & 0 \\ 1 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)}, \tag{3.3.6}$$

where I_{N-1} is the identity matrix of size $N-1$, and then the algebraic equation (3.3.4) can be written in a compact form:

$$\tilde{a}A - ABA = 0, \tag{3.3.7}$$

where $\tilde{a}$ is defined by (3.3.5).

Corollary 3.3.2. *If a positive-semidefinite matrix A satisfies the condition (3.3.7), then $|A| = 0$.*

Proof. If $|A| \neq 0$, then $\tilde{a} I_{N+1} - AB = 0$, and so $A = \tilde{a} B^{-1}$. The matrix B has two eigenvalues ± 1 (and an eigenvalue 2σ of multiplicity $N-1$), and thus B^{-1} also has two eigenvalues ± 1. Therefore, A is not positive-semidefinite unless $\tilde{a} = 0$. In this case, $ABA = 0$, and then $|ABA| = |A|^2 |B| = 0$, which leads to $|A| = 0$. A contradiction! □

This corollary is also a consequence of Theorem 3.2.9. For the $(N+1)$-dimensional KP equations, since the corresponding symmetric coefficient matrix $P^{(2)}$ is not zero, their corresponding Hirota bilinear equations do not possess any quadratic function solution which can be written as a sum of squares of $N+1$ linearly independent linear functions.

We remark that it is not easy to find all solutions to the system of quadratic equations in (3.3.7). The following examples show us that the gKPI equations have lump or lump-type solutions. It is also direct to observe that any lump or lump-type solution to an $(N+1)$-dimensional gKPI equation is a lump-type solution to an $((N+1)+1)$-dimensional gKPI equation of the same type as well.

Example 3.3.3. *Let us consider the simplest case: $N = 2$. This corresponds to the (2+1)-dimensional KPI and KPII equations:*

$$(u_t + 6uu_x + u_{xxx})_x + \sigma u_{yy} = 0, \ \sigma = \mp 1, \tag{3.3.8}$$

where we set $x_1 = x$ and $x_2 = y$.

By using Maple, we can have

$$A = \begin{bmatrix} a & b & \sigma(ac - 2b^2)/a \\ b & c & -\sigma bc/a \\ \sigma(ac - 2b^2)/a & -\sigma bc/a & \sigma^2 c^2/a \end{bmatrix} \text{ with } a > 0, \ c > 0, \ ac - b^2 > 0.$$

This leads to

$$\begin{aligned} f(x, y, t) &= ax^2 + cy^2 + \frac{\sigma^2 c^2}{a} t^2 + 2bxy - \frac{2\sigma bc}{a} yt + \frac{2\sigma}{a}(ac - 2b^2)xt + d \\ &= a[x + \frac{b}{a} y + \frac{\sigma}{a^2}(ac - 2b^2)t]^2 + \frac{ac - b^2}{a}(y - \frac{2\sigma b}{a} t)^2 + d, \end{aligned} \tag{3.3.9}$$

which reduces to

$$f(x, y, t) = ax^2 + cy^2 + \frac{\sigma^2 c^2}{a} t^2 + 2\sigma cxt + d = a(x + \frac{\sigma ct}{a})^2 + cy^2 + d,$$

when $b=0$. The condition (3.3.3) now reads

$$6a^2+d[\frac{\sigma(ac-2b^2)}{a}+\sigma c]=6a^2+2d\frac{\sigma(ac-b^2)}{a}=0,$$

which yields

$$d=-\frac{3a^3}{\sigma(ac-b^2)}>0. \tag{3.3.10}$$

By Corollary 3.2.8, for any constants $\gamma_1,\gamma_1,\gamma_3\in\mathbb{R}$, we have the following quadratic function solutions:

$$\begin{aligned} f(x,y,t) &= a[(x-\gamma_1)+\frac{b}{a}(y-\gamma_2)+\frac{\sigma}{a^2}(ac-2b^2)(t-\gamma_3)]^2 \\ &\quad +\frac{ac-b^2}{a}[(y-\gamma_2)-\frac{2\sigma b}{a}(t-\gamma_3)]^2+d \\ &= a[x+\frac{b}{a}y+\frac{\sigma}{a^2}(ac-2b^2)t-\delta_1]^2 \\ &\quad +\frac{ac-b^2}{a}(y-\frac{2\sigma b}{a}t-\delta_2)^2+d, \end{aligned} \tag{3.3.11}$$

with δ_1 and δ_2 being defined by

$$\delta_1=\gamma_1+\frac{b}{a}\gamma_2+\frac{\sigma}{a^2}(ac-2b^2)\gamma_3,\ \delta_2=\gamma_2-\frac{2\sigma b}{a}\gamma_3.$$

Because $\gamma_1,\gamma_2,\gamma_3$ are arbitrary, so are δ_1 and δ_2. Furthermore, the corresponding lump solutions to the (2+1)-dimensional KPI equation in (3.3.8) read

$$\begin{aligned} & u(x,y,t)=2(\ln f)_{xx} \\ =& \frac{4\big\{-a^2[x+\frac{b}{a}y+\frac{\sigma}{a^2}(ac-2b^2)t-\delta_1]^2+(ac-b^2)(y-\frac{2\sigma b}{a}t-\delta_2)^2+ad\big\}}{\big\{a[x+\frac{b}{a}y+\frac{\sigma}{a^2}(ac-2b^2)t-\delta_1]^2+\frac{ac-b^2}{a}(y-\frac{2\sigma b}{a}t-\delta_2)^2+d\big\}^2}, \end{aligned}$$

where d is defined by (3.3.10), $a,b,c\in\mathbb{R}$ satisfy $a>0$, $c>0$, $ac-b^2>0$, and δ_1 and δ_2 are arbitrary. When taking

$$a=1,\ b=\sqrt{3}\,a,\ c=3(a^2+b^2),\ d=\frac{1}{b^2},\ \delta_1=\delta_2=0,\ y\to\frac{1}{\sqrt{3}}y,$$

the resulting lump solutions reduce to the solutions in (3.1.2).

Remark 3.3.4. *The condition in (3.3.10) implies that $\sigma=-1$ in order to have lump solutions generated from positive quadratic functions. This shows that the (2+1)-dimensional KPI equation ($\sigma=-1$) possesses the discussed lump solutions whereas the (2+1)-dimensional KPII equation ($\sigma=1$) does not.*

The gKP equations in $(N+1)$-dimensions with $N \geq 3$ have no lump solutions generated from quadratic functions.

Theorem 3.3.5. *Let $N \geq 3$. Then there is no symmetric matrix solution $A \in \mathbb{R}^{(N+1)\times(N+1)}$ to the matrix equation* (3.3.7) *with rank$(A) = N$, which implies that the $(N+1)$-dimensional gKP equations (3.3.1) have no lump solution generated from quadratic functions under the transformation $u = 2(\ln f)_{xx}$.*

Proof. Suppose that there is a symmetric matrix $A \in \mathbb{R}^{(N+1)\times(N+1)}$ which solves the equation (3.3.7) and whose rank is N. Then, since A is symmetric and rank$(A) = N$, there exists an orthogonal matrix $U \in \mathbb{R}^{(N+1)\times(N+1)}$ such that

$$\hat{A} = U^T A U = \begin{bmatrix} \hat{A}_1 & 0 \\ 0 & 0 \end{bmatrix},\ \hat{A}_1 = \mathrm{diag}(\lambda_1, \cdots, \lambda_N),$$

where $\lambda_i \neq 0,\ 1 \leq i \leq N$. Set

$$\hat{B} = U^T B U = \begin{bmatrix} \hat{B}_1 & \hat{B}_2 \\ \hat{B}_3 & \hat{B}_4 \end{bmatrix},\ \hat{B}_1 = (\hat{b}_{ij})_{N\times N} \in \mathbb{R}^{N\times N}.$$

Upon noting that $\tilde{a}$ is an invariant under an orthogonal similarity transformation, it follows from (3.3.7) that

$$\tilde{a}\hat{A}_1 - \hat{A}_1\hat{B}_1\hat{A}_1 = 0,\ \tilde{a} = \sum_{i,j=1}^{N+1} a_{ij}p_{ij} = \frac{1}{2}\sum_{k=1}^{N} \lambda_k \hat{b}_{kk}.$$

Then, based on this sub-matrix equation, using the same idea in the proof of Theorem 3.2.9 shows that $N\tilde{a} = 2\tilde{a}$, which leads to $\tilde{a} = 0$ since $N \geq 3$. Further, we have $\hat{B}_1 = 0$, and thus, rank$(\hat{B}) \leq 2$, which is a contradiction to rank$(\hat{B})$ = rank$(B) = N + 1$. Therefore, there is no symmetric matrix solution A to the equation (3.3.7) with rank$(A) = N$.

Finally, note that the existence of a non-zero $(N+1, N+1)$ minor of A implies that rank$(A) \geq N$, and thus, by Theorem 3.2.9, we have rank$(A) = N$. Now, it follows that there is no symmetric matrix solution A to the equation (3.3.7) with a non-zero $(N+1, N+1)$ minor. This means that the gKP equations, defined by (3.3.1), in $(N+1)$-dimensions with $N \geq 3$ have no lump solution, which are generated from quadratic functions under the transformation $u = 2(\ln f)_{xx}$. The proof is finished. □

Generalized KP and BKP equations with general 2nd-order derivatives has been studied in [59].

3.4 Generalized bilinear equations

3.4.1 Generalized bilinear equations and polynomial solutions

Let $P(x)$ be a polynomial in $x \in \mathbb{R}^M$ with degree d_P and $P(0) = 0$ (P may not be even). Suppose that $p \geq 2$ is an integer. Formulate a generalized bilinear equation as follows:

$$P(D_p)f \cdot f = 0, \tag{3.4.1}$$

where $D_p = (D_{p,1}, D_{p,2}, \cdots, D_{p,M})$.

We consider polynomial solutions $f(x)$ with the independent variable $x \in \mathbb{R}^M$. For a monomial $P_k(x) = x_1^{n_1} \cdots x_M^{n_M}$, noting that (i) $\deg(f) = 0$ if and only if $f =$ const. $\neq 0$ and (ii) $\deg(0) = -\infty$, we have

$$\deg(P_k(D_{\langle p \rangle})f \cdot f) \leq 2 \deg(f) - \deg(P_k),$$

since

$$P_k(D_p)f \cdot f = \sum_J \alpha_J \frac{\partial^{j_1}}{\partial x_1^{j_1}} \cdots \frac{\partial^{j_M}}{\partial x_M^{j_M}} f(x) \frac{\partial^{n_1 - j_1}}{\partial x_1^{n_1 - j_1}} \cdots \frac{\partial^{n_M - j_M}}{\partial x_M^{n_M - j_M}} f(x)$$

for some real constants α_J with $J = (j_1, \cdots, j_M)$. As a corollary, for a linear or quadratic function f (i.e., $\deg(f) \leq 2$), we have

$$P_k(D_p)f \cdot f = 0, \tag{3.4.2}$$

when $\deg(P_k) \geq 5$.

In general, if f is a polynomial solution to the generalized bilinear equation (3.4.1), then the coefficients of f satisfy a group of nonlinear algebraic equations. For example, if $f(x,t) = ax^2 + 2bxt + ct^2 + dx + et + g$ is a solution of the bilinear KdV equation (1.3.39), then we have

$$ab = ac = ae = bc = cd = 12a^2 + 2bg - de = 0, \tag{3.4.3}$$

which leads to the following three classes of solutions:

(i) $f(x,t) = 2bxt + dx + et + de/(2b)$, (ii) $f(x,t) = ct^2 + et + g$, (iii) $f(x,t) = dx + g$.

However, if f is quadratic, then we can easily find that

$$D_{p,i}^n f \cdot f = D_i^n f \cdot f, \; n \geq 1, \tag{3.4.4}$$

where $1 \leq i \leq M$ and $p \geq 2$, and thus, the same quadratic function f can solve all generalized bilinear differential equations with different values of $p \geq 2$, (see Example 1.3.1 and 1.3.2 for

$p = 2, 3$). For $k < p$, the following equality holds for $p > 4$:

$$D_{p,n_1} \cdots D_{p,n_k} f \cdot f = D_{2,n_1} \cdots D_{2,n_k} f \cdot f.$$

We list this result as follows.

Theorem 3.4.1. *The generalized bilinear equations* (3.4.1) *with a given polynomial P and different integers $p \geq 2$ possess the same set of quadratic function solutions.*

Let us consider a quadratic function f defined by (3.2.1) and write the polynomial $P(x)$ defining the generalized bilinear equation (3.4.1) as follows:

$$P(x) = \sum_{k=1}^{N} \sum_{i_1,\cdots,i_k=1}^{M} p_{i_1 \cdots i_k} x_{i_1} \cdots x_{i_k}, \tag{3.4.5}$$

where $N = \deg(P) \geq 1$ is an integer, and $p_{i_1 \cdots i_k},\ 1 \leq i_1, \cdots, i_k \leq M,\ 1 \leq k \leq N$, are real constants.

Noting the properties in (3.4.2) and (3.4.4), we can see that only the coefficients p_{ij} and p_{ijkl} take effect in computing quadratic function solutions. Necessary and sufficient conditions on quadratic function solutions to Hirota bilinear equations have been presented in the Theorem 3.2.7, indeed. Based on Theorem 3.4.1, we can have the same criterion on quadratic function solutions to generalized bilinear equations.

We point out that for distinct p, the generalized bilinear equations by (3.4.1) may have different polynomial solutions of higher order than two. For example, any C^3-differentiable function is a solution to the equation $D_x^3 f \cdot f = 0$. But if $f = f(x,t) = x^4 + t^2$, we can have

$$D_{3,x}^3 f \cdot f = 48x(x^4 + t^2) \neq 0,$$

which means that this quartic function f does not solve $D_{3,x}^3 f \cdot f = 0$.

In general, it is difficult to find rational function solutions to nonlinear differential equations. But using Mathematical software such as Maple, we can find polynomial solutions to generalized bilinear differential equations.

In this section, we'll try to search for positive quadratic or quartic polynomial solutions to generalized bilinear equations. From those polynomial solutions f, we will be able to construct lump-type solutions to nonlinear differential equations, via the transformation of dependent variables $u = (2 \ln f)_x$.

3.4.2 A class of nonlinear equations

Example 3.4.2. *Let us begin with the following polynomial (see formula (18) in Ref.[44]):*

$$P = c_1x^5 + c_2x^3y + c_3x^2z + c_4xt + c_5yz,$$

where the coefficients $c_i,\ 1 \le i \le 5$, *are free real constants. The associated generalized bilinear differential equation with* $p = 3$ *reads (see (19) in Ref.[44]):*

$$\begin{aligned} & P(D_{3,x}, D_{3,y}, D_{3,z}, D_{3,t})f \cdot f \\ = \ & 2c_1(f_{xxxxx}f - 5f_{xxxx}f_x + 10f_{xxx}f_{xx}) + 6c_2f_{xx}f_{xy} \\ & +2c_3f_{xxz}f + 2c_4(f_{xt}f - f_xf_t) + 2c_5(f_{yz}f - f_yf_z) = 0. \end{aligned} \tag{3.4.6}$$

Taking $u = 2(\ln f)_x$ *generates the corresponding nonlinear differential equation:*

$$\begin{aligned} & \frac{\partial}{\partial x}\frac{P(D_{3,x}, D_{3,y}, D_{3,z}, D_{3,t})f \cdot f}{f^2} \\ = \ & c_1\big(\frac{15}{2}u_x^3 + \frac{5}{2}u^3u_{xx} + \frac{15}{8}u^4u_x + 10u_xu_{xxx} + \frac{15}{2}u^2u_x^2 + 15uu_xu_{xx} \\ & +10u_{xx}^2 + u_{xxxxx}\big) + c_2\big[\frac{3}{8}u^3u_y + \frac{3}{2}u_xu_{xy} + \frac{3}{4}u^2u_{xy} + \frac{3}{2}u_{xx}u_y \\ & +\frac{9}{4}uu_xu_y + \frac{3}{8}(3u^2u_x + 2uu_{xx} + 2u_x^2)v\big] + c_3\big[uu_{xz} + u_{xxz} \\ & +\frac{3}{2}u_xu_z + \frac{1}{4}u^2u_z + \frac{1}{2}(u_{xx} + uu_x)w\big] + c_4u_{xt} + c_5u_{yz} = 0, \end{aligned} \tag{3.4.7}$$

where $u_y = v_x$ *and* $u_z = w_x$. *Therefore, if* f *solves the generalized bilinear equation (3.4.6), then* $u = 2(\ln f)_x$ *solves the nonlinear differential equation (3.4.7).*

Quadratic function solutions

Let us first consider quadratic function solutions to the generalized bilinear equation (3.4.6), which involve a sum of two squares. Based on the discussion in the aforementioned section, we know that such solutions have nothing to do with c_1 and c_3. Therefore, the coefficients c_1 and c_3 will be arbitrary real constants. Three cases of such solutions by symbolic computation with Maple are displayed as follows.

(1) When $c_4 \neq 0$, but c_2 and c_5 are arbitrary, we have

$$\begin{aligned} f = \ & \big(\frac{a_4a_7a_8c_5}{{a_2}^2c_4}t + a_2x - \frac{a_7a_8}{a_2}y + a_4z + a_5\big)^2 \\ & +\big(-\frac{a_4a_8c_5}{a_2c_4}t + a_7x + a_8y + \frac{a_4a_7}{a_2}z + a_{10}\big)^2 + a_{11}, \end{aligned}$$

where $a_2, a_4, a_5, a_7, a_8, a_{10}$ and a_{11} are arbitrary real constants satisfying $a_2 \neq 0$ and $a_{11} > 0$.

(2) When $c_2c_4 \neq 0$, but c_5 is arbitrary. we have

$$f = \Big[\frac{a_3(a_4{}^2a_{11}c_5 - 3\,a_7{}^3a_9c_2)c_5}{3\,a_7{}^4c_2c_4}t + a_3y + a_4z + a_5\Big]^2 + \Big[\frac{a_3a_4\left(3\,a_7{}^3c_2 + a_9a_{11}c_5\right)c_5}{3a_7{}^4c_2c_4}t + a_7x - \frac{a_3a_4a_{11}c_5}{3a_7{}^3c_2}y + a_9z + a_{10}\Big]^2 + a_{11},$$

where $a_3, a_4, a_5, a_7, a_9, a_{10}$ and a_{11} are arbitrary real constants satisfying $a_7 \neq 0$ and $a_{11} > 0$.

(3) When $c_4c_5 \neq 0$, but c_2 is arbitrary, we have

$$f = \big[-(a_2a_3a_4 - a_2a_8a_9 + a_3a_7a_9 + a_4a_7a_8)\,d_1t + a_2x + a_3y + a_4z + a_5\big]^2 + \big[-(a_2a_3a_9 + a_2a_4a_8 - a_3a_4a_7 + a_7a_8a_9)\,d_1t + a_7x + a_8y + a_9z + a_{10}\big]^2 + d_2$$

with

$$d_1 = \frac{c_5}{(a_2{}^2 + a_7{}^2)c_4},\ d_2 = -\frac{3(a_2{}^2 + a_7{}^2)^2(a_2a_3 + a_7a_8)c_2}{(a_2a_9 - a_4a_7)(a_2a_8 - a_3a_7)c_5},$$

where $a_i, i = 2, \cdots, 5, 7, \cdots, 10$ are arbitrary real constants satisfying $a_2a_9 - a_4a_7 \neq 0$ and $a_2a_8 - a_3a_7 \neq 0$.

The discussion of lump and lump-type solutions to another class (2+1) of nonlinear PDE can be found in [61].

Quartic function solutions

Let us now consider quartic function solutions to the generalized bilinear equation (3.4.6). A direct symbolic computation with Maple tells us seven classes of positive quartic function solutions.

(1) Solutions independent of y:

$$f = (a_2x + a_4z + a_5)^2 + (a_7x + a_9z + a_{10})^2 + (a_{14}z + a_{15})^4 + a_{16},$$

where $a_2, a_4, a_5, a_7, a_9, a_{10}, a_{14}, a_{15}$ and $a_{16} > 0$ are arbitrary real constants.

(2) Solutions independent of z:

$$f = \Big(-\frac{a_7a_8}{a_3}x + a_3y + a_5\Big)^2 + (a_7x + a_8y + a_{10})^2 + (a_{13}y + a_{15})^4 + a_{16},$$

where $a_3, a_5, a_7, a_8, a_{10}, a_{13}, a_{15}$ and $a_{16} > 0$ are arbitrary real constants.

(3) Solutions independent of x and y:

$$f = (a_1t + a_4z + a_5)^2 + (a_6t + a_9z + a_{10})^2 + (a_{11}t + a_{14}z + a_{15})^4 + a_{16},$$

where $a_1, a_4, a_5, a_6, a_9, a_{10}, a_{11}, a_{14}, a_{15}$ and $a_{16} > 0$ are arbitrary real constants.

(4) Solutions independent of x and z:

$$f = (a_1 t + a_3 y + a_5)^2 + (a_6 t + a_8 y + a_{10})^2 + (a_{11} t + a_{13} y + a_{15})^4 + a_{16},$$

where $a_1, a_3, a_5, a_6, a_8, a_{10}, a_{11}, a_{13}, a_{15}$ and $a_{16} > 0$ are arbitrary real constants.

(5) When $c_5 \neq 0$, but $c_k,\ 1 \leq k \leq 4$, are arbitrary, we have

$$\begin{aligned} f = & \ \big(-\frac{a_7 a_8 a_{11}}{a_2 a_{13}} t + a_2 x - \frac{a_7 a_8}{a_2} y - \frac{a_2 a_{11} c_4}{a_{13} c_5} z + a_5\big)^2 + \big(\frac{a_8 a_{11} t}{a_{13}} + a_7 x \\ & + a_8 y - \frac{a_7 a_{11} c_4}{a_{13} c_5} z + a_{10}\big)^2 + (a_{11} t + a_{13} y + a_{15})^4 + a_{16}, \end{aligned}$$

where $a_2 a_{13} \neq 0$, $a_{16} > 0$ and all other involved parameters are arbitrary real constants.

(6) When $c_4 \neq 0$, but $c_k, 1 \leq k \leq 3$, we have

$$f = \big(-\frac{a_4 a_{13} c_5}{a_{11} c_4} x + a_4 z + a_5\big)^2 + (a_{11} t + a_{13} y + a_{15})^4 + a_{16},$$

where $a_{11} \neq 0$, $a_{16} > 0$ and all other involved parameters are arbitrary real constants.

(7) When $c_4 \neq 0$, but $c_k, 1 \leq k \leq 3$, we have

$$\begin{aligned} f = & \ \big(-\frac{a_3 a_9 c_5}{a_7 c_4} t + a_3 y + a_5\big)^2 + (a_7 x + a_9 z + a_{10})^2 \\ & + \big(-\frac{a_9 a_{13} c_5}{a_7 c_4} t + a_{13} y + a_{15}\big)^4 + a_{16}, \end{aligned}$$

where $a_7 \neq 0$, $a_{16} > 0$ and all other involved parameters are arbitrary real constants.

Discussions

Lump solutions are rationally localized in all directions in the space. For the exact solutions we discussed above, this characteristic property equivalently requires

$$\lim_{x^2+y^2+z^2 \to \infty} u(x, y, z, t) = 0,\ \forall t \in \mathbb{R},$$

where $u = 2(\ln f)_x$, and obviously, a sufficient condition for u to be a lump solution is

$$\lim_{x^2+y^2+z^2 \to \infty} f(x, y, z, t) = \infty,\ \forall t \in \mathbb{R}. \tag{3.4.8}$$

We claim that all the above solutions do not satisfies the critierion (3.4.8), but function $f(x, y, z, t)$ does go to ∞ in some directions, and thus functions $u(x, y, z, t)$ are lump-type solutions.

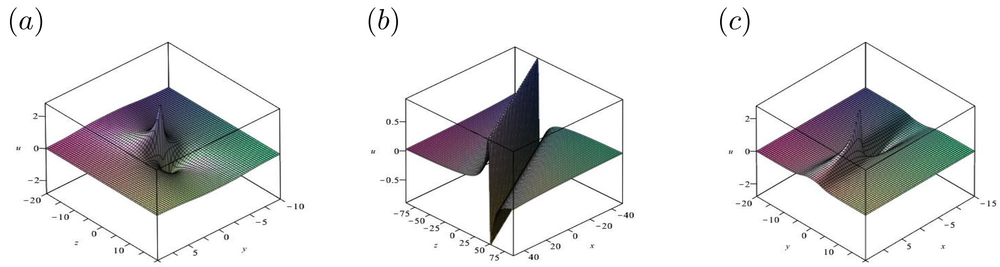


Figure 1.: *Plots of (3.4.9) at* $t = 0$ *with (a)* $x = 0$*, (b)* $y = -1$ *and (c)* $z = 1$.

We consider a special case of the parameters and coefficients for the class of solutions in the case (5). Choose $a_2 = a_{11} = a_{13} = a_{16} = 1, a_5 = a_{10} = a_{15} = 0, a_7 = -1,\ a_8 = 2$ and $c_4 = c_5 = 2,$. Then, we have the following positive quartic function solution to the generalized bilinear equation (3.4.6):

$$f = (2t + x + 2y - z)^2 + (2t - x + 2y + z)^2 + (t + y)^4 + 1,$$

and the corresponding lump-type solution to the nonlinear differential equation (3.4.7):

$$u = 2(\ln f)_x = \frac{8(x - z)}{(2t + x + 2y - z)^2 + (2t - x + 2y + z)^2 + (t + y)^4 + 1}. \tag{3.4.9}$$

The figure 1 shows three 3d plots of this lump-type solution at $t = 0$ with $x = 0$, $y = -1$ and $z = 1$, respectively.

3.5 Higher-degree polynomial solutions and Hilbert's seventeenth problem

Suppose f is a polynomial on $\mathbb{R}^M$ with deg$(f) = 2N$. We know when f is quadratic (i.e. N=1), f is non-negative if and only if it is a sum of squares of polynomials (SOS). Is it true for a general case? Unfortunately, in 1888, D. Hilbert proved [30] the existence of a real polynomial in two variables($M = 2$) of degree six ($N = 3$) which is nonnegative but not a sum of squares of real polynomials. Hilbert's proof used some basic results from the theory of algebraic curves. Five years later, in 1893, the second pioneering paper of Hilbert [31] in this area appeared. He proved that each nonnegative polynomial f with two variables is a finite sum of squares of rational functions. Afterwords he posed it in his famous 23 problems at the International Congress of Mathematicians in Paris (1900):

Hilbert's 17th problem:

Suppose that f is a nonnegative polynomial. Is f a finite sum of squares of rational functions?

Hilbert's 17th problem was solved in the affirmative by E. Artin [5] in 1927.

Theorem 3.5.1 (E. Artin)**.** *If a polynomial f is nonnegative on $\mathbb{R}^M$, then there are polynomials $q, p_1, \cdots, p_k$ on $\mathbb{R}^M$ with $q \neq 0$, such that*

$$f \cdot q^2 = p_1^2 + \cdots + p_k^2. \tag{3.5.1}$$

In 1967, T. S. Motzkin [66] gave the first explicit example of nonnegative polynomial in two variables

$$f(x,y) = x^4y^2 + x^2y^4 + 1 - 3x^2y^2 \tag{3.5.2}$$

which is not a sum of squares of polynomials. However, it can be represented by the sum of four squares of rational functions of (x, y):

$$f(x,y) = \frac{x^2y^2(x^2+y^2+1)(x^2+y^2-2)^2 + (x^2-y^2)^2}{(x^2+y^2)^2}.$$

In Motzkin's example, f is a two variables polynomial of degree 6. $f \cdot (x^2+y^2)^2$ can be expressed as sum of 4 squares of polynomials. Let $z = (x^3, x^2y, xy^2, y^3, x^2, xy, y^2, x, y, 1)^T$, then $f(x,y) = z^T Az$, where A is a constant matrix of order 10:

$$A = \begin{bmatrix} 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-2\alpha & 0 & \beta & 0 & 0 & 0 & 0 & a & 0 \\ \alpha & 0 & 1-2\beta & 0 & 0 & 0 & 0 & b & 0 & 0 \\ 0 & \beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{3.5.3}$$

where $d = -3 - 2a - 2b - 2c$, α, β, a, b, c are arbitrary real numbers. We claim that matrix A is not positive semidefinite for any parameters α, β, a, b and c. Otherwise, if $A \geq 0$ then $\alpha = \beta = a = b = c = 0$ and thus $d = -3$, a contradiction!

Therefore, a nonnegative polynomial may be not a sum of squares of polynomials. However, its multiplication by a square of some polynomial can be a sum of squares of polynomials.

We have the following result for a polynomial to be a sum of squares of polynomials.

Proposition 3.5.2. *Suppose f is a polynomial on $\mathbb{R}^M$ of degree no large than $2N$ for integer $N > 0$. Let $y = (x_1^N, x_1^{N-1}x_2, \cdots, x_M^N, x_1^{N-1}, x_1^{N-2}x_2, \cdots, x_M^2, x_1, \cdots, x_M, 1)^T$. Then f is a sum of squares of polynomials if and only if there exists a matrix $A \geq 0$ such that $f(x) = y^T Ay$.*

Proof. If: Let $f(x) = P_1(x)^2 + \cdots + P_r(x)^2$ where $\deg(P_j) \leq N, j = 1, \cdots, r$. Obiously P_j must a linear combination of components of y, that is, $P_j(x) = b_j^T y$ for some column vectors b_j, $j = 1, \cdots, r$. Therefore

$$f(x) = \sum_{j=1}^{r} P_j(x)^2 = \sum_{j=1}^{r}(b_j^T y)^2 = \sum_{j=1}^{r} y^T b_j b_j^T y = y^T \sum_{j=1}^{r}(b_j b_j^T)y.$$

We can simply take $A = \sum_{j=1}^{r}(b_j b_j^T)$, which is non-negative.

Only if: If $A \geq 0$ then there is a matrix B such that $A = B^T B$ and B is of full row rank. So $f(x) = (By)^T(By) = y^T B^T By = y^T Ay$. □

We now consider a special case when $M = N = 2$. By [30], we know in this class, a polynomial is nonnegative if and only if it is a sum of squares of polynomials.

Let $f(x, t)$ be a quartic polynomial, and then it can be written as $y^T Ay$, where $A \in \mathbb{R}^{6\times 6}, y = (x^2, xt, t^2, x, t, 1)^T$. Note for a given f, A is not unique even if we require that the matrix A is symmetric.

Example 3.5.3. *Let $f(x, t) = x^2t^2 + 1$ and then $f(x, t) = y^T Ay$ for*

$$A = \begin{bmatrix} 0 & 0 & \alpha & 0 & 0 & 0 \\ 0 & 1-2\alpha & 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and $\alpha \in \mathbb{R}$. When $\alpha = 1$, the matrix A is not positive semidefinite whereas when $\alpha = 0$, the matrix A is indeed positive semidefinite.

When $f(x,t)$ is nonnegative, it is a sum of squares of polynomials with degree no large than 2. Thus we have a matrix $A \geq 0$ such that $f(x,t) = y^T Ay$. Consider a given bilinear equation (3.2.9) with an even polynomial P satisfying $P(0) = 0$. Substituting $f(x,t) = y^T Ay$ into (3.2.9), we will get a system of quadratic equations. If we can get an nonnegative solution of A, then by the logarithm transformation, $u = 2(\ln f)_x$ or $u = 2(\ln f)_{xx}$, we get a lump or lump type solution provided that f is positive, and otherwise u is a rational solution with singularity.

An algorithm

In general, we propose an algorithm for finding rational solutions to an evolution equation.

Step 1: Suppose (3.2.9) is the bilinear form of a nonlinear PDE which we are considering.

By proposition 3.5.2 we know that any polynomial f of degree no more than $2N$ being a sum of squares of polynomials can be written as a quadratic form which is parametrized by a symmetric matrix A.

Step 2. Solve a matrix A by substituting f into the bilinear form.

Step 3. Check the positivity of A, and note for a fixed function f such a matrix A is not unique. If we can find some $A \geq 0$ then f is non-negative. Since f can be decomposed as finite sum of square of some polynomials, f is positive if there is no common zeros of these polynomials.

Step 4. Computer u by the logarithmic or rational transformation. We usually get a rational solution. When f is positive, then u is a lump or lump-type solution. When f is nonnegative but not positive, then u is a rational solution with singularity.

For a more general nonnegative polynomial solution f, we may not be able to find $A \geq 0$ like the example provided by T. S. Motzkin. We need to develop some algorithm to check the positivity of a polynomial. We will discuss it in the future to study quartic and higher order nonnegative polynomial solutions to the bilinear KdV and KP equations.

3.6 Concluding remarks

In this chapter, we studied positive quadratic function solutions to Hirota bilinear equations. Sufficient and necessary conditions for the existence of such polynomial solutions were given. In turn, positive quadratic function solutions generate lump or lump-type solutions to nonlinear partial differential equations possessing Hirota bilinear forms. Applications were made for a few generalized

KP and BKP equations. We also considered quartic function solutions.

Proposition 3.2.3 provides a criterion for the positivity of quadratic functions. It, however, still remains open how to determine the positivity of higher-order multivariate polynomials. It should be also interesting to look for positive polynomial solutions to generalized bilinear equations [44], which generate exact rational function solutions to novel types of nonlinear partial differential equations [75, 82, 83] .

Chapter 4

Complexitons to nonlinear PDEs

4.1 Introduction

It is always of great interest to find exact solutions to nonlinear partial differential equations, especially soliton equations. The Hirota direct method [10, 32] is a powerful tool of finding exact soliton solutions to nonlinear partial differential equations, which takes advantage of bilinear derivatives. There are also some other works successfully extending the direct method [29, 44, 47, 48]. For some kind of soliton equations like the KdV equation, mKdV equation, sG equation, etc. [80], Hirota established nonlinear superposition principles to find multi-soliton solutions and the method was extended to find multi-complexiton solutions [86].

The linear superposition principle is very important in studying differential equations. However, solutions to a nonlinear differential equation do not form a linear space. It is valuable that we can find some subset of solutions to a nonlinear differential equation which forms a linear space. In the references [53, 57] , the authors discussed the linear superposition principle of exponential traveling waves. Recently the papers [69, 84] extended the results to a special hyperbolic function ($\cosh(\eta)$) waves and a trigonometric function ($\cos(\eta)$) waves under certain conditions. Complexiton solutions, which are combinations of trigonometric function waves and exponential traveling waves, are introduced by W. X. Ma in 2002 [41]. The Wronskian technique is an effective method to find complexitons [56]. However, the complexitons obtained by the Wronskian technique are very complicated and for high dimensions and for general nonlinear differential equations we do not know how to find Wronskian solutions. In this chapter, we first propose an algorithm [85] to find complex valued multi-solitons and the result turns out to be multi-comlexitons or mixed solitons and complexitons when suitable coefficients are chosen. We then establish the linear superposition principle [86] for complexitons which is a generalization of [69, 84].

4.2 Using the Hirota method to find complexitons

4.2.1 General bilinear equations

In this section, we apply the Hirota method to solve nonlinear PDEs based on bilinear forms.

Suppose P is a real polynomial of M variables with the properties that $P(0) = 0$ and $P(-x) = P(x)$ and functions f is differentiable on $\mathbb{R}^M$. We consider the following bilinear equation

$$P(D)f \cdot f = 0. \tag{4.2.1}$$

In order to find traveling wave solutions of (4.2.1), let

$$\eta = \beta_0 + \sum_{k=1}^{M} \beta_k x_k, \tag{4.2.2}$$

where $\beta_k \in \mathbb{C}, k = 0, \cdots, M$.

Applying the Hirota method, we consider the expansion

$$f = 1 + \varepsilon f_1 + \varepsilon^2 f_2 + \cdots . \tag{4.2.3}$$

For one soliton solution, suppose $f_1 = \exp(\eta)$. A complex function $f = 1 + \exp(\eta)$ is a solution to (4.2.1) if and only if the following nonlinear dispersion relation holds:

$$P(\beta_1, \cdots, \beta_M) = 0. \tag{4.2.4}$$

Since P and x are real we have (the bar denoting complex conjugation)

$$\overline{P(\beta_1, \cdots, \beta_M)} = P(\bar{\beta}_1, \cdots, \bar{\beta}_M).$$

Therefore, the functions η and $\bar{\eta}$ satisfy the same dispersion relation.

By the same approach as that in chapter 1, we have if $\eta_j = \beta_{j0} + \sum_{k=1}^{M} \beta_{jk} x_k, j = 1, 2$, satisfy the dispersion relation (4.2.4) and $P(\beta_{21} + \beta_{11}, \cdots, \beta_{2M} + \beta_{1M}) \neq 0$, then the (complex) function

$$f = 1 + \exp(\eta_1) + \exp(\eta_2) + a_{12} \exp(\eta_1 + \eta_2), \tag{4.2.5}$$

where $a_{12} = -\dfrac{P(\beta_{21} - \beta_{11}, \cdots, \beta_{2M} - \beta_{1M})}{P(\beta_{21} + \beta_{11}, \cdots, \beta_{2M} + \beta_{1M})}$, is a solution of (4.2.1). Taking $\eta_2 = \bar{\eta}_1$, we get

$$\begin{aligned} f &= 1 + \exp(\eta) + \exp(\bar{\eta}) + a_{12} \exp(\eta + \bar{\eta}) \\ &= 1 + 2\mathrm{Re}(\exp(\eta)) \cos(\mathrm{Im}(\exp(\eta))) + a_{12} \exp(2\mathrm{Re}(\eta)) \in \mathbb{R}, \end{aligned} \tag{4.2.6}$$

since P is a real and even polynomial,

$$a_{12} = -\frac{P(2i\mathrm{Im}(\beta_1), \cdots, 2i\mathrm{Im}(\beta_M))}{P(2\mathrm{Re}(\beta_1), \cdots, 2\mathrm{Re}(\beta_M))} \in \mathbb{R}. \tag{4.2.7}$$

Now we consider $N \geq 3$, according to [32], N–soliton solutions can be written as (1.3.25) where $e^{A_{jk}} := a_{jk}$ denoted by

$$a_{jk} := -\frac{P(\beta_{k1} - \beta_{j1}, \cdots, \beta_{kM} - \beta_{jM})}{P(\beta_{k1} + \beta_{j1}, \cdots, \beta_{kM} + \beta_{jM})} = a_{kj}, \quad 1 \leq j < k \leq N. \tag{4.2.8}$$

However, the polynomial P must satisfies the Hirota condition to have N-soliton solutions:

$$\sum P\Big(\sum_{j=1}^{N} \sigma_j \beta_{j1}, \cdots, \sum_{j=1}^{N} \sigma_j \beta_{jM}\Big) \prod_{k<j} P(\sigma_k \beta_{k1} - \sigma_j \beta_{j1}, \cdots, \sigma_k \beta_{kM} - \sigma_j \beta_{jM}) \sigma_k \sigma_j = 0, \tag{4.2.9}$$

where the summation over all possible combinations of $\sigma_j = \pm 1, j, k = 1, 2, \cdots, N$.

We have the following result.

Theorem 4.2.1. *Let P be a real coefficient polynomial on $\mathbb{R}^M$ satisfying $P(0) = 0, P(-x) = P(x)$, and N be an positive integer. Assume that the complex functions $\eta_j = \beta_{j0} + \sum_{l=1}^{M} \beta_{jl} x_l$, $j = 1, 3, \cdots, 2N-1$, satisfy the dispersion relation* (4.2.4) *and the Hirota condition* (4.2.9)*. Suppose $\eta_{2j} = \bar{\eta}_{2j-1}, j = 1, \cdots, N$. Then the function*

$$f = 1 + \sum_{n=1}^{2N} \sum_{\sum_{j=1}^{2N} \mu_j = n} \exp\left(\sum_{j=1}^{2N} \mu_j \eta_j + \sum_{k<j} A_{kj} \mu_k \mu_j \right), \tag{4.2.10}$$

where $\mu_j = 0$ or 1 for $j = 1, 2, \cdots, 2N$, and $a_{kj} = e^{A_{kj}}$, $j, k = 1, 2, \cdots, 2N$, dertermined by (4.2.8)*, presents a complexiton solution to* (4.2.1).

Proof. We only need to show that function f given by (4.2.10) is real and we use the mathematical induction. We have proved the case of $N = 1$. Suppose $N' \geq 1$ is an integer and we assume for $1 \leq n \leq 2N'$,

$$\sum_{\sum \mu_j = n} \exp\left(\sum_{j=1}^{2N'} \mu_j \eta_j + \sum_{m<j} A_{mj} \mu_m \mu_j \right) \in \mathbb{R}. \tag{4.2.11}$$

When $N = N' + 1$, for any fixed $n : 1 \leq n \leq 2N$, we want to show

$$\sum_{\sum \mu_j = n} \exp\left(\sum_{j=1}^{2N'+2} \mu_j \eta_j + \sum_{m<j} A_{mj} \mu_m \mu_j \right) \in \mathbb{R}. \tag{4.2.12}$$

For fixed $n \geq 1$, the sum in (4.2.12) consists of three parts: $\sum_1^{2N'} \mu_j = n, n-1, n-2$. In the first case $\mu_{2N'+1} = \mu_{2N'+2} = 0$, by induction we know the sum is real.

In the second case $\mu_{2N'+1} = 1, \mu_{2N'+2} = 0$ or $\mu_{2N'+1} = 0, \mu_{2N'+2} = 1$. Since we take all the possible sum, this part of sum equals

$$\sum_{\sum_{j=1}^{2N'} \mu_j = n-1} \exp\left(\sum_{j=1}^{2N'} \mu_j \eta_j + \sum_{m<j} A_{mj}\mu_m\mu_j\right) \sum_{m=1}^{2} \exp\left(\eta_{2N'+m} + \sum_{j=1}^{2N'} \mu_j A_{j,2N'+m}\right). \tag{4.2.13}$$

By (4.2.8) we have for $1 \leq j < k \leq N' + 1$,

$$a_{2j-1,2k-1} = \bar{a}_{2j,2k}, \qquad a_{2j,2k-1} = \bar{a}_{2j-1,2k} \tag{4.2.14}$$

and

$$a_{2k-1,2k} := -\frac{P(2i\mathrm{Im}(\beta_{2k-1,1}), \cdots, 2i\mathrm{Im}(\beta_{2k-1,M}))}{P(2\mathrm{Re}(\beta_{2k-1,1}), \cdots, 2\mathrm{Re}(\beta_{2k-1,M}))} \in \mathbb{R}. \tag{4.2.15}$$

We introduce a map $\star : \mathbb{N} \to \mathbb{N}$

$$(2j-1)^\star = 2j, \quad (2j)^\star = 2j-1, \qquad \forall j \in \mathbb{N}.$$

This map has the property

$$(j^\star)^\star = j, \quad \forall j \in \mathbb{N}$$

and

$$\sum_{j=1}^{2N} \mu_j = \sum_{j^\star=1}^{2N} \mu_{j^\star}, \quad \forall N \in \mathbb{N}.$$

Case I. If $(\mu_{1^\star}, \mu_{2^\star}, \cdots, \mu_{(2N'-1)^\star}, \mu_{(2N')^\star}) = (\mu_1, \mu_2, \cdots, \mu_{2N'-1}, \mu_{2N'})$, then $\mu_{2j-1} = \mu_{2j}$ for $1 \leq j \leq N'$,

$$\mu_{2j-1}\eta_{2j-1} + \mu_{2j}\eta_{2j} = \mu_{2j-1}(\eta_{2j-1} + \bar{\eta}_{2j-1}) \in \mathbb{R}. \tag{4.2.16}$$

When $\mu_{2j-1} = \mu_{2k-1} = 1$, we get

$$a_{2j-1,2k-1} + a_{2j-1,2k} + a_{2j,2k-1} + a_{2j,2k} = 2\mathrm{Re}(a_{2j-1,2k-1} + a_{2j-1,2k}). \tag{4.2.17}$$

Therefore

$$\exp\left(\sum_{j=1}^{2N'} \mu_j \eta_j + \sum_{m<j} A_{mj}\mu_m\mu_j\right) \in \mathbb{R}. \tag{4.2.18}$$

On the other hand

$$\eta_{2N'+m} + \sum_{j=1}^{2N'} \mu_j A_{j,2N'+m} = \eta_{2N'+m} + \sum_{j=1}^{N'} \mu_{2j-1}(A_{2j-1,2N'+m} + A_{2j,2N'+m}) \tag{4.2.19}$$

implies

$$\exp\left(\eta_{2N'+1} + \sum_{j=1}^{2N'} \mu_j A_{j,2N'+1}\right) = \overline{\exp\left(\eta_{2N'+2} + \sum_{j=1}^{2N'} \mu_j A_{j,2N'+2}\right)} \tag{4.2.20}$$

and this concludes

$$\sum_{m=1}^{2} \exp\left(\eta_{2N'+m} + \sum_{j=1}^{2N'} \mu_j A_{j,2N'+m}\right) \in \mathbb{R}$$

and hence (4.2.13) is real.

Case II. If $(\mu_{1^\star}, \mu_{2^\star}, \cdots, \mu_{(2N'-1)^\star}, \mu_{(2N')^\star}) \neq (\mu_1, \mu_2, \cdots, \mu_{2N'-1}, \mu_{2N'})$, then $\mu_{2j-1} \neq \mu_{2j}$ for some $1 \leq j \leq N'$. Because we have $\mu_{j^\star}\eta_{j^\star} = \mu_j \bar{\eta}_j$. Suppose $\mu_m = \mu_j = 1$ and $m < j, j \neq m^\star$ then by (4.2.14) and (4.2.15) we have

$$a_{j,j^\star} = a_{j^\star,j} \in \mathbb{R}, \quad a_{m,j} = \bar{a}_{m^\star,j^\star}, \quad a_{m,j^\star} = \bar{a}_{m^\star,j}. \tag{4.2.21}$$

Therefore

$$\exp\left(\sum_{j^\star=1}^{2N'} \mu_{j^\star}\eta_{j^\star} + \sum_{m^\star<j^\star} A_{m^\star,j^\star}\mu_{m^\star}\mu_{j^\star}\right) = \exp\left(\sum_{j=1}^{2N'} \mu_j\bar{\eta}_j + \sum_{m<j} \bar{A}_{mj}\mu_m\mu_j\right). \tag{4.2.22}$$

In the same way, it is east to see

$$\sum_{m=1}^{2} \exp\left(\eta_{(2N'+m)^\star} + \sum_{j^\star=1}^{2N'} \mu_{j^\star} A_{j^\star,(2N'+m)^\star}\right) = \sum_{m=1}^{2} \exp\left(\bar{\eta}_{2N'+m} + \sum_{j=1}^{2N'} \mu_j \bar{A}_{j,2N'+m}\right),$$

which means

$$\begin{aligned} & \exp\left(\sum_{j^\star=1}^{2N'} \mu_{j^\star}\eta_{j^\star} + \sum_{m^*<j^*} A_{m^\star j^\star}\mu_{m^\star}\mu_{j^\star}\right) \sum_{m=1}^{2} \exp\left(\eta_{2N'+m} + \sum_{j^\star=1}^{2N'} \mu_{j^\star} A_{j^\star,2N'+m}\right) \\ = & \overline{\exp\left(\sum_{j=1}^{2N'} \mu_j\eta_j + \sum_{m<j} A_{mj}\mu_m\mu_j\right) \sum_{m=1}^{2} \exp\left(\eta_{2N'+m} + \sum_{j=1}^{2N'} \mu_j A_{j,2N'+m}\right)}. \end{aligned} \tag{4.2.23}$$

So we know (4.2.13) is real.

In the third case $\mu_{2N'+1} = \mu_{2N'+2} = 1$. Let

$$C_0 := \exp\left(\eta_{2N'+1} + \eta_{2N'+2} + A_{2N'+1,2N'+2}\right) = a_{2N'+1,2N'+2}\exp(2\mathrm{Re}(\eta_{2N'+1})) \in \mathbb{R}.$$

And we have also

$$\exp\sum_{m^\star=1}^{2N'}\mu_{m^\star}(A_{m^\star,2N'+1}+A_{m^\star,2N'})=\overline{\exp\sum_{m=1}^{2N'}\mu_m(A_{m,2N'+1}+A_{m,2N'})}.$$

Therefore

$$\begin{aligned}&\exp\left(\sum_{j^\star=1}^{2N'}\mu_{j^\star}\eta_{j^\star}+\sum_{m^\star<j^\star}A_{m^\star,j^\star}\mu_{m^\star}\mu_{j^\star}\right)\exp\left(\sum_{j^\star=1}^{2N'}\mu_{j^\star}(A_{j^\star,2N'+1}+A_{j^\star,2N'+2})\right)\\&=\overline{\exp\left(\sum_{j=1}^{2N'}\mu_j\eta_j+\sum_{m<j}A_{m,j}\mu_m\mu_j\right)\exp\left(\sum_{j=1}^{2N'}\mu_j(A_{j,2N'+1}+A_{j,2N'+2})\right)}.\end{aligned}\tag{4.2.24}$$

This tells us

$$\begin{aligned}&\sum_{\sum_{j=1}^{2N'}\mu_j=n-2}\exp\left(\sum_{j=1}^{2N'}\mu_j\eta_j+\sum_{m<j}A_{mj}\mu_m\mu_j\right)\exp\left(\sum_{m=1}^{2}\eta_{2N'+m}\right.\\&\left.+A_{2N'+1,2N'+2}+\sum_{m=1}^{2}\sum_{j=1}^{2N'}\mu_jA_{j,2N'+m}\right)\\=\;&C_0\sum_{\sum_{j=1}^{2N'}\mu_j=n-2}\exp\left(\sum_{j=1}^{2N'}\mu_j\eta_j+\sum_{m<j}A_{mj}\mu_m\mu_j\right)\exp\left(\sum_{m=1}^{2}\sum_{j=1}^{2N'}\mu_jA_{j,2N'+m}\right)\end{aligned}$$

is real.

Combining with the above proofs, we get (4.2.12) is real for any $n\geq 1$, that concludes the function f is real for $N=N'+1$. This completes the proof by the induction. □

In general, we can reformulate Theorem 4.2.1 to get N-complexiton solutions.

Theorem 4.2.2. *Let N be a positive number. The real function η_j, $j=1,2,\cdots,2N$, are defined by* (1.3.11) *and complex valued functions $\tilde{\eta}_{2j-1}=\eta_{2j-1}+i\eta_{2j},\tilde{\eta}_{2j}=\eta_{2j-1}-i\eta_{2j}$, for $j=1,2,\cdots,N$. Assume that the dispersion relation* (4.2.27) *is true for $\tilde{\eta}_j, 1\leq j\leq 2N$. Then the function*

$$f=1+\sum_{n=1}^{2M}\sum_{\sum_{j=1}^{2N}\mu_j=n}\exp\left(\sum_{j=1}^{2N}\mu_j\tilde{\eta}_j+\sum_{m<j}A_{mj}\mu_m\mu_j\right),\tag{4.2.25}$$

where $\mu_j=0$ or 1 for $j=1,2,\cdots,2N$, a_{mj} denoted by (4.2.31) *and $a_{mj}=e^{A_{mj}}$ for $1\leq m<j\leq 2N$, presents a complexiton solution to* (1.3.9).

4.2.2 Example: the bilinear KP equation

It is well known that KdV, KP, sG, nlS equations possess multi-soliton solutions. In this section, we will apply Theorem 4.2.1 to multi-complexiton solutions to the bilinear KP equation. By the references [32, 29], we know that the bilinear KP equation satisfies the Hirota condition.

We use the notation $i := \sqrt{-1}$. Suppose the function

$$\eta(x, y, t) = kx + ly + wt + \eta^0 = \eta_1 + i\eta_2, \tag{4.2.26}$$

where k, l, w, η^0 are constants and $\eta_1 = \mathrm{Re}(\eta), \eta_2 = \mathrm{Im}(\eta)$. Then $f := 1 + e^\eta$ is a complex valued solution to (1.3.9) if and only if the following dispersion relation holds (recall for the KP equation, $P(x, y, t) = x^4 + y^2 + xt$)

$$P(k, l, w) = P(\bar{k}, \bar{l}, \bar{w}) = 0. \tag{4.2.27}$$

Let $k_1 := \mathrm{Re}(k), k_2 := \mathrm{Im}(k), l_1 := \mathrm{Re}(l), l_2 := \mathrm{Im}(l), w_1 := \mathrm{Re}(w), w_2 := \mathrm{Im}(w)$. Under the condition $k_1^2 + k_2^2 > 0$, equation (4.2.27) is equavalent to

$$\begin{cases} w_1 = -k_1^3 + 3k_1k_2^2 - \dfrac{k_1(l_2^2 - l_1^2) + 2k_2l_1l_2}{k_1^2 + k_2^2}, \\ w_2 = k_2^3 - 3k_1^2k_2 - \dfrac{k_2(l_1^2 - l_2^2) + 2k_1l_1l_2}{k_1^2 + k_2^2}. \end{cases} \tag{4.2.28}$$

Now suppose η stisfies (4.2.27) or (4.2.28). By the two solition formulation and the above discussion, we get a one complexiton solution to (1.3.9):

$$f = 1 + e^\eta + e^{\bar{\eta}} + a_{12}e^{\eta+\bar{\eta}} = 1 + 2e^{\eta_1}\cos(\eta_2) + a_{12}e^{2\eta_1}, \tag{4.2.29}$$

where

$$a_{12} = -\frac{P(2ik_2, 2il_2, 2iw_2)}{P_1(2k_1, 2l_1, 2w_1)} = -\frac{4k_2^4 - l_2^2 - k_2w_2}{4k_1^4 + l_1^2 + k_1w_1} \in \mathbb{R}.$$

The solution for the KP equation reads

$$\begin{aligned} u &= 2(\ln f)_{xx} \\ &= \frac{1}{(1 + 2e^{\eta_1}\cos(\eta_2) + a_{12}e^{2\eta_1})^2}\{4e^{\eta_1}[(k_1^2 - k_2^2)\cos(\eta_2) - 2k_1k_2\sin(\eta_2)] \\ &\quad +8e^{2\eta_1}(a_{12}k_1^2 - k_2^2) + 4a_{12}e^{3\eta_1}[(k_1^2 - k_2^2)\cos(\eta_2) + 2k_1k_2\sin(\eta_2)]\}. \end{aligned} \tag{4.2.30}$$

Due to the relation (4.2.28), we have four free real parameters in this simplest complexiton solution, whereas the solutions in [79] only possess two parameters α and β. Our solutions can't be covered by those solutions.

Now we consider $N = 4$. Suppose that the dispersion relation (4.2.27) is true for $\tilde{\eta}_1 = \eta_1 + i\eta_2, \tilde{\eta}_2 = \eta_1 - i\eta_2$ and $\tilde{\eta}_3 = \eta_3 + i\eta_4, \tilde{\eta}_4 = \eta_3 - i\eta_4$, where $\eta_j = \eta_j^0 + k_j x + l_j y + w_j t, j = 1, \cdots, 4$, are all real.

Let

$$a_{j'j} = -\frac{P(\tilde{k}_{j'} - \tilde{k}_j, \tilde{l}_{j'} - \tilde{l}_j, \tilde{w}_{j'} - \tilde{w}_j)}{P(\tilde{k}_{j'} + \tilde{k}_j, \tilde{l}_{j'} + \tilde{l}_j, \tilde{w}_{j'} + \tilde{w}_j)}, \qquad 1 \leq j' < j \leq N. \tag{4.2.31}$$

Then we have

$$\begin{aligned}
a_{12} &= \frac{-4k_2^4 + l_2^2 + k_2 w_2}{4k_1^4 + l_1^2 + k_1 w_1} \in \mathbb{R}, \\
a_{13} &= -\frac{P(k_1 - k_3 + i(k_2 - k_4), l_1 - l_3 + i(l_2 - l_4), w_1 - w_3 + i(w_2 - w_4))}{P(k_1 + k_3 + i(k_2 + k_4), l_1 + l_3 + i(l_2 + l_4), w_1 + w_3 + i(w_2 + w_4))}, \\
a_{14} &= -\frac{P(k_1 - k_3 + i(k_2 + k_4), l_1 - l_3 + i(l_2 + l_4), w_1 - w_3 + i(w_2 + w_4))}{P(k_1 + k_3 + i(k_2 - k_4), l_1 + l_3 + i(l_2 - l_4), w_1 + w_3 + i(w_2 - w_4))}, \\
a_{23} &= -\frac{P(k_1 - k_3 - i(k_2 + k_4), l_1 - l_3 - i(l_2 + l_4), w_1 - w_3 - i(w_2 + w_4))}{P(k_1 + k_3 - i(k_2 - k_4), l_1 + l_3 - i(l_2 - l_4), w_1 + w_3 - i(w_2 - w_4))} \\
&= \overline{a_{14}}, \\
a_{24} &= -\frac{P(k_1 - k_3 - i(k_2 - k_4), l_1 - l_3 - i(l_2 - l_4), w_1 - w_3 - i(w_2 - w_4))}{P(k_1 + k_3 - i(k_2 + k_4), l_1 + l_3 - i(l_2 + l_4), w_1 + w_3 - (w_2 + w_4))} \\
&= \overline{a_{13}}, \\
a_{34} &= \frac{-4k_4^4 + l_4^2 + k_4 w_4}{4k_3^4 + l_3^2 + k_3 w_3} \in \mathbb{R}.
\end{aligned} \tag{4.2.32}$$

Let the function f be defined by

$$\begin{aligned}
f \ &= 1 + e^{\tilde{\eta}_1} + e^{\tilde{\eta}_2} + e^{\tilde{\eta}_3} + e^{\tilde{\eta}_4} + a_{12} e^{\tilde{\eta}_1 + \tilde{\eta}_2} + a_{13} e^{\tilde{\eta}_1 + \tilde{\eta}_3} + a_{14} e^{\tilde{\eta}_1 + \tilde{\eta}_4} \\
&+ a_{23} e^{\tilde{\eta}_2 + \tilde{\eta}_3} + a_{24} e^{\tilde{\eta}_2 + \tilde{\eta}_4} + a_{34} e^{\tilde{\eta}_3 + \tilde{\eta}_4} + a_{123} e^{\tilde{\eta}_1 + \tilde{\eta}_2 + \tilde{\eta}_3} + a_{124} e^{\tilde{\eta}_1 + \tilde{\eta}_2 + \tilde{\eta}_4} \\
&+ a_{134} e^{\tilde{\eta}_1 + \tilde{\eta}_3 + \tilde{\eta}_4} + a_{234} e^{\tilde{\eta}_2 + \tilde{\eta}_3 + \tilde{\eta}_4} + a_{1234} e^{\tilde{\eta}_1 + \tilde{\eta}_2 + \tilde{\eta}_3 + \tilde{\eta}_4}
\end{aligned} \tag{4.2.33}$$

with

$$\begin{aligned}
a_{1234} &= a_{12} a_{13} a_{14} a_{23} a_{24} a_{34} = a_{12} \overline{a_{24} a_{23}} a_{23} a_{24} a_{34} \in \mathbb{R}, \\
a_{123} &= a_{12} a_{13} a_{23} = \overline{a_{12} a_{24} a_{14}} = \bar{a}_{124}, \\
a_{134} &= a_{13} a_{14} a_{34} = \overline{a_{24} a_{23} a_{34}} = \bar{a}_{234}.
\end{aligned} \tag{4.2.34}$$

Then it is a two complexiton solution to (1.3.9).

The function f can be simplified as

$$\begin{aligned} f &= 1 + 2e^{\eta_1}\cos(\eta_2) + 2e^{\eta_3}\cos(\eta_4) + a_{12}e^{2\eta_1} + a_{34}e^{2\eta_3} \\ &+2\mathrm{Re}\{a_{13}e^{\eta_1+\eta_3+i(\eta_2+\eta_4)} + a_{14}e^{\eta_1+\eta_3+i(\eta_2-\eta_4)}\} \\ &+2\mathrm{Re}\{a_{123}e^{2\eta_1+\eta_3+i\eta_4} + a_{134}e^{\eta_1+2\eta_3+i\eta_2}\} \\ &+a_{1234}e^{2\eta_1+2\eta_3}. \end{aligned} \tag{4.2.35}$$

In particular, if $\eta_4 = 0$ then $\tilde{\eta}_3 = \tilde{\eta}_4 = \eta_3$. By (4.2.31) we get $a_{34} = 0$. It is clear $a_{14} = a_{13}, a_{134} = a_{1234} = 0$. Therefore (4.2.35) can be written as

$$\begin{aligned} f &= 1 + 2e^{\eta_1}\cos(\eta_2) + 2e^{\eta_3} + a_{12}e^{2\eta_1} + 2\mathrm{Re}\{2a_{13}e^{\eta_1+\eta_3+i\eta_2}\} \\ &+2\mathrm{Re}\{a_{123}e^{2\eta_1+\eta_3}\}. \end{aligned} \tag{4.2.36}$$

This is a mixed one-soliton and two-complexiton solution.

In general, we can have a mixed N_1-soliton and $2N_2$-complexiton solution for the KP equation.

Remark 4.2.3. *In this section, we study Hirota bilinear forms. For generalized cases (i.e.* $p = 3, 5, \cdots$*) the problem is still open.*

4.3 Applications of linear superposition principles to complexitons

4.3.1 Exponential wave solutions and linear superposition principle

In this section, suppose that P is a real polynomial in M variables. We will study the bilinear equation with D_p-operators:

$$P(D_p)f \cdot f = 0. \tag{4.3.1}$$

Let N be a positive integer. We define N-wave variables

$$\eta_m := \beta_{m,0} + \beta_{m,1}x_1 + \beta_{m,2}x_2 + \cdots \beta_{m,M}x_M, \quad 1 \le m \le N \tag{4.3.2}$$

and exponential wave functions

$$f_m := e^{\eta_m}, \quad 1 \le m \le N, \tag{4.3.3}$$

where $\beta_{m,j}, 1 \le m \le N,\ 0 \le j \le M$, are all real constants.

It is not difficult to prove that for any positive integer k, we have

$$D_j^k e^{\eta_1} \cdot e^{\eta_2} = (\beta_{1,j} - \beta_{2,j})^k e^{\eta_1+\eta_2}, \tag{4.3.4}$$

where D_j is a Hirota derivative. Therefore

$$P(D)e^{\eta_1} \cdot e^{\eta_2} = P(\beta_{1,1} - \beta_{2,1}, \cdots, \beta_{1,M} - \beta_{2,M})e^{\eta_1+\eta_2}. \quad (4.3.5)$$

In particular, we have

$$P(D)e^{\eta_1} \cdot e^{\eta_1} = P(0, \cdots, 0)e^{2\eta_1}. \quad (4.3.6)$$

Hence function $f_1 = e^{\eta_1}$ with arbitrary constant coefficients $\beta_{1,j}, 0 \leq j \leq M$, is a solution to a bilinear differential equation

$$P(D)f \cdot f = 0, \quad (4.3.7)$$

provided that $P(0) = 0$.

Similarly, for D_p-operators, we have

$$P(D_p)e^{\eta_1} \cdot e^{\eta_2} = P(\beta_{1,1} + \alpha_p\beta_{2,1}, \cdots, \beta_{1,M} + \alpha_p\beta_{2,M})e^{\eta_1+\eta_2}. \quad (4.3.8)$$

Therefore $f_1 = e^{\eta_1}$ solves (4.3.1) if and only if

$$P(\beta_{1,1} + \alpha_p\beta_{1,1}, \cdots, \beta_{1,M} + \alpha_p\beta_{1,M}) = 0.$$

Our problem is: assuming $f_1 = e^{\eta_1}$ and $f_2 = e^{\eta_2}$ are all solutions of (4.3.1), is any linear combination of f_1 and f_2 still a solution? We are going to answer the question later.

From the previous discussion, we know the bilinear equation (4.2.1) possesses infinitely many exponential function solutions. We are interested in constructing some subset of such solutions which forms a nonzero linear space. The idea is to find certain conditions such that for some given solutions, their linear combinations will still be solutions.

Suppose that $P(x)$ is a polynomial in $x \in \mathbb{R}^M$ (P may be not an even function). Assume that p is a positive integer.

We are interested in finding the conditions for a linear combination of functions

$$f = \varepsilon_1 f_1 + \varepsilon_2 f_2 + \cdots + \varepsilon_N f_N, \quad (4.3.9)$$

where $\varepsilon_m, 1 \leq m \leq N$, are arbitrary real constants, and $f_m, 1 \leq m \leq N$, are solutions of bilinear equation (4.3.1).

As in [44], we can compute that

$$\begin{aligned}
& P(D_p)f\cdot f \\
= \ & \sum_{j,k=1}^{N} \varepsilon_j\varepsilon_k P(D_p)e^{\eta_j}\cdot e^{\eta_k} \\
= \ & \sum_{j,k=1}^{N} \varepsilon_j\varepsilon_k P(\beta_{j,1}+\alpha\beta_{k,1},\cdots,\beta_{j,M}+\alpha\beta_{k,M})e^{\eta_j+\eta_k} \\
= \ & \sum_{j=1}^{N} \varepsilon_j^2 P(\beta_{j,1}+\alpha\beta_{j,1},\cdots,\beta_{j,M}+\alpha\beta_{j,M})e^{2\eta_j} + \sum_{1\le j<k\le N} \varepsilon_j\varepsilon_k e^{\eta_j+\eta_k}[P(\beta_{j,1}+\alpha\beta_{k,1}, \\
& \quad \cdots,\beta_{j,M}+\alpha\beta_{k,M}) + P(\beta_{k,1}+\alpha\beta_{j,1},\cdots,\beta_{k,M}+\alpha\beta_{j,M})].
\end{aligned}$$

Because ε_j and ε_k, $1\le j,k\le N$ are arbitrary constants, the function f, defined by (4.3.9), solves bilinear equation (4.3.1) if and only if for $1\le j\le k\le N$, it is true that

$$P(\beta_{j,1}+\alpha\beta_{k,1},\cdots,\beta_{j,M}+\alpha\beta_{k,M}) + P(\beta_{k,1}+\alpha\beta_{j,1},\cdots,\beta_{k,M}+\alpha\beta_{j,M}) = 0. \tag{4.3.10}$$

The above conditions are a collection of nonlinear algebraic equations, solving these equations paves a way of finding exact solutions by solving a group of algebraic equations rather than a group of nonlinear partial differential equations.

We now summarize our discussion as follows.

Theorem 4.3.1 (Linear superposition principle[44]). *Let $P(x)$ be a polynomial in $x\in\mathbb{R}^M$ and η_m, $1\le m\le N$, be the N-wave variables $\eta_m=\sum_{j=1}^{M}\beta_{m,j}x_j$, $1\le m\le N$, where $\beta_{m,j}$'s are all constants. Then any linear combination of the exponential waves $f_m=e^{\eta_m}$, $1\le m\le N$, solves the bilinear differential equation* (4.3.1) *if and only if the conditions in* (4.3.10) *are satisfied.*

Remark 4.3.2. *In the above linear superposition principle if $\beta_{m,j}$'s are complex, then linear combinations provides a large class of complex valued function solutions.*

In the situation of $p=2$, the conditions (4.3.10) can be simplified. We list the related results in the following two corollaries.

Corollary 4.3.3 (Linear superposition principle I[53]). *Let $P(x)$ be a polynomial satisfying $P(0)=0$ and the wave variables $\eta_m, 1\le m\le N$ be defined by* (4.3.2)*. Then any linear combination of the exponential waves $f_m=e^{\eta_m}$, $1\le m\le N$, solves the bilinear differential equation* (4.2.1) *if and only if*

$$P(\beta_{j,1}-\beta_{k,1},\cdots,\beta_{j,M}-\beta_{k,M})=0, \tag{4.3.11}$$

for $1 \le j < k \le N$.

Corollary 4.3.4 (Linear superposition principle II)**.** *Let* $P(x)$ *be a polynomial satisfying* $P(0) = 0$ *and the wave variables* $\eta_m, 1 \le m \le N$ *be defined by* (4.3.2)*. Then any linear combination of the exponential waves* $f_m = e^{\eta_m}$*,* $f_{m+N} = e^{-\eta_m}$*,* $1 \le m \le N$*, solves the bilinear differential equation* (4.2.1) *if and only if*

$$P(\beta_{j,1} \pm \beta_{k,1}, \cdots, \beta_{j,M} \pm \beta_{k,M}) = 0, \tag{4.3.12}$$

for $1 \le j < k \le N$.

The prove is apply the Corollary 4.3.3 for $2N$ exponential waves. Note

$$\sum_{m=1}^{2N} \varepsilon_m f_m = \sum_{m=1}^{N} [(\varepsilon_m + \varepsilon_{m+N})\mathrm{ch}(\eta_m) + (\varepsilon_m - \varepsilon_{m+N})\mathrm{sh}(\eta_m)] \tag{4.3.13}$$

The condition is the same as that in the Theorem 1 of [84] and we have much strong results: any linear combination of $\mathrm{ch}(\eta_m), \mathrm{sh}(\eta_m), 1 \le m \le N$ solves the bilinear differential equation (4.2.1).

By the proved linear superposition principle, we know for some bilinear partial differential equations under certain conditions, there exist some infinite dimensional linear subspaces of solutions although all the solutions do not form a linear space. Such a kind of examples can be seen in the works [53, 57]. In subsets of the solutions to bilinear PDEs (4.3.1), we are particularly interested in that spanned by $f_m = e^{\eta_m}$ with

$$\eta_m = b_1 k_m^{\alpha_1} x_1 + b_2 k_m^{\alpha_2} x_2 + \cdots + b_M k_m^{\alpha_M} x_M,\ 1 \le m \le N, \tag{4.3.14}$$

where $b_j \in \mathbb{R}, \alpha_j \in \mathbb{Z}$, for $j = 1, \cdots, M$, are fixed numbers, and $k_m \in \mathbb{R}, m = 1, \cdots, N$, are arbitrary.

Theorem 4.3.5. *Let* $P(x)$ *be a polynomial in* $x \in \mathbb{R}^M$ *and* $\eta_m, 1 \le m \le N$*, be the* N*-wave variables* $\eta_m = \sum_{j=1}^{M} b_j k_m^{\alpha_j} x_j$*, for* $1 \le m \le N$*, where* b_j*'s are all real constant. Then any linear combination of the exponential waves* $f_m = e^{\eta_m}, 1 \le m \le N$*, solves the bilinear differential equation* $P(D_p) f \cdot f = 0$ *if and only if*

$$\begin{aligned} &P(b_1(k_j^{\alpha_1} + \alpha k_m^{\alpha_1}), \cdots, b_M(k_j^{\alpha_M} + \alpha k_m^{\alpha_M})) \\ &+P(b_1(k_m^{\alpha_1} + \alpha k_j^{\alpha_1}), \cdots, b_M(k_m^{\alpha_M} + \alpha k_j^{\alpha_M})) = 0, \end{aligned} \tag{4.3.15}$$

for $1 \le j \le m \le N$.

Moreover, (4.3.15) *is true for arbitrary* $k_1, k_2 \in \mathbb{R}$ *if and only if it is also true for arbitrary constants* $k_1, k_2 \in \mathbb{C}$.

Remark 4.3.6. *Theorem 4.3.5 works also for complex* k_m*'s, which leads to complex-valued linear combination. However, we are interested in only real solutions. Is it possible for us to choose suitable coefficients to get real solutions?*

From the second part of the Theorem 4.3.5, we see if (4.3.15) is true for arbitrary $k_1, k_2 \in \mathbb{R}$. Then N can be any positive integer and k_m, $1 \leq k_m \leq N$ are arbitrary complex numbers.

Let $N \in \mathbb{N}, i = \sqrt{-1}$. Suppose $k_m = \gamma_m + i\delta_m, \gamma_m, \delta_m, \theta_m \in \mathbb{R}, m = 1, 2, \cdots, N$. Suppose that (4.3.15) is satisfied for any $k_1, k_2, \in \mathbb{R}$. Let the functions $\eta_m = \sum_{j=1}^{M} b_j k_m^{\alpha_j} x_j$, $f_m = e^{\eta_m + i\theta_m}$, for $m = 1, 2, \cdots, N$. It is easy to check that $P(D_p) f_m \cdot f_m = 0$ if and only if $P(D_p)\bar{f}_m \cdot \bar{f}_m = 0$.

By the superposition principle, we know

$$\sum_{m=1}^{N} \left(\frac{\varepsilon_m}{2} f_m + \frac{\varepsilon_m}{2} \bar{f}_m \right) = \sum_{m=1}^{N} \varepsilon_m e^{\mathrm{Re}(\eta_m)} \cos(\mathrm{Im}(\eta_m) + \theta_m) \tag{4.3.16}$$

is a real valued solution to (4.3.1) for arbitrary $\varepsilon_m \in \mathbb{R}, 1 \leq m \leq N$. For some $m \in \{1, 2, \cdots, N\}$, it may happen that $\delta_m = 0$. This implies that $k_m \in \mathbb{R}$ and η_m is a real wave function. Then we have a real solution of equation (4.3.1):

$$\varepsilon_m e^{\mathrm{Re}(\eta_m)} \cos(\mathrm{Im}(\eta_m) + \theta_m) = \varepsilon'_m e^{\eta_m}, \tag{4.3.17}$$

where $\varepsilon'_m = \varepsilon_m \cos(\theta_m)$ is a real constant.

In general, we have following result.

Theorem 4.3.7 (Linear superposition principle for complexitons)**.** *Let* $P(x)$ *be a polynomial in* $x \in \mathbb{R}^M$. *Suppose that* $N \in \mathbb{N}, b_j \in \mathbb{R}, \alpha_j \in \mathbb{Z}$ *for* $j = 1, \cdots, M$*, are all fixed. The* $2N$*-wave variables* $\eta_m = \displaystyle\sum_{j=1}^{M} b_j k_m^{\alpha_j} x_j$, $\eta_{m+N} = \displaystyle\sum_{j=1}^{M} b_j \bar{k}_m^{\alpha_j} x_j, \eta_{m,1} = \mathrm{Re}(\eta_m)$, $\eta_{m,2} = \mathrm{Im}(\eta_m)$, $1 \leq m \leq N$.

If (4.3.15) *is true for arbitrary* $k_j, k_m, 1 \leq j, m \leq 2N$*. Then any linear (real) combination of the waves* $e^{\eta_{m,1}} \cos(\eta_{m,2})$, $e^{\eta_{m,1}} \sin(\eta_{m,2})$, $1 \leq m \leq N$

$$\sum_{m=1}^{N} e^{\eta_{m,1}} \Big[a_m \cos(\eta_{m,2}) c_m \sin(\eta_{m,2}) \Big]. \tag{4.3.18}$$

solves the bilinear differential equation (4.3.1) *with arbitrary* $a_m, c_m \in \mathbb{R}, 1 \leq m \leq N$.

Proof. From the Theorem 4.3.5, we know any linear combination of $f_m = e^{\eta_m}$, $1 \le m \le 2N$, with complex coefficients is a complex solution to the bilinear differential equation (4.3.1)

$$\begin{aligned}\sum_{m=1}^{2N} \varepsilon_m f_m &= \sum_{m=1}^{N} \Big[(\varepsilon_m + \varepsilon_{m+N})\frac{f_m + \bar{f}_m}{2} + (\varepsilon_m - \varepsilon_{m+N})\frac{f_m - \bar{f}_m}{2}\Big] \\ &= \sum_{m=1}^{N} \Big[(\varepsilon_m + \varepsilon_{m+N})e^{\eta_{m,1}}\cos(\eta_{m,2}) + i(\varepsilon_m - \varepsilon_{m+N})e^{\eta_{m,1}}\sin(\eta_{m,2})\Big].\end{aligned}$$

The algebraic equations

$$\begin{cases} \varepsilon_m + \varepsilon_{m+N} = a, \\ i(\varepsilon_m - \varepsilon_{m+N}) = b, \end{cases} \tag{4.3.19}$$

has a unique (complex) solution $(\varepsilon_m, \varepsilon_{m+N})$ for any fixed real numbers a and b. Our theorem follows. □

Remark 4.3.8. *For any fixed* m, $1 < m \le N$, *if* $k_m \in \mathbb{R}$ *or* $\mathrm{Im}(\eta_m) = 0$ *then* $f_m = f_{m+N}$ *is an exponential wave.*

Suppose $p = 2$ *and* $\mathrm{Re}(\eta_m) = 0, 1 \le m \le N$. *We have a linear combination of* $\cos(\eta_{m,2})$ *and* $\sin(\eta_{m,2})$, $1 \le m \le N$. *Obviously, this is a generalization of Theorem 2 of [84].*

When $\eta_{m,1}\eta_{m,2} \ne 0$, *we get at least a complexiton.*

In general, we may have a linear combination of exponential waves, trigonometric waves and multiplication of exponential and trigonometric waves.

4.3.2 Some examples

First, we consider Hirota bilinear equations.

Example 4.3.9. *We consider the bilinear KdV equation*

$$D_x(D_x^3 + D_t)f \cdot f = 0. \tag{4.3.20}$$

For a fixed positive integer $N \ge 2$, consider

$$\eta_j = k_j x + w_j t, \qquad 1 \le j \le N. \tag{4.3.21}$$

then by N- wave solution condition in the Theorem 4.3.1,

$$(k_j - k_m)[(k_j - k_m)^3 + (w_j - w_m)] = 0, \qquad 1 \le j < m \le N. \tag{4.3.22}$$

The only nontrivial solution is in the case of $N = 2$ and $(k_1 - k_2)^3 + w_1 - w_2 = 0$. Let

$$k_1 = k + a,\ k_2 = k - a,\ w_1 = w + b,\ w_2 = w - b. \tag{4.3.23}$$

Then we have

$$(2a)^3 + 2b = 0, \tag{4.3.24}$$

therefore $b = -4a^3$. By Theorem 4.3.1, we know for any $k, a \neq 0, w \in \mathbb{R}$,

$$f = \varepsilon_1 e^{(k+a)x+(w-4a^3)t} + \varepsilon_2 e^{(k-a)x+(w+4a^3)t} \tag{4.3.25}$$

is a solution to the bilinear KdV equation for any real coefficients ε_1 and ε_2. We get a solution to the KdV equation from f:

$$\begin{aligned} u(x,t) &:= 2\ln(f(x,t))_{xx} = \frac{8a^2\varepsilon_1\varepsilon_2 e^{2kx+2wt}}{[\varepsilon_1 e^{(k+a)x+(w-4a^3)t} + \varepsilon_2 e^{(k-a)x+(w+4a^3)t}]^2} \\ &= \frac{8a^2\varepsilon_1}{[\varepsilon_1 e^{ax-4a^3t} + \varepsilon_2 e^{-ax+4a^3t}]^2}, \end{aligned} \tag{4.3.26}$$

which is just a one-soliton solution.

Now we turn to find complexitons. Suppose that $k_2 = \bar{k}_1$ and $w_2 = \bar{w}_1$, then

$$k_1 + k_2 = 2k = 2\mathrm{Re}(k_1),\ k_1 - k_2 = 2a = 2i\mathrm{Im}(k_1). \tag{4.3.27}$$

Hence $k = \mathrm{Re}(k_1), \tilde{a} = \mathrm{Im}(k_1) = a/i$. With the same discussion, we have $w = \mathrm{Re}(w_1), \tilde{b} = \mathrm{Im}(w_1) = b/i$. By (4.3.24) , we get $\tilde{b} = 4\tilde{a}^3$. Fix $\theta \in \mathbb{R}$, define

$$\begin{aligned} f &= \varepsilon[e^{(k+a)x+(w-4a^3)t+i\theta} + e^{(k-a)x+(w+4a^3)t-i\theta}] \\ &= 2\varepsilon e^{kx+wt}\cos(\tilde{a}x + 4\tilde{a}^3t + \theta). \end{aligned} \tag{4.3.28}$$

Then f is a solution to the bilinear KdV equation for any real coefficients ε. The solution to the KdV equation reads

$$u(x,t) = 2\ln(f(x,t))_{xx} = \frac{-2a^2}{[\cos(ax + 4a^3t + \theta)]^2}. \tag{4.3.29}$$

In this example, we proved that there are no vector spaces with dimension large than two which can be subset of solutions to the bilinear KdV equation.

Example 4.3.10. *We study the bilinear KP equation*

$$(D_x^4 + D_xD_t + D_y^2)f \cdot f = 0. \tag{4.3.30}$$

For a fixed positive integer $N \geq 2$, let

$$\eta_m = k_m x + b_1 k_m^2 y + b_2 k_m^3 t, \ 1 \leq m \leq N. \tag{4.3.31}$$

Then by the linear superposition principle, we get

$$(k_j - k_m)[(k_j - k_m)^3 + b_2(k_j^3 - k_m^3)] + b_1^2(k_j^2 - k_m^2)^2 = 0, \ 1 \leq j < m \leq N. \tag{4.3.32}$$

The parameters b_1 and b_2 should satisfy

$$\begin{cases} b_1^2 + b_2 + 1 = 0, \\ -b_2 - 4 = 0. \end{cases} \tag{4.3.33}$$

Therefore, we have solutions $b_1 = \sqrt{3}, b_2 = -4$ and $b_1 = -\sqrt{3}, b_2 = -4$.

Suppose $N \in \mathbb{N}$. The N-wave variables $\eta_m = k_m x + b_1 k_m^2 y + b_2 k_m^3 t, \ 1 \leq m \leq N$ with parameters $k_m = \gamma_m + i\delta_m$, $1 \leq m \leq N$. Moreover, for $1 \leq m \leq N$, we have

$$\begin{aligned} \eta_m &= k_m x + b_1 k_m^2 y + b_2 k_m^3 t \\ &= \gamma_m x + (\gamma_m^2 - \delta_m^2) b_1 y - 4(\gamma_m^3 - 3\gamma_m \delta_m^2) t \\ &\quad + i[\delta_m x + 2\gamma_m \delta_m b_1 y - 4(3\gamma_m^2 \delta_m - \delta_m^3) t]. \end{aligned} \tag{4.3.34}$$

Therefore

$$\begin{cases} \eta_{m,1} = \gamma_m x + (\gamma_m^2 - \delta_m^2) b_1 y - 4(\gamma_m^3 - 3\gamma_m \delta_m^2) t, \\ \eta_{m,2} = \delta_m x + 2\gamma_m \delta_m b_1 y - 4(3\gamma_m^2 \delta_m - \delta_m^3) t \end{cases} \tag{4.3.35}$$

with $b_1 = \pm\sqrt{3}$. By Theorem 4.3.7, any linear (real) combination of the exponential and trigonometric waves (4.3.18) solves the bilinear KP equation.

If $\eta_{m,1} = 0$ then $\gamma_m = \delta_m = 0$ and thus $\eta_m = 0, f_m = 1$. If $\eta_{m,2} = 0$ then $\delta_m = 0$, γ_m is arbitrary and $f_m = \exp(\gamma_m x + \gamma_m^2 b_1 y - 4\gamma_m^3 t)$.

Example 4.3.11. *We introduce a polynomial*

$$P(x, y, z, t) = c_1 x^4 + c_2 x^2 y + c_3 xz + c_4 xt + c_5 y^2 \tag{4.3.36}$$

and define weights of the independent variables:

$$(w(x), w(y), w(z), w(t)) = (1, 2, 3, 3). \tag{4.3.37}$$

The corresponding Hirota bilinear equation reads

$$\begin{aligned} & P(D_x, D_y, D_z, D_t) f\cdot f \\ = \;& 2c_1(ff_{4x} - 4f_xf_{3x} + 3f_{xx}^2) + 2c_3(ff_{xz} - f_xf_z) \\ & \quad +2c_4(ff_{xt} - f_xf_t) + 2c_5(ff_{yy} - f_y^2) \\ = \;& (c_1D_x^4 + c_3D_xD_z + c_4D_xD_t + c_5D_y^2) f\cdot f = 0. \end{aligned} \tag{4.3.38}$$

Let us take the wave variables

$$\eta_m = k_m x + b_1k_m^2 y + b_2k_m^3 z + b_3k_m^3 t,\ 1 \leq m \leq N. \tag{4.3.39}$$

Suppose that a linear subspace of N-wave solutions is given by

$$f = \sum_{m=1}^{N} \varepsilon_m f_m = \sum_{m=1}^{N} \varepsilon_m e^{\eta_m}, \tag{4.3.40}$$

Since the solutions of (4.3.38) do not depend on c_2, taking $\tilde{P}(x, y, z, t) = c_1x^4 + c_3xz + c_4xt + c_5y^2$, applying Theorem 4.3.5 for $\tilde{P}$, $b_m,\ m = 1, 2, 3$, are need satisfy

$$\begin{cases} c_5b_1^2 = 3c_1, \\ c_3b_2 + c_4b_3 = -4c_1. \end{cases} \tag{4.3.41}$$

We know if $c_1c_5 > 0, c_3, c_4$ not all zero, then (4.3.41) has real solutions. Thus the corresponding equation (4.3.15) is satisfied. Suppose that $k_m, \eta_m, 1 \leq m \leq N$ are defined by the Theorem 4.3.7. Moreover, for $N_1 < m \leq N$, we have

$$\begin{aligned} \eta_m &= k_m x + b_1k_m^2 y + b_2k_m^3 z + b_3k_m^3 t \\ &= \gamma_m x + (\gamma_m^2 - \delta_m^2) b_1 y + (\gamma_m^3 - 3\gamma_m\delta_m^2)(b_2 z + b_3 t) \\ &\quad + i[\delta_m x + 2\gamma_m\delta_m b_1 y + (3\gamma_m^2\delta_m - \delta_m^3)(b_2 z + b_3 t)]. \end{aligned} \tag{4.3.42}$$

So we get

$$\begin{cases} \eta_{m,1} = \gamma_m x + (\gamma_m^2 - \delta_m^2) b_1 y + (\gamma_m^3 - 3\gamma_m\delta_m^2)(b_2 z + b_3 t), \\ \eta_{m,2} = \delta_m x + 2\gamma_m\delta_m b_1 y + (3\gamma_m^2\delta_m - \delta_m^3)(b_2 z + b_3 t) \end{cases} \tag{4.3.43}$$

We can apply Theorem 4.3.7. Any linear (real) combination of the exponential and trigonometric waves (4.3.18) solves the bilinear differential equation (4.3.38).

Note this example comes from [53] with $c_2 = 0$.

This example shows that there are infinite (actually uncountable) dimensional vector spaces as subset of solutions to the bilinear KPI equation.

Example 4.3.12. *In this example, we have weights* $(w(x), w(y), w(z), w(t)) = (1, -1, -2, 3)$ *with negative weight components. Then, consider a homogeneous polynomial of weight 2*

$$P = c_1 ty + c_2 x^3 y + c_3 x^4 y^2 + c_4 x^4 z + c_5 z^2 t^2 + c_6 x^2. \tag{4.3.44}$$

This example is similar to but different from Example 5 in [53]. Let us take the wave variables

$$\eta_m = k_m x + b_1 k_m^{-1} y + b_2 k_m^{-2} z + b_3 k_m^3 t,\ 1 \leq m \leq N. \tag{4.3.45}$$

The corresponding Hirota bilinear equation reads

$$\begin{aligned}
& P(D_x, D_y, D_z, D_t) f \cdot f \\
= \ & 2c_1(ff_{ty} - f_t f_y) + 2c_2(ff_{3xy} - 3f_x f_{xxy} + 3f_{xx} f_{xy} - f_{xxx} f_y) + 2c_3(ff_{4xyy} \\
& -4f_x f_{3x2y} + 6f_{xx} f_{xxyy} - 4f_{3x} f_{xyy} + f_{4x} f_{yy} - 2f_y f_{4xy} + 8f_{xy} f_{3xy} - 6f_{xxy}^2) \\
& +2c_5(ff_{ttzz} - 2f_z f_{ttz} + f_{zz} f_{tt} - 2f_t f_{zzt} + 2f_{tz}^2) + 2c_6(ff_{xx} - f_x^2).
\end{aligned} \tag{4.3.46}$$

It possesses an N-wave solution

$$f = \sum_{m=1}^{N} \varepsilon_m f_m = \sum_{m=1}^{N} \varepsilon_m e^{\eta_m}, \tag{4.3.47}$$

where ε_m's and k_m's are arbitrary, and $b_j, 1 \leq j \leq 3$, satisfy

$$\begin{cases} 3c_2 b_1 = -c_6, \\ c_1 b_1 b_3 + c_2 b_1 = 0, \\ c_3 b_1^2 = c_5 (b_2 b_3)^2 = 0. \end{cases} \tag{4.3.48}$$

We have a solution $b_1 = -\frac{c_6}{3c_2}, b_3 = -\frac{c_2}{c_1}$ and b_2 is an arbitrary real number when $c_3 = c_5 = 0$ and $c_1 c_2 \neq 0$. (Note $c_4 D_x^4 D_z f \cdot f = 0$) The equation (4.3.46) reduces to

$$(c_1 D_t D_y + c_2 D_x^3 D_y + c_6 D_x^2) f \cdot f = 0. \tag{4.3.49}$$

Suppose that $k_m, \eta_m, 1 \leq m \leq N$, are defined by the Theorem 4.3.7. Then for $N_1 < m \leq N$, we compute

$$\begin{aligned}
\eta_m \ & = k_m x + b_1 k_m^{-1} y + b_2 k_m^{-2} z + b_3 k_m^3 t \\
& = \gamma_m x + \frac{\gamma_m}{\sqrt{\gamma_m^2 + \delta_m^2}} b_1 y + \frac{\gamma_m^2 - \delta_m^2}{\gamma_m^2 + \delta_m^2} b_2 z + (\gamma_m^3 - 3\gamma_m \delta_m^2) b_3 t \\
& +i[\delta_m x - \frac{\delta_m}{\sqrt{\gamma_m^2 + \delta_m^2}} b_1 y - \frac{2\gamma_m \delta_m}{\gamma_m^2 + \delta_m^2} b_2 z + (3\gamma_m^2 \delta_m - \delta_m^3) b_3 t].
\end{aligned} \tag{4.3.50}$$

We use Theorem 4.3.7. Any linear (real) combination of waves (4.3.18) solves the bilinear differential equation (4.3.49).

Now we consider two examples of generalized bilinear equations with the D_p-operators under $p = 3$.

Example 4.3.13. *We consider a polynomial*

$$P(x, y, z, t) = c_1x^5 + c_2x^3y + c_3x^2z + c_4xy^2 + c_5xt + c_6yz \tag{4.3.51}$$

with weights

$$(w(x), w(y), w(z), w(t)) = (1, 2, 3, 4). \tag{4.3.52}$$

This example is similar to but different from Example 1 in [44]. The corresponding Hirota bilinear equation is

$$\begin{aligned} & P(D_{3,x}, D_{3,y}, D_{3,z}, D_{3,t})f \cdot f \\ = \; & 2c_1(f_{5x}f - 5f_{4x}f_x + 10f_{3x}f_{2x}) + 6c_2f_{xx}f_{xy} + 2c_3f_{xxz}f + 2c_4f_{xyy} \\ & +2c_5(f_{xt}f - f_xf_t) + 2c_6(f_{yz}f - f_yf_z) = 0. \end{aligned} \tag{4.3.53}$$

Let us take the wave variables

$$\eta_m = k_mx + b_1k_m^2y + b_2k_m^3z + b_3k_m^4t, \quad 1 \le j \le N. \tag{4.3.54}$$

Suppose that the linear subspace of N-wave solutions is

$$f = \sum_{m=1}^{N} \varepsilon_mf_m = \sum_{m=1}^{N} \varepsilon_me^{\eta_m}, \tag{4.3.55}$$

where ε_m, $1 \le m \le N$, are arbitrary constants. The parameters b_1, b_2, b_3 satisfy

$$\begin{cases} c_6b_1b_2 - 3c_2b_1 = 10c_1, \\ c_5b_3 = -5c_1, \\ c_4b_1^2 + c_5b_1b_2 + c_3b_2 + c_5b_3 = -c_1. \end{cases} \tag{4.3.56}$$

By symbolic computation, we get the following result.

1. If $c_1 = 0$, then we have the solution

$$b_1 = 0, \; b_2 = \begin{cases} 0, & if\ c_3 \neq 0, \\ arbitrary, & if\ c_3 = 0, \end{cases} \quad b_3 = \begin{cases} 0, & if\ c_5 \neq 0, \\ arbitrary, & if\ c_5 = 0, \end{cases} \tag{4.3.57}$$

Moreover, when $c_6 \neq 0$ and $\Delta = 9c_2^2 - 12c_2c_3c_4/c_6 \geq 0$, we have another solution

$$b_1,\ b_2 = \frac{3c_2}{c_6},\ b_3 = \begin{cases} 0, & if\ c_5 \neq 0, \\ arbitrary, & if\ c_5 = 0, \end{cases} \tag{4.3.58}$$

where b_1 is a real solution to the quadratic equation

$$c_4 b_1^2 + 3c_2 b_1 + 3c_2c_3/c_6 = 0. \tag{4.3.59}$$

2. If $c_1c_4c_5c_6 \neq 0$, then we have the solution

$$b_2 = \frac{3c_2b_1 + 10c_1}{c_6b_1},\ b_3 = -\frac{5c_1}{c_5}, \tag{4.3.60}$$

where b_1 is a real solution to the equation

$$c_4c_6x^3 + 3c_2c_6x^2 + (3c_2c_3 + 6c_1c_6)x + 10c_1c_3 = 0. \tag{4.3.61}$$

3. If $c_1c_2c_3c_5 \neq 0$, and $c_6 = 0$, then we have the solution

$$b_1 = -\frac{10c_1}{3c_2},\ b_2 = \frac{4c_1}{c_3} - \frac{100c_1^2c_4}{9c_2^2c_3},\ b_3 = -\frac{5c_1}{c_5}. \tag{4.3.62}$$

Moreover, for $N_1 < m \leq N$ we have

$$\begin{aligned} \eta_m &= k_m x + b_1 k_{N+m}^2 y + b_2 k_m^3 z + b_3 k_m^4 t \\ &= \gamma_m x + (\gamma_m^2 - \delta_m^2) b_1 y + (\gamma_m^3 - 3\gamma_m\delta_m^2) b_2 z + (\gamma_m^4 + \delta_m^4 - 6\gamma_m^2\delta_m^2) b_3 t \\ &\quad + i[\delta_m x + 2\gamma_m\delta_m b_1 y + (3\gamma_m^2\delta_m - \delta_m^3) b_2 z + 4(\gamma_m^3\delta_m - \gamma_m\delta_m^3) b_3 t]. \end{aligned} \tag{4.3.63}$$

We can apply Theorem 4.3.7. Any linear (real) combination of waves (4.3.18) solves the bilinear differential equation (4.3.53).

Note We have more examples in [86].

4.4 Concluding remarks

In this chapter, we presented a general scheme for constructing multi-complexitons to Hirota bilinear equations satisfying the Hirota condition. We also established the linear superposition principle to solitons and complexitons of both Hirota and generalized bilinear PDEs. The key is to take pairs of conjugate complex wave variables in formulating real solutions.

Chapter 5

Algebro-geometric solutions to a soliton hierarchy

5.1 Introduction

The modern theory on the integrability of evolution systems was greatly developed after the inverse scattering transform (IST) method been invented. However, IST requires that the potential functions be spatially fast decaying. In the middle of 1970s, with the study of periodic, quasi-periodic and almost periodic solutions to the KdV equation, the new approach to obtain solutions based on algebro-geometric data [6, 11, 13, 22] was developed by B. A. Dubrovin, A. R. Its, I. M. Krichever, V.B. Matveev, S. P. Novikov and others.

There are many discussions about quasi-periodic solutions of different soliton hierarchies [17, 21, 23, 62, 71, 72, 81] for associate with elliptic and hyper-elliptic curves. Recently, explicit quasi-periodic solutions of the entire Kaup-Newell hierarchy were constructed [18]. There are also many researches for quasi-periodic solutions on trigonal curves [20, 27, 28, 51].

In the paper [60], we considered a generalized Kaup-Newell spectral problem possessing two potential functions associated with sl(2,$\mathbb{R}$). In this chapter, we will construct algebro-geometric solutions to the corresponding soliton hierarchy.

5.2 Riemann Surfaces

We need some preliminary knowledge of Riemann surfaces. In this section we will list the most important definitions and theorems about Riemann surfaces. For details, readers can check references [6, 11, 13, 34].

A Riemann surface is a one complex dimensional connected analytic manifold $\mathcal{K}$. The simplest Riemann surfaces are $\mathbb{C}$ and $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. In what follows, we introduce a special class of Riemann surfaces called algebraic curves.

Definition 5.2.1 (Algebraic curve)**.** *Suppose $F(\cdot,\cdot)$ is a nonconstant irreducible polynomial on $\mathbb{C}^2$. Equation $F(\lambda, y) = 0$ defines a Riemann surface $\mathcal{K} := \{(\lambda, y) : F(\lambda, y) = 0\}$, which is also called an algebraic curve. It is nonsingular at $P_0 = (\lambda_0, y_0) \in \mathcal{K}$ if*

$$\mathbf{grad}(F)\big|_{(\lambda_0,y_0)} = \Big(\frac{\partial F}{\partial \lambda}(\lambda_0, y_0), \frac{\partial F}{\partial y}(\lambda_0, y_0)\Big) \neq 0. \tag{5.2.1}$$

The algebraic curve $\mathcal{K}$ is called nonsingular, or smooth, if it is nonsingular at every point of $\mathcal{K}$.

Example 5.2.2 (Elliptic and hyperelliptic curve)**.** *For any integer $N \geq 1$,*

$$y^2 = \prod_{j=1}^{N} (\lambda - E_j), \qquad E_j \neq E_k, j \neq k, j, k = 1, 2, \cdots, N, \tag{5.2.2}$$

defines a quadratic algebraic curve $\mathcal{K}$. It is called an elliptic curve for $N = 3, 4$ and a hyperelliptic curve for $N > 4$.

The nonnegative integer $g = [(N-1)/2]$ is called the genus of the algebraic curve, where $[x]$ is the integer part of x.

In this example, $F(\lambda, y) = y^2 - \prod_{j=1}^{N}(\lambda - E_j)$. When $F_y(\lambda, y) = 0$, we have $y = 0, \lambda = E_j, j = 1, \cdots, N$. The algebraic curve is nonsingular if and only if $E_j, j = 1, \cdots, N$, are distinct.

For any $z = re^{i\theta} \in \mathbb{C}, 0 \leq \theta < 2\pi, r > 0$, we define $\sqrt{z} = \sqrt{r}e^{i\theta/2}$. Then $\sqrt{r} \neq -\sqrt{r}$ and $z = (\pm\sqrt{z})^2$. Let $\lambda \in \mathbb{C}$ and $\lambda \neq E_j, j = 1, \cdots, N$, then there are exact two points $P = (\lambda, \pm\sqrt{z}) \in \mathcal{K}$ with $z = \prod_{j=1}^{N}(\lambda - E_j)$. However, for each $\lambda = E_j, j = 1, \cdots, N$, there is only one point $(E_j, 0) \in \mathcal{K}$. We call $(E_j, 0)$ a branch point of $\mathcal{K}$, $j = 1, \cdots, N$.

We will only study the compact Riemann surface and we will compactify an algebraic curve by joining some points at infinity. For the quadratic algebraic curve, there are one such point for $N = 2g + 1$, and two points denoted by $\infty^{\pm}$ when $N = 2g + 2$, here positive integer g is the genus of $\mathcal{K}$.

We use a biholomorphic map (holomorphic map with holomorphic inverse map) $(\lambda, y) \mapsto (\mu, z)$ in a neighbourhood of infinity $U_\infty = \big\{(\lambda, y) \in \mathcal{K} : |\lambda| > \max_{j=1,\cdots,N} |E_j|\big\}$:

$$z = \frac{y}{\lambda^{g+1}}, \quad \mu = \frac{1}{\lambda}. \tag{5.2.3}$$

The image will be a punctured neighbourhood of the point $(\mu, z) = (0, 0)$ of the curve when $N = 2g + 1$

$$z^2 = \mu \prod_{j=1}^{2g+1} (1 - E_j\mu), \tag{5.2.4}$$

or two punctured neighbourhoods of the point $(\mu, z) = (\pm 1, 0)$ of the curve when $N = 2g + 2$

$$z^2 = \prod_{j=1}^{2g+2} (1 - E_j \mu), \tag{5.2.5}$$

Usually, we can distinguish the two points $\infty^{\pm}$ by

$$P \equiv (\lambda, y) \to \infty^{\pm} \Leftrightarrow \lambda \to \infty, y \sim \pm \lambda^{g+1}. \tag{5.2.6}$$

Definition 5.2.3 (Holomorphic and meromorphic function)**.** *A function $f : \mathcal{K} \to \bar{\mathbb{C}}$ on a Riemann surface is said a meromorphic function if the local notation $f(z) = f \circ \varphi^{-1}(z)$ is a meromorphic function of $z = \varphi(U)$.*

A holomorphic function is a meromorphic function with range in $\mathbb{C}$.

By Liouville's theorem, a holomorphic function on a compact Riemann surface must be a constant and also a meromorphic function on a compact Riemann surface can not have infinitely many poles and thus a meromorphic on a compact Riemann surface is a rational function.

Definition 5.2.4 (Abelian differential)**.** *An Abelian differential on Riemann surfaces is a meromorphic 1-form ω on X, namely, $\omega = f(z)dz$ locally for a meromorphic function of z.*

A holomorphic 1-form is called a differential of the first kind, a meromorphic 1-form with all of its residues vanishing is called a differential of the second kind, and a meromorphic 1-form whose poles are all simple is called a differential of the third kind.

A fundamental property of an Abelian differential ω on a compact Riemann surface is [34, Lemma 5.3.1]

$$\sum_{all\ poles} \mathrm{res}(\omega) = 0. \tag{5.2.7}$$

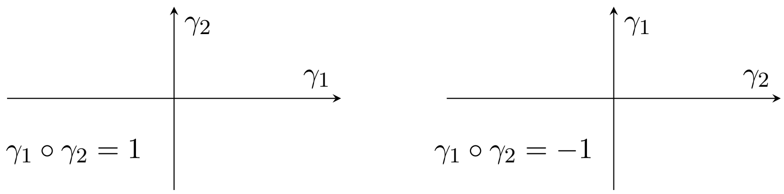
Theorem 5.2.5 (Riemann)**.** *Let $\mathcal{K}$ be a Riemann surface of genus g. Then*

a) the dimension of the space of differentials holomorphic on $\mathcal{K}$ is equal to g;

b) for any finite set $\mathcal{P} := \{P_j \in \mathcal{K}, j = 1, \cdots, k\}$, there is an Abelian differential which is holomorphic on $\mathcal{K} \setminus \mathcal{P}$ and has the poles at $P_j, j = 1, \cdots, k$, with arbitrary preassigned principal parts satisfying (5.2.7)*.*

Since any Abelian differential ω on a Riemann surface is closed. Then a primitive function for ω always exists locally

$$\Omega(P) = \int_{P_0}^{P} \omega.$$



An oriented closed curve on a Riemann surface is called a cycle. At each intersection point of cycles γ_1 and γ_2, we define the intersection number $\gamma_1 \circ \gamma_2 = \pm 1$ according to above figures. We assign $\gamma_1 \circ \gamma_2 = 0$ if cycles γ_1 and γ_2 do not intersect. Note that we always have $\gamma_1 \circ \gamma_2 = -\gamma_2 \circ \gamma_1$.

A canonical basis of the cycles of a Riemann surface $\mathcal{K}$ of genus g consists of cycles: $a_1, b_1, \cdots, a_g$, b_g with the intersection numbers

$$a_j \circ a_k = b_j \circ b_k = 0, \quad a_j \circ b_k = \delta_{jk}, \quad j, k = 1, 2 \cdots, g.$$

Let $\Omega(P)$ be a holomophic function. Then we have A-periods and B-periods of differential $d\Omega$:

$$A_j := \int_{a_j} d\Omega, \quad B_j := \int_{b_j} d\Omega, \quad 1 \leq j \leq g. \tag{5.2.8}$$

If $\gamma = \sum_j (n_j a_j + m_j b_j), m_j, n_j \in \mathbb{Z}$, is a cycle then define

$$\int_\gamma d\Omega = \sum_j (n_j A_j + m_j B_j). \tag{5.2.9}$$

Basically, we have the Riemann bilinear relation

Theorem 5.2.6 (Riemann's bilinear identity)**.** *Let $a_1, b_1, \cdots, a_g, b_g$ be a canonical basis of the cycles of a Riemann surface $\mathcal{K}$ of genus g. For any two Abelian integrals Ω and Ω', let A_j, B_j, A'_j, B'_j, $j = 1, \cdots, g$ be their periods. We have*

$$\int_{\partial X \setminus \cup_{k=1}^g (a_k \cup b_k)} \Omega' d\Omega = \sum_{k=1}^g (A'_k B_k - A_k B'_k). \tag{5.2.10}$$

Further more, if Ω and Ω' are all holomorphic. Then

$$\sum_{k=1}^g (A'_k B_k - A_k B'_k) = 0. \tag{5.2.11}$$

We select a basis for Abelian differentials

$$\tilde{\omega}_j := \frac{\lambda^{j-1} d\lambda}{y}, \quad 1 \leq j \leq g. \tag{5.2.12}$$

Then the period matrices $\tilde{A} = [\tilde{A}_{kj}]_{g\times g}$ and $\tilde{B} = [\tilde{B}_{kj}]_{g\times g}$ have the following entries:

$$\tilde{A}_{kj} = \int_{a_j} \tilde{\omega}_k, \quad \tilde{B}_{kj} = \int_{b_j} \tilde{\omega}_k. \tag{5.2.13}$$

Since $\tilde{A}$ is invertible [6], set

$$C := \tilde{A}^{-1} = [C_{kj}]_{g\times g}, \tag{5.2.14}$$

let $B := C\tilde{B}$. We normalize $\tilde{\omega}_j$ with

$$\omega_j = \sum_{k=1}^{g} C_{jk}\tilde{\omega}_k, \quad 1 \le j \le g, \tag{5.2.15}$$

and then we have

$$\int_{a_k} \omega_j = \sum_{l=1}^{g} C_{jl} \int_{a_k} \tilde{\omega}_l = \delta_{jk}, \quad \int_{b_k} \omega_j = B_{jk}. \tag{5.2.16}$$

We can choose a basis for Abelian differentials $\omega_1, \cdots, \omega_g$ such that the period matrix are I_g and B. The period matrix B has following properties.

Theorem 5.2.7 ([34]). *The period matrix B is symmetric ($B = B^T$) and the imaginary part is positive definite ($\mathrm{Im}(B) > 0$), i.e. $x^T\mathrm{Im}(B)x > 0, \forall x \in \mathbb{R}^g, x \neq 0$.*

Based on the period matrix B of the Riemann surface X, we define a lattice in $\mathbb{C}^g$

$$L_g := \{N + BM : N, M \in \mathbb{Z}^g\},$$

and propose an equivalence relation in $\mathbb{C}^g$

$$x \sim x' \Leftrightarrow x - x' \in L_g.$$

The g-dimensional torus $J(\mathcal{K}) := \mathbb{C}^g/L_g$ is named as a Jacobian variety.

Let $P_0 \in \mathcal{K}$ be a base point. The Abelian mapping is defined by

$$\mathcal{A}(P) := \int_{P_0}^{P} \omega = \Big(\int_{P_0}^{P} \omega_1, \cdots, \int_{P_0}^{P} \omega_g\Big)^T \in J(\mathcal{K}). \tag{5.2.17}$$

Definition 5.2.8 (Divisor). *A divisor on a Riemann surface $\mathcal{K}$ is a formal finite sum of points on it $D = \sum_{k=1}^{m} n_k P_k, P_k \in \mathcal{K}, n_k \in \mathbb{Z}$. The number $\deg(D) := \sum_{k=1}^{m} n_k$ is called the degree of D.*

It is not difficult to show that the divisors on $\mathcal{K}$ form a group under pointwise addition denoted $\mathrm{Div}(\mathcal{K})$.

The Abelian mapping of a divisor $D = \sum_{k=1}^{m} n_k P_k$ is defined by

$$\mathcal{A}(D) := \sum_{k=1}^{m} n_k \mathcal{A}(P_k).$$

A positive divisor

$$D = \sum_{k=1}^{m} n_k P_k \geq 0 \Leftrightarrow n_k \geq 0, k = 1, \cdots, g,$$

and a partial order of divisors is determined by

$$D \leq D' \Leftrightarrow D' - D \geq 0.$$

The divisor of a meromorphic function f on $\mathcal{K}$ is denoted by $(f) := \sum_k n_k P_k$ where $n_k > 0$ if P_k is a zero of f with multiplicity of n_k and $n_k < 0$ if P_k is a pole of f with multiplicity of $-n_k$.

A divisor D is said to be principal if $D = (f)$ for some function f on $\mathcal{K}$.

Theorem 5.2.9 (Abel's theorem)**.** *Suppose D is a divisor of $\mathcal{K}$ with deg$(D) = 0$. Then the divisor is principal if and only if $\mathcal{A}(D) = 0$.*

Theorem 5.2.10 (Riemann-Roch theorem)**.** *If $\mathcal{K}$ is a Riemann surface of genus g and $D \in Div(\mathcal{K})$. Then*

$$\dim F_{-D} - \dim(d\Omega_D) = 1 - g + \deg(D), \tag{5.2.18}$$

where F_D is a linear space of meromorphic functions f on $\mathcal{K}$ and divisible by D (i.e. $(f) \geq D$); $d\Omega_D$ is a linear space of meromorphic differentials ω on $\mathcal{K}$ and divisible by D (i.e. $(\omega) \geq D$).

Definition 5.2.11. *The divisor $D > 0, \deg(D) \leq g$, is said special if $\dim F_{-D} > 1$. Otherwise $\dim F_{-D} = 1$, it is called non-special.*

Suppose that the matrix B satisfies $B = B^T$ and $\mathrm{Im}(B) > 0$. A Riemann theta function is defined by its Fourier series for $z \in \mathbb{C}^g$

$$\theta(z; B) := \sum_{m \in \mathbb{Z}^g} \exp\Big\{\pi i \langle Bm, m\rangle + 2\pi i \langle z, m\rangle\Big\}, \tag{5.2.19}$$

where $\langle x, y\rangle = \sum_{j=1}^{g} x_j \bar{y}_j$ is the inner product in $\mathbb{C}^g$.

The Riemann theta function satisfies following properties

$$\theta(z_1, \cdots, z_{j-1}, -z_j, z_{j+1}, \cdots, z_g; B) = \theta(z; B), \tag{5.2.20}$$

for $j = 1, \cdots, g$, $z = (z_1, \cdots, z_g)^T \in \mathbb{C}^g$, and

$$\theta(z + n + Br; B) = e^{-\pi i <Br,r> - 2\pi i <z,r>} \theta(z; B), \tag{5.2.21}$$

for $n, r \in \mathbb{Z}^g$.

Theorem 5.2.12 (Riemann Theorem)**.** *Let a Riemann surface $\mathcal{K}$ of genus g be equipped with a canonical basis $a_1, b_1, \cdots, a_g, b_g$ and a vector of Riemann constants K with components*

$$K_j = \frac{2\pi i + B_{jj}}{2} - \frac{1}{2\pi i} \sum_{k \neq j} \Big(\int_{a_k} \omega_k(P) \int_{P_0}^{P} \omega_j \Big), \quad j = 1, \cdots, g. \tag{5.2.22}$$

Let $\zeta = (\zeta_1, \cdots, \zeta_g)^T \in J(\mathcal{K})$ such that

$$F(P) := \theta(\int_{P_0}^{P} \omega - \zeta - K; B)$$

does not vanish identically on $\mathcal{K}$. Then

a) the Riemann theta function F has exactly g zeros $P_1, \cdots, P_g$ that solve the Jacobi inverse problem

$$\sum_{k=1}^{g} \int_{P_0}^{P_k} \omega_j = \zeta_j, \quad j = 1, \cdots, g; \tag{5.2.23}$$

b) the divisor $D = P_1 + \cdots + P_g$ is non-special;

c) the points $P_j, j = 1, \cdots, g$, are uniquely determined up to a permutation.

5.3 The general Kaup-Newell hierarchy from sl(2,$\mathbb{R}$)

In this section we will derive a soliton hierarchy from the matrix loop algebra $\widetilde{\mathrm{sl}}(2, \mathbb{R})$. We begin with a spectral problem and let e_1, e_2, e_3 be defined by (2.1.11).

Let α be an arbitrary real constant. Let us introduce a spectral matrix

$$U = U(u, \lambda) = (\lambda + \alpha)e_1 + pe_2 + qe_3 = \begin{bmatrix} \lambda + \alpha & \lambda p \\ q & -\lambda - \alpha \end{bmatrix}, \tag{5.3.1}$$

and consider the following isospectral problem

$$\phi_x = U\phi, \ u = \begin{bmatrix} p \\ q \end{bmatrix}, \ \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}, \tag{5.3.2}$$

associated with $\widetilde{\mathrm{sl}}(2, \mathbb{R})$.

Define

$$W = ae_1 + be_2 + ce_3 = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \in \widetilde{\mathrm{sl}}(2, \mathbb{R}), \tag{5.3.3}$$

and then, the stationary zero curvature equation $W_x = [U, W]$ becomes

$$\begin{cases} a_x = \lambda pc - qb, \\ b_x = 2(\lambda + \alpha)b - 2\lambda pa, \\ c_x = 2qa - 2(\lambda + \alpha)c. \end{cases} \tag{5.3.4}$$

We further assume that

$$a = \sum_{i\geq 0} a_i \lambda^{-i},\ b = \sum_{i\geq 0} b_i \lambda^{-i},\ c = \sum_{i\geq 0} c_i \lambda^{-i-1}, \tag{5.3.5}$$

and take the initial values

$$a_0 = 1,\ b_0 = p,\ c_0 = q.$$

Now based on (5.3.4), we have

$$\begin{cases} a_{i,x} = pc_i - qb_i, \\ b_{i,x} = 2\alpha b_i + 2b_{i+1} - 2pa_{i+1}, \quad i \geq 0. \\ c_{i,x} = 2qa_{i+1} - 2\alpha c_i - 2c_{i+1}, \end{cases} \tag{5.3.6}$$

From this, we can derive the recursion relations for $i \geq 0$:

$$\begin{cases} a_{i+1,x} = \alpha qb_i - \alpha pc_i - \frac{q}{2} b_{i,x} - \frac{p}{2} c_{i,x}, \\ b_{i+1} = \frac{1}{2} b_{i,x} - \alpha b_i + pa_{i+1}, \\ c_{i+1} = qa_{i+1} - \frac{1}{2} c_{i,x} - \alpha c_i. \end{cases} \tag{5.3.7}$$

Note The recursion relations (5.3.7) we derived here is the same as that in [60].

We impose the conditions for integration:

$$a_i|_{u=0} = b_i|_{u=0} = c_i|_{u=0} = 0,\ i \geq 1, \tag{5.3.8}$$

to determine the sequence of $\{a_i, b_i, c_i | i \geq 1\}$ uniquely. Based on the recursion relations (5.3.7), we can have

$$\begin{bmatrix} c_{i+1} \\ b_{i+1} \end{bmatrix} = \Psi \begin{bmatrix} c_i \\ b_i \end{bmatrix},\ i \geq 0, \tag{5.3.9}$$

where

$$\Psi = \begin{bmatrix} -\alpha - \frac{1}{2}\partial - \alpha q\partial^{-1}p - \frac{1}{2}q\partial^{-1}p\partial & \alpha q\partial^{-1}q - \frac{1}{2}q\partial^{-1}q\partial \\ -\alpha p\partial^{-1}p - \frac{1}{2}p\partial^{-1}p\partial & -\alpha + \frac{1}{2}\partial + \alpha p\partial^{-1}q - \frac{1}{2}p\partial^{-1}q\partial \end{bmatrix}, \tag{5.3.10}$$

in which $\partial = \frac{\partial}{\partial x}$. We will see that all vectors $(c_i, b_i)^T,\ i \geq 0$, are gradient, and will generate conserved functionals.

Now for each $m \geq 0$, we introduce

$$\begin{aligned} V^{[m]} &= (\lambda^m W)_+ + c_m e_3 \\ &= \sum_{i=0}^{m} [a_i \lambda^{m-i+1} e_1 + b_i \lambda^{m-i+1} e_2 + c_i \lambda^{m-i} e_3], \end{aligned} \tag{5.3.11}$$

and the corresponding zero curvature equations

$$U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0,\ m \geq 0, \tag{5.3.12}$$

engender a hierarchy of solution equations

$$u_{t_m} = K_m = \begin{bmatrix} b_{m,x} - 2\alpha b_m \\ c_{m,x} + 2\alpha c_m \end{bmatrix} = J \begin{bmatrix} c_m \\ b_m \end{bmatrix},\ m \geq 0, \tag{5.3.13}$$

where

$$J = \begin{bmatrix} 0 & \partial - 2\alpha \\ \partial + 2\alpha & 0 \end{bmatrix}. \tag{5.3.14}$$

It is obvious that J is a Hamiltonian operator, since it is skew-adjoint and does not depend on the potentials [68]. When $m = 0, 1, 2$, we have the systems

$$\begin{aligned} u_{t_0} &= \begin{bmatrix} p \\ q \end{bmatrix}_{t_0} = \begin{bmatrix} p_x - 2\alpha p \\ q_x + 2\alpha q \end{bmatrix}, \\ u_{t_1} &= \begin{bmatrix} p \\ q \end{bmatrix}_{t_1} = \begin{bmatrix} 2\alpha^2 p + \alpha p^2 q + \frac{1}{2}p_{xx} - 2\alpha p_x - pqp_x - \frac{1}{2}p^2 q_x \\ -2\alpha^2 q - \alpha pq^2 - \frac{1}{2}q_{xx} - 2\alpha q_x - pqq_x - \frac{1}{2}p_x q^2 \end{bmatrix}, \\ u_{t_2} &= \begin{bmatrix} p \\ q \end{bmatrix}_{t_2} = \begin{bmatrix} -2\alpha^3 p + 3\alpha p^2 p_x - \frac{3}{2}\alpha p_{xx} + \frac{1}{4}p_{xxx} + \frac{9}{2}\alpha pp_x q + \frac{3}{2}\alpha p^2 q_x - \frac{3\alpha^2}{2}p^2 q \\ -\frac{3\alpha}{4}p^3 q^2 + \frac{3}{4}(p_x^2 q + pp_{xx}q + pp_x q_x) + \frac{3p^2 q}{8}(3p_x q + 2pq_x) \\ 2\alpha^3 q + 3\alpha^2 q_x + \frac{3\alpha}{2}q_{xx} + \frac{1}{4}q_{xxx} + \frac{9}{2}pqq_x + \frac{3\alpha}{2}p_x q^2 + \frac{3\alpha}{2}pq^2 \\ +\frac{3}{8}p^2 q^3 + \frac{3}{4}(p_x qq_x + pq_x^2 + pqq_{xx}) + \frac{2pq^2}{8}(2p_x q + 3pq_x) \end{bmatrix}. \end{aligned}$$

5.4 Hyperelliptic curve and Baker-Akhiezer functions

5.4.1 Hyperelliptic curve

For a fixed integer $m \in \mathbb{N}$, let us introduce a Lax matrix (see [17]) $W = \begin{bmatrix} G & F \\ H & -G \end{bmatrix}$ satisfying

$$W_x = [U, W], \quad W_{t_m} = [V^{[m]}, W]. \tag{5.4.1}$$

Lemma 5.4.1. *Suppose $U(x) \in \mathbb{C}^{n\times n}$ is continuous on (a, b) and $W(x)$ is a matrix solution of $W_x = UW - WU$. Then* $\det(W)_x = 0$.

Proof. Take $x_0 \in (a, b)$. Let $X_1(x, x_0), X_2(x, x_0)$ be the fundamental solutions of linear equations $\frac{d}{dx}y(x) = U(x)y(x)$ and $\frac{d}{dx}y(x) = -y(x)U(x)$ respectively. Let $A \in \mathbb{C}^{n\times n}$ be a constant matrix. Then it is easy to verify that $X(x) := X_1(x, x_0)AX_2(x, x_0)$ solves $W_x = UW - WU$ with $X(x_0) = A$. By the Liouville's formula

$$\det(X_1(x, x_0)) = \exp\left(\int_{x_1}^{x} \mathrm{tr}(U(s))ds\right), \quad \det(X_2(x, x_0)) = \exp\left(\int_{x_1}^{x} \mathrm{tr}(-U(s))ds\right). \tag{5.4.2}$$

Therefore $\det(X(x)) = \exp(\int_{x_1}^{x} \mathrm{tr}(U(s))ds) \det(A) \exp(\int_{x_1}^{x} \mathrm{tr}(-U(s))ds) = \det(A)$. Since A is a constant matrix, we get

$$[\det(W)]_x = 0, \tag{5.4.3}$$

and thus $\det(W)$ is independent of x. □

Similarly, we have

$$[\det(W)]_{t_m} = 0.$$

Therefore, when (5.4.1) is true, then $\det(W)$ is a function of λ, which is independent of (x, t).

By (5.4.1), we have

$$\begin{aligned} G_x &= \lambda pH - qF, \\ F_x &= 2(\lambda + \alpha)F - 2\lambda pG, \\ H_x &= 2qG - 2(\lambda + \alpha)H. \end{aligned} \tag{5.4.4}$$

Let $n \in \mathbb{N}$ also be fixed in this section, and assume that

$$G = \sum_{j=0}^{n} g_j \lambda^{n+1-j}, \quad F = \sum_{j=0}^{n} f_j \lambda^{n+1-j}, \quad H = \sum_{j=0}^{n} h_j \lambda^{n-j}.$$

We denote

$$W_j = \begin{bmatrix} g_j & f_j \\ h_j & -g_j \end{bmatrix}, \quad j = 0, 1, \cdots, n.$$

It is easy to see

$$W_0 = \alpha_0 \begin{bmatrix} 1 & p \\ q & -1 \end{bmatrix} \tag{5.4.5}$$

and

$$W_n = \begin{bmatrix} \beta_1 + \partial^{-1}(\beta_3 p e^{-2\alpha x} - \beta_2 q e^{2\alpha x}) & \beta_2 e^{2\alpha x} \\ \beta_3 e^{-2\alpha x} & -\beta_1 - \partial^{-1}(\beta_3 p e^{-2\alpha x} - \beta_2 q e^{2\alpha x}) \end{bmatrix}, \tag{5.4.6}$$

where $\beta_j, j = 1, 2, 3$, are arbitrary constants. Without loss of generality, we take $\alpha_0 = 1$ and we obtain from (5.3.4) and (5.4.4),

$$W_k = \sum_{j=0}^{k} \alpha_j V_{k-j}, \quad 0 \leq k \leq n, \tag{5.4.7}$$

where $V_j = \begin{bmatrix} a_j & b_j \\ c_j & -a_j \end{bmatrix}$ and $\alpha_1, \cdots, \alpha_n$ are integration constants.

Since $-\det(W) = G^2 + FH$ is a polynomial of λ with degree $2n+2$ and leading coefficient 1. We denote

$$R(\lambda) := G^2 + FH = \lambda \prod_{j=1}^{2n+1} (\lambda - \lambda_j) = \prod_{j=0}^{2n+1} (\lambda - \lambda_j), \tag{5.4.8}$$

where $\lambda_0 = 0$. Consider the characteristic function of W

$$|yI_2 - W| = y^2 - G^2 - FH = y^2 - R(\lambda). \tag{5.4.9}$$

We introduce the algebraic curve of genus n

$$\mathcal{K}_n = \{(\lambda, y) : \ y^2 - R(\lambda) = 0\} \tag{5.4.10}$$

with $P_0 = (0, 0) \in \mathcal{K}_n$ and the curve is compactified by joining two distinct infinity points $P_{\infty+}$ and $P_{\infty-}$. We further assume that $\lambda_j, j = 0, \cdots, 2n+1$, are distinct and thus $\mathcal{K}_n$ is non-singular. For notational simplicity, the Riemann surface after compactification is still denoted by $\mathcal{K}_n$. The set of branch points of $\mathcal{K}_n$ is given by $\{(\lambda_k, 0) : k = 0, 1, \cdots, 2n+1\}$.

The branch of y near $P_{\infty\pm}$ is fixed, according to

$$\lim_{P \to P_{\infty\pm}} \frac{y(P)}{G(P, x, t_m)} = \mp 1. \tag{5.4.11}$$

We define the holomorphic sheet exchange map (involution)

$$*: \mathcal{K}_n \to \mathcal{K}_n, \quad P = (\lambda, y) \mapsto P^* = (\lambda, -y), \;\; P_{\infty\pm} \mapsto P^*_{\infty\pm} = P_{\infty\mp}. \tag{5.4.12}$$

We define the Baker-Akhiezer (BA) function $\Psi(P, x, x_0, t_m, t_{m,0})$ to be the spectral function satisfying

$$\Psi_x(P, x, x_0, t_m, t_{m,0}) = U(u(x, t_m), \lambda(P))\Psi(P, x, x_0, t_m, t_{m,0}), \tag{5.4.13}$$

$$\Psi_{t_m}(P, x, x_0, t_m, t_{m,0}) = V^{[m]}(u(x, t_m), \lambda(P))\Psi(P, x, x_0, t_m, t_{m,0}), \tag{5.4.14}$$

$$W(u(x, t_m), \lambda(P))\Psi(P, x, x_0, t_m, t_{m,0}) = y(P)\Psi(P, x, x_0, t_m, t_{m,0}). \tag{5.4.15}$$

Since the above three equations are all linear, we assume that the first component of $\Psi = (\psi_1, \psi_2)^T$ satisfies

$$\psi_1(P, x_0, x_0, t_{m,0}, t_{m,0}) = 1$$

for fixed $(x_0, t_{m,0}) \in \mathbb{R}^2$ to achieve the uniqueness.

5.4.2 Characteristic variables and Dubrovin type equations

By the initial conditions, we introduce the elliptic variables $\{\mu_j : j = 1, \cdots, n\}$ and $\{\nu_j : j = 1, \cdots, n\}$ and take

$$F = p\lambda \prod_{j=1}^{n} (\lambda - \mu_j), \quad H = q \prod_{j=1}^{n} (\lambda - \nu_j). \tag{5.4.16}$$

We can lift the elliptic variables to $\mathcal{K}_n$ by

$$\hat{\mu}_j(x, t_m) = (\mu_j(x, t_m), -G(\mu_j(x, t_m), x, t_m)), \tag{5.4.17}$$

$$\hat{\nu}_j(x, t_m) = (\nu_j(x, t_m), G(\nu_j(x, t_m), x, t_m)) \tag{5.4.18}$$

for $j = 1, 2, \cdots, n$ and $(x, t_m) \in \mathbb{R}^2$. Hence by (5.4.8)

$$G|_{\lambda=\mu_j} = \sqrt{R(\mu_j)}, \quad G|_{\lambda=\nu_j} = \sqrt{R(\nu_j)}. \tag{5.4.19}$$

Now we study the dynamics of elliptic variables $\{\mu_j : j = 1, \cdots, n\}$ and $\{\nu_j : j = 1, \cdots, n\}$.

Theorem 5.4.2. *The elliptic variables satisfy the Dubrovin-type equations for $j = 1, \cdots, n$:*

$$\mu_{j,x} = \frac{2\sqrt{R(\mu_j)}}{\prod_{k=1, k\neq j}^{n} (\mu_j - \mu_k)}, \quad \nu_{j,x} = -\frac{2\sqrt{R(\nu_j)}}{\prod_{k=1, k\neq j}^{n} (\nu_j - \nu_k)}, \tag{5.4.20}$$

and

$$\mu_{j,t_m} = \frac{2b^{[m]}(\mu_j)\sqrt{R(\mu_j)}}{p\mu_j \prod_{k=1,k\neq j}^{n}(\mu_j-\mu_k)}, \quad \nu_{j,t_m} = -\frac{2c^{[m]}(\nu_j)\sqrt{R(\nu_j)}}{q \prod_{k=1,k\neq j}^{n}(\nu_j-\nu_k)}. \tag{5.4.21}$$

Proof. Taking derivative with respect to x, we get

$$F_x|_{\lambda=\mu_j} = -p\mu_j\mu_{j,x}\prod_{k=1,k\neq j}^{n}(\mu_j-\mu_k) = -2p\mu_j G|_{\lambda=\mu_j}, \tag{5.4.22}$$

$$H_x|_{\lambda=\nu_j} = -q\nu_{j,x}\prod_{k=1,k\neq j}^{n}(\nu_j-\nu_k) = 2q\nu_j G|_{\lambda=\nu_j}. \tag{5.4.23}$$

We can easily get (5.4.20)

With the same approach and by

$$\begin{aligned} G_{t_m} &= \lambda b^{[m]}H - c^{[m]}F, \\ F_{t_m} &= 2(a^{[m]}F - b^{[m]}G), \\ H_{t_m} &= 2(c^{[m]}G - a^{[m]}H), \end{aligned} \tag{5.4.24}$$

we have (5.4.21) □

In order to straighten out of the corresponding flows, we equip $\mathcal{K}_n$ with the canonical basis cycles: $a_1, \cdots, a_n; b_1, \cdots, b_n$. We can select a basis ω_j, $j = 1, \cdots, n$ such that

$$\int_{a_k}\omega_j = \delta_{jk}, \quad \int_{b_k}\omega_j = B_{jk}, \tag{5.4.25}$$

where $B = (B_{jk})$ is a period matrix. By section 5.2, we know when the period matrix $\tilde{A}$ is defined by (5.2.13), we define matrix C is inverse of $\tilde{A}$. We can define the period lattice Λ_g and the Jacobian variety of $J(\mathcal{K}_n)$.

We choose a branch point $Q_0 = (\lambda_{j_0}, 0)$ for some $1 \leq j_0 \leq 2n+1$ as a base point, and assume that $\lambda(Q_0)$ is its local coordinate.

Denote the Abel map for $P \in \mathcal{K}_n$:

$$\mathcal{A}(P) \quad : \quad Div(\mathcal{K}_n) \to J(\mathcal{K}_n), \tag{5.4.26}$$

$$\mathcal{A}(P) \quad = \quad \int_{Q_0}^{P}\omega \mod \Lambda_g. \tag{5.4.27}$$

For two special divisors $\sum_{k=1}^{n} P_k^{(l)}, l = 1, 2, k = 1, \cdots, n$, with $P_k^{(1)} = \hat{\mu}_k(x, t_m)$ and $P_k^{(2)} = \hat{\nu}_k(x, t_m)$, we define the Abel-Jacobi coordinates as

$$\rho^{(l)} := \mathcal{A}\left(\sum_{k=1}^{n} P_k^{(l)}\right) = \sum_{k=1}^{n} \mathcal{A}\left(P_k^{(l)}\right) = \sum_{k=1}^{n} \int_{Q_0}^{P_k^{(l)}} \omega. \tag{5.4.28}$$

Theorem 5.4.3. *Assume that $\mu_j \neq \mu_k$ and $\nu_j \neq \nu_k$ for $j \neq k, 1 \leq j, k \leq n$. Then*

$$\rho^{(1)} = 2C_n x + \Omega_m t_m + \rho_0^{(1)}, \tag{5.4.29}$$

$$\rho^{(2)} = -2C_n x - \Omega_m t_m + \rho_0^{(2)}, \tag{5.4.30}$$

where $\rho_0^{(1)}, \rho_0^{(2)} \in \mathbb{R}^n$, and $C_k = (C_{1k}, \cdots, C_{nk})^T, 1 \leq k \leq n$, with components C_{jk} defiend by (5.2.14) *and*

$$\Omega_m = 2\sum_{j=0}^{m} \gamma_j C_{n-m+l}, \quad 0 \leq m \leq n-1, \tag{5.4.31}$$

with

$$\gamma_0 = 1, \quad \gamma_1 = -\alpha_1, \cdots, \gamma_k = -\sum_{j=1}^{k} \alpha_j \gamma_{k-j}. \tag{5.4.32}$$

Proof. Since for $1 \leq l \leq n$ it is true that

$$\sum_{k=1}^{n} \frac{\mu_k^{l-1}}{\prod_{r=1, r\neq k}^{n} (\mu_k - \mu_r)} = \delta_{ln},$$

we then have

$$\partial_x \rho_j^{(1)} = \sum_{k=1}^{n}\sum_{l=1}^{n} C_{jl} \frac{\mu_k^{l-1}\mu_{k,x}}{\sqrt{R(\mu_k)}} = \sum_{l=1}^{n}\sum_{k=1}^{n} \frac{2C_{jl}\mu_k^{l-1}}{\prod_{r=1, r\neq k}^{n} (\mu_k - \mu_r)} = 2C_{jn}, \quad 1 \leq j \leq n. \tag{5.4.33}$$

By the relation $f_k = \sum_{j=0}^{k} \alpha_j b_{k-j}^{[m]}$ and $\alpha_0 = 1$, we have (which can be verified by mathematical induction)

$$b_k^{[m]} = \sum_{j=0}^{k} \gamma_j f_{k-j} \tag{5.4.34}$$

with $\gamma_j = 1, 0 \leq j \leq k$ are defined by (5.4.32).

Since $f_0 = b_0^{[m]} = p$, we get for $1 \le j \le n$,

$$\begin{aligned}
\partial_{t_m}\rho_j^{(1)} &= \sum_{k=1}^{n}\sum_{l=1}^{n} C_{jl}\frac{\mu_k^{l-1}\mu_{k,t_m}}{\sqrt{R(\mu_k)}} = \sum_{l=1}^{n}\sum_{k=1}^{n}\frac{2C_{jl}\mu_k^{l-2}b^{[m]}}{p\prod_{r=1,r\neq k}^{n}(\mu_k-\mu_r)} \\
&= \sum_{l=1}^{n}\sum_{k=1}^{n}\frac{2C_{jl}\mu_k^{l-2}}{p\prod_{r=1,r\neq k}^{n}(\mu_k-\mu_r)}\sum_{i=0}^{m} b_i^{[m]}\mu_k^{m+1-i} \\
&= \sum_{l=1}^{n}\sum_{k=1}^{n}\frac{2C_{jl}}{p\prod_{r=1,r\neq k}^{n}(\mu_k-\mu_r)}\sum_{i=0}^{m}\left(\sum_{i'=0}^{i}\gamma_{i'}f_{i-i'}\right)\mu_k^{m+l-i-1} \\
&= 2\sum_{l=0}^{m}\gamma_l C_{j,n-m+l}. \qquad (5.4.35)
\end{aligned}$$

Similarly

$$\partial_x\rho_j^{(2)} = \sum_{k=1}^{n}\sum_{l=1}^{n} C_{jl}\frac{\nu_k^{l-1}\nu_{k,x}}{\sqrt{R(\nu_k)}} = -\sum_{l=1}^{n}\sum_{k=1}^{n}\frac{2C_{jl}\nu_k^{l-1}}{\prod\limits_{r=1,r\neq k}^{n}(\nu_k-\nu_r)} = -2C_{jn}, \quad 1\le j\le n, \quad (5.4.36)$$

and

$$\begin{aligned}
\partial_{t_m}\rho_j^{(2)} &= \sum_{k=1}^{n}\sum_{l=1}^{n} C_{jl}\frac{\nu_k^{l-1}\nu_{k,t_m}}{\sqrt{R(\nu_k)}} = -\sum_{l=1}^{n}\sum_{k=1}^{n}\frac{2C_{jl}\nu_k^{l-1}b^{[m]}}{q\prod\limits_{r=1,r\neq k}^{n}(\nu_k-\nu_r)} \\
&= -\sum_{l=1}^{n}\sum_{k=1}^{n}\frac{2C_{jl}\nu_k^{l-1}}{q\prod\limits_{r=1,r\neq k}^{n}(\nu_k-\nu_r)}\sum_{i=0}^{m} c_i\nu_k^{m-i} \\
&= -2\sum_{l=0}^{m}\gamma_l C_{j,n-m+l}. \qquad (5.4.37)
\end{aligned}$$

The proof is finished. □

5.4.3 Algebro-geometric solutions

In this subsection, we will find quasi-periodic solutions to the gKN hierarchy.

By $y^2 = G^2 + FH$, we have $(y-G)(y+G) = FH$, and we consider the meromorphic function $\phi(\cdot, x, t_m)$ on $\mathcal{K}_n$

$$\phi(P, x, t_m) := \frac{\psi_2(P, x, x_0, t_m, t_{m,0})}{\psi_1(P, x, x_0, t_m, t_{m,0})} \qquad (5.4.38)$$

By (5.4.15), we have

$$\phi(x, t_m) = \frac{y-G}{F} = \frac{H}{y+G},$$

where $P=(\lambda,y)\in\mathcal{K}_n\setminus\{P_{\infty+},P_{\infty-}\}$.

It is easy to verify the following lemma.

Lemma 5.4.4. *Suppose that (p,q) satisfies the nth generalized KN (gKN) equation* (5.4.4)*, and let $P=(\lambda,y)\in\mathcal{K}_n\setminus\{P_0,P_{\infty+},P_{\infty-}\}$. Then ϕ satisfies the Riccati-type equation*

$$\phi_x(P,x,t_m)=q(x,t_m)-2(\lambda+\alpha)\phi_x(P,x,t_m)-\lambda p(x,t_m)\phi_x(P,x,t_m)^2, \tag{5.4.39}$$

and

$$\begin{aligned}\phi(P,x,t_m)\phi(P^*,x,t_m) &= -\frac{H(\lambda,x,t_m)}{F(\lambda,x,t_m)}, &(5.4.40)\\ \phi(P,x,t_m)+\phi(P^*,x,t_m) &= -\frac{2G(\lambda,x,t_m)}{F(\lambda,x,t_m)}, &(5.4.41)\\ \phi(P,x,t_m)-\phi(P^*,x,t_m) &= \frac{2y}{F(\lambda,x,t_m)}. &(5.4.42)\end{aligned}$$

Now we discuss the asymptotics of the function ϕ at $P_0,P_{\infty+}$ and $P_{\infty-}$.

Theorem 5.4.5. *Suppose that (p,q) satisfies the nth gKN equation* (5.4.4)*, and let $P=(\lambda,y)\in\mathcal{K}_n\setminus\{P_0,P_{\infty+},P_{\infty-}\}$. Then*

$$\phi=\begin{cases}-\dfrac{2}{p}+\Big(-\dfrac{q}{2}+\dfrac{p_x}{p}-\dfrac{2\alpha}{p}\Big)\zeta^2+O(\zeta^2), & P\to P_{\infty+},\zeta=\lambda^{-1},\\ -\dfrac{q}{2}+\Big(-\dfrac{\alpha q}{2}+\dfrac{q_x}{4}+\dfrac{pq^2}{8}\Big)\zeta^2+O(\zeta^3), & P\to P_{\infty-},\zeta=\lambda^{-1},\\ \sqrt{\dfrac{\beta_3}{\beta_2}}e^{-2\alpha x}\zeta^{-1}+O(1), & P\to P_0,\ \zeta^2=\lambda.\end{cases}$$

Proof. Firstly, we consider $P\to P_{\infty\pm}$, and let $\zeta=1/\lambda$.

$$y=\mp\sqrt{R(\lambda)}=\mp\zeta^{-n-1}(1+\alpha_1\zeta+\alpha_2\zeta^2+O(\zeta^3)),\ as\ P\to P_{\infty\pm}, \tag{5.4.43}$$

and

$$\begin{aligned}F^{-1} &= \zeta^{n+1}(f_0+f_1\zeta+O(\zeta^2))^{-1}=\zeta^{n+1}(f_0^{-1}-f_0^{-2}f_1\zeta+O(\zeta^2)),\\ G &= \zeta^{-n-1}(g_0+g_1\zeta+g_2\zeta^2+O(\zeta^3)). &(5.4.44)\end{aligned}$$

Then when $P\to P_{\infty+}$, we have

$$\begin{aligned}\phi &= \frac{y-G}{F}\\ &= -((g_0+1)+(g_1+\alpha_1)\zeta+O(\zeta^2))(f_0^{-1}-f_0^{-2}f_1\zeta+O(\zeta^2))\\ &= -\frac{2}{p}+\Big(-\frac{q}{2}+\frac{p_x}{p^2}-\frac{2\alpha}{p}\Big)\zeta+O(\zeta^2). &(5.4.45)\end{aligned}$$

When $P \to P_{\infty-}$, we get

$$\begin{aligned}
\phi &= \frac{y-G}{F} \\
&= ((g_1-\alpha_1)\zeta + (g_2-\alpha_2)\zeta + O(\zeta^3))(f_0^{-1} - f_0^{-2} f_1 \zeta + O(\zeta^2)) \\
&= -\frac{q}{2}\zeta + \Big(\frac{\alpha q}{2} + \frac{q_x}{4} + \frac{pq^2}{8}\Big)\zeta^2 + O(\zeta^3).
\end{aligned} \tag{5.4.46}$$

When $P \to P_0 = (0,0)$, take the local coordinate $\zeta^2 = \lambda$. We have

$$\begin{aligned}
\phi &= \frac{y-G}{F} = \frac{\zeta(f_n h_n + O(\zeta^2))^{1/2} - \zeta^2(g_n + O(\zeta^2))}{\zeta^2(f_n + O(\zeta^2))} \\
&= \sqrt{\frac{h_n}{f_n}}\zeta^{-1} + O(1) = \sqrt{\frac{\beta_3}{\beta_2}} e^{-2\alpha x}\zeta^{-1} + O(1).
\end{aligned} \tag{5.4.47}$$

Proof finished. □

Since the divisor of ϕ

$$(\phi(P,x,t_m)) = \mathcal{D}_{P_{\infty-},\hat{\nu}_1(x,t_m),\cdots,\hat{\nu}_n(x,t_m)} - \mathcal{D}_{P_0,\hat{\mu}_1(x,t_m),\cdots,\hat{\mu}_n(x,t_m)}. \tag{5.4.48}$$

In order to represent the functions p and q in terms of the Riemann theta function, we denote $\omega^{(3)}_{P_{\infty-},P_0}$ as the normal differential of the third kind being holomorphic on $\mathcal{K}_n \setminus \{P_{\infty-}, P_0\}$ with simple poles at $P_{\infty-}$ and P_0 and residues 1 and -1, respectively. Then

$$\omega^{(3)}_{P_{\infty-},P_0} = -\frac{1}{2\lambda}d\lambda - \frac{1}{2y}\prod_{j=1}^{n}(\lambda - \delta_i)d\lambda, \tag{5.4.49}$$

where $\delta_i \in \mathbb{C}, j = 1, \cdots, n$, are constants that are determined by

$$\int_{a_j} \omega^{(3)}_{P_{\infty-},P_0}(P) = 0, \quad j = 1, \cdots, n. \tag{5.4.50}$$

If the local coordinate near $P_{\infty\pm}$ is taken by $\zeta = \lambda^{-1}$, then the asymptotic expansion

$$\omega^{(3)}_{P_{\infty-},P_0} = \begin{cases} \frac{1}{2}\Big(\alpha_1 + \sum_{j=1}^{n}\delta_j + O(\zeta)\Big)d\zeta, & as\ P \to P_{\infty+}, \\ \Big[\zeta^{-1} - \frac{1}{2}\Big(\alpha_1 + \sum_{j=1}^{n}\delta_j + O(\zeta)\Big)\Big]d\zeta, & as\ P \to P_{\infty-}. \end{cases} \tag{5.4.51}$$

We use the local coordinate $\zeta^2 = \lambda$ near P_0, and then

$$\omega^{(3)}_{P_{\infty-},P_0} = (-\zeta^{-1} + O(1))d\zeta, \quad as\ P \to P_0. \tag{5.4.52}$$

Therefore, we get

$$\int_{Q_0} \omega^{(3)}_{P_{\infty-},P_0}(P) = \begin{cases} \ln \omega_{\infty+} + O(\zeta), & as\ P \to P_{\infty+}, \\ \ln \zeta + \ln \omega_{\infty-} + O(\zeta), & as\ P \to P_{\infty-}, \\ -\ln \zeta + \ln \omega_0 + O(\zeta), & as\ P \to P_0, \end{cases} \tag{5.4.53}$$

where $\omega_{\infty\pm}$ and ω_0 are integration constants.

Let the Riemann theta function $\theta(z)$ [6, 11] associated with $\mathcal{K}_n$ equipped with homology basis and holomorphic differentials be defined by

$$\theta(z) := \sum_{N\in\mathbb{Z}^n} \exp\{2\pi i\langle z, N\rangle + \pi i\langle BN, N\rangle\}. \tag{5.4.54}$$

Our main result of this chapter is the following theorem.

Theorem 5.4.6. *Let $P = (\lambda, y) \in \mathcal{K}_n \setminus \{P_{\infty+}, P_{\infty-}\}$, $(x, t_m) \in M$, where $M \subset \mathbb{R}^2$ is open and connected. Suppose $p(x, t_m), q(x, t_m) \in C^\infty(M)$ satisfies the hierarchy equation. Assume that $\lambda_j, 1 \le j \le 2n + 1$ are none-zero, distinct complex numbers. Moreover, suppose that $\mathcal{D}_{\hat{\mu}(x,t_m)}$ or equivalently, $\mathcal{D}_{\hat{\nu}(x,t_m)}$, are nonspecial for $(x, t_m) \in M$. Then p, q admit the following expression*

$$p = \eta_1 \frac{\theta(\Delta_1 + 2C_n x + \Omega_m t_m)\theta(\Delta_2 - 2C_n x - \Omega_m t_m)}{\theta(\Delta_3 - 2C_n x - \Omega_m t_m)\theta(\Delta_4 + 2C_n x + \Omega_m t_m)}, \tag{5.4.55}$$

$$q = \eta_2 \frac{\theta(\Delta_4 + 2C_n x + \Omega_m t_m)\theta(\Delta_5 - 2C_n x - \Omega_m t_m)}{\theta(\Delta_2 - 2C_n x - \Omega_m t_m)\theta(\Delta_6 + 2C_n x + \Omega_m t_m)}, \tag{5.4.56}$$

where

$$\Delta_1 = K + \rho_0^{(1)} - \mathcal{A}(P_{\infty+}), \quad \Delta_2 = K + \rho_0^{(2)} - \mathcal{A}(P_0), \quad \Delta_3 = K + \rho_0^{(2)} - \mathcal{A}(P_{\infty+}),$$
$$\Delta_4 = K + \rho_0^{(1)} - \mathcal{A}(P_0), \quad \Delta_5 = K + \rho_0^{(2)} - \mathcal{A}(P_{\infty-}), \quad \Delta_6 = K + \rho_0^{(1)} - \mathcal{A}(P_{\infty-}),$$
$$\eta_1 = -2\Big(\frac{\beta_2}{\beta_3}\Big)^{\frac{1}{2}} \frac{\omega_0}{\omega_{\infty+}} e^{2\alpha x}, \quad \eta_2 = -2\Big(\frac{\beta_3}{\beta_2}\Big)^{\frac{1}{2}} \frac{\omega_{\infty-}}{\omega_0} e^{-2\alpha x},$$

where K is the vector of Riemann constants defined by (5.2.22).

Proof. Let us define a function $z : \mathcal{K}_n \times \sigma^n\mathcal{K}_n \to \mathbb{C}$

$$z(P, Q) = K - \mathcal{A}(P) + \sum_{Q'\in Q} D(Q')\mathcal{A}(Q'), \tag{5.4.57}$$

whree $\sigma^n\mathcal{K}_n$ denote the g-th symmetric power of $\mathcal{K}^n$. Then we get

$$\begin{aligned}\theta(z(P,\hat{\mu}(x,t_m))) &= \theta(K-\mathcal{A}(P)+\rho^{(1)}),\\ \theta(z(P,\hat{\nu}(x,t_m))) &= \theta(K-\mathcal{A}(P)+\rho^{(2)}).\end{aligned} \tag{5.4.58}$$

By the Riemann vanishing theorem and the divisor of meromorphic function ϕ

$$\phi(P,x,t_m)=N(x,t_m)\frac{\theta(z(P,\hat{\nu}(x,t_m)))}{\theta(z(P,\hat{\mu}(x,t_m)))}\exp\left(\int_{Q_0}^{P}\omega^{(3)}_{P_{\infty-},P_0}(P)\right), \tag{5.4.59}$$

where $N(x,t_m)$ is independent of $P\in\mathcal{K}_n$.

Let us assume that $\mu_j(x,t_m)$ are distinct for $j=1,\cdots,n$ and $(x,t_m)\in\tilde{M}$ then consider the expansion near $P_0, P_{\infty\pm}$.

$$\sqrt{\frac{h_n}{f_n}} = N(x,t_m)\omega_0\frac{\theta(\Delta_2-2C_nx-\Omega_mt_m)}{\theta(\Delta_4+2C_nx+\Omega_mt_m)}, \tag{5.4.60}$$

$$-\frac{2}{p} = N(x,t_m)\omega_{\infty+}\frac{\theta(\Delta_3-2C_nx-\Omega_mt_m)}{\theta(\Delta_1+2C_nx+\Omega_mt_m)}, \tag{5.4.61}$$

$$-\frac{q}{2} = N(x,t_m)\omega_{\infty-}\frac{\theta(\Delta_5-2C_nx-\Omega_mt_m)}{\theta(\Delta_6+2C_nx+\Omega_mt_m)}. \tag{5.4.62}$$

Theorm 5.4.6 follows by solving functions p and q. □

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