May 2017

Contributions to Quandle Theory: A Study of f-Quandles, Extensions, and Cohomology

Indu Rasika U. Churchill
University of South Florida, udyanganiesi@mail.usf.edu

Contributions to Quandle Theory: A Study of f-Quandles, Extensions, and
Cohomology

by

Indu Rasika U Churchill

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
Department of Mathematics and Statistics
College of Arts and Sciences
University of South Florida

Co-Major Professor: Mohamed Elhamdadi, Ph.D.
Co-Major Professor: Abdenacer Makhlouf, Ph.D.
Xiang-dong Hou, Ph.D.
Masahiko Saito, Ph.D.
David Rabson, Ph.D.

Date of Approval:
May 03, 2017

Keywords: Quandle, f-Quandle, Extension, Cohomology, Cocycle

This doctoral dissertation is dedicated to my mother, father, brother, and husband.

List of Figures

ABSTRACT

Quandles are distributive algebraic structures that were introduced by David Joyce [24] in his Ph.D. dissertation in 1979 and at the same time in separate work by Matveev [34]. Quandles can be used to construct invariants of the knots in the 3-dimensional space and knotted surfaces in 4-dimensional space. Quandles can also be studied on their own right as any non-associative algebraic structures.

In this dissertation, we introduce f-quandles which are a generalization of usual quandles. In the first part of this dissertation, we present the definitions of f-quandles together with examples, and properties. Also, we provide a method of producing a new f-quandle from a given f-quandle together with a given homomorphism. Extensions of f-quandles with both dynamical and constant cocycles theory are discussed. In Chapter 4, we provide cohomology theory of f-quandles in Theorem 4.1.1 and briefly discuss the relationship between Knot Theory and f-quandles.

In the second part of this dissertation, we provide generalized $2, 3$, and 4-cocycles for Alexander f-quandles with a few examples.

Considering "Hom-algebraic Structures" as our nutrient enriched soil, we planted "quandle" seeds to get f-quandles. Over the last couple of years, this f-quandle plant grew into a tree. We believe this tree will continue to grow into a larger tree that will provide future fruit and contributions.

In this chapter, we give a brief review of quandles, their history, connections to knot theory, and motivation towards f-quandles. After briefly discussing the history, we will give an overview of the structure and organization of this dissertation. We also state the definitions that we use throughout this dissertation.

It is well known that quandles are strongly related to knot theory while Hom-type algebras which led to f-quandles have their origin in physics and they are related to quantum deformations of some algebras of vector fields like Virasoro and oscillator algebras.

To answer natural questions like "what is a quandle?" and "who invented quandles?" or "what is the history behind quandles?", we will go back to the 1880s.

1.1   History

The term *quandle* first appeared in the 1979 Ph.D. thesis of David Joyce, which was published in 1982, [24]. It also appeared in a separate work by Sergey V. Matveev [34]. David Joyce introduced the *knot quandle*, as an invariant of the knot as well as a classifying invariant of the knot [24]. It has been found that quandles appeared earlier in history with different names:

- In 1942, Mituhisa Takasaki [45] used the word *kei* as an abstraction of the notion of symmetric transformation. ("Kei" are called "involutory quandles" in David Joyce's work.)

- In 1950, Conway and Wraith discussed similar structures called "wracks" [14].

- In 1982, motivated by colorings of knots, Matveev used "distributive groupoids" to construct invariants of knots [34].

- Later, Louis Kauffman used the word "crystal" [25].

- In mid 1980s, Brieskorn named the structures as "automorphic sets" [5].

- In 1992, Roger Fenn and Colin Rourke used the word "racks" [19] in their generalization of the quandle idea.

The concepts of "Hom-Lie algebras, quasi-Hom-Lie and quasi-Lie algebras, and Hom-algebra structures" were introduced by Hartwig, Larsson and Silvestrov, in [20, 26, 27]. This is in order to provide general frameworks to handle q-deformations of some Lie algebras of vector fields such as deformations and quasi-deformations of the Heisenberg Lie algebra, $\mathfrak{sl}_2(\mathbb{K})$, oscillator algebras and other finite-dimensional Lie algebras and infinite-dimensional Lie algebras of Witt and Virasoro type. These algebras play an important role in Physics within the string theory, vertex operator models, quantum scattering, lattice models and other contexts, as well as various classes of quadratic and sub-quadratic algebras arising in connection to non-commutative geometry, twisted derivations and deformed difference operators and non-commutative differential calculi. The main initial motivation for this investigation was the goal of creating a unified general approach to examples of q-deformations of Witt and Virasoro algebras constructed in 1990–1992 in pioneering works by physists, where in particular it was observed that in these examples the Jacobi identity is no longer satisfied, but some q-deformations of ordinary Lie algebra Jacobi identities hold. Motivated by these examples Hartwig, Larsson and Silvestrov introduced Hom-Lie algebras which generalize usual Lie algebras. In fact they introduced a more general class called quasi-Lie algebras including quasi-Hom-Lie algebras and Hom-Lie algebras as subclasses. In the subclass of Hom-Lie algebras skew-symmetry is untwisted, whereas the Jacobi identity is twisted by a single linear map and contains three terms as in Lie algebras, reducing to ordinary Lie algebras when the twisting linear map is

the identity map. Later, Makhlouf and Silvestrov introduced Hom-associative algebras in [30], generalizing associative algebras. They proved that the commutator of a Hom-associative algebra defines a Hom-Lie algebra with the same structure map, showing that there is a functor between the category of Hom-associative algebras and the category of Hom-Lie algebras. The adjoint functor leading to the enveloping algebra of a Hom-Lie algebra was constructed by Yau. Various algebraic structures and results have been extended to this Hom-type framework. The main feature of Hom-type algebras is that the usual defining identities are twisted by one or several deforming twisting maps. Notice that a significant research activity was developed about Hom-type algebras in the past few years.

Quandles are very useful structures not just in Mathematics, but also in connecting Mathematics with Physics [49, 50].

Since quandles are algebraic structures, one can think of applying the algebraic "Deformation Theory" to it. Twisting quandles with linear maps gives the notion of f-quandles.

This dissertation mainly consists of two parts: From chapters one to four, we introduce generalized quandles, which we call f-quandles, their extensions and cohomologies. In the second part, we give an application of f-quandles to Alexander f-quandles by computing low dimensional cocycles.

In Chapter 2, we introduce generalized quandles, f-quandles, and give examples of f-quandles. Furthermore, we provide a method for constructing a new f-quandle using a given f-quandle and an f-quandle morphism. Extensions of f-quandles with dynamical cocycles and with constant cocycles are presented in Chapter 3 together with modules. Chapter 4 is devoted to the Cohomology Theory of f-racks and f-quandles. We give a couple examples of f-rack cohomologies. At the end, we give some remarks on the relationship between f-quandles and Knot Theory.

In Chapter 5, we give some generalized computations on 2, 3, and 4 dimensional cocycles of Alexander f-quandles and provide some examples if the given conjectures hold.

In Appendix A, we present a useful computer code, the $Maple$ program code we used to verify the cohomology of the f-racks in Example 4.1.5.

1.2   Knot Diagrams

In this section we will review some basics about knots.

Definition 1.2.1 *A knot is an embedding of the circle S^1 into $\mathbb{R}^3$ or S^3. In other words, it is a simple closed curve. A link is a finite, ordered collection of disjoint knots.*

Two knots K and K' are equivalent if they are ambient isotopic, that is, K can be deformed continuously to K'. Precisely, we have the following definition.

Definition 1.2.2 *[16] K is ambient isotopic to K' if there is a continuous map $H :$ $\mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ such that $H(K,0) = K$, $H(K,1) = K'$ and $H(x,t)$ is injective for all $t \in [0,1]$. Such a map is called an "ambient isotopy."*

One way of changing a knot K in $\mathbb{R}^3$ is to consider a projection of the knot on to a plane; $p : K \subset \mathbb{R}^3 \to \mathbb{R}^2$. A point $x \in \mathbb{R}^2$ is called a double point of $\mathbb{R}^3$. A double point is called a crossing. When considering projections of knots on to planes, we require that the projection has only finitely many double points.

We usually indicate over crossings and under crossings by drawing the under strand broken on the projected diagram. Figure 1.1 shows some knot diagrams.
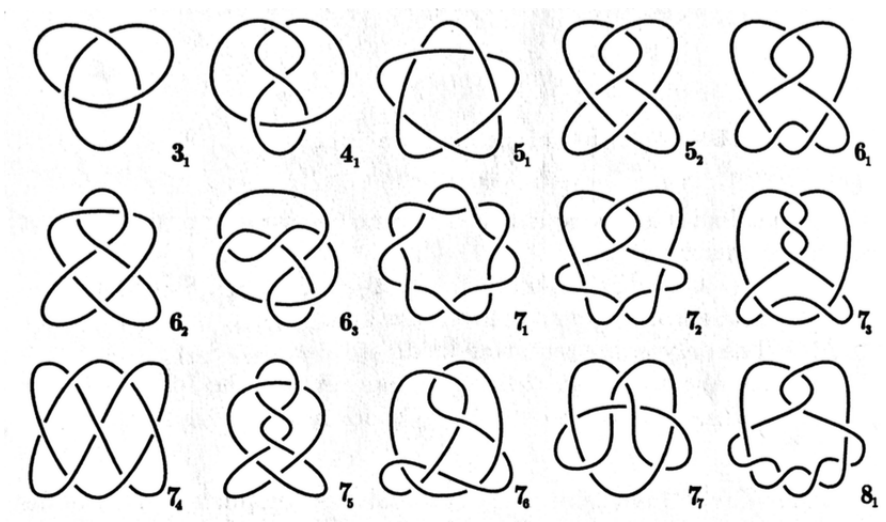
Figure 1.1: Some knot diagrams *

Recall that an orientation of a knot is defined by choosing a direction to travel around the knot. So we have two different crossings; called positive crossing and negative crossing as shown in Figure 1.2 below.
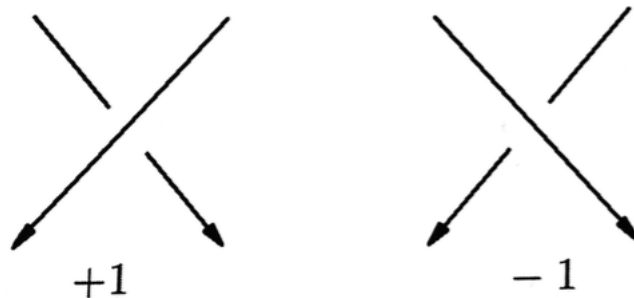
Figure 1.2: Positive and negative crossings

Obviously a knot has many diagrams. So mathematicians had to figure out a way to distinguish knots apart using diagrams. To overcome this situation, they came up with "knot invariants".

---

*Appendix (I), [37]

A knot invariant is an object (number, group, etc.) of a knot that does not change under ambient isotopy. So in order to show that a given object of a knot is a knot invariant, one must show that it is invariant under any ambient isotopy. In 1927, Reidemeister proved that any ambient isotopy can be achieved via a finite sequence of three moves that we call the "Reidemeister moves".

Definition 1.2.3 *[2] A Reidemeister move is one of three ways to change a projection of a knot that will change the relation between the crossings. The first move allows us to put in or out a twist in the knot, the second move allows us to add two crossings or remove two crossings, and the third move allows us to slide a strand of the knot from one side of a crossing to the other side of the crossing as in the diagram below:*
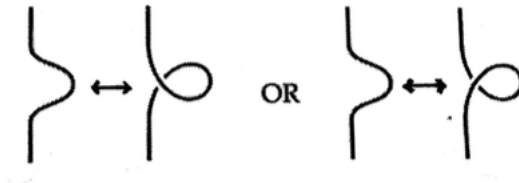
Figure 1.3: Reidemeister move type I
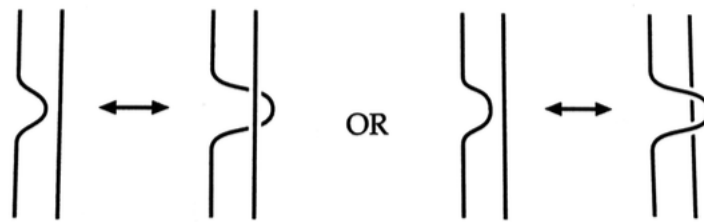
Figure 1.4: Reidemeister move type II
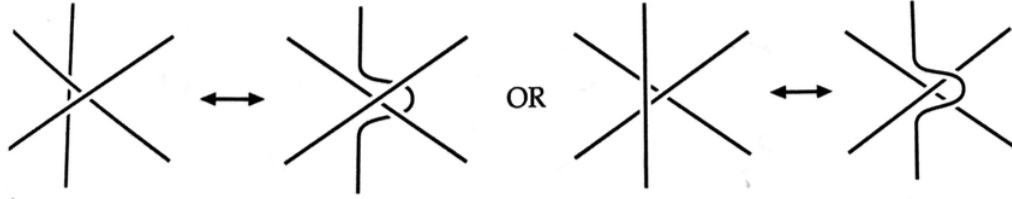
Figure 1.5: Reidemeister move type III

Definition 1.2.4 *Two knot diagrams, K_1 and K_2, are ambient isotopic if and only if one can be changed into the other by a finite sequence of planar isotopies and Reidemeister moves.*

As an example in [16], we can transfer the following given knot diagram into a standard diagram of the unknot by using a sequence of Reidemeister moves as follows:
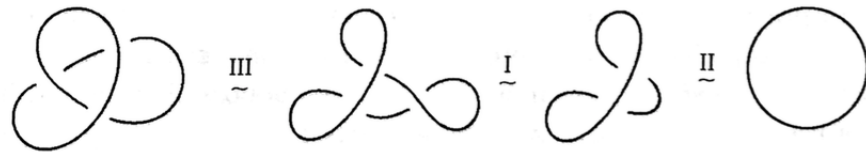
Figure 1.6: Sequence of Reidemeister moves transferring a given knot into an unknot

In order to construct invariants of knots, the notion of quandles was introduced by Joyce and Matveev [24, 34]. How quandles relate to knots will be investigated in our next section.

1.3 Quandles

In this section, we will show a bridge between knot theory and quandle theory via algebraic structures. Moreover, in this section we will review the notions of shelves, racks, and quandles and give some examples. For future convenience, we will use the following notation:

7

Notation 1.3.1 *Throughout this dissertation, we use the notation $\triangleright$ to denote the binary operation on "quandles" and $*$ to denote the binary operation on "f-quandles" unless otherwise stated.*

Recall that, an algebraic structure is a set X with one or more operations defined on it that satisfies a list of axioms.

In order to build up a bridge relating knot theory to quandle theory, let X be a set of labels and $\triangleright$ and $\triangleright^{-1}$ be two binary operations. We consider the following labeling scheme convention:
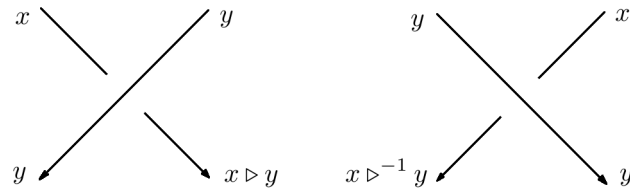
Figure 1.7: Positive and negative crossings.

According to this labeling scheme, the Reidemeister moves can be labeled as follows:
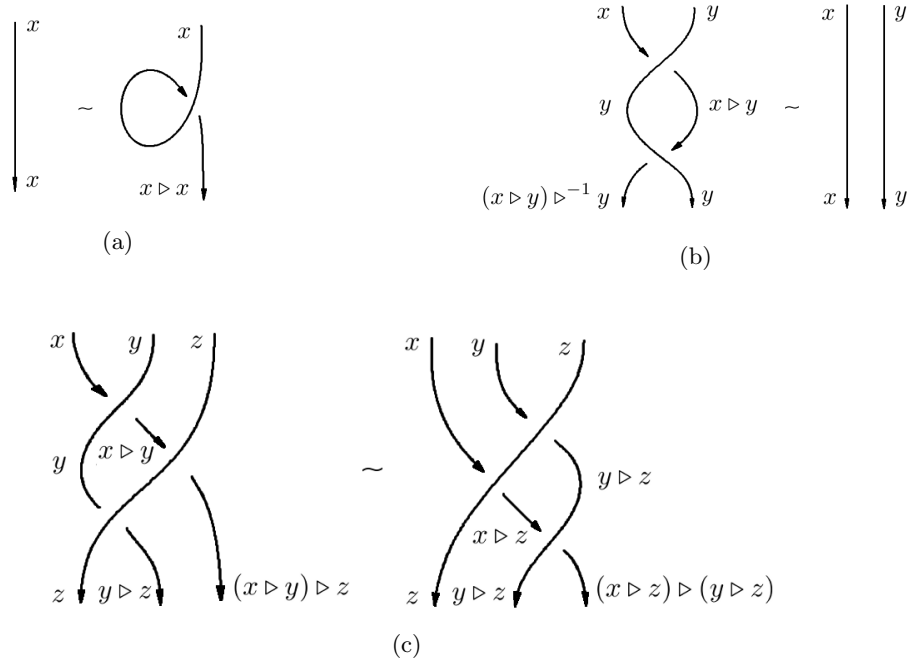
Figure 1.8: Relationship between the quandle axioms and the Reidemeister moves.

8

From Reidemeister move III, we obtain the following equation

$$(x \rhd y) \rhd z = (x \rhd z) \rhd (y \rhd z)$$

allowing the following definition.

Definition 1.3.2 *[4] A shelf is a pair $(X, \rhd)$, where X is a non-empty set with a binary operation $\rhd$ satisfying the following identity:*

$$(a \rhd b) \rhd c = (a \rhd c) \rhd (b \rhd c), \quad \forall a, b, c \in X. \tag{1.3.1}$$

Reidemeister move II gives the equation $(x \rhd y) \rhd^{-1} y = x$. This gives the notion of a rack.

Definition 1.3.3 *[4] A rack is a shelf such that, for any $b, c \in X$, there exists a unique $a \in X$ such that*

$$a \rhd b = c. \tag{1.3.2}$$

Allowing Reidemeister move I gives the equation $x \rhd x = x$ giving the notion of a quandle.

Definition 1.3.4 *[4] A quandle is a rack such that, for each $a \in X$, the identity*

$$a \rhd a = a \tag{1.3.3}$$

holds.

Together with an extra condition, we will get the notion of a crossed set.

Definition 1.3.5 *[4] A crossed set is a quandle $(X, \rhd)$ such that $a \rhd b = a$ whenever $b \rhd a = b$ for any $a, b \in X$.*

Remark 1.3.6 *Using the right translation $R_x : X \to X$ defined by $R_x(a) = a \rhd x$, the identity (1.3.1) can be written as $R_c(R_b(a)) = R_{b \rhd c}(R_c(a))$ for any $a, b, c \in X$.*

9

The extra condition in crossed sets can be written as $R_b(a) = a$ whenever $R_a(b) = b$ for any $a, b \in X$.

Definition 1.3.7 *Let $(X, \rhd_1)$ and $(Y, \rhd_2)$ be two racks. Let $\phi : (X, \rhd_1) \to (Y, \rhd_2)$ be a function. Then ϕ is a morphism of racks if $\phi(x \rhd_1 y) = \phi(x) \rhd_2 \phi(y)$ for all $x, y \in X$.*

Typical examples of quandles include the following:

- Given any non-empty set X with the operation $x \rhd y = x$ for any $x, y \in X$, then $(X, \rhd)$ is a quandle called the *trivial quandle*.

- A group $X = G$ with n-fold conjugation as the operation:

$$a \rhd b = b^{-n} a b^n.$$

  Then $(X, \rhd)$ is a quandle.

- For $a, b \in \mathbb{Z}_n$ (integers modulo n), where n be a positive integer, define

$$a \rhd b \equiv 2b - a \pmod{n}.$$

  Then the operation $\rhd$ defines a quandle structure called the *dihedral quandle*, R_n. The reason it is called "dihedral" is that this quandle can be identified with the set of reflections of a regular n-gon with conjugation as the quandle operation.

- Any $\mathbb{Z}[t, t^{-1}]$-module M is a quandle with the operation $a \rhd b = ta + (1 - t)b$, $a, b \in M$, called an *Alexander quandle*.

Sometimes quandles can have some additional properties.

Definition 1.3.8 *A quandle is Latin if for each $x \in X$, the map $L_x : X \to X$ (we use L for left multiplication) defined by $L_x(a) = x \rhd a$ is a bijection.*

Definition 1.3.9 *A quandle X is medial if for all $x, y, u, v \in X$, we have*

$$(x \rhd y) \rhd (u \rhd v) = (x \rhd u) \rhd (y \rhd v).$$

1.4   Cohomology Theory of Quandles

In this section, we will review the definition of quandle homology and cohomology for quandles, a new invariant introduced by Carter, Jelsovsky, Kamada, Langford, and Saito [10], and provide formulas which are useful for later chapters.

Notation 1.4.1 *In the following, the subscripts/superscript R, Q, and D represents rack, quandle, and degenerate cochain complexes respectively and W can be any of them.*

Let $C_n^R(X)$ be the free abelian group generated by n-tuples $(x_1, \ldots, x_n)$ of elements of a quandle X. Define a homomorphism $\partial_n : C_n^R(X) \to C_{n-1}^R(X)$ by

$$
\begin{aligned}
\partial_n(x_1, \ldots, x_n) &= \sum_{i=2}^{n}(-1)^i \left[(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \right. \quad\quad (1.4.4) \\
&\quad - \left. (x_1 \rhd x_i, x_2 \rhd x_i, \ldots, x_{i-1} \rhd x_i, x_{i+1}, \ldots, x_n) \right]
\end{aligned}
$$

for $n \geq 2$ and $\partial_n = 0$ for $n \leq 1$ in $[10]$.

Then $C_\bullet^R(X) = \{C_n^R(X), \partial_n\}$ is a chain complex.

Let $C_n^D(X)$ be the subset of $C_n^R(X)$ generated by the (n)-tuples $(x_1, \cdots, x_n)$ where $x_i = x_{i+1}$ for some $i \in \{1, \cdots, n-1\}$ if $n \geq 2$; otherwise let $C_n^D(X) = 0$. The subcomplex $C_n^D(X)$ is called the *degenerate subcomplex*. If X is a quandle, then $\partial_n(C_n^D(X) \subset C_{n-1}^D(X)$ and $C_\bullet^D(X) = \{C_n^D(X), \partial_n\}$ is a sub-complex of $C_\bullet^R(X)$. By taking $C_n^Q(X) = C_n^R(X)/C_n^D(X)$ and $C_\bullet^Q(X) = \{C_n^Q(X), \partial'_n\}$, where ∂'_n is the induced homomorphism. Therefore, all boundary maps will be denoted by ∂_n.

For an abelian group A, define the chain and cochain complexes

$$C_\bullet^W(X; A) = C_\bullet^W(X) \otimes A, \quad \partial = \partial \otimes \mathrm{id}$$

$$C_W^\bullet(X; A) = \mathrm{Hom}(C_\bullet^W(X), A), \quad \delta = \mathrm{Hom}(\partial, \mathrm{id})$$

in the usual way. The groups of cycles and boundaries are denoted by $\ker(\partial) = Z_n^W(X; A) \subset C_n^W(X; A)$ and $\mathrm{Im}(\partial) = B_n^W(X; A) \subset C_n^W(X; A)$.

The cocycles and coboundaries are denoted by $\ker(\delta) = Z_W^n(X; A) \subset C_W^n(X; A)$ and $\mathrm{Im}(\delta) = B_W^n(X; A) \subset C_W^n(X; A)$.

Definition 1.4.2 *The nth quandle homology group and the nth quandle cohomology group of a quandle $(X, \rhd)$ with coefficients in group A are as follows:*

$$H_n^Q(X; A) = H_n(C_\bullet^Q(X; A)) = Z_n^Q(X; A)/B_n^Q(X; A)$$

$$H_Q^n(X; A) = H^n(C_Q^\bullet(X; A)) = Z_Q^n(X; A)/B_Q^n(X; A).$$

1.5   Quandle Cocycles

Let us consider low dimensional cocycles. Given an abelian group A, a 2-cocycle with coefficients in A is a function $\phi : X \times X \to A$ satisfying the following equations:

$$\phi(x, y) + \phi(x \rhd y, z) = \phi(x, z) + \phi(x \rhd z, y \rhd z).$$

This equation can be obtained from Reidemeister moves. First color the arcs of crossing as in the following figure.
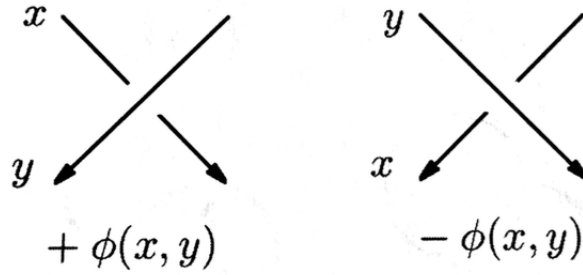
Figure 1.9: Quandle Cocycle Coloring.

Then $\phi(x, y)$ can be thought of as a weight at the crossing as shown in Figure 1.9 above.

Using this labeling scheme, we can color the three Reidemeister moves and obtain the 2-cocycle condition as follows:
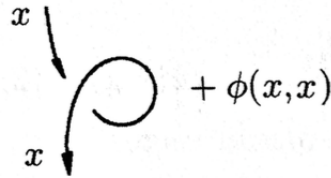
Figure 1.10: Reidemeister move I on cocycle coloring.

and results

$$\phi(x, x) = 0.$$

In a similar fashion, Reidemeister move Type II can be labeled and notice that the two crossings below cancel each other out as follows:

13
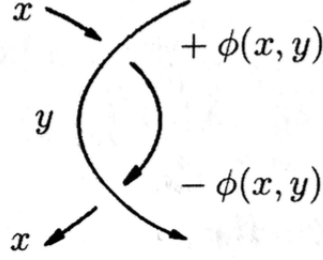
Figure 1.11: Reidemeister move II on cocycle coloring.

Finally, the Reidemeister move Type III yields the following diagram and the equation that we call the 2-cocycle condition:
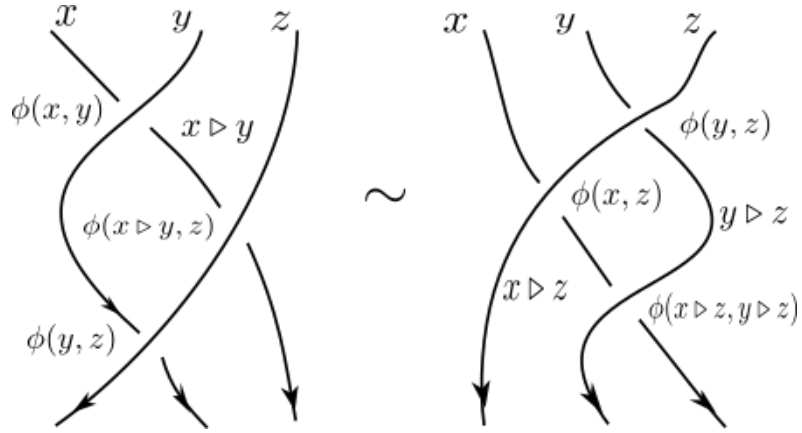
Figure 1.12: Reidemeister move III on cocycle coloring.

$$\phi(x, y) + \phi(x \triangleright y, z) = \phi(x, z) + \phi(x \triangleright z, y \triangleright z), \qquad (1.5.5)$$

where ϕ gives an invariant sum called the *Boltzmann weight* together with $\phi(x, x) = 0$ for all $x \in X$.

In a similar way, a 3-cocycle $\theta : X \times X \times X \to A$ satisfies the equation:

$$\theta(p, q, r) + \theta(p \triangleright r, q \triangleright r, s) + \theta(p, r, s) \; =$$
$$\theta(p \triangleright q, r, s) + \theta(p, q, s) + \theta(p \triangleright s, q \triangleright s, r \triangleright s) \qquad (1.5.6)$$

14

where $\theta(p, p, r) = 0$ and $\theta(p, q, q) = 0$ for all $p, q \in X$.

1.6   Hom-Algebras

In this section, we recall the definitions of Hom-Lie algebras which were introduced by Hartwig, Larsson and Silvestrov [20, 26, 27] and Hom-associative algebras, introduced by Makhlouf and Silvestrov in [30]. In particular, they showed that there is a functor from the category of Hom-associative algebras into the category of Hom-Lie algebras. The adjoint functor corresponding to enveloping algebra was given by Yau who defined also a free Hom-associative algebra.

We will take $\mathbb{K}$ to be an algebraically closed field of characteristic 0 and V to be a linear space over $\mathbb{K}$. An *Algebra* is a pair (V, α), where V is a linear space and α is a linear self-map of V. A *Hom-algebra* is a triple (V, μ, α) consisting of a linear space V, a bilinear map $\mu : V \times V \to V$, and a linear transformation $\alpha : V \to V$.

Definition 1.6.1 ([30]) *A Hom-associative algebra is a triple (V, μ, α) consisting of a linear space V, a bilinear map $\mu : V \times V \to V$, and a linear space homomorphism $\alpha : V \to V$ satisfying*

$$\mu(\alpha(x), \mu(y, z)) = \mu(\mu(x, y), \alpha(z)).$$

We recover associative algebras when the structure map α is the identity map. Recall that an associative algebra is a pair (V, μ) consisting of a linear space V and a bilinear map $\mu : V \times V \to V$ satisfying

$$\mu(x, \mu(y, z)) = \mu(\mu(x, y), z). \tag{1.6.7}$$

This equation can be written as $\mu(id \otimes \mu) = \mu(\mu \otimes id)$, where id is the identity map. Similarly, one obtains Hom-Lie algebras which are natural generalization of Lie algebras.

Definition 1.6.2 ([20]) *A Hom-Lie algebra is a triple $(V, [\cdot, \cdot], \alpha)$ consisting of a*

linear space V, bilinear map $[\cdot,\cdot] : V \times V \to V$, and a linear space homomorphism
$\alpha : V \to V$ satisfying

1. *$[x,y] = -[y,x]$,*

2. *$[[x,y],\alpha(z)] + [[y,z],\alpha(x)] + [[z,x],\alpha(y)] = 0$.*

Remark: One recovers Lie algebras by taking α being the identity map. Recall that a Lie algebra is a pair $(V,[\cdot,\cdot])$ consisting of a linear space V over a field $\mathbb{K}$ and a bilinear map $[\cdot,\cdot] : V \times V \to V$, that we call the "Lie bracket", satisfying

1. $[x,y] = -[y,x]$,

2. $[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$.

Proposition 1.6.3 (Functor Hom-Lie, [30]) *To any Hom-associative algebra defined by the multiplication μ and a homomorphism α over a $\mathbb{K}$-vector space A, one may associate a Hom-Lie algebra defined for all $x,y \in A$ by the bracket*

$$[x,y] = \mu(x,y) - \mu(y,x).$$

The enveloping algebra was studied by D. Yau.

Example 1.6.4 *[31] Let $\{x_1, x_2, x_3\}$ be a basis of a 3-dimensional vector space A over $\mathbb{K}$. The following multiplication μ and linear map α on $A = \mathbb{K}^3$ define a Hom-associative algebra over $\mathbb{K}$:*

$$
\begin{aligned}
\mu(x_1,x_1) &= ax_1, & \mu(x_2,x_2) &= ax_2, \\
\mu(x_1,x_2) &= \mu(x_2,x_1) = ax_2, & \mu(x_2,x_3) &= bx_3, \\
\mu(x_1,x_3) &= \mu(x_3,x_1) = bx_3, & \mu(x_3,x_2) &= \mu(x_3,x_3) = 0,
\end{aligned}
$$

$$\alpha(x_1) = ax_1, \quad \alpha(x_2) = ax_2, \quad \alpha(x_3) = bx_3$$

where a, b are parameters in $\mathbb{K}$. This algebra is not associative when $a \neq b$ and $b \neq 0$, since

$$\mu(\mu(x_1, x_1), x_3)) - \mu(x_1, \mu(x_1, x_3)) = (a - b)bx_3.$$

Example 1.6.5 (Jackson $\mathfrak{sl}_2$, [29]) *The Jackson $\mathfrak{sl}_2$ is a q-deformation of the classical Lie algebra $\mathfrak{sl}_2$. It carries a Hom-Lie algebra structure but not a Lie algebra structure by using Jackson derivations. It is defined with respect to a basis $\{x_1, x_2, x_3\}$ by the brackets and a linear map α such that*

$$
\begin{aligned}
\left[x_1, x_2\right] &= -2qx_2, & \alpha(x_1) &= qx_1, \\
\left[x_1, x_3\right] &= 2x_3, & \alpha(x_2) &= q^2 x_2, \\
\left[x_2, x_3\right] &= -\tfrac{1}{2}(1 + q)x_1, & \alpha(x_3) &= qx_3,
\end{aligned}
$$

where q is a parameter in $\mathbb{K}$. If $q = 1$ we recover the classical $\mathfrak{sl}_2$.

Hom-analogues of various classical structures and results have been introduced and discussed by many authors. For instance, representation theory, cohomology and deformation theory for Hom-associative algebras and Hom-Lie algebras have been developed. Moreover, the dual concept of Hom-associative algebras, called Hom-coassociative coalgebras, as well as Hom-bialgebras and Hom-Hopf algebras, have been introduced and studied. Furthermore Hom-Lie bialgebras have been studied using a dual version of Hom-Lie algebras.

These "Hom-algebraic structures" motivated us to define and explore f-quandles, a generalized version of quandles. We attach a map $f : X \to X$ to a quandle $(X, \rhd)$ which results in a triple $(X, *, f)$ that satisfies certain conditions twisting the usual conditions.

2 f-QUANDLES [*]

Motivated by Quandles and Hom-algebras, it is naturel to address the question about Hom-type quandles. In this chapter we introduce so called f-quandles, which are a twisted version of quandles. Indeed, we give the definitions of f-quandle, f-quandle morphism, and f-crossed set. Also, we provide some of their properties. This chapter is divided into two sections: In section one, we give the definition of an f-quandle, state some properties, and give some examples. Proposition 2.2.1 in section 2.2 provides a method to construct a new f-quandle when an f-quandle and an f-quandle morphism are given. This provides a way to produce examples.

2.1   Definitions and Properties of f-Quandles

Definition 2.1.1 *An f-shelf is a triple $(X, *, f)$ in which X is a non-empty set, $*$ is a binary operation on X, and $f \colon X \to X$ is a map such that, for any $x, y, z \in X$, the identity*

$$(x * y) * f(z) = (x * z) * (y * z) \tag{2.1.1}$$

holds. An f-rack is an f-shelf such that, for any $x, y \in X$, there exists a unique $z \in X$ such that

$$z * y = f(x). \tag{2.1.2}$$

An f-quandle is an f-rack such that, for each $x \in X$, the identity

$$x * x = f(x) \tag{2.1.3}$$

holds.

An f-crossed set is an f-quandle $(X, *, f)$ such that $f : X \to X$ satisfies $x * y = f(x)$ whenever $y * x = f(y)$ for any $x, y \in X$.

Remark 2.1.2 *Notice that a quandle (resp. rack, shelf) may be viewed as an f-quandle (resp. f-rack, f-shelf) when the structure map f is the identity map.*

Remark 2.1.3 *Using the right translation $R_a : X \to X$ defined as $R_a(x) = x * a$, identity (2.1.1) can be written as*

$$R_{f(z)} R_y = R_{R_z(y)} R_z$$

for any $y, z \in X$.

The extra condition in the f-crossed set definition means that, for any $x, y \in X$, $R_x(y) = f(y)$ is equivalent to $R_y(x) = f(x)$.

The notion of homomorphism of f-quandles is given in the following definition.

Definition 2.1.4 *Let $(X_1, *_1, f_1)$ and $(X_2, *_2, f_2)$ be two f-quandles. A map $\phi : X_1 \to X_2$ is an f-quandle morphism if it satisfies $\phi(a *_1 b) = \phi(a) *_2 \phi(b)$.*

Note that $\phi \circ f_1 = f_2 \circ \phi$ is automatically holds from the definition.

Here we will give some examples of f-quandles:

Example 2.1.5 *Given any set X and an injective map $f : X \to X$, then the operation $x * y = f(x)$ for any $x, y \in X$ gives an f-quandle $(X, *, f)$.*

One can check the conditions as follows:

The left-hand side of equation (2.1.1) gives us

$$(x * y) * f(z) = f(x * y) = f(f(x)).$$

The right-hand side of equation (2.1.1) gives us

$$(x * z) * (y * z) = f((x * z)) = f(f(x)).$$

Therefore, equation (2.1.1) is satisfied. To check the second condition, assume there exists $z, z' \in X$ such that $z * y = f(x)$ and $z' * y = f(x)$. By definition, $z * y = f(z)$ and $z' * y = f(z')$, thus $f(x) = f(z) = f(z')$ implies that $z = z'$, so that the uniqueness requirement is satisfied. The third condition is straight forward since $x * x = f(x)$. We call $(X, *, f)$ a trivial f-quandle structure on X.

Example 2.1.6 *For any group G and any group endomorphism f of G, the operation $x * y = f(y)xy^{-1}$ defines an f-quandle structure on G.*

To check the first condition: Let $x, y, z \in G$. Starting from the left-hand side of (2.1.1),

$$
\begin{aligned}
\text{Left-hand side} &= (x * y) * f(z) \\
&= f(f(z))(x * y)f^{-1}(z) \\
&= f^2(z)f(y)xy^{-1}f(z)^{-1}
\end{aligned}
$$

$$
\begin{aligned}
\text{Right-hand side} &= (x * z) * (y * z) \\
&= f(y * z)(x * z)(y * z)^{-1} \\
&= f(f(z)yz^{-1})(f(z)xz^{-1})(f(z)yz^{-1})^{-1} \\
&= f^2(z)f(y)xy^{-1}f(z)^{-1}
\end{aligned}
$$

To check the second condition, assume there exist $z_1, z_2 \in G$ such that $z_1 * y = f(x)$ and $z_2 * y = f(x)$. By definition, $f(y)z_1 y^{-1} = f(x) = f(y)z_2 y^{-1} \implies z_1 = z_2$. To show that condition (2.1.3) holds, we have $x * x = f(x)xx^{-1} = f(x)$. Therefore, $(G, *, f)$, where $*$ is defined by $x * y = f(y)xy^{-1}$, is an f-quandle.

Example 2.1.7 *Consider the Dihedral quandle R_n, where $n \geq 2$, and let f be an automorphism of R_n. Then f is given by $f(x) = ax + b$, for some invertible element $a \in \mathbb{Z}_n$ and some $b \in \mathbb{Z}_n$ [17]. The binary operation $x * y = f(2y - x) = 2ay - ax + b$ (mod n) gives an f-quandle structure called the f-Dihedral quandle.*

We verify condition (2.1.1):

The left-hand side of this condition is given by:

$$
\begin{aligned}
(x * y) * f(z) &= 2af(z) - a(x * y) + b \quad \mathrm{mod}\ (n) \\
&= 2a(az + b) - a(2ay - ax + b) + b \quad \mathrm{mod}\ (n) \\
&= 2a^2 z + 2ab - 2a^2 y + a^2 x - ab + b \quad \mathrm{mod}\ (n) \\
&= 2a^2 z - 2a^2 y + a^2 x + ab + b \quad \mathrm{mod}\ (n)
\end{aligned}
$$

While the right-hand side is given by:

$$
\begin{aligned}
(x * z) * (y * z) &= 2a(y * z) - a(x * z) + b \quad \mathrm{mod}\ (n) \\
&= 2a(2az - ay + b) - a(2az - ax + b) + b \quad \mathrm{mod}\ (n) \\
&= 4a^2 z - 2a^2 y + 2ab - 2a^2 z + a^2 x - ab + b \quad \mathrm{mod}\ (n) \\
&= 2a^2 z - 2a^2 y + a^2 x + ab + b \quad \mathrm{mod}\ (n)
\end{aligned}
$$

Thus $(x * y) * f(z) = (x * z) * (y * z)$.

To show the existence, given y, there exists an element z such that $z = a^{-1}(b-k)+2y$, where $k = ax + b$ so that $z * y = ax + b = f(x)$.

To verify condition (2.1.2), assume there exists $z, z' \in X$ such that condition (2.1.2) holds. Then

$$
z * y = 2ay - az + b = f(x) = 2ay - az' + b = z' * y
$$

which shows that $z = z'$. It is straight forward that $x * x = f(x)$. Therefore, $(R_n, *, f)$, with the binary operator $*$ defined by $x * y = 2ay - ax + b \pmod{n}$, is an f-Dihedral quandle.

Example 2.1.8 *Any $\mathbb{Z}[T^{\pm 1}, S]$-module M is an f-quandle with $x * y = Tx + Sy$, $x, y \in M$, with $TS = ST$ and $f(x) = (S + T)x$, called an Alexander f-quandle.*

21

The left-hand side of (2.1.1) is

$$
\begin{aligned}
(x * y) * f(z) &= T(x * y) + S(f(z)) \\
&= T(Tx + Sy) + S((S + T)z) \\
&= TTx + TSy + SSz + STz
\end{aligned}
$$

and the right-hand side of (2.1.1) is

$$
\begin{aligned}
(x * z) * (y * z) &= T(x * z) + S(y * z) \\
&= T(Tx + Sz) + S(Ty + Sz) \\
&= TTx + TSz + STy + SSz
\end{aligned}
$$

which confirms that (2.1.1) holds under the condition $TS = ST$.

Assume there exist $z, z' \in M$ such that $z * y = Tz + Sy = f(x) = Tz' + Sy = z' * y$ which implies $z = z'$. It is straight forward that $x * x = Tx + Sx = f(x)$. Hence, $(M, *, f)$, with the operation defined above together with the condition $ST = TS$, is an Alexander f-quandle.

To give a slightly different interpretation to equation $(x * y) * f(z) = (x * z) * (y * z)$, we need to recall that a quandle $(X, \triangleright)$ is *medial* if,

$$
(x \triangleright y) \triangleright (u \triangleright v) = (x \triangleright u) \triangleright (y \triangleright v), \ \forall x, y, u, v \in X.
$$

Remark 2.1.9 *Axioms (2.1.1) and (2.1.3) of Definition 2.1.1 give the following equation:*

$$
(x * y) * (z * z) = (x * z) * (y * z).
$$

We note that the two medial terms in this equation are swapped (resembling the mediality condition of a quandle). Note also that the mediality in the general context may not be satisfied for f-quandles.

For example, we can check that the f-quandle given in Example 2.1.6 is NOT medial:

since

$$(x * y) * (z * z) = (z * z)^{-1}(x * y)f(z * z) = f^{-1}(z)y^{-1}xf(y)z$$

is not equal to

$$(x * z) * (y * z) = (y * z)^{-1}(x * z)f(y * z) = f^{-1}(z)y^{-1}xf(y)(f \circ f)(z).$$

On the other hand, one can check that Example 2.1.7 is medial.

Remark 2.1.10 *Let* $(X, *, f)$ *be an* f*-quandle. Then* f *is a homomorphism since*

$$
\begin{aligned}
f(x) * f(y) &= (x * x) * f(y) \quad (\because (2.1.3)) \\
&= (x * y) * (x * y) \quad (\because (2.1.1)) \\
&= f(x * y). \quad (\because (2.1.3))
\end{aligned}
$$

2.2 Construction of new f-quandles

In this section, we provide a method that shows how to construct a new f-quandle with a given f-quandle and an f-quandle morphism. We give a couple of examples before we finish this section. At the end, we discuss a functoriality property between f-racks and groups; see [4, 19] for the classical case.

Proposition 2.2.1 *Let* $(X, *, f)$ *be a finite* f*-quandle and* $\phi : X \to X$ *be an* f*-quandle morphism; that is,* $(\phi(x * y) = \phi(x) * \phi(y))$ *for all* $x, y \in X$*. If* ϕ *is an automorphism with* $a *_\phi b = \phi(a * b)$ *and* $f_\phi(a) = \phi(f(a))$ *then* $(X, *_\phi, f_\phi)$ *is an* f_ϕ*-quandle.*
If $(X, *_\phi, f_\phi)$ *is an* f_ϕ*-quandle where* f *is injective then* ϕ *is an automorphism.*

We will refer to $(X, *_\phi, f_\phi)$ as a *twist* of $(X, *, f)$.

Proof. Assume that ϕ is an automorphism. Let $a, b, c \in X$.

$$
\begin{aligned}
(a *_\phi b) *_\phi f_\phi(c) &= (a *_\phi b) *_\phi \phi(f(c)) = \phi(a * b) *_\phi \phi(f(c)) \\
&= \phi[(a * b) *_\phi f(c)] = \phi^2((a * c) * (b * c)) \\
&= \phi[\phi(a * c) * \phi((b * c))] = \phi[(a *_\phi c) * (b *_\phi c)] \\
&= (a *_\phi c) *_\phi (b *_\phi c).
\end{aligned}
$$

Obviously $a * a = f(a)$ implies $f_\phi(a) = \phi(f(a)) = \phi(a * a) = a *_\phi a$. Now given $a, b \in X$ there exists a unique $c \in X$ such that $c * b = f(a)$; therefore,

$$
c *_\phi b = \phi(c * b) = \phi(f(a)) = f_\phi(a).
$$

Hence $(X, *_\phi, f_\phi)$ is an f_ϕ-quandle.

In oder to show ϕ is an automorphism, assume that $(X, *_\phi, f_\phi)$ is an f_ϕ-quandle with $a *_\phi b = \phi(a * b)$ and $f_\phi(a) = \phi(f(a))$.

Assume $\phi(a) = \phi(b)$. Then there exist a unique element $x \in X$ such that $x * a = f(b)$. Then, $\phi(x * a) = \phi(f(b))$.

$$
\begin{aligned}
x *_\phi a &= f(\phi(b)) \\
&= f(\phi(a)) \\
&= \phi(f(a)) \\
&= \phi(a * a).
\end{aligned}
$$

That is $x *_\phi a = a *_\phi a = f(\phi(a))$ implies that $x = a$.

Thus $x * a = f(b) \implies f(a) = f(b) \implies a = b$ if f is injective. Since X is finite and ϕ is an injective homomorphism, ϕ is an automorphism. $\blacksquare$

Corollary 2.2.2 *In the case where f is the identity map, Proposition 2.2.1 shows that any usual quandle along with any automorphism gives rise to an f-quandle.*

Example 2.2.3 *Recall from [17] that any automorphism of the dihedral quandle $\mathbb{Z}_n$ is*

of the form $\phi_{a,b}(x) = ax + b$ for some $a, b \in \mathbb{Z}_n$. Using Proposition (2.2.1), we recover the Dihedral f-quandle in Example 2.1.7. To see this, let $(X, *, f) = (\mathbb{Z}_n, *, id)$, so that $(\mathbb{Z}_n, *)$ is a quandle, where $x * y = 2y - x \pmod{n}$ and define $x *_\phi y = \phi(x * y)$. It is easy to see that

$$x *_\phi y = \phi(x * y) = \phi(2y - x) = 2ay - ax + b \pmod{n}.$$

Therefore, $(\mathbb{Z}_n, *_\phi, \phi)$ is the ϕ-quandle given in Example 2.1.7.

Example 2.2.4 *Let $S_3 = \; < s, t : s^2 = t^3 = e, ts = st^2 >$ be the symmetric group on three letters. Let $*$ be conjugation on S_3. We know that $(S_3, *)$ is a quandle (so that $(S_3, *, id)$ is an f-quandle where $f = id$). Let ϕ be the group automorphism on S_3 defined by $s \mapsto st, t \mapsto t^2$. Then $(S_3, *_\phi, \phi)$ is a $\phi-$quandle where the operator $*_\phi$ is defined by $x *_\phi y = \phi(y)^{-1}\phi(x)\phi(y)$. For convenience, we give the Cayley table below:*

For example:

$$
\begin{aligned}
t *_\phi st &= \phi(st)^{-1}\phi(t)\phi(st) \\
&= \phi(t^{-1}s^{-1})\phi(t)\phi(st) \quad (\because \phi \text{ is a homomorphism}) \\
&= \phi(t^2 s)\phi(t)\phi(st) \quad (\because t^{-1} = t^2, s^{-1} = s) \\
&= (t^4 st)(t^2)(stt^2) \\
&= t^4 st^3 st^3 \\
&= ts^2 \\
&= t
\end{aligned}
$$

The Cayley table of the ϕ-quandle $(S_3, *_\phi, \phi)$ is given by:

↻*	e	s	t	t^2	st	st^2
e	e	e	e	e	e	e
s	st	st	st^2	s	st^2	s
t	t^2	t	t^2	t^2	t	t
t^2	t	t^2	t	t	t^2	t^2
st	s	st^2	st	st^2	s	st
st^2	st^2	s	s	st	st	st^2

Table 2.1: Cayley Table for $x *_\phi y = \phi(y)^{-1}\phi(x)\phi(y)$

We now define the concept of enveloping groups of f-racks.

Definition 2.2.5 *Let $(X, *, f)$ be an f-rack and let $F(X)$ denote the free group generated by X. Then there is a natural map $\iota : X \to G_X$, where G_X is called the enveloping group of the f-rack of X, and is defined as*

$$G_X = F(X)/ < x * y = f(y)xy^{-1}, x, y \in X > . \tag{2.2.4}$$

In the following, we discuss a functoriality property between f-racks and groups.

Proposition 2.2.6 *Let $(X, *, f)$ be an f-rack and G be a group. Given any f-rack homomorphism $\varphi : X \to G_{conj}$, where G_{conj} is a group together with an f-rack structure along a group homomorphism g, where the operation is defined as $a *_G b = g(b)ab^{-1}$, there exists a unique group homomorphism $\widetilde{\varphi} : G_X \to G$ which makes the following diagram commutes:*

$$
\begin{array}{ccc}
(X, *, f) & \xrightarrow{\ \iota\ } & G_X \\
{\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \widetilde{\varphi}} \\
(G_{conj}, *_G, g) & \xrightarrow[\ id\ ]{} & G
\end{array}
$$

Proof. Let $\bar{\varphi} : F(X) \to G$ be an f-rack homomorphism extension of φ to the free group $F(X)$. Then

$$\bar{\varphi}(yx^{-1}f(y)^{-1}(x * y)) = 1.$$

26

Indeed,

$$
\begin{aligned}
\bar{\varphi}(yx^{-1}f(y)^{-1}(x*y)) &= \varphi(y)\varphi(x^{-1})\varphi(f(y)^{-1})\varphi(x*y) \\
&= \varphi(y)\varphi(x)^{-1}\varphi(f(y))^{-1}(\varphi(x)*_G\varphi(y)) \\
&= \varphi(y)\varphi(x)^{-1}\varphi(f(y))^{-1}g(\varphi(y))\varphi(x)\varphi(y)^{-1} \\
&= 1
\end{aligned}
$$

since $\varphi \circ f = g \circ \varphi$ (f-rack morphism). It follows that φ factors through a unique homomorphism $\tilde{\varphi} : G_X \to G$. The commutativity of the diagram is straightforward. ∎

Corollary 2.2.7 *The functor $(X,*,f) \to G_X$ is left-adjoint to the functor $G \to G_{conj}$ from the category of groups to that of f-racks. That is,*

$$
Hom_{groups}(G_X, G) \simeq Hom_{f-racks}(X, G_{conj})
$$

by the natural isomorphism.

3 EXTENSIONS OF f-QUANDLES AND MODULES [‡]

In this chapter, we investigate extensions of f-quandles motivated by the work of [7]. We provide a general construction, as showed in [4], for an *extension* of X by a dynamical cocycle α, denoted by $X \times_\alpha A$. We define generalized f-quandle 2-cocycles and give examples. We give an explicit formula relating group 2-cocycles to f-quandle 2-cocycles when the f-quandle is constructed from a group.

3.1 Extensions with Dynamical Cocycles and Extensions with Constant Cocycles

Let $(X, *, f)$ be an f-quandle and A be a non-empty set. Let $\alpha : X \times X \to \mathrm{Fun}(A \times A, A)$ be a function, $g : A \to A$ a map, and $F(x, a) = (f(x), g(a))$. Our aim in this section is to give conditions on α that guarantee that $(X \times A, *, F)$ is an F-quandle. To come up with such conditions, we begin with the given condition that $(X, *, f)$ is an f-quandle where $\alpha(x, y) : A \times A \to A$ is defined by $\alpha(x, y)(a, b) = \alpha_{x,y}(a, b)$. For $(x, a), (y, b), (z, c) \in X \times A$, the operation $*$ is defined as

$$(x, a) * (y, b) = (x * y, \alpha_{x,y}(a, b)).$$

In order to prove that the operation $*$ satisfies the condition (2.1.1) of Definition 2.1.1, we need to show that

$$[(x, a) * (y, b)] * F(z, c) = [(x, a) * (z, c)] * [(y, b) * (z, c)]. \tag{3.1.1}$$

The left-hand side of (3.1.1) can be written as

$$[(x, a) * (y, b)] * F(z, c) = [(x, a) * (y, b)] * (f(z), g(c))$$
$$= (x * y, \alpha_{x,y}(a, b)) * (f(z), g(c))$$
$$= [(x * y) * f(z); \ \alpha_{(x*y), f(z)}(\alpha_{x,y}(a, b), g(c))]$$

The right-hand side of (3.1.1) is given by

$$[(x, a) * (z, c)] * [(y, b) * (z, c)] = [(x * z), \ \alpha_{x,z}(a, c)] * [(y * z), \ \alpha_{y,z}(b, c)])$$
$$= [(x * z) * (y * z); \ \alpha_{(x*z), (y*z)}(\alpha_{x,z}(a, c), \alpha_{y,z}(b, c)]$$

Equating the right-hand side and the left-hand side forces the following condition:

$$\alpha_{x*y, f(z)}(\alpha_{x,y}(a, b), g(c)) = \alpha_{x*z, y*z}(\alpha_{x,z}(a, c), \alpha_{y,z}(b, c))$$

for all $x, y, z \in X$ and $a, b, c \in A$.

For the operation $*$ to satisfy condition (2.1.3), we need to show that $(x, a) * (x, a) = F(x, a)$. So we have

$$(x, a) * (x, a) = (x * x, \ \alpha_{x,x}(a, a))$$
$$= (f(x); \ \alpha_{x,x}(a, a)).$$

On the other hand, $F(x, a) = (f(x), g(a))$ by definition. So the result is

$$\alpha_{x,x}(a, a) = g(a).$$

For the bijectivity of the map in (2.1.2), we need to show that given (y, b) and (x, a) there exists a unique $(z, c) \in X \times A$ such that $(z, c) * (y, b) = F(x, a)$. The left-hand side can be simplified to $(z, c) *_F (y, b) = (z * y; \alpha_{z,y}(c, b))$. By definition, $F(x, a) = (f(x), g(a))$. Therefore, we get the two equations $z * y = f(x)$ and

29

$\alpha_{x,y}(c, b) = g(a)$. Thus the uniqueness of z and c. This result is better stated in the following proposition:

Proposition 3.1.1 *Let $(X, *, f)$ be an f-quandle and A be a non-empty set. Let $\alpha : X \times X \to \text{Fun}(A \times A, A)$ be a function and $g : A \to A$ a map. Then, $(X \times A, *, F)$ is an F-quandle where $F(x, a) = (f(x), g(a))$ and the operation is defined as*

$$(x, a) * (y, b) = (x * y, \ \alpha_{x,y}(a, b)), \tag{3.1.2}$$

where $x*y$ denotes the f-quandle operation in X, if and only if α satisfies the following three conditions:

1. *$\alpha_{x,x}(a, a) = g(a)$ for all $x \in X$ and $a \in A$;*

2. *For all $x, y \in X$ and for all $b \in A$, $\alpha_{x,y}(-, b) : A \to A$ is a bijection.*

3. *$\alpha_{x*y, f(z)}(\alpha_{x,y}(a, b), g(c)) = \alpha_{x*z, y*z}(\alpha_{x,z}(a, c), \alpha_{y,z}(b, c))$ for all $x, y, z \in X$ and $a, b, c \in A$.*

If $(X, *, f)$ is a f-crossed set, then $(X \times A, *, F)$ is a f-crossed set if and only if the map α further satisfies $\alpha_{x,y}(a, b) = g(b)$ whenever $y * x = f(y)$ and $\alpha_{y,x}(b, a) = g(a)$.

Such a function α is called a dynamical f-quandle cocycle or dynamical f-rack cocycle (when it satisfies the above conditions).

The F-quandle constructed above is denoted by $X \times_\alpha A$, and it is called the *extension* of X by a dynamical cocycle α.

Remark 3.1.2 *When $x = y$ in item (3) in Proposition 3.1.1, we have*

$$
\begin{aligned}
\alpha_{f(x), f(z)}(\alpha_{x,x}(a, b), g(c)) &= \alpha_{x*z, x*z}(\alpha_{x,z}(a, c), \alpha_{x,z}(b, c)) \\
&= \alpha_{x,z}(a, c) *_{x*z} \alpha_{x,z}(b, c), \quad \forall a, b, c \in A.
\end{aligned}
$$

When f and g are both the identity map, we have

$$\alpha_{x,z}(\alpha_{x,x}(a,b),\,c) \;=\; \alpha_{x*z,x*z}\big(\alpha_{x,z}(a,c),\,\alpha_{x,z}(b,c)\big)$$

$$\alpha_{x,z}(a *_x b,\,c) \;=\; \alpha_{x*z}(a,c) *_{x*z} \alpha_{x,z}(b,c) \quad \forall a,b,c \in A$$

and thus it reduces to the classical case where

$$\alpha_{x,z}(c) : (A, *_x) \to (A, *_{x*z})$$

is an isomorphism as in [7].

Now, we discuss extensions with constant cocycles.

If for all $x,y \in X$, the map $\alpha_{x,y} : A \times A \to A$ is a constant map where $\alpha_{x,y}(a,b) = a + \phi(x,y)$. Then equation 3.1.2 of Proposition 3.1.1 can be written as $(x,a) * (y,b) = (x * y, a + \phi(x,y))$. Recall that this situation was discussed in [9].

When $\phi(x,y) = \beta_{x,y}$, then we have the following.

Definition 3.1.3 *[4] Let $(X, \triangleright)$ be a rack and $\beta : X \times X \to S_X$. β is a constant rack cocycle if*

$$\beta_{x \triangleright y, z} \beta_{x,y} = \beta_{x \triangleright z, y \triangleright z} \beta_{x,z}.$$

If X is a quandle, then β is called a constant quandle cocycle if β further satisfy $\beta_{x,x} = id$.

Definition 3.1.4 *Let $(X, *, f)$ be an f-rack and $\lambda : X \times X \to S_A$, where S_A is the group of permutations of A. If,*

$$\lambda_{x*y, f(z)} \lambda_{x,y} = \lambda_{x*z, y*z} \lambda_{x,z}$$

we say λ is a constant f-rack cocycle.

If $(X, *, f)$ is an f-quandle and further satisfies $\lambda_{x,x} = g$ for all $x \in X$, then we say λ is a constant f-quandle cocycle.

3.2 Modules over f-Racks

In this section we will provide an equation for generalized f-rack cocycles and provide a couple examples.

Definition 3.2.1 *Let $(X, *, f)$ be an f-rack. Let A be an abelian group and let $g : A \to A$ be a homomorphism. A structure of an X-module on A consists of a family of automorphisms $(\eta_{ij})_{i,j \in X}$ and a family of endomorphisms $(\tau_{ij})_{i,j \in X}$ of A satisfying the following conditions:*

$$\eta_{x*y,f(z)}\eta_{x,y} = \eta_{x*z,y*z}\eta_{x,z} \tag{3.2.3}$$

$$\eta_{x*y,f(z)}\tau_{x,y} = \tau_{x*z,y*z}\eta_{y,z} \tag{3.2.4}$$

$$\tau_{x*y,f(z)}g = \eta_{x*z,y*z}\tau_{x,z} + \tau_{x*z,y*z}\tau_{y,z} \tag{3.2.5}$$

Remark 3.2.2 *If X is an f-quandle and we set $x = y = z$, then the f-quandle structure of the $\mathbb{Z}(X)$-module on A is a structure of an X-module that further satisfies*

$$\tau_{f(x),f(x)}g = \big(\eta_{f(x),f(x)} + \tau_{f(x),f(x)}\big)\tau_{x,x}.$$

Furthermore, if f and g are identity maps, then η and τ satisfy the following condition:

$$\eta_{x,x} + \tau_{x,x} = id$$

as shown in [7].

Recall that α is called a dynamical cocycle if it satisfies the conditions in Proposition 3.1.1. Assume that $\alpha_{x,y}(a,b) = \eta_{x,y}(a) + \tau_{x,y}(b) + \kappa_{x,y}$, then let $\Omega(X)$ be the free $\mathbb{Z}$-algebra generated by $\eta_{x,y}, \tau_{x,y}$ for $x, y \in X$, where $\eta_{x,y}$ is invertible for every $x, y \in X$. We define $\mathbb{Z}(X)$ to be the quotient $\mathbb{Z}(X) = \Omega(X)/R$ where R is the ideal generated by the above relations given in the definition 3.2.1. The algebra $\mathbb{Z}(X)$ is called the f-quandle algebra over X.

Recall from [7], a generalized rack 2-cocycle condition is given by

$$\eta_{x\triangleright y,z}(\kappa_{x,y}) + \kappa_{x\triangleright y,z} = \eta_{x\triangleright z,y\triangleright z}(\kappa_{x,z}) + \tau_{x\triangleright z,y\triangleright z}(\kappa_{y,z}) + \kappa_{x\triangleright z,y\triangleright z}$$

for any $x, y, z \in X$, where $\kappa_{x,y}$ means $\kappa(x, y)$ for $(x, y) \in X^2$.

Example 3.2.3 *Let A be a non-empty set, (X, f) be an f-quandle, and κ be a generalized 2-cocycle. For $a, b \in A$, let*

$$\alpha_{x,y}(a, b) = \eta_{x,y}(a) + \tau_{x,y}(b) + \kappa_{x,y}.$$

We substitute $\alpha_{x,y}(a, b)$ in item (3) of Proposition 3.1.1 to obtain

$$
\begin{aligned}
\textit{Left-hand side} \quad &= \quad \alpha_{x*y,f(z)}(\alpha_{x,y}(a, b) \,, \; g(c)) \\
&= \quad \alpha_{x*y,f(z)}\big(\eta_{x,y}(a) + \tau_{x,y}(b) + \kappa_{x,y} \,, \; g(c)\big) \\
&= \quad \eta_{x*y,f(z)}\big(\eta_{x,y}(a) + \tau_{x,y}(b) + \kappa_{x,y}\big) + \tau_{x*y,f(z)}(g(c)) + \kappa_{x*y,f(z)}.
\end{aligned}
$$

$$
\begin{aligned}
\textit{Right-hand side} \quad &= \quad \alpha_{x*z,y*z}(\alpha_{x,z}(a, c) \,, \; \alpha_{y,z}(b, c)) \\
&= \quad \alpha_{x*z,y*z}\big(\eta_{x,z}(a) + \tau_{x,z}(c) + \kappa_{x,z} \,, \; \eta_{y,z}(a) + \tau_{y,z}(c) + \kappa_{y,z}\big) \\
&= \quad \eta_{x*z,y*z}\big(\eta_{x,z}(a) + \tau_{x,z}(c) + \kappa_{x,z}\big) + \tau_{x*z,y*z}\big(\eta_{y,z}(a) + \tau_{y,z}(c) + \kappa_{y,z}\big) \\
&+ \quad \kappa_{x*z,y*z}.
\end{aligned}
$$

By comparison, it can be verified directly that α is a dynamical cocycle and the following relations hold:

$$\eta_{x*y,f(z)}\eta_{x,y} = \eta_{x*z,y*z}\eta_{x,z} \tag{3.2.6}$$

$$\eta_{x*y,f(z)}\tau_{x,y} = \tau_{x*z,y*z}\eta_{y,z} \tag{3.2.7}$$

$$\tau_{x*y,f(z)}g = \eta_{x*z,y*z}\tau_{x,z} + \tau_{x*z,y*z}\tau_{y,z} \tag{3.2.8}$$

$$\eta_{x*y,f(z)}\kappa_{x,y} + \kappa_{x*y,f(z)} = \eta_{x*z,y*z}\kappa_{x,z} + \tau_{x*z,y*z}\kappa_{y,z} + \kappa_{x*z,y*z} \tag{3.2.9}$$

where $\kappa_{x,y}$ means $\kappa(x,y)$ for $(x,y) \in X \times X$, as in [7]. The condition in (3.2.9) is called the *generalized f-rack 2-cocycle condition*.

Definition 3.2.4 *When κ further satisfies $\kappa_{z,z} = 0$ in (3.2.9) for any $z \in X$, we call it a generalized f-quandle 2-cocycle.*

Example 3.2.5 *Let $(X, *_f, f)$ be an f-quandle and A be an abelian group. Set $\eta_{x,y} = id, \tau_{x,y} = 0, \kappa_{x,y} = \phi(x,y)$. Then ϕ satisfies the following condition:*

$$\phi(x,y) + \phi(x *_f y, f(z)) = \phi(x,z) + \phi(x *_f z, y *_f z).$$

If f further satisfy $f = id$, then $\phi(x,y) + \phi(x \triangleright y, z) = \phi(x,z) + \phi(x \triangleright z, y \triangleright z)$ as in [9].

Example 3.2.6 *Let $\Gamma = \mathbb{Z}[T^{\pm 1}, S]$ denote the ring of Laurent polynomials. Then any Γ-module M is a $\mathbb{Z}(X)$-module for any f-quandle $(X, *, f)$ with $ST = TS$ where*

$$\eta_{x,y}(a) = Ta, \quad \tau_{x,y}(b) = Sb \quad and \quad f(c) = (T + S)(c)$$

for any $x, y \in X$.

Example 3.2.7 *For any f-quandle $(X, *, f)$, the enveloping group*

$$G_X = < x \in X | x * y = f(y)xy^{-1} >$$

with $\eta_{x,y}(a) = f(y)a$ and $\tau_{x,y}(b) = f(b) - (x * y)(b)$ where $x, y \in X$ and $a, b \in G$, is an X-module.

Example 3.2.8 *Here we provide an example of an f-quandle module and the explicit formula for the f-quandle 2-cocycle obtained from a group coboundary.*

Let G be a group and A be an abelian group. Let $0 \to A \to E \to G \to 1$ be a short exact sequence of groups where $E = A \rtimes_\theta G$ is an extension of G by A with a group 2-cocycle θ. Note that θ is a coboundary as well.

The group multiplication in E is given by

$$(a, x) \cdot (b, y) = (a + x(b) + \theta(x, y),\ xy),$$

where $x(b)$ means the action of A on G. Recall that the group 2-cocycle condition is

$$\theta(x, y) + \theta(xy, z) = x\theta(y, z) + \theta(x, yz).$$

So we obtain that

$$(b, y)^{-1} = (-y^{-1}(b) - \theta(y^{-1}, y),\ y^{-1})$$

Now, let $X = G$ be an f-quandle with the operation $x * y = y^{-1}xf(y)$ and let $g : A \to A$ be a map on A so that we have a map $F : E \to E$ given by $F(a, x) = (g(a), f(x))$. Therefore, the group E becomes an f-quandle with the operation

$$(a, x) * (b, y) = (b, y)^{-1}(a, x)F(b, y).$$

Then one can compute,

$$
\begin{aligned}
(a, x) * (b, y) &= (b, y)^{-1}(a, x)F(b, y) \\
&= (b, y)^{-1}(a, x)(g(b), f(y)).
\end{aligned}
$$

Explicit computations yield

$$\eta_{x,y}(a) = y^{-1}a$$

$$\tau_{x,y}(b) = y^{-1}xf(b) - y^{-1}b$$

and

$$\kappa_{x,y} = -\theta(y^{-1}, y) + \theta(y^{-1}, x) + \theta(y^{-1}x, g(y)).$$

4   Generalized Cohomology Theory of f-racks and f-quandles

In this chapter, our aim is to introduce a generalized cohomology theory of f-quandles and give some examples. Quandle cohomology groups will be computed for some Alexander f-quandles. We first start by giving the general formula of the boundary map δ^n showing that we obtain a cochain complex. Then we will give the formula of the boundary map in the simplest case when η is the identity map and τ is the zero map as in Example (3.2.5). We will also give the formula in the case when η is multiplication by T and τ is multiplication by S as in Example (2.1.8).

In the second half of this chapter, we make a short comment on some connections to Knot Theory. We will give a diagrammatic notion for the twisted operator and then show how it relates to the Reidemeister moves.

4.1   Cohomology Theory of f-quandles

Let $(X, *, f)$ be an f-rack where $f : X \to X$ is an f-rack morphism. We start by introducing some notation so that we can define the most generalized cohomology theories of f-racks as follows.

For a sequence of elements $(x_1, x_2, x_3, x_4, \ldots, x_n) \in X^n$, define

$$[x_1, x_2, x_3, x_4, \ldots, x_n] = ((\ldots (x_1 * x_2) * f(x_3)) * f^2(x_4)) * \ldots) * f^{n-2}(x_n).$$

Notice that, for $i < n$, by applying the first axiom of f-racks $(n - i)$ times, first grouping the first $(i - 1)$ terms together, then iterating this process, again grouping

and iterating each time, we obtain the following formula which we will use in our computations.

$$[x_1, x_2, x_3, x_4, \ldots, x_n] = [x_1, \ldots, \hat{x}_i, \ldots, x_n] * f^{i-2}[x_i, \ldots, x_n].$$

Let $C_n^R(X, f)$ be the free abelian group generated by n-tuples $(x_1, \ldots, x_n)$ of elements of an f-quandle (X, f). Define $C_R^n((X, f), A) := \mathrm{Hom}(C_n^R(X, f), A)$.

The following theorem provides a cohomology complex for f-racks.

Theorem 4.1.1 *The following family of operators $\delta^n : C_R^n((X, f), A) \to C_R^{n+1}((X, f), A)$ defines a cohomology complex:*

$$\delta^n \phi(x_1, \ldots, x_{n+1})$$

$$= (-1)^{n+1} \sum_{i=2}^{n+1} (-1)^i \Big(\eta_{[x_1, \ldots, \hat{x}_i, \ldots, x_{n+1}], f^{i-2}[x_i, \ldots, x_{n+1}]} \phi(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1})$$

$$- \phi(x_1 * x_i, x_2 * x_i, \ldots, x_{i-1} * x_i, f(x_{i+1}), \ldots, f(x_{n+1})) \Big)$$

$$+ (-1)^{n+1} \tau_{[x_1, x_3, \ldots, x_{n+1}], [x_2, \ldots, x_{n+1}]} \phi(x_2, \ldots, x_{n+1})$$

for $n \geq 2$ and $\delta^n = 0$ for $n \leq 1$. Then $C_R^*(X, f) = \{C_R^n(X, f), \delta^n\}$ is a cochain complex.

For $n \geq 2$, let $C_n^D(X, f)$ be the subgroup of $C_n^R(X, f)$ generated by $(n + 1)$-tuples $\vec{x} = (x_1, x_2, \ldots, x_{n+1})$ where $x_i = x_{i+1}$ for some $2 \leq i \leq n$.
Define $P^n(X, f, A) = \{\phi \in C_R^n(X, f, A) \mid \phi(\vec{x}) = 0, \forall \vec{x} \in C_n^D(X)\}$, otherwise define $C_D^n(X, f) = 0$. Then $\delta^n(C_D^n(X, f)) \subset C_D^{n+1}(X, f)$ and thus $C_D^*(X, f) = \{C_D^n(X, f), \delta^n\}$ is called degenerate subcomplex of $C_R^*(X, f)$. The cohomology $H_D^n(X, f, A)$ is called as degenerate f-quandle cohomology of X with coefficients in A.

Before we give the general proof of the above theorem, we would like to show a low

dimensional case, that is for all $x_1, x_2, x_3 \in X$,

$$\delta^2 \delta^1 \phi(x_1, x_2, x_3) = 0$$

which will help to understand the general proof. For $\psi : X \to A$, according to Theorem 4.1.1, we have the map $\delta^1 \psi : X \times X \to A$ given by

$$\delta^1 \psi(x_1, x_2) = \eta_{[x_1], f^0[x_2]} \psi(x_1) - \psi(x_1 * x_2) + \tau_{[x_1],[x_2]} \psi(x_2).$$

Now for a map $\phi : X \times X \to A$, we have the map $\delta^2 \phi : X \times X \times X \to A$ such that

$$\begin{aligned}
\delta^2 \phi(x_1, x_2, x_3) &= -\eta_{[x_1,x_3], f^0[x_2,x_3]} \phi(x_1, x_3) + \eta_{[x_1,x_2], f[x_3]} \phi(x_1, x_2) \\
&+ \phi(x_1 * x_2, f(x_3)) - \phi(x_1 * x_3, x_2 * x_3) - \tau_{[x_1,x_3],[x_2,x_3]} \phi(x_2, x_3).
\end{aligned}$$

Now taking the composition $\delta^2 \delta^1 \phi(x_1, x_2, x_3)$, we get the following:

$$\begin{aligned}
&\delta^2 \delta^1 \psi(x_1, x_2, x_3) \\
&= -\eta_{[x_1,x_3], f^0[x_2,x_3]} \left(\eta_{[x_1], f^0[x_3]} \psi(x_1) - \psi(x_1 * x_3) + \tau_{[x_1],[x_3]} \psi(x_3) \right) \\[2mm]
&+ \eta_{[x_1,x_2], f[x_3]} \left(\eta_{[x_1], f^0[x_2]} \psi(x_1) - \psi(x_1 * x_2) + \tau_{[x_1],[x_2]} \psi(x_2) \right) \\[2mm]
&+ \left(\eta_{[x_1 * x_2], f^0[f(x_3)]} \psi(x_1 * x_2) - \psi((x_1 * x_2) * f(x_3)) + \tau_{[x_1 * x_2],[f(x_3)]} \psi(f(x_3)) \right) \\[2mm]
&- \left(\eta_{[x_1 * x_3], f^0[x_2 * x_3]} \psi(x_1 * x_3) - \psi((x_1 * x_3) * (x_2 * x_3)) + \tau_{[x_1 * x_3],[x_2 * x_3]} \psi(x_2 * x_3) \right) \\[2mm]
&- \tau_{[x_1,x_3],[x_2,x_3]} \left(\eta_{[x_2], f^0[x_3]} \psi(x_2) - \psi(x_2 * x_3) + \tau_{[x_2],[x_3]} \psi(x_3) \right).
\end{aligned}$$

Notice that the coefficient of $\psi(x_1)$ is

$$-\eta_{[x_1,x_3],f^0[x_2,x_3]}\eta_{[x_1],f^0[x_3]} + \eta_{[x_1,x_2],f[x_3]}\eta_{[x_1],f^0[x_2]}$$

which can be rewritten as

$$-\eta_{x_1*x_3,x_2*x_3}\eta_{x_1,x_3} + \eta_{x_1*x_2,f(x_3)}\eta_{x_1,x_2} = 0$$

which follows from (3.2.3). Similarly, the coefficient of $\psi(x_2)$ is

$$
\begin{aligned}
\eta_{[x_1,x_2],f[x_3]}\tau_{[x_1],[x_2]} - \tau_{[x_1,x_3],[x_2,x_3]}\eta_{[x_2],f^0[x_3]} &= \eta_{x_1*x_2,f(x_3)}\tau_{x_1,x_2} - \tau_{x_1*x_3,x_2*x_3}\eta_{x_2,x_3} \\
&= 0 \quad \text{from (3.2.4)}
\end{aligned}
$$

and the coefficient of $\psi(x_3)$ is

$$
\begin{aligned}
-\eta_{[x_1,x_3],f^0[x_2,x_3]}\tau_{[x_1],[x_3]} + \tau_{[x_1*x_2],[f(x_3)]}f &- \tau_{[x_1,x_3],[x_2,x_3]}\tau_{[x_2],[x_3]} \\
= -\eta_{x_1*x_3,x_2*x_3}\tau_{x_1,x_3} + \tau_{x_1*x_2,f(x_3)}f &- \tau_{x_1*x_3,x_2*x_3}\tau_{x_2,x_3} \\
&= 0 \quad \text{from (3.2.5)}
\end{aligned}
$$

The coefficient of $\psi(x_1 * x_3)$ is given by

$$\eta_{[x_1,x_3],f^0[x_2,x_3]} - \eta_{[x_1*x_3],f^0[x_2*x_3]} = 0$$

since $[x_1, x_3] = x_1 * x_3$ and $[x_2, x_3] = x_2 * x_3$. The rest of the terms cancel out by (2.1.1) and we see that $\delta^2\delta^1\phi(x_1, x_2, x_3) = 0$.

Now we will give the general proof that, for all $n \geq 1$, $\delta^{n+1}\delta^n = 0$.

Proof. In order to prove that $\delta^{n+1}\delta^n = 0$, we use the linearity of η and τ, and break the composition $\delta^{n+1}\delta^n = 0$ into two pieces.

First, for $i \leq j$, we will show that the composition of the ith term of the first summand of δ^n with the jth term of the first summand of δ^{n+1} cancels out with the composition of the $(j+1)^{th}$ term of the first summand of δ^n with the ith term of the first summand

39

of δ^{n+1}. As the signs of these terms are opposite, we need only to show that the compositions are equal up to their signs. Now, we can see that the composition of the ith term of the first summand of δ^n with the jth term of the first summand of δ^{n+1} can be rewritten as follows:

$$\eta_{[x_1,\ldots,\hat{x}_i,\ldots,x_{n+1}],f^{i-1}[x_i,\ldots,x_{n+1}]}\eta_{[x_1,\ldots,\hat{x}_i,\ldots,\hat{x}_{j+1},\ldots,x_{n+1}],f^j[x_{j+1},\ldots,x_{n+1}]}$$

$$= \eta_{[x_1,\ldots,\hat{x}_i,\ldots,\hat{x}_{j+1},\ldots,x_{n+1}]*f^{j-1}[x_j,\ldots,x_{n+1}],f^{i-1}[x_i,\ldots,\hat{x}_{j+1},\ldots,x_{n+1}]*f^{i-1}f^{j-i+1}[x_{j+1},\ldots,x_{n+1}]}$$

$$\eta_{[x_1,\ldots,\hat{x}_i,\ldots,\hat{x}_{j+1},\ldots,x_{n+1}],f^{j-1}[x_{j+1},\ldots,x_{n+1}]}$$

$$= \eta_{[x_1,\ldots,\hat{x}_i,\ldots,\hat{x}_{j+1},\ldots,x_{n+1}]*f^{i-1}[x_i,\ldots,\hat{x}_{j+1},\ldots,x_{n+1}],f^j[x_{j+1},\ldots,x_{n+1}]}$$

$$\eta_{[x_1,\ldots,\hat{x}_i,\ldots,\hat{x}_{j+1},\ldots,x_{n+1}],f^{i-1}[x_i,\ldots,\hat{x}_{j+1},\ldots,x_{n+1}]}$$

$$= \eta_{[x_1,\ldots,\hat{x}_{j+1},\ldots,x_{n+1}],f^j[x_{j+1},\ldots,x_{n+1}]}\eta_{[x_1,\ldots,\hat{x}_i,\ldots,\hat{x}_{j+1},\ldots,x_{n+1}],f^{i-1}[x_i,\ldots,\hat{x}_{j+1},\ldots,x_{n+1}]},$$

which is precisely the $(j+1)^{th}$ term of the first summand of δ^n with the ith term of the first summand of δ^{n+1}. For the simplicity, let's denote this by

$$\eta_i\eta_j \quad = \quad \eta_{j+1}\eta_i.$$

Similar computations show that for $i \leq j$, the composition of τ from δ^n with the ith term of the first sum of δ^{n+1} cancels out with the composition of the $(i+1)^{th}$ term of the first sum of δ^n with τ from δ^{n+1} (with the same relation holding for the compositions of τ and the second summands). That can be abbreviate by

$$\eta_i\tau \quad = \quad \tau\eta_{i+1}.$$

The composition of the ith term of the second summand of δ^n with the jth term of the second summand of δ^{n+1} cancels with the $(j+1)^{th}$ term of the second summand of δ^n with the ith term of the second summand of δ^{n+1}, the composition of the ith term of the second summand of δ^n with the jth term of the first summand of δ^{n+1} cancels with the $(j+1)^{th}$ term of the first summand of δ^n with the ith term of the second summand of δ^{n+1} for $i \leq j$. For the sake of brevity we will omit showing these

computations.

All these relations leave three remaining terms, which cancel via the third axiom in Definition 3.2.1.

∎

The Table below presents all the above relations in an easier to read manner. In the table η_i represents the ith summand of the first sum, $\circ_i$ represents the ith summand of the second sum, with order of composition determining its origin in δ^n or δ^{n+1}.

$$\eta_i \eta_j = \eta_{j+1} \eta_i$$
$$\eta_i \circ_j = \circ_{j+1} \eta_i$$
$$\eta_i \tau = \tau \eta_{i+1}$$
$$\tau \circ_i = \circ_{i+1} \tau$$
$$\circ_i \circ_j = \circ_{j+1} \circ_i .$$

Now we will give some examples of f-quandles when η and τ are precisely defined.

Example 4.1.2 *Let $(X, *, f)$ be an f-quandle where η is the identity map and τ is the zero map. For $n \geq 2$, we compute δ^n :*

$$
\begin{aligned}
\delta^n &\phi(x_1, x_2, \ldots, x_{n+1}) \\
&= (-1)^{n+1} \sum_{i=2}^{n+1} (-1)^i \phi \left\{ (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}) \right. \\
&\quad \left. - (x_1 * x_i, x_2 * x_i, \ldots, x_{i-1} * x_i, f(x_{i+1}), \ldots, f(x_{n+1})) \right\}.
\end{aligned}
$$

Example 4.1.3 Let η be multiplication by T and τ be multiplication by S as in Example 2.1.8 with $TS = ST$ and $f(z) = (S + T)z$ for all $z \in X$. Then for $n \geq 2$, the map δ^n is

$$\delta^n \phi(x_1, x_2, \ldots, x_{n+1})$$

$$= (-1)^{n+1} \sum_{i=2}^{n+1} (-1)^i T\phi(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1})$$

$$- (-1)^{n+1} \sum_{i=2}^{n+1} (-1)^i \phi(x_1 * x_i, x_2 * x_i, \ldots, x_{i-1} * x_i, f(x_{i+1}), \ldots, f(x_{n+1}))$$

$$+ (-1)^{n+1} S\phi(x_2, \ldots, x_{n+1}).$$

In particular, the 1-cocycle condition is written for a function $\psi : X \to A$ as

$$T\psi(x) + S\psi(y) - \psi(x * y) = 0.$$

Note that this means $\psi : X \to A$ is a quandle homomorphism.

For $\phi : X \times X \to A$, the 2-cocycle condition must be written as

$$T\phi(x_1, x_2) + \phi(x_1 * x_2, f(x_3))$$

$$= T\phi(x_1, x_3) + S\phi(x_2, x_3) + \phi(x_1 * x_3, x_2 * x_3).$$

Remark 4.1.4 *Suppose that the coefficient group of A is written additively as $\mathbb{Z}$ or $\mathbb{Z}_n$. Define a characteristic function*

$$\chi_x(y) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{if } x \neq y \end{cases}$$

from the free abelian group generated by X^n to the group A. The set $\{\chi_x : x \in X^n\}$ of such functions spans the group $C(X, f, A)$ of cochains. Thus, if $h \in C_R^1(X, f, A)$, then

$$h = \sum_{x \in X^n} \lambda_x \chi_x$$

where $\lambda_x \in A$ are the coefficients of χ_x.

Similarly, define a characteristic function

$$\chi_{i,j}(k,l) = \begin{cases} 1, & \text{if } (i,j) = (k,l) \\ 0, & \text{if } (i,j) \neq (k,l) \end{cases}$$

from the free abelian group generated by X^n to the group A. The set $\{\chi_{i,j} : (i,j) \in X^n \times X^n\}$ of such functions spans the group $C^2(X,f,A)$ of cochains. Thus, if $g \in C_R^2(X,f,A)$, then

$$g = \sum_{(i,j) \in X^n} \lambda_{i,j} \chi_{i,j}$$

where $\lambda_{i,j} \in A$ are the coefficients of $\chi_{i,j}$.

Now we will give an explicit example.

Example 4.1.5 *We compute the first and second cohomology groups H^1 and H^2, with coefficients in the abelian group $\mathbb{Z}_3$, of the f-quandle $X = \mathbb{Z}_3$, $T = 1, S = 2$. Notice that then $f(x) = 0$. A direct computation shows $H_R^1(\mathbb{Z}_3, \mathbb{Z}_3)$ is 1-dimensional with basis $\{\chi_1 + 2\chi_2\}$. Now consider the 2–cocycle $\phi = \sum_{i,j \in \mathbb{Z}_3} \lambda_{(i,j)} \chi_{(i,j)}$ where $\chi(i,j)$ denotes the characteristic function. Then, for all $(i,j,k) \in \mathbb{Z}_3$, ϕ satisfies: $[\phi(i,j) + \phi(i+2j,0) - \phi(i,k) - 2\phi(j,k) - \phi(i+2k,j+2k) = 0.]$ Therefore, a direct computation shows that $H^2(\mathbb{Z}_3, \mathbb{Z}_3)$ is 1-dimension with basis $\{\chi_{(1,2)} - \chi_{(2,1)}\}$.*

Proof. The f-rack $\mathbb{Z}_3$ has the operation

$$i * j = i + 2j \pmod 3.$$

First, note that $h \in Z^1(\mathbb{Z}_3, f, \mathbb{Z}_3)$. Then the 1-cocycle condition is

$$h(i) + 2h(j) - h(i * j) = 0$$

which can be rewritten as

$$h(i) + 2h(j) - h(i + 2j) = 0.$$

43

This implies

$$h(0) = 0$$

$$h(2) + h(1) = 0$$

By taking $h(1) = \beta$, we get $h(2) = 2\beta$.

A direct computation shows $Z^1(X = \mathbb{Z}_3, A = \mathbb{Z}_3)$ is 1-dimensional with basis $\{\chi_1 + 2\chi_2\}$.

Now let $g \in Z^2(\mathbb{Z}_3, f, \mathbb{Z}_3)$ be expressed as

$$g = \sum_{i,j \in \mathbb{Z}_3} \lambda_{(i,j)} \chi_{(i,j)}$$

Then, the two cocycle condition can be written as

$$\lambda_{(i,j)} + \lambda_{(i*j,f(k))} - \lambda_{(i,k)} - 2\lambda_{(j,k)} - \lambda_{(i*k,j*k)} = 0 \ \text{ for } \ i, j, k \in \mathbb{Z}_3.$$

It simplifies to the equation

$$\lambda_{(i,j)} + \lambda_{(i+2j,0)} - \lambda_{(i,k)} - 2\lambda_{(j,k)} - \lambda_{(i+2k,j+2k)} = 0 \ \text{ for } \ i, j, k \in \mathbb{Z}_3$$

and

$$\lambda_{(i,i)} = 0 \ \text{ for } \ i \in \mathbb{Z}_3.$$

Recall that the f-rack $\mathbb{Z}_3$ has the operation

$$i * j = i + 2j \pmod 3.$$

Substituting the elements for all possibilities for the variables i, j in the above expres-

sion: as i and j vary over all values of $\mathbb{Z}_3$, we get the following simplified equations:

$$\lambda_{(2,0)} + \lambda_{(1,0)} = 0$$
$$\lambda_{(0,2)} - \lambda_{(0,1)} - \lambda_{(2,0)} = 0$$
$$\lambda_{(0,1)} - \lambda_{(1,0)} - \lambda_{(0,2)} = 0$$
$$\lambda_{(1,2)} + \lambda_{(2,0)} + \lambda_{(2,1)} - \lambda_{(0,1)} = 0$$
$$\lambda_{(2,1)} + \lambda_{(1,0)} + \lambda_{(1,2)} - \lambda_{(0,2)} = 0$$
$$\lambda_{(i,i)} = 0 \text{ for } i \in \{0,1,2\}.$$

Therefore, by taking $\lambda_{(1,2)} = \alpha$, $\lambda_{(2,0)} = \beta$ and $\lambda_{(2,1)} = \gamma$, we get

$$\lambda_{(0,0)} = 0 , \quad \lambda_{(0,1)} = \alpha + \beta + \gamma , \quad \lambda_{(0,2)} = \alpha - \beta + \gamma ,$$
$$\lambda_{(1,0)} = -\beta , \quad \lambda_{(1,1)} = 0 , \quad \lambda_{(1,2)} = \alpha ,$$
$$\lambda_{(2,0)} = \beta , \quad \lambda_{(2,1)} = \gamma , \quad \lambda_{(2,2)} = 0.$$

Then

$$g = \alpha[\chi_{(0,1)} + \chi_{(0,2)} + \chi_{(1,2)}] + \beta[\chi_{(0,1)} - \chi_{(0,2)} - \chi_{(1,0)} + \chi_{(2,0)}] + \gamma[\chi_{(0,1)} + \chi_{(0,2)} + \chi_{(2,1)}].$$

Since

$$\delta\chi_0 = \chi_{(0,1)} + \chi_{(0,2)} - \chi_{(1,0)} - \chi_{(2,0)}$$
$$\delta\chi_1 = -\chi_{(0,1)} - \chi_{(0,2)} + \chi_{(1,2)} + \chi_{(2,1)}$$

$\dim(H^2) = \dim\left(Ker\,\delta^2\right) - \dim(Im\,\delta^1) = 3 - 2 = 1$. Second cohomology has the basis $\{\chi_{(1,2)} - \chi_{(2,1)}\}$.

∎

4.2 Relation to Knot Theory

The relationship between quandles and knots leads to the natural question of whether f-quandles could be used to define similar knot invariants. However, the introduction

of the f-map in the axioms causes the standard labeling schemes for the crossing diagrams to fail to produce labeling invariants under the Reidemeister moves. Instead, we found the following crossing diagram to be of greater interest if we consider the crossing to be as follows:
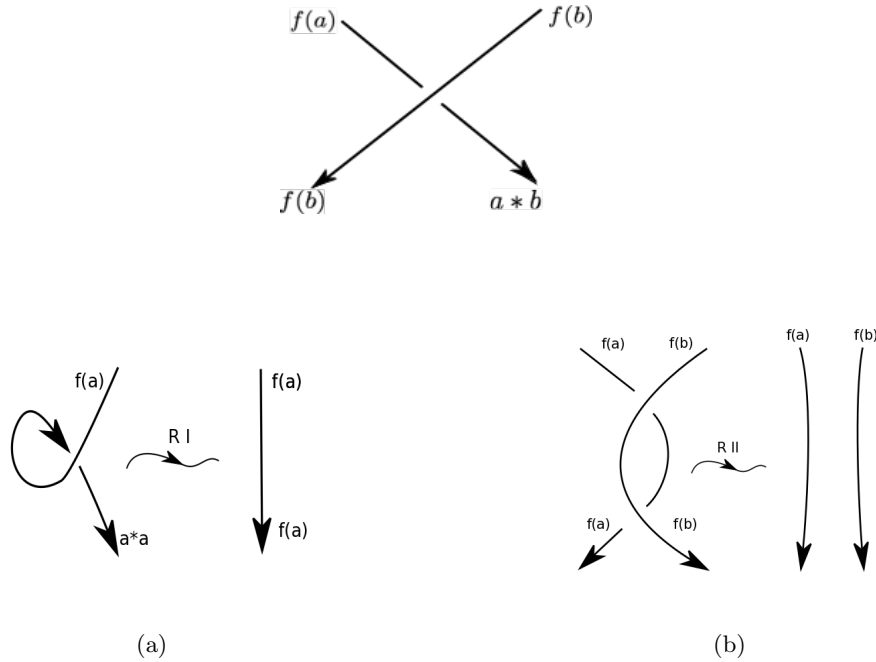
Figure 4.1: Relationship between f-quandle axioms and Reidemeister move I (a) and II(b)
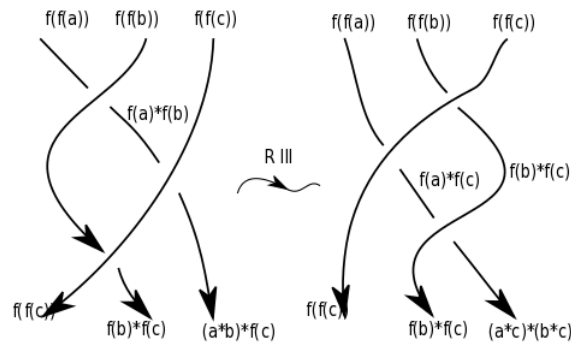
Figure 4.2: Relationship between f-quandle axioms and Reidemeister move III.

Unfortunately, we see that change-of-label at an under crossing is not a closed operation unless the f-map is bijective. That is, for an f-quandle $(X, *, f)$, the label of the arc entering a crossing must be an element of $f(X)$, while the label of the out going arc need not be. Indeed, to ensure a consistent labeling, it is clear one must only use labels from $f^n(X)$, where n is such that $f^n(X) = f^{n+m}(X)$ for every m, to ensure one may return to the same labeling upon completing a circuit of the knot. If a particular subset is selected though, one is restricted to a sub-f-quandle on which the f-map is bijective.

While these issues are resolved when labeling via an f-quandle with a bijective f-map, it appeared that this reduces the labeling by the f-quandle to precisely that of a standard quandle.

In this chapter, we focus on Alexander f-quandles introduced in the previous chapter and their f-quandle cohomology groups, particularly their low dimensional cohomology groups. Precisely, we determine the second, third, and fourth cohomology groups of Alexander f-quandles of the form $\mathbb{F}_q[T, S]/(T - \omega, S - \beta)$, where $\mathbb{F}_q$ denotes the finite field of order q, $\omega \in \mathbb{F}_q \backslash \{0, 1\}$ and $\beta \in \mathbb{F}_q$. Throughout this chapter, as in standard quandle cohomology theory, we will factor by the degenerate subcomplex and call it the f-quandle cohomology of X and denote it by $H^*(X, *, f)$ (See the precise definition below).

5.1 Preliminaries

We first recall the definition of Alexander f-quandles. Any $\mathbb{Z}[T^{\pm 1}, S]$-module M with the binary operation $x * y = Tx + Sy$ for $x, y \in M$, with $TS = ST$ and $f(x) = (S+T)x$, is an *Alexander f-quandle*.

In Example 4.1.2 of the previous chapter, when $\eta = id$ and $\tau = 0$, we obtained a cohomology theory where the differential is given by

$$
\begin{aligned}
\delta^n &\phi(x_1, \ldots, x_{n+1}) \\
&= (-1)^{n+1} \sum_{i=2}^{n+1} (-1)^i \phi(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1}) \\
&\quad - (-1)^{n+1} \sum_{i=2}^{n+1} (-1)^i \phi(x_1 * x_i, x_2 * x_i, \ldots, x_{i-1} * x_i, f(x_{i+1}), \ldots, f(x_{n+1})).
\end{aligned}
\tag{5.1.1}
$$

---

[*]Sections of this chapter are taken from [13], which has been submitted to "J. of Knot Theory and Its Ramification", 2017.

As in standard quandle homology theory, the degenerate subcomplex is given by $C_n^D = \{(x_1, x_2, ..., x_n) \in X^n \,;\, x_i = x_{i+1} \text{ for } i \geq 2\}$. A similar degenerate subcomplex appeared in the work of Niebrzydowski and Przytycki, [39] under the name *late degenerate complex*. Also, recall Example 2.1.8: Any $\mathbb{Z}[\omega^{\pm 1}, \beta]$-module M is an Alexander f-quandle when

$$x * y = \omega \cdot x + \beta \cdot y$$

for $x, y \in M$ and $\omega\beta = \beta\omega$. Notice that $f(x) = (\omega + \beta)x$.

Remark 5.1.1 *When f is the identity map and $\beta = 1 - \omega$ above, then $(X, *)$ is a quandle and $(M, *)$ is an Alexander quandle as usual.*

We obtain similar results as in [3]: Consider $X = \mathbb{Z}_p[T^{\pm 1}, S]/h(t)$ and $A = \mathbb{Z}_p[T^{\pm 1}, S]/g(t)$ are module rings where $h(t), g(t) \in \mathbb{Z}_p[T^{\pm 1}, S]$ and $g(t)$ divides $h(t)$. Also there exists a quotient homomorphism $X \to A$.

Proposition 5.1.2 *Let $a_i = p^{m_i}$ for $i = 1, \cdots, n-1$, where p is a prime and the m_i are non-negative integers. For a positive integer n, define $\phi : X^n \to A$ by*

$$\phi(x_1, x_2, \cdots, x_n) = (x_1 - x_2)^{a_1}(x_2 - x_3)^{a_2} \cdots (x_{n-1} - x_n)^{a_{n-1}} x_n^{a_n},$$

where the $x_i \in A$ via the quotient map. Then

1. *If $a_n = 0$, then ϕ is an n-cocycle in $\mathbb{Z}^n(X, *, f; A)$ where $f(x) = (\omega + \beta)x$.*

2. *If $a_n = p^{m_n}$, then ϕ is an n-cocycle if $g(t)$ divides $1 - \omega^a$, where $a = a_1 + a_2 + \cdots + a_n$.*

Proof. 1. By definition of δ^n on the function ϕ, we obtain,

$$\delta^n \phi(x_1, \ldots, x_{n+1})$$

$$= (-1)^{n+1} \sum_{i=2}^{n+1} (-1)^i [\phi(x_1, \ldots, x_{i-1}, x_{i+1}, x_{i+2}, \ldots, x_{n+1})$$

$$- \phi(x_1 * x_i, \ldots, x_{i-1} * x_i, f(x_{i+1}), f(x_{i+2}), \ldots, f(x_{n+1}))]$$

$$= \bigstar$$

$$\bigstar = (-1)^{n+1} \sum_{i=2}^{n} (-1)^i (x_1 - x_2)^{a_1} \cdots (x_{i-1} - x_{i+1})^{a_{i-1}} (x_{i+1} - x_{i+2})^{a_i} \cdots$$

$$\cdots (x_n - x_{n+1})^{a_{n-1}} x_{n+1}^{a_n} (-1)^{2n+2} (x_1 - x_2)^{a_1} \cdots (x_{n-1} - x_n)^{a_{n-1}} x_n^{a_n}$$

$$- (-1)^{n+1} \sum_{i=2}^{n} (-1)^i (x_1 * x_i - x_2 * x_i)^{a_1} \cdots (x_{i-1} * x_1 - f(x_{i+1}))^{a_{i-1}}$$

$$(f(x_{i+1}) - f(x_{i+2}))^{a_i} \cdots \quad (f(x_n) - f(x_{n+1}))^{a_{n-1}} f^{a_n}(x_{n+1})$$

$$- (-1)^{2n+2} (x_1 * x_{n+1} - x_2 * x_{n+1})^{a_1} \cdots (x_{n-1} * x_{n+1} - x_n * x_{n+1})^{a_{n-1}} (x_n * x_{n+1})^{a_n}$$

$$(5.1.2)$$

By substituting $y_i = x_i - x_{i+1}$ and $f(x) = (\omega + \beta)x$, we have

$$\bigstar = \sum_{i=2}^{n} (-1)^i y_1^{a_1} y_2^{a_2} \cdots (y_{i-1} + y_i)^{a_{i-1}} y_{i+1}^{a_i} \cdots y_n^{a_{n-1}} x_{n+1}^{a_n}$$

$$+ (-1)^{n+1} y_1^{a_1} y_2^{a_2} \cdots y_{n-1}^{a_{n-1}} x_n^{a_n}$$

$$- \sum_{i=2}^{n} (-1)^i (\omega y_1)^{a_1} (\omega y_2)^{a_2} \cdots \{\omega y_{i-1} + (\omega + \beta) y_i\}^{a_{i-1}} \{(\omega + \beta) y_{i+1}\}^{a_i} \cdots$$

$$\cdots \{\omega + \beta) y_n\}^{a_{n-1}} f^{a_n}(x_{n+1})$$

$$- (-1)^{n+1} (\omega y_1)^{a_1} (\omega y_2)^{a_2} \cdots (\omega y_{n-1})^{a_{n-1}} \{\omega y_n + (\omega + \beta) x_{n+1}\}^{a_n}$$

Since each a_i is a power of p, and the coefficients are in $\mathbb{Z}_p$, we obtain

$$\delta^n\phi(x_1,\ldots,x_{n+1})$$

$$= \sum_{i=2}^{n}(-1)^i y_1^{a_1} y_2^{a_2} \cdots y_{i-1}^{a_{i-1}} y_{i+1}^{a_i} \cdots y_n^{a_{n-1}} x_{n+1}^{a_n} + (-1)^{n+1} y_1^{a_1} y_2^{a_2} \cdots y_{n-1}^{a_{n-1}} x_n^{a_n}$$

$$+ \sum_{i=2}^{n}(-1)^i y_1^{a_1} y_2^{a_2} \cdots y_i^{a_{i-1}} y_{i+1}^{a_i} \cdots y_n^{a_{n-1}} x_{n+1}^{a_n}$$

$$- \sum_{i=2}^{n}(-1)^i \omega^{a_1+a_2+\cdots+a_{i-1}}(\omega+\beta)^{a_i+\cdots+a_{n-1}} y_1^{a_1} y_2^{a_2} \cdots y_{i-1}^{a_{i-1}} y_{i+1}^{a_i} \cdots y_n^{a_{n-1}} f(x_{n+1})^{a_n}$$

$$- \sum_{i=2}^{n}(-1)^i \omega^{a_1+a_2+\cdots+a_{i-2}}(\omega+\beta)^{a_{i-1}+\cdots+a_{n-1}} y_1^{a_1} y_2^{a_2} \cdots y_i^{a_{i-1}} y_{i+1}^{a_i} \cdots y_n^{a_{n-1}} f(x_{n+1})^{a_n}$$

$$- (-1)^{n+1} \omega^{a_1+a_2+\cdots+a_{n-2}} y_1^{a_1} y_2^{a_2} \cdots y_{n-1}^{a_{n-1}} \{\omega y_n + (\omega+\beta)x_{n+1}\}^{a_n}.$$

$$(5.1.3)$$

By changing the indices in the second and fourth sum we obtain

$$\delta^n\phi(x_1,\ldots,x_{n+1})$$

$$= \sum_{i=2}^{n}(-1)^i y_1^{a_1} y_2^{a_2} \cdots y_{i-1}^{a_{i-1}} y_{i+1}^{a_i} \cdots y_n^{a_{n-1}} x_{n+1}^{a_n}$$

$$+ \sum_{i=1}^{n-1}(-1)^i y_1^{a_1} y_2^{a_2} \cdots y_{i-1}^{a_{i-1}} y_{i+1}^{a_i} \cdots y_n^{a_{n-1}} x_{n+1}^{a_n}$$

$$- \sum_{i=2}^{n}(-1)^i \omega^{a_1+a_2+\cdots+a_{i-1}}(\omega+\beta)^{a_i+\cdots+a_n} y_1^{a_1} y_2^{a_2} \cdots y_{i-1}^{a_{i-1}} y_{i+1}^{a_i} \cdots y_n^{a_{n-1}} x_{n+1}^{a_n}$$

$$- \sum_{i=1}^{n-1}(-1)^i \omega^{a_1+a_2+\cdots+a_{i-1}}(\omega+\beta)^{a_i+\cdots+a_n} y_1^{a_1} y_2^{a_2} \cdots y_{i-1}^{a_{i-1}} y_{i+1}^{a_i} \cdots y_n^{a_{n-1}} x_{n+1}^{a_n}$$

$$+ (-1)^{n+1} y_1^{a_1} y_2^{a_2} \cdots y_{n-1}^{a_{n-1}} x_n^{a_n}$$

$$- (-1)^{n+1} \omega^{a_1+a_2+\cdots+a_{n-1}} y_1^{a_1} y_2^{a_2} \cdots y_{n-1}^{a_{n-1}} \{\omega y_n + (\omega+\beta)x_{n+1}\}^{a_n}$$

which can be simplified to

$$\delta^n \phi(x_1, \ldots, x_{n+1})$$
$$= (-1)^n y_1^{a_1} y_2^{a_2} \cdots y_{n-1}^{a_{n-1}} x_{n+1}^{a_n}$$
$$- (-1)^n \omega^{a_1 + a_2 + \cdots + a_{n-1}} (\omega + \beta)^{a_n} y_1^{a_1} y_2^{a_2} \cdots y_{n-1}^{a_{n-1}} x_{n+1}^{a_n}$$
$$+ (-1)^{n+1} y_1^{a_1} y_2^{a_2} \cdots y_{n-1}^{a_{n-1}} x_n^{a_n}$$
$$- (-1)^{n+1} \omega^{a_1 + a_2 + \cdots + a_{n-1}} y_1^{a_1} y_2^{a_2} \cdots y_{n-1}^{a_{n-1}} \{\omega y_n + (\omega + \beta) x_{n+1}\}^{a_n}.$$

Notice that when $a_n = 0$, then the term $(\omega + \beta)^0 = 1$, and therefore $\delta^n \phi(x_1, \ldots, x_{n+1}) = 0$ which implies ϕ is an n-cocycle.

2. If $a_n = p^{m_n}$, then

$$\delta^n \phi(x_1, \ldots, x_{n+1})$$
$$= (-1)^n y_1^{a_1} y_2^{a_2} \cdots y_{n-1}^{a_{n-1}} x_{n+1}^{a_n} - (-1)^n \omega^{a_1 + a_2 + \cdots + a_{n-1}} (\omega + \beta)^{a_n} y_1^{a_1} y_2^{a_2} \cdots y_{n-1}^{a_{n-1}} x_{n+1}^{a_n}$$
$$+ (-1)^{n+1} y_1^{a_1} y_2^{a_2} \cdots y_{n-1}^{a_{n-1}} x_n^{a_n} - (-1)^{n+1} \omega^{a_1 + a_2 + \cdots + a_n} y_1^{a_1} y_2^{a_2} \cdots y_{n-1}^{a_{n-1}} y_n^{a_n}$$
$$- (-1)^{n+1} \omega^{a_1 + a_2 + \cdots + a_{n-1}} (\omega + \beta)^{a_n} y_1^{a_1} y_2^{a_2} \cdots y_{n-1}^{a_{n-1}} x_{n+1}^{a_n}$$

$$= (-1)^{n+1} y_1^{a_1} y_2^{a_2} \cdots y_{n-1}^{a_{n-1}} (x_n - x_{n+1})^{a_n} - (-1)^{n+1} \omega^{a_1 + a_2 + \cdots + a_n} y_1^{a_1} y_2^{a_2} \cdots y_{n-1}^{a_{n-1}} y_n^{a_n}$$
$$= (-1)^{n+1} (1 - \omega^{a_1 + a_2 + \cdots + a_n}) y_1^{a_1} y_2^{a_2} \cdots y_{n-1}^{a_{n-1}} y_n^{a_n}$$
$$= (-1)^{n+1} (1 - \omega^a) y_1^{a_1} y_2^{a_2} \cdots y_{n-1}^{a_{n-1}} y_n^{a_n} = 0 \quad \in A$$

by assumption. Hence, ϕ is an n-cocycle.

■

For convenience of calculations, we will reformulate the f-quandle cohomology as follows: For $i = 1, \cdots, n$, let $U_i = x_i - x_{i+1}$. Then (5.1.1) becomes

$$\delta^n \phi(U_1, \ldots, U_{n+1})$$

$$= (-1)^{n+1} \sum_{i=1}^{n} (-1)^i \phi(U_1, \ldots, U_{i-1}, U_i + U_{i+1}, U_{i+2}, \ldots, U_{n+1})$$

$$- (-1)^{n+1} \sum_{i=1}^{n} (-1)^i \phi(\omega\, U_1, \omega\, U_2, \ldots, \omega\, U_{i-1}, \omega\, U_i + (\omega + \beta)\, U_{i+1}, f(U_{i+2}), \ldots, f(U_{n+1})).$$

$$(5.1.4)$$

The following formula is a generalization of [41, Eq. (3)] when $\eta = id$ and $\tau = 0$. We will decompose the complex $C^n(X)$ by the homogeneous degree as in [35], that is,

$$C_d^n(X) := \{ \sum a_{i_1, \cdots, i_n} \cdot U_1^{i_1} \cdots U_n^{i_n} \in C^n(X) \mid \sum_{1 \leq k \leq n} i_k = d \}.$$

Since $\delta^n(C_d^n(X)) \subset C_d^{n+1}(X)$, we obtain a direct sum decomposition of the complex as $(C^n(X), \delta^n) = (\oplus C_d^n(X), \delta^n)$. Also we will decompose $\phi \in C_d^n(X)$ as $\phi = \sum_{0 \leq a \leq p-1} \phi_a(U_1, \cdots, U_{n-1}) \cdot U_n^a$ and $\text{degree}(\phi_a) = d_a$.

$$\delta^n(\phi)(U_1, \ldots, U_n, U_{n+1}) = \sum_{0 \leq a \leq p-1} \delta^{n-1}(\phi_a)(U_1, \ldots, U_n) \cdot U_{n+1}^a$$

$$+ (-1)^{n-1} \sum_{0 \leq a \leq p-1} \phi_a(U_1, \ldots, U_{n-1})(U_n + U_{n+1})^a$$

$$- (-1)^{n-1} \sum_{0 \leq a \leq p-1} \phi_a(U_1, \ldots, U_{n-1})\, \omega^{d_a} (\omega + \beta)^{d - d_a - a} (\omega\, U_n + (\omega + \beta)\, U_{n+1})^a.$$

$$(5.1.5)$$

5.2 The 2-cocycles

In this section, we investigate the 2-cocycles by using equation (5.1.5). More precisely, we provide a basis for the second cohomology $H_Q^2((X, *, f); \mathbb{F}_q)$.

Proposition 5.2.1 *If $\omega^{p^t + p^s} = 1$ and $(\omega + \beta)^{p^t + p^s} = 1$, where s and t are non-negative integers, then $U_1^{p^t} U_2^{p^s}$ is a 2-cocycle.*

53

Proof. By (5.1.4), we have

$$\delta(U_1^{p^t}) = (U_1 + U_2)^{p^t} - (\omega\, U_1 + (\omega + \beta)\, U_2)^{p^t}.$$

Then it follows from (5.1.4) and (5.1.5) that

$$\begin{aligned}
\delta(U_1^{p^t} U_2^{p^s}) &= \delta(U_1^{p^t})U_3^{p^s} - U_1^{p^t}(U_2 + U_3)^{p^s} \\
&\quad + U_1^{p^t}\omega^{d_a}(\omega + \beta)^{d-d_a-a}(\omega\, U_2 + (\omega + \beta)\, U_3)^{p^s}.
\end{aligned} \tag{5.2.6}$$

Also, note that $d_a = p^t$, $a = p^s$, and $d = p^t + p^s$. Then, we have from (5.2.6),

$$\begin{aligned}
\delta(U_1^{p^t} U_2^{p^s}) &= (U_1 + U_2)^{p^t} U_3^{p^s} - (\omega + \beta)^{p^s}(\omega\, U_1 + (\omega + \beta)\, U_2)^{p^t} U_3^{p^s} - U_1^{p^t}(U_2 + U_3)^{p^s} \\
&\quad + U_1^{p^t}\omega^{p^t}(\omega + \beta)^0(\omega\, U_2 + (\omega + \beta)\, U_3)^{p^s} \\
&= (1 - \omega^{p^t}(\omega + \beta)^{p^s})U_1^{p^t} U_3^{p^s} + (1 - (\omega + \beta)^{p^s + p^t})U_2^{p^t} U_3^{p^s} \\
&\quad - (1 - \omega^{p^t}(\omega + \beta)^{p^s})U_1^{p^t} U_3^{p^s} - (1 - \omega^{p^s + p^t})U_2^{p^t} U_3^{p^s}.
\end{aligned} \tag{5.2.7}$$

Since $\omega^{p^t + p^s} = 1$ and $(\omega + \beta)^{p^t + p^s} = 1$, the right-hand side of (5.2.7) vanishes. This completes the proof. ∎

We have the following:

Conjecture 5.2.2 *Fix $\omega, \beta \in \mathbb{F}_q$ with $\omega \neq 0, 1$. Let X be the corresponding Alexander f-quandle on $\mathbb{F}_q$. Then the set*

$$\{U_1^{p^v} U_2^{p^u} \mid \omega^{p^v + p^u} = 1, (\omega + \beta)^{p^v + p^u} = 1;\ 0 \leq v < u < m\}$$

is a basis for the second cohomology $H_Q^2((X, *, f); \mathbb{F}_q)$.

Example 5.2.3 *Let p be an odd prime and u, v be non-negative integers. Let $\omega = -1$ and $\beta = 2$. Then we have $\omega^{p^v + p^u} = 1$ and $(\omega + \beta)^{p^v + p^u} = 1$. Hence, the set defined in Conjecture 5.2.2 is a basis for 2-cocycles.*

Let us recall the definition of a primitive element.

Definition 5.2.4 *Let F be a finite dimensional extension field of K. An element α such that $F = K(\alpha)$ is said to be primitive.*

Example 5.2.5 *Let $f(x) = x^2 + x + 1 \in \mathbb{F}_2[x]$ and consider $\mathbb{F}_4 = \mathbb{F}_2[x]/(f)$. Let ω be a primitive element of $\mathbb{F}_4$. (Then the order of ω is 3). Let $\beta = \omega^2$. Note that $\omega^2 = \omega + 1$ and that ω^2 is also a primitive element of $\mathbb{F}_4$ since it is a conjugate of ω with respect to $\mathbb{F}_2$. We have*

$$\omega^{2^0 + 2^1} = 1 \text{ and } (\omega + \beta)^{2^0 + 2^1} = 1.$$

If Conjecture 5.2.2 true, then $\{U_1^{2^0} U_2^{2^1}\}$ will be a basis for the second cohomology $H_Q^2((X, *, f); \mathbb{F}_4)$.

5.3   The 3-Cocycles

In this section, we give a basis for the third cohomology group $H_Q^3((X, *, f); \mathbb{F}_q)$. For positive integers a and b, let

$$\mu_a(x, y) = (x + y)^a - x^a - y^a$$

and define

$$\psi(a, b) := \left(\mu_a(U_1, U_2) - \mu_a(\omega\, U_1, (\omega + \beta)\, U_2) \right) \cdot U_3^b.$$

Then we have the following proposition.

Proposition 5.3.1 *If $\omega^{a + p^s} = 1$ and $(\omega + \beta)^{a + p^s} = 1$, then $\Psi(a, p^s)$ is a 3-cocycle.*

55

Proof. Define

$$h(U_1, U_2) = \mu_a(U_1, U_2) - \mu_a(\omega\, U_1, (\omega + \beta)\, U_2).$$

Note that

$$\psi(a, b) := h(U_1, U_2) \cdot U_3^b.$$

Then by equation (5.1.4), we have

$$\delta(U_1^a) = (U_1 + U_2)^a - (\omega\, U_1 + (\omega + \beta)U_2)^a,$$

which implies the following equality.

$$h(U_1, U_2) = \delta(U_1^a) - (1 - \omega^a) \cdot U_1^a - (1 - (\omega + \beta)^a) \cdot U_2^a.$$

Also, from equation (5.1.4), we have

$$
\begin{aligned}
&\delta(h(U_1, U_2)) \\
&= -h(U_1 + U_2, U_3) + h(\omega U_1 + (\omega + \beta)U_2, (\omega + \beta)U_3) + h(U_1, U_2 + U_3) \\
&\quad - h(\omega U_1, \omega U_2 + (\omega + \beta)U_3) \\
&= (1 - \omega^a)\, h(U_1, U_2) - (1 - (\omega + \beta)^a)\, h(U_2, U_3) \\
&= \big(h(U_1, U_2) - h(U_2, U_3)\big) - \big(\omega^a\, h(U_1, U_2) - (\omega + \beta)^a\, h(U_2, U_3)\big).
\end{aligned}
\tag{5.3.8}
$$

Since

$$\psi(a, b) = h(U_1, U_2) \cdot U_3^b,$$

from equations (5.1.5) and (5.3.8) we have

$$
\begin{aligned}
&\delta(\Psi(a, b)) \\
&= \delta(h(U_1, U_2)) \cdot U_4^b - h(U_1, U_2)\, \delta(U_3^b) \\
&= \left[\big(h(U_1, U_2) - h(U_2, U_3)\big) - (\omega + \beta)^b \big(\omega^a\, h(U_1, U_2) - (\omega + \beta)^a\, h(U_2, U_3)\big) \right] U_4^b \\
&\quad - h(U_1, U_2) \big((U_3 + U_4)^b - \omega^a\, (\omega\, U_3 + (\omega + \beta)\, U_4)^b\big).
\end{aligned}
$$

$$\tag{5.3.9}$$

Let $b = p^s$. Then, from equation (5.3.9) we obtain

$$\delta(\Psi(a, p^s))$$
$$= (1 - \omega^a (\omega + \beta)^{p^s}) h(U_1, U_2) U_4^{p^s} - (1 - (\omega + \beta)^{a+p^s}) h(U_2, T_3) U_4^{p^s} \quad (5.3.10)$$
$$- (1 - \omega^{a+p^s}) h(U_1, U_2) U_3^{p^s} - (1 - \omega^a (\omega + \beta)^{p^s}) h(U_1, U_2) U_4^{p^s}.$$

Since $\omega^{a+p^s} = 1$ and $(\omega + \beta)^{a+p^s} = 1$, the right-hand side of (5.3.10) is zero. This completes the proof.

∎

Let $\chi(x, y) = \sum_{i=1}^{p-1} (-1)^{i-1} \cdot i^{-1} \cdot x^{p-i} \cdot y^i \equiv \frac{1}{p}((x + y)^p - x^p - y^p) \pmod{p}$. Define

$$E_0(a \cdot p, b) = \left(\chi(U_1, U_2)^a - (\omega + \beta)^b \chi(\omega U_1, (\omega + \beta) U_2)^a \right) \cdot U_3^b.$$

Also, define $h(U_1, U_2) := \chi(U_1, U_2)^a - (\omega + \beta)^b \chi(\omega U_1, (\omega + \beta) U_2)^a$.

Then we have

$$E_0(a \cdot p, b) = h(U_1, U_2) \cdot U_3^b.$$

Hence, we have the following proposition:

Proposition 5.3.2 *If $\omega^{p^s+p^h} = 1$ and $(\omega + \beta)^{p^s+p^h} = 1$ with $s > 0$, then $E_0(p^s, p^h)$ is a $3-cocycle.*

Proof. We compute the following coboundary of $E_0(a \cdot p, b)$.

$$\delta(E_0(a \cdot p, b)) = \delta(h(U_1, U_2)) \cdot U_4^b - h(U_1, U_2) \delta(U_3^b)$$
$$= (1 - \omega^{ap} (\omega + \beta)^b) h(U_1, U_2) U_4^b - (1 - (\omega + \beta)^{ap+b}) h(U_2, U_3) U_4^b$$
$$- h(U_1, U_2) \left((U_3 + U_4)^b - \omega^{ap} (\omega U_3 + (\omega + \beta) U_4)^b \right).$$

$$(5.3.11)$$

Let $a = p^{s-1}$ and $b = p^h$. Then from equation (5.3.11) we have

$$\delta(E_0(p^s, p^h)) = (1 - \omega^{p^s} (\omega + \beta)^{p^h}) h(U_1, U_2) \cdot U_4^{p^h}$$
$$- (1 - (\omega + \beta)^{p^s + p^h}) h(U_2, U_3) \cdot U_4^{p^h}$$
$$- (1 - \omega^{p^s + p^h}) U_3^{p^h} h(U_1, U_2)$$
$$- (1 - \omega^{p^s} (\omega + \beta)^{p^h}) h(U_1, U_2) \cdot U_4^{p^h}.$$

(5.3.12)

Since $\omega^{p^s + p^h} = 1$ and $(\omega + \beta)^{p^s + p^h} = 1$, the right hand side of (5.3.12) is zero. This completes the proof.

∎

Again, let $\chi(x, y) = \sum_{i=1}^{p-1} (-1)^{i-1} \cdot i^{-1} \cdot x^{p-i} \cdot y^i \equiv \frac{1}{p}((x+y)^p - x^p - y^p) \pmod{p}$.

Define

$$E_1(a, b \cdot p) = U_1^a \cdot \left(\chi(U_2, U_3)^b - \omega^a \chi(\omega U_2, (\omega + \beta) U_3)^b \right).$$

Also, define

$$h(U_2, U_3) := \chi(U_2, U_3)^b - \omega^a \chi(\omega U_2, (\omega + \beta) U_3)^b.$$

Then we have the following proposition.

Proposition 5.3.3 *If $\omega^{p^s + p^t} = 1$ and $(\omega + \beta)^{p^s + p^t} = 1$ with $s > 0$, then $E_1(p^t, p^s)$ is a 3−cocycle.*

Proof. Note that

$$E_1(a, b \cdot p) = U_1^a \cdot h(U_2, U_3).$$

We have

$$\delta(E_1(a, b \cdot p))$$
$$= \delta(U_1^a \cdot h(U_2, U_3))$$
$$= \delta(U_1^a) \, h(U_3, U_4) - U_1^a \, \delta(h(U_2, U_3))$$
$$= \left((U_1 + U_2)^a - (\omega + \beta)^{p \cdot b} \, (\omega \, U_1 + (\omega + \beta) \, U_2)^a\right) h(U_3, U_4)$$
$$- U_1^a \left(h(U_2 + U_3, U_4) - \omega^a \, h(\omega \, U_2 + (\omega + \beta) \, U_3, \, (\omega + \beta) \, U_4)\right)$$
$$+ U_1^a \left(h(U_2, U_3 + U_4) - \omega^a \, h(\omega \, U_2, \omega \, U_3 + \, (\omega + \beta) \, U_4)\right).$$

(5.3.13)

Let $a = p^t$ and $b = p^{s-1}$. Then from (5.3.13) we have

$$\delta(E_1(p^t, p^s))$$
$$= U_1^{p^t} \cdot \left[(1 - \omega^{p^t}(\omega + \beta)^{p^s}) \, h(U_3, U_4) - h(U_2 + U_3, \, U_4)\right.$$
$$+ \omega^{p^t} \, h(\omega \, U_2 + (\omega + \beta) \, U_3, \, (\omega + \beta) \, U_4) + h(U_2, U_3 + U_4)$$
$$\left. - \omega^{p^t} \, h(\omega \, U_2, \omega \, U_3 + \, (\omega + \beta) \, U_4)\right] + (1 - (\omega + \beta)^{p^t + p^s}) \, U_2^{p^t} \, h(U_3, U_4).$$

(5.3.14)

Since $h(U_i, U_{i+1}) = \chi(U_i, U_{i+1})^{p^{s-1}} - \omega^{p^t} \chi(\omega \, U_i, \, (\omega + \beta) \, U_{i+1})^{p^{s-1}}$, $\omega^{p^s + p^h} = 1$ and $(\omega + \beta)^{p^s + p^t} = 1$, straightforward computation yields that the right-hand side of (5.3.14) vanishes. This completes the proof.

∎

Let p be a prime and let u, v, and t be non-negative integers. Define $F(p^v, p^u, p^t) = U_1^{p^v} U_2^{p^u} U_3^{p^t} \in C^3$ where $p^v, p^u, p^t < q$.

Proposition 5.3.4   *1. If $\omega^{p^v + p^u + p^t} = 1$ and $(\omega + \beta)^{p^v + p^u + p^t} = 1$, then $F(p^v, p^u, p^t)$ is a 3-cocycle.*

2. If $\omega^{p^v + p^u} = 1$ and $(\omega + \beta)^{p^v + p^u} = 1$, then $F(p^v, p^u, 0)$ is a 3-cocycle.

Proof. We first prove (1).

$$\delta(F(p^v, p^u, p^t))$$

$$= \delta(U_1^{p^v} U_2^{p^u} U_3^{p^t})$$

$$= \left((U_1 + U_2)^{p^v} - (\omega + \beta)^{p^u + p^t}(\omega U_1 + (\omega + \beta)U_2)^{p^v}\right) \cdot U_3^{p^u} \cdot U_4^{p^t}$$

$$- U_1^{p^v} \cdot \left((U_2 + U_3)^{p^u} - \omega^{p^v}(\omega + \beta)^{p^t}(\omega U_2 + (\omega + \beta)U_3)^{p^u}\right) \cdot U_4^{p^t}$$

$$+ U_1^{p^v} \cdot U_2^{p^u}\left((U_3 + U_4)^{p^t} - \omega^{p^v + p^u}(\omega U_3 + (\omega + \beta)U_4)^{p^t}\right) \tag{5.3.15}$$

$$= (1 - \omega^{p^v}(\omega + \beta)^{p^u + p^t})U_1^{p^v} U_3^{p^u} U_4^{p^t} + (1 - (\omega + \beta)^{p^v + p^u + p^t})U_2^{p^v} U_3^{p^u} U_4^{p^t}$$

$$- (1 - \omega^{p^v + p^u}(\omega + \beta)^{p^t})U_1^{p^v} U_2^{p^u} U_4^{p^t} - (1 - \omega^{p^v}(\omega + \beta)^{p^t + p^u})U_1^{p^v} U_3^{p^u} U_4^{p^t}$$

$$+ (1 - \omega^{p^v + p^u + p^t})U_1^{p^v} U_2^{p^u} U_3^{p^t} + (1 - \omega^{p^v + p^u}(\omega + \beta)^{p^t})U_1^{p^v} U_2^{p^u} U_4^{p^t}$$

$$= 0.$$

Since $\omega^{p^v + p^u + p^t} = 1$ and $(\omega + \beta)^{p^v + p^u + p^t} = 1$, the right hand side of (5.3.15) is 0. In (2), by taking p^t as zero in (5.3.15), and with $\omega^{p^v + p^u} = 1$ and $(\omega + \beta)^{p^v + p^u} = 1$, it can be shown in a similar manner that $\delta(F(p^v, p^u, 0)) = 0$.

∎

As in [33, 35], let Q be the set of all tuples (p^v, p^u, p^t, p^s) where p is a prime, $v < t, u < s, u \le t$, and $\omega^{p^v + p^t} = \omega^{p^u + p^s} = (\omega + \beta)^{p^v + p^t} = (\omega + \beta)^{p^u + p^s} = 1$, and where one of the following conditions hold:

Case I. $\omega^{p^v + p^u} = (\omega + \beta)^{p^v + p^u} = 1$.

Case II. $\omega^{p^v + p^u}, (\omega + \beta)^{p^v + p^u} \ne 1$ and $t > s$.

Case III. $\omega^{p^v + p^u} \ne 1, (\omega + \beta)^{p^v + p^u} \ne 1, t = s$, and $p \ne 2$.

Case IV. $\omega^{p^v + p^u} \ne 1, (\omega + \beta)^{p^v + p^u} \ne 1, u \le v < t < s, \omega^{p^v} = \omega^{p^u}$, and $(\omega + \beta)^{p^v} = (\omega + \beta)^{p^u}$ when $p \ne 2$.

Case V. $\omega^{p^v + p^u} \ne 1, (\omega + \beta)^{p^v + p^u} \ne 1, u < v < t \le s, \omega^{p^v} = \omega^{p^u}$, and $(\omega + \beta)^{p^v} = (\omega + \beta)^{p^u}$ when $p = 2$. Moreover, if $p = 2$, we need $u < t$ as well.

For each $(p^v, p^u, p^t, p^s) \in Q$, we denote a cocycle by Γ. We have the following proposition related to case I, $\omega^{p^v + p^u} = (\omega + \beta)^{p^v + p^u} = 1$.

Proposition 5.3.5 $\Gamma(p^v, p^u, p^t, p^s) = F(p^v, p^u + p^t, p^s)$ *is a 3-cocycle.*

Proof. We compute $\delta(F(p^v, p^u + p^t, p^s))$.

$$\delta(F(p^v, p^u + p^t, p^s))$$
$$= \delta(U_1^{p^v} U_2^{p^u+p^t} U_3^{p^s})$$
$$= \delta(U_1^{p^v}) U_3^{p^u+p^t} U_4^{p^s} - U_1^{p^v} \delta(U_3^{p^u+p^t}) U_4^{p^s} + U_1^{p^v} U_3^{p^u+p^t} \delta(U_4^{p^s})$$
$$= ((U_1 + U_2)^{p^v} - (\omega + \beta)^{p^u+p^t+p^s}(\omega U_1 + (\omega + \beta)U_2)^{p^v} U_3^{p^u+p^t} U_4^{p^s}$$
$$- U_1^{p^v} \cdot ((U_2 + U_3)^{p^u+p^t} - \omega^{p^v}(\omega + \beta)^{p^s}(\omega + \beta)^{p^v}(\omega U_2 + (\omega + \beta)U_3)^{p^u+p^t})) U_4^{p^s}$$
$$+ U_1^{p^v} \cdot U_2^{p^u+p^t} \cdot ((U_3 + U_4)^{p^s} - \omega^{p^v+p^u+p^t}(\omega + \beta)^{p^v+p^u+p^t}(\omega U_3 + (\omega + \beta)U_4)^{p^s}).$$

$$(5.3.16)$$

Note that $(x + y)^{p^u+p^t} = (x^{p^u} + y^{p^u})(x^{p^t} + y^{p^t})$, in which case the above can be reduced to

$$\delta(U_1^{p^v} U_2^{p^u+p^t} U_3^{p^s})$$
$$= (1 - \omega^{p^v}(\omega + \beta)^{p^u+p^t+p^s}) U_1^{p^v} U_3^{p^u+p^t} U_4^{p^s} + (1 - (\omega + \beta)^{p^u+p^t+p^s+p^v}) U_2^{p^v} U_3^{p^u+p^t} U_4^{p^s}$$
$$- (1 - \omega^{p^v+p^u+p^t}(\omega + \beta)^{p^s}) U_1^{p^v} U_2^{p^u+p^t} U_4^{p^s} - (1 - \omega^{p^v}(\omega + \beta)^{p^s+p^u+p^t}) U_1^{p^v} U_3^{p^u+p^t} U_4^{p^s}$$
$$- (1 - \omega^{p^v+p^u}(\omega + \beta)^{p^s+p^t}) U_1^{p^v} U_2^{p^u} U_3^{p^t} U_4^{p^s} - (1 - \omega^{p^v+p^t}(\omega + \beta)^{p^s+p^u}) U_1^{p^v} U_2^{p^t} U_3^{p^u} U_4^{p^s}$$
$$+ (1 - \omega^{p^v+p^u+p^t+p^s}) U_1^{p^v} U_2^{p^u+p^t} U_3^{p^s} + (1 - \omega^{p^v+p^u+p^t}(\omega + \beta)^{p^s}) U_1^{p^v} U_2^{p^u+p^t} U_4^{p^s}$$
$$= 0.$$

$$(5.3.17)$$

■

For case II, where we have the following proposition $\omega^{p^v+p^u}, (\omega + \beta)^{p^v+p^u} \neq 1$ and $t > s$.

Proposition 5.3.6

$$\Gamma(p^v, p^u, p^t, p^s) = F(p^v, p^u + p^t, p^s) - F(p^u, p^v + p^s, p^t)$$
$$- (\omega^{p^u}(\omega + \beta)^{p^s} - 1)^{-1}(1 - \omega^{p^u+p^v}(\omega + \beta)^{p^t+p^s}) F(p^v, p^u, p^t + p^s) + F(p^v + p^u, p^s, p^t)$$

is a 3-cocycle.

Proof.

$$\delta(F(p^v, p^u + p^t, p^s)) - \delta(F(p^u, p^v + p^s, p^t))$$

$$- (\omega^{p^u}(\omega + \beta)^{p^s} - 1)^{-1}(1 - \omega^{p^u + p^v}(\omega + \beta)^{p^t + p^s})\delta(F(p^v, p^u, p^t + p^s)$$

$$+ \delta(F(p^v + p^u, p^s, p^t)))$$

$$= -(1 - \omega^{p^v + p^u}(\omega + \beta)^{p^s + p^t})U_1^{p^v}U_2^{p^u}U_3^{p^t}U_4^{p^s} + (1 - \omega^{p^v + p^u}(\omega + \beta)^{p^s + p^t})U_1^{p^u}U_2^{p^v}U_3^{p^s}U_4^{p^t}$$

$$- (\omega^{p^u}(\omega + \beta)^{p^s} - 1)^{-1}(1 - \omega^{p^u + p^v}(\omega + \beta)^{p^t + p^s})[(1 - \omega^{p^v + p^u + p^t}(\omega + \beta)^{p^s})U_1^{p^v}U_2^{p^u}U_3^{p^t}U_4^{p^s}$$

$$- (1 - \omega^{p^u}(\omega + \beta)^{p^s + p^t + p^v})U_1^{p^u}U_2^{p^v}U_3^{p^s}U_4^{p^t}]$$

$$= 0.$$

$$(5.3.18)$$

■

For case III, where we have the following proposition, $\omega^{p^v + p^u}, (\omega + \beta)^{p^v + p^u} \neq 1, t = s$ and $p \neq 2$. In [42], it is shown that we can present this as follows:

Proposition 5.3.7 $\Gamma(p^v, p^u, p^t, p^s) = F(p^v, p^t + p^s, p^u)$ *is a 3-cocycle.*

Proof. The proof is similar to that of Proposition 5.3.5.

■

Finally, we have the following proposition concerning the cases IV and V, $\omega^{p^v + p^u} \neq 1, (\omega + \beta)^{p^v + p^u} \neq 1, u \leq v < t < s$ and $\omega^{p^v} = \omega^{p^u}, (\omega + \beta)^{p^v} = (\omega + \beta)^{p^u}$ when $p \neq 2$ and $\omega^{p^v + p^u}, (\omega + \beta)^{p^v + p^u} \neq 1, u < v < t \leq s$ and $\omega^{p^v} = \omega^{p^u}, (\omega + \beta)^{p^v} = (\omega + \beta)^{p^u}$ when $p = 2$. In [42], can be present as:

Proposition 5.3.8 $\Gamma(p^v, p^u, p^t, p^s) = F(p^t, p^v + p^u, p^s)$ *is a 3-cocycle.*

Proof. The proof is similar to that of Proposition 5.3.5.

■

We will have the following.

Conjecture 5.3.9 *Fix* $\omega, \beta \in \mathbb{F}_q$ *with* $\omega \neq 0, \pm 1$. *Let* X *be the corresponding Alexander* f-*quandle on* $\mathbb{F}_q$ *where* $H_Q^2((X, *, f); \mathbb{F}_q) \cong 0$. *Then the set*

$$I = \{F(p^v, p^u, p^t) \mid \omega^{p^v + p^u + p^t} = (\omega + \beta)^{p^v + p^u + p^t} = 1, p^v < p^u < p^t < q\}$$

$$\cup \{F(p^v, p^u, 0) \mid \omega^{p^v + p^u} = (\omega + \beta)^{p^v + p^u} = 1, p^v < p^u < q\}$$

$$\cup \{E_0(p \cdot p^v, p^u) \mid \omega^{p^{v+1} + p^u} = (\omega + \beta)^{p^{v+1} + p^u} = 1, p^v < p^u < q\}$$

$$\cup \{E_1(p^v, p \cdot p^u) \mid \omega^{p^v + p^{u+1}} = (\omega + \beta)^{p^v + p^{u+1}} = 1, p^v \leq p^u < q\}$$

$$\cup \{\Gamma(p^v, p^u, p^t, p^s) \mid (p^v, p^u, p^t, p^s) \in Q(q)\}$$

is a basis for the third cohomology $H_Q^3((X, *, f); \mathbb{F}_q)$.

Example 5.3.10 *Let* p *be an odd prime and let* u, v *and* t *be non-negative integers.* *Let* $\omega = -1$ *and* $\beta = 2$. *Then* $\omega^{p^v + p^u + p^t} \neq 1$ *and* $(\omega + \beta)^{p^v + p^u + p^t} = 1$, *and we have the following:*

1. $F(p^v, p^u, p^t)$ *is not a 3-cocycle.*

2. $F(p^v, p^u, 0)$ *is a 3-cocycle since* $\omega^{p^v + p^u} = 1$ *and* $(\omega + \beta)^{p^v + p^u} = 1$. *Also,* $E_0(p^{v+1}, p^u)$ *and* $E_1(p^v, p^{u+1})$ *are 3-cocycles. Moreover,*

$$Q(q) = \{(p^v, p^u, p^t, p^s) \mid p^u \leq p^t, p^v < p^t, p^u < p^s\},$$

and $\omega^{p^v + p^u} = (\omega + \beta)^{p^v + p^u} = 1$ *for any* $(p^v, p^u, p^t, p^s) \in Q(q)$.

Therefore,

$$\{F(p^v, p^u, 0) \mid 0 < p^v < p^u < q\} \cup \{E_0(p^{v+1}, p^u) \mid p^v < p^u < q\}$$

$$\cup \{E_1(p^v, p^{u+1}) \mid p^v < p^u < q\}$$

$$\cup \{F(p^v, p^u + p^t, p^s) \mid p^u \leq p^t, p^v < p^t, p^u < p^s, p^i < q, \text{ for all } i \in \{v, u, t, s\}\}$$

is a basis for the cohomology group $H_Q^3((X, *, f); \mathbb{F}_q)$.

Remark 5.3.11 *Example 5.3.10 shows that when $\beta = 1 - \omega$, the basis for the cohomology group $H_Q^3((X, *, f); \mathbb{F}_q)$ above is the same as the basis for the cohomology group $H_Q^3((X, *); \mathbb{F}_q)$ as in [35, Subsection 2.4.1].*

Example 5.3.12 *Let $f(x) = x^3 + x^2 + 1 \in \mathbb{F}_2[x]$ and consider $\mathbb{F}_8 = \mathbb{F}_2[x]/(f)$. Let ω be a primitive element of $\mathbb{F}_8$. (Then the order of ω is 7). Let $\beta = \omega^{2^2}$. Note that $\omega^{2^2} = \omega^3 + \omega$ and that ω^{2^2} is also a primitive element of $\mathbb{F}_8$ since it is a conjugate of ω with respect to $\mathbb{F}_2$. We have*

$$\omega^{2^0 + 2^1 + 2^2} = 1 \quad and \quad (\omega + \beta)^{2^0 + 2^1 + 2^2} = 1,$$

but $\omega^{2^i + 2^j} \neq 1$ for $i, j \in \{0, 1, 2\}$. If Conjecture 5.3.9 is true, then $\{F(2^0, 2^1, 2^2)\}$ will be a basis for the third cohomology group $H_Q^3((X, *, f); \mathbb{F}_8)$.

Note: There are many Alexander quandles satisfying the condition $H^2(X, \mathbb{F}_q) \cong 0$ as shown in [40] with $\mathbb{F}_q$ is an extension of $\mathbb{F}_p$ of the odd degree with $\omega \neq -1$. Under the assumption of $H_Q^2(X, f, \mathbb{F}_q) \cong 0$, basis in Conjecture 5.3.9 is given in the following remark.

Remark 5.3.13 (Theorem A.2, [1]) *Fix $\omega \in \mathbb{F}_q$ with $\omega \neq 0, 1$. Let X be the Alexander f-quandle on $\mathbb{F}_q$ as above. Assume $H_Q^2(X, f, \mathbb{F}_q) \cong 0$. Then the following set provides a basis of the third cohomology $H_Q^3(X, f, \mathbb{F}_q)$:*

$$\{U^{p^v} U^{p^u} U^{p^t} \mid \omega^{p^v + p^u + p^t} = 1, (\omega + \beta)^{p^v + p^u + p^t} = 1, 0 \leq v < u < t < m\}.$$

5.4 The 4-Cocycles

In this section, we present some polynomials and show that they are 4-cocycles. The main theorem gives a basis for the cohomology group $H_Q^4((X, *, f); \mathbb{F}_q)$ under the condition that $H_Q^2((X, *, f); \mathbb{F}_q)$ is trivial. Also we have given some 4-cocycles in Propositions 5.4.5, 5.4.6, 5.4.7, and 5.4.8 without the assumption of $H_Q^2((X, *, f); \mathbb{F}_q) \cong 0$.

Proposition 5.4.1 *If* $\omega^{p^v+p^u+p^t+p^s} = 1 = (\omega + \beta)^{p^v+p^u+p^t+p^s}$, *then the polynomial* $U_1^{p^v} U_2^{p^u} U_3^{p^t} U_4^{p^s}$ *is a 4-cocycle.*

Proof.

$$\delta(U_1^{p^v} U_2^{p^u} U_3^{p^t} U_4^{p^s})$$

$$= ((U_1 + U_2)^{p^v} - (\omega + \beta)^{p^u+p^t+p^s} (\omega U_1 + (\omega + \beta)U_2)^{p^v})U_3^{p^u} U_4^{p^t} U_5^{p^s}$$

$$- U_1^{p^v} \cdot ((U_2 + U_3)^{p^u} - \omega^{p^v} (\omega + \beta)^{p^t+p^s} (\omega U_2 + (\omega + \beta)U_3)^{p^u}) U_4^{p^t} U_5^{p^s}$$

$$+ U_1^{p^v} \cdot U_2^{p^u} \cdot ((U_3 + U_4)^{p^t} - \omega^{p^v+p^u} (\omega + \beta)^{p^s} (\omega U_3 + (\omega + \beta)U_4)^{p^t}) U_5^{p^s}$$

$$- U_1^{p^v} \cdot U_2^{p^u} \cdot U_3^{p^t} \cdot ((U_4 + U_5)^{p^s} - \omega^{p^v+p^u+p^t} (\omega U_4 + (\omega + \beta)U_5)^{p^s})$$

$$= (1 - \omega^{p^v} (\omega + \beta)^{p^u+p^t+p^s}) U_1^{p^v} U_3^{p^u} U_4^{p^t} U_5^{p^s} + (1 - (\omega + \beta)^{p^v+p^u+p^t+p^s})U_2^{p^v} U_3^{p^u} U_4^{p^t} U_5^{p^s}$$

$$- (1 - \omega^{p^v+p^u} (\omega + \beta)^{p^t+p^s})U_1^{p^v} U_2^{p^u} U_4^{p^t} U_5^{p^s} - (1 - \omega^{p^v} (\omega + \beta)^{p^u+p^t+p^s})U_1^{p^v} U_3^{p^u} U_4^{p^t} U_5^{p^s}$$

$$+ (1 - \omega^{p^v+p^u+p^t} (\omega + \beta)^{p^s}) U_1^{p^v} U_2^{p^u} U_3^{p^t} U_5^{p^s} + (1 - \omega^{p^v+p^u} (\omega + \beta)^{p^s+p^t})U_1^{p^v} U_2^{p^u} U_4^{p^t} U_5^{p^s}$$

$$- (1 - \omega^{p^v+p^u+p^t+p^s})U_1^{p^v} U_2^{p^u} U_3^{p^t} U_4^{p^s} - (1 - \omega^{p^v+p^u+p^t} (\omega + \beta)^{p^s})U_1^{p^v} U_2^{p^u} U_3^{p^t} U_5^{p^s}.$$

$$(5.4.19)$$

Since $\omega^{p^v+p^u+p^t+p^s} = 1 = (\omega + \beta)^{p^v+p^u+p^t+p^s} = 1$, the right-hand side of (5.4.19) is zero. This completes the proof. ∎

We recall $\chi(x, y) = \sum_{i=1}^{p-1} (-1)^{i-1} \cdot i^{-1} \cdot x^{p-i} \cdot y^i \equiv \frac{1}{p}((x + y)^p - x^p - y^p) \pmod{p}$.

Proposition 5.4.2 *If* $\omega^{p^{u+1}+p^t+p^s} = 1 = (\omega + \beta)^{p^{u+1}+p^t+p^s}$, *then the polynomial* $\left(\chi(U_1, U_2)^{p^u} - (\omega + \beta)^{p^t+p^s}\chi(\omega U_1, (\omega + \beta) U_2)^{p^u}\right)U_3^{p^t} U_4^{p^s}$ *is a 4-cocycle.*

Proof. Let

$$h(U_1, U_2) = \chi(U_1, U_2)^{p^u} - (\omega + \beta)^{p^t+p^s}\chi(\omega U_1, (\omega + \beta) U_2)^{p^u}.$$

Then we will get the following.

$$\delta(h(U_1, U_2)\, U_3^{p^t} U_4^{p^s})$$

$$= \delta(h(U_1, U_2)\, U_4^{p^t} U_5^{p^s}$$

$$- h(U_1, U_2)\left((U_3 + U_4)^{p^t} - \omega^{p^{u+1}}(\omega + \beta)^{p^s}\left(\omega\, U_3 + (\omega + \beta)\, U_4\right)^{p^t}\right) U_5^{p^s}$$

$$+ h(U_1, U_2)\, U_3^{p^t}\left((U_4 + U_5)^{p^s} - \omega^{p^{u+1}+p^t}\left(\omega\, U_4 + (\omega + \beta)\, U_5\right)^{p^s}\right)$$

$$= (1 - \omega^{p^{u+1}}(\omega + \beta)^{p^s + p^t})\, h(U_1, U_2)\, U_4^{p^t} U_5^{p^s} - (1 - (\omega + \beta)^{p^{u+1}+p^t+p^s})\, h(U_2, U_3)\, U_4^{p^t} U_5^{p^s}$$

$$- (1 - \omega^{p^{u+1}+p^t}(\omega + \beta)^{p^s})\, h(U_1, U_2)\, U_3^{p^t} U_5^{p^s} - (1 - \omega^{p^{u+1}}(\omega + \beta)^{p^s + p^t})\, h(U_1, U_2)\, U_4^{p^t} U_5^{p^s}$$

$$+ (1 - \omega^{p^{u+1}+p^t+p^s})\, h(U_1, U_2)\, U_3^{p^t} U_4^{p^s} + (1 - \omega^{p^{u+1}+p^t}(\omega + \beta)^{p^s})\, h(U_1, U_2)\, U_3^{p^t} U_5^{p^s}.$$

$$(5.4.20)$$

Since $\omega^{p^{u+1}+p^t+p^s} = 1 = (\omega + \beta)^{p^{u+1}+p^t+p^s} = 1$, the right hand side of (5.4.20) is vanishes. This completes the proof. $\blacksquare$

Proposition 5.4.3 *If $\omega^{p^v + p^{t+1} + p^s} = 1 = (\omega + \beta)^{p^v + p^{t+1} + p^s} = 1$, then the polynomial $U_1^{p^v}\left(\chi(U_2, U_3)^{p^t} - \omega^{p^v}(\omega + \beta)^{p^s}\chi(\omega U_2, (\omega + \beta)U_3)^{p^t}\right) U_4^{p^s}$ is a 4-cocycle.*

Proof. Let $h(U_2, U_3) = \chi(U_2, U_3)^{p^t} - \omega^{p^v}(\omega + \beta)^{p^s}\chi(\omega U_2, (\omega + \beta)U_3)^{p^t}$. Then

$$\delta(U_1^{p^v}\, h(U_2, U_3)\, U_4^{p^s})$$

$$= \left((U_1 + U_2)^{p^v} - (\omega + \beta)^{p^{t+1}+p^s}\left(\omega\, U_1 + (\omega + \beta)\, U_2\right)^{p^v}\right) h(U_3, U_4)\, U_5^{p^s}$$

$$- U_1^{p^v}\left(h(U_2 + U_3, U_4) - \omega^{p^v}(\omega + \beta)^{p^s} h(\omega\, U_2 + (\omega + \beta)\, U_3, (\omega + \beta)\, U_4)\right) U_5^{p^s}$$

$$+ U_1^{p^v}\left(h(U_2, U_3 + U_4) - \omega^{p^v}(\omega + \beta)^{p^s} h(\omega\, U_2, \omega\, U_3 + (\omega + \beta)\, U_4)\right) U_5^{p^s}$$

$$- U_1^{p^v}\, h(U_2, U_3)\left((U_4 + U_5)^{p^s} - \omega^{p^v + p^{t+1}}\left(\omega\, U_4 + (\omega + \beta)\, U_5\right)^{p^s}\right)$$

$$= U_1^{p^v}\left[(1 - \omega^{p^v}(\omega + \beta)^{p^{t+1}+p^s})\, h(U_3, U_4) - h(U_2 + U_3, U_4) + h(U_2, U_3 + U_4)\right.$$

$$+ \omega^{p^v}(\omega + \beta)^{p^s} h(\omega\, U_2 + (\omega + \beta)\, U_3, (\omega + \beta)\, U_4)$$

$$\left. - \omega^{p^v}(\omega + \beta)^{p^s} h(\omega\, U_2, \omega\, U_3 + (\omega + \beta)\, U_4) - (1 - \omega^{p^v + p^{t+1}}(\omega + \beta)^{p^s})\, h(U_2, U_3)\right] U_5^{p^s}$$

$$+ (1 - (\omega + \beta)^{p^v + p^{t+1} + p^s})\, U_2^{p^v}\, h(U_3, U_4)\, U_5^{p^s} - (1 - \omega^{p^v + p^{t+1} + p^s})\, U_1^{p^v}\, h(U_2, U_3)\, U_4^{p^s}.$$

$$(5.4.21)$$

66

Since

$$h(U_i, U_{i+1}) = \chi(U_i, U_{i+1})^{p^t} - \omega^{p^v}(\omega + \beta)^{p^s}\chi(\omega U_i, (\omega + \beta) U_{i+1})^{p^t},$$

and $\omega^{p^v + p^{t+1} + p^s} = 1 = (\omega + \beta)^{p^v + p^{t+1} + p^s} = 1$, it can be shown that the right hand side of (5.4.21) is zero. This completes the proof.

$\blacksquare$

Proposition 5.4.4 *If $\omega^{p^v + p^u + p^{s+1}} = 1$ and $(\omega + \beta)^{p^v + p^u + p^{s+1}} = 1$, then the polynomial*

$$U_1^{p^v} U_2^{p^u} \left(\chi(U_3, U_4)^{p^s} - \omega^{p^v + p^u} \chi(\omega U_3, (\omega + \beta) U_4)^{p^s} \right)$$

is a 4-cocycle.

Proof. Let $h(U_3, U_4) = \chi(U_3, U_4)^{p^s} - \omega^{p^v + p^u} \chi(\omega U_3, (\omega + \beta) U_4)^{p^s}$. Then

$$\delta(U_1^{p^v} U_2^{p^u} h(U_3, U_4))$$

$$= \delta(U_1^{p^v})U_3^{p^u} h(U_4, U_5) - U_1^{p^v}\delta(U_2^{p^u})h(U_4, U_5)) + U_1^{p^v} U_2^{p^u} \delta(h(U_3, U_4))$$

$$= \left((U_1 + U_2)^{p^v} - (\omega + \beta)^{p^u + p^{s+1}} (\omega U_1 + (\omega + \beta) U_2)^{p^v} \right)U_3^{p^u} h(U_4, U_5)$$

$$- U_1^{p^v} \left((U_2 + U_3)^{p^u} - \omega^{p^v}(\omega + \beta)^{p^{s+1}} (\omega U_2 + (\omega + \beta) U_3)^{p^u} \right)h(U_4, U_5)$$

$$+ U_1^{p^v} U_2^{p^u} \left(h(U_3 + U_4, U_5) - \omega^{p^v + p^u} h(\omega U_3 + (\omega + \beta) U_4, (\omega + \beta) U_5) \right)$$

$$- U_1^{p^v} U_2^{p^u} \left(h(U_3, U_4 + U_5) - \omega^{p^v + p^u} h(\omega U_3, \omega U_4 + (\omega + \beta) U_5) \right)$$

$$= (1 - (\omega + \beta)^{p^v + p^u + p^{s+1}}) U_2^{p^v} U_3^{p^u} h(U_4, U_5) + U_1^{p^v} U_2^{p^u} \Big[h(U_3 + U_4, U_5) - h(U_3, U_4 + U_5)$$

$$- (1 - \omega^{p^v + p^u} (\omega + \beta)^{p^{s+1}}) h(U_4, U_5) + \omega^{p^v + p^u} h(\omega U_3, \omega U_4 + (\omega + \beta) U_5)$$

$$- \omega^{p^v + p^u} h(\omega U_3 + (\omega + \beta) U_4, (\omega + \beta) U_5) \Big].$$

$$(5.4.22)$$

Since

$$h(U_i, U_{i+1}) = \chi(U_i, U_{i+1})^{p^s} - \omega^{p^v + p^u} \chi(\omega U_i, (\omega + \beta) U_{i+1})^{p^s}$$

and $\omega^{p^v+p^u+p^{s+1}} = 1 = (\omega + \beta)^{p^v+p^u+p^{s+1}}$, it can be shown that the right-hand side of (5.4.22) is vanishes. This completes the proof.

∎

In following Propositions 5.4.5, 5.4.6, 5.4.7, and 5.4.8), the condition $H^2_Q((X, *, f); \mathbb{F}_q) = 0$ is not needed.

Proposition 5.4.5 *If* $\omega^{p^i+p^j+p^u+p^t+p^s} = 1$, $(\omega+\beta)^{p^i+p^j+p^u+p^t+p^s} = 1$, $\omega^{p^i+p^j} = \omega^{p^i+p^u} = 1$ *and* $(\omega + \beta)^{p^u+p^t+p^s} = (\omega + \beta)^{p^j+p^t+p^s} = 1$, *then the polynomial* $U_1^{p^i} U_2^{p^j+p^u} U_3^{p^t} U_4^{p^s}$ *is a 4-cocycle.*

Proof. We will compute $\delta(U_1^{p^i} U_2^{p^j+p^u} U_3^{p^t} U_4^{p^s})$.

$$
\begin{aligned}
&\delta(U_1^{p^i} U_2^{p^j+p^u} U_3^{p^t} U_4^{p^s}) \\
&= ((U_1 + U_2)^{p^i} - (\omega + \beta)^{p^j+p^u+p^t+p^s} (\omega U_1 + (\omega + \beta)U_2)^{p^i}) U_3^{p^j+p^u} U_4^{p^t} U_5^{p^s} \\
&\quad - U_1^{p^i} \cdot ((U_2 + U_3)^{p^j+p^u} - \omega^{p^i} (\omega + \beta)^{p^t+p^s} (\omega U_2 + (\omega + \beta)U_3)^{p^j+p^u})) U_4^{p^t} U_5^{p^s} \\
&\quad + U_1^{p^i} \cdot U_2^{p^j+p^u} \cdot ((U_3 + U_4)^{p^t} - \omega^{p^i+p^j+p^u} (\omega + \beta)^{p^s} (\omega U_3 + (\omega + \beta)U_4)^{p^t})) U_5^{p^s} \\
&\quad - U_1^{p^i} \cdot U_2^{p^j+p^u} \cdot U_3^{p^t} \cdot ((U_4 + U_5)^{p^s} - \omega^{p^i+p^j+p^u+p^t} (\omega U_4 + (\omega + \beta)U_5)^{p^s}).
\end{aligned}
$$

$$(5.4.23)$$

Note that $(x + y)^{p^j+p^u} = (x^{p^j} + y^{p^j})(x^{p^u} + y^{p^u})$. Hence, from (5.4.23) we have

$$\delta(U_1^{p^i} U_2^{p^j+p^u} U_3^{p^t} U_4^{p^s}) = (1 - \omega^{p^i}(\omega+\beta)^{p^j+p^u+p^t+p^s})U_1^{p^i} U_3^{p^j+p^u} U_4^{p^t} U_5^{p^s}$$
$$+ (1 - (\omega+\beta)^{p^i+p^j+p^u+p^t+p^s})U_2^{p^i} U_3^{p^j+p^u} U_4^{p^t} U_5^{p^s}$$
$$- (1 - \omega^{p^i+p^j+p^u}(\omega+\beta)^{p^t+p^s})U_1^{p^i} U_2^{p^j+p^u} U_4^{p^t} U_5^{p^s}$$
$$- (1 - \omega^{p^i}(\omega+\beta)^{p^j+p^u+p^t+p^s})U_1^{p^i} U_3^{p^j+p^u} U_4^{p^t} U_5^{p^s}$$
$$- (1 - \omega^{p^i+p^j}(\omega+\beta)^{p^u+p^t+p^s})U_1^{p^i} U_2^{p^j} U_3^{p^u} U_4^{p^t} U_5^{p^s}$$
$$- (1 - \omega^{p^i+p^u}(\omega+\beta)^{p^j+p^t+p^s})U_1^{p^i} U_2^{p^u} U_3^{p^j} U_4^{p^t} U_5^{p^s}$$
$$\text{(5.4.24)}$$
$$+ (1 - \omega^{p^i+p^j+p^u+p^t}(\omega+\beta)^{p^s})U_1^{p^i} U_2^{p^j+p^u} U_3^{p^t} U_5^{p^s}$$
$$+ (1 - \omega^{p^i+p^j+p^u}(\omega+\beta)^{p^s+p^t})U_1^{p^i} U_2^{p^j+p^u} U_4^{p^t} U_5^{p^s}$$
$$- (1 - \omega^{p^i+p^j+p^u+p^t+p^s})U_1^{p^i} U_2^{p^j+p^u} U_3^{p^t} U_4^{p^s}$$
$$- (1 - \omega^{p^i+p^j+p^u+p^t}(\omega+\beta)^{p^s})U_1^{p^i} U_2^{p^j+p^u} U_3^{p^t} U_5^{p^s}.$$

Since $\omega^{p^i+p^j+p^u+p^t+p^s} = 1$, $(\omega+\beta)^{p^i+p^j+p^u+p^t+p^s} = 1$, $\omega^{p^i+p^j} = \omega^{p^i+p^u} = 1$ and $(\omega+\beta)^{p^u+p^t+p^s} = (\omega+\beta)^{p^j+p^t+p^s} = 1$, the right hand side of (5.4.24) is zero. This completes the proof.

∎

Proposition 5.4.6 *If $\omega^{p^{u+1}+p^{s+1}} = 1$ and $(\omega+\beta)^{p^{u+1}+p^{s+1}} = 1$, then the polynomial*

$$\left(\chi(U_1,U_2)^{p^u} - (\omega+\beta)^{p^{s+1}} \chi(\omega\, U_1, (\omega+\beta)\, U_2)^{p^u}\right)\left(\chi(U_3,U_4)^{p^s} - \omega^{p^{u+1}} \chi(\omega\, U_3, (\omega+\beta)\, U_4)^{p^s}\right)$$

is a 4-cocycle.

Proof. Let $h(U_1,U_2) = \chi(U_1,U_2)^{p^u} - (\omega+\beta)^{p^{s+1}} \chi(\omega\, U_1, (\omega+\beta)\, U_2)^{p^u}$ and

$$h^*(U_3,U_4) = \chi(U_3,U_4)^{p^s} - \omega^{p^{u+1}} \chi(\omega\, U_3, (\omega+\beta)\, U_4)^{p^s}.$$

Then

$$\delta(h(U_1, U_2)\, h^*(U_3, U_4))$$

$$= \delta(h(U_1, U_2))\, h^*(U_4, U_5) - h(U_1, U_2)\, \delta(h^*(U_3, U_4))$$

$$= h(U_1, U_2)\Big[h^*(U_3, U_4 + U_5) - h^*(U_3 + U_4, U_5) + (1 - \omega^{p^{u+1}}(\omega + \beta)^{p^{s+1}})h^*(U_4, U_5)$$

$$- \omega^{p^{u+1}}\, h^*(\omega\, U_3, \omega\, U_4 + (\omega + \beta)\, U_5) + \omega^{p^{u+1}}\, h^*(\omega\, U_3 + (\omega + \beta)\, U_4, (\omega + \beta)\, U_5)\Big]$$

$$- (1 - (\omega + \beta)^{p^{u+1}+p^{s+1}})\, h(U_2, U_3)h^*(U_4, U_5).$$

$$(5.4.25)$$

Since $h(U_i, U_{i+1}) = \chi(U_i, U_{i+1})^{p^u} - (\omega + \beta)^{p^{s+1}}\, \chi(\omega\, U_i, (\omega + \beta)\, U_{i+1})^{p^u}$,

$h^*(U_i, U_{i+1}) = \chi(U_i, U_{i+1})^{p^s} - \omega^{p^{u+1}}\, \chi(\omega\, U_i, (\omega + \beta)\, U_{i+1})^{p^s}$, $\omega^{p^{u+1}+p^{s+1}} = 1$, and

$(\omega + \beta)^{p^{u+1}+p^{s+1}} = 1$, it can be shown that the right-hand side of (5.4.25) is zero. This

completes the proof.

∎

Proposition 5.4.7 *If* $\omega^{p^i+p^j+p^u+p^{s+1}} = 1$, $(\omega+\beta)^{p^i+p^j+p^u+p^{s+1}} = 1$, $\omega^{p^i+p^j} = \omega^{p^i+p^u} = 1$, *and* $(\omega + \beta)^{p^u+p^{s+1}} = (\omega + \beta)^{p^j+p^{s+1}} = 1$, *then the polynomial*

$$U_1^{p^i} U_2^{p^j+p^u}\Big(\chi(U_3, U_4)^{p^s} - \omega^{p^i+p^j+p^u}\, \chi(\omega\, U_3, (\omega + \beta)\, U_4)^{p^s}\Big)$$

is a 4-cocycle.

Proof. Let $h(U_3, U_4) = \chi(U_3, U_4)^{p^s} - \omega^{p^i+p^j+p^u}\, \chi(\omega\, U_3, (\omega + \beta)\, U_4)^{p^s}$. Then

$$\delta(U_1^{p^i} \, U_2^{p^j+p^u} \, h(U_3, U_4))$$

$$= \delta(U_1^{p^i}) \, U_3^{p^j+p^u} \, h(U_4, U_5) - U_1^{p^i} \, \delta(U_2^{p^j+p^u}) \, h(U_4, U_5) + U_1^{p^i} \, U_2^{p^j+p^u} \, \delta(h(U_3, U_4))$$

$$= ((U_1 + U_2)^{p^i} - (\omega + \beta)^{p^j+p^u+p^{s+1}} \, (\omega \, U_1 + (\omega + \beta)U_2)^{p^i}) \, U_3^{p^j+p^u} \, h(U_4, U_5)$$

$$- U_1^{p^i} \cdot ((U_2 + U_3)^{p^j+p^u} - \omega^{p^i} (\omega + \beta)^{p^{s+1}} (\omega \, U_2 + (\omega + \beta) \, U_3)^{p^j+p^u}) \, h(U_4, U_5)$$

$$+ U_1^{p^i} \, U_2^{p^j+p^u} \left(h(U_3 + U_4, U_5) - \omega^{p^i+p^j+p^u} \, h(\omega \, U_3 + (\omega + \beta) \, U_4, (\omega + \beta) \, U_5) \right)$$

$$- U_1^{p^i} \, U_2^{p^j+p^u} \left(h(U_3, U_4 + U_5) - \omega^{p^i+p^j+p^u} \, h(\omega \, U_3, \omega \, U_4 + (\omega + \beta) \, U_5) \right)$$

$$= (1 - \omega^{p^i} \, (\omega + \beta)^{p^j+p^u+p^{s+1}}) \, U_1^{p^i} \, U_3^{p^j+p^u} \, h(U_4, U_5)$$

$$+ (1 - (\omega + \beta)^{p^i+p^j+p^u+p^{s+1}}) \, U_2^{p^i} \, U_3^{p^j+p^u} \, h(U_4, U_5)$$

$$- (1 - \omega^{p^i+p^j+p^u} \, (\omega + \beta)^{p^{s+1}}) \, U_1^{p^i} \, U_2^{p^j+p^u} \, h(U_4, U_5)$$

$$- (1 - \omega^{p^i+p^j} \, (\omega + \beta)^{p^u+p^{s+1}}) \, U_1^{p^i} \, U_2^{p^j} \, U_3^{p^u} \, h(U_4, U_5)$$

$$- (1 - \omega^{p^i+p^u} \, (\omega + \beta)^{p^j+p^{s+1}}) \, U_1^{p^i} \, U_2^{p^u} \, U_3^{p^j} \, h(U_4, U_5)$$

$$- (1 - \omega^{p^i} \, (\omega + \beta)^{p^j+p^u+p^{s+1}}) \, U_1^{p^i} \, U_3^{p^j+p^u} \, h(U_4, U_5)$$

$$+ U_1^{p^i} \, U_2^{p^j+p^u} \left(h(U_3 + U_4, U_5) - \omega^{p^i+p^j+p^u} \, h(\omega \, U_3 + (\omega + \beta) \, U_4, (\omega + \beta) \, U_5) \right)$$

$$- U_1^{p^i} \, U_2^{p^j+p^u} \left(h(U_3, U_4 + U_5) - \omega^{p^i+p^j+p^u} \, h(\omega \, U_3, \omega \, U_4 + (\omega + \beta) \, U_5) \right).$$

$$(5.4.26)$$

Since $h(U_i, U_{i+1}) = \chi(U_i, U_{i+1})^{p^s} - \omega^{p^i+p^j+p^u} \chi(\omega \, U_i, (\omega+\beta) \, U_{i+1})^{p^s}$, $\omega^{p^i+p^j+p^u+p^{s+1}} = 1$, $(\omega + \beta)^{p^i+p^j+p^u+p^{s+1}} = 1$, $\omega^{p^i+p^j} = \omega^{p^i+p^u} = 1$, and $(\omega + \beta)^{p^u+p^{s+1}} = (\omega + \beta)^{p^j+p^{s+1}} = 1$, the right hand side of (5.4.26) vanishes. This completes the proof. ∎

Proposition 5.4.8 *If* $\omega^{p^i+p^j+p^v+p^u+p^t+p^s} = 1, (\omega + \beta)^{p^i+p^j+p^v+p^u+p^t+p^s} = 1,$
$\omega^{p^i+p^j} = \omega^{p^i+p^v} = \omega^{p^v+p^u} = \omega^{p^v+p^t} = 1,$ *and*
$(\omega + \beta)^{p^s+p^t} = (\omega + \beta)^{p^s+p^u} = (\omega + \beta)^{p^v+p^u} = (\omega + \beta)^{p^j+p^u} = 1,$
then the polynomial $U_1^{p^i} U_2^{p^j+p^v} U_3^{p^u+p^t} U_4^{p^s}$ *is a 4-cocycle.*

Proof.

$$\delta(U_1^{p^i} U_2^{p^j+p^v} U_3^{p^u+p^t} U_4^{p^s})$$

$$= ((U_1 + U_2)^{p^i} - (\omega + \beta)^{p^j+p^v+p^u+p^t+p^s} (\omega\, U_1 + (\omega + \beta)U_2)^{p^i}) U_3^{p^j+p^v} U_4^{p^u+p^t} U_5^{p^s}$$

$$- U_1^{p^i} \cdot ((U_2 + U_3)^{p^j+p^v} - \omega^{p^i}(\omega + \beta)^{p^u+p^t+p^s}(\omega\, U_2 + (\omega + \beta)U_3)^{p^j+p^v})U_4^{p^u+p^t} U_5^{p^s}$$

$$+ U_1^{p^i} \cdot U_2^{p^j+p^v} \cdot ((U_3 + U_4)^{p^u+p^t} - \omega^{p^i+p^j+p^v}(\omega + \beta)^{p^s}(\omega U_3 + (\omega + \beta)U_4)^{p^u+p^t}) U_5^{p^s}$$

$$- U_1^{p^i} \cdot U_2^{p^j+p^v} \cdot U_3^{p^u+p^t} \cdot ((U_4 + U_5)^{p^s} - \omega^{p^i+p^j+p^v+p^u+p^t} (\omega\, U_4 + (\omega + \beta)U_5)^{p^s}).$$

$$(5.4.27)$$

Note that $(x + y)^{p^j+p^u} = (x^{p^j} + y^{p^j})(x^{p^u} + y^{p^u})$. Hence, from (5.4.27) we have

$$\delta(U_1^{p^i} U_2^{p^j+p^v} U_3^{p^u+p^t} U_4^{p^s})$$

$$= (1 - \omega^{p^i}(\omega + \beta)^{p^j+p^v+p^u+p^t+p^s})U_1^{p^i} U_3^{p^j+p^v} U_4^{p^u+p^t} U_5^{p^s}$$

$$+ (1 - (\omega + \beta)^{p^i+p^j+p^v+p^u+p^t+p^s}) U_2^{p^i} U_3^{p^j+p^v} U_4^{p^u+p^t} U_5^{p^s}$$

$$- (1 - \omega^{p^i+p^j+p^v}(\omega + \beta)^{p^u+p^t+p^s})U_1^{p^i} U_2^{p^j+p^v} U_4^{p^u+p^t} U_5^{p^s}$$

$$- (1 - \omega^{p^i}(\omega + \beta)^{p^j+p^v+p^u+p^t+p^s})U_1^{p^i} U_3^{p^j+p^v} U_4^{p^u+p^t} U_5^{p^s}$$

$$- (1 - \omega^{p^i+p^j}(\omega + \beta)^{p^v+p^u+p^t+p^s})U_1^{p^i} U_2^{p^j} U_3^{p^v} U_4^{p^u+p^t} U_5^{p^s}$$

$$- (1 - \omega^{p^i+p^v}(\omega + \beta)^{p^j+p^u+p^t+p^s})U_1^{p^i} U_2^{p^v} U_3^{p^j} U_4^{p^u+p^t} U_5^{p^s} \qquad (5.4.28)$$

$$+ (1 - \omega^{p^i+p^j+p^v+p^u+p^t}(\omega + \beta)^{p^s}) U_1^{p^i} U_2^{p^j+p^v} U_3^{p^u+p^t} U_5^{p^s}$$

$$+ (1 - \omega^{p^i+p^j+p^v}(\omega + \beta)^{p^u+p^t+p^s}) U_1^{p^i} U_2^{p^j+p^v} U_4^{p^u+p^t} U_5^{p^s}$$

$$+ (1 - \omega^{p^i+p^j+p^v+p^u}(\omega + \beta)^{p^s+p^t}) U_1^{p^i} U_2^{p^j+p^v} U_3^{p^u} U_4^{p^t} U_5^{p^s}$$

$$+ (1 - \omega^{p^i+p^j+p^v+p^t}(\omega + \beta)^{p^s+p^u}) U_1^{p^i} U_2^{p^j+p^v} U_3^{p^t} U_4^{p^u} U_5^{p^s}$$

$$- (1 - \omega^{p^i+p^j+p^v+p^u+p^t+p^s}) U_1^{p^i} U_2^{p^j+p^v} U_3^{p^u+p^t} U_4^{p^s}$$

$$- (1 - \omega^{p^i+p^j+p^v+p^u+p^t}(\omega + \beta)^{p^s}) U_1^{p^i} U_2^{p^j+p^v} U_3^{p^u+p^t} U_5^{p^s}.$$

Since $\omega^{p^i+p^j+p^v+p^u+p^t+p^s} = 1$, $(\omega+\beta)^{p^i+p^j+p^v+p^u+p^t+p^s} = 1$, $\omega^{p^i+p^j} = \omega^{p^i+p^v} = \omega^{p^v+p^u} = \omega^{p^v+p^t} = 1$, and $(\omega + \beta)^{p^s+p^t} = (\omega + \beta)^{p^s+p^u} = (\omega + \beta)^{p^v+p^u} = (\omega + \beta)^{p^j+p^u} = 1$, the right hand side of (5.4.28) is zero. This completes the proof.

Let $q = p^m$, where m is a positive integer. Let

$A = \left\{ U_1^{p^v} U_2^{p^u} U_3^{p^t} U_4^{p^s} \mid \omega^{p^v + p^u + p^t + p^s} = 1, (\omega + \beta)^{p^v + p^u + p^t + p^s} = 1, 0 \leq v < u < t < s < m \right\}$,

$B = \left\{ \left(\chi(U_1, U_2)^{p^u} - (\omega + \beta)^{p^t + p^s} \chi(\omega\, U_1, (\omega + \beta)\, U_2)^{p^u} \right) U_3^{p^t} U_4^{p^s} \mid \omega^{p^{u+1} + p^t + p^s} = 1, (\omega + \beta)^{p^{u+1} + p^t + p^s} = 1, 0 \leq u < t < s < m \right\}$,

$C = \left\{ U_1^{p^v} \left(\chi(U_2, U_3)^{p^t} - \omega^{p^v} (\omega + \beta)^{p^s} \chi(\omega U_2, (\omega + \beta)U_3)^{p^t} \right) U_4^{p^s} \mid \omega^{p^v + p^{t+1} + p^s} = 1, (\omega + \beta)^{p^v + p^{t+1} + p^s} = 1, 0 \leq v \leq t < s < m \right\}$,

$D = \left\{ U_1^{p^v} U_2^{p^u} \left(\chi(U_3, U_4)^{p^s} - \omega^{p^v + p^u} \chi(\omega U_3, (\omega + \beta)\, U_4)^{p^s} \right) \mid \omega^{p^v + p^u + p^{s+1}} = 1, (\omega + \beta)^{p^v + p^u + p^{s+1}} = 1, 0 \leq v < u \leq s < m \right\}$, and

$E = \Gamma(p^v, p^u, p^t, 0) = \left\{ U_1^{p^v} U_2^{p^u} U_3^{p^t} \mid \omega^{p^v + p^u + p^t} = 1, (\omega + \beta)^{p^v + p^u + p^t} = 1, 0 \leq v < u < t < m \right\}$.

We have the following Conjecture:

Conjecture 5.4.9 *Fix $\omega, \beta \in \mathbb{F}_q$ with $\omega \neq 0, 1$. Let X be the corresponding Alexander f-quandle on $\mathbb{F}_q$ where $H_Q^2((X, *, f); \mathbb{F}_q) \cong 0$. Then $A \cup B \cup C \cup D \cup E$ (defined above) is a basis for the fourth cohomology $H_Q^4((X, *, f); \mathbb{F}_q)$.*

Example 5.4.10 *Let $f(x) = x^4 + x + 1 \in \mathbb{F}_2[x]$ and consider $\mathbb{F}_{16} = \mathbb{F}_2[x]/(f)$. Let ω be a primitive element of $\mathbb{F}_{16}$. (Then the order of ω is 15). Let $\beta = \omega^{2^2}$. Note that $\omega^{2^2} = \omega + 1$ and that ω^{2^2} is also a primitive element of $\mathbb{F}_{16}$ since it is a conjugate of ω with respect to $\mathbb{F}_2$. We have*

$$\omega^{2^0 + 2^1 + 2^2 + 2^3} = 1 \quad and \quad (\omega + \beta)^{2^0 + 2^1 + 2^2 + 2^3} = 1,$$

but $\omega^{2^i + 2^j + 2^k} \neq 1$ for $i, j, k \in \{0, 1, 2, 3\}$. If Conjecture 5.4.9 is true, then A is a basis for fourth cohomology group $H_Q^4((X, *, f); \mathbb{F}_{16})$ where A defined above.

6   CONCLUSION AND FUTURE WORK

The main goal of this dissertation was to generalize quandles. We have introduced a new hom-algebraic structure on quandles, an f-quandle. In Chapters $2, 3$, and 4, we have given definitions, properties, examples, classification, extensions, and cohomologies. Since Quandles and hom-algebraic structures have been investigated separately in a thoroughly manner by numerous authors, one can investigate more properties and connections of f-quandles towards different areas of Mathematics and Physics, such as the Yang-Baxter Equation, String Theory, quantum scattering, lattice models, and other contexts.

We have given generalized $2, 3$, and 4-cocycles on Alexander f-quandles in chapter 5. Furthermore, we have given three Conjectures, $(5.2.2), (5.3.9)$, and $(5.4.9)$. We plan to work on their proofs. Also, one can investigate the results of attaching not only one homomorphism, but several homomorphisms to an f-quandle.

We believe we have opened up a new concept that can be more and more fruitful in Mathematics and Physics.

References

[1] Abe, S., *On 4-cocycles of Alexander quandles on finite fields*, J. Knot Theory Ramifications 23 (2014), no. 8, 1450043, 30 pp.

[2] Adams, C., *The Knot book*, W. H. Freeman and Company, New York, (1994).

[3] Ameur, K., Saito, M., *Polynomial cocycles of Alexander quandles and applications*, J. Knot Theory Ramifications 18 (2009), no. 2, 151165.

[4] Andruskiewitsch, N., Graña, M., *From racks to pointed Hopf algebras*, Adv. Math., **178**, (2003), no. 2, 177–243.

[5] Brieskorn, E., *Automorphic sets and braids and singularities*, Braids(Santa Cruz, CA, 1986, (1988), 45–115.

[6] Carter, J. S., Crans, A. S., Elhamdadi, M., Saito, M., *Cohomology of categorical self-distributivity*, J. Homotopy Relat. Struct., **3**, (2008), no. 1, 13–63.

[7] Carter, J. S., Elhamdadi, M., Graña, M., Saito, M., *Cocycle knot invariants from quandle modules and generalized quandle homology*, Osaka J. Math., **42**, (2005), no. 3, 499–541.

[8] Carter, J. S., Elhamdadi, M., Saito, M., *Quandle homology theory and cocycle knot invariants*, Algebr. Geom. Topol., **2**, (2002), no. 95–135 (electronic).

[9] Carter, J. S., Elhamdadi, M., Nikiforou, M. A., Saito, M., *Extensions of quandles and cocycle knot invariants*, J. Knot Theory Ramifications, **12**, (2003), no. 6, 725–738.

[10] Carter, J. S., Jelsovsky, D., Kamada, S., Langford, L., Saito, M., *Quandle cohomology and state-sum invariants of knotted curves and surfaces.* Trans. Amer. Math. Soc. 355 (2003), no. 10, 3947-3989.

[11] Clark, W. E., Elhamdadi, M., Saito, M., Yeatman, T., *Quandle colorings of knots and applications.* J. Knot Theory Ramifications 23 (2014), no. 6, 1450035, 29 pp.

[12] Churchill, I. R. U., Elhamdadi, M., Green, M., Makhlouf, A., *f-racks, f-quandles, their extensions and cohomology*, J. Algebra Appl. and Its Applications, (2017), Vol. 16, no. 11.

[13] Churchill, I. R. U., Elhamdadi, M., Fernando, N., *The cocycle structure of the Alexander f-quandles on finite fields*, submitted to J. of Knot Theory and Its Ramifications, https://arxiv.org/pdf/1612.08968.pdf.

[14] Conway, J. H., Wraith, G., unpublished correspondence, (1959).

[15] Crans, A. S., Wagemann, F., *Crossed modules of racks*, Homology Homotopy Appl., **16**, (2014), no. 2, 85–106.

[16] Elhamdadi, M., Nelson, S., *Quandles—an introduction to the algebra of knots*, American Mathematical Society, Providence, RI, **74** , (2015).

[17] Elhamdadi, M., Macquarrie, J., Restrepo, R., *Automorphism groups of quandles*, J. Algebra Appl., **11**, (2012), no. 1.

[18] Elhamdadi, M., Moutuou, E., *Foundations of topological racks and quandles*, J. Knot Theory Ramifications 25 (2016), no. 3, 1640002, 17 pp.

[19] Fenn, R.,Rourke, C., *Racks and links in codimension two*, J. Knot Theory Ramifications, **1**, (1992), no. 4, 343–406.

[20] Hartwig, J. T., Larsson, D., Silvestrov, S. D., *Deformations of Lie algebras using σ-derivations*, J. Algebra, **295**, (2006), no. 2, 314–361.

[21] Hou, X., *Automorphism groups of Alexander quandles*, J. Algebra 344 (2011), 373-385.

[22] Hou, X., *Finite modules over* $\mathbb{Z}[t, t^{-1}]$, J. Knot Theory Ramifications 21 (2012), no. 8, 1250079, 28 pp.

[23] Hulpke, A., Stanovsky, D., Vojtechovsky, P., *Connected quandles and transitive groups*, J. Pure Appl. Algebra 220 (2016), no. 2, 735-758.

[24] Joyce, D., *A classifying invariant of knots, the knot quandle*, J. Pure Appl. Algebra, **23**, (1982), no. 1, 37–65.

[25] Kauffman, L. H., *Knots and physics, Series on Knots and Everything*, Volume 1, World Scientific Publishing Co. Inc., (1991).

[26] Larsson, D., Silvestrov, S. D., *Quasi-hom-Lie algebras, central extensions and 2-cocycle-like identities*, J. Algebra, **288**, (2005), no. 2, 321–344.

[27] Larsson, D., Silvestrov, S. D., *Quasi-Lie algebras, in Noncommutative Geometry and Representation Theory in Mathematical Physics*, Contemp. Math., **391**, Amer. Math. Soc., Providence, RI, 241–248 (2005).

[28] Lickorish., W. B. R., *An introduction to Knot Theory*, Springer, (1997).

[29] Makhlouf, A., *Hom-Dendriform Algebras and Rota-Baxter Hom-Algebras*, Nankai Ser Pure Appl Math Theor Phys 9 (2012): 147-171.

[30] Makhlouf, A., Silvestrov, S. D., *Hom-algebra Structures*, J. of Generalized Lie Theory and Applications, **2**, (2008), no. 2, 51–64.

[31] Makhlouf, A., Silvestrov, S. D., *Hom-algebras and Hom-coalgebras*, J. Algebra Appl., **9**, (2010), no. 4, 553–589.

[32] Makhlouf, A., Silvestrov, S. D., *Notes on 1-parameter formal deformations of Hom-associative and Hom-Lie algebras*, Forum Math., **22**, (2010), no. 4, 715–739.

[33] Mandemaker, J., *Various Topics in Rack and Quandle Homology,* Master's Thesis, Radboud University Nijmegen (2010).

[34] Matveev, S. V., *Distributive groupoids in knot theory*, Mat. Sb. (N.S.), **119(161)**, (1982), no. 1, 78–88, 160.

[35] Mochizuki, T., *The 3-cocycles of the Alexander quandles $\mathbb{F}_q[T]/(T-\omega)$*, Algebr. Geom. Topol. 5 (2005), 183205.

[36] Mochizuki, T., *Some calculations of cohomology groups of finite Alexander quandles*, J. Pure Appl. Algebra (2003), 287-330.

[37] Murasugi, K., *Knot theory and its applications*: translate by Bohdan Kurpita, Birkhäuser Boston (1996).

[38] Nelson, S., *The combinatorial revolution in knot theory*, Notices Amer. Math. Soc. 58 (2011), no. 11, 1553-1561.

[39] Niebrzydowski, M., Przytycki, J. H., *Homology operations on homology of quandles*, J. of Algebra (2010), 1529-1548.

[40] Nosaka, T., *Quandle homotopy invariants of knotted surfaces*, Mathematische Zeitschrift., (2013), no. 274, 341–365.

[41] Nosaka, T., *On quandle homology groups of Alexander quandles of prime order*, Trans. Amer. Math. Soc. 365 (2013), no. 7, 34133436.

[42] Nosaka, T., *On third homologies of groups and of quandles via the Dijkgraaf-Witten invariant and Inoue-Kabaya map*, Algebr. Geom. Topol., (2014), no. 5, 2655–2691.

[43] Rubinsztein, R., *Topological quandles and invariants of links* , J. Knot Theory Ramifications 16 (2007), no. 6, 789-808.

[44] Takahashi, N., *Quandle varieties, generalized symmetric spaces, and ϕ-spaces*, Transform. Groups 21 (2016), no. 2, 555-576.

[45] Takasaki, M., *Abstraction of symmetric transformations*, Tohoku Math. J., (1942), 145–207.

[46] Yau, D., *The Hom-Yang-Baxter equation and Hom-Lie algebras*, J. Math. Phys., **52**, (2011), no. 5, 053502, 19.

[47] Yau, D., *The Hom-Yang-Baxter equation, Hom-Lie algebras, and quasi-triangular bialgebras*, J. Phys. A, **42**, (2009), no. 16, 165202, 12.

[48] Yau, D., *Hom-algebras and homology*, J. Lie Theory, **19**, (2009), no. 2, 409–421.

[49] Zucchini, R., *On higher holonomy invariants in higher Gauge Theory I*, https://arxiv.org/pdf/1505.02121.pdf, (2015).

[50] Zucchini, R., *On higher holonomy invariants in higher Gauge Theory II*, https://arxiv.org/pdf/1505.02122.pdf, (2015).

APPENDICES

Appendix A - Programming Codes

Here we present a Maple code that we used to double-check our hand calculations on Example 4.1.5.

Maple Code 1

```
#clean up everything and initialize used libraries
restart:
with(group):
with(LinearAlgebra[Modular]):


#gives the dig-th digit of the number a over base pp
used for indexing purposes
digt:=proc(a,dig)
  floor(a/pp^(dig-1)) mod pp: end:
pp:=3; # This is ''p'' of Z_p over which the the twisted quandle
is defined.
SS:=2;
TT:=1;
Zn:=3; #the n of Zn being mapped too


#defines the operation of the twisted quandle
h:=proc(a1,a2)
 (TT*a1 + SS*a2) mod pp : end:
 #1-cocycle acting on b1 and b2 with phi set to be the characteristic
 function for Charic
 cyc1:=proc(b1,b2,Charic)
 local map1;
 map1:=[seq(0,i=1..pp)];
 map1[Charic+1]:=1;
```

```
(TT*map1[b1+1]+SS*map1[b2+1]-map1[h(b1,b2)+1]) mod Zn: end:


#runs 1-cocycle for all b1,b2 with characteristic function for
 c1, output given as a list
Imag1:=proc(c1)
  [seq(cyc1(digt(i,2),digt(i,1),c1),i=0..pp^2-1)]: end:


#2-cocycle acting on d1,d2,d3, with phi set as the characteristic
function for pair (Char1,Char2)
cyc2:=proc(d1,d2,d3,Char1,Char2)


 local map1:=[seq([seq(0,i=1..pp)],j=1..pp)]:
 map1[Char1+1,Char2+1]:=1:
 (TT*map1[d1+1][d3+1]+SS*map1[d2+1][d3+1]+
  map1[h(d1,d3)+1][h(d2,d3)+1]-TT*map1[d1+1][d2+1]-
  map1[h(d1,d2)+1][h(d3,d3)+1]) mod Zn: end:


#runs 2-cocycle for all d1,d2,d3 with characteristic function
 for (e1,e2), output as list
Imag2:=proc(e1,e2)
  [seq(cyc2(digt(i,3),digt(i,2),digt(i,1),e1,e2),i=0..pp^3-1)]: end:
  for i from 1 to pp-1 do
  ImgCyc1[i,1]:=0;
  ImgCyc1[i,5]:=0;
  ImgCyc1[i,9]:=0;
od:
#Creates a matrix whose i-th row is the values of the
1-cocycle condition acting on the
```

```
#i-th characteristic function, evaluated on each pair of

elements in lexicographic order

ImgCyc1:=Matrix([seq(Imag1(i),i=0..pp-1)]);


#The above matrix thus gives the image of each characteristic

function, and thus a generating

#set for the Image of the 1-cocycle


#Dimension of Img of the 1-cocycle condition

Rank(pp,ImgCyc1);


#Creates a matrix like a above for the second cocycle condition

ImgCyc2:=Matrix([seq(Imag2(digt(i,2),digt(i,1)),i=0..pp^2-1)]):


#implements 2-cocycle secondary condition \xi[i][i]=0

for i from 1 to pp^3 do

  ImgCyc2[1,i]:=0;

  ImgCyc2[5,i]:=0;

  ImgCyc2[9,i]:=0;

od:


ImgCyc2;


#Dimension of Img of the 2-cocycle condition

Rank(pp,ImgCyc2);
```

```
#These give the basis vectors for the Img of 1-cocycle and
 2-cocycle conditions
Basis(Zn,ImgCyc1,row,row,false);
Basis(Zn,ImgCyc2,row,row,false);


#These give the basis vectors for the ker of 1-cocycle
and 2-cocycle conditions
Nullspace(ImgCyc1^+) mod 3; Nullspace(ImgCyc2^+) mod 3;
```

Appendix B - Copyright and Permissions

Appendix B (Continued)

USF

Indu Rasika Churchill <udyanganiesi@mail.usf.edu>

Requesting for copyright permission

Tan Rok Ting <rttan@wspc.com>
To: "udyanganiesi@mail.usf.edu" <udyanganiesi@mail.usf.edu>

Tue, Feb 21, 2017 at 4:36 AM

Dear Dr Churchill

Thanks for your email. It is fine if you cite the original paper.

Best regards

TAN Rok Ting (Ms)

Editor

From: Indu Rasika Churchill [mailto:udyanganiesi@mail.usf.edu]
Sent: Friday, 17 February, 2017 6:23 AM
To: Nur Wahdiah Binte Abd Rahman <nurwahdiah@wspc.com>
Subject: Requesting for copyright permission

ABOUT THE AUTHOR

Indu Rasika Churchill (Hamudra) was born in Colombo, Sri Lanka. She received her primary and secondary school education from Seeduwa Roman Catholic Primary School and Newstead Girls College, in Negombo, Sri Lanka. In 2009, she earned her BSc (special) degree majoring in Mathematics from the University of Kelaniya, Sri Lanka. She entered the Mathematics Ph.D. program at the University of South Florida in Spring 2012 and obtained her Masters in Fall 2013 while continuing her Ph.D. studies at USF.

Rasika studied f-quandles and their cohomologies under the supervision of Dr. Mohamed Elhamdadi (USF) and Dr. Abdenacer Makhlouf (University of Haute Alsace, France). She taught several undergraduate courses at USF as a Graduate Teaching Associate. Her research interests lie in Knot Theory, Quandles, Low-dimensional Topology, Mathematical Physics and Finite Fields.