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Prevalence of Typical Images in High School Geometry Textbooks

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Prevalence of Typical Images in High School

Geometry Textbooks

by

Megan N. Cannon

A thesis submitted in partial fulfillment
of the requirements for the degree of
Master of Arts
Department of Mathematics and Statistics
College of Arts and Sciences
University of South Florida

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ABSTRACT

Visualization in mathematics can be discussed in many ways; it is a broad term that references physical visualization objects as well as the process in which we picture images and manipulate them in our minds. Research suggests that visualization can be a powerful tool in mathematics for intuitive understanding, providing and/or supporting proof and reasoning, and assisting in comprehension. The literature also reveals some difficulties related to the use of visualization, particularly how illustrations can mislead students if they are not comfortable seeing concepts represented in varied ways. However, despite the extensive research on the benefits and challenges of visualization there is little research into what types of figures students are exposed to through their textbooks.

This study examines 14 high school geometry textbooks in total, comprised of eight physical textbooks from the top three major textbook publishers in the United States and six FlexBooks created by a non-profit organization developing free and customizable textbooks online. In each textbook the printed images from four topics were classified: Parallel Lines and Transversals, Classifying Triangles, Parallelograms, and Trapezoids. The 'typical' images in each of the four topics were defined and the percentages of images that were typical for each textbook in both the lesson and exercise portions were calculated. Results indicate that lesson portions of sections generally contain more typical images than exercise portions and that the total percentage of typical images in an average section varies from 51.9% typical images in the Parallel Lines and Transversals section to 75.2% typical images in the Trapezoid section. Based on these results we list possible avenues for further research in this area.

CHAPTER ONE:

INTRODUCTION AND RESEARCH QUESTION

Introduction

Pictures and illustrations are used regularly in mathematics to solve problems and turn abstract ideas into concrete visual representations. Figures and diagrams are present in many fields of mathematics, from learning basic arithmetic in elementary school to more complex subjects in graduate mathematics courses. The prevalence of visual images leads us to question what aspects of a diagram or figure are helpful, and what characteristics can potentially lead to misconceptions and misunderstandings. This study of images leads to the broader study of visualization in mathematics.

Visualization can be thought of as a comprehensive notion with many distinct definitions. In fact, Phillips, Norris, and Macnab (2010) found 23 explicit definitions of visualization in published research and divided these into three categories: visualization objects, introspective visualization, and interpretive visualization. This variety in definitions allows us to describe visualization as physical or tangible representations (visualization objects), mental pictures or internal visual objects (introspective visualization), and the process of deriving meaning from these images (interpretive visualization), depending on which category of definition we use. Research in mathematics education demonstrates that visualization as a process can aid students and mathematicians in problem solving (Lakin & Simon, 1987), reasoning (Presmeg, 1992), showing relationships between concepts (Phillips, Norris, & Macnab, 2010), discovering mathematical ideas (Zimmerman & Cunningham, 1991), offering intuitive visual evidence (Duval, 2006), and possibly even providing basic proofs (Arcavi, 2003). However, visualization in mathematics education is a relatively new field of study and some mathematicians still do not regard visualizations as an effective form of proof and reasoning (Inglis & Mejía-Ramos, 2008).

For more than a century, visualization has been studied as a "psychological phenomenon" (Phillips et al., 2010, p. 9); however, the process of visualization is not bound to one single field. This thesis specifically deals with visualization in relation to mathematics, but the process of visualization can be beneficial in the fields of engineering, sciences, psychology, education, and technology. Mathematicians naturally use many methods of communicating ideas and representing concepts from rigorous mathematical writing to visual methods (Alshwaikh, 2010). Examining presentations of mathematical ideas throughout history reveals that visual representations have always been crucial to developing mathematics (Duval, 2006). Nevertheless, research on the use of visual representation of mathematical objects did not begin until the late 1970s, and it wasn't until the 1990s that this became a significant field of study (Presmeg, 2006). Quoting Zimmerman and Cunningham, in 1991 "the sciences, engineering, and to a more limited extent mathematics enjoyed a renaissance in visualization" (p. 1).

Currently, there is no clear idea of what the role of visualization in mathematics and mathematics education should be. Visual images are becoming more pervasive in mathematics textbooks and more complicated in their presentation (Dimmel & Herbst, 2015), so there is a clear need for understanding the use of visual materials. The use of visual objects in mathematics was once thought to be only for illustrative purposes, but is now believed to provide assistance in reasoning and some forms of proof (Arcavi, 2003). This expansion of purpose is in part due to growth in the field of computer graphics (Zimmerman & Cunningham, 1991). Computers have offered renewed possibilities for educators to incorporate visualization, but have not eliminated all of the challenges in determining what elements of a visual representation are helpful and what elements are potentially misleading (Phillips et al., 2010). Despite the revitalized interest in visualization, the specific relationship between spatial abilities and mathematical performance has not been explained (Battista, Wheatley, & Talsma, 1982).

Research Question

Research in visualization suggests that prototypical images can be important to students' understanding necessary properties of figures (Presmeg, 1992). Yet, there has been little research into the visual content of textbooks. We could find no study showing what 'typical' images students encounter through the use of their textbooks. We decided to focus in on four topics of a typical geometry course: parallel lines and transversals, classifying triangles, parallelograms, and trapezoids. Therefore, our research question is: *What are the 'typical' images in High School Geometry textbooks in the sections covering parallel lines and transversals, classifying triangles, parallelograms, and trapezoids?*

CHAPTER TWO:

LITERATURE REVIEW

In this Chapter, I will present a cross-section of the research relating to the multiple definitions surrounding the word *visualization.* Specifically we will address the following:

Defining Visualization

The Benefits of Visualization

The Limitations of Visualization

Suggestions for Overcoming Limitations of Visualization in the Curriculum

The Impact of Teacher Beliefs About Visualization on Student Learning

Defining Visualization

There is a broad spectrum of ideas relating to the definition of visualization. It can refer to objects that are realistic or solely representational and can also refer to anything that can be seen or imagined (Phillips et al., 2010). Zimmerman and Cunningham provide one possible definition of visualization to "describe the process of producing or using geometrical or graphical representations of mathematical concepts, principles or problems, whether hand drawn or computer generated" (1991, p. 1). Visualization is also described as "something which someone does in one's mind—it is a personal process that assumes that the person involved is developing or using a mental image" (Clements, 2014, p. 181). Presmeg (2006) states that the method of "visualization is taken to include processes of constructing and transforming both visual mental imagery and all of the inscriptions of a spatial nature that may be implicated in doing mathematics" (p. 2). Visualization is further described as:

> (1) A mode of mathematical thinking
>
> (2) A group of *signs* and relationships among them ("a language"), by which mathematical thinking, including the visual one, might be developed, limited, ex- pressed and communicated to oneself and to others.
>
> (Nardi, 2014, p. 198)

For the remainder of this paper, we will refer to visualization as a process of using or producing visual representations either physically or mentally.

Components of visualization can be discussed in terms of the process of visualizing, visual methods of solutions, visualizers, and visualization objects. According to Presmeg (1986), "A visual method of solution is one that involves visual imagery, with or without a diagram, as an essential part of the method of solution, even if reasoning or algebraic methods are also employed" (p. 42). This definition is contrary to what one might expect of as a "visual method" in that a visual representation does not need to be drawn, only imagined as assistance to the solution. Therefore visual processing could be necessary for solving some problems that are seemingly nonvisual (Presmeg, 1992).

As defined by Phillips, Norris, and Macnab (2010), "A visualization object is any object that a student observes to assist in the learning or understanding of some topic of educational importance. A Visualization object could be a picture, a schematic diagram, a computer simulation, or a video" (p.3). While a student is using some visualization object to solve a problem, they are said to be *visualizing,* and when the student is using visual means with no visualization object they are *introspectively visualizing* (Phillips et al., 2010). A visual image is any mental model that portrays visual or spatial knowledge (Presmeg, 1992).

Based on Tall and Vinner's research (1981), a particularly important category of visual images is that of concept images. A *concept image* is the visual information we associate with a certain mathematical idea. This is gained through personal experiences in working with the concept. This idea contrasts a *concept definition,* which is what students are explicitly taught about the idea (Vinner, 1983). Tall and Vinner (1981) suggest that the concept image of a specific idea or object is built up over many years of experience and exposures, and includes all of the relevant mental pictures, properties, and processes involved (p. 2). For example, when students learn about slope of a line the concept definition will be what is explicitly taught in the textbook or class. This may be the idea of 'rise over run' or the formula for finding the slope of the line between two points. The concept image is the picture that forms in their head when they hear the word slope. This concept image could be different for every student depending on what visual representation of slope they have deemed the most relevant, but the concept definition of slope should be the same for every student.

When students begin to create concept images, they may struggle trying to sort out what examples are and are not relevant (Vinner & Dreyfus, 1989). This difficulty in determining relevance is problematic because a concept image is only considered functional when it allows the student to classify an example or non-example (Gutierrez & Jaime, 1999). When we think about a particular topic, the activated portion of the concept image is called the *evoked concept image.* Depending on the activity at hand, these evoked concept images may sometimes be conflicting, although as long as the conflicting concept images are not evoked simultaneously there should be no confusion or uncertainty (Tall & Vinner, 1981). For example, a student's concept image of a line may have positive or negative slope, but these two concept images should not be evoked at the same time because the student should only be visualizing the image that is relevant to the current problem.

A specific type of concept image is the prototype. A prototype is what we think of as the most generic representation of a category of objects; it is the first image we envision when we think of a specific topic. Prototypes are particularly useful because they allow students to attribute properties for a category of objects to one single representative of the group. Figure 1 illustrates prototypical images of three common geometric figures.

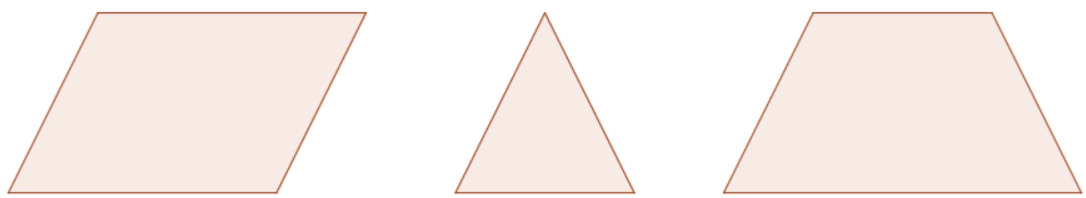

Figure 1: Example prototypes of a parallelogram, triangle, and trapezoid

The Advantages of Visualization

The process of visualization involves forming images in our heads, on paper, or using technology to assist in mathematical breakthroughs and comprehension (Zimmerman & Cunningham, 1991). Visualization promotes a deeper conceptual understanding of certain mathematical ideas. For example, in Figure 2, by taking a parallelogram and reconfiguring it to produce a rectangle, we can construct a visual explanation for the area formula for parallelograms. For some students, this method of thinking comes naturally, but for others it takes time and practice. Based on the idea of “opportunity to learn”, students who have been exposed to visual methods have a better opportunity to learn visualization than those who have not (Hiebert & Grouws, 2007). The process of visualization is a powerful tool for comprehension (Zimmerman & Cunningham, 1991), and therefore needs to be taught to students.

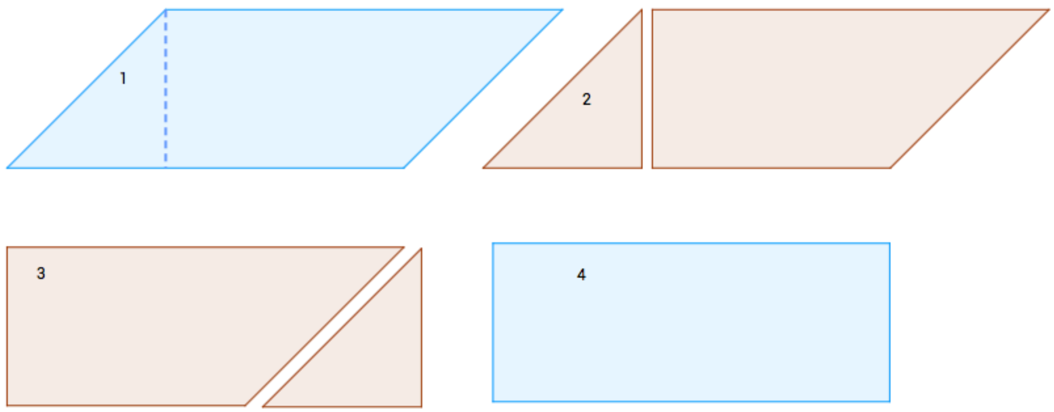

Figure 2: Visual method for demonstrating how area of a trapezoid relates to area of a rectangle

The process of visualization is a natural part of the critical thinking process. When working on a mathematical puzzle or problem solving, many students innately tend to use representation as a reasoning tool (Lakin & Simon, 1987). These representations may take the form of sketches on paper, drawings on a whiteboard or chalkboard, or visualizations using dynamic geometry software. Students may also use visualization without a physical visualization object by transforming images in their mind.

Using visualization to supplement mathematics education can support students' comprehension of symbolic results, provide a method of proof, assist students in understanding when their intuitions are misleading, and help students appreciate concepts without the use of a formal solution (Arcavi, 2003). One important feature of teaching visualization is the idea of the concept image and concept definition. When reading a textbook, it is expected that students understand the concept definition the same way; readers may not necessarily share the same concept image, but they can agree on whether or not a particular image fits the concept definition (Shepard, A. Selden, & J. Selden, 2012). Eisenburg (2014) states that when a particular mathematics topic is discussed, he immediately envisions the pictures in his mind that capture the concept for him, and this idea of visuality should be a crucial part of education in the mathematics classroom (p. 41). Polya (1945) suggests visualizing geometric relations as a strategy for problem solving in *How to solve it: A new aspect of mathematical model*. Using visual methods to solve mathematical problems can help develop students' conceptual understanding of the material.

Inglis and Mejia-Ramos (2008) conducted two studies to examine how persuasive mathematical arguments are when expressed visually. The first study involved 58 university students and 56 active mathematicians. The participants were provided visual arguments (some with a short description of the argument and some without) and asked what level of proof they believe was being provided. They found that a visual argument accompanied by a short description was fully convincing to over half of the participants. The second study involved 24 active mathematicians and 39 mathematics undergraduate students. Participants were again provided with visual arguments (some with a short description accompanying it and some without) but the image for the no-description category was designed to point out noticeable parts of the visual argument. The results were about the same as the first experiment, suggesting that visual arguments can be powerful tools for proof when accompanied by a short description (Inglis & Mejía-Ramos, 2008).

The Limitations of Visualization

The arguments against the use of visualization in the mathematics curriculum stem from under-valuing the impact of visualization on students' ability to think critically and reason. The common view of visual representations in mathematics, held by many mathematicians, is that while images can be helpful in understanding proof and reasoning, they are inappropriate when used to provide evidence to support a claim (Inglis & Mejía-Ramos, 2008).

According to Aspinwall, Shaw, and Presmeg (1997), in the 1990s, textbook authors and mathematics instructors began using more graphs in calculus textbooks with the rationale that visual materials increase conceptual understanding. With the increase in use of visual materials, it will be important for educators to understand possible difficulties students may face when using visual methods. One problem facing students using visualization methods is the phenomenon of *compartmentalization*, which occurs when students have conflicting schemes or concept images and therefore may provide inconsistent answers or solutions. Vinner and Dreyfus completed a quantitative study on concept images and definitions of functions involving 271 college students and 36 mathematics teachers. In one particular example their study found that 56% of students struggled with compartmentalization, causing them to provide somewhat conflicting responses to questions. Respondents gave a particular definition for the concept of a function, but then did not use that definition when answering the remaining questions in the study.

Nardi (2014) expanded on the issue of compartmentalization and added a list of challenges most commonly encountered by students:

> "The one-case concreteness of an image may be tied to irrelevant details or introduce false information, a prototypical image may induce inflexible thinking, an

uncontrollable image may persist, thus preventing more fruitful avenues of thought, and imagery needs to link with rigorous analytical thought processes to be effective" (p. 213).

Presmeg (1992) also asserted that many of the difficulties experienced by visualizers are related to the one-case concreteness they associate with certain images (p.603). This refers to when a student observes that a particular property holds true for a particular case and then incorrectly generalizes this property to other cases. For example, if a student finds on a particular problem that the altitude of a triangle lies inside the triangle, they might incorrectly assume that this is always true and become confused when they try to do the same for an obtuse triangle. The problems associated with this one-case concreteness can be overcome using methods discussed in a later section.

Another potential problem for students who are using visual methods is inappropriate use of prototypical images. A prototypical image is one that the student creates in his or her mind as the most generic example to represent a category of objects. This can be helpful when a student wants to quickly classify objects, but can interfere with identification of somewhat atypical examples (Presmeg, 1986). For example, as shown in Figure 3, if students are used to seeing most trapezoids drawn to look isosceles, they may not recognize other trapezoids still belong to the same category of objects. There are certain 'classical' drawings in textbooks, which have been featured as representations of objects; these traditional drawings allow students to recognize figures at first glance but also limit students' imagination of alternate drawings that still fit the concept (Parzysz, 1988). This implies that textbook creators can have an impact on students' ability to successfully use visualization.

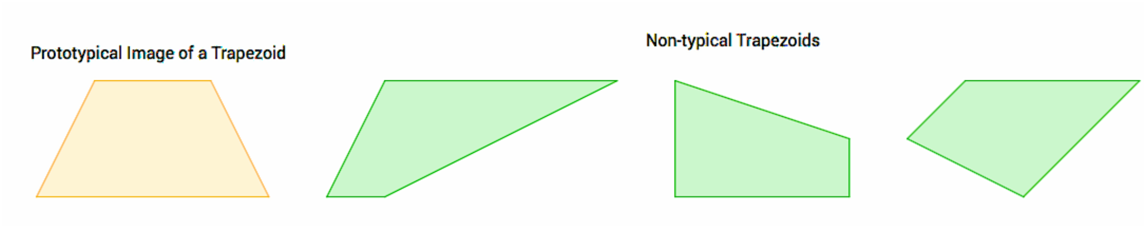

Figure 3: Examples of prototypical trapezoid and non-typical trapezoids

Given a mathematical problem, students sometimes introspectively visualize a mental image of a figure that may not be appropriate for the solution they are working towards. This type of mental figure, when persistent, is called an uncontrollable image, and can interfere with the students' ability to analyze the problem. The student cannot regulate uncontrollable images, which can prevent a student from successfully making mathematical generalizations. These images persist in spite of being inappropriate and proven to be unrelated to the given exercise (Aspinwall, Shaw, & Presmeg, 1997). When an uncontrollable image is vivid, it can prevent an individual from pursuing other strategies or methods for solution (Presmeg, 1986).

Visualization needs to be combined with critical thinking to be useful. If imagery is not linked to analytical thinking, the result can be non-beneficial (Presmeg, 1986). Use of imagery will reach its full potential when it is used to benefit the abstraction of mathematics (Presmeg, 1992). As quoted by Duval (2006), "When we focus on visualization we are facing a strong discrepancy between the common way to see the figures, generally in an iconic way, and the mathematical way they are expected to be looked at."

Suggestions for Overcoming Limitations of Visualization in the Curriculum

Visualization improves the conceptual understanding of mathematics, but not without obstacles. One method for overcoming limitations of visual methods is using a concrete case to display abstract information about an image. This can assist students in understanding what portions of an image can be changed while maintaining the same mathematical properties. In this case, dynamic imagery, designed to help students to recognize relevant properties, can be very useful.

Computers provide a unique opportunity to explore and benefit from dynamic imagery by using visualization with incorporated movement (Duval, 1998). Dynamic Geometry software, such as The Geometer's Sketchpad (www.dynamicgeometry.com) or GeoGebra (www.geogebra.org), can use movement to demonstrate how properties in images are preserved or changed (Guvan &Kosa, 2008). Figure 4 shows how GeoGebra can be used to illustrate the Inscribed Angle Theorem. GeoGebra allows students to construct an inscribed angle with endpoints on a circle corresponding to the endpoints of the related central angle, and drag the vertex of the angle to different points along the circle's arc. It can also display the angle measure as students move the vertex along the arc of the circle to show that the measure is maintained.

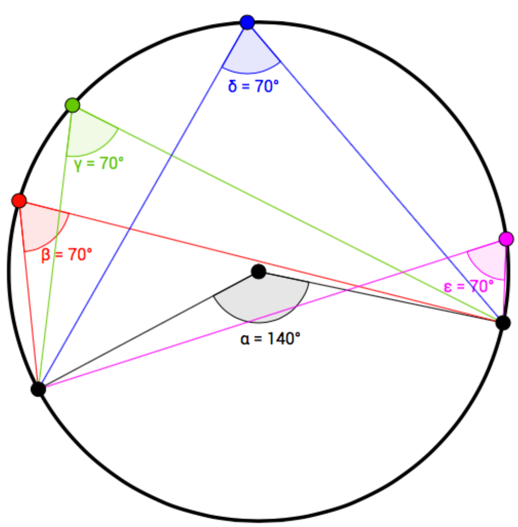

Figure 4: GeoGebra illustration of Inscribed Angle Theorem

These software packages can be used as tools for students to manipulate figures and examine certain properties which helps combat the problem of one-case correctness. Dynamic Geometry Software can also link visualization to critical thinking by allowing students to see how changing certain parameters or how repositioning of objects affects the properties of the image. Not all teachers are taught to teach mathematics with technology resources (Niess, 2005), meaning there is a good chance that students may never have the opportunity to interact with visual diagrams in a dynamic way.

The Impact of Teacher Beliefs About Visualization on Student Learning

According to the *Report of the 2012 National Survey of Science and Mathematics Education* (2013), "Mathematics teachers' beliefs about effective instruction are, in some ways, in line with current recommendations from research and, in other ways, are not well aligned" (p.31). This can be troublesome when considering that teachers' beliefs about education and mathematics, even if those views are unconscious, can have a critical impact on instructional processes (Thompson, 1984). Teachers' preferences and knowledge significantly influence their pedagogy (Frykholm, 2004). Pre-service teachers already have developed certain ideas and beliefs that filter the experiences and knowledge they encounter in their training (Feiman-Nemser, 2001), meaning they may have opinions about the use of visualization in the classroom that impact the way they use visual materials. The implication is that any research designed to improve the quality of mathematics education needs to begin by looking at teachers' beliefs and the influence on their instructional practice, since teachers are the main intermediary between students and the subject they are learning (Thompson, 1984).

Teachers' beliefs are not the sole influence on student learning; instructional materials (textbooks in particular), and how teachers use the materials, are also crucial in shaping the learning experience for the student. Teachers use instructional materials as a resource for developing lessons; however the instructional materials do not embody the entirety of the learning opportunities presented to students (Rosaen, 1992). Furthermore, teachers may not use the textbook or other instructional materials effectively, possibly omitting important topics or misrepresenting key concepts (Ball, 2000). According to the *Report of the 2012 National Survey of Science and Mathematics Education,* 74% of high school mathematics teachers use textbooks to guide the overall structure and content of the class. Teachers sometimes deviate from the given instructional materials by skipping over certain lessons that they believe to be insufficiently developed, and supplementing with materials they find more suitable (Banilower et al., 2013). Therefore, teachers' beliefs, ideas, and pedagogy (including their ideas about the use of visualization), along with classroom instructional material, have a significant impact on a student's learning.

Opportunity to learn is a concept that explains the connection between the information taught to students with the information students learn. Yet, the concept is much more complex and multi-faceted than simple exposure of materials to students (Hiebert & Grouws, 2007). The way that teachers present concepts, what ideas teachers believe are important, and the knowledge of the subject matter that teachers have, all influence what students recognize as significant mathematical ideas. For students to benefit fully from using visualization as a learning tool in mathematics, the learning environment created by the teacher would need to encourage visual methods. If teachers are highly visual, then they may have a pre-disposition to use visual presentations when presenting mathematical concepts or justifications (Presmeg, 1986), and they are more likely to include visual materials and representations in their lessons. Students naturally differ in how visual they are, but teachers and instructional materials can still develop or stifle this ability.

CHAPTER THREE:

RATIONALE FOR TEXTBOOK EXAMINATION

In the field of mathematics education, studies have looked at the impact textbooks have on curriculum (Nicol & Crepso, 2006), and separately at the effectiveness of visualization in learning mathematics (Presmeg, 1986), (Inglis & Mejía-Ramos, 2008), (Arcavi, 2003). Despite extensive research on the importance of textbooks in student learning, more effort is needed to analyze textbook content, including visual images in textbooks (Zorin, 2011). Dogbey (2010) states that when it comes to curriculum, textbooks are an expert representation of a subject, leading to little questioning of the content. During the 1970s through 1980s, many research projects were conducted on curriculum; their findings suggested that there is a need for more research to analyze the content of textbooks (Consuelo, 2012). During the 20th century, textbooks experienced a 300% growth in the average number of content pages (Knipe et al., 2010), making it even more difficult to analyze what content is actually used during instruction.

Textbooks are usually developed and produced by one of a few major publishing companies (Consuelo, 2012). Public school textbooks make up a $4.3 billion dollar a year market, yet they are surprisingly unexamined. This is likely because they are not purchased or retailed in a free market (Finn & Ravitch, 2004). States with textbook adoption laws and large buying power have substantial influence over what is included in textbooks, with a small number of publishers providing the supply for the country, leading to a market that is heavily saturated with a small number of products. This means there is less competition and choices amongst textbooks which makes it easier to examine the content of these particular textbooks, but harder to study how a variety of options compare.

Textbooks' Influence on Curriculum

The curriculum for a class is a sequence of instruction for the topics in a subject, most often in line with some specific learning goals. The instructional materials assigned to a course are designed to include the curriculum (Zorin, 2011). Curriculum materials, which are usually textbooks, largely determine what is actually taught in most classrooms (Dogbey, 2010). Often the curriculum presented in textbooks is referred to as the "intended curriculum" and represents the scope and sequence of the topics to be covered in a course (Consuelo, 2012).

Textbooks also influence how teachers see the material for a course. In a study on how pre-service teachers use textbooks, Crespo & Nicol (2006) found that many prospective teachers use their textbooks to answer their questions about the course concepts. Many teachers depend on the textbook and might not supplement the class text with any other materials (Vincent & Stacey, 2008). The textbook allows teachers to quickly determine what should be taught, how it should be taught, and offers built in practice for students (Thompson, Senk, & Johnson, 2012). In other words, textbooks have extensive influence over the classroom curriculum (Banilower et al., 2013).

In a mathematics classroom, textbooks provide a structure for the sequencing of course concepts along with how and what will be taught (Nicol & Crepso, 2006). Textbooks are the most basic way in which the curriculum is shared with the student (Consuelo, 2012), so it is important that they are appropriate for the use of the students and the teachers (Dogbey, 2010).

Even something not directly related to the content, such as writing style, can influence what information students identify as important in their textbooks (Vincent & Stacey, 2008).

Textbooks can influence students in a variety of ways, therefore it is not a stretch to think that the visual information in textbooks can change the way students learn concepts. For example, if the visual representations of a certain object are always printed aligned with the text, then students may begin to see that alignment as a property of the object and not recognize the object represented in other ways. In geometry, the altitude of a triangle is often presented as in left triangle of Figure 5. So when students are presented with an obtuse triangle, like the middle triangle of figure 5, they may struggle to understand how to find the altitude if it falls outside the triangle. If they have seen examples of altitudes in obtuse triangles, but only in examples aligned horizontally with the text, they may struggle with an example printed like the triangle on the right in figure 5. It may be hard for them to see how the altitude still falls outside the triangle when the triangle is not aligned with the text.

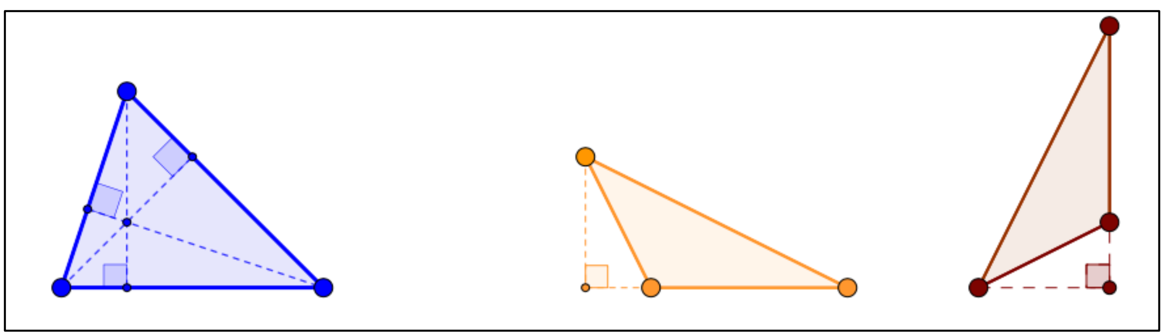

Figure 5: Representations of altitudes of triangles with different alignments

Visual Materials in Textbooks

While textbooks usually have pictures in them, they do not necessarily present methods for interpreting visual material. Geometry textbooks often have only the most prototypical shapes and lack a sufficient quantity of non-standard examples (Cunningham & Roberts, 2010). This can lead to students incorrectly forming concept images by attributing things like alignment of a figure to the properties of the object. Also, the number of images or pictures used in a book generally varies inversely with grade level, so as students move through grades, they receive less and less exposure to visual materials (Phillips et al., 2010). Visuals in textbooks are often considered unreliable for the reason that students are warned not to trust information given from diagrams; the textbook authors may be trying to 'trick' them by incorrectly representing scale of objects or positioning (Herbst & Arbor, 2004). Currently, there is a trend for newer textbooks to have a more varied semiotic catalog, or features used to represent objects, than previous textbooks; this suggests that teaching visual literacy (the ability to interpret and create meaning from an image) is an important feature of mathematics education (Dimmel & Herbst, 2015).

Geometry often contains more visual demands than any other subject in high school mathematics. Each exercise and problem in geometry naturally requires a 'seeing' process that happens when students physically look at the image (Gal & Linchevski, 2010). Diagrams in geometry use objects that are drawn in a specific way to convey a specific meaning (Dimmel & Herbst, 2015). Geometry involves using at least two representations, visual and verbal/numerical expression of properties of figures (Duval, 2006). The focus in the geometry curriculum is usually on figures and their properties along with mental transformations like reflections and rotations. The more involved the mental transformation is, the more difficult it is for students to internally visualize (Gal & Linchevski, 2010). Diagrams in geometry offer 'spatio-graphical' properties that activate students' visual and perceptive abilities (Laborde, 2005). This visual demand in geometry makes geometry textbooks an appropriate place to analyze images.

CHAPTER FOUR:

METHODOLOGY AND CONCEPTUAL FRAMEWORK

Initial Study

We begin by defining what we mean by the term *typical image.* Our definition of a typical image is similar to the previously given definition of a prototypical image.

A typical image of a particular mathematical object is a visual representation of that object that is drawn a certain way in the majority of the instances with no content-based reason.

For example, in a section about equilateral triangles, if we notice that over 50% (a majority) of the triangles are positioned such that one of their sides is horizontally aligned with the text, we can say that an equilateral triangle with this horizontal alignment is a typical image. However, the fact that every triangle in the section is equilateral would not mean that equilateral triangles are typical because there is a content-based reason for this representation.

To determine what are the typical images, we looked at a relatively small sample of geometry textbooks to document what kind of images were being presented. We found that a majority of the polygons were drawn such that at least one side of the polygon was horizontal, with no content-based reason for this alignment. These typical images can lead to students assuming that this type of figure must always be drawn with this particular alignment, and thus may affect the concept image a student has of this particular polygon. Figure 6 shows some examples of typical and non-typical triangles.

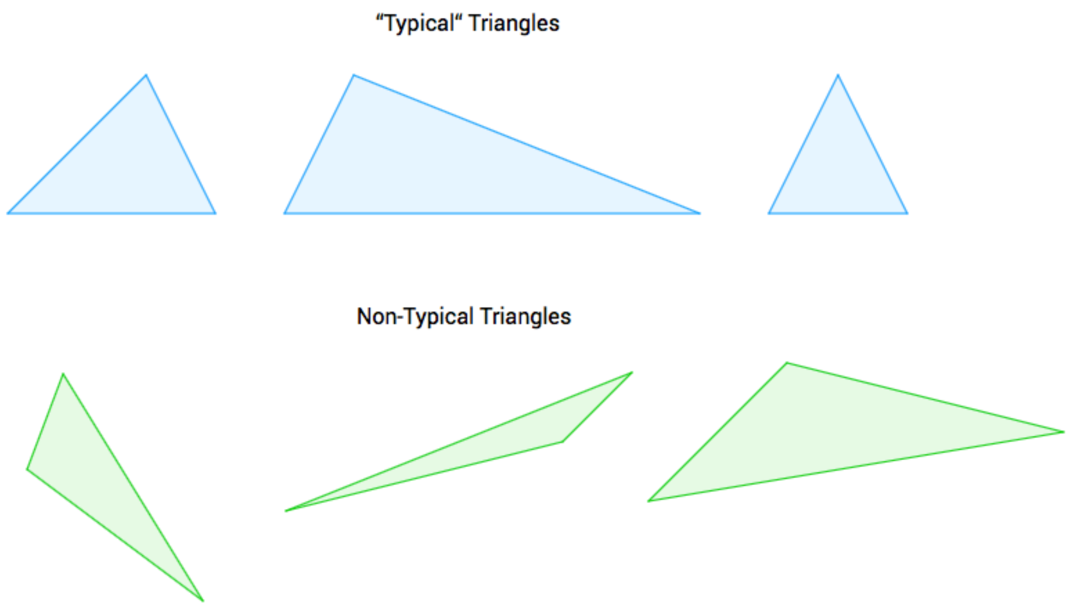

Figure 6: Typical and non-typical triangle examples

The Sample

In mathematics classrooms across the nation, three major publishing houses, Houghton Mifflin Harcourt, McGraw-Hill, and Pearson, provide the majority of secondary mathematics textbooks accounting for approximately 75% of the market (Banilower et al., 2013). Two other publishers, the University of Chicago School Mathematics Project (UCSMP) and the CK-12 Foundation were included in the sample even though they are not as commonly adopted. UCSMP is a University based curriculum project currently being used by 4.5 million students nationwide; they are student-centered and research driven. The goal of the project has been to raise standards for mathematics education (Usiskin). UCSMP uses a transformational prospective in teaching geometry and encourages dynamic approaches making their textbook an interesting addition to the study. The CK-12 Foundation is a non-profit organization developing free and customizable textbooks that are designed to align with state standards. They develop what they call FlexBooks, which are interactive online textbooks that can include multimedia education. For example, a teacher can imbed a figure directly into the textbook lesson that uses dynamic geometry software so students can manipulate the image.

To display the information in an easier to read format, we gave each textbook a short code as shown below in Table 1.

Table 1: Short codes used for textbooks.

Textbook and Publisher	Short Code
Pearson: *Geometry Common Core* (Charles et al., 2015)	P1
Pearson: *CME Project Geometry* (Cuoco et al., 2009)	P2
Pearson: *Informal Geometry-Classics Edition* (Cox, 2006)	P3
Pearson: *Blitzer, Thinking Mathematically* (Blitzer, 2015)	P4
McGraw Hill: *Glencoe Geometry* (Carter, Cuevas, Day, Malloy, & Cummins 2010)	M1
McGraw Hill: *Geometry Concepts and Applications* (Cummins, Kanold, Kenney, Malloy, & Mojica, 2006)	M2
Houghton Mifflin Harcourt: *Holt McDougal Geometry Common Core Edition* (Burger, Chard, Kennedy, Leinwand, Roby, Seymour & Waits, 2012)	H1
University of Chicago School Mathematics Project (UCSMP): *Geometry* (Benson, Klien, Miller, Capuzzi-Feurstein, Fletcher, Marino, & Usiskin, 2009)	U1
CK-12 Foundation: *Geometry 2nd Edition* (Jordan & Dirga, 2015)	C1
CK-12 Foundation: *Basic Geometry Concepts* (Greenberg, Jordan, Gloag, Cifarelli, Sconyers, & Zahner, 2015)	C2
CK-12 Foundation: *Geometry Concepts-Honors* (Spong, 2016)	C3
CK-12 Foundation: *Geometry-Basic* (Jordan, Zahner, Cifarelli, Gloag, Greenberg, & Sconyers, 2014)	C4
CK-12 Foundation: *Geometry-Concepts* (Dirga & Jordan, 2015)	C5
CK-12 Foundation: *Foundation and Leadership Public Schools, College Access Reader, Geometry* (Fauteux & Zapata, 2015)	C6

From Pearson publishing we chose four textbooks, each with a slightly different method of presenting the material. Pearson's *Geometry Common Core* (Charles et al., 2015) textbook is the publisher's main series aligned with the Common Core standards. The *CME Project Geometry* (Cuoco et al., 2009) textbook is published by Pearson, but is very different from their main series; it is an NSF-funded high school mathematics textbook series developed by the Center for Mathematics Education (Center for Mathematics Education- CME Project, 2016). The *Informal Geometry-Classics Edition* (Cox, 2006) presents material in a less customary format, focusing more on concepts and less on formal definitions, proofs, and theorems (Informal Geometry- Classics Edition, 2016). Pearson's *Blitzer, Thinking Mathematically* (Blitzer, 2015) is a more inclusive textbook that contains content from algebra, geometry, set theory, probability, and more. It is designed for students to use in a liberal arts mathematics course with the assumption that those students will not be continuing on to become math majors and therefore focuses on real world applications and building mathematical confidence.

We chose two textbooks from McGraw Hill publishing, one with more formal presentations of proofs and one with conceptual proofs. McGraw Hill's *Glencoe Geometry* (Carter et al., 2010) is their most widely used series that has formal presentations of ideas and is aligned with Common Core standards. Their *Geometry Concepts and Applications* (Cummins et al., 2006) is also widely used but is a less formal textbook focusing on conceptual understanding and applications in the real world.

From Houghton Mifflin Harcourt and University of Chicago Mathematics School Project (UCSMP) we chose one book each: *Holt McDougal Geometry* (Burger et al., 2012) from Houghton Mifflin Harcourt and *Geometry* (Benson, et al., 2009) from UCSMP. *Holt McDougal*

Geometry is Houghton Mifflin Harcourt's main series and *Geometry* is UCSMP's only geometry textbook.

CK-12 Foundation had a variety of geometry textbooks for different purposes so we chose six books to examine. We analyzed their most general geometry book, a book for basic geometry learning, a book based on more conceptual learning, a basic concept book, an honors geometry book with conceptual understanding, and a geometry book for college preparation.

Topics Examined

The topics we examined included sections that should be taught in every geometry course and would potentially have many images that do not have a content-based reason for being drawn in a particular alignment. We focused on the visual representations of the following four topics: parallel lines and transversals, classifying triangles, parallelograms, and trapezoids. For each textbook, we found the sections that most thoroughly covered these four topics.

The researchers looked briefly at each of the topics to determine what would be considered typical and noticed a common theme. Images are consistently drawn with at least one line segment horizontal. For each of the sections, we defined the typical images (shown in Figure 7) as follows:

- *Parallel lines and transversals*- A transversal is a line that intersects two other lines at two distinct points. Transversals are particularly interesting when they pass through parallel lines, as there are many useful properties about angle measures. We defined the typical image to be an image in which the parallel lines aligned with the text.
- *Classifying triangles*- A triangle is a polygon that is three sided. We defined the typical image of a triangle as a triangle drawn with one side aligned with the text in the book.

- *Parallelograms*- A parallelogram is a quadrilateral with two pairs of parallel sides. We define the typical image of a parallelogram to be an image that is drawn with one pair of parallel sides aligned with the text.
- *Trapezoids*- A trapezoid is a quadrilateral with at least one pair of sides parallel. We define the typical image of a trapezoid to be a trapezoid that is drawn with the pair of parallel sides aligned with the text.

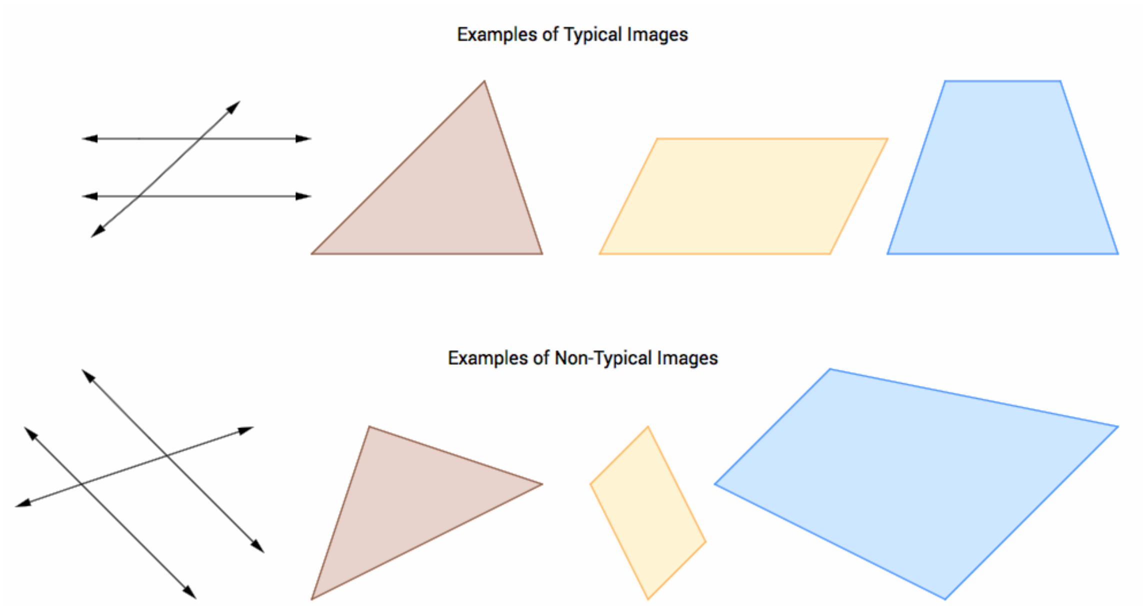

Figure 7: Examples of typical and non-typical images in each section

Procedure

For each of the textbooks, we examined the four previously listed sections. Each section contains a lesson portion and an exercises portion, usually around 2-6 pages for the lesson and around 2-4 pages for the exercises. We decided to look at these two parts of each section separately to determine if there was any difference in the number of typical images presented.

Each image in the textbook section was coded as either typical or atypical according to the previously discussed specifications and a tally was kept.

When an image presented ambiguity in coding because it was not a single figure but a composition of more than one figure, (example shown in Figure 8), we examined the questions asked about the figure. So, for example, in Figure 8, if the question asked about triangle ABD then we would code triangle ABD as a typical image. If the question asked about parallelogram AEFB then the image would not be typical because the parallelogram is not aligned horizontally. This was done for each question asked about the compilation image to avoid inconsistencies.

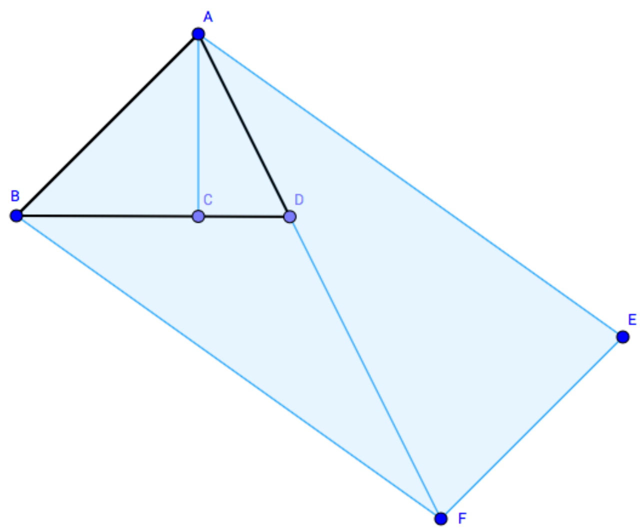

Figure 8: Example of a compilation image needing interpretation for coding

CHAPTER FIVE:

ANALYSIS OF RESULTS

Table 2: Number of images examined in each textbook by section.

	Parallel Lines and Transversals		**Classifying Triangles**		**Parallelograms**		**Trapezoids**		**Totals**		
Textbook	**Lesson**	**Exercises**	**Lesson**	**Exercises**	**Lesson**	**Exercises**	**Lesson**	**Exercises**	**Lesson**	**Exercises**	**Total**
P1	10	20	18	23	18	17	11	21	57	81	138
P2	2	8	1	15	6	3	1	9	10	35	45
P3	7	22	15	14	4	15	7	22	33	73	106
P4	5	9	29	71	1	3	1	2	36	85	121
M1	10	23	17	42	16	32	13	17	56	114	170
M2	10	11	13	15	11	7	6	16	40	49	89
H1	14	25	13	17	15	17	14	23	56	82	138
U1	3	8	10	12	5	3	6	5	24	28	52
C1	14	13	30	30	13	16	13	9	70	68	138
C2	4	0	29	15	13	17	12	9	58	41	99
C3	4	3	10	6	2	2	2	0	18	11	29
C4	17	11	32	27	23	16	15	12	87	66	153
C5	2	0	29	12	13	13	12	9	56	34	90
C6	9	6	30	1	13	2	13	2	65	11	76
Overall	111	159	276	300	153	163	126	156	666	778	1444

In total, we examined 1444 images in 14 geometry textbooks as shown in Table 2 and Figure 9. For example, we can see that in the lesson section of the Parallel Lines and Transversals we found 10 images in P1 and only 2 images in P2. Most of the sections examined were about the same number of pages but differed in the number of images.

Typical Images by Topic

Table 3 shows the percent of images that are typical by topic for each of the four topics examined: parallel lines and transversals, classifying triangles, parallelograms, and trapezoids. Sections on parallelograms and trapezoids were much more likely to contain typical images than sections on classifying triangles and parallel lines and transversals.

Table 3: Percent of images analyzed that are typical by section

Topic	Number of Images Analyzed	Number of Images that are Typical	Percent of Images that are Typical
Parallel Lines and Transversals	270	140	51.9
Classifying Triangles	576	333	57.8
Parallelograms	316	236	74.7
Trapezoids	282	212	75.2
Total	1444	921	63.8

Typical Images by Topic and Textbook

Figures 9-12 show the percentage of typical images for each of the four topics: parallel lines and transversals, classifying triangles, parallelograms, and trapezoids. Each topic is then broken down to examine the percent of images typical for each textbook.

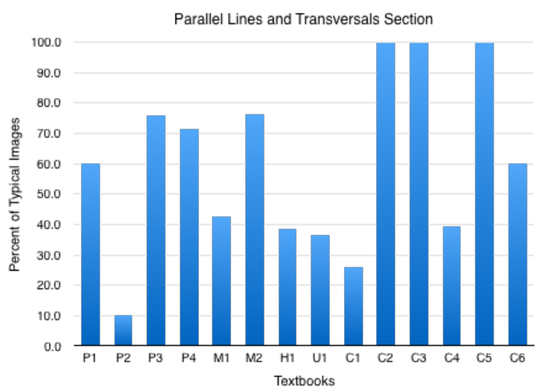

Figure 9: Percent of images in Parallel Lines and Transversals section that are typical by textbook

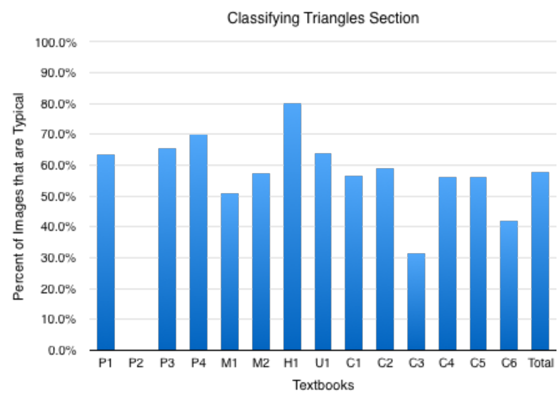

Figure 10: Percent of images in Classifying Triangles section that are typical by textbooks

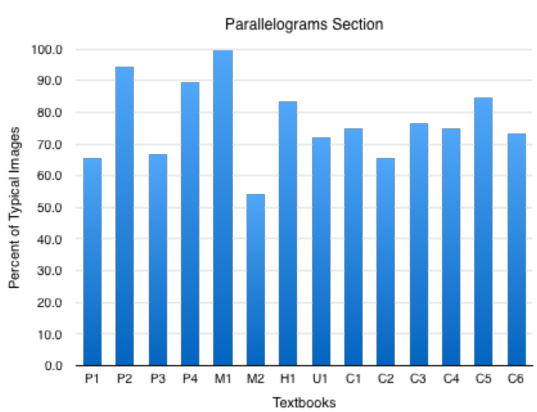

Figure 11: Percent of images in Parallelograms section that are typical by textbook

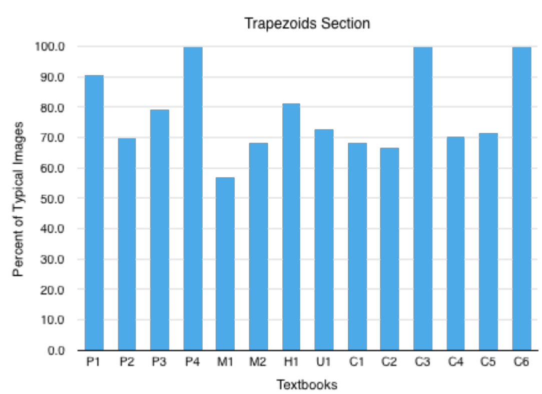

Figure 12: Percent of images in Trapezoids section that are typical by textbook

Figures 9 and 10 show a large discrepancy in the percent of typical images in the sections on parallel lines and transversals and classifying triangles. The sections on trapezoids and parallelograms had a more consistent percentage of typical images in all of the textbooks as shown in figures 11 and 12.

Typical Images in Lessons Versus Exercises

Table 4: Percent of images that are typical in lessons versus exercises by textbook.

Textbooks	Percent of Typical Images in Lessons	Percent of Typical Images in Exercises
P1	89.5	67.9
P2	60.0	22.9
P3	90.9	69.9
P4	83.3	98.8
M1	50.0	54.4
M2	80.0	61.2
H1	67.9	65.9
U1	66.7	53.6
C1	51.4	57.4
C2	67.2	68.3
C3	66.7	45.5
C4	63.2	62.1
C5	62.5	64.7
C6	61.5	72.7

We examined each section's lesson and exercise portions separately in order to see if there were more typical images in one or the other. Table 4 and Figure 13 below show the results of this inquiry. In P2, there is a striking difference between the percent of typical images in the lessons portion compared to the exercise portion of the section. There are nearly 40% more typical images in the lessons portion, suggesting that students will be taught the materials using a majority of typical images, but will be expected to complete problems that have non-typical images. The P1 lessons portion, the P3 lessons portion, and the P4 exercise portion all contain nearly all-typical images. In fact, the P4 exercise portion has 98.8% typical images.

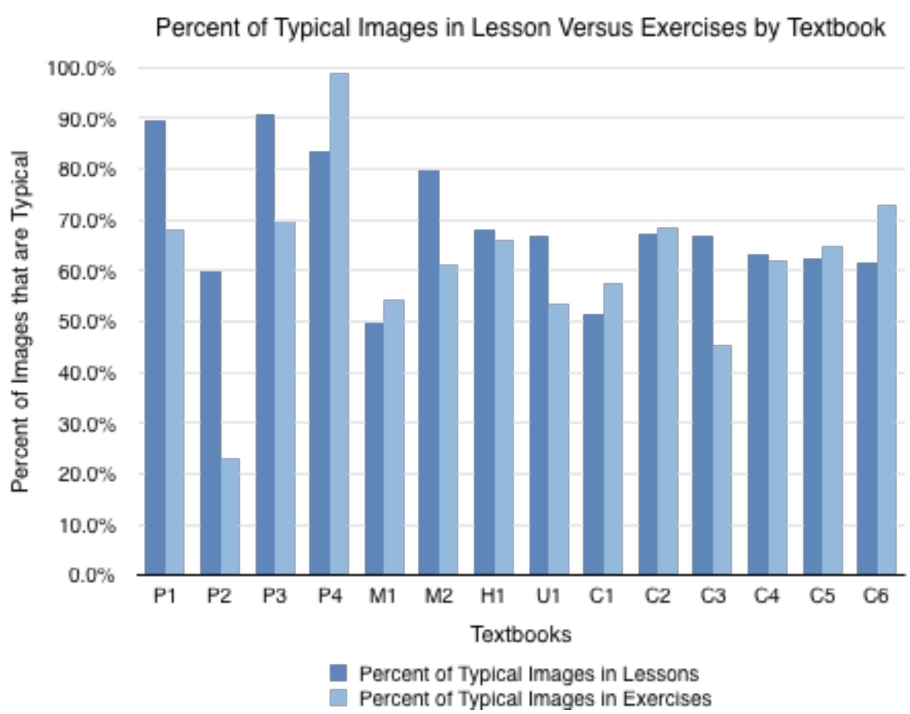

Figure 13: Graph showing percent of images that are typical in lessons versus exercises by textbook

Typical Images in Physical Textbooks versus FlexBooks

Of the 14 high school geometry textbooks examined, 8 were physical printed textbooks and 6 were FlexBooks, which are open sourced, web-based, customizable, and interactive textbooks. One of the suggestions for overcoming limitations in visualization coming from the static nature of images is to use interactive figures that the FlexBooks have the ability to provide. However, the CK-12 FlexBooks contained image content similar to that of a typical printed textbook. This seems like a missed opportunity to provide content that is interactive and dynamic. FlexBooks could easily differentiate themselves from physical textbooks, but it does not seem that they are fully utilizing that capability.

Table 5 compares the percent of typical images found in the physical textbooks versus the CK-12 FlexBooks.

Table 5: Percent of typical images in physical textbooks versus FlexBooks.

	Number of Images Analyzed	Number of Images that are Not Typical	Number of Images that are Typical	Percent of Images that are Typical
Physical Textbooks	859	298	561	65.3
FlexBooks	585	225	360	61.5

CK-12 Foundation is relatively new to the textbook publishing industry, so it is possible that they have not yet developed their FlexBooks to their full potential. They provide an opportunity through which high school teachers can start with a basic FlexBook then add to it and customize. So, it is feasible that a teacher could download a CK-12 FlexBook as it is given and change all of the static images to interactive ones, thereby making the book a more effective tool for teaching mathematics with a focus on visual methods.

CHAPTER SIX:

DISCUSSION AND CONCLUSIONS

Discussion

Visualization in mathematics has recently been given a significant increase in research attention, however the extent and nature of images in textbooks has not been researched thoroughly. Textbooks play a decisive role in developing the visual literacy of students, so it is appropriate to pay more attention to the role of images in textbooks. If students are not given the opportunity to learn visual strategies for analyzing images, then they are less likely to understand material that is highly visual or use the power of visualization for cognitive processes.

The textbook image analysis we provided was based on 14 high school geometry textbooks, 8 physical printed textbooks from publishers who dominate the textbook market and 6 FlexBooks from a relatively new and innovative publisher (CK-12 Foundation). We examined the alignment of images in four major geometry topics: parallel lines and transversals, classifying triangles, parallelograms, and trapezoids. For each topic, we looked at the section in the book that was most relevant to the concept and found the ratio of typical images to the total number of images presented for the lesson portion and exercise portion separately. Our intention was to establish what percent of images seen by students are typical images.

In each section, the percent of images that are typical images represent a majority of the images. Of the four topics, parallel lines and transversals has the lowest percentage of typical images with 51.9% of images classified as typical. It may seem as though 51.9% is an insignificant percentage, but two parallel lines can be given an uncountable number of alignments. So having 51.9% all aligned the same way represents a majority, which is significant.

While examining the percent of typical images by section, we found that *CME Project Geometry* often had a lower percentage of typical images compared to similar textbooks. *CME Project Geometry* is funded by the NSF and developed by the Center for Mathematics Education giving it a unique perspective for a textbook. This difference in authors and contributors may explain why *CME Project Geometry* consistently has a lower percentage of typical images, particularly for parallel lines and transversals and classifying triangles.

Pearson's *Blitzer, Thinking Mathematically* had very few images to analyze in the sections for parallelograms and trapezoids, which could be a reason why they included only typical images in these two sections. Similarly, some of the CK-12 books had relatively few images to examine; for example, *Geometry Concepts-Honors* only had 29 images in the four sections, which gives a small sample size to examine, possibly explaining why some books had 100% typical images. Also, the CK-12 books are made with the idea that teachers will adapt the books to their particular classroom needs, so the publisher may not feel it necessary to add as much variety.

Of the 14 textbooks examined, 9 did not have much difference in the percent of typical images shown in the lessons versus the percent of typical images in the exercises. However, Pearson *Geometry Common Core*, Pearson *CME Project Geometry*, Pearson *Informal Geometry-Classics Edition*, and McGraw Hill *Geometry Concepts and Applications* had significantly more (more than 15%) typical images found in the lessons portion than in the exercise portions of the sections. This leads students to learn concepts using images that generally have the same alignment but are then asked to answer questions that require them to analyze images that may be in an unfamiliar alignment. If the lesson portion of a section shows only typical images, then when asked to solve a problem in the exercise portion that contains a non-typical image students may be confused. When students see an image with an unfamiliar alignment, they may not immediately recognize the image as having the same properties as the typical form of the image. Contrastingly, Pearson's *Blitzer, Thinking Mathematically* had significantly more typical images in the exercise portions. This can create a different problem for students. In this situation, students may not have an adequate opportunity to practice problems with non-typical images.

Summary

Examination of the textbooks revealed that approximately 63.8% of the images used in corresponding sections of these textbooks are typical images. This average can be refined further to show 65.3% typical images in the physical and more widely used textbooks and 61.5% typical images in the relatively new CK-12 FlexBooks. With such a large portion of the images being typical, students' visual literacy could potentially be negatively affected. So in a case where a figure is shown with an unusual alignment or representation, students may not recognize the figure.

Of the four topics we analyzed, in all sections at least 50% of the images were typical. The sections on parallel lines and transversals and classifying triangles were on the lower end, having 51.9% and 57.8% typical images respectively. Of the parallelograms examined, 74.7% were drawn as typical images and 75.2% of trapezoids were. This indicates that if students were given a problem with a parallelogram or trapezoid with all pairs of parallel sides non-horizontal, they may not immediately recognize the shape and the properties inherited.

Limitations

Our sample for the study used a range of textbooks but the sample is still not fully representative of all the textbooks currently in use in the country. While these choices provide a wide range of textbook types, the sample of books could be expanded. Our study also examined four topics: parallel lines and transversals, classifying triangles, parallelograms, and trapezoids. The percentage of typical images in these topics may not represent the percentage of typical images in other topic areas.

In this study we focused on the alignment of certain images in textbooks. We did not consider more what other aspects of a figure may be typical.

We focused only on geometry textbooks because we expected to see a large number of images for analysis based on the geometric nature of the subject, but the results could be different for books such as Algebra or Calculus.

The examination of the CK-12 FlexBooks occurred until March 1, 2016. The FlexBooks open-sourced web-based format allows for quick changes and revisions to the content. It is conceivable that these books may have changed from the time they were originally analyzed.

Future Work

Visualization in mathematics is an important research topic as visual methods can help to dispel misconceptions, provide intuitive understanding, and assist in the proving process. However, there does not seem to be much research on the visual content of textbooks.

One avenue of such research could catalogue all of the images in a textbook to get a better understanding of what kind of visual literacy is required of students. Another direction would be to look at different subjects of mathematics other than geometry to see if, for example, algebra textbooks also have a majority of typical images. Specifically, it would be exciting to look at sections like slope, to determine if points used to find slope are always indexed from left to right in the distance formula, or to determine if points used to illustrate the distance formula are always in the first quadrant. We can examine the images of tangent lines in sections of calculus books presenting the derivative, to determine if the tangent lines intersect the curve at a second point. It would also be interesting to look at other topics in geometry, such as right triangles, to determine if the right angle is often drawn on the same side, or central angles and inscribed angles, to determine if the central angle is always drawn opening upward, and re-examine trapezoids and triangles, to determine if they are often drawn appearing to be isosceles. Examining images from these areas may provide a better understanding of the extent to which images are presented in a typical alignment or format.

The CK-12 Foundation FlexBooks provide an interesting opportunity for research because they are customizable and interactive. Using interactive diagrams is recommended by Duval (1998) as a method for overcoming limitations of visualization and the CK-12 FlexBooks provide the prospect of building that functionality directly into the curriculum materials. An interesting research topic would be to use a customized CK-12 FlexBook to create all interactive images (no static images) and determine if it is possible to create a textbook with much more limited typical images.

Implications and Conclusions

This study shows that visual material presented to students needs more variation. The lack of diversity of images could lead to misconceptions if students are presented with images that are not aligned in the typical fashion or have an unusual orientation (Nardi, 2014).

For educators, this means they should actively incorporate more varied imagery into their classroom lessons and accurately show the necessary properties of geometric figures. Teachers can also use interactive dynamic geometry software to present concepts and properties that hold for a multitude of alignments, potentially giving students deeper conceptual understanding. For textbook authors and curriculum developers, this research shows they should be sensitive to the images included to prevent a lack of non-typical images in the visual materials.

It is our belief that there is potential for FlexBooks to be used successfully in the classroom. While they do not yet differ much from physical textbooks, FlexBooks have the ability to adapt and change rapidly with the demands of the learning environment. In addition, teachers can customize them to the needs of their classroom and students. More interactive material can be added to the FlexBooks to provide opportunities for visualization to enrich the mathematical learning process.

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APPENDIX A:

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