3-23-2017

Schreier Graphs of Thompson's Group T

Allen Pennington
University of South Florida, penningtona@mail.usf.edu

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by

Allen C. Pennington

A thesis submitted in partial fulfillment
of the requirements for the degree of
Master of Arts
Department of Mathematics & Statistics
College of Arts and Sciences
University of South Florida

Major Professor: Dmytro Savchuk, Ph.D.
Mile Krajcevski, Ph.D.
Mohamed Elhamdadi, Ph.D.

Date of Approval:
March 22, 2017

Keywords: amenable, gromov, cayley, action, homeomorphism

Table of Contents

List of Tables

List of Figures

Abstract

Thompson's groups F, T, and V represent crucial examples of groups in geometric group theory that bridge it with other areas of mathematics such as logic, computer science, analysis, and geometry. One of the ways to study these groups is by understanding the geometric meaning of their actions. In this thesis we deal with Thompson's group T that acts naturally on the unit circle S^1, that is identified with the segment $[0, 1]$ with the end points glued together. The main result of this work is the explicit construction of the Schreier graph of T with respect to the action on the orbit of $\frac{1}{2}$. This is done by careful examination of patterns in how the generators of T act on binary words. As a main application, the nonamenability of the action of T on S^1 is proved by defining injections on the set of vertices of the constructed graph that satisfy Gromov's doubling condition. This gives an alternative proof of the known fact that T is nonamenable.

Chapter 1

Introduction

In this thesis we study Thompson's group T, which is one of the three Thompson's groups (F, T, and V) that were discovered by Richard Thompson in 1965. A lot of fascinating properties of these groups were discovered later on, many of which are surveyed nicely in [CFP96]. For example, Thompson's group T was one of the first (and rare) examples of infinite, but finitely-presented simple groups.

Geometric group theory studies groups via associated metric spaces. One of the main objects of study for a finitely generated group G is its Cayley graph. Given a finite generating set S for G, the vertices of the Cayley graph $Cay(G, S)$ are just elements of G, and for each $g \in G$ and $s \in S$ there is an oriented edge in the graph of the form $g \to gs$ labeled by the generator s. It turns out, that there is a deep connection between the geometric structure of $Cay(G, S)$ and the algebraic structure of G. For example, by the celebrated theorem of Gromov, a group G has a polynomial growth (i.e., the sequence of sizes of balls in $Cay(G, S)$ centered at the identity of radius n grows as a polynomial function) if and only if G is virtually nilpotent. Another connection that we will mention here is that the amenability of a finitely generated group G is equivalent to the amenability of its Cayley graph (with respect to any finite generating system).

Even though Cayley graphs give a lot of information about the group, in most (non-trivial) cases they are hard to construct and work with. In particular, sometimes they carry too much information that may be useless for intended purposes. One way to fight this problem is to consider the Schreier graphs. Given a group G generated by set S, and a subgroup H of G, Schreier graph $Sch(G, H, S)$ of $H < G$ is an "approximation" of the Cayley graph of G that can be defined in two essentially equivalent ways. In the first definition the set of vertices of $Sch(G, H, S)$ is the set G/H of all right cosets of H in G, and the edges are of the form

$$Hg \xrightarrow{s} Hgs$$

for each $Hg \in G/H$ and $s \in S$. Thus, the Schreier graph of G with respect to the trivial subgroup is the Cayley graph of G. In other words, a Schreier graph can be obtained from a Cayley graph simply by gluing all vertices in each coset of H to one vertex and gluing all multiple edges labeled by the same generator to one edge.

Alternatively, given the action of G on a set X, we can define the Schreier graph $Sch(G, X, S)$ of this action as follows. The set of vertices of $Sch(G, X, S)$ is X and the edges are of the form

$$x \xrightarrow{s} x^s$$

for each $x \in X$ and $s \in S$ (where x^s denotes the action of s on x). If the action is transitive, then the Schreier graph of this action is naturally isomorphic to the Schreier graph of the stabilizer of any point of X in G. Conversely, Schreier graph $Sch(G, H, S)$ is canonically isomorphic to the Schreier graph of the action of G on the set G/H of right cosets of H in G by right multiplication. In the thesis we will use both of these equivalent views.

There are several motivations for studying Schreier graphs. They have been used occasionally as of now for more than 80 years (sometimes called Schreier coset diagrams [CM80], action graphs, or orbit graphs). Recently they became important in the connection to problems in analysis [BG00], holomorphic dynamics [Nek05], ergodic theory [GKN12], probability [Kap02], etc. Many of the recent constructions and applications of Schreier graphs are related to groups acting by automorphisms on rooted trees (see, for example, [DDMN10, GNS00, GŠ06, Bon07]). Every such action induces a sequence of finite Schreier graphs from the action on the levels of the tree, and an uncountable family of infinite (but countable) Schreier graphs of the action on the orbits of elements of the boundary of the tree.

Schreier graphs also allow us to better understand the actions of groups. For example, they produce lower bounds on the lengths of elements of a group G with respect to the generating set S, give a simple way to construct elements of G with prescribed action X, better understand the dynamics of elements of G, etc.

Finally, another motivation to study Schreier graphs is their direct relation to amenability of the group. A group is nonamenable if and only if its Cayley graph is nonamenable (changing generating set does not affect amenability of the Cayley graph). The situation is not that simple for Schreier graphs, but the nonamenability of any Schreier graph of a group implies nonamenability of a group. For example, one of the most famous open questions about Thompson's groups is whether F is amenable. This question was first posed by Richard Thompson in 1960s, but gained attention after Geoghegan popularized it in 1979 (see p.549 of [GS87]). Thus, to prove nonamenability of F it suffices to construct a nonamenable Schreier graph of F. Initially, this question arose in an attempt to construct a finitely generated non-amenable group without non-abelian free subgroups. (It was proved by Brin and Squier in [BS85] that F does not have non-abelian free subgroups). Since the time the question was posed, many examples of such groups have been already found (see [Adi82], and [OS02]), but the question of amenability of F is now mainly interesting on its own. Arguably, Thompson's group F is the most famous group whose amenability is not determined yet.

Schreier graphs of the action of Thompson's group F on the unit interval were constructed by Savchuk in [Sav10] and [Sav15]. It has been shown that all of these graphs are amenable, have exponential growth, are nonisomorphic for points from different orbits. The structure of these graphs was later used in [Mis15] to construct a nontrivial Poisson boundary of a simple random walk on this graph.

The main goal of this thesis is to extend the results of [Sav15] to the case of Thompson's groups T. Namely, in Chapter 4 we explicitly construct the Schreier graph of the action of T on the orbit of $\frac{1}{2}$. As the main application we show in Chapter 5 that the corresponding graph is nonamenable, thus giving another proof of nonamenability of T.

Chapter 2

Thompson's Groups F and T

Thompson's Groups F, T, and V were first defined by Richard Thompson in 1965. The groups have unusual properties and were used as counterexamples to disprove various conjectures. In this paper, we will focus on studying T, but to those new to Thompson's groups, it will be easier to first understand F, and then to view T as an extension of F. The proofs of all theorems below can be found in the Introductory Notes on Richard Thompson's Groups survey paper [CFP96].

DEFINITION 2.1 *Thompson's group* F is the set of piecewise linear homeomorphisms from the closed unit interval $[0, 1]$ to itself that are differentiable except at finitely many dyadic rational numbers and such that on intervals of differentiability the derivatives are powers of 2. The group operation is composition of homeomorphisms.

THEOREM 2.1 *The following two homeomorphisms A and B generate Thompson's Group F.*

$$A(x) = \begin{cases} \frac{x}{2} & 0 \le x \le \frac{1}{2} \\ x - \frac{1}{4} & \frac{1}{2} \le x \le \frac{3}{4} \\ 2x - 1 & \frac{3}{4} \le x \le 1 \end{cases} \qquad B(x) = \begin{cases} x & 0 \le x \le \frac{1}{2} \\ \frac{x}{2} + \frac{1}{4} & \frac{1}{2} \le x \le \frac{3}{4} \\ x - \frac{1}{8} & \frac{3}{4} \le x \le \frac{7}{8} \\ 2x - 1 & \frac{7}{8} \le x \le 1 \end{cases}$$

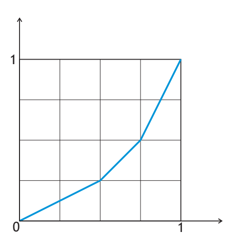

Figure 1: $A(x)$

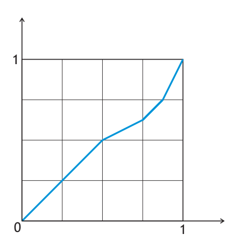

Figure 2: $B(x)$

Note, that here we follow notation from [CFP96]. In many sources the homeomorphisms A and B are denoted by x_0 and x_1 respectively.

DEFINITION 2.2 Consider S^1 as the interval [0, 1] with its endpoints identified. *Thompson's group* T is the group of piecewise linear homeomorphisms from S^1 to itself that map images of dyadic rational numbers to images of dyadic rational numbers and that are differentiable except at finitely many dyadic rational numbers and such that on intervals of differentiability the derivatives are powers of 2. The group operation is composition of homeomorphisms.

THEOREM 2.2 *Homeomorphisms A, B, and C generate Thompson's Group T, with C given below.*

$$C(x) = \begin{cases} \frac{x}{2} + \frac{3}{4} & 0 \leq x \leq \frac{1}{2} \\ 2x - 1 & \frac{1}{2} \leq x \leq \frac{3}{4} \\ x - \frac{1}{4} & \frac{3}{4} \leq x \leq 1 \end{cases}$$

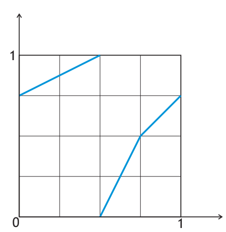

Figure 3: $C(x)$

Since F and T map dyadic rationals to dyadic rationals, to describe elements of these groups we will often have to address these numbers. For simplicity reasons, we will define them by binary words. Regarding notation, we will eliminate the decimal in expressing these binary numbers, since they are all less than 1 (e.g., $\frac{1}{4}$ will be represented by 01 as opposed to .01). Additionally, binary numbers followed by a binary word W will denote a word that begins with those binary numbers and has W as a tail (e.g., $01W$ denotes a binary word that begins with 01 and ends with W).

The following theorem is straightforward from the definition of C, and we give its proof here for completeness.

THEOREM 2.3 *C generates a cyclic group of order 3.*

Proof. If $0 \leq x \leq \frac{1}{2}$, then x can be denoted by the binary word $0W$. If $\frac{1}{2} \leq x \leq \frac{3}{4}$, then x can be denoted by the binary word $10W$. And if $\frac{3}{4} \leq x \leq 1$, then x can be denoted by the binary word

$11W$. It is easy to verify that $C(0W) = 11W$, $C(11W) = 10W$, and $C(10W) = 0W$. Thus C generates a cyclic group of order 3. $\square$

We also note, that both groups F and T are finitely presented. The group T is simple, and F has a simple commutator subgroup. We will not use these facts below, so we do not give more detalis on this here and refer the reader to [CFP96].

Chapter 3

Schreier Graphs

In this chapter we formally introduce the notion of a Schreier graph and discuss related concepts. As mentioned in the introduction, there are two ways to define them. We will start from (a bit more general) definition based on a group action. Below we will use the following notation. If a group G acts on a set M, $x \in M$ is an arbitrary element of M, and $g \in G$, then we denote by x^g the image of x under the action of g.

DEFINITION 3.1 Let G be a group generated by a finite generating set S acting on the set M. The *Schreier graph* $\Gamma(G, M, S)$ of the action of G on M with respect to the generating set S is an oriented labeled graph defined as follows: The set of vertices of $\Gamma(G, M, S)$ is M and there is an edge from $x \in M$ to $y \in M$ labeled by $s \in S$ if and only if $x^s = y$.

A standard and special example of a Schreier graph comes from the action of G by multiplication on the right on the set G/H of right cosets of a subgroup H of G. Such Schreier graphs are denoted by $\Gamma(G, H, S)$ and are often called *coset graphs*. Note, that if the action of G on M is transitive, then $\Gamma(G, M, S)$ is canonically isomorphic to $\Gamma(G, Stab_G(x), S)$ for any $x \in X$ under the isomorphism that sends $y \in M$ to the coset of $Stab_G(x)$ that consists of all elements of G that map x to y. Here by $Stab_G(x)$ we denote the stabilizer of x in G, i.e., the subgroup of G consisting of all elements that fix x.

Thus, for transitive actions, the notions of Schreier graphs and cosets graphs coincide. We will use both of these languages in the rest of the text.

Schreier graphs are used to study group actions, and one common application of Schreier graphs is to obtain bounds on the lengths of elements. One of the main motivations that will concern us in this paper is the implications that Schreier graphs have on the amenability of a group. It is known that a group is nonamenable if and only if its Cayley graph is nonamenable. But as we mentioned in the introduction, for many groups, it turns out to be difficult to construct and analyze their Cayley graphs explicitly. However, the nonamenability of any Schreier graph of a group also implies the nonamenability of a group. It is well-known that T and V are nonamenable, and it is an open question as to whether or not F is amenable. However, showing that one of the Schreier graphs of T is nonamenable would provide an alternate proof of its nonamenability [Sav15].

There are many equivalent definitions that can be used to define the amenability of a group. Amenability is a concept that is important in many different areas of mathematics, and often the definition that is used depends on the context. Historically speaking, one of the first motives for looking into the amenability of a group was the Banach-Tarski paradox, which stated that a group with a paradoxical decomposition could be used to break apart a unit ball into a finite number of parts (which need to be non-measurable) and reassemble these parts using rigid motions of $\mathbb{R}^3$ into two unit spheres identical to the original. Von Neumann was the first one to describe the property of the group that acts on a space which is responsible for such a *paradoxical decomposition* [vN29]. He used a term *measurable* groups. In particular, he proved that free non-abelian groups are non-amenable, and that the class of amenable groups is closed under taking subgroups, quotients, extensions, and direct limits. The term *amenable* was coined by M. Day in [Day57], where he also extended the notion of amenability to semigroups.

DEFINITION 3.2 Let G be a group acting on a set X. A subset $E \subset X$ is *paradoxical* if there exist pairwise disjoint subsets $A_1, \ldots, A_n$ and $B_1, \ldots, B_m$ in E and there exist $g_1, \ldots, g_n$ and $h_1, \ldots, h_m$ in G such that

$$E = \bigsqcup_{i=1}^{n} g_i A_i = \bigsqcup_{j=1}^{m} h_j B_j$$

DEFINITION 3.3 A group itself is said to be *paradoxical* or *nonamenable* if the action of the group on itself by left multiplication is paradoxical. A group which is not paradoxical is said to be *amenable.*

Another common (and equivalent) definition for amenability involves Følner sequences.

DEFINITION 3.4 Given a group G that acts on a countable set X, a *Følner sequence* for the action is a sequence of finite subsets $F_1, F_2, \ldots$ of X which exhaust X such that for every $x \in X$ there exists some i such that $x \in F_j$ for all $j > i$, and

$$\lim_{i\to\infty} \frac{|gF_i \triangle F_i|}{|F_i|} = 0$$

where $\triangle$ is the symmetric difference operator. A group that has a Følner sequence is said to be *amenable.* Otherwise, it is *nonamenable.*

The notion of amenability can also be extended to graphs, allowing us to get new ways to prove nonamenability of a group.

DEFINITION 3.5 Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For $A \subseteq V(G)$, let $\partial A = \{(x, y) \in E(G) : x \in A, y \in V(G) \setminus A\}$. Then the *Cheeger constant* of G, denoted $h(G)$, is defined by: $h(G) = \min\{\frac{|\partial A|}{|A|} : A \subseteq V(G), 0 < |A| < \infty\}$. If the Cheeger constant of a graph is 0, then the graph is said to be *amenable.* Otherwise, it is *nonamenable.*

The following theorem states a well-known connection between the amenability of graphs and amenability of groups.

THEOREM 3.1 A group G is amenable if and only if the Cayley graph of G is amenable.

Additionally, if H is a subgroup of G, then another common application of Schreier graphs is to use the Reidemeister-Schreier procedure to find the generators and relations that define H. It is currently an open problem as to whether or not the stabilizers of points in S^1 under the action of T are finitely generated.

Chapter 4

Schreier Graph of $Stab_T(\frac{1}{2})$ in T

In this chapter we give a complete description of the Schreier graph of T with respect to the stabilizer $Stab_T(\frac{1}{2})$. We describe its structure in several theorems below and include several figures to show all parts of the graph.

Since F is generated by A and B, and T is generated by A, B, and C, we can construct the Schreier graph of $Stab_T(\frac{1}{2})$ by taking the Schreier graph of $Stab_F(\frac{1}{2})$, adding a vertex for 0, that is in the orbit under T but not in the orbit under F, and drawing edges for the extra generator, C. The Schreier graph of $Stab_F(\frac{1}{2})$ was constructed in [Sav15] and is shown below. We will use the structure of this graph without repeating the proofs given in [Sav15].

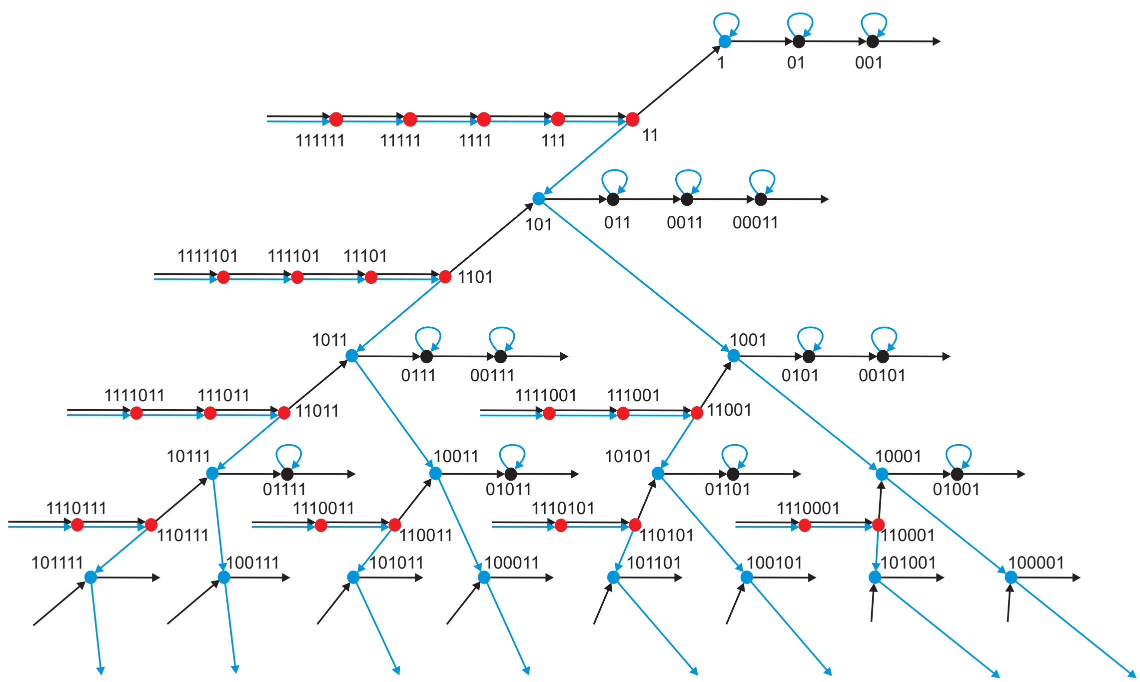

Figure 4: The Schreier graph of $Stab_F(\frac{1}{2})$

Recall, that the vertices are labeled by binary strings that correspond to the digits after the period

Table 1: Changing the vertex labels while moving along the edges in the Schreier graph of $Stab_F(\frac{1}{2})$

A move in the graph				**Change of vertex label**
initial vertex	terminal vertex	direction	generator	
black	*black*	$\rightarrow$	A	$0W \longrightarrow 00W$
blue	*black*	$\rightarrow$	A	$10W \longrightarrow 01W$
red	*blue*	$\nearrow$	A	$110W \longrightarrow 10W$
red	*red*	$\rightarrow$	A, B	$111W \longrightarrow 11W$
red	*blue*	$\swarrow$	B	$110W \longrightarrow 101W$
blue	*blue*	$\searrow$	B	$10W \longrightarrow 100W$
black	*black*	none	B	$0W \longrightarrow 0W$

in the binary expansions of corresponding dyadic rational numbers. It will be important for us to keep track how labels of vertices change when we move along the edges in the graph. Table 1 provided in [Sav15] encodes concisely exactly this information.

Note that the graph is infinite, as denoted by the arrows (drawing more of the graph will just extend the same pattern). The edges of the graph has been given color in order to better visualize how each generator acts on the elements of the group. Vertices are colored blue only if they are on the binary tree. These vertices correspond to the dyadic rational numbers from the interval $[\frac{1}{2}, \frac{3}{4})$. Vertices are colored black only if they are on a ray extending rightward. These vertices correspond to the dyadic rational numbers from the interval $(0, \frac{1}{2})$. Finally, vertices are colored red only if they are on a ray extending leftward. These vertices correspond to the dyadic rational numbers from the interval $[\frac{3}{4}, 1)$. Edges corresponding to the generator A are drawn in black, and edges corresponding to B are drawn in blue. When we add the edges corresponding to the generator C to the graph, these edges will be labeled in red.

One can immediately notice that all black vertices are of the binary form $0W$, all blue vertices are of the binary form $10W$, and all red vertices are of the binary form $11W$. This will prove to be quite convenient when we go to add the edges corresponding to the generator C to the graph.

Rather than give the entire graph of $Stab_T(\frac{1}{2})$ at once, which would be quite overwhelming and chaotic, the set of edges corresponding to the generator C will be broken down and organized into chunks, so that one can see the patterns of the graph amidst the chaos. The graph in Figure 5 is not a full graph of $Stab_T(\frac{1}{2})$; rather, it is the graph of $Stab_F(\frac{1}{2})$ with extra edges drawn from the generator C acting only on blue vertices that just branched left from the level above them on the binary tree (i.e., the vertex labeled by 1 and the blue vertices that can be expressed by $101W$).

Since the orbit of $\frac{1}{2}$ under T differs from the one under F only by adding a point $0 = C(\frac{1}{2})$, an extra vertex needs to be added to the Schreier graph $Stab_F(\frac{1}{2})$ to represent this number. Edges for the generators A and B were added to this newly added point as well ($A(0) = 0$ and $B(0) = 0$).

We start from observing that $C(1) = 0$, so there is a red edge from vertex labeled by 1 to vertex labeled by 0. Now we describe several infinite families of red edges that describe what happens with each vertex of the graph after the generator C is applied to it.

For simplicity, we will call blue vertices in the graph whose labels start from 101 by *branching to the left*, and those blue vertices with labels starting from 100 by *branching to the right*.

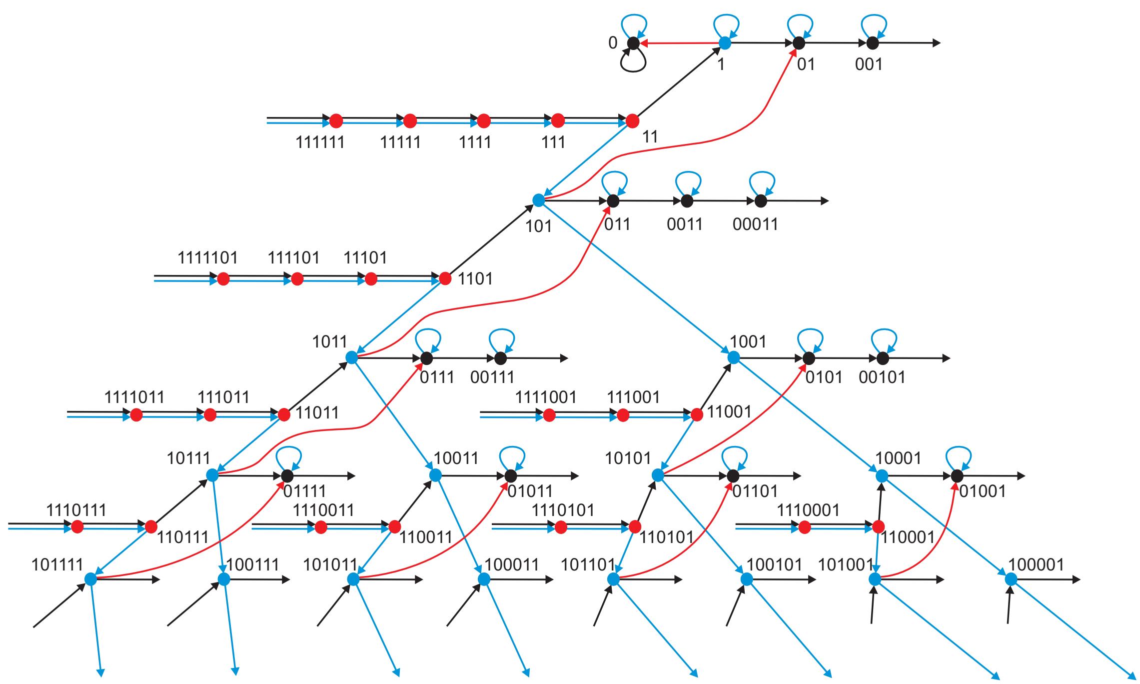

Figure 5: Schreier graph with edges from C acting on blue vertices that branched to the left

THEOREM 4.1 (Blue-left to Black) *For each branching to the left blue vertex, there is an edge labeled by C from this vertex to the black vertex obtained by moving one level up to the next blue* *vertex and then moving to the right once (see Figure 5).*

Proof. We need to show that $C(101W) = (A^2 \circ B^{-1})(101W)$, since "moving up to the next level and to the right once" according to Table 1 corresponds to the application of a homeomorphism $A^2 \circ B^{-1}$. From the proof of Theorem 2.3, we can see $C(101W) = 01W$. From examining Table 1, we can see that $B^{-1}(101W) = 110W$. Also from reading the table, $A(110W) = 10W$ and $A(10W) = 01W$. Hence, $(A^2 \circ B^{-1})(101W) = 01W = C(101W)$. □

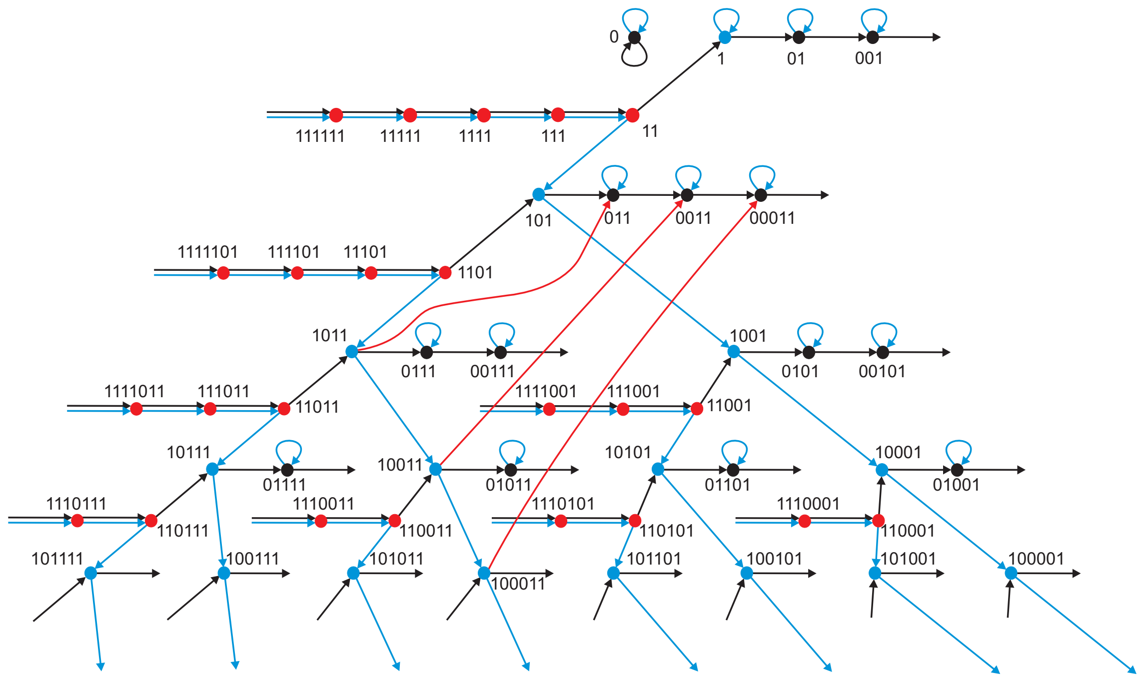

Figure 6: Schreier graph with edges from C acting on blue vertices that branched to the right

THEOREM 4.2 (Blue-right to Black) *For each branching to the left blue vertex, there is an edge labeled by C from this vertex to the black vertex obtained by moving up in the blue tree until the blue vertex branching to the left is reached in, say, k steps (counting the steps between the blue vertices), moving up to the right to the next blue vertex, and then moving $k+1$ times to the right along the ray of black vertices (see Figure 6).*

Proof. The proof is done by induction on k. The case $k = 0$ is covered in Theorem 4.1. For $k > 0$, a branching to the right blue vertex has the label $100W$. We want to show that $C(100W) = (A \circ C \circ B^{-1})(100W)$. Indeed, by Theorem 2.3 we have $C(100W) = 00W$ and $C(10W) = 0W$.

From Table 1, $B^{-1}(100W) = 10W$ and $A(0W) = 00W$. So we get $(A \circ C \circ B^{-1})(100W) = 00W = C(100W)$. $\square$

At this point, we have completely described how C acts on all blue vertices. Now we will describe the action of C on red vertices. We notice that $C(11) = 1$ and describe the patterns of the rest of the edges with the theorems below.

THEOREM 4.3 (Red-right to Blue) *For each red vertex on the far-right of a red ray, there is an edge labeled by C from this vertex to the blue vertex obtained by going up one level to the blue vertex and then going down and right one level to the next blue vertex (see Figure 7).*

Proof. We need to show that $C(110W) = (B \circ A)(110W)$. By Theorem 2.3, $C(110W) = 100W$. From Table 1, $A(110W) = 10W$ and $B(10W) = 100W$. So we get $(B \circ A)(110W) = 100W = C(110W)$. $\square$

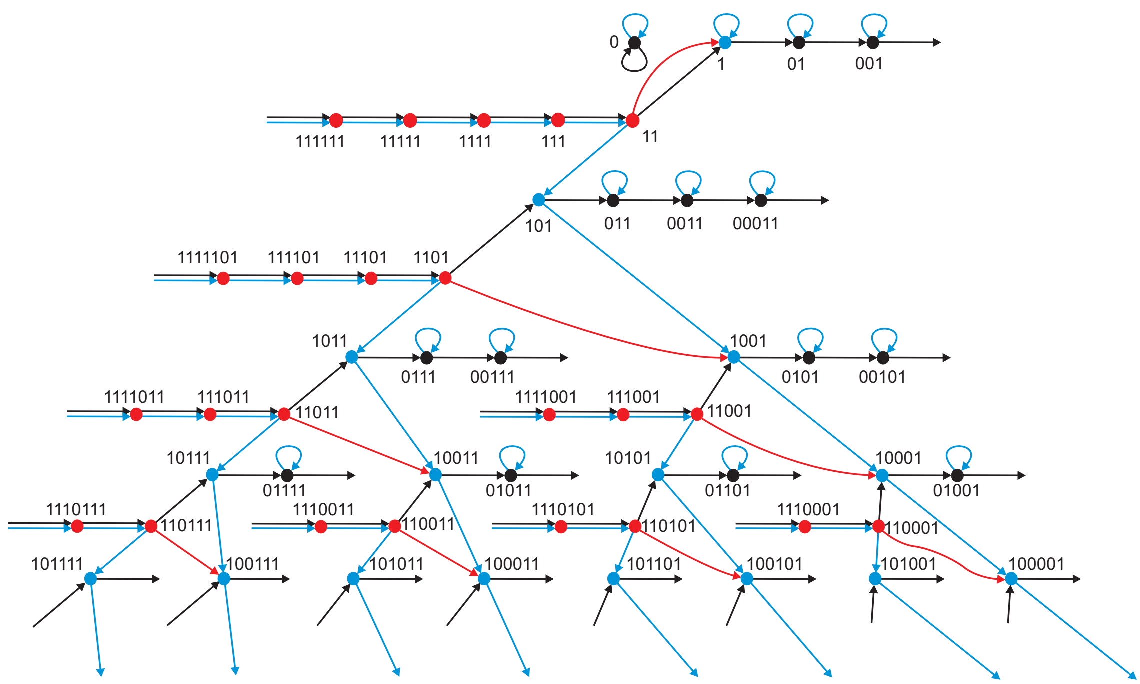

Figure 7: Schreier graph with edges from C acting on right-most red vertices of rays

THEOREM 4.4 (Red-other to Blue) *For each red vertex on a red ray (not on the far right), there is an edge labeled by C from this vertex to the blue vertex obtained by moving right on the red ray* *until the far-right vertex is reached in k steps, then moving up one level to the blue vertex, going down and right one level to the next blue vertex, and then going down and left to reach a blue vertex k times (see Figure 8).*

Proof. The proof is done by induction on k. The case $k = 0$ is covered by Theorem 4.3. For $k > 0$, a red vertex has the label $111W$. We want to show that $C(111W) = (B \circ A^{-1} \circ C \circ A)(111W)$. By Theorem 2.3, $C(111W) = 101W$ and $C(11W) = 10W$. From Table 1, $A(111W) = 11W$, $A^{-1}(10W) = 110W$, and $B(110W) = 101W$. So we get $(B \circ A^{-1} \circ C \circ A)(111W) = 101W = C(111W)$. $\square$

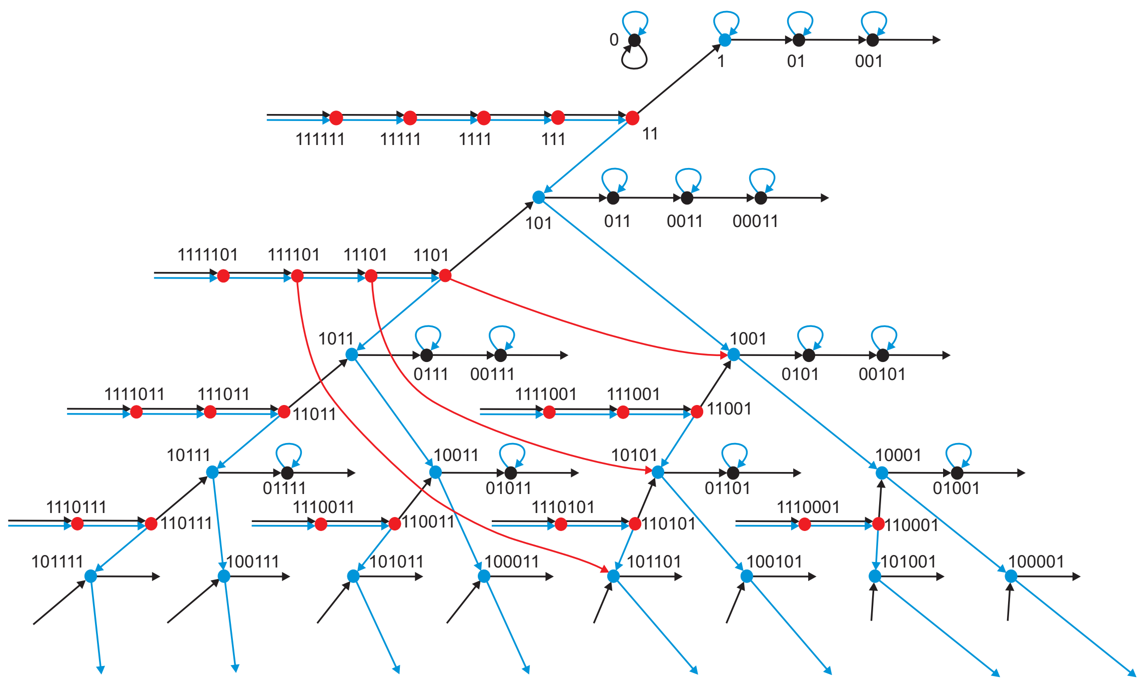

Figure 8: Schreier graph with edges from C acting on the other red vertices on a ray

If we wanted to, we could now deduce the action of C on any black vertex by using the fact that C is cyclic of order 3. That is, we know that red vertices map to blue vertices, and blue vertices map to black vertices. One could easily construct a cycle by picking a red vertex, identifying which blue vertex that it maps to, identifying which black vertex that blue vertex maps to, and then connecting the cycle by drawing an edge from that black vertex back to the starting red vertex. The disadvantage to doing this however is that does not become clear what pattern there is among the edges which connect black vertices to red vertices. The patterns of the edges which connect black vertices to red vertices are more complex than the previous patterns, and in the interest of completeness, I will present these patterns with more figures rather than finish the graph by just completing each cycle.

THEOREM 4.5 (Black-left-vertex-right-branch to Red) *For each black vertex on the far-left of a ray attached to a branching to the right blue vertex, there is an edge labeled by C from this vertex to the red vertex obtained by moving one to left to a blue vertex, up one level to another blue vertex, down and left to a red vertex, and then one to the left to the next red vertex (see Figure 9).*

Proof. We need to show that $C(010W) = (A^{-2} \circ B^{-1} \circ A^{-1})(010W)$. By Theorem 2.3, $C(010W) = 1110W$. From Table 1, $A^{-1}(010W) = 100W$, $B^{-1}(100W) = 10W$, $A^{-1}(10W) = 110W$, and $A^{-1}(110W) = 1110W$. So we get $(A^{-2} \circ B^{-1} \circ A^{-1})(010W) = 1110W = C(010W)$. □

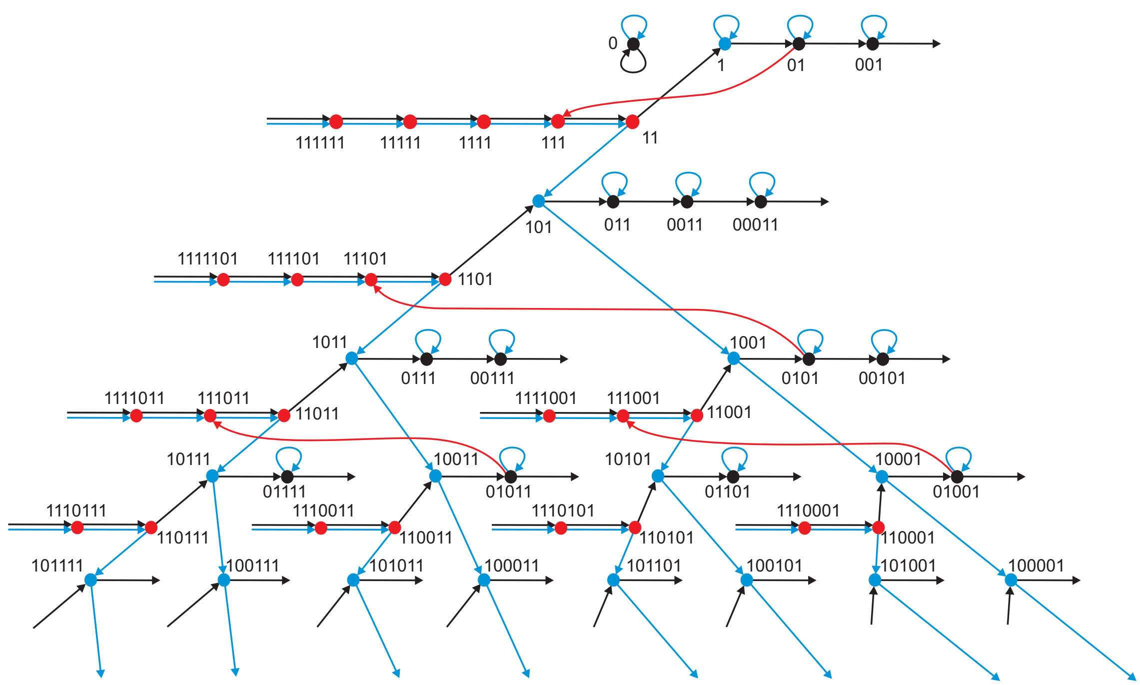

Figure 9: Schreier graph with edges from C acting on left-most vertices of black rays that branched to the right

THEOREM 4.6 (Black-left-vertex-left-branch to Red) *For each black vertex on the far left of a ray attached to a branching to the left blue vertex, there is an edge labeled by C from this vertex to the red vertex obtained by moving up and right levels until the black vertex branching right is reached* *in k steps, moving one to the left to a blue vertex, moving one level up to the next blue vertex, moving one level down and left to a red vertex, and then moving left $k+1$ times to reach a final red vertex (see Figure 10).*

Proof. The proof is done by induction on k. The case $k = 0$ is covered by Theorem 4.6. For $k > 0$, a black vertex on the far left of a ray attached to a branching to the left blue vertex has the label $011W$. We want to show that $C(011W) = (A^{-1} \circ C \circ A^2 \circ B^{-1} \circ A^{-1})(011W)$. By Theorem 2.3, $C(011W) = 1111W$ and $C(01W) = 111W$. From Table 1, $A^{-1}(011W) = 101W$, $B^{-1}(101W) = 110W$, $A(110W) = 10W$, $A(10W) = 01W$, and $A^{-1}(111W) = 1111W$. So we get $(A^{-1} \circ C \circ A^2 \circ B^{-1} \circ A^{-1})(011W) = 1111W = C(011W)$. $\square$

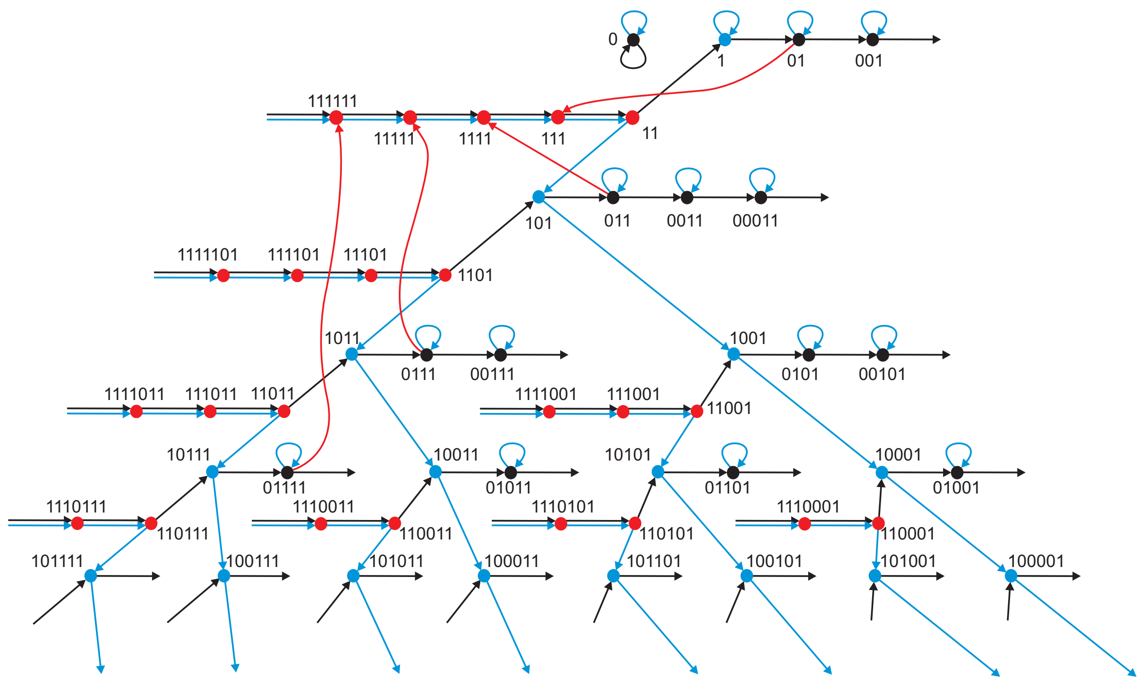

Figure 10: Schreier graph with edges from C acting on left-most black vertices on a ray that branched to the left

With these two figures, we now know where each left-most vertex on a black ray is mapped to. The action of C on vertices on right branches is defined explicitly, while the action of C on vertices on left branches is defined recursively.

THEOREM 4.7 (Black-second-from-left to Red) *For each black vertex second from the left of a black ray, there is an edge labeled by C from this vertex to the red vertex obtained by moving* *left twice to a blue vertex, moving down and left one level to the next blue vertex, and then moving down and left again to the red vertex (see Figure 11).*

Proof. We need to show that $C(001W) = (A^{-1} \circ B \circ A^{-3})(001W)$. By Theorem 2.3, $C(001W) = 1101W$. From Table 1, $A^{-1}(001W) = 01W$, $A^{-1}(01W) = 10W$, $A^{-1}(10W) = 110W$, $B(110W) = 101W$, and $A^{-1}(101W) = 1101W$. So we get $(A^{-1} \circ B \circ A^{-3})(001W) = 1101W = C(001W)$. $\square$

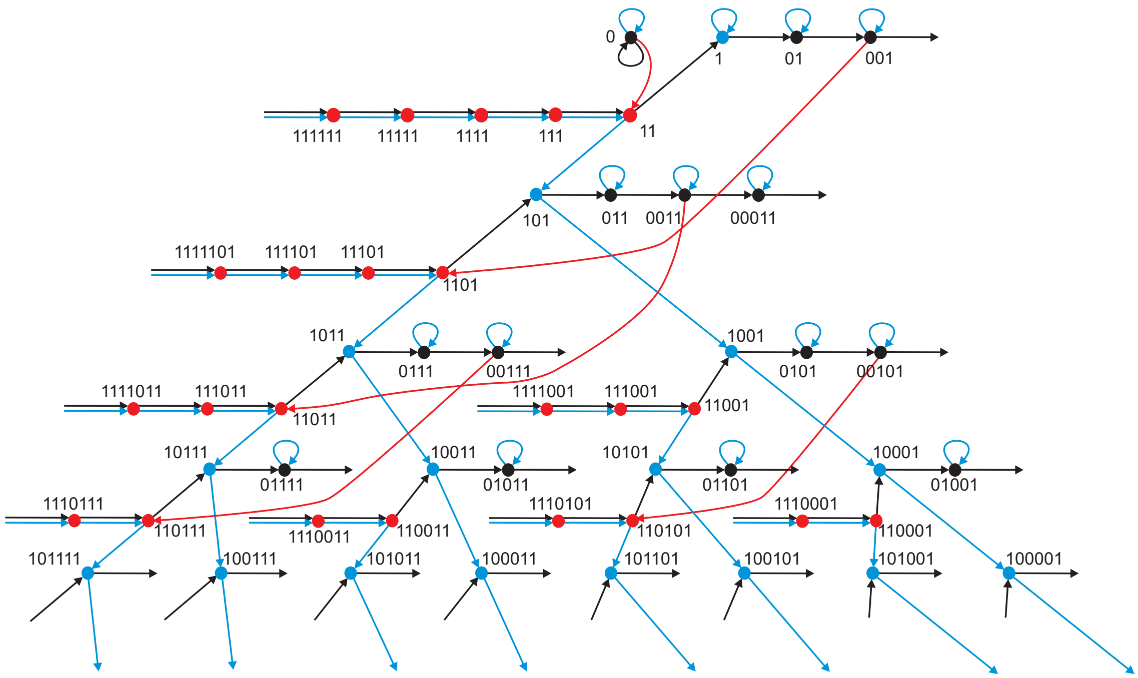

Figure 11: Schreier graph with edges from C acting on second from the left vertices of black rays

In Figure 12, the top-most black ray has been extended so that we can see how C acts on the remaining black vertices of black rays (third from the left, fourth from the left, and so on).

THEOREM 4.8 (Black-other to Red) *For each black vertex that is neither the first nor second from the left of a black ray, there is an edge labeled by C from this vertex to the red vertex obtained by moving left k times to the left until a blue vertex is reached, moving down and left one level to the next blue vertex, moving down and to the right to another blue vertex $k-2$ times, and then moving down and left to the red vertex (see Figure 12).*

Proof. The proof is done by induction on k. The case $k = 2$ is covered by Theorem 4.7. For

$k > 2$, a black vertex that is neither the first nor second from the left of a black ray has the label $000W$. We want to show that $C(000W) = (A^{-1} \circ B \circ A \circ C \circ A^{-1})(000W)$. By Theorem 2.3, $C(000W) = 1100W$ and $C(00W) = 110W$. From Table 1, $A^{-1}(000W) = 00W$, $A(110W) = 10W$, $B(10W) = 100W$, $A^{-1}(100W) = 1100W$. So we get $(A^{-1} \circ B \circ A \circ C \circ A^{-1})(000W) = 1100W = C(000W)$. $\square$

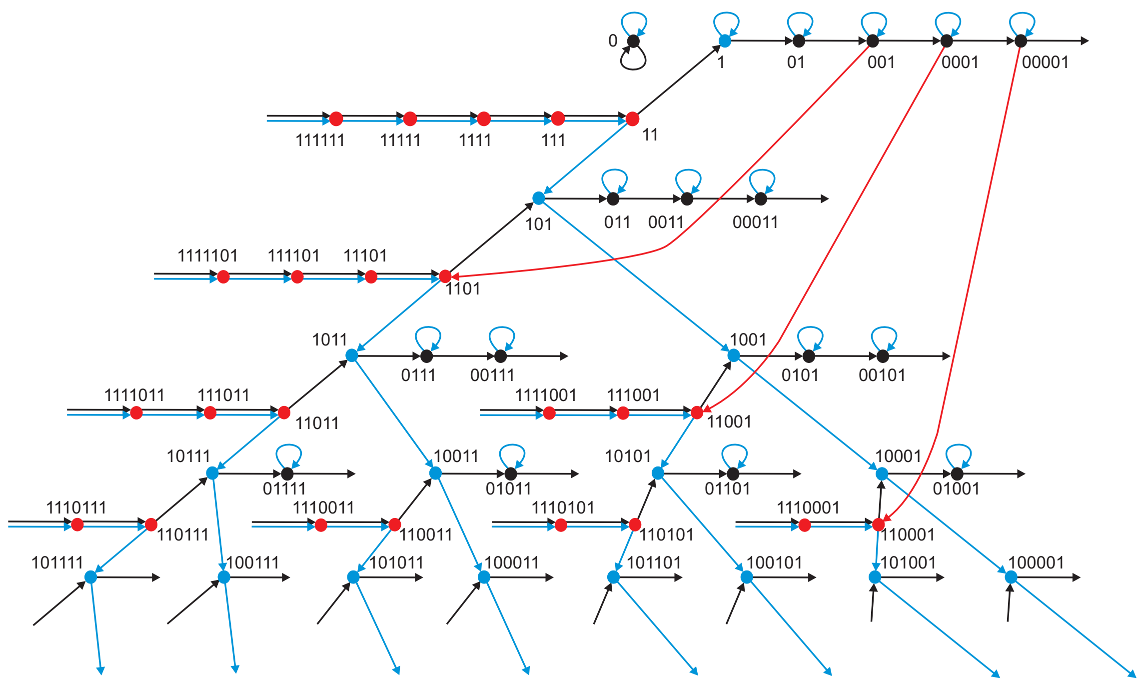

Figure 12: Schreier graph with edges from C acting on the other black vertices on a ray

Using these 9 figures, we have now identified how C acts on any point of the graph. Combining all of these figures gives us a complete (but quite chaotic) graph for $Stab_T(\frac{1}{2})$. If you look carefully, you can pick out individual 3-cycles in the graph.

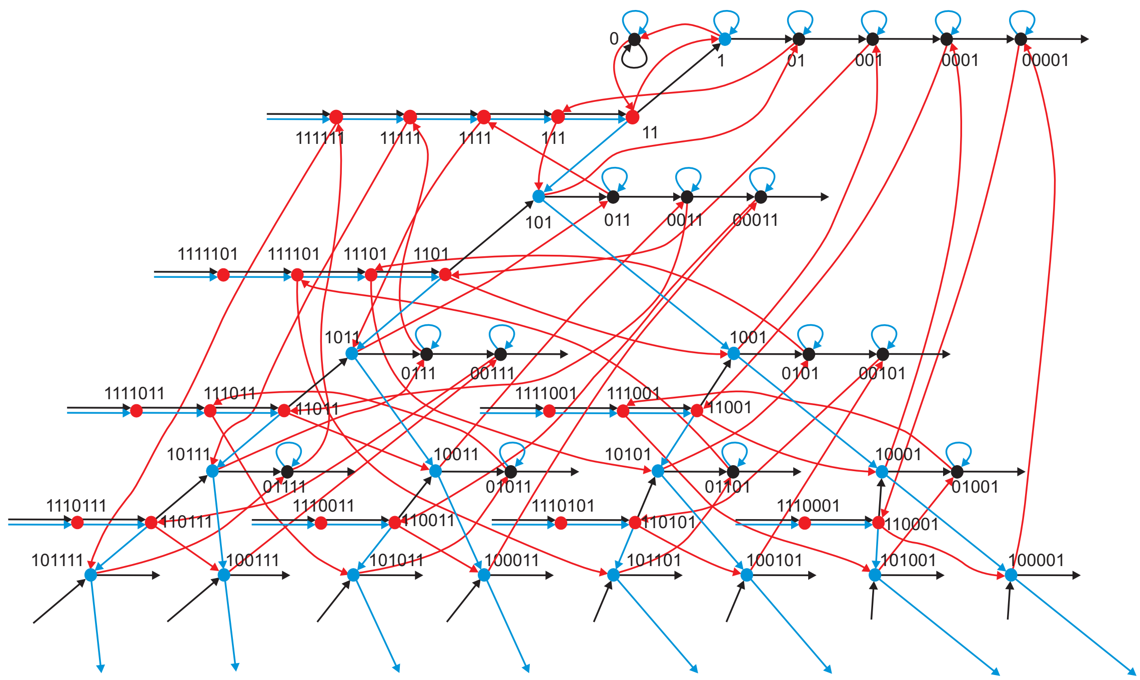

Figure 13: A complete Schreier graph of $Stab_T(\frac{1}{2})$

Chapter 5

Application: Nonamenability of the Action of T on S^1

The goal of this chapter is to show that the graph of $Stab_T(\frac{1}{2})$ (shown partially in Figure 13) is nonamenable. There are multiple ways to prove that a graph is or is not amenable. Here we will use Gromov's doubling condition, which often works when the structure of the graph is completely described. It can only prove nonamenability of the graph.

Note, that the nonamenability of the action of T on S^1 can be proved by other methods. For example, this follows from the proof that T contains a free nonabelian group F_2. However, our approach illustrates one possible application of the structure of the Schreier graph constructed in Chapter 4.

The definition of an amenable graph was given in Chapter 3 (see Definition 3.5). We start from formulating the Gromov's doubling condition for graphs (see, for example, [dlAGCSn99]). Recall that for a graph Γ the set of vertices of Γ is denoted by $V(\Gamma)$.

THEOREM 5.1 (Gromov's doubling condition) *Let Γ be a graph. If there exist two injections, φ_0 and $\varphi_1\colon V(\Gamma) \to V(\Gamma)$, such that:*
1) $\mathrm{Im}\varphi_0 \cap \mathrm{Im}\varphi_1 = \emptyset$
2) $\exists R\, \forall v \in V(\Gamma)\colon (d(\varphi_0(v), v) < R) \wedge (d(\varphi_1(v), v) < R)$
Then Γ is nonamenable. [dlAGCSn99]

Note, that the second condition simply means that both injections can move vertices of Γ by a uniformly bounded distance.

Throughout this section let Γ denote the Schreier graph of $Stab_T(\frac{1}{2})$. In order to apply Gromov's doubling condition to Γ we explicitly define φ_0 and φ_1 mentioned in Theorem 5.1. However, it will be easier to prove that φ_0 and φ_1 satisfy Gromov's doubling condition by defining them in terms of some simpler functions.

DEFINITION 5.1 Define $\varphi_b\colon S^1 \to [\frac{1}{2}, \frac{3}{4})$ to be the following:

$$\varphi_b(x) = \begin{cases} C^{-1}(x) & 0 \leq x < \frac{1}{2} \\ x & \frac{1}{2} \leq x < \frac{3}{4} \\ C(x) & \frac{3}{4} \leq x < 1 \end{cases}$$

A good way to think about how φ_b works is with the following: It takes a vertex and sends it to the binary tree (blue vertices) using the shortest path possible. For a black vertex, this will be by applying C^{-1}. For a red vertex, this will be by applying C. And for a blue vertex, it is already on the binary tree, so it is simply left alone.

DEFINITION 5.2 Define $\varphi_l : [\frac{1}{2}, \frac{3}{4}) \to [\frac{1}{2}, \frac{3}{4})$ to be the following:

$$\varphi_l(x) = (B \circ A^{-1})(x)$$

It is easy to see from the graph that φ_l takes a blue vertex on the binary tree (recall that the blue vertices are exactly dyadic rationals that belong to $[\frac{1}{2}, \frac{3}{4})$) and moves it one level down and to the left on the binary tree.

DEFINITION 5.3 Define $\varphi_r\colon [\frac{1}{2}, \frac{3}{4}) \to [\frac{1}{2}, \frac{3}{4})$ to be the following:

$$\varphi_r(x) = B(x)$$

The map φ_r is similar to φ_l. It takes a vertex on the binary tree and moves it one level down and to the right on the binary tree. Now we are ready to define φ_0 and φ_1.

DEFINITION 5.4 Define $\varphi_0\colon S^1 \to S^1$ and $\varphi_1\colon S^1 \to S^1$ to be the following:

$$\varphi_0(x) = \begin{cases} (\varphi_l \circ \varphi_l \circ \varphi_l \circ \varphi_b)(x) & 0 \leq x < \frac{1}{2} \\ (\varphi_r \circ \varphi_l \circ \varphi_l \circ \varphi_b)(x) & \frac{1}{2} \leq x < \frac{3}{4} \\ (\varphi_l \circ \varphi_r \circ \varphi_l \circ \varphi_b)(x) & \frac{3}{4} \leq x < 1 \end{cases}$$

$$\varphi_1(x) = \begin{cases} (\varphi_l \circ \varphi_l \circ \varphi_r \circ \varphi_b)(x) & 0 \leq x < \frac{1}{2} \\ (\varphi_r \circ \varphi_l \circ \varphi_r \circ \varphi_b)(x) & \frac{1}{2} \leq x < \frac{3}{4} \\ (\varphi_l \circ \varphi_r \circ \varphi_r \circ \varphi_b)(x) & \frac{3}{4} \leq x < 1 \end{cases}$$

Note that φ_0 and φ_1 are both similar. Even though they are defined for all points in S^1, we will be interested only in their restrictions to the set of dyadic integers identified with $V(\Gamma)$. They take a vertex, send it to the binary tree, and then send it somewhere 3 levels down. Where they are sent 3 levels down is dependent upon the starting vertex (whether it is black, blue, or red) and whether φ_0 or φ_1 is being applied (φ_0 always goes left initially, whereas φ_1 goes right).

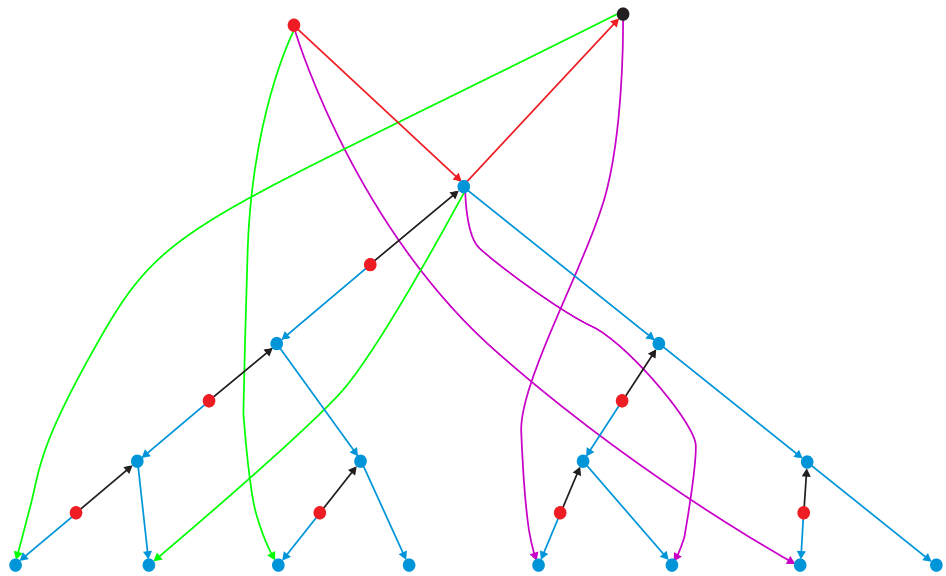

Figure 14: A visual representation of how φ_0 (green) and φ_1 (purple) act on vertices.

LEMMA 5.1 *The map* $\varphi_0 \colon V(\Gamma) \to V(\Gamma)$ *is an injection.*

Proof. We want to show that if $x_0 \neq x_1$ then $\varphi_0(x_0) \neq \varphi_0(x_1)$. Let $x_0, x_1 \in S^1$ such that $x_0 \neq x_1$. If $\varphi_b(x_0) \neq \varphi_b(x_1)$, then it is obvious that $\varphi_0(x_0) \neq \varphi_0(x_1)$ since x_0 and x_1 are moved to two different points of the binary tree initially, and thus there will be no way for them to meet up at the end regardless of how they move 3 levels down. If $\varphi_b(x_0) = \varphi_b(x_1)$, then the vertex associated with x_0 must be of a different color than the vertex associated with x_1 (otherwise the only way for $\varphi_b(x_0) = \varphi_b(x_1)$ is if $x_0 = x_1$, which would contradict our initial assumption). That is, x_0 and x_1 are not from the same interval $[0, \frac{1}{2})$, $[\frac{1}{2}, \frac{3}{4})$, or $[\frac{3}{4}, 1)$. Let $y = (\varphi_l \circ \varphi_b)(x_0) = (\varphi_l \circ \varphi_b)(x_1)$. Since we know that x_0 and x_1 are from different intervals and $\varphi_b(x_0) = \varphi_b(x_1)$, the only way that it is possible that $\varphi_0(x_0) = \varphi_0(x_1)$ is if one of the following is true:

1) $(\varphi_l \circ \varphi_l)(y) = (\varphi_r \circ \varphi_l)(y)$

2) $(\varphi_l \circ \varphi_l)(y) = (\varphi_l \circ \varphi_r)(y)$

3) $(\varphi_r \circ \varphi_l)(y) = (\varphi_l \circ \varphi_r)(y)$

It is clear that none of these statements can be true, because there are not multiple ways to travel down a binary tree and reach the same point. Thus we have shown that $x_0 \neq x_1 \Rightarrow \varphi_0(x_0) \neq \varphi_0(x_1)$, so φ_0 is an injection. □

LEMMA 5.2 *The map $\varphi_1 \colon V(\Gamma) \to V(\Gamma)$ is an injection.*

Proof. The proof is completely analogous to the proof of Lemma 5.1. □

LEMMA 5.3 $\mathrm{Im}\varphi_0 \cap \mathrm{Im}\varphi_1 = \emptyset$.

Proof. This is clear from the way that φ_0 and φ_1 are defined. After sending x to the binary tree by applying φ_b, φ_0 always first branches left by applying φ_l, whereas φ_1 first branches right by applying φ_r. So it is impossible for their images to overlap at all. □

LEMMA 5.4 *For $R = 8, \forall v \in V(\Gamma) \colon d(\varphi_0(v), v) < R \wedge d(\varphi_1(v), v) < R$*

Proof. $d(\varphi_b(v), v) \leq 1$, $d(\varphi_l(v), v) = 2$, and $d(\varphi_r(v), v) = 1$. So using the definitions of φ_0 and φ_1 we can see that $d(\varphi_0(v), v) \leq 7$ for $v \in [0, \frac{1}{2})$, $d(\varphi_0(v), v) \leq 6$ for $v \in [\frac{1}{2}, \frac{3}{4})$, $d(\varphi_0(v), v) \leq 5$ for $v \in [\frac{3}{4}, 1)$, $d(\varphi_1(v), v) \leq 6$ for $v \in [0, \frac{1}{2})$, $d(\varphi_1(v), v) \leq 5$ for $v \in [\frac{1}{2}, \frac{3}{4})$, and $d(\varphi_1(v), v) \leq 5$ for $v \in [\frac{3}{4}, 1)$. □

Now we are ready to conclude the previous lemmata with the main result of this chapter.

THEOREM 5.2 *The Schreier graph Γ of $Stab_T(\frac{1}{2})$ is nonamenable.*

Proof. The prior definitions and lemmas show that it is possible to find two injections on the graph $Stab_T(\frac{1}{2})$, φ_0 and φ_1, that satisfy Gromov's doubling condition. Therefore, the graph is nonamenable. □

In particular, we obtain an alternative simple proof of the following well-known result. Otherwise it can be proved using the, so-called, ping-pong lemma, that T contains a copy of a free non-abelian group and, hence, must be nonamenable.

COROLLARY 5.2.1 *Thompson's group T is nonamenable.*

Proof. It is known that the nonamenability of one of a group's Schreier graphs implies the nonamenability of its Cayley graph as well. Because T has a nonamenable Cayley graph, T is nonamenable. $\square$

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