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Dynamics of Multicultural Social Networks

by

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A thesis submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
in Mathematics
with a concentration in Pure and Applied Mathematics
Department of Mathematics & Statistics
College of Arts and Sciences
University of South Florida

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Yuncheng You, Ph.D.

Dedication

To my love, Drew.

Acknowledgments

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Abstract

Historically human endeavors around the globe are in search of bilateral relationships. Knowledge and commerce has played a very significant role in increasing the ability for humans to connect for the betterment of the human species. As the means of communication improve, mutual benefits to the community as a whole also increase. Moreover, the benefits are filtered down to members of the overall community. Recent advancement in electronic communication technologies and in knowledge, in particular, physical, chemical, engineering and medical sciences and philosophies, have facilitated nearly instantaneous multicultural interactions. Local problems and solutions have become global. This has generated a need for cooperation, coordination, and co-management at local and global levels. This instant communication and easy access to almost all members of the global community, with minimal cost and effort, causes an increase in uncertainty and lack of clarity in communication and misunderstanding between the members of the community. This leads to a fuzzy and stochastic environment. In short, the 21st century has seen a significant increase in the need to consider all human endeavors as being subject to random environmental fluctuations.

More precisely, currently the human species is highly mobile. In this work, an attempt is made (**1**) to balance communities working cooperatively and cohesively with one another while preserving member ability to retain individuality and fostering an environment of cultural state diversity. We develop (**2**) constructive analytic algorithms that provide tools to study qualitative and quantitative properties of multicultural diverse dynamic social networks. Under network parametric space/set conditions (**3**) cohesion and co-existence of members of multicultural dynamic network are insured. The parametric conditions (**4**) are algebraically simple, easy to verify, and robust. Moreover, the presented study of parametric representations of cohesion, co-existence and consensus attributes and robustness of multicultural dynamic networks provides a quantitative tool for planning, policy and performance of human mobility processes for members of a multicultural dynamic network.

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We develop and investigate (**5**) a deterministic dynamic multicultural network. To exhibit the significance of the analysis, ideas, the complexity and limitations, we present a specific prototype model. This serves to establish the framework for finding explicit sufficient conditions in terms of system parameters for studying a complex dynamic network. Further, we decompose the cultural state domain into invariant subsets (**6**) and consider the behavior of members within each cultural state subset. Moreover, we analyze the relative cultural affinity between individual members relative to the center of the social network. We then (**7**)outline the general method for investigating a multicultural cultural network. We also demonstrate the degree of conservatism of the estimates using Euler type numerical approximation schemes. We are then able to consider how changes in the various parametric effects are reflected on the dynamics of the network.

The deterministic multicultural dynamic model and analysis is extended (**8**) to a multicultural dynamic network under random environmental perturbations. We present a detailed prototype model for the purpose of investigation. Introducing the concept of stochastic cohesion and consensus in the context of probabilistic modes of convergence (**9**), the explicit sufficient conditions in terms of system parameters are given to exhibit the cohesive property of the stochastic network. The effects of random fluctuations are characterized.

We then extend the stochastic model (**10**) to a multicultural hybrid stochastic dynamic network model. By considering a hybrid dynamic, we can explore the properties of a multicultural dynamic under the influence of both continuous-time and discrete-time cultural dynamic systems. This model captures external influences and internal changes that may have an impact on the members and system parameters of the dynamic network. We extend the ideas of global cohesion and consensus to local cohesion and consensus (**11**). Again, the detailed study is focused on a prototype hybrid multicultural dynamic network. Using the ideas and tools developed from the stochastic network (**12**), we are able to establish conditions on the network parameters for which the cultural network is locally cohesive. Using Euler-Maruyama type numerical approximation schemes to model the network, we better understand to what extent the analytically developed estimates are feasible.

<div align="center">

Chapter 1

Preliminary Concept and Tools

</div>

1.1   General Notations and Results

In this section, we provide some basic notations, definitions, and important results which will be used in later chapters. To this end, below are the general notations used throughout this work.

$$I(1,m) : \{i \in \mathbb{Z} : 1 \le i \le m\}$$

$$a.s. : \text{almost surely}$$

$$w.p.1 : \text{with probability 1}$$

$$\|x\| : \text{Euclidean norm of x}$$

$$x^T : \text{transpose of x}$$

$$tr(G) : \text{trace of a square matrix G}$$

$$B(x,\delta) : \{y \in \mathbb{R}^n : \|x - y\| < \delta\}$$

$$B^c(x,\delta) : \{y \in \mathbb{R}^n : \|x - y\| \ge \delta\}$$

Let $t \ge t_0$ and $x(t) \in \mathbb{R}^n$. Let

$$dx = f(t,x)dx, \qquad x(t_0) = x_0 \tag{1.1}$$

be an initial value problem such that

(i.) f is a continuous function;

(ii.) f satisfies the growth condition $\|f(t,x)\| \le K\left(1 + \|x\|\right)$ and the Lipschitz condition $\|f(t,x) - f(t,y)\| \le L\|x - y\|$ for $(t,x),(t,y) \in [t_0, \infty) \times \mathbb{R}^n$.

We introduce the definitions of invariant sets as given by [21] as stated below which will be used throughout this work.

Definition 1.1 *Let $A, B \in C[\mathbb{R}^+, \Omega]$ such that $A(t) \subset B(t)$ for $t \in \mathbb{R}^+$. The set $B(t)$ is said to be*

(i) equi conditionally invariant *relative to the set $A(t)$ and the differential system (1.1) if for some $t_0 \geq 0$, $x_0 = x(t_0) \in A(t_0)$ implies*

$$x(t, t_0, x_0) \in B(t), \qquad t \geq t_0; \tag{1.2}$$

(ii) uniformly conditionally invariant *relative to the set $A(t)$ and the differential system if (i) holds for every $t_0 \in \mathbb{R}^n$.*

Definition 1.2 *The set $S(t)$ is said to be* equi self-invariant *relative to the differential equation (1.1) if for some $t_0 \geq 0$, $u_0 \in S(t_0)$ implies*

$$u(t, t_0, u_0) \in S(t), \qquad t \geq t_0. \tag{1.3}$$

Next, we introduce the definition of the maximal and minimal solutions to a scalar comparison differential equation

$$du = g(t, u), \qquad u(t_0) = u_0, \tag{1.4}$$

where $g \in C[E, R]$, with E an open (t, u) set in $\mathbb{R}^2$ as defined in [19].

Definition 1.3 *A solution $r(t)$ to the scalar comparison differential equation (1.4) on $[t_0, t_0 + a)$ is said to be the maximal solution if every solution $u(t)$ existing on $[t_0, t_0 + a)$ satisfies the inequality*

$$u(t) \leq r(t), \tag{1.5}$$

for $t \in [t_0, t_0 + a)$.

Definition 1.4 *A solution $\rho(t)$ to the scalar comparison differential equation (1.4) on $[t_0, t_0+)$ is said to be the minimal solution if every solution $u(t)$ existing on $[t_0, t_0 + a)$ satisfies the inequality*

$$\rho(t) \leq u(t), \tag{1.6}$$

for $t \in [t_0, t_0 + a)$.

Below we note a comparison theorem for deterministic differential equations (1.1) as given in [26].

Theorem 1 *Let E be on open (t, u)-set in $\mathbb{R}^2$ and $g \in C[E, \mathbb{R}]$. Suppose that $[t_0, t_0 + a)$ is the largest interval in which the maximal solution $r(t)$ exists. Let $m \in C[(t_0, t_0 + a), \mathbb{R}], (t, m(t)) \in E$ for $t \in [t_0, t_0 + a), m(t_0) \leq u_0$ and for a fixed Dini derivative*

$$Dm(t) \leq g(t, m(t)), \tag{1.7}$$

$t \in [t_0, t_0 + a) - S.$ *Then*

$$m(t) \leq r(t), \qquad t \in (t_0, t_0 + a). \tag{1.8}$$

1.2 Stochastic Differential Equations

In this section we outline sufficient conditions for the existence of a solution for a system of stochastic differential equations. Let $t \geq t_0$ and $x(t) \in \mathbb{R}^n$ be a random vector on the complete probability space (Ω, F, P). An Itô-Doob type stochastic differential equation is given by

$$dx = f(t, x)dt + \sigma(t, x)d\xi, \qquad x(t_0) = x_0, \tag{1.9}$$

where

(i.) f and σ are the drift and diffusion rates respectively;

(ii.) $\xi = (\xi_1, \xi_2, \ldots, \xi_m)^T$ is an m-dimensional normalized Wiener process of independent increments;

(iii.) For $t \geq t_0$, F_t is an increasing family of sub-σ algebras of σ-algebra F.

(iv.) f and σ satisfy the linear growth condition: there exist some positive constants N and M such that for $t \in J \times \mathbb{R}^n$ where $J \subset [t_0, \infty)$

$$\|f(t, x)\| + \|\sigma(t, x)\| \leq N + M\|x\|; \tag{1.10}$$

(v.) f and σ satisfy the Lipschitz condition: for $(t, x), (t, y) \in J \times \mathbb{R}^n$,

$$\|f(t, x) - f(t, y)\| + \|\sigma(t, x) - \sigma(t, y)\| \leq L\|x - y\|. \tag{1.11}$$

From $(iv.)$ and $(v.)$, (1.9) has a unique solution.

We now state the Itô-Doob differential formula as stated in [20].

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Theorem 2 *Let $J \subset [t_0, \infty)$ and $V(t, x) \in C[J \times \mathbb{R}, \mathbb{R}]$ be a continuously differentiable function with respect to t and twice continuously differentiable with respect to x in (1.9). Then,*

$$dV(t, x(t)) = LV(t, x(t))dt + \sigma(t)\frac{\partial}{\partial x}V(t, x(t))d\xi(t), \tag{1.12}$$

where ξ is a Wiener process and LV is the differential operator associated with (1.9) defined by

$$LV(t, x(t)) = \frac{\partial}{\partial t}V(t, x(t)) + f(t, x)\frac{\partial}{\partial x}V(t, x(t))dx(t) + \frac{1}{2}tr\left(\frac{\partial^2}{\partial x^2}V(t, x(t))\sigma(t, x)\sigma^T(t, x(t))\right). \tag{1.13}$$

1.3   Sufficient Conditions for Qualitative Properties of Stochastic Differential Equations

In this section, we state the comparison theorems which serve as one of the basic principals for studying non-linear stochastic differential equations for which we either have or do not have a closed form solution.

Let $\mathcal{H}$ denote the class of functions $b \in C[[0, \rho), \mathbb{R}^+]$ such that $b(0) = 0$ and $b(r)$ is strictly increasing in r, where $0 \leq \rho \leq \infty$. Further, let $\mathcal{CH}$ be the class of functions $a \in C[\mathbb{R}^+ \times [0, \rho), \mathbb{R}^+]$ such that $b(0) = 0$ and $b(r)$ is convex and strictly increasing in r. We state a comparison theorem related to the use of Lyapunov-like functions as given in [23].

Theorem 3 *Assume that*

(i.) $g \in C[\mathbb{R}^+ \times \mathbb{R}_N^+, \mathbb{R}_N]$, $g(t, 0) \equiv 0$, $g(t, u)$ is concave and quasi-monotone nondecreasing in u for each $t \in \mathbb{R}^+$;

(ii.) $V \in C[\mathbb{R}^+ \times B(\rho), \mathbb{R}_N]$, $V_t(t, x), V_x(t, x)$ and $V_{xx}(t, x)$ exist and are continous on $\mathbb{R}^+ \times B(\rho)$ and for $(t, x) \in \mathbb{R}^+ \times B(\rho)$,

$$LV(t, x) \leq g(t, V(t, x)), \tag{1.14}$$

where $B(\rho) = \{x \in \mathbb{R}^n : \|x\| < \rho\}$;

(iii.) for $(t, x) \in \mathbb{R}^+ \times B(\rho)$,

$$b(\|x\|) \leq \sum_{i=1}^{N} V_i(t, x) \leq a(t, \|x\|), \tag{1.15}$$

where $b \in \mathcal{H}$ and $a \in \mathcal{CH}$.

Then the stability of the equilibrium solution of

$$u' = g(t, u), \qquad u(t_0) = u_0, \tag{1.16}$$

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implies the stability in probability of the equilibrium solution of (1.9).

Chapter 2

Deterministic Multicultural Dynamic Network

2.1 Introduction

The goal of this work is to explore the properties of a dynamic network of agents under the influence of internal and external perturbations [6]. Cohesion within a social network is a current topic of great interest and many authors have done research within this area [8, 4]. The concepts of cohesion and cooperation within a group are often multi-faceted, dynamic and complex but are important concepts when trying to better understand how nations or human groups interact and function [3]. As Knoke and Yang note [18], it is social cohesion which enables information to spread and allows a group to act as a unit rather than individuals.

Dynamic networks are often useful for modeling a variety of situations. For example, dynamic networks can be used in creating dynamic models of the traffic flow of cars or airplanes (as agents) where a minimum or threshold distance is desired between two cars or airplanes to avoid ground or air collisions. In addition, we can use a dynamic network to model a community with individual members of community as agents and a distances that need to be maintained to control the spread of an infectious disease. Further, social and cultural dynamics within a group are also often represented by such a network to safe-guard or maintain their self-respect or identity or comfort zone. In particular, we are interested in the cultural shifts of members within culturally diverse groups. We seek to better understand the internal and environmental factors that may foster a sense of cooperation between members of the network while allowing individuality and diversity to be maintained and enhanced.

We use the term multicultural social network to describe a social network in which the members have a diverse cultural and/or ethnical background and are actively seeking to enhance and to maintain diversity with harmony and prosperity. In such a network, the goal of members is not approaching a consensus but rather the ability to live and work cooperatively with one another for common goods and goals. For example, consider a population in an area for which there exists a sub-populace of immigrants. In such a situation, the

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subgroups or sub-communities of immigrants desire to be an integral part of the community and seek to be respectable productive members of the community and the society in general while retaining their cultural diversity.

We wish to model a network that is cohesive but for which there is not a consensus of cultural unity, that is to say the network does not develop a singular cultural identity. In doing so, we are interested in better understanding the cohesive properties of a multicultural social network. We present a prototype of a dynamic model for which we explore the features of such a network. The presented example is used to exhibit the quantitative and qualitative properties of the network. Further, the techniques used are computationally attractive, easy to verify and algebraically simple. In addition, the presented results are in terms of network parameters that characterize the attributes of the network. The byproduct of this provides tools for planning and decision making policies regarding a dynamic network.

We first consider a cultural network dynamic model experiencing both attractive and repulsive forces in the absence of stochastic perturbations. In Section 2.2, we develop the general dynamic model as well as assumptions and notations used throughout this chapter. In Section 2.3, we present an example of such a network. In Section 2.4, analytical tools and results are creatively developed for the usage of Lyapunov's Second Method and the comparison method [26] for the dynamics of individual members within the network. In Section 2.5, long and short term behavior of group members and invariant cultural state sets are investigated. In Section 2.7, we consider numerical simulations of the network to better understand the extent of the role and scope of the estimates developed in Section 2.5. Finally, in Section 2.8, we consider parametric variations within the model affecting the dynamics of the network. Further, we will consider how the model relates to a multicultural network.

2.2   Problem Formulation

The network consists of m agents/members whose position at time t is represented by $x_i(t)$, $i \in I(1, m)$ with $x_i(t) \in \mathbb{R}^n$. For each member, $x_i \in \mathbb{R}^n$ is a cultural position at time $t > t_0$. In our model, this vector does not represent a geographical location but rather a cultural state position of the ith member. That is to say, the vector x_i is a numerical representation of the ith member's belief or background on certain cultural or ethnic characteristics relevant to the network and questions being considered. We then consider a cultural

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state dynamic model described by a system of differential equations:

$$dx_i = \sum_{j=1}^{m} f(t, x_i - x_j) dt, \qquad x_i(t_0) = x_i^0. \qquad (2.1)$$

Further, let us define a relative cultural state of ith member with a jth member of the community as $x_{ij} = x_i - x_j$, and a center of cultural state of the network

$$\bar{x} = \frac{1}{m} \sum_{j=1}^{m} x_j. \qquad (2.2)$$

We assume that f is a function such that

(i.) f is continuous function;

(ii.) f satisfies the growth condition $\|f(t, x)\| \leq K (1 + \|x\|)$ and the Lipschitz condition $\|f(t, x) - f(t, y)\| \leq L\|x - y\|$ for $(t, x), (t, y) \in [t_0, \infty) \times \mathbb{R}^n$;

(iii.) $\bar{x}$ is a stationary center of f.

In the following, we introduce a few definitions with regard to the quantitative and qualitative behavior of a center $\bar{x}$ of a cultural state network dynamic system (2.1).

Definition 2.1 *We say that the network is cohesive if there exist constants T and M such that $t_0 \leq T \leq t$ implies $\left\|x_i(t, t_0, x_i^0) - \bar{x}\right\| \leq M$.*

Definition 2.2 *We say that the network reaches a consensus if $\left\|x_i(t, t_0, x_i^0) - \bar{x}\right\| \to 0$ as $t \to \infty$ for all $i \in I(1, m)$.*

Definition 2.3 *We define the term relative cultural state affinity to be $\|x_{ij}(t)\| = \left\|x_i(t, t_0, x_i^0) - x_j(t, t_0, x_j^0)\right\|$, the distance between the cultural state vectors of members x_i and x_j. We define the relative cultural state change to be $x_{ij}(t) = x_i(t, t_0, x_i^0) - x_j(t, t_0, x_j^0)$.*

Remark 2.1: Definition 2.1 signifies that one can find a time after the initial state time such that the cultural states of members of the network remain within a certain distance from the network center after some time. In the case of Definition 2.2, each member of the network draws closer to each other and the cultural state network center.

In the following sections, we establish the framework and exhibit the tools, ideas, and methods for gaining insight and working with nonlinear and non-stationary multicultural dynamic network (2.1). We develop a detailed nontrivial model which provides an understanding for the qualitative and quantitative analysis of a multicultural dynamic network.

2.3 Prototype Dynamic Model

We seek to develop and analyze a prototype dynamic multicultural network model that captures the behavior of individual members who are seeking to belong to the group but also wanting to retain individuality and diversity from other network members. Therefore, we consider dynamic equations subjected to both attractive and repulsive forces. In [10], one such function considered when modeling biological dynamic networks is given by

$$g(y) = -y \left(a - b \exp \left[\frac{-\|y\|^2}{c} \right] \right), \tag{2.3}$$

where a, b and c are positive constants and $y \in \mathbb{R}^n$. The function g has long range attraction and short range repulsion. In the following, we formulate a modified version of a network dynamic model in which individuals seek to retain a balance between individual member identity and a group/community membership. In the following, we consider a network whose dynamics are described by incorporating a long range attraction and short range repulsion similar to that in (2.3).

Consider the network whose dynamic is given by

$$dx_i = \left[a \sum_{j=1}^{m} x_{ij} - q \|x_i - \bar{x}\|^2 \sum_{j=1}^{m} x_{ij} \right.$$
$$\left. + b \sin\|x_i - \bar{x}\| \sum_{j=1}^{m} x_{ij} \exp \left[-\frac{\|x_{ij}\|^2}{c} \right] \right] dt; \qquad x_i(t_0) = x_i^0. \tag{2.4}$$

The constant coefficient parameters, $a, b, c,$ and $q \in \mathbb{R}^+$ represent the weight of the social moderation attractiveness (q), the repulsive forces (a), the rate of decay of the long range attractiveness (c), and the long-range attractiveness (b) between individual members and social groups. Moreover, for each $i \in I(1, m)$, the first term in (2.4) characterizes the aggregate repulsive force driving the ith member in the presence of network members. This repulsive force is generated by the relative cultural state change of the ith member with all other network members. The second term in (2.4) signifies the limitations of the repulsive force causing the generation of a retardation force influencing a short-range attraction to the ith member in the presence of network members. Finally, the third term of (2.4) naturally characterizes the long-range influence of the relative cultural affinity to the ith member due to the interactions with members of the network.

Attractive influences can be thought of as attributes that bring people to active membership within the group. Social acceptance, gaining social status, economic opportunity, career growth, common purpose and membership, personal development, and a sense of mutual respect, trust and understanding are examples

of attractive influences within a social cultural network. Repulsive forces are attributes that create some desire for individuals to leave or be less involved in the group or to preserve some personal identity from one other with their individual magnitude of inner repulsive force. A desire to retain a sense of individuality, economic or emotional cost, interpersonal conflict within the group, or disagreement with parts of the overall philosophies of the group are forces that may be considered as repulsive forces. In short, economic, educational, and social inequalities coupled with the race, gender, ethical and religious bias are sources of repulsive forces. A balance between the total attraction and repulsive forces attributes to a general sense of individual agents maintaining a "live and let live" philosophy for the greater benefit of the community and the common good of society.

2.4   Characteristics of the Network

In this section, we wish to explore the dynamics of the agents with the network dynamic described by (2.4). We will be considering the cohesion, qualitative and quantitative properties such as the overall stability of the network center, and various types of invariant sets. While exploring these ideas, we will also consider what happens as the size of the network increases and what roles the parameters $a, b,$ and c play within the model. Moreover, the presented example is utilized to exhibit the quantitative and qualitative properties of the network. In order to accomplish such a task, we utilize Lyapunov's Second Method [26]. This method is algebraically simple, easy to verify and computationally attractive. Furthermore, the results depend on the system parameters a, b, c and q.

Let us first examine the dynamic of the network center, $\bar{x}$, as defined in (2.2). We note that, that $\sum_i \sum_j x_{ij} = 0$, and

$$
\begin{aligned}
d\bar{x} &= \left[a \sum_{j=1}^{m} (\bar{x} - x_j) - q\|\bar{x} - \bar{x}\|^2 \sum_{j=1}^{m} (\bar{x} - x_j) \right. \\
&\qquad \left. b \sin\|\bar{x} - \bar{x}\| \sum_{j=1}^{m} (\bar{x} - x_j) \exp\left[-\frac{\|\bar{x} - x_j\|^2}{c} \right] \right] dt \\
&= 0
\end{aligned}
\tag{2.5}
$$

and $\bar{x}$ is a stationary center of the network. We define the transformation of the network by $z_i = x_i - \bar{x}$,

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noting that

$$mz_i = \sum_{j=1}^{m} x_{ij} \tag{2.6}$$

$$x_{ij} = z_{ij} = z_i - z_j. \tag{2.7}$$

Therefore, the dynamics of the transformed network are given by

$$
\begin{aligned}
dz_i &= d\left(x_i - \bar{x}\right) \\
&= dx_i \\
&= \left[amz_i - qm\|z_i\|^2 z_i \right. \\
&\quad \left. + b\sin\|z_i\| \sum_{j=1}^{m} z_{ij} \exp\left[-\frac{\|z_{ij}\|^2}{c} \right] \right] dt, \qquad z_i(t_0) = z_i^0.
\end{aligned}
\tag{2.8}
$$

Dynamic equation (2.8) can be useful in modeling a variety of multicultural social networks. Again, we note that in (2.8), the magnitude of the repulsive force is represented by $am\|z_i\|$ and the magnitude of the long range attractive force is described by $b\left\|\sum z_{ij} \exp[-\|z_{ij}\|^2/c]\right\|$. Furthermore, $\sin\left(\|z_i\|\right)$ is the sine-cyclical influence due to the magnitude of the deviation of the ith agent's cultural state from the center of the network.

In order to better understand the dynamics of (2.8), we need to creatively develop necessary tools and results to apply Lyapunov's Second Method in conjunction with the comparison method [26]. These methods will provide a computationally attractive means to better understand the movement of members within the network. To that end, let us begin with a choice of a candidate for energy function defined by

$$V(z_i) = \frac{1}{2}\|z_i\|^2. \tag{2.9}$$

Then the differential of V along the vector field generated by (2.8) is given by

$$
\begin{aligned}
dV(z_i) &= z_i^T \cdot dz_i \\
&= \left[am\|z_i\|^2 - qm\|z_i\|^4 + b\sin\|z_i\| \sum_{j=1}^{m} z_i^T z_{ij} \exp\left[-\frac{\|z_{ij}\|^2}{c} \right] \right] dt \\
&= LV(z_i)dt, \tag{2.10}
\end{aligned}
$$

where

$$LV(z_i) = am\|z_i\|^2 - qm\|z_i\|^4 + b\sin\left(\|z_i\|\right) \sum_{j=1}^{m} z_i^T z_{ij} \exp\left[-\frac{\|z_{ij}\|^2}{c} \right]. \tag{2.11}$$

In subsections 2.4.1 and 2.4.2, using Lyapunov's Second Method and the comparison method [26],we will find upper and lower estimates for $LV(z_i)$ respectively.

2.4.1 Upper Estimate of $LV(z_i)$

In this subsection, we seek constraints on a, b, c, and q such that for z_i outside of a given ball, we can establish an upper estimate of $LV(z_i)$. We will then use these assumptions in conjunction with the Lyapunov method and comparison theorem [26] to establish the case for which

$$V(z_i) \leq r(t, t_0, u_0), \tag{2.12}$$

where $r(t, t_0, u_0), r(t_0) = u_0$ is the maximal solution of a comparison differential equation through (t_0, u_0).

By considering the derivative of the function $f(r) = r \exp\left[-\frac{r^2}{c}\right]$, we note that

$$\|z_{ij}\| \exp\left[-\frac{\|z_{ij}\|^2}{c}\right] \tag{2.13}$$

has a global maximum when $\|z_{ij}\| = \sqrt{\frac{c}{2}}$ with a maximum value of

$$\sqrt{\frac{c}{2}} \exp\left[-\frac{1}{2}\right]. \tag{2.14}$$

From (2.13), (2.14), and the fact that $\sin\|z_i\| \leq 1$, for $i \in I(1, m)$, (2.11) reduces to:

$$
\begin{aligned}
LV(z_i) &\leq am\|z_i\|^2 - qm\|z_i\|^4 + b\sum_{j\neq i}^{m}\|z_i\|\|z_{ij}\| \exp\left[-\frac{\|z_{ij}\|^2}{c}\right] \\
&\leq am\|z_i\|^2 - qm\|z_i\|^4 + b(m-1)\|z_i\|\sqrt{\frac{c}{2}} \exp\left[-\frac{1}{2}\right] \\
&= am\|z_i\|^2 - (qm-1)\|z_i\|^4 - \|z_i\|^4 + b(m-1)\|z_i\|\sqrt{\frac{c}{2}} \exp\left[-\frac{1}{2}\right] \\
&= am\|z_i\|^2 - \|z_i\|^4 - (qm-1)\|z_i\|\left(\|z_i\|^3 - \frac{b(m-1)\sqrt{\frac{c}{2}} \exp\left[-\frac{1}{2}\right]}{qm-1}\right).
\end{aligned}
\tag{2.15}
$$

Assumption H_1: Suppose $qm - 1 > 0$. Let us define

$$\beta_1 = \left(\frac{b(m-1)\sqrt{\frac{c}{2}} \exp\left[-\frac{1}{2}\right]}{qm-1}\right)^{\frac{1}{3}}, \tag{2.16}$$

and let $B(0, \beta_1) = \{x \in \mathbb{R}^n : \|x\| < \beta_1\}$. Further, let us denote the compliment of the $B(0, \beta_1)$ by $B^c(0, \beta_1)$. For any $z_i \in B^c(0, \beta_1), i \in I(1, m)$, (2.15) yields the following inequality:

$$
\begin{aligned}
LV(z_i) &\leq am\|z_i\|^2 - \|z_i\|^4 \\
&= 4V(z_i)\left(\frac{am}{2} - V(z_i)\right).
\end{aligned}
\tag{2.17}
$$

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Using (2.17) along with the comparison theorems [26], we establish the following result.

LEMMA 2.1 *Let V be the energy function defined in* (2.9), *z_i be a solution of the initial value problem defined in* (2.8). *Further, let*

$$du = 4u \left(\frac{am}{2} - u \right) dt, \qquad r(t_0) = u_0. \qquad (2.18)$$

For each $i \in I(1, m)$ satisfying the differential inequality (2.17) *and $V(z_i(t_0)) \leq u_0$, it follows that the network is cohesive and*

$$V(z_i(t)) \leq r(t, t_0, u_0), \qquad (2.19)$$

where $r(t)$ is the maximal solution of the initial value problem (2.18).

Proof. Under the assumptions of the lemma and using the standard argument [26] combined with the above discussion, the proof of the lemma follows from (2.17). The cohesiveness of the network follows by definition as the solution to (2.18) is bounded. □

Remark 2.2: We remark that the assumption H_1 is an alternative sufficient condition as: From (2.15), we have

$$
\begin{aligned}
LV &\leq (am + r_1)\|z_i\|^2 - qm\|z_i\|^4 - r_1\|z_i\|^2 + b(m-1)\|z_i\| \sqrt{\frac{c}{2}} \exp\left[-\frac{1}{2} \right] \\
&= qm\|z_i\|^2 \left(\frac{am + r_1}{qm} - \|z_i\|^2 \right) - r_1\|z_i\| \left(\|z_i\| - \frac{b(m-1)}{r_1} \sqrt{\frac{c}{2}} \exp\left[-\frac{1}{2} \right] \right) \\
&\leq 4qmV(z_i) \left(\frac{am + r_1}{qm} - V(z_i) \right), \qquad z_i \in B^c(0, \beta_1), \qquad (2.20)
\end{aligned}
$$

where $B(0, \beta_1) = \{x \in \mathbb{R}^n : \|x\| < \beta_1\}$, $\beta_1 = \frac{b(m-1)}{r_1} \sqrt{\frac{c}{2}} \exp\left[-\frac{1}{2} \right]$ for any $r_1 > 0$.

2.4.2 Lower Estimate of $LV(z_i)$

Next, we look to establish a lower estimate of $LV(z_i)$ such that

$$LV(z_i) \geq \rho(t, t_0, u_0), \qquad (2.21)$$

where $\rho(t)$ is the minimal solution to a comparison equation through (t_0, u_0).

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Imitating the argument used to arrive at (2.15) and noting that, for $\alpha > 0$, $\|x\| < \alpha$ if and only if $-\alpha < \|x\| < \alpha$, for $i \in I(1, m)$, (2.11) reduces to the inequality

$$
\begin{aligned}
LV(z_i) &\geq am\|z_i\|^2 - qm\|z_i\|^4 - b \sum_{j \neq i}^m \|z_i\| \sqrt{\frac{c}{2}} \exp\left[-\frac{1}{2}\right] \\
&= am\|z_i\|^2 - qm\|z_i\|^4 - b(m-1)\sqrt{\frac{c}{2}} \exp\left[-\frac{1}{2}\right] \|z_i\| \\
&= a\|z_i\|^2 + a(m-1)\|z_i\|^2 - qm\|z_i\|^4 - b(m-1)\sqrt{\frac{c}{2}} \exp\left[-\frac{1}{2}\right] \|z_i\| \\
&= a\|z_i\|^2 - qm\|z_i\|^4 + a(m-1)\|z_i\| \left(\|z_i\| - \frac{b\sqrt{\frac{c}{2}} \exp\left[-\frac{1}{2}\right]}{a} \right).
\end{aligned}
\tag{2.22}
$$

Assumption H_2: Let us define

$$
\beta_2 = \frac{b\sqrt{\frac{c}{2}} \exp\left[-\frac{1}{2}\right]}{a},
\tag{2.23}
$$

and $B(0, \beta_2) = \{x \in \mathbb{R}^n : \|x\| < \beta_2\}$, with its complement being $B^c(0, \beta_2)$. For $z_i \in B^c(0, \beta_2), i \in I(1, m)$, (2.22), reduces to the following differential inequality:

$$
\begin{aligned}
LV(z_i) &\geq a\|z_i\|^2 - qm\|z_i\|^4 \\
&= 4qmV(z_i) \left(\frac{a}{2qm} - V(z_i) \right).
\end{aligned}
\tag{2.24}
$$

Using (2.24) along with the comparison theorems [26], we establish the following result.

LEMMA 2.2 *Let V be the energy function defined in (2.9) and z_i be a solution of the initial value problem defined in (2.8). Further, let*

$$
du = \left(4qmu \left(\frac{a}{2qm} - u \right) \right) dt, \qquad u(t_0) = u_0,
\tag{2.25}
$$

For each $i \in I(1, m)$ satisfying the differential inequality (2.24) and $V(z_i(t_0)) \geq u_0$, it follows that

$$
V(z_i(t)) \geq \rho(t, t_0, u_0),
\tag{2.26}
$$

where $\rho(t)$ is the minimal solution of the initial value problem (2.25).

Proof. Under the assumptions of the lemma and using the standard argument [26] combined with the above discussion, the proof of the lemma follows from (2.24). □

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Remark 2.3: A remark similar to Remark 2.2 is as follows: From (2.22), we have

$$LV \geq (am - r_2)\|z_i\|^2 - qm\|z_i\|^4 - r_2\|z_i\|^2 + b(m-1)\|z_i\|\sqrt{\frac{c}{2}}\exp\left[-\frac{1}{2}\right]$$

$$= qm\|z_i\|^2\left(\frac{am - r_2}{qm} - \|z_i\|^2\right) + r_2\|z_i\|\left(\|z_i\| - \frac{b(m-1)}{r_2}\sqrt{\frac{c}{2}}\exp\left[-\frac{1}{2}\right]\right)$$

$$\geq 4qmV(z_i)\left(\frac{am - r_2}{qm} - V(z_i)\right), \qquad z_i \in B^c(0, \beta_2), \tag{2.27}$$

where $B(0, \beta_2) = \{x \in \mathbb{R}^n : \|x\| < \beta_2\}$, where $\beta_2 = \frac{b(m-1)}{r_2}\sqrt{\frac{c}{2}}\exp\left[-\frac{1}{2}\right]$, and any $r_2 > 0$.

We note that the upper and lower comparison equations, (2.18) and (2.25) respectively, each have a unique solution. Therefore, the maximal and minimal solutions are the unique solutions to the respective comparison equations.

2.5   Long and Short Term Behavior of Members and Invariant Sets

Using Lyapunov's Second Method and the comparison method [26], we consider the behavior of the members over time t and consider the stability and invariant sets of the network. First, let us note from $\rho(t)$, the minimal solution to the initial value problem (2.25) in Lemma 2.2, we find

$$\lim_{t \to \infty} \rho(t) = \lim_{t \to \infty} \frac{au_0}{2m\left(u_0 + \left(\frac{a}{2m} - u_0\right)\exp\left[-2a(t - t_0)\right]\right)}$$

$$= \frac{a}{2qm}. \tag{2.28}$$

Similarly, from the solution of the comparison differential equation (2.18), and Lemma 2.1, we note that

$$\lim_{t \to \infty} r(t) = \lim_{t \to \infty} \frac{amu_0}{2\left(u_0 + \left(\frac{am}{2} - u_0\right)\exp\left[-2a(t - t_0)\right]\right)}$$

$$= \frac{am}{2}. \tag{2.29}$$

Therefore, by Lemmas 2.1 and 2.2, when $z_i \in B^c(0, \beta_1) \cap B^c(0, \beta_2)$, it follows that

$$\sqrt{\frac{a}{qm}} \leq \lim_{t \to \infty}\|z_i(t)\| \leq \sqrt{am}. \tag{2.30}$$

From (2.29), (2.28), and (2.30), we consider one case and the associated invariant sets. First, let us consider the case for which $\beta_2 \leq \beta_1$. That is, let us suppose that

$$\frac{b\sqrt{\frac{c}{2}}\exp\left[-\frac{1}{2}\right]}{a} \leq \left(\frac{b(m-1)\sqrt{\frac{c}{2}}\exp\left[-\frac{1}{2}\right]}{qm - 1}\right)^{1/3}. \tag{2.31}$$

Let us further suppose that it is the case that

$$\frac{b\sqrt{\frac{c}{2}}\exp\left[-\frac{1}{2}\right]}{a} \leq \sqrt{\frac{a}{qm}} \leq \left(\frac{b(m-1)\sqrt{\frac{c}{2}}\exp\left[-\frac{1}{2}\right]}{qm-1}\right)^{1/3}. \tag{2.32}$$

For β_1 and β_2, let us define the following sets:

$$A = B(0, \beta_2)$$

$$B = B^c(0, \beta_2) \cap B(0, \sqrt{\frac{a}{qm}})$$

$$C = B^c(0, \sqrt{\frac{a}{qm}}) \cap B(0, \beta_1)$$

$$D = B^c(0, \beta_1) \cap B(0, \sqrt{am})$$

$$E = B^c(0, \sqrt{am}). \tag{2.33}$$
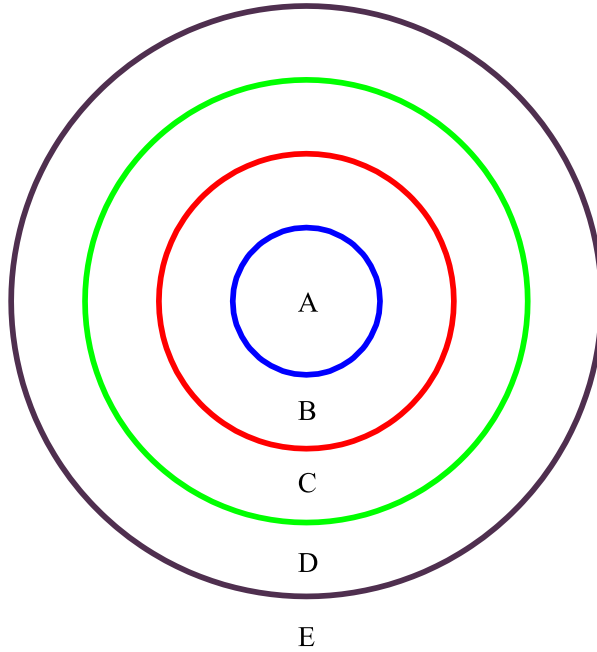
Figure 2.1.: An example in R^2 of the sets defined in (2.33). Under the assumptions in (2.32), the sets form concentric annuli.

In the following, we state and prove a few qualitative properties of the solution process of the center of

the multi-agent determinist dynamic network described by (2.4). The following result exhibits the major influence of long range attractive forces.

THEOREM 2.1 *For $0 < \epsilon < 1$, if for all $i \in I$, $z_{i0} \in B(0, \sqrt{2\epsilon})$, a neighborhood of the center $\bar{x}$, then (2.11) reduces to the inequality*

$$LV(z_i) \geq 4qmV(z_i)\left(\frac{a}{2q} - V(z_i)\right) - \frac{b}{2}(m-1)\epsilon. \tag{2.34}$$

Further, if $\frac{a}{2q} > u_0$, there exists $0 < \bar{\epsilon} \leq 1$ such that $\|z_i(t)\| > 0$ for $t \geq t_0$ when, for all $i \in I(1,m)$, $z_{i0} \in B(0, \sqrt{2\bar{\epsilon}})$.

Proof. Let $0 < \epsilon < 1$ and $\|z_i\|^2 < \epsilon$ for all $i \in I$. Then,

$$
\begin{aligned}
LV &= am\|z_i\|^2 - qm\|z_i\|^4 \\
&\quad + b\sin(\|z_i\|)\sum_{\substack{j\neq i}}^{m}\frac{1}{2}\left[\|z_i\|^2 - \|z_j\|^2 + \|z_{ij}\|^2\right]\exp\left[-\frac{\|z_{ij}\|^2}{c}\right] \\
&\geq am\|z_i\|^2 - qm\|z_i\|^4 - \frac{b}{2}\sum_{\substack{j\neq i}}^{m}\epsilon \\
&= qm\|z_i\|^2\left(\frac{a}{q} - \|z_i\|^2\right) - \frac{b}{2}(m-1)\epsilon \\
&= 4qmV(z_i)\left(\frac{a}{2q} - V(z_i)\right) - \frac{b}{2}(m-1)\epsilon.
\end{aligned}
\tag{2.35}
$$

Considering the non-homogeneous comparison equation,

$$du = \left[4qmu\left(\frac{a}{2q} - u\right) - \frac{b}{2}(m-1)\epsilon\right]dt, \qquad u(t_0) = v_0, \tag{2.36}$$

it follows that

$$V(z_i) \geq u(t), \tag{2.37}$$

where $u(t)$ is the minimal solution of (2.36) when $V(z_{i0}) \geq u_0$. Let $\hat{u}(t)$ be the solution of the homogeneous differential equation

$$d\hat{u} = 4mq\hat{u}\left(\frac{a}{2q} - \hat{u}\right)dt, \qquad \hat{u}(t_0) = u_0 \tag{2.38}$$

Then, by using the method of variation of parameters, the solution to the non-homogenous differential equation (2.36) is given by

$$u(t) = \hat{u}(t) - \frac{b}{2}(m-1)\epsilon\int_{t_0}^{t}\Phi(t, s, \hat{u}(s))ds, \tag{2.39}$$

where

$$\Phi(t, t_0, u_0) = \frac{\partial \hat{u}}{\partial u_0}(t, t_0, u_0). \tag{2.40}$$

Using separation of variables, the solution of the homogeneous differential equation is given by

$$\hat{u}(t) = \frac{u_0 a}{2q \left(u_0 + \left(\frac{a}{2q} - u_0 \right) \exp\left[-2am(t - t_0) \right] \right)}, \tag{2.41}$$

and

$$\frac{\partial \hat{u}}{\partial u_0}(t, t_0, u_0) = \frac{a^2 \exp\left[-2am(t - t_0) \right]}{4q^2 \left(u_0 + \left(\frac{a}{2q} - \hat{u}_0 \right) \exp\left[-2am(t - t_0) \right] \right)^2}. \tag{2.42}$$

Therefore, from (2.39),

$$
\begin{aligned}
u(t) &= \frac{u_0 a}{2q \left(u_0 + \left(\frac{a}{2q} - u_0 \right) \exp\left[-2am(t - t_0) \right] \right)} \\
&\quad - \frac{b}{2}(m - 1)\epsilon \int_{t_0}^{t} \frac{a^2 \exp\left[-2am(t - t_0) \right]}{4q^2 \left(u_0 + \left(\frac{a}{2q} - \hat{u}_0 \right) \exp\left[-2am(t - t_0) \right] \right)^2} \\
&= \frac{u_0 a}{2q \left(u_0 + \left(\frac{a}{2q} - u_0 \right) \exp\left[-2am(t - t_0) \right] \right)} \\
&\quad - \frac{b(m - 1)\epsilon a^2}{2\alpha} \left[\frac{1}{2am} \left(1 - \exp\left[-2am(t - t_0) \right] \right) \right. \\
&\quad + 2u_0 \left(\frac{a}{2q} - u_0 \right)(t - t_0) \exp\left[-2am(t - t_0) \right] \\
&\quad \left. - \frac{\left(\frac{a}{2q} - u_0 \right)^2}{2am} \left(\exp\left[-4am(t - t_0) \right] - \exp\left[-2am(t - t_0) \right] \right) \right], \tag{2.43}
\end{aligned}
$$

where

$$\alpha = \left(u_0 a + \left(\frac{a}{2q} - u_0 \right) \exp\left[-2am(t - t_0) \right] \right)^2. \tag{2.44}$$

Let $g(t)$ be the function defined as

$$
\begin{aligned}
g(t) &= \frac{b(m - 1)a^2}{2\alpha} \left[\frac{1}{2am} \left(1 - \exp\left[-2am(t - t_0) \right] \right) \right. \\
&\quad + 2u_0 \left(\frac{a}{2q} - u_0 \right)(t - t_0) \exp\left[-2am(t - t_0) \right] \\
&\quad \left. - \frac{\left(\frac{a}{2q} - u_0 \right)^2}{2am} \left(\exp\left[-4am(t - t_0) \right] - \exp\left[-2am(t - t_0) \right] \right) \right]. \tag{2.45}
\end{aligned}
$$

We note that $g(t)$ is continuous on $[t_0, \infty)$, $g(t_0) = 0$ and

$$\lim_{t \to \infty} g(t) = \frac{b(m-1)}{2amu_0^2}. \tag{2.46}$$

As the limit as $t \to \infty$ of $g(t)$ is finite, for any given $\delta > 0$, there exists a $T > t_0$ such that

$$\left| g(t) - \frac{b(m-1)}{2amu_0^2} \right| < \delta, \qquad \text{for } t > T, \tag{2.47}$$

and so $g(t)$ has an upper bound on (T, ∞), say M_1. Further, as $g(t)$ is continuous on $[t_0, T]$, $g(t)$ has an upper bound on this interval, say M_2. Let $M = \max\{M_1, M_2\}$. As $0 = g(t_0) \leq M$, it must be the case that $M > 0$ and hence (2.39) reduces to

$$u(t) \geq \frac{u_0 a}{2q \left(u_0 + \left(\frac{a}{2q} - u_0 \right) \exp\left[-2am(t - t_0) \right] \right)} - M\epsilon. \tag{2.48}$$

Suppose that it is the case that $\frac{a}{2q} > u_0$ and so the solution $u(t)$ is monotonically increasing as $t \to \infty$. Choosing $0 < \bar{\epsilon} < 1$, such that $\bar{\epsilon} < \frac{a}{2q}$ and $\bar{\epsilon} < \frac{u_0}{M}$, it follows that (2.48) has the lower bound

$$u(t) > 0, \qquad \text{for } t > t_0. \tag{2.49}$$

Thus, $\|z_i(t)\| > 0$ for all $t \geq t_0$ when $\|z_{i0}\| \leq \bar{\epsilon}$ for all $i \in I$. $\qquad \square$

THEOREM 2.2 *Let the hypotheses of Lemmas 2.1 and 2.2 be satisfied. Then*

(i.) the set $C \cup D \cup E = B^c \left(0, \sqrt{\frac{a}{qm}} \right)$ is conditionally invariant relative to E;

(ii.) the set D is either self-invariant or $C \cup D = B^c \left(0, \sqrt{\frac{a}{qm}} \right) \cap B \left(0, \sqrt{am} \right)$ is conditionally invariant relative to D;

(iii.) the set C is either self-invariant or is $C \cup D$ is conditionally invariant relative to C;

(iv.) the set $C \cup D$ is self-invariant;

(v.) the set $B \cup C \cup D = B^c \left(0, \beta_2 \right) \cap B \left(0, \sqrt{am} \right)$ is conditionally invariant relative to B.

Proof. For $z_i \in E$, $i \in I(1, m)$, the hypotheses of Lemmas 2.1 and 2.2 are satisfied. Thus by the application of these Lemmas, we have

$$\rho(t, t_0, \rho_0) \leq V\left(z_i(t, t_0, z_0) \leq r(t, t_0, r_0), \tag{2.50}$$

for $t > t_0$, $z_{i0} \in \bar{B}^c(0, \sqrt{am})$, where $\bar{B}^c(0, \sqrt{am}) = \{x \in \mathbb{R}^n : \|x\| > \sqrt{am}\}$; $\rho(t, t_0, , \rho_0)$ and $r(t, t_0, r_0)$ are the minimal and maximal solutions of the comparison differential equations (2.25) and (2.18) respectively. Moreover, for $z_i \in B^c(0, \sqrt{am})$, with $r_0 = \rho_0 = V(z_{i0}) = \frac{1}{2}\|z_{i0}\|^2$ and $z_i(t_0) = z_{i0}$, the solutions $r(t, t_0, r_0)$ and $\rho(t, t_0, \rho_0)$ are both monotonically decreasing and approaching to $\frac{am}{2}$ and $\frac{a}{2qm}$ respectively. This yields

$$2\rho(t, t_0, \rho_0) \leq \|z_i(t, t_0, z_{i,0})\|^2 \leq 2r(t, t_0, r_0), \tag{2.51}$$

for $t \geq t_0$. From (2.51), $z_{i0} \in E$, and the definitions of self-invariant and conditionally invariant [21], it follows that statement (i) is valid. The proofs of $(ii), (iii)$ and (iv) follow by imitating the argument used in the proof of (i). For $z_{i0} \in D$, we note that $\rho(t, t_0, \rho_0)$ is monotonically decreasing and $r(t, t_0, r_0)$ is monotonically increasing to $\frac{a}{2qm}$ and $\frac{am}{2}$ as $t \rightarrow \infty$ respectively. This together with (2.51) establishes that $z_i(t, t_0, z_{i0}) \in C \cup D$ proving statement (ii). For $z_{i0} \in C \cup D$, $\rho(t, t_0, \rho_0)$ is decreasing and the proof of (iii) and (iv) follows from (i) and (ii). Similarly, the proof for statement (v) also follows by imitating the argument used in (i). For $z_{i0} \in B$, the solutions to the comparison equation (2.25), $\rho(t, t_0, \rho_0)$ is monotonically increasing to $\frac{a}{2qm}$ as $t \rightarrow \infty$ respectively. Therefore, by (2.51), $z_i(t, t_0, z_{i0}) \in B \cup C \cup D$ proving statement (v). $\qquad\square$

2.5.1 Interpretations of Results

Let us expand upon the results of Theorems 2.1 and 2.2. First, let us note that these two theorems provide the qualitative and quantitative requirements on the cultural state parameters to insure that the model is cohesive (Theorem 2.2) and simultaneously does not reach a cultural consensus (Theorem 2.1). We introduce the definition of *cultural bound* to describe the boundary between two cultural sets, dividing the degree of individual versus community level interaction domains of the cultural state. Suppose $z_i \in A$. It can be shown that there exists a neighborhood, $B(0, \sqrt{2\bar{\epsilon}})$ of the center such that for $z_i \in B(0, \sqrt{2\bar{\epsilon}})$, the individual member cultural state is pushed out/repulsed from the cultural state center $\bar{x}$ at some time T depending on $\bar{\epsilon} > 0$. Therefore, if the cultural state of the ith member x_i of the network is such that the relative cultural affinity between x_i and the center, $\bar{x}$, of the network is sufficiently close to zero, then the agent's cultural state is repulsed from the center. That is to say, the membership of the social network will obtain and then maintain a relative cultural affinity between members and the center that is bounded below by a value strictly greater than zero. Once the state of the ith member z_i has moved away from the center, it may be the case that z_i remains in A or the case that the state z_i moves to the cultural set B, at which time the agent's

cultural state behavior will follow that of another category of membership described by the cultural state set B discussed below.

Suppose the initial value, that is the function of the magnitude of the cultural state, ρ_0 of the comparison equation is such that $\rho_0 \leq \frac{a}{2qm}$. Then the solution to the lower comparison equation grows as t grows and approaches asymptotically to the threshold limit $\frac{a}{2qm}$ from below resulting in stronger ties with the community center state, $\bar{x}$. If the initial value of the lower comparison equation is such that $\rho_0 \geq \frac{a}{2qm}$, the solution decays and asymptotically approaches to the threshold limit $\frac{a}{2qm}$ from above. Therefore, if z_i, a member of the transformed social network such that $z_i \in B$, by Theorem 2.2, over time, z_i moves to the cultural bound of the set C. It may also cross the cultural bound or it may be the case that z_i approaches asymptotically to the cultural bound of C. Similarly, if $z_i \in C$, z_i may stay in C, approaching the cultural bounds of sets B and/or D or it may be the case that z_i crosses the cultural bound of D from which point the member will behave as other members of D. However, if $z_i \in C$, eventhough it may approach the cultural bound of B, it will never cross the bound. In terms of a given social network, this implies that members with a distinct enough cultural state from the weighted average of cultural states will retain that distinctiveness of culture. Thus, if the relative cultural affinity between a member x_i and the center of the network is at least $\sqrt{\frac{a}{qm}}$ initially, then the relative cultural affinity will always be at least that value.

Turning to the upper comparison equation, we can consider the behavior of the transformed network members whose initial positions are in the sets D and E. Let r_0 be the initial position of the solution $r(t, t_0, r_0)$ to the upper comparison equation given in Lemma 2.1. If $r_0 < \frac{am}{2}$, then the solution $r(t, t_0, r_0)$ grows and approaches asymptotically to the value $\frac{am}{2}$ from below. If $r_0 > \frac{am}{2}$, the solution decays and approaches asymptotically to the limit from above. Therefore, if $z_i \in D$, z_i can approach and cross the cultural boundary of C (but will remain in $C \cup D$) or z_i may approach but not cross the cultural boundary of E. For $z_i \in E$, z_i may either cross the cultural boundary of D or the members cultural state will approach asymptotically to the cultural boundary of D. Thus, for agents x_i within the network whose initial relative cultural affinity with respect to the center is sufficiently large, as $t \to \infty$, the relative cultural affinity will remain large and the although the agent is attracted back towards the center of the network, the relative affinity is bounded below by $\sqrt{\frac{a}{qm}}$.

Further, it follows from Lemmas 2.1 and 2.2, if all parameters other than the size of the network are held constant, then as the size of the network increases, so also the difference between the upper and lower bounds on the relative cultural affinity between agents and the center of the network increases. Naturally,

increasing the size of the network leads to the concept of the crowding effect. Competition over ideology or cultural traits create a stronger desire for agents to retain more of their individuality within the society or group. Cultural subgroups who have a high degree of separation in terms of their relative cultural affinities are an emergent characteristic of such large scale multi-cultural networks. In the modeling for members whose cultural state is in $\mathbb{R}$ (so one aspect of culture/interest being considered), we see the network dividing into two subgroups with agents converging to states that are symmetric with respect to the time axis. One can think of situations like a large urban environment in which there exists multiple communities, each with a distinct cultural identity. In such a case, members within the individual communities may seek to retain their cultural diversity. Thus, it may be the case that there is a larger relative cultural affinity between members of different communities than the relative cultural affinity between members within the same community.

2.6 A Brief Procedure of Multicultural Dynamic Networks

The detailed development and qualitative and quantitative analysis of a prototype model for a multicultural dynamic network in Sections 2.3, 2.4, and 2.5 sets a stage and provides a complete underlying working insight and understanding regarding the analytical algorithm for analyzing a nonlinear and non-stationary multicultural dynamic network (2.1). First, we note that the presented development and analysis can be directly extended to (a) time-varying coefficient rates in (2.4), and (b) both constant and time-varying coefficient matrices. We now give the procedure for studying the multicultural dynamic network (2.4).

Step 1: **Choose an Energy Function.**

We choose an appropriate energy function $V(t, x)$ [19] such that

(i.) $V(t, x)$ is continuous on $[t_0, \infty) \times \mathbb{R}^n$ into $\mathbb{R}$;

(ii.) For $(t, x) \in [t_0, \infty) \times \mathbb{R}$, $V(t, x)$ is monotonic in x for each t;

(iii.) V is continuously differentiable with respect to t and x.

Step 2: **Aggregation of Cultural State via Energy Function.**

We next find the differential LV of the energy function V along the vector field generated by (2.1) given by

$$LV(t, x(t)) = \frac{\partial}{\partial t} V(t, x(t)) + f(t, x(t)) \frac{\partial}{\partial x} V(t, x(t)). \tag{2.52}$$

Step 3: **Construction of Upper and Lower Differential Inequalities.**

Using differential inequalities, we find the upper estimate $g(x, t)$ and lower estimate $h(t, x)$ of LV such that

$$h(t, V(t, x)) \leq LV(t, x) \leq g(t, V(t, x)), \tag{2.53}$$

where g and h are simpler functions than $f(t, x)$ and g, h are continuous.

Step 4: **Formation of Comparison Theorems.**

Employing the upper and lower estimates in (2.53), we formulate the comparison initial value problems as:

$$du = h(t, u)dt \qquad u(t_0) = u_0 \tag{2.54}$$

and

$$dv = g(t, v)dt \qquad v(t_0) = v_0 \tag{2.55}$$

Let $\rho(t, t_0, v_0)$ and $r(t, t_0, v_0$ be the minimal and maximal solutions of the lower and upper comparison differential equations respectively and $r(t, x)$ [26].

Step 5: **Quantitative and Qualitative Analysis of Comparison Equations.**

We study the behavior and characteristics of the maximal and minimal solutions of the simpler comparison differential equations.

Step 6: **Quantitative and Qualitative Analysis of Original Dynamic System.**

Either by solving or analyzing the simpler differential equations (2.54) and (2.55) and determining the behavior of ρ and r, we are able to analyze the behavior of the solution to (2.1) without knowing an explicit solution. As (2.1) is bounded by the solutions of the comparison equations, we are able to establish quantitative and qualitative properties of (2.1) by considering the quantitative and qualitative properties of ρ and r.

Step 7: **Interpretations.**

Based on the quantitative and qualitative properties of (2.1) found in Step 6, we draw a few interpretations of the characteristics of the multicultural dynamic network.

2.7 Numerical Simulation

In this section, using Euler's type numerical to approximation scheme applied to (2.8), we consider the numerical simulations for the network dynamics governed by (2.8). We consider a network consisting of 50 members with parameters $a = 0.5, q = 0.04, b = 0.41$, and $c = 2$. Further, we note that in this case,

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$\beta_1 = 2.3, \beta_2 \approx 0.5$ and

$$\sqrt{\frac{a}{qm}} = 0.5 \qquad \sqrt{am} = 5. \qquad (2.56)$$

In this example, the conditions for the invariant sets given in Section 2.5 are satisfied. Hence, for z_i such that $0.5 \le |z_i|$, it is the case that after some time, $|z_i| \ge 0.5$; that is, the member does not move towards the center of the network. Further, for z_i such that $2.3 \le |z_i|$, after some time, $0.5 \le |z_i| \le 5$. Figure 2.1a is a plot of the approximate solutions for the full membership of the network. In order to make the dynamics of the network clearer, Figure 2.1b is a plot of the approximate solution of (2.8) for six of the members of the network.

Cultural positions of full network   Cultural position for six members

Figure 2.2.: Euler approximation of the solution to the differential equation given by (2.8) with parameters $a = 0.5, b = 0.41$, and $q = 0.04$.

Next, we consider the network with the same initial values with the parameters $a = 0.25, b = 0.14$ and $q = 0.04$. In this case $\beta_1 \approx 1.61$, $\beta_2 \approx 0.35$ and

$$\sqrt{\frac{a}{qm}} \approx 0.35 \qquad \sqrt{am} \approx 3.54. \qquad (2.57)$$

For z_i such that $|z_i| \ge 0.35$, the member does not move towards the center of the network and for z_i such that $|z_i| \ge 1.61$, after some time, $0.35 \le |z_i| \le 3.54$. Similar to above, we have plotted the approximate solution for the full network in 2.2a and the approximate solution for the same six members as in Figure 2.1b in Figure 2.2b.
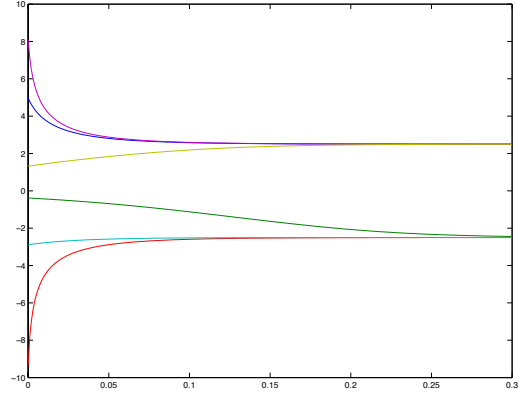
Cultural positions of full network   Cultural position for six members

Figure 2.3.: Euler approximation of the solution to the differential equation given by (2.8) with parameters $a = 0.25, b = 0.14$, and $q = 0.04$.

The last case we considered is the network with the same initial positions with the parameters $a = 0.5, b = .18$ and $q = 0.2$. Thus, with the given parameters, $\beta_1 \approx 0.84, \beta_2 \approx 0.22$, and

$$\sqrt{\frac{a}{qm}} \approx 0.22 \qquad \sqrt{am} = 5. \tag{2.58}$$

For z_i such that $|z_i| \geq 0.22$, the member does not move towards the center of the network and for z_i such that $|z_i| \geq 0.84$, after some time, $0.22 \leq |z_i| \leq 5$. Similar to above we have plotted the approximate solution for the full network in 2.3a and the approximate solution for the same six members in Figure 2.3b.

2.8 Conclusion

We have considered requirements on network parameters for long term qualitative properties of the network. We developed a model and established conditions on the parameters that ensure a balance between cohesion and consensus. Further, we have considered how the initial cultural state of a network member affects the behavior of that member over time. The presented conditions of the system are algebraically simple, easily verifiable and computationally attractive. The developed results provide a tool for planning, decision making, and performance. Furthermore, the presented sufficient conditions are conservative but robust, verifiable, and reliable. From the above conditions, we are able to consider certain dynamic properties of the social networks governed by (2.4).
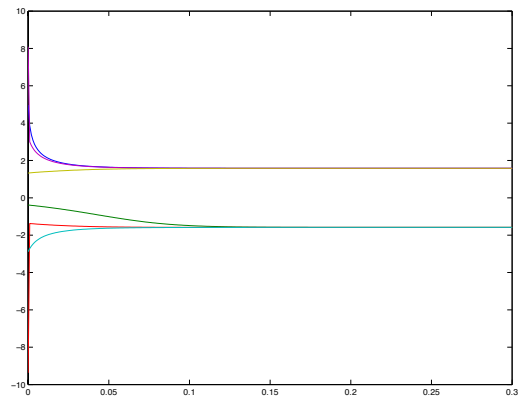
Cultural positions of full network   Cultural position for six members
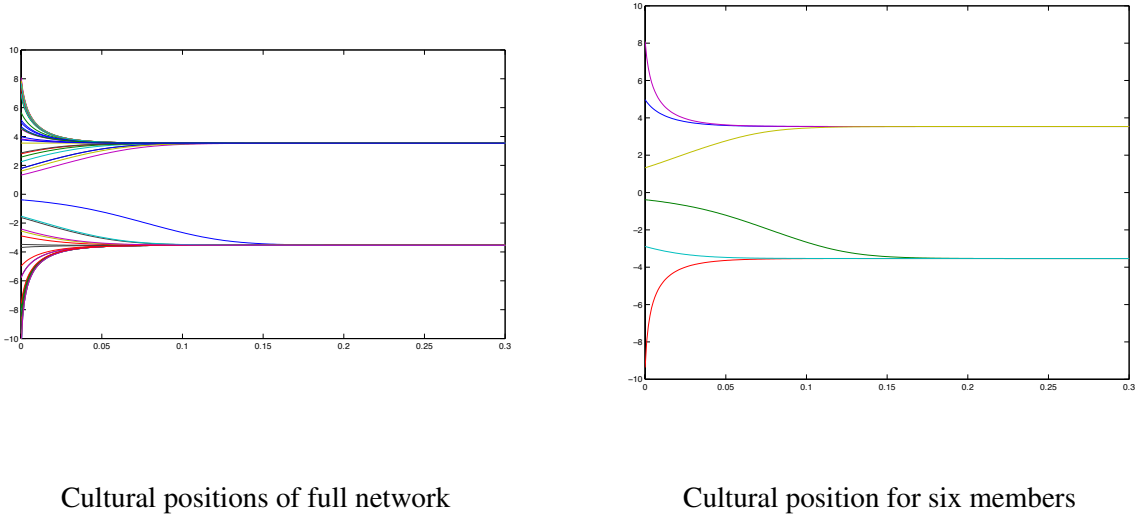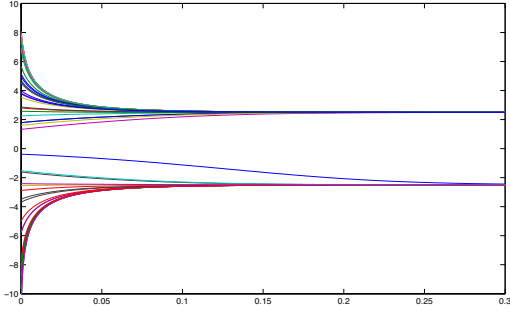
Figure 2.4.: Euler approximation of the solution to the differential equation given by (2.8) with parameters $a = 0.5, b = 0.18$, and $q = 0.2$.

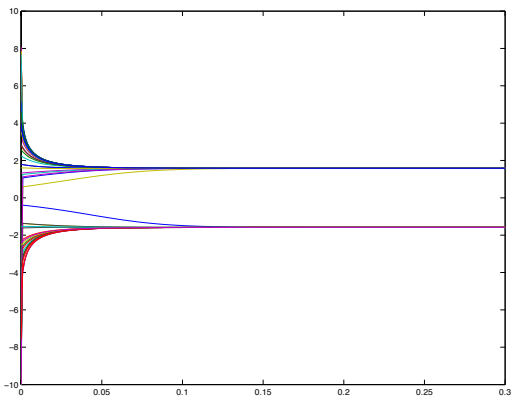# Chapter 3

Stochastic Multicultural Dynamic Network

3.1   Introduction

In this chapter we examine the cohesive properties of a dynamic network of agents/members under the influence of internal and external perturbations [6, 13]. We extend the deterministic modeling of a multicultural dynamic model to a stochastic model.

We often seek to create situations for which people of different backgrounds and beliefs are able to coexists and create a thriving sense of community. In exploring the stochastic dynamics of a multi-cultural network, we are looking to better understand the delicate balance between a culturally diverse cohesive social structure and a social structure for which cohesion does not exists. For when cohesion is lacking in the social network, cooperation may not be as prevalent and we begin to see features such as segregation, violence, economic destabilization and crime within the network.

Uncertainties and destabilizing factors are forces which generate random environmental perturbations. Moreover, through the centuries human societies across the globe have progressively established bilateral relationships and contacts [29]. With recent advancement in electronic technologies in the areas of communications, transportation, advancements in science and technology, and fundamental services, multicultural interactions have been facilitated. Local problems and solution have become global. This has generated a need of cooperation, coordination, co-existence, and understanding at all levels. Naturally, this has generated a complexity and the complexity leads to the generation of random internal and external perturbations.

We seek to model such a situation and better understand the social dynamics of a group seeking to find such a balance between environment and conditions. In particular, we are looking to model a dynamic social network for which there is a balance between consensus and cohesion under stochastic environmental perturbations. We present a prototype of a dynamic model experiencing stochastic perturbations for which we initiate the basic features, components, and analytic tools of such a network. The perturbations reflect the randomness that exist for the model over time as people consider and seek a balance between individuality

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and belonging to a cultural group. The presented example is used to exhibit the quantitative and qualitative properties of a stochastic network. Further, the techniques used are computationally attractive, easy to verify and algebraically simple. In addition, the presented results are in terms of network parameters that characterize the attributes of the network. The byproduct of this provides tools for planning and decision making policies regarding a dynamic network.

In this chapter, we consider a cultural network dynamic in the presence of random environmental perturbations by exploring a cultural state stochastic dynamic model described by a system of Itô-Doob type stochastic differential equations. In Section 3.2, we present the general problem under consideration and the underlining assumptions. We then present an example of such a network in Section 3.3. By creatively developing and applying an appropriate energy function and the comparison method [23], upper and lower estimates on cultural states are established in Sections 3.4 and 3.5, respectively. In Section 3.6, the long-term behavior of the solutions to the comparison equations are examined. Then, in Section 3.7, we explore the study of the cultural state invariant sets in the context of the illustration presented in Section 3.3 and using the long-term behavior of the comparison solutions described in Section 3.6. In additions, using the cohesive property of the network, we examine the dynamic properties of the network. In Section 3.9, we use numerical simulations to model the network and better understand to what extent the estimates in Sections 3.4, 3.5, and 3.6 are feasible. Using the cohesive property of the network, we examine the dynamic properties of the network in Section 3.10.

3.2 Problem Formulation

The network consists of m agents whose position at time t is represented by $x_i(t), i \in I(1, m) = \{1, 2, \ldots, m\}$, with $x_i(t) \in \mathbb{R}^n$. In our model, this vector does not represent a geographical location but rather a cultural state position of the ith member. That is to say, the vector x_i is a numerical representation of the ith member's beliefs or background on certain cultural or ethnic characteristics relevant to the network and question being considered. Further, we assume that $\xi_{ij}, i, j \in I(1, m)$ is a normalized Wiener process with $\xi_{ij} = \xi ji$ and for $j \neq k$, ξ_{ij} and ξ_{ik} are independent. We then consider a cultural state stochastic dynamic model described by a system of Itô-Doob type stochastic differential equation:

$$dx_i = \sum_{j=1}^{m} f(t, x_i - x_j)dt + \sum_{j \neq i}^{m} \sigma(t, x_i - x_j)d\xi_{ij}(t) \qquad x_i(t_0) = x_i^0, \qquad (3.1)$$

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where $i \in I(1, m)$; f and σ are drift and diffusion rate coefficient functions, respectively. Let

$$\bar{x} = \frac{1}{m} \sum_{j=1}^{m} x_i \tag{3.2}$$

be a center of the multicultural state dynamic network (3.1).We will also make the following assumptions:

Assumption H_1: For $t_0 \in [0, \infty)$,

(i.) $x_i(t_0) = x_{i,0}$ is an n-dimensional initial cultural state random vector defined on the complete probability space $(\Omega, F, P) \equiv \Omega$;

(ii.) For $t \geq t_0$, F_t is an increasing family of sub-σ algebras of σ-algebra F, i.e. $F_s \subset F_t$ if $t_0 < s < t$;

(iii.) For $i, j \in I(1, m)$, $\xi_i(t) = (\xi_{i1}, \xi_{i2}, \ldots, \xi_{im})^T$ is a m-dimensional normalized Wiener process of independent increments for $i \in I(1, m)$;

(iv.) $\xi_{ij}(t)$ is F_t-measurable for $t \geq t_0$ and $x_i(t_0)$ is F_{t_0} measurable;

(v.) $x_i(t_0)$ and $\xi_{ij}(t)$ are independent for each $t \geq t_0$ for $i \neq j$, $i, j \in I(1, m)$.

(vi.) f and σ satisfy the growth and Lipschitz condtitions;

(vii.) $\bar{x}$ is a stationary center of (3.1)

It is assumed that the initial value problem (3.1) for the stochastic system of differential equations has a solution process [20].

In the following, we extend the Definitions 2.1, 2.2, and 2.3 to the stochastic multicultural dynamic network.

Definition 3.1 *Let r_1 and r_2 be non-negative random functions for $t \in [t_0, \infty)$ such that $r_1 \leq r_2$. We say that a stochastic multicultural dynamic network is*

(a.) cohesive with probability 1 if for any $N \in F_t$ such that $P(N) = 0$, $N \subset \Omega$ and for all $t \in [t_0, T], T \in \mathbb{R}^+$

$$r_1(t) \leq \|x_i(t) - x_j(t)\| \leq r_2(t), \tag{3.3}$$

for all $i, j \in I(1, m)$ (or cohesive a.s);

(b.) *cohesive in probability if for all* $i, j \in I(1, m)$ *and any* ϵ *such that* $0 < \epsilon < 1$

$$P\left(\{\omega : \|x_i(t) - x_j(t)\| < r_1(t) \, or \, \|x_i(t) - x_j(t)\| > r_2(t)\}\right) < \epsilon, \tag{3.4}$$

for all $i \in I(1, m)$;

(c.) *cohesive in pth mean if for all* $i, j \in I(1, m)$

$$E\left[r_1(t)\right] \le E\left[\|x_i(t) - x_j(t)\|^p\right] \le E\left[r_2(t)\right], \tag{3.5}$$

for all $i \in I(1, m)$.

Definition 3.2 *We say that a stochastic multicultural dynamic network*

(a.) *reaches a consensus with probability 1 if there exists* $N \subset F$ *such that* $P(N) = 0$ *and for all* $\omega \in \Omega \setminus N$,

$$\lim_{t \to \infty} \|x_i(t) - \bar{x}\| = 0, \tag{3.6}$$

for all $i, j \in I(1, m)$ *(or consensus a.s.)*;

(b.) *reaches a consensus in probability if for* $\epsilon > 0$,

$$\lim_{\omega \to \infty} P\left(\{\omega : \|x_i(t) - \bar{x}\| > \epsilon\}\right) = 0, \tag{3.7}$$

for all $i \in I(1, m)$;

(c.) *reaches a consensus in the pth mean if*

$$\lim_{t \to \infty} E\left[\|x_i(t) - \bar{x}\|^p\right] = 0, \tag{3.8}$$

for all $i \in I(1, m)$.

Definition 3.3 *Let* x_i *and* x_j *be cultural state random vectors for* $i, j \in I(1, m)$. *We define the relative cultural state affinity with probability 1 sense by*

$$\|x_i(t) - x_j(t)\|; \tag{3.9}$$

We note that the existence of the relative cultural state affinity in the a.s. sense is trivial as $\|\bullet\|$ is a Borel function. In general a composite function of a random variable need not to be a random variable.

3.3 Prototype Dynamic Model

Let us define a prototype multicultural network dynamic model under the stochastic environmental perturbations described by the Itô-Doob type stochastic system of differential equations

$$
\begin{cases}
dx_i &= \Bigg[a \sum_{j=1}^{m} x_{ij} - q\|x_i - \bar{x}\|^2 \sum_{j=1}^{m} x_{ij} \\
&\quad + b\sin\|x_i - \bar{x}\| \sum_{j=1}^{m} x_{ij} \exp\left[-\frac{\|x_{ij}\|^2}{c}\right] \Bigg] dt \\
&\quad + \beta\sin\|x_i - \bar{x}\| \sum_{j=1}^{m} x_{ij} \exp\left[-\frac{\|x_{ij}\|^2}{c}\right] d\xi_{ij}, \\
x_i(t_0) &= x_i^0,
\end{cases}
\tag{3.10}
$$

where a, q, b, c and β are positive real numbers; and ξ_{ij}'s are Weiner processes that are mutually independent for $i \neq j$, for $i, j \in I(1, m)$, and

$$
x_{ij} = x_i - x_j.
\tag{3.11}
$$

Here, $\bar{x}$ is the center of the multicultural dynamic system (3.10) defined by:

$$
\bar{x} = \frac{1}{m} \sum_{j=1}^{m} x_j,
\tag{3.12}
$$

and note that by substituting $x_i = \bar{x}$ into (3.10),

$$
\begin{aligned}
d\bar{x} &= \Bigg[a \sum_{j=1}^{m} (\bar{x} - x_j) - q\|\bar{x} - \bar{x}\|^2 \sum_{j=1}^{m} (\bar{x} - x_j) \\
&\quad + b\sin\|\bar{x} - \bar{x}\| \sum_{j=1}^{m} (\bar{x} - x_j) \exp\left[-\frac{\|\bar{x} - x_j\|^2}{c}\right] \Bigg] dt \\
&\quad + \beta\sin\|\bar{x} - \bar{x}\| \sum_{j=1}^{m} (\bar{x} - x_j) \exp\left[-\frac{\|\bar{x} - x_j\|^2}{c}\right] d\xi_{\bar{x}j}, \\
&= am\bar{x} - a \sum_{j=1}^{m} x_j \\
&= am\bar{x} - am\bar{x} \\
&= 0,
\end{aligned}
\tag{3.13}
$$

and thus $\bar{x}$ defined in (3.12) is a stationary center of the multicultural dynamic network. We define the transformation $z_i = x_i - \bar{x}$ and observe that $x_{ij} = z_i - z_j = z_{ij}$. Then the transformed network dynamic

model corresponding to (3.10) is reduced to:

$$
\begin{cases}
dz_i & = \left[amz_i - q\|z_i\|^2 mz_i + b\sin\|z_i\| \sum_{j=1}^{m} z_{ij}\exp\left[-\frac{\|z_{ij}\|^2}{c}\right] \right] dt \\
& \quad + \beta\sin\|z_i\| \sum_{j=1}^{m} z_{ij}\exp\left[-\frac{\|z_{ij}\|^2}{c}\right] d\xi_{ij}, \\
z_i(t_0) & = z_{i0}.
\end{cases}
\tag{3.14}
$$

The center $\bar{x}$ of the multicultural dynamic model (3.10) is reduced to the center zero in (3.14). Here a, b, c and q are as described and characterized in the deterministic network dynamic model (2.4). The parameter β characterizes the random environmental perturbations. It exhibits both attractive and repulsive forces that are centered at the center of the network. The magnitude of the repulsive force is described by $am\|z_i\|$. Repulsive forces are attributes that create some desire for individuals to leave or be less involved in the group or to preserve some personal identity from one other with their individual magnitude of inner repulsive force. A desire to retain a sense of individuality, economic or emotional cost, interpersonal conflict within the group, or disagreement with parts of the overall philosophies of the group are forces that may be considered as repulsive forces. The magnitude of the long range deterministic attractive force is characterized by $b\left\|\sum z_{ij}\exp\left[-\frac{\|z_{ij}\|^2}{c}\right]\right\|$. Attractive influences can be thought of as attributes that bring people to active membership within the group. Social acceptance, gaining social status, economic opportunity, career growth, common purpose and membership, personal development, and a sense of mutual respect, trust and understanding are examples of attractive influences within a social cultural network. Further, $\sin\|z_i\|$ is the sine-cyclical influence of the ith member's relative distance to the center of the network. The stochastic term represents the environmental influence due to long-range attractive forces. In particular, in the case of a multi-cultural network, the noise captures the uncertainty generated due to the membership interactions and deliberations under the influence of the long-range cultural forces.

In order to study the multicultural dynamics (3.14), we use Lyapunov's Second Method in conjunction with the comparison method [23]. These methods are computationally attractive and provide a means of better understanding the movement and behavior of the cultural state memberships of the network. By utilizing these methods, we are able to establish conditions for which we have both upper and lower estimates on the members cultural state positions. We assume that all the inequalities presented below are with probability 1.

3.4 Upper Comparison Equation

Using Lyapunov's Second Method and differential inequalities, we first seek a function $r(t, t_0, u_0)$ such that

$$\|z_i(t)\| \le r(t, t_0, r_0). \tag{3.15}$$

From Definition 3.1, relation (3.15) generate concepts of a upper-cohesive cultural network in the almost sure, probability, and pth moment sense.

To this end, let us choose an energy function V as:

$$V(z_i) = \|z_i\| = \left(z_i^T z_i\right)^{\frac{1}{2}}, \tag{3.16}$$

and let us denote

$$\phi_1(z_i) = amz_i - q\|z_i\|^2 mz_i + b\sin\|z_i\| \sum_{j=1}^{m} z_{ij} \exp\left[-\frac{\|z_{ij}\|^2}{c}\right], \tag{3.17}$$

and

$$\phi_2(z_{ij}) = \beta \sin\|z_i\| z_{ij} \exp\left[-\frac{\|z_{ij}\|^2}{c}\right]. \tag{3.18}$$

Then applying Itô-Doob differential formula [20]to (3.16), the differential of V in the direction of the vector field represented by (3.14) is

$$
\begin{aligned}
dV &= \frac{z_i^T dz_i}{\|z_i\|} + \frac{1}{2}\left[\frac{dz_i^T dz_i}{\|z_i\|} - \frac{\left(z_i^T dz_i\right)^2}{\|z_i\|^3}\right] \\
&= \frac{z_i^T\left(\phi_1(z_i)dt + \sum_{j=1}^{m}\phi_2(z_{ij})d\xi_{ij}\right)}{\|z_i\|} \\
&\quad + \frac{\left(\phi_1^T(z_i)dt + \sum_{j=1}^{m}\phi_2^T(z_{ij})d\xi_{ij}\right)\left(\phi_1(z_i)dt + \sum_{j=1}^{m}\phi_2(z_{ij})d\xi_{ij}\right)}{2\|z_i\|} \\
&\quad - \frac{\left(z_i^T\left(\phi_1(z_i)dt + \sum_{j=1}^{m}\phi_2(z_{ij})d\xi_{ij}\right)\right)^2}{2\|z_i\|^3} \\
&= \frac{z_i^T \sum_{j=1}^{m}\phi_2(z_{ij})d\xi_{ij}}{\|z_i\|} + LV(z_i)dt,
\end{aligned} \tag{3.19}
$$

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where

$$LV(z_i) = \frac{z_i^T \phi_1(z_i)dt}{\|z_i\|} + \frac{\sum_{j=1}^m \phi_2^T(z_{ij})\phi_2(z_{ij})}{2\|z_i\|} - \frac{\left(z_i^T \sum_{j=1}^m \phi_2(z_{ij})\right)^2}{2\|z_i\|^3} \qquad (3.20)$$

$$= \left[am\|z_i\| - qm\|z_i\|^3 + \frac{b\sin\|z_i\|}{\|z_i\|} \sum_{j=1}^m z_i^T z_{ij} \exp\left[-\frac{\|z_{ij}\|^2}{c}\right] \right.$$

$$+ \frac{\beta^2 \sin^2\|z_i\| \sum_{j=1}^m z_{ij}^T z_{ij} \exp\left[-\frac{2\|z_{ij}\|^2}{c}\right]}{2\|z_i\|}$$

$$\left. - \frac{\beta^2 \sin^2\|z_i\| \sum_{j=1}^m (z_i^T z_{ij})^2 \exp\left[-\frac{2\|z_i\|^2}{c}\right]}{2\|z_i\|^3} \right] dt.$$

We seek constraints on the parameters a, b, c, q and β for which we have an upper estimate on the first moment of $V(z_i)$. Thus, let us consider an upper estimate on LV defined in (3.20). We first note that the function

$$f(r) = r \exp\left[-\frac{r^2}{c}\right] \qquad (3.21)$$

has a maximum value of $\sqrt{\frac{c}{2}} \exp\left[-\frac{1}{2}\right]$ when $r = \sqrt{\frac{c}{2}}$. Further the function

$$g(r) = r^2 \left(\exp\left[-\frac{r^2}{c}\right]\right)^2 = r^2 \exp\left[-\frac{2r^2}{c}\right] \qquad (3.22)$$

that has a maximum value of $\frac{c}{2} \exp\left[-1\right]$ when $r = \sqrt{\frac{c}{2}}$. Therefore, from (3.20),

$$LV \le am\|z_i\| - qm\|z_i\|^3 + \frac{b\sin\|z_i\|}{\|z_i\|} \sum_{j \ne i}^m z_i^T z_{ij} \exp\left[-\frac{\|z_{ij}\|^2}{c}\right] \qquad (3.23)$$

$$+ \frac{\beta^2 \sin^2\|z_i\|}{2\|z_i\|} \sum_{j \ne i}^m \|z_{ij}\|^2 \exp\left[-\frac{\|z_{ij}\|^2}{c}\right]$$

$$\le am\|z_i\| - qm\|z_i\|^3 + b \sum_{j \ne i}^m \|z_i\|\|z_{ij}\| \exp\left[-\frac{\|z_{ij}\|^2}{c}\right]$$

$$+ \frac{\beta^2 \|z_i\| \sin^2\|z_i\|}{2\|z_i\|^2} \sum_{j \ne i}^m \|z_{ij}\|^2 \exp\left[-\frac{\|z_{ij}\|^2}{c}\right]$$

$$\le am\|z_i\| - qm\|z_i\|^3 + b\|z_i\| \sum_{j \ne i}^m \|z_{ij}\| \exp\left[-\frac{\|z_{ij}\|^2}{c}\right]$$

$$+ \frac{\beta^2 \|z_i\|}{2} \sum_{j \ne i}^m \|z_{ij}\|^2 \exp\left[-\frac{\|z_{ij}\|^2}{c}\right]$$

$$\le am\|z_i\| - qm\|z_i\|^3 + b\|z_i\| (m-1) \sqrt{\frac{c}{2}} \exp\left[-\frac{1}{2}\right]$$

$$+\frac{\beta^2\|z_i\|\,(m-1)\,c\exp[-1]}{4}$$

$$=\|z_i\|\left(am+b\,(m-1)\,\sqrt{\frac{c}{2}}\exp\left[-\frac{1}{2}\right]+\frac{\beta^2\,(m-1)\,c\exp[-1]}{4}\right)$$

$$-qm\|z_i\|^3$$

$$=qm\|z_i\|\left(\frac{a}{q}+\frac{4b\,(m-1)\,\sqrt{\frac{c}{2}}\exp\left[-\frac{1}{2}\right]+\beta^2\,(m-1)\,c\exp[-1]}{4qm}-\|z_i\|^2\right)$$

$$=qmV\left(\eta^2-V^2\right)$$

$$=qmV\left(\eta-V\right)\left(\eta+V\right),$$

where

$$\eta=\left(\frac{a}{q}+\frac{4b\,(m-1)\,\sqrt{\frac{c}{2}}\exp\left[-\frac{1}{2}\right]+\beta^2\,(m-1)\,c\exp[-1]}{4qm}\right)^{\frac{1}{2}}. \tag{3.24}$$

In the following, we present a result that will be used subsequently.

LEMMA 3.1 *Let V be the energy function defined in* (3.16) *and z_i be a solution of the initial value problem defined in* (3.14). *Then, for each $i\in I(1,m)$,*

$$E\left[V(z_i(t+\Delta t))-V(z_i(t))|F_t\right]=LV(z_i(t))\Delta t, \tag{3.25}$$

where E stands for the conditional expected value for given F_t and Δt, a positive increment to t.

Proof. Let $z_i(t,t_0,z_i(t_0))$ be the solution process of (3.14). Let F_t be an increasing family of sub-σ algebras as defined in Assumption H_1 and set

$$m(t)=E\left[V\left(z_i(t)\right)|F_t\right]=V\left(z_i(t)\right), \tag{3.26}$$

where the last equality holds as $z_i(t)$ is F_t measurable. Similarly, we set

$$m(t+\Delta t)=E\left[V\left(z_i(t+\Delta t)\right)|F_t\right], \tag{3.27}$$

for all $\Delta t>0$. We consider

$$m(t+\Delta t)-m(t)=E\left[V\left(z_i(t+\Delta t)-V\left(z_(t)\right)\right)|F_t\right] \tag{3.28}$$

$$=E\left[\frac{\partial V}{\partial z}\left(z_i(t)\right)\Delta z_i(t)\right.$$

$$\left.+\frac{1}{2}tr\left(\frac{\partial^2 V}{\partial z^2}\left(\Delta z_i(t)\right)\left(\Delta z_i(t)\right)^T\right)|F_t\right]$$

$$=E\left[dV\left(z_i(t)\right)|F_t\right].$$

This together with the F_t measurability of $z_i(t)$, (3.19), and (3.20) yields

$$m(t + \Delta t) - m(t) = E\left[LV(z_i(t))\Delta t | F_t\right] \tag{3.29}$$

$$= LV(z_i(t))\Delta t,$$

as $z_i(t)$ is F_t measurable. We note that for small Δt, we have

$$dm(t) = LV(z_i(t))\,dt. \tag{3.30}$$

From (3.26) and (3.27), (3.30) reduces to (3.25). This completes the proof of the Lemma. $\quad\square$

From inequality (3.23), using the comparison method [23] and Lemma 3.1, we establish the following lemma. The presented result establishes not only an upper bound but also the upper cohesive property almost surely. Hereafter, all inequalities and equalities are assumed to be valid with probability one.

LEMMA 3.2 *Let V be the energy function defined in* (3.16) *and z_i be a solution of the initial value problem defined in* (3.14)*. Further, let*

$$du = [qmu(\eta - u)(\eta + u)]\,dt, \qquad r(t_0) = u_0, \tag{3.31}$$

where η is as defined in (3.24)*. For each $V(z_i)$, $i \in I(1, m)$ satisfying the differential inequality* (3.23) *and $V(z_i(t_0)) \leq u_0$, it follows that the multicultural dynamic network* (3.14) *is upper cohesive with probability 1 and*

$$V(z_i(t)) \leq r(t, t_0, u_0), \tag{3.32}$$

where $r(t)$ is the maximal solution of the scalar non-linear deterministic comparison differential equation random initial value problem (3.31)*.*

Proof. From (3.23), Lemma 3.1, and following the standard argument used in proofs of comparison theorems [23] in the frame-work of the Lyapunov method, with probability 1, it follows that

$$V(z_i(t)) \leq r(t, t_0, u_0), \tag{3.33}$$

whenever $V(z_i(t_0)) \leq u_0$. We note that the maximal solution of (3.31) is an upper bound. Hence, the network is upper cohesive almost surely. $\quad\square$

Remark 3.1: If the solution processes of (3.14) and (3.31) have a first moment, then the solution process of (3.14) is upper 1st moment cohesive. Furthermore, under the current inequality, it is indeed upper cohesive in the sense of probability.

3.5 Lower Comparison Equation

Using Lyapunov's Second Method and differential inequalities, we next seek a function $\rho(t, t_0, u_0)$ such that

$$\|z_i(t)\| \geq \rho(t, t_0, \rho_0). \tag{3.34}$$

Again, from Definition 3.1, relation (3.34) initiates a notion of a lower cohesive cultural dynamic network in the almost sure sense.

Using the energy function defined in (3.16), (3.19) and relation (3.20), it follows that

$$LV \geq am\|z_i\| - qm\|z_i\|^3 - b\sum_{j\neq i}^{m}\|z_i\|\|z_{ij}\| \exp\left[-\frac{\|z_{ij}\|^2}{c}\right] \tag{3.35}$$

$$-\frac{\beta^2}{2\|z_i\|}\sum_{j\neq i}^{m}\|z_i\|^2\|z_{ij}\|^2 \exp\left[-\frac{2\|z_{ij}\|^2}{c}\right]$$

$$= am\|z_i\| - qm\|z_i\|^3 - b\|z_i\|\sum_{j\neq i}^{m}\|z_{ij}\| \exp\left[-\frac{\|z_{ij}\|^2}{c}\right]$$

$$-\frac{\beta^2\|z_i\|}{2}\sum_{j\neq i}^{m}\|z_{ij}\|^2 \exp\left[-\frac{2\|z_{ij}\|^2}{c}\right]$$

$$\geq amV - qmV^3 - V(m-1)b\sqrt{\frac{c}{2}} \exp\left[-\frac{1}{2}\right]$$

$$-\frac{\beta^2(m-1)c\exp[-1]}{4}V$$

$$= qmV\left(\frac{a}{q} - \frac{4(m-1)b\sqrt{\frac{c}{2}}\exp\left[-\frac{1}{2}\right] + \beta^2 c(m-1)\exp[-1]}{4qm} - V^2\right).$$

Assumption H_2: Assume there exists a positive number α such that

$$\alpha \leq \left(\frac{a}{q} - \frac{4(m-1)b\sqrt{\frac{c}{2}}\exp\left[-\frac{1}{2}\right] + \beta^2(m-1)c\exp[-1]}{4qm}\right)^{\frac{1}{2}}. \tag{3.36}$$

From (3.35), and noticing the fact that assumption H_2 implies

$$\frac{a}{q} > \frac{4(m-1)b\sqrt{\frac{c}{2}}\exp\left[-\frac{1}{2}\right] + \beta^2(m-1)\exp[-1]}{4qm}, \tag{3.37}$$

it follows that

$$LV \geq qmV(\alpha - V)(\alpha + V). \tag{3.38}$$

From inequality (3.38) in conjunction with the comparison method [23] and Lemma 3.1, we establish the following lemma. The presented result provides the lower estimate which in turn establishes the stochastic lower cohesive property of (3.14).

LEMMA 3.3 *Let V be the energy function defined in (3.16) and z_i be a solution of the initial value problem defined in (3.14). Further, let*

$$du = qmu \, (\alpha - u) \, (\alpha + u) \, dt, \qquad u(t_0) = u_0, \tag{3.39}$$

where α is as defined in (3.36). For each $V(z_i)$, $i \in I(1, m)$ satisfying the differential inequality (3.38) and $V(z_i(t_0)) \geq u_0$, it follows that is lower cohesive with probability 1 and

$$V(z_i(t)) \geq \rho(t, t_0, u_0), \tag{3.40}$$

where $\rho(t)$ is the minimal solution of the deterministic non-linear comparison random initial value problem (3.39).

Proof. From inequality (3.38) and Lemma 3.1, imitating the outline of the proof of Lemma 3.2, it follows that

$$V(z_i(t)) \geq \rho(t, t_0, u_0) \tag{3.41}$$

provided that $V(z_i(t_0)) \geq u_0$. As the minimal solution of (3.39) is a lower bound, the network is lower cohesive almost surely. Moreover, a remark similar to Remark 3.1 establishes the stochastic mean and probability of (3.14) $\qquad \square$

We note that comparison differential equations (3.31) and (3.39) each have a unique solution process. Therefore the maximal and minimal solutions of (3.31) and (3.39) are indeed the unique solution of the respective random initial value problems.

3.6 Long-term Behavior of the Comparison Differential Equation

To appreciate the role and scope of Lemmas 3.2 and 3.3, we seek to better understand the long-term behavior of the network. For this purpose, we find the closed form solutions of the comparison random initial value problems (3.31) and (3.39). Moreover, we analyze the qualitative properties of the solutions to the comparison equations. Using the comparison method [23], we are able to establish, computationally, the overall long-term behavior of both individual member cultural dynamic states within the network as well as multicultural network state as a whole.

Let us first begin with the solution of the comparison differential equation

$$du = qmu \, (\nu - u) \, (\nu + u) \, dt, \qquad u(t_0) = u_0, \tag{3.42}$$

where ν is a positive real number. Following the method of finding the closed form solution process of the initial value problem [20], the solution of (3.42) is represented by

$$u(t, t_0, u_0) = \frac{u_0 \nu}{\sqrt{u_0^2 + (\nu^2 - u_0^2) \exp\left[-2\nu^2 qm(t - t_0)\right]}}. \tag{3.43}$$

We note that both ν and u_0 in (3.43) are positive. If $\nu > u_0$, then $\nu^2 > u_0^2$, and hence the term under the radical is positive. Suppose it is the case that $\nu < u_0$. Then we note that

$$0 < \exp\left[-2\nu^2 qm(t - t_0)\right] < 1, \tag{3.44}$$

for $t > t_0$. From (3.43) and (3.44), it follows that

$$u_0^2 + (\nu^2 - u_0^2) \exp\left[-2\nu^2 qm(t - t_0)\right] = u_0^2 - u_0^2 \exp\left[-2\nu^2 qm(t - t_0)\right] \tag{3.45}$$

$$+ \nu^2 \exp\left[-2\nu^2 qm(t - t_0)\right]$$

$$= u_0^2 \left(1 - \exp\left[-2\nu^2 qm(t - t_0)\right]\right)$$

$$+ \nu^2 \exp\left[-2\nu^2 qm(t - t_0)\right]$$

$$> 0.$$

Hence, the term under the radical in (3.43) is positive in both cases: $\nu > u_0$ and $\nu < u_0$. Thus, under either of the conditions, $\nu > u_0$ or $\nu < u_0$,

$$\lim_{t \to \infty} u(t, t_0, u_0) = \lim_{t \to \infty} \frac{u_0 \nu}{\sqrt{u_0^2 + (\nu^2 - u_0^2) \exp\left[-2\nu^2 qm(t - t_0)\right]}} \tag{3.46}$$

$$= \nu \tag{3.47}$$

From (3.24), Lemma 3.2 and (3.46), for $\nu = \eta$ it follows that the limit of the upper comparison solution $r(t)$ as t grows large is

$$\eta = \left(\frac{a}{q} + \frac{4b(m - 1)\sqrt{\frac{1}{2}}\exp\left[-\frac{1}{2}\right] + \beta^2(m - 1)c\exp\left[-1\right]}{4qm}\right)^{\frac{1}{2}}. \tag{3.48}$$

Similarly, from (3.36), Lemma 3.3 and (3.46), for $\nu = \alpha$, the limit of the lower comparison solution $\rho(t)$ as t grows large is α, where,

$$\alpha \leq \left(\frac{a}{q} - \frac{4(m - 1)b\sqrt{\frac{c}{2}}\exp\left[-\frac{1}{2}\right] + \beta^2(m - 1)c\exp\left[-1\right]}{4qm}\right)^{\frac{1}{2}}. \tag{3.49}$$

From the solution of the comparison equations in conjunction with Lemmas 3.2 and 3.3, we establish the following theorem.

THEOREM 3.1 *Let the hypotheses of Lemmas 3.2 and 3.3 be satisfied. Then the network is cohesive in the almost surely.*

Proof. From Lemmas 3.2 and 3.3,

$$\rho(t, t_0, \rho_0) \leq V(z_i(t)) \leq r(t, t_0, r_0) \tag{3.50}$$

with probability 1. Moreover, as the solution to the upper comparison equation is bounded above by η and the solution to the lower comparison equation is bounded below by α, the network is cohesive almost surely. In addition, under the conditions in Remark 3.1, the solution process of (3.14) is cohesive in probability and mean sense. $\square$

In the following section, we provide various characterizations of cultural state dynamics. This is achieved by the nature of the initial cultural state parameters and the behavior of the upper and lower comparison cultural state dynamic processes.

3.7 Invariant Sets and Interpretations

In this section, we analyze various types of invariant states of the multicultural dynamic network. This is achieved by using the behavior of the solutions to both the upper and lower comparison equations. Let us denote

$$r_2 = \left(\frac{a}{q} - \frac{4(m-1)b\sqrt{\frac{c}{2}} \exp\left[-\frac{1}{2}\right] + \beta^2 (m-1)c \exp\left[-1\right]}{4qm} \right)^{\frac{1}{2}} \tag{3.51}$$

and

$$r_1 = \left(\frac{a}{q} + \frac{4b(m-1)\sqrt{\frac{c}{2}} \exp\left[-\frac{1}{2}\right] + \beta^2 (m-1)c \exp\left[-1\right]}{4qm} \right)^{\frac{1}{2}} \tag{3.52}$$

We note that the parameters $a, b, q, c,$ and β imply the following relation:

$$r_2 < r_1. \tag{3.53}$$

Further, let us define the following sets:

$$\begin{cases} A & = B\left(0, r_2\right) \\ B & = B^c\left(0, r_2\right) \cap B\left(0, r_1\right) \\ C & = B^c\left(0, r_1\right) \end{cases} \tag{3.54}$$

Under the obvious relation (3.53), we develop and establish the following result.

THEOREM 3.2 *Let the hypotheses of Lemmas 3.2 and 3.3 be satisfied. Then almost surely,*

(i) the set $A \cup B$ is conditionally invariant relative to A;

(ii) the set B is self-invariant;

(iii) the set $B \cup C$ is conditionally invariant relative to C.

Proof. For $z_i \in C$, $i \in I(1, m)$, the hypotheses of Lemmas 3.2 and 3.3 are satisfied. Thus by the application of these Lemmas, we have

$$\rho(t, t_0, \rho_0) \le V(z_i(t, t_0, z_0) \le r(t, t_0, r_0), \tag{3.55}$$

for $t > t_0$, $z_{i0} \in \bar{B}^c(0, r_1)$, and $\rho(t, t_0, , \rho_0)$ and $r(t, t_0, r_0)$ are the minimal and maximal solutions of the comparison differential equations (3.39) and (3.31) respectively. Moreover, for $z_i \in B^c(0, r_1)$, with $r_0 = \rho_0 = V(z_{i0}) = \|z_{i0}\|$, the solutions $r(t, t_0, r_0)$ and $\rho(t, t_0, \rho_0)$ are both monotonically decreasing and approaching to r_1 and r_2 respectively. Hence, we have

$$\rho(t, t_0, \rho_0) \le \|z_i(t, t_0, z_{i,0})\| \le r(t, t_0, r_0), \tag{3.56}$$

for $t \ge t_0$. From the definitions of self-invariant and conditionally invariant sets [21], it follows that statement (iii) is valid. The proofs of (i) and (ii) follow by imitating the argument used in the proof of (iii). For $z_{i0} \in B$, we note that $\rho(t, t_0, \rho_0)$ is monotonically decreasing and $r(t, t_0, r_0)$ is monotonically increasing to r_2 and r_1 as $t \to \infty$, respectively. This establishes that $z_i(t, t_0, z_{i0}) \in B$ proving statement (ii). For $z_{i0} \in A$, the solutions to the comparison equation (3.39), $\rho(t, t_0, \rho_0)$ is monotonically increasing to r_2 as $t \to \infty$. Therefore $z_i(t, t_0, z_{i0}) \in A \cup B$ proving statement (i). $\qquad\square$

Let us examine the results of Theorems 3.2. First, we note that this theorem provides sufficient conditions for the qualitative and quantitative behavior of the cultural state dynamics. In particular, the model is cohesive and simultaneously, it does not reach a cultural consensus.

We introduce the definition of *cultural threshold bound* to describe the boundary between two cultural state sets. It is based on the degree of individual versus community level interaction domains of the cultural states. Suppose $z_i \in A$. It is the case that the individual member cultural state is pushed out/repulsed from the cultural state center $\bar{x}$ at some time $T \geq t_0$. That is to say, the membership of the social network will support and then maintain a relative cultural affinity between members and the cultural center that is bounded below by the quantity r_2. Once the state of the ith member z_i has moved away from the center, it is the case that the state z_i moves to the cultural state set B, at which time the agent's cultural state behavior will follow that of another category of membership described by the cultural state set B discussed below.

Suppose that the ith member initial cultural state z_i of the transformed social network is such that $z_i \in B$. Then by Theorem 3.2, over time, z_i may stay in B, approaching the cultural threshold bounds of sets C and/or A. However, if $z_i \in B$, even though it may approach the cultural bound of A and/or C, it will never cross either of the boundaries. In terms of a given social network, this implies that members with a distinct enough cultural states from the weighted average of cultural states will retain that distinctiveness of culture while maintaining a certain level of closeness to the average cultural state. Thus, if the relative cultural affinity between a member x_i and the center of the network is at least r_2 and less than r_1, initially, then the relative cultural state affinity will always be at least the quantity r_2 but no more than the value r_1.

If it is the case that z_i is a member of the transformed network such that $z_i \in C$. By Theorem 3.2, z_i may either cross the cultural boundary of B or the members cultural state will approach asymptotically to the cultural state network boundary of B. Thus, for agents x_i within the network whose initial relative cultural state affinity with respect to the cultural state center is sufficiently large, as $t \rightarrow \infty$, the relative cultural affinity will remain large and although the agent is attracted back towards the center of the network, the relative cultural state affinity is bounded below by r_2.

3.8 A Brief Procedure of Stochastic Multicultural Dynamic Networks

The detailed development and qualitative and quantitative analysis of a prototype model for a multicultural dynamic network in Sections 3.3, 3.4, 3.5, and 3.6, provides a framework regarding the analytical algorithm for analyzing a nonlinear and non-stationary stochastic multicultural dynamic network (3.1). Similarly to the remarks from Section 2.6, we note that the presented development and analysis can be directly extended to (a) time-varying coefficient rates in (3.10), (b) both constant and time-varying coefficient matrices, and (c) the drift and diffusion rate functions may also be functions of a right continuous Markov chain with a

finite number of states. We now give the procedure for studying the multicultural dynamic network (3.10),

Step 1: **Choose an Energy Function.**

First, we choose an appropriate energy function $V(t, x)$ [20] such that

(i.) $V(t, x)$ is continuous on $[t_0, \infty) \times \mathbb{R}^n$ into $\mathbb{R}$;

(ii.) For $(t, x) \in [t_0, \infty) \times \mathbb{R}$, $V(t, x)$ is monotonic in x for each t;

(iii.) V is continuously differentiable with respect to t and x.

Step 2: **Aggregation of Cultural State via Energy Function.**

We next find the differential of V along the vector field generated by (3.1) given by

$$dV(t, x(t)) = LV(t, x(t))dt + \sigma(t, x)\frac{\partial}{\partial x}V(t, x(t))d\xi(t), \tag{3.57}$$

where

$$LV(t, x(t)) = \frac{\partial}{\partial t}V(t, x(t)) + f(t, x(t))\frac{\partial}{\partial x}V(t, x(t)) + \frac{1}{2}\sigma^2(t, x)\frac{\partial^2}{\partial x^2}V(t, x(t)). \tag{3.58}$$

Step 3: **Construction of Upper and Lower Differential Inequalities.**

Using differential inequalities, we find the upper estimate $g(x, t)$ and lower estimate $h(t, x)$ of LV such that

$$h(t, V(t, x)) \leq LV(t, x) \leq g(t, V(t, x)), \tag{3.59}$$

where g and h are simpler functions than $f(t, x)$ and g, h are continuous.

Step 4: **Formation of Comparison Theorems.**

From the upper and lower estimates in (3.59), we develop the comparison random differential initial value problems:

$$du = h(t, u)dt \qquad h(t_0) = u_0 \tag{3.60}$$

and

$$dv = g(t, v)dt \qquad g(t_0) = v_0. \tag{3.61}$$

Let $\rho(t, t_0, u_0)$ and $r(t, t_0, v_0)$ be the minimal and maximal solutions of the lower and upper comparison equations respectively.and $r(t, x)$. Taking the expected value of (3.19), we then apply comparison theorems [23].

Step 5: **Quantitative and Qualitative Analysis of Comparison Equations.**

We study of the behavior of the maximal and minimal solutions of the comparison equations.

Step 6: **Quantitative and Qualitative Analysis of Original Stochastic Dynamic Network.**

By either solving or analyzing the simpler differential equations (3.60) and (3.61) and determining the behavior of ρ and r, we are able to analyze the behavior of the solution to (3.1) without knowing an explicit solution.

Step 7: **Interpretations.**

Based on the quantitative and qualitative properties of (3.1) found in Step 6, we draw interpretations of the characteristics of the stochastic multicultural dynamic network.

3.9   Numerical Simulations

In this section, we consider numerical simulations for the multicultural dynamic network governed by the stochastic differential equation (3.14) using a Euler-Maruyama [17, 11, 12] type numerical approximation scheme. We consider a network of fifty members, using the same initial position and varying the parameters a, b, q, c, and β. Further, we consider the case such that $\xi_{ij}(t)$ for $i, j \in I(1, 50)$ are a one dimensional Brownian motion process with mean of zero and variance of 1 over the interval $[0, 1]$. To generate each member state cultural trajectory, we average the position for fifty simulations for each of the various cases, and then plot the average position, $z_i(t)$ for each member.

In order to consider the effects of changing the parametric value β, we consider various models for which $a = 2, b = 1$, and $c = 2$ are held constant and we vary both β and q. First, in Figure 3.1, we consider a network in which $q = \frac{2}{1.7}$ and $\beta = .5$. With the given parameters, $r_1 \approx 1.5$ and $r_2 \approx 1.1$. Therefore, using the upper and lower limits of the comparison equations, the long run behavior of the network has the approximate bounds given by

$$1.1 \leq \|z_i\| \leq 1.5, \tag{3.62}$$

as demonstrated in Figure 3.1. In the simulation, we see that members whose cultural state start close to the center shift away from the center over time. Further, in the simulation, members whose cultural state start farther away from the center are attracted back towards the center over time.

Next, in Figure 3.2, we consider the case with the parameters $q = \frac{2}{5.4}$ and $\beta = 1$. In this case, $r_1 \approx 2.7$, and $r_2 \approx 1.8$. In this case, using the bounds on the limits of the solutions of the comparison equations yield the approximate bounds on the long term behavior of the network

$$1.8 \leq \|z_i\| \leq 2.7. \tag{3.63}$$

Figure 3.1.: Euler-Maruyama approximation of the differential equation given by (3.14) with fifty members and parameters $a = 2, b = 1, c = 2, q = \frac{2}{1.7}$, and $\beta = .5$.
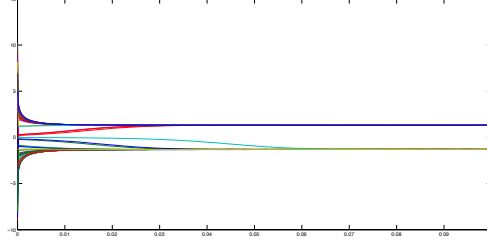

Figure 3.2.: Euler-Maruyama approximation of the differential equation given by (3.14) with fifty members and parameters $a = 2, b = 1, c = 2, q = \frac{2}{5.4}$, and $\beta = 1$.

We note that in this case by increasing β and decreasing q, the upper and lower bounds, as well as the distance between them, increase. In the simulation, we observe a similar behavior of members within the network; those starting close to the center are repulsed away and those starting away from the center are attracted back towards it. In Figure 3.3, we consider the case with parameters $q = \frac{1}{12}$ and $\beta = 2$. Further, we note that in this case, $r_1 \approx 6.3$, and $r_2 \approx 2.9$. In this case, the approximate bounds on the long term behavior of the network are given by

$$2.9 \leq \|z_i\| \leq 6.3. \tag{3.64}$$

By increasing β and decreasing q, we have again increased the values of the upper and lower bounds as well as the distance between the bounds. Further, in the simulations, we see a strong repulsion from the center of the network and that over time, the memberships cultural states settle relatively far from the cultural state of the center.

We now consider the case with the parameters $q = \frac{1}{7}$ and $\beta = 2$. In this case, $r_1 \approx 4.8$, and $r_2 \approx 2.2$. We note that using the limits of the upper and lower comparison equations, we compute the long term approximate bounds as
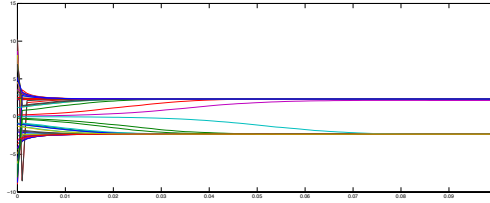
$$2.2 \leq \|z_i\| \leq 4.8, \tag{3.65}$$

Figure 3.3.: Euler-Maruyama approximation of the differential equation given by (3.14) with fifty members and parameters $a = 2, b = 1, c = 2, q = \frac{1}{12}$, and $\beta = 2$.
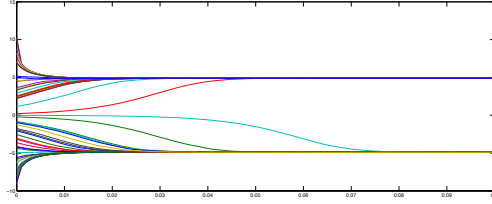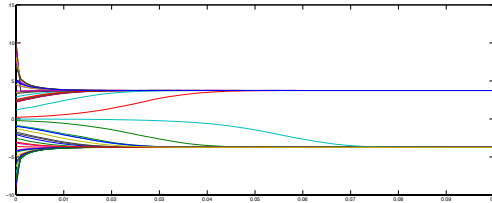

Figure 3.4.: Euler-Maruyama approximation of the differential equation given by (3.14) with fifty members and parameters $a = 2, b = 1, c = 2, q = \frac{1}{7}$, and $\beta = 2$.

as seen in Figure 3.4. By increasing q in this simulation, the upper and lower bounds are smaller than those from Figure 3.3. We also note that the distance between the bounds has decreased from that in Figure 3.3.

3.10 Conclusion

We have considered requirements on the parameters that allow the perturbed multicultural dynamic network to remain cohesive while retaining a cultural state that is distinctive from the cultural state center of the network. We established qualitative and quantitative conditions that are computationally attractive and veri-fiable. Further, we have analyzed cultural state invariant sets and long-term cultural states of members within the multicultural dynamic network. We also conducted simulations of the multicultural network that exhibit the influence of the random perturbations as well as demonstrate the long-term behavior of the multicultural network.

Chapter 4
Stochastic Hybrid Dynamic Network

4.1 Introduction

The aim of this chapter is to explore and extend the cohesive properties of a dynamic network of multi-agents/members with a desired minimum distance between the members of the network [6, 13, 14] under the influence of both continuous and discrete-time stochastic perturbations. One of the concepts studied using a dynamic social network is that of consensus [7, 9, 27, 1]. In such models, the conditions under which a group collectively comes to an agreement are studied. Another question of interest for such a network is when the group may divide into subgroups with an agreement reached within the subgroup but never reaching a consensus at an overall group level.

Dynamic network models play an important role in a variety of modeling applications. For example, economics, finance, engineering, management sciences, and biological networks have considered such large scale dynamic models to investigate connectivity, stability, dynamic reliability, and convergence [22, 24, 2, 28]. Much of the work done in these areas look to develop consensus seeking algorithms and consider long term stability of the network in consideration [5, 30, 15, 16]. The concepts of cohesion, coordination, and cooperation within a group are often multi-faceted, dynamic and complex, but are important concepts when trying to better understand how nations or communities function [3]. We seek to better understand the group dynamics of such a society in order to create policies and practices which encourage a sense of community among individuals from a variety of cultural backgrounds.

In fact, we systematically initiated the study of this issue [13, 14] to better understand the social dynamics of a group seeking to find such a balance under the influence of both continuous and discrete-time stochastic perturbations. In doing so, we are interested in better understanding the cohesive properties of a multi-cultural social network. In this work, we further extend the developed results in the framework of hybrid stochastic dynamic model for which we explore the features of the a network. By considering a hybrid dynamic [25], we are able to consider the impact that events both from external and internal stochastic

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fluctuations on the network have on the cultural dynamics. The presented work is used to exhibit the quantitative and qualitative properties of the network. Further, the techniques used are computationally attractive and algebraically simple relating with the underlying network parameters.

In Section 4.2, we present the general problem under consideration and the underlining assumptions. We then present an illustration of such a network in Section 4.3 to exhibit the role and scope of the underlying complexity with the simplicity without loss of generality. Using an appropriate energy function and the comparison method, upper and lower estimates on cultural states are established in Section 4.4. In Section 4.5, the long-term behavior of the solutions to the comparison equations are examined and we explore the study of the cultural state invariant sets in the context of the illustration presented in Section 4.3. In Section 4.6, we use numerical simulations to model the network and to better understand to what extent the analytically developed estimates in Section 4.5 are feasible.

4.2   Problem Formulation

The network consists of m agents whose position at time t is represented by $x_i(t), i \in I(1,m) = \{1, 2, \ldots, m\}$, with $x_i(t) \in \mathbb{R}^n$. In our model, this vector does not represent a geographical location but rather a cultural position of the ith member. That is to say, the vector x_i is a numerical representation of the ith member's beliefs or background on certain cultural or ethnic practices relevant to the network under study. Further, we assume that $\xi_{ij}, i, j \in I(1,m)$ is a normalized Wiener process such that $\xi_{ij} = \xi_{ji}$ and for $j \neq k$, ξ_{ij} and ξ_{ik} are independent. We then consider a system of Itô-Doob type stochastic system of differential equations that describes the cultural state dynamic process:

$$
\begin{cases}
dx_i & = \sum_{j=1}^{m} f(t, x_i, x_i - x_j, k - 1)dt + \sum_{j \neq i}^{m} \sigma(t, x_i - x_j, k - 1)d\xi_{ij}(t), \\
\Delta x_i^k & = I\left(x_i^{k-1}(t_k^-, t_{k-1}, x_i^{k-1}), k\right), \qquad x_i^0(t_0) = x_i^0,
\end{cases}
\tag{4.1}
$$

for $(t, x_i) \in [t_{k-1}, t_k) \times \mathbb{R}^n$ and $k \in I(1, \infty)$, where $x_i, x_j \in \mathbb{R}^n$ are continuous time dynamic states; $i, j \in I(1, m)$; f and σ are drift and diffusion rate coefficient functions, respectively; and $\Delta x_i^k = x_i^k - x_i^{k-1}$, where I in (4.1) stands for a discrete time intervention dynamic process. We will also make the following assumptions: *Assumption H_1*: For

i) $x_i^{k-1}(t_{k-1}) = x_i^{k-1}$ is an n-dimensional initial cultural state random vector defined on the complete probability space (Ω, F, P) and $k \in I(0, \infty)$ at the kth intervention time;

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ii) x_i^{k-1} and $\xi_{ij}(t)$ are mutually independent for each $t_{k-1} \le t < t_k$ for $i \ne j$, $i, j \in I(1, m)$ and $k \in I(0, \infty)$;

iii) For $i, j \in I(1, m)$, $\xi_i(t) = (\xi_{i1}, \xi_{i2}, \dots, \xi_{ij} \dots, \xi_{im})^T$ is a m-dimensional normalized Wiener process of independent increments for $i \in I(1, m)$;

iv) ξ_{ij}s are F_t-measurable for all $t \ge t_0$ and $\xi_{ij}(t + h) - \xi_{ij}(t)$ is independent of F_t, where F_t represents an increasing family of the smallest sub-σ algebra of F, i.e. $F_s \subset F_t$ if $t_0 < s < t$;

v) $x_i(t_0)$ is F_{t_0} measurable;

vi) $\{t_k\}_{k=1}^{\infty}$ is a sequence of intervention time, and $t_k \to \infty$ as $k \to \infty$;

vii) f and σ are defined on : $\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \times I(1, \infty)$ into $\mathbb{R}^n$, and continuous on $[t_{k-1}, t_k) \times \mathbb{R}^n \times \mathbb{R}^n$ for each $(t, x, y) \in [t_{k-1}, t_k) \times \mathbb{R}^n \times \mathbb{R}^n$;

viii) f and σ satisfy for each $k \in I(1, \infty)$ and for each $(t, x, y, k) \in [t_{k-1}, t_k) \times \mathbb{R}^n \times \mathbb{R}^n \times I(1, \infty)$,

$$f(t, x, y, k - 1) \to f(t_k^-, x, y, k - 1) \tag{4.2}$$

$$\sigma(t, x, y, k - 1) \to \sigma(t_k^-, x, y, k - 1) \tag{4.3}$$

as $t \to t_k^-$;

ix) $I : \mathbb{R}^n \times I(1, \infty) \to \mathbb{R}^n$ is a Borel measurable discrete time intervention function.

It is assumed that the initial value problem (4.1) for the system of stochastic differential equations has a solution process.

We wish to investigate the stochastic cohesive property of such a network. Further, we will explore the behavior of a member of the network based on the cultural state distance between a network member cultural state and the cultural state center of the network.

In the following, we extend Definitions 3.1, 3.2, and 3.3 to the hybrid stochastic multicultural dynamic network.

Definition 4.1 *Let r_1 and r_2 be non-negative random functions for $t \in [t_{k-1}, t_k)$, $k \in I(1, \infty)$ such that $r_1 \le r_2$. We say that a stochastic multicultural dynamic network is*

(a.) locally cohesive with probability 1 if for any $N \in F_t$ such that $P(N) = 0, N \subset \Omega$ and for all $t \in [t_{k-1}, t_k)$

$$r_1(t) \leq \left\| x_i^{k-1}(t) - x_j^{k-1}(t) \right\| \leq r_2(t), \tag{4.4}$$

for all $i, j \in I(1, m)$;

(b.) locally cohesive in probability if for all $i, j \in I(1, m)$, $t \in [t_{k-1}, t_k)$, $\omega \in \Omega$ and any $0 < \epsilon < 1$

$$P\left(\left\{\omega : \left\| x_i^{k-1}(t) - x_j^{k-1}(t) \right\| < r_1(t) \, or \left\| x_i^{k-1}(t) - x_j^{k-1}(t) \right\| > r_2(t)\right\}\right) < \epsilon, \tag{4.5}$$

for all $i \in I(1, m)$;

(c.) locally cohesive in pth mean if for all $i, j \in I(1, m)$ and $t \in [t_{k-1}, t_k)$

$$E\left[r_1(t)\right] \leq E\left[\left\| x_i^{k-1}(t) - x_j^{k-1}(t) \right\|^p\right] \leq E\left[r_2(t)\right], \tag{4.6}$$

for all $i \in I(1, m)$.

If $(i.), (ii.),$ or $(iii.)$ exist for all $t \in [t_0, \infty)$, we say the network is globally cohesive with probability 1, in probability or in pth mean respectively.

Definition 4.2 *We say that a stochastic multicultural dynamic network*

(a.) locally reaches a consensus with probability 1 if there exists $N \subset F$ such that $P(N) = 0$ and for all $\omega \in \Omega \setminus N$,

$$\lim_{t \to \infty} \left\| x_i^{k-1}(t) - \bar{x}^{k-1} \right\| = 0, \tag{4.7}$$

for $k \in I(1, \infty)$ and all $i, j \in I(1, m)$;

(b.) locally reaches a consensus in probability if for $\epsilon > 0$ and $t \in [t_{k-1}, t_k)$, $k \in I(1, \infty)$,

$$\lim_{t \to \infty} P\left(\left\{\left\| x_i^{k-1}(t) - \bar{x}^{k-1} \right\| > \epsilon\right\}\right) = 0, \tag{4.8}$$

for all $i \in I(1, m)$;

(c.) locally reaches a consensus in the pth mean if for $t \in [t_{k-1}, t_k)$, $k \in I(1, \infty)$,

$$\lim_{t \to \infty} E\left[\|x_i(t) - \bar{x}\|^p\right] = 0, \tag{4.9}$$

for all $i \in I(1, m)$.

If (i.), (ii.), or (iii.) exist for all $t \in [t_0, \infty)$, we say the network is reaches a global consensus with probability 1, in probability or in pth mean respectively.

Definition 4.3 *Let x_i^{k-1} and x_j^{k-1} be cultural state random vectors for $i, j \in I(1, m)$ and $k \in I(1, \infty)$. For $t \in [t_{k-1}, t_k)$, we define the relative cultural state affinity with probability 1 by*

$$\left\| x_i^{k-1}(t) - x_j^{k-1}(t) \right\|. \tag{4.10}$$

We note that the relative cultural state affinity in the a.s. sense exists as $\|\bullet\|$ is Borel measurable.

4.3 Prototype Dynamic Model

Let us define a prototype multicultural network dynamic model under the stochastic environmental perturbations described by the Itô-Doob type stochastic system of differential equations

$$
\begin{cases}
dx_i^{k-1} &= \left[a_{k-1} \sum_{j=1}^m x_{ij}^{k-1} - q_{k-1} \left\| x_i^{k-1} - \bar{x}^{k-1} \right\|^2 \sum_{j=1}^m x_{ij}^{k-1} \right. \\
&\quad + b_{k-1} \sin \left\| x_i^{k-1} - \bar{x}^{k-1} \right\| \sum_{j=1}^m x_{ij}^{k-1} \exp \left[-\frac{\|x_{ij}^{k-1}\|^2}{c_{k-1}} \right] \Bigg] dt \\
&\quad + \beta_{k-1} \sin \left\| x_i^{k-1} - \bar{x}^{k-1} \right\| \sum_{j=1}^m x_{ij}^{k-1} \exp \left[-\frac{\|x_{ij}^{k-1}\|^2}{c_{k-1}} \right] d\xi_{ij}, \\
x_i^k &= (1 + \delta_i^{k-1}) x_i^{k-1}(t_k^-, t_{k-1}, x_i^{k-1}), \qquad x_i^0(t_0) = x_i^0,
\end{cases}
\tag{4.11}
$$

for $t \in [t_{k-1}, t_k)$, $k \in I(0, \infty)$ and where $a_{k-1}, q_{k-1}, b_{k-1}, c_{k-1}$ and β_{k-1} are positive real numbers,

$$x_{ij}^{k-1} = x_i^{k-1} - x_j^{k-1}, \tag{4.12}$$

We note that the solution process x_i of (4.11) is defined by

$$
x_i(t, t_0, x_i^0) =
\begin{cases}
x_i^0(t, t_0, x_i^0) & t_0 \leq t < t_1, \\
x_i^1(t, t_1, x_i^1 & t_1 \leq t < t_2, \\
\vdots & \vdots \\
x_i^{k-1}(t, t_{k-1}, x_i^{k-1}) & t_{k-1} \leq t < t_k, \\
\vdots & \vdots
\end{cases}
\tag{4.13}
$$

Here, $\bar{x}^{k-1}$ is the center of the multicultural dynamic system (4.11) defined by:

$$\bar{x}^{k-1} = \frac{1}{m} \sum_{j=1}^m x_j^{k-1}(t), \qquad t \in [t_{k-1}, t_k), \tag{4.14}$$

and note that by substituting for x_i^{k-1} by $\bar{x}^{k-1}$ into (4.11), we have

$$d\bar{x}^{k-1} = \left[a_{k-1} \sum_{j=1}^{m} \left(\bar{x}^{k-1} - x_j^{k-1} \right) - q_{k-1} \left\| \bar{x}^{k-1} - \bar{x}^{k-1} \right\|^2 \sum_{j=1}^{m} \left(\bar{x}^{k-1} - x_j^{k-1} \right) \right.$$

$$+ b_{k-1} \sin \left\| \bar{x}^{k-1} - \bar{x}^{k-1} \right\| \sum_{j=1}^{m} \left(\bar{x}^{k-1} - x_j^{k-1} \right) \exp \left[-\frac{\left\| \bar{x}^{k-1} - x_j^{k-1} \right\|^2}{c_{k-1}} \right] \Bigg] dt$$

$$+ \beta_{k-1} \sin \left\| \bar{x}^{k-1} - \bar{x}^{k-1} \right\| \sum_{j=1}^{m} \left(\bar{x}^{k-1} - x_j^{k-1} \right) \exp \left[-\frac{\left\| \bar{x}^{k-1} - x_j^{k-1} \right\|^2}{c_{k-1}} \right] d\xi_{\bar{x}^{k-1}j},$$

$$= a_{k-1} m \bar{x}^{k-1} - a_{k-1} \sum_{j=1}^{m} x_j^{k-1}$$

$$= a_{k-1} m \bar{x}^{k-1} - a_{k-1} m \bar{x}^{k-1}$$

$$= 0, \tag{4.15}$$

for $t \in [t_{k-1}, t_k)$, $k \in I(1, \infty)$, and thus $\bar{x}^{k-1}$ defined in (4.14) is a stationary center of the multicultural dynamic network on each interval $[t_{k-1}, t_k)$. We define the transformation $z_i^{k-1} = x_i^{k-1} - \bar{x}^{k-1}$ and observe that $x_{ij}^{k-1} = z_i^{k-1} - z_j^{k-1} = z_{ij}^{k-1}$. Then the transformed network dynamic model corresponding to (4.11) is reduced to:

$$\begin{cases} dz_i^{k-1} &= \left[a_{k-1} m z_i^{k-1} - q_{k-1} \left\| z_i^{k-1} \right\|^2 m z_i^{k-1} + b_{k-1} \sin \left\| z_i^{k-1} \right\| \sum_{j=1}^{m} z_{ij}^{k-1} \exp \left[-\frac{\| z_{ij}^{k-1} \|^2}{c_{k-1}} \right] \right] dt \\ &\quad + \beta_{k-1} \sin \left\| z_i^{k-1} \right\| \sum_{j=1}^{m} z_{ij}^{k-1} \exp \left[-\frac{\| z_{ij}^{k-1} \|^2}{c_{k-1}} \right] d\xi_{ij}, \qquad t \in [t_{k-1}, t_k) \\ z_i^k &= \left(1 + \delta_i^{k-1} \right) z_i^{k-1} \left(t_k^-, t_{k-1}, z_i^{k-1} \right), \qquad z_i^0(t_0) = z_i^0. \end{cases}$$

$$\tag{4.16}$$

The center $\bar{x}^{k-1}$ of the multicultural dynamic model (4.11) is reduced to the center zero in (4.16) over each interval $[t_{k-1}, t_k)$ and $k \in I(1, m)$. For each $k \in I(1, \infty)$, a_k, b_k, c_k, q_k and β_k are as described and characterized in (3.14). It exhibits both attractive and repulsive forces that are centered at the center of the network. The magnitude of the repulsive forces over $[t_{k-1}, t_k)$ are described by $a_{k-1} m \left\| z_i^{k-1} \right\|$ and the magnitude of the long range deterministic attractive forces are characterized by $b_{k-1} \left\| \sum z_{ij}^{k-1} \exp \left[-\frac{\| z_{ij}^{k-1} \|^2}{c} \right] \right\|$. Further, $\sin \left\| z_i^{k-1} \right\|$ is the sine-cyclical influence of the ith member's relative distance to the center of the network. The stochastic term represents the environmental influence due to long-range attractive forces. In particular, in the case of a multi-cultural network, the noise captures the uncertainty generated due to the membership interactions and deliberations under the influence of the long-range cultural forces.

We remark that the solution process of (4.16) can be re-casted as (4.13). In order to study the multicultural dynamics (4.16), we use Lyapunov's Second Method in conjunction with the comparison method [23]. These methods are computationally attractive and provide a means of better understanding the movement and behavior of the state memberships of the network. By utilizing these methods, we are able to establish conditions for which we have both upper and lower estimates on the members cultural state positions on the interval $[t_{k-1}, t_k)$ for $k \in I(1, m)$. In this work, we assume that all inequalities are with probability 1.

4.4   Upper and Lower Comparison Equations4

Using Lyapunov's Second Method and differential inequalities, we first seek a function $r_{k-1}(t, t_{k-1}, u_{k-1})$ such that

$$\left\| z_i^{k-1}(t) \right\| \leq r_{k-1}(t, t_{k-1}, r_{k-1}), \qquad t \in [t_{k-1}, t_k). \tag{4.17}$$

From Definition 4.1, relation (4.17) generates a concept of a locally upper-cohesive cultural network in the almost surely on the $k - 1$th interval for $k \in I(1, \infty)$.

To this end, for $t \in [t_{k-1}, t_k)$ let us choose an energy function V_{k-1} as:

$$V_{k-1}(z_i^{k-1}) = \left\| z_i^{k-1} \right\| = \left(\left(z_i^{k-1} \right)^T z_i^{k-1} \right)^{\frac{1}{2}}. \tag{4.18}$$

We have previously shown that the differential of V_{k-1} in the direction of the vector field represented by (4.16) is

$$dV_{k-1} = \frac{\left(z_i^{k-1} \right)^T dz_i^{k-1}}{\left\| z_i^{k-1} \right\|} + \frac{1}{2} \left[\frac{\left(dz_i^{k-1} \right)^T dz_i^{k-1}}{\left\| z_i^{k-1} \right\|} - \frac{\left(\left(z_i^{k-1} \right)^T dz_i^{k-1} \right)^2}{\left\| z_i^{k-1} \right\|^3} \right]$$

$$= \frac{\left(z_i^{k-1} \right)^T \sum_{j=1}^{m} \phi_2(z_{ij}^{k-1}) d\xi_{ij}}{\left\| z_i \right\|^{k-1}} + LV(z_i^{k-1}) dt, \tag{4.19}$$

where

$$\phi_1(z_i^{k-1}) = a_{k-1} m z_i^{k-1} - q_{k-1} \left\| z_i^{k-1} \right\|^2 m z_i + b_{k-1} \sin \left\| z_i^{k-1} \right\| \sum_{j=1}^{m} z_{ij}^{k-1} \exp \left[-\frac{\left\| z_{ij}^{k-1} \right\|^2}{c_{k-1}} \right], \tag{4.20}$$

$$\phi_2(z_{ij}^{k-1}) = \beta_{k-1} \sin \left\| z_i^{k-1} \right\| z_{ij}^{k-1} \exp \left[-\frac{\left\| z_{ij}^{k-1} \right\|^2}{c_{k-1}} \right], \tag{4.21}$$

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and

$$LV_{k-1}(z_i^{k-1}) = \frac{\left(z_i^{k-1}\right)^T \phi_1(z_i^{k-1})dt}{\left\|z_i^{k-1}\right\|} + \frac{\sum_{j=1}^m \phi_2^T(z_{ij}^{k-1})\phi_2(z_{ij}^{k-1})}{2\left\|z_i^{k-1}\right\|} - \frac{\left(\left(z_i^{k-1}\right)^T \sum_{j=1}^m \phi_2(z_{ij}^{k-1})\right)^2}{2\left\|z_i^{k-1}\right\|^3}$$

$$= \left[a_{k-1}m\left\|z_i^{k-1}\right\| - q_{k-1}m\left\|z_i^{k-1}\right\|^3 + \frac{b_{k-1}\sin\left\|z_i^{k-1}\right\|}{\left\|z_i^{k-1}\right\|}\sum_{j=1}^m \left(z_i^{k-1}\right)^T z_{ij}^{k-1}\exp\left[-\frac{\|z_{ij}\|^2}{c_{k-1}}\right]\right.$$

$$+\frac{\beta_{k-1}^2\sin^2\left\|z_i^{k-1}\right\|\sum_{j=1}^m \left(z_{ij}^{k-1}\right)^T z_{ij}^{k-1}\exp\left[-\frac{2\|z_{ij}^{k-1}\|^2}{c_{k-1}}\right]}{2\left\|z_i^{k-1}\right\|}$$

$$\left.-\frac{\beta_{k-1}^2\sin^2\left\|z_i^{k-1}\right\|\sum_{j=1^m}\left(\left(z_i^{k-1}\right)^T z_{ij}^{k-1}\right)^2\exp\left[-\frac{2\|z_i^{k-1}\|^2}{c_{k-1}}\right]}{2\left\|z_i^{k-1}\right\|^3}\right]dt. \tag{4.22}$$

We seek constraints on the parameters $a_{k-1}, b_{k-1}, c_{k-1}, q_{k-1}$ and β_{k-1}, $k \in I(1,\infty)$ for which we have an upper estimate on $V_{k-1}(z_i^{k-1})$. To this end, imitating the argument made in [14], an upper estimate of LV_{k-1} in (4.22) is

$$LV_{k-1} \leq q_{k-1}m\|z_i\|\left(\frac{a_{k-1}}{q_{k-1}}\right.$$

$$+\frac{4b_{k-1}(m-1)\sqrt{\frac{c_{k-1}}{2}}\exp\left[-\frac{1}{2}\right] + \beta_{k-1}^2(m-1)c_{k-1}\exp[-1]}{4q_{k-1}m} - \|z_i\|^2\Bigg)$$

$$\leq q_{k-1}mV_{k-1}\left(\eta_{k-1}^2 - V_{k-1}^2\right)$$

$$\leq q_{k-1}mV_{k-1}\left(\eta_{k-1} - V_{k-1}\right)\left(\eta_{k-1} + V_{k-1}\right), \tag{4.23}$$

where

$$\eta_{k-1} = \left(\frac{a_{k-1}}{q_{k-1}} + \frac{4b_{k-1}(m-1)\sqrt{\frac{c_{k-1}}{2}}\exp\left[-\frac{1}{2}\right] + \beta_{k-1}^2(m-1)c_{k-1}\exp[-1]}{4q_{k-1}m}\right)^{\frac{1}{2}}. \tag{4.24}$$

In the following, we present a result that will be used subsequently.

Lemma 4.1 *Let V_{k-1} be the energy function defined in (4.18) and z_i^{k-1} be a solution of the initial value problem defined in (4.16). Then, for each $i \in I(1,m)$, $k \in I(1,\infty)$, and $t \in [t_{k-1}, t_k)$,*

$$E\left[V_{k-1}(z_i^{k-1}(t+\Delta t)) - V_{k-1}(z_i^{k-1}(t))|F_t\right] = LV_{k-1}(z_i^{k-1}(t))\Delta t, \tag{4.25}$$

where E stands for the expected value.

Proof. For each $k \in I(1, \infty)$, let $z_i^{k-1}(t, t_{k-1}, z_i(t_{k-1}))$ be the solution process of (4.16). Let F_t be an increasing family of sub-σ algebras as previously defined and set

$$m(t) = E\left[V_{k-1}\left(z_i^{k-1}(t)\right)|F_t\right] = V\left(z_i^{k-1}(t)\right), \quad (4.26)$$

where the last equality holds as $z_i^{k-1}(t)$ is F_t measurable. Similarly, we have set

$$m(t + \Delta t) = E\left[V_{k-1}\left(z_i^{k-1}(t + \Delta t)\right)|F_t\right], \quad (4.27)$$

for all $\Delta t > 0$ sufficiently small such that $(t + \Delta t) \in [t_{k-1}, t_k)$. We consider

$$m(t + \Delta t) - m(t) = E\left[V_{k-1}\left(z_i^{k-1}(t + \Delta t) - V_{k-1}\left(z_i^{k-1}(t)\right)\right)|F_t\right]$$

$$= E\left[\frac{\partial V_{k-1}}{\partial z}\left(z_i^{k-1}(t)\right)\Delta z_i^{k-1}(t) + \frac{1}{2}tr\left(\frac{\partial^2 V_{k-1}}{\partial z^2}\left(\Delta z_i^{k-1}(t)\right)\left(\Delta z_i^{k-1}(t)\right)^T\right)|F_t\right]$$

$$= E\left[dV_{k-1}\left(z_i^{k-1}(t)\right)|F_t\right]. \quad (4.28)$$

This together with (4.19), yields

$$m(t + \Delta t) - m(t) = E\left[LV_{k-1}\left(z_i^{k-1}(t)\right)\Delta t|F_t\right]$$

$$= LV_{k-1}\left(z_i^{k-1}(t)\right)\Delta t, \quad (4.29)$$

as $z_i^{k-1}(t)$ is F_t measurable. We note that for small Δt, we have

$$dm(t) = LV_{k-1}\left(z_i^{k-1}(t)\right)dt. \quad (4.30)$$

$\square$

From the inequality (4.23) utilizing the comparison method [23] and Lemma 4.1, we establish the following lemma. For each interval $[t_{k-1}, t_k)$ and $k \in I(1, \infty)$, the presented result establishes not only an upper bound but also the locally upper cohesive property almost surely. Hereafter, all inequalities and equalities are assumed to be valid with probability one.

Lemma 4.2 *Let V_{k-1} be the energy function defined in (4.18), $k \in I(1, \infty)$, $t \in [t_{k-1}, t_k)$, and z_i^{k-1} be a solution of the initial value problem defined in (4.16). Let $r_{k-1}(t)$ be the maximal solution of a random initial value problem [23]*

$$du_{k-1} = [q_{k-1}mu_{k-1}(\eta_{k-1} - u_{k-1})(\eta_{k-1} + u_{k-1})]\,dt, \qquad u_{k-1}(t_{k-1}) = u_{k-1}, \quad (4.31)$$

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where η_{k-1} is defined as in (4.24). For each $V_{k-1}(z_i^{k-1})$, $i \in I(1, m)$, $k \in I(1, \infty)$ satisfying the differential inequality (4.23) and $V_{k-1}(z_i^{k-1}(t_{k-1})) \leq u_{k-1}$, it follows that the multicultural dynamic network (4.16) is upper cohesive on $[t_{k-1}, t_k)$ with probability 1 and

$$V_{k-1}(z_i^{k-1}(t)) \leq u_{k-1}(t, t_{k-1}, u_{k-1}), \tag{4.32}$$

Proof. From Lemma 4.1, (4.23), and the application of stochastic comparison theorem [23], with probability 1, it follows that

$$V_{k-1}(z_i^{k-1}(t)) \leq r(t, t_{k-1}, u_{k-1}), \tag{4.33}$$

when $V_{k-1}(z_i(t_{k-1})) \leq u_{k-1}$. As the solution to (4.31) has an upper bound, the network is upper cohesive almost surely. $\qquad\square$

Remark 4.1: For each $k \in I(1, \infty)$, if the solution processes of (4.16) and (4.31) have a first moment, then the solution process of (4.16) is locally upper 1st moment cohesive. Furthermore, under the current inequality, it is indeed locally upper cohesive in the sense of probability.

Next we consider the lower comparison equation. Using Lyapunov's Second Method and differential inequalities, we next seek a function $\rho_{k-1}(t, t_{k-1}, u_{k-1})$ such that

$$\|z_i(t)\| \geq \rho(t, t_{k-1}, \rho_{k-1}), \qquad t \in [t_{k-1}, t_k). \tag{4.34}$$

Again, from Definition 4.1, relation (4.34) initiates a notion of a locally lower cohesive cultural dynamic network in the almost sure sense.

Using the energy function defined in (4.18) and relation (4.22), for $t \in [t_{k-1}, t_k)$ it follows that

$$\begin{aligned}
Lv_{k-1} &\geq a_{k-1} m V_{k-1} - q_{k-1} m V_{k-1}^3 - V_{k-1}(m-1) b_{k-1} \sqrt{\frac{c_{k-1}}{2}} \exp\left[-\frac{1}{2}\right] \\
&\quad - \frac{\beta_{k-1}^2 (m-1) c_{k-1} \exp[-1]}{4} V_{k-1} \\
&= q_{k-1} m V_{k-1} \left(\frac{a_{k-1}}{q_{k-1}} \right. \\
&\quad \left. - \frac{4(m-1) b_{k-1} \sqrt{\frac{c_{k-1}}{2}} \exp\left[-\frac{1}{2}\right] + \beta_{k-1}^2 c_{k-1}(m-1) \exp[-1]}{4 q_{k-1} m} - V_{k-1}^2 \right).
\end{aligned} \tag{4.35}$$

Assumption H_2: Assume there exists a positive number α_{k-1} such that

$$\alpha_{k-1} \leq \left(\frac{a_{k-1}}{q_{k-1}} - \frac{4(m-1) b_{k-1} \sqrt{\frac{c_{k-1}}{2}} \exp\left[-\frac{1}{2}\right] + \beta_{k-1}^2 (m-1) c_{k-1} \exp[-1]}{4 q_{k-1} m} \right)^{\frac{1}{2}}. \tag{4.36}$$

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From (4.35), and noticing the fact that assumption H_2 implies

$$\frac{a_{k-1}}{q_{k-1}} > \frac{4\,(m-1)\,b_{k-1}\sqrt{\frac{c_{k-1}}{2}}\exp\left[-\frac{1}{2}\right] + \beta_{k-1}^2\,(m-1)\exp\left[-1\right]}{4q_{k-1}m}, \tag{4.37}$$

it follows that

$$LV_{k-1} \geq q_{k-1}mV_{k-1}(\alpha_{k-1} - V_{k-1})(\alpha_{k-1} + V_{k-1}). \tag{4.38}$$

By inequality (4.38) and the comparison method [23] and Lemma 4.1, we establish the following lemma. The presented result provides the lower estimate which in turn establishes the locally lower cohesive property of (4.16).

Lemma 4.3 *Let V_{k-1} be the energy function defined in (4.18), $k \in I(1,\infty)$, $t \in [t_{k-1}, t_k)$, and z_i^{k-1} be a solution of the initial value problem defined in (4.16). Let $\rho_{k-1}(t)$ be the minimal solution of a random initial value problem [23]*

$$du_{k-1} = q_{k-1}mu_{k-1}\left(\alpha_{k-1} - u_{k-1}\right)\left(\alpha_{k-1} + u_{k-1}\right)dt, \qquad u_{k-1}(t_{k-1}) = u_{k-1}, \tag{4.39}$$

where α_{k-1} is as defined in (4.36). For each $V_{k-1}(z_i^{k-1})$, $i \in I(1,m)$, $k \in I(1,\infty)$ satisfying the differential inequality (4.38) and $V(z_i^{k-1}(t_{k-1})) \geq u_{k-1}$, it follows that the multicultural dynamic network (4.11) is lower cohesive on $[t_{k-1}, t_k)$ with probability 1 and

$$V_{k-1}(z_i^{k-1}(t)) \geq \rho_{k-1}(t, t_{k-1}, u_{k-1}). \tag{4.40}$$

Proof. From inequality (4.38) and Lemma 4.1 and the imitating the outline of the proof of Lemma 4.2, it follows that

$$V_{k-1}(z_i^{k-1}(t)) \geq \rho(t, t_{k-1}, u_{k-1}) \tag{4.41}$$

provided that $V_{k-1}(z_i^{k-1}(t_{k-1})) \geq u_{k-1}$. As the minimal solution of of (4.39) is a lower bound, the network is lower cohesive almost surely. Moreover, a remark similar to Remark 4.1 establishes the locally stochastic mean and probability of (4.16) □

We note that comparison differential equations (4.31) and (4.39) each have a unique solution process. Therefore the maximal and minimal solutions of (4.31) and (4.39) are the unique solutions of the respective random initial value problems.

4.5 Long-term Behavior of Comparison Differential Equations and Invariant Sets

To appreciate the role and scope of Lemmas 4.2 and 4.3, we seek to better understand both the behavior of the network on each interval $[t_{k-1}, t_k)$ and the long-term behavior of the network. For this purpose, for $k \in (1, \infty)$, we find the closed form solutions of the comparison random initial value problems (4.31) and (4.39). Moreover, we analyze the qualitative properties of the solutions to the comparison equations. Using the comparison method [23], we are able to establish, quantitatively, the behavior of the individual member cultural dynamic states on the interval $[t_{k-1}, t_k)$. Using this, we also establish the overall long-term behavior of both individual member cultural dynamic states in the network as well as multicultural network state as a whole.

Following the method of finding the closed form solution process of the initial value problem [20], the solution of (4.39) is represented by

$$u_{k-1}(t, t_{k-1}, u_{k-1}) = \frac{u_{k-1}\nu}{\sqrt{u_{k-1}^2 + \left(\nu^2 - u_{k-1}^2\right)\exp\left[-2\nu^2 q_{k-1} m(t - t_{k-1})\right]}}. \tag{4.42}$$

As $z_i^k(t_k) = (1 + \delta_i^{k-1})z_i^{k-1}(t_k^-, t_{k-1}, x_i^{k-1})$ for $k \in I(0, \infty)$, we seek to write the initial position u_k in terms of u_0.

By squaring both sides and rearranging the terms, we can write the above as

$$\frac{u_{k-1}^2(t, t_{k-1}, u_{k-1})}{\nu^2 - u_{k-1}^2(t, t_{k-1}, u_{k-1})} = \frac{u_{k-1}^2 \exp\left[2\nu^2 q_{k-1} m(t - t_{k-1})\right]}{\nu^2 - u_{k-1}^2}. \tag{4.43}$$

We now set

$$y_{k-1}(t, t_{k-1}, y_{k-1}) = \frac{u_{k-1}^2(t, t_{k-1}, u_{k-1})}{\nu^2 - u_{k-1}^2(t, t_{k-1}, u_{k-1})}, \tag{4.44}$$

where $y(t_{k-1}) = y_{k-1}$ on the interval $[t_{k-1}, t_k)$. Next, we take the derivative of both sides

$$
\begin{aligned}
dy_{k-1} &= \frac{2u_{k-1}(t, t_{k-1}, u_{k-1})\left[\left(\nu^2 - u_{k-1}^2(t, t_{k-1}, u_{k-1})\right) + 2u_{k-1}(t, t_{k-1}, u_{k-1})(u_{k-1}^2(t, t_{k-1}, u_{k-1})\right] du_{k-1}}{\left(\nu^2 - u_{k-1}^2(t, t_{k-1}, u_{k-1})\right)}^2 \\
&= \frac{2\nu^2 u_{k-1}(t, t_{k-1}, u_{k-1}) du_{k-1}}{\left(\nu^2 - u_{k-1}^2(t, t_{k-1}, u_{k-1})\right)} \\
&= \frac{2\nu^2 u_{k-1}(t, t_{k-1}, u_{k-1})\left(q_{k-1} m u_{k-1}(t, t_{k-1}, u_{k-1})(\nu^2 - u_{k-1}^2(t, t_{k-1}, u_{k-1}))\right) dt}{\left(\nu^2 - u_{k-1}^2(t, t_{k-1}, u_{k-1})\right)^2} \\
&= 2\nu^2 q_{k-1} m \left(\frac{u_{k-1}^2(t, t_{k-1}, u_{k-1})}{\nu^2 - u_{k-1}^2(t, t_{k-1}, u_{k-1})}\right) dt \\
&= \left(2\nu^2 q_{k-1} m\right) y_{k-1} dt. \tag{4.45}
\end{aligned}
$$

Therefore, on the interval $[t_{k-1}, t_k)$, the solution of (4.45) is

$$y_{k-1}(t, t_{k-1}, y_{k-1}) = y_{k-1} \exp\left[2\nu^2 m q_{k-1}(t - t_{k-1})\right], \qquad y_0(t_0) = y_0. \tag{4.46}$$

Let $\Delta t_k = t_k - t_{k-1}$. When $k = 1$, the solution of (4.46) on $[t_0, t_1)$ is

$$y_0(t, t_0, u_0) = y_0 \exp\left[2\nu^2 m q_0(t - t_0)\right]$$

$$y_0(t_1^-, t_0, u_0) = y_0 \exp\left[2\nu^2 m q_0 \Delta t_1\right]$$

$$y_1(t_1) = \left|1 + \delta_i^0\right| y_0 \exp\left[2\nu^2 m q_0 \Delta t_1\right]. \tag{4.47}$$

We assume that for $k - 1 \in I(1, \infty)$, the solution of (4.46) on $[t_{k-1}, t_k)$ is

$$y_{k-1}(t, t_{k-1}, y_{k-1}) = \prod_{j=1}^{k-1}\left|1 + \delta_i^{j-1}\right| y_0 \exp\left[2\nu^2 m \left(\sum_{j=1}^{k-1} q_{j-1}\Delta t_{j-1} + (t - t_{k-1})\right)\right]$$

$$y_{k-1}(t_k^-, t_{k-1}, u_{k-1}) = \prod_{j=1}^{k-1}\left|1 + \delta_i^{j-1}\right| y_0 \exp\left[2\nu^2 m \sum_{j=1}^{k-1} q_{j-1}\Delta t_j\right]$$

$$y_k(t_k) = \left|1 + \delta_i^{k-1}\right| y_{k-1}(t_k^-, t_{k-1}, u_{k-1})$$

$$= \prod_{j=1}^{k}\left|1 + \delta_i^{j-1}\right| y_0 \exp\left[2\nu^2 m \sum_{j=1}^{k} q_{j-1}\Delta t_j\right]. \tag{4.48}$$

Then for $k \in I(1, m)$, the solution of (4.46) on $[t_k, t_{k-1})$ is

$$y_k(t, t_k, y_k) = y_k \exp\left[2\nu^2 m q_k(t - t_k)\right]$$

$$= \prod_{j=1}^{k}\left|1 + \delta_i^{j-1}\right| y_0 \exp\left[2\nu^2 m \left(\sum_{j=1}^{k} q_{j-1}\Delta t_j + (t - t_k)\right)\right], \tag{4.49}$$

and

$$y_k(t_{k+1}^-, t_k, y_k) = \prod_{j=1}^{k}\left|1 + \delta_i^{j-1}\right| y_0 \exp\left[2\nu^2 m \sum_{j=1}^{k+1} q_{j-1}\Delta t_j\right], \tag{4.50}$$

so

$$y_{k+1}(t_{k+1}) = \left|1 + \delta_i^{k+1}\right| y_k(t_{k+1}^-, t_k, y_k)$$

$$= \prod_{j=1}^{k+1}\left|1 + \delta_i^{j-1}\right| y_0 \exp\left[2\nu^2 m \sum_{j=1}^{k+1} q_{j-1}\Delta t_j\right]. \tag{4.51}$$

Therefore, using mathematical induction, it follows that for any $k \in I(1, \infty)$,

$$y_{k-1}(t, t_{k-1}, y_{k-1}) = \prod_{j=1}^{k-1} \left|1 + \delta_i^{j-1}\right| y_0 \exp \left[2\nu^2 m \left(\sum_{j=1}^{k-1} q_{j-1}\Delta t_{j-1} + (t - t_{k-1})\right)\right]$$

$$y_k(t_k) = \prod_{j=1}^{k} \left|1 + \delta_i^{j-1}\right| y_0 \exp \left[2\nu^2 m \sum_{j=1}^{k} q_{j-1}\Delta t_j\right]. \tag{4.52}$$

From the definition of y_k and (4.52), for $k \in I(1, \infty)$ and $t \in [t_{k-1}, t_k)$,

$$u_{k-1}^2(t, t_{k-1}, u_{k-1}) = \frac{\nu^2 y_{k-1}(t, t_{k-1}, y_{k-1})}{1 + y_{k-1}(t, t_{k-1}, y_{k-1})}$$

$$= \frac{\nu^2 \prod_{j=1}^{k-1} \left|1 + \delta_i^{j-1}\right| u_0^2}{\prod_{j=1}^{k-1} \left|1 + \delta_i^{j-1}\right| u_0^2 + (\nu^2 - u_0^2) \exp\left[-2\nu^2 m \left(\sum_{j=1}^{k-1} q_{j-1}\Delta t_j + (t - t_{k-1})\right)\right]}$$

$$= \frac{\nu^2 u_0^2}{u_0^2 + (\nu^2 - u_0^2) \exp\left[-2\nu^2 m \left(\sum_{j=1}^{k-1} q_{j-1}\Delta t_j + (t - t_{k-1})\right)\right] \prod_{j=1}^{k-1} \left|1 + \delta_i^{j-1}\right|^{-1}} \tag{4.53}$$

and

$$u(t, t_k, u_k) = \frac{\nu u_0}{\left(u_0^2 + (\nu^2 - u_0^2) \exp\left[-2\nu^2 m \left(\sum_{j=1}^{k} q_{j-1}\Delta t_{j-1} + (t - t_k)\right)\right] \prod_{j=1}^{k-1} \left|1 + \delta_i^{j-1}\right|^{-1}\right)^{\frac{1}{2}}}. \tag{4.54}$$

Further, for $k \in (1, \infty)$,

$$u(t_k) = \frac{u_0 \nu}{\left(u_0^2 + (\nu^2 - u_0^2) \exp\left[-2\nu^2 m \sum_{j=1}^{k+1} q_{j-1}\delta t_{j-1}\right] \prod_{j=1}^{k} \left|1 + \delta_i^{j-1}\right|^{-1}\right)^{\frac{1}{2}}}. \tag{4.55}$$

By (4.55), taking the limit as $k\infty$, it follows that the initial positions u_{k-1} will converge and

$$\lim_{k \to \infty} u_{k-1} = \nu. \tag{4.56}$$

Further, by (4.54)

$$\lim_{k \to \infty} u(t, t_{k-1}, u_{k-1}) = \nu. \tag{4.57}$$

Therefore, taking the limit of the upper comparison solution $r(t, t_0, u_0)$ at $t \to \infty$, the long term behavior of is such that

$$\lim_{t \to \infty} r(t, t_0, u_0) = \eta, \tag{4.58}$$

where

$$\eta = \limsup_{k \to \infty} \eta_{k-1}, \tag{4.59}$$

if it exists and

$$\eta_{k-1} = \left(\frac{a_{k-1}}{q_{k-1}} + \frac{4b_{k-1}\,(m-1)\,\sqrt{\frac{1}{2}}\exp\left[-\frac{1}{2}\right] + \beta_{k-1}^2\,(m-1)\,c_{k-1}\exp\left[-1\right]}{4q_{k-1}m} \right)^{\frac{1}{2}}. \tag{4.60}$$

Thus, if the limit superior η exists, the solution process of (4.16) is globally upper cohesive a.s. on $[0, \infty)$.

Similarly, the limit of the solution of the lower comparison equation (4.39) as $t \to \infty$ is

$$\lim_{t \to \infty} \rho(t, t_0, u_0) = \alpha, \tag{4.61}$$

where

$$\alpha = \liminf_{k \to \infty} \alpha_{k-1} \tag{4.62}$$

and

$$\alpha_{k-1} \geq \left(\frac{a_{k-1}}{q_{k-1}} - \frac{4(m-1)b_{k-1}\sqrt{\frac{c_{k-1}}{2}}\exp\left[-\frac{1}{2}\right] + \beta_{k-1}^2(m-1)c_{k-1}\exp\left[-1\right]}{4q_{k-1}m} \right)^{\frac{1}{2}}. \tag{4.63}$$

Moreover, the solution process of (4.16) is globally lower cohesive a.s. on $[t_0, \infty)$.

Using the long term behavior of the comparison equations in conjunction with Lemmas 4.2 and 4.3, we establish the following theorem.

Theorem 4 *Let the hypotheses of Lemmas 4.2 and 4.3 be satisfied. Then the network is locally cohesive in the almost surely on $[t_{k-1}, t_k)$ for $k \in I(1, \infty)$. If additionally η exists and is finite, then the network is globally cohesive almost surely on $[t_0, \infty)$.*

Proof. From Lemmas 4.2 and 4.3,

$$\rho_{k-1}(t, t_{k-1}, \rho_{k-1}) \leq V_{k-1}(z_i^{k-1}(t)) \leq r_{k-1}(t, t_{k-1}, r_{k-1}) \tag{4.64}$$

with probability 1. Moreover, as the solution to the upper comparison equation is bounded above by η_{k-1} and the solution to the lower comparison equation is bounded below by α_{k-1}, the network is cohesive almost surely. Suppose that η exist and is finite. Then, we have

$$\rho(t, t_0, u_0) \leq V(z_i(t, t_0, z_i^0)) \leq r(t, t_0, u_0) \tag{4.65}$$

for $t \geq t_0$. As the solutions ρ and r are bounded, the network is globally cohesive with probability 1. $\quad\square$

4.5.1 Invariant Sets

In the case of the hybrid stochastic dynamical network, we can first consider the behavior of the solution process on the interval $[t_{k-1}, t_k)$. In this situation the invariant sets can be found in the same manner as those in Section invariant sets3. For $k \in I(1, \infty)$, let us denote

$$r_2 = \left(\frac{a_{k-1}}{q_{k-1}} - \frac{4(m-1)b_{k-1}\sqrt{\frac{c_{k-1}}{2}}\exp\left[-\frac{1}{2}\right] + \beta_{k-1}^2(m-1)c_{k-1}\exp\left[-1\right]}{4q_{k-1}m} \right)^{\frac{1}{2}} \tag{4.66}$$

and

$$r_1 = \left(\frac{a_{k-1}}{q_{k-1}} + \frac{4b_{k-1}(m-1)\sqrt{\frac{c_{k-1}}{2}}\exp\left[-\frac{1}{2}\right] + \beta^2(m-1)c_{k-1}\exp\left[-1\right]}{4q_{k-1}m} \right)^{\frac{1}{2}}. \tag{4.67}$$

Further, let us define the following sets:

$$\begin{cases} A_{k-1} &= B\left(0, r_2\right) \\ B_{k-1} &= B^c\left(0, r_2\right) \cap B\left(0, r_1\right) \\ C_{k-1} &= B^c\left(0, r_1\right) \end{cases} \tag{4.68}$$

From the analysis developed in that section, we establish the following theorem for the solution on the interval $[t_{k-1}, t_k)$.

THEOREM 4.4 *Let the hypotheses of Lemmas 4.2 and 4.3 be satisfied. Then almost surely,*

(i) the set $A_{k-1} \cup B_{k-1}$ is conditionally invariant relative to A_{k-1};

(ii) the set B_{k-1} is self-invariant;

(iii) the set $B_{k-1} \cup C_{k-1}$ is conditionally invariant relative to C_{k-1}.

Proof. Following the proof outlined in Theorem 3.2, the result follows directly. □

By considering the limit as $k \to \infty$, we also establish the following result for the long-range invariant sets of (4.16). For $k \in (1, \infty)$

$$\lim_{k \to \infty} u(t_{k-1}) = \lim_{k \to \infty} u(t, t_{k-1}, u_{k-1}) \tag{4.69}$$

for both the upper and lower comparison equations, then as $k \to \infty$

$$\alpha \leq \left\| z_i^{k-1}(t_{k-1}) \right\| \leq \eta \tag{4.70}$$

and

$$\alpha \leq \left\| z_i^{k-1}(t, t_{k-1}, z_i^{k-1}) \right\| \leq \eta \tag{4.71}$$

for sufficiently large $k \in (1, \infty)$. Thus, (4.16) exhibits long-range self-invariance for every member of the network.

In Section 4.6, we use numerical simulations to better understand the estimates and network behavior on the intervals $[t_{k-1}, t_k)$ for a finite number k.

4.6   Numerical Simulations

In this section, we consider numerical simulations for the multicultural dynamic network governed by the stochastic differential equation (4.16). We use a Euler-Maruyama [17, 11, 12] type numerical approximation scheme. We consider a network of six members, using the same initial position and varying the parameters a_{k-1}, b_{k-1}, and β_{k-1}, $k \in I(1, \infty)$. Further, we consider the case such that $\xi_{ij}^{k-1}(t)$ for $i, j \in I(1, 6)$ are a one dimensional Brownian motion process with mean of zero and variance of 1 over the interval $[0, 1]$.

Often in a cultural network, events such as natural disasters, sudden political or economic changes, etc., can cause rippling effects in the cultural network. These changes can be characterized by the parametric changes in the stochastic differential equation (4.16). Therefore, we choose to simulate such a situation in the models in this section. Here, we choose 5 arbitrary times t_k on the interval $(0, 1)$ for which the model experiences an intervention on the dynamic. Further, for each t_k, $k \in I(1, 5)$, we set $x_i^k(t_k) = (1 + \delta_i^k)x_i^k(t^-)$, δ_i^k is a constant for fixed i and $k \in I(1, 5)$, and consider the various scenarios based on changing the parameters a_k, b_k and β_k.

In order to consider the effects of changing the parametric quantity a_{k-1}, we consider various models for which $\beta_{k-1} = 2, b_{k-1} = 1, c_{k-1} = 2$, and $q_{k-1} = \frac{1}{7}$ are held constant for $k \in I(1, 5)$ and $a_k = a_{k-1} + 1$, $a_0 = 2$. The plot of the position $z_i(t)$ for $t \in [0, 1]$ is given in Figure 4.1 and Figure 4.2 is the plot of the positions on the interval $[0.2, 0.4]$.

In order to consider the effects of changing the parametric quantity b_{k-1}, we consider the model for which $a_{k-1} = 2, \beta_{k-1} = 2, c_{k-1} = 2$, and $q_{k-1} = \frac{1}{7}$ are held constant for $k \in I(1, 5)$ and $b_k = b_{k-1} + 1, b_0 = 1$. Figure 4.3 exhibits the simulated positions of the members z_i.

In order to consider the effects of changing the parametric quantity β_{k-1}, we consider the model for which $a_{k-1} = 2, b_{k-1} = 1, c_{k-1} = 2$, and $q_{k-1} = \frac{1}{7}$ are held constant for $k \in I(1, 5)$ and $\beta_k = \beta_{k-1} + 1$,
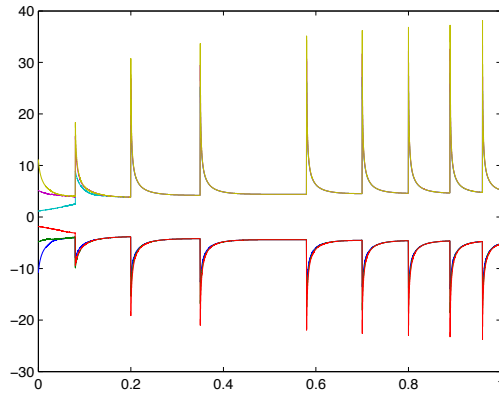
Figure 4.1.: Euler-Maruyama approximation of the differential equation with six members and parameter $a_k = a_{k-1} + 1$.
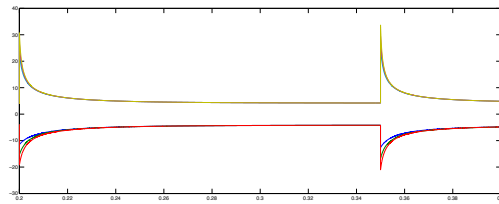

Figure 4.2.: Euler-Maruyama approximation of the differential equation with six members and parameter $a_k = a_{k-1} + 1$ for t in the interval $[0.2, 0.3]$.
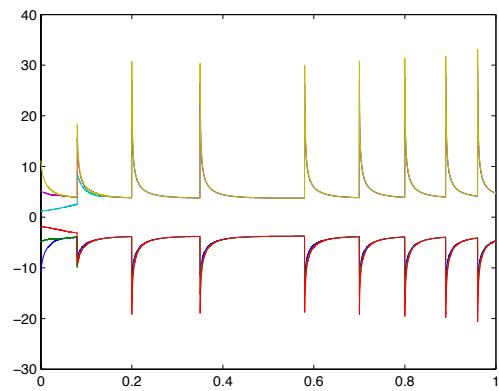

Figure 4.3.: Euler-Maruyama approximation of the differential equation with six members and parameter $b_k = b_{k-1} + 1$.
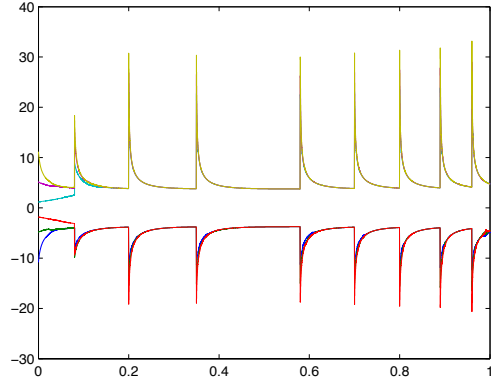
Figure 4.4.: Euler-Maruyama approximation of the differential equation with six members and parameter $\beta_k = \beta_{k-1} + 1$.
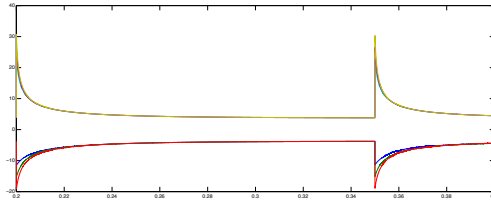

Figure 4.5.: Euler-Maruyama approximation of the differential equation with six members and parameter $\beta_k = \beta_{k-1} + 1$.

$\beta_0 = 2$. In Figure 4.4, we plot the positions of the members for $t \in [0, 1]$ and Figure 4.5 is the position for $t \in [0.2, 0.4]$.

In order to consider the effects of a change in the parametric quantity a_{k-1}, and β_{k-1}, we consider the model for which $b_k = 1, c_{k-1} = 2$, and $q_{k-1} = \frac{1}{7}$ are held constant for $k \in I(1, 5)$ and $a_k = a_{k-1} + 1$, and $\beta_k = \beta_{k-1} + 1, \beta_0 = 2$. The plot of the positions of the members of the simulated network are given in 4.6

4.7 Conclusion

Maintaining diversity while simultaneously fostering a sense of community membership, individual cultural identity, and cohesion is currently a goal among communities worldwide. It is important for members in a society to both feel as a part of the community in which they live and interact as well as feel free to embrace a strong sense of self and individuality. We seek to better understand the factors that play a role in obtaining such a balance by considering the impact of the repulsive and attractive forces influencing
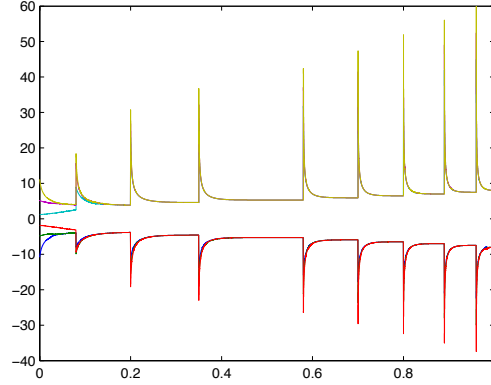
Figure 4.6.: Euler-Maruyama approximation of the differential equation with six members and parameters $a_k = a_{k-1} + 1$, $b_k = b_{k-1} + 1$ and $\beta_k = \beta_{k-1} + 1$.

the multicultural network as in the previous work [13, 14]. Attractive influences can be thought of as attributes that bring people to active membership within the group. Social acceptance, gaining social status, economic opportunity, career growth, common purpose and membership, personal development, and a sense of mutual respect, trust and understanding are examples of attractive influences within a social cultural network. Repelling forces are attributes that create some desire for individuals to leave or be less involved in the group or to preserve some personal identity from one other with their individual magnitude of inner repulsive force. A desire to retain a sense of individuality, economic or emotional cost, interpersonal conflict within the group, or disagreement with parts of the overall philosophies of the group are forces that may be considered as repulsive forces. The goal of the presented multicultural dynamic network is model the balance sought by members of the network in achieving these type of objectives. By doing so, we can consider the impact that policies and environmental factors may have on such a network.

By considering a hybrid dynamic model, we are able to better understand the impacts of outside influences that occur within a community members and the cultural impacts such events have on the modeled cultural network. We have considered change based on the parameters that allow the perturbed multicultural dynamic network to remain cohesive while retaining a cultural state that is distinctive from the cultural state center of the network. We established qualitative and quantitative conditions that are computationally attractive and verifiable. We also conducted simulations of the multicultural network that exhibit the influence of the random perturbations as well as demonstrate the long-term behavior of the multicultural network.

We are interested in further exploring similar multicultural networks in the context of better understanding the relative cultural affinity $\|x_{ij}\|$ between members within the network and not just the cultural affinity

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between the cultural state of a member relative to the center of the network. The goal is to better understand the environmental factors that help foster a sense of individuality and diversity between all members within the network while maintaining a cohesive structure.

Chapter 5

Conclusion and Future Work

Maintaining diversity while simultaneously fostering a sense of community membership, individual cultural identity, and cohesion is currently a goal among communities worldwide. It is important for members in society to both feel as a part of the community in which they live and interact as well as feel free to embrace a strong sense of self and individuality. We seek to better understand the factors that play a role in obtaining such a balance by considering the impact of the repulsive and attractive forces influencing the multicultural network. The goal of the presented multicultural dynamic network is model the balance sought by members of the network in achieving these type of objectives. By doing so, we can consider the impact that policies and environmental factors may have on such a network.

The presented work provides a framework for considering cultural dynamic networks. It can be noted that in the presented prototypes, the parameters are considered constant. In future work, we wish to consider the case that the parameters are not constant which can be built up from the presented work. In this work, we explored the features of a multicultural network with dynamics described by a specific differential equation and the long term stability and behaviors of individual members within such a network. We are interested in further exploring social networks in the context of better understanding the relative cultural state affinity between agents $\|x_{ij}\|$ and not just the cultural affinity between an agent and the center of the network. The presented work may be utilized to create a coupled dynamic system in which relative cultural state affinity can be further explored. Our hopes are to better understand what factors may lead to preserving a lower bound on the relative cultural state affinity $\|x_{ij}\|$ that is strictly greater than zero as $t \to \infty$. In modeling such a network, we are looking to better understand how diversity between all members may be maintained over the long term within a culturally diverse network.

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