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# Modeling in Finance and Insurance With Levy-It'o Driven Dynamic Processes under Semi Markov-type Switching Regimes and Time Domains

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Modeling in Finance and Insurance with Levy-Itó Driven Dynamic Processes under Semi Markov-type Switching Regimes and Time Domains.

by

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**Dedication**

In memory of my late Mother Julienne Eken Assonken and
to my Father Benoit Assonken.

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**Table of Contents**

**List of Tables**

## List of Figures

## Abstract

Mathematical and statistical modeling have been at the forefront of many significant advances in many disciplines in both the academic and industry sectors. From behavioral sciences to hard core quantum mechanics in physics, mathematical modeling has made a compelling argument for its usefulness and its necessity in advancing the current state of knowledge in the 21rst century. In Finance and Insurance in particular, stochastic modeling has proven to be an effective approach in accomplishing a vast array of tasks: risk management, leveraging of investments, prediction, hedging, pricing, insurance, and so on. However, the magnitude of the damage incurred in recent market crisis of 1929 (the great depression), 1937 (recession triggered by lingering fears emanating from the great depression), 1990 (1 year recession following a decade of steady expansion) and 2007 (the great recession triggered by the sub-prime mortgage crisis) has suggested that there are certain aspects of financial markets not accounted for in existing modeling. Explanations have abounded as to why the market underwent such deep crisis and how to account for regime change risk. One such explanation brought forth was the existence of regimes in the financial markets. The basic idea of market regimes underscored the principle that the market was intrinsically subjected to many different states and can switch from one state to another under unknown and uncertain internal and external perturbations. Implementation of such a theory has been done in the simplifying case of Markov regimes. The mathematical simplicity of the Markovian regime model allows for semi-closed or closed form solutions in most financial applications while it also allows for economically interpretable parameters. However, there is a hefty price to be paid for such practical conveniences as many assumptions made on the market behavior are quite unreasonable and restrictive. One assumes for instance that each market regime has a constant propensity of switching to any other state irrespective of the age of the current state. One also assumes that there are no íntermediateśtates as regime changes occur in a discrete manner from one of the finite states to another. There is therefore

no telling how meaningful or reliable interpretation of parameters in Markov regime models are.

In this thesis, we introduced a sound theoretical and analytic framework for Levy driven linear stochastic models under a semi Markov market regime switching process and derived Itó formula for a general linear semi Markov switching model generated by a class of Lévy Itó processes (**1**). Itó formula results in two important byproducts, namely semi closed form formulas for the characteristic function of log prices and a linear combination of duration times (**2**). Unlike Markov markets, the introduction of semi Markov markets allows a time varying propensity of regime change through the conditional intensity matrix. This is more in line with the notion that the market's chances of recovery (respectively, of crisis) are affected by the recession's age (respectively, recovery's age). Such a change is consistent with the notion that for instance, the longer the market is mired into a recession, the more improbable a fast recovery as the the market is more likely to either worsens or undergo a slow recovery. Another interesting consequence of the time dependence of the conditional intensity matrix is the interpretation of semi Markov regimes as a pseudo-infinite market regimes models. Although semi Markov regime assume a finite number of states, we note that while in any give regime, the market does not stay the same but goes through an infinite number of changes through its propensity of switching to other regimes. Each of those separate íntermediateśtates endows the market with a structure of pseudo-infinite regimes which is an answer to the long standing problem of modeling market regime with infinitely many regimes.

We developed a version of Girsanov theorem specific to semi Markov regime switching stochastic models, and this is a crucial contribution in relating the risk neutral parameters to the historical parameters (**3**). Given that Levy driven markets and regime switching markets are incomplete, there are more than one risk neutral measures that one can use for pricing derivative contracts. Although much work has been done about optimal choice of the pricing measure, two of them jump out of the current literature: the minimal martingale measure and the minimum entropy martingale measure. We first presented a general version of Girsanov theorem explicitly accounting for semi Markov regime. Then we presented Siu and Yang pricing kernel. In addition, we developed the conditional and unconditional minimum entropy martingale measure which minimized the dissimilarity between the historical and risk neutral probability measures through a version of Kulbach Leibler distance (**4**).

Estimation of a European option price in a semi Markov market has been attempted before in the restricted case of the Black Scholes model. The problems encountered then were twofold: First, the author employed a Markov chain Monte Carlo methods which relied much on the tractability of the likelihood function of the normal random sequences. This tractability is unavailable for most Levy processes, hence the necessity of alternative pricing methods is essential. Second, the accuracy of the parameter estimates required tens of thousands of simulations as it is often the case with Metropolis Hasting algorithms with considerable CPU time demand. Both above outlined issues are resolved by the development of a semi-closed form expression of the characteristic function of log asset prices, and it opened the door to a Fourier transform method which is derived on the heels of Carr and Madan algorithm and the Fourier time stepping algorithm (**5**).

A round of simulations and calibrations is performed to better capture the performance of the semi Markov model as opposed to Markov regime models. We establish through simulations that semi Markov parameters and the backward recurrence time have a substantial effect on option prices (**6**). Differences between Markov and Semi Markov market calibrations are quantified and the CPU times are reported. More importantly, interpretation of risk neutral semi Markov parameters offer more insight into the dynamic of market regimes than Markov market regime models (**7**). This has been systematically exhibited in this work as calibration results obtained from a set of European vanilla call options led to estimates of the shape and scale parameters of the Weibull distribution considered, offering a deeper view of the current market state as they determine the in-regime dynamic crucial to determining where the market is headed.

After introducing semi Markov models through linear Levy driven models, we consider semi Markov markets with nonlinear multidimensional coupled asset price processes (**8**). We establish that the tractability of linear semi Markov market models carries over to multidimensional nonlinear asset price models. Estimating equations and pricing formula are derived for historical parameters and risk neutral parameters respectively (**9**). The particular case of basket of commodities is explored and we provide calibration formula of the model parameters to observed historical commodity prices through the LLGMM method. We also study the case of Heston model in a semi Markov switching market where only one parameter is subjected to semi Markov regime changes. Heston model is one the most popular model in option pricing as it reproduces many more stylized facts than Black Scholes model while retaining tractability. However, in addition to having a faster deceasing smiles than observed, one of the most damning shortcomings of most diffusion models such as Heston model, is their inability to accurately reproduce short term options prices. An avenue for solving these issues consists in generalizing Heston to account for semi Markov market regimes. Such a solution is implemented and a semi analytic formula for options is obtained.

# Chapter 1

## A linear Levy-Itó switching model for log asset prices under semi Markov regimes

### 1.1 Introduction

The well known Black Scholes model [6], despite its slew of laureates has long shown well documented weaknesses as it pertains to its consistency with stylized facts of financial asset returns and option prices. Smiles, smirks and skews are well documented empirical features of implied volatilities [10, 26, 60, 66], unexplained in the context of Black Scholes model. Moreover, stylized facts of financial time series also cast a doubt on the appropriateness of the normal log return distribution assumption. The literature goes about solving these issues in two main ways. The first approach uses time dependent deterministic volatility models [39, 46] to capture most of option market empirical properties. However, it has been shown [31] that risk neutral volatilities behave in a random manner. This leads to the development of the second modeling approach consisting of stochastic volatility, local volatility and regime switching models. Stochastic volatility models [33, 40] are based on the assumption that volatility is a dynamic process in itself. In local volatility models [5, 24], the volatility depends on time and stock price through a deterministic functional. In both cases, in addition to the possibility of a misspecified functional form of the volatility [16], the volatility surface often lacks smoothness and at times, takes nonsensical and counterintuitive forms. One of the main advantages of regime switching models as noted by [16], is the interpretability of the market states while disassociating with the very restrictive functional form assumption of the local and stochastic volatility models.
The present chapter is an attempt to extend the currently existing semi Markov switching models for stock price and at setting up a general theoretical framework to study qualitative and quantitative properties of asset price processes. To the best of our knowledge, mostly Markovian interventions on price processes are being successfully studied and have been investigated in many papers [12, 13, 14, 16, 35, 37, 53, 60]. Recently, stochastic models [30, 41, 42, 59, 65] under the influ-

ence of a semi Markov process have been examined. However, these attempts leave two gaping holes. First they are limited to Black Scholes switching models, hence failing to take advantage of the proven flexibility of Levy driven models. In fact, most continuous models, fail to appropriately price short maturity options because existence of such contracts in the market is an acknowledgement that significant market movements can be observed in short period of time. Second, the estimation method of [42] is an MCMC based estimation method, which by his own admission requires tens of thousands of samples, as well as it depends highly on the tractability of the normal likelihood and efficiency of the candidate distribution chosen to implement the Metropolis Hasting algorithm. A Stochastic Maximum Principle for semi Markov switching jump diffusion models [21] has been established, leaving out the class of Lévy processes with infinite activity such as the variance Gamma model or the Normal Inverse Gamma. We propose a theoretical setting for a more inclusive model.

The organizational outline of this Chapter is as follows: in Section 1.2, we introduce the necessary definitions and notations and we present known results. In Section 1.3, we find a closed form solution of a Lévy type of SDE. In Section 1.4, we derive Itó differential formula and the infinitesimal generator for a class of stochastic linear hybrid models under semi Markovian and Lévy-type structural perturbations. Section 1.5 is concerned with the derivation of a closed form characteristic function of the log price process. This is useful for recovering risk neutral densities to estimate option prices. Moreover, this provides an alternative tool to the computationally extensive continuous time MCMC and the two-step numerical integration procedure [30, 42], for simulating option prices and calibrating model parameters to market option prices.

## 1.2 Preliminary Definitions and Results

In this work, $T^\star < \infty$ and $T \in [0, T^\star]$. $T^\star$ and $T$ stand for the time horizon of the market and the maturity time of a contingent claim, respectively; $\big(\Omega, \mathbb{F}, P\big)$ is a complete probability space; $\theta$ is a semi Markov process defined on $\mathbb{R}^+ \times \big(\Omega, \mathbb{F}, P\big)$ into $E$, where $E$ is an at most countable subset of the set of natural numbers $\mathbb{N}$ and $\mathbb{R}^+ = [0, \infty)$. For each $n \in \mathbb{N}$, $T_n$ stands for the $n-th$ jump time of $\theta$. For $s \in [0,T]$ and $\theta_{s^-} = j$, $(L^j_{s-})_{s\in[0,T]}$ is the Lévy process with Lévy triplet $(\mu(j), \sigma(j), \nu(j, ))$, where $\mu(j), \sigma(j)$ and $\nu(j, )$ are the drift rate, the diffusion rate and the Lévy measure, respectively. $(\mathbb{L}_t)_{t\in[0,T]}$, $(\mathbb{H}_t)_{t\in[0,T]}$ and $(\mathbb{H}_t \vee \mathbb{L}_t)_{t\in[0,T]}$ are sub-sigma algebras of $\mathbb{F}$ generated by the collection of Lévy processes $L^j$, $\forall j \in E$, the semi Markov process $\theta$ and $(L_t^\theta, \theta_t)_{t\in[0,T]}$, respectively. Let $(\beta_n)_{n\geq 0}$ and $(\mathbb{B}_t)_{t\in[0,T]}$ be a discrete-time real valued stochastic process and the sub-sigma algebra of $\mathbb{F}$ adapted to the discrete process $\beta_n$, respectively. We denote the enlarged filtrations $(\mathbb{L}_t \vee \mathbb{B}_t)_{t\in[0,T]}$ and $(\mathbb{H}_t \vee \mathbb{B}_t)_{t\in[0,T]}$ by $(\bar{\mathbb{L}}_t)_{t\in[0,T]}$ and $(\bar{\mathbb{H}}_t)_{t\in[0,T]}$, respectively. Let $\psi(j, ., .)$ be the Poisson random measure with compensator $\nu(j, dz)$. It is also assumed that the sequence $(\beta_n)_{n\geq 0}$ is independent of both $\psi(j, ., .)$ and the Brownian process $B_t$, for $j \in E$.

DEFINITION 1.2.1 *[17] Let $\theta$ and $\{T_n\}_{n=1}^{\infty}$ be a semi Markov process and its jump time sequence with $T_0 = 0$, respectively. A couple $(\theta_n, T_n)$ is called a Markov renewal process with kernel $Q$ induced by the semi Markov Process $(\theta_t)$, if it satisfies:*

$$\begin{aligned} &P\big(\theta_n = j, T_n \leq t | (\theta_k, T_k),\ k = 1, 2...n-1\big) \\ &= P\big(\theta_n = j, T_n - T_{n-1} \leq t - T_{n-1} | \theta_{n-1}, T_{n-1}\big) \\ &= Q(\theta_{n-1}, j, t - T_{n-1}), \end{aligned} \tag{1.2.1}$$

*where $\theta_n$ stands for $\theta_{T_n}$.*

REMARK 1.2.1 $\tau_n = T_{n+1} - T_n$ denotes a holding time at $T_n$. The holding times conditional on the current state are independent [17]. The kernel in (1.2.1) can be represented as:

$$Q(i, j, t - T_n) = P\big(\theta_n = j, T_n - T_{n-1} \leq t - T_{n-1} | \theta_{n-1} = i\big). \tag{1.2.2}$$

Moreover, for $(\theta_n, T_n) = (\theta_{n(t)}, T_{n(t)}), \forall t \in [0, T]$, where

$$n(t) = max\{k \in \mathbb{N}, T_k \leq t\} \tag{1.2.3}$$

In particular,

$$Q(i, j, t) = P(\theta_n = j, T_n - T_{n-1} \leq t | \theta_{n-1} = i). \tag{1.2.4}$$

Furthermore, we define

$$p_{ij} = \lim_{t\to\infty} Q(i, j, t), \tag{1.2.5}$$

where $p_{ij}$ is called the steady state transition probability of the embedded Markov chain from state $i$ to state $j$ with $i, j \in E$ and $n(E) = m$.

For the sake of completeness, we present survival distribution and sojourn time distributions associated with the semi Markov process $\theta$.

DEFINITION 1.2.2 *The conditional cumulative distribution of the holding/sojourn/residence time (respectively, survival function) given that $\theta$ transits from a state $i$ to state $j$ is defined by $F(t|i,j) = P(\tau_n \leq t \big| \theta_n = j, \theta_{n-1} = i)$ (respectively, $S(.|i,j) = 1 - F(.|i,j)$).*

In the following lemma, we outline a few well known properties of semi Markov processes [17, 30].

LEMMA 1.2.1 *The kernel of the semi Markov process defined in* (1.2.4) *is represented by*

$$Q(i,j,t) = p_{i,j} F(t|i,j). \tag{1.2.6}$$

*Moreover,*

$$S(t|i) = 1 - \sum_{j \in E} p_{i,j} F(t|i,j),, \tag{1.2.7}$$

$$\frac{f(r|\theta_0 = i)}{P(T_1 > s|\theta_0 = i)} = \frac{-\frac{dS}{dr}(t|i,j)}{S(s^-|i)}, \text{ for } r \geq s \tag{1.2.8}$$

$$\lambda_{i,j}(t) = p_{i,j} \frac{-\frac{dS}{dt}(t|i,j)}{S(t^-|i)}. \tag{1.2.9}$$

*Proof.* We first establish (1.2.6). From (1.2.4) we have

$$\begin{aligned} Q(i,j,t) &= P(\theta_n = j, T_n - T_{n-1} \leq t | \theta_{n-1} = i), \\ &= P(\theta_n = j | \theta_{n-1} = i) P(T_n - T_{n-1} \leq t | \theta_{n-1} = i, \theta_n = j) \\ &= p_{i,j} F(t|i,j). \end{aligned}$$

For the proof of (1.2.7), we use Definition 1.2.2 and (1.2.6).

$$\begin{aligned} F(t|i) &= P(\tau_n \leq t | \theta_n = i) \\ &= \sum_{j \in E} p_{i,j} F(t|i,j) \end{aligned} \tag{1.2.10}$$

Hence,

$$S(t|i) = 1 - \sum_{j \in E} p_{i,j} F(t|i,j).$$

This completes the proof of (1.2.7). For the proof of (1.2.8), for $r > s$, we have

$$\begin{aligned} &\frac{f(r|\theta_0 = i)}{P(T_1 > s|\theta_0 = i)}, (\text{ definition of conditional density}) \\ &= \frac{-\frac{dS}{dr}(r|i)}{P(T_1 > s|\theta_s = i)}, (\text{ definition of survival function}) \\ &= \frac{-\frac{dS}{dr}(r|i)}{S(s^-|i)}. \end{aligned} \tag{1.2.11}$$

This completes the proof of (1.2.8). The proof of (1.2.9) follows from the definition of Hazard functions and (1.2.6). This completes the proof of the lemma. □

REMARK 1.2.2 A homogeneous Markov process is a particular case of semi Markov process. Hence, $Q_{ij}(t) = p_{ij}(1 - e^{q(i)})$. We also have the following relationship:

$$q_{ij} = p_{ij}q(i), \tag{1.2.12}$$

where $(q_{ij})_{m\times m}$ is the Infinitesimal generator (intensity matrix) of a Markov process, and $(p_{ij})_{m\times m}$ is its transition probability matrix defined in (1.2.5). The following can also be inferred from (1.2.6) and (1.2.7),

$$S(t|i) = 1 - \sum_{j\in E} Q(i,j,t). \tag{1.2.13}$$

DEFINITION 1.2.3 *Let $y_t$ be the backward recurrence time of the semi Markov process $\theta$ at time t. $y_t$ is defined as follows:*

$$y_t = \sum_{n\geq 0}(t - T_n)1_{(T_n\leq t<T_{n+1})}, \tag{1.2.14}$$

*where the sequence $\{T_n\}_{n=0}^{\infty}$ is introduced in Definition 1.2.1.*

DEFINITION 1.2.4 *Let $\psi : \mathbb{R}^+ \times \mathbb{R} \times \mathcal{R}^+ \mapsto \mathbb{R}$ be the random Poisson measure with intensity measure $\nu$, $H$ and $G$ smooth functions defined on $\mathbb{R}^+ \times \mathbb{R}$ into $\mathbb{R}$, satisfying the condition: $\int_{z\in\mathbb{R}} \Big(\big(1 + H^2(z,j)\big)1_{|z|>1} + G^2(z,j)1_{|z|\leq 1}\Big)\nu(j,dz) < \infty$, $\forall j \in E$. Moreover; $\bar{\psi} = \psi - \nu$ denotes the compensated Poisson measure associated with $\psi$.*

In the following we present a lemma, which would be used, subsequently.

LEMMA 1.2.2 *Let $(a_n, b_n)$ and $(c_n, d_n)$ be two renewal processes defined on the same probability space $(\Omega, \mathbb{F}, P)$ and state space E. Then the renewal processes have identical transition probability matrices and sojourn time distributions, respectively.*

*Proof.* From (1.2.5), it's clear that the transition probability and the holding time distribution are completely defined by the kernel matrix. In fact, we have,

$$\lim_{t\to\infty} Q(i,j,t) = \lim_{t\to\infty} p_{ij}F(t|i,j) = p_{ij},$$

and hence,

$$F(t|i,j) = \frac{Q(i,j,t)}{p_{ij}}.$$

This establishes the result. □

LEMMA 1.2.3 *Let $n(t)$ be defined as in* (1.2.3). *The pair $(\theta_t, t - T_{n(t)})$ is a Markov process.*

*Proof.* Let be $s \leq t$ with $t \in [T_n, T_{n+1})$ and $s \in [T_m, T_{m+1})$ $(m < n)$. For $u \leq s$, we have:

$$\begin{aligned}
&P\big(\theta_t = i, t - T_{n(t)} \leq a|(\theta_u, u - T_{n(u)})\big) \text{ , (for some a} \in \mathbb{R}^+)\\
&= P\big(\theta_{T_n} = i, T_{n+1} - T_n \leq a|(\theta_u, u - T_{n(u)})\big), (\text{ for t } \in [T_n, T_{n+1}))\\
&= P\big(\theta_{T_n} = i, T_{n+1} - T_n \leq a|(\theta_{T_k}, T_k), k \leq m\big), (\text{ for s} \in [T_m, T_{m+1}))\\
&= P\big(\theta_{T_n} = i, T_{n+1} - T_n \leq a|\theta_{T_m}\big), \big(\text{ Markov renewal process property}\big)\\
&= P\big(\theta_t = i, t - T_n(t) \leq a|\theta_s\big), \big(\text{ definition of n(t)}\big).
\end{aligned}$$

Hence, the probability at a future time depends only on the most current information at time s. This shows that $(\theta_t, t - T_{n(t)})$ is a Markov process. □

REMARK 1.2.3 For the remainder of this paper $\theta$ is a semi Markov process with jump time $T_n$, with sojourn time $\tau_n = T_{n+1} - T_n \sim f(|\theta_n, \theta_{n+1})$ with CDF $F(|\theta_n, \theta_{n+1})$ and with survival CDF $S(|\theta_n, \theta_{n+1}) = 1 - F(|\theta_n, \theta_{n+1})$. The semi Markov kernel is denoted $Q(i,j,t)$, the backward recurrence time $y_t$ is defined in (1.2.14) and $(p_{i,j})_{m\times m}$ is the transition probability matrix of the embedded Markov chain.

### 1.3 Method for Finding Closed Form Solutions

In this section, we find a closed form solution of a Lévy-type Linear Stochastic Differential Equation under semi Markovian structural perturbations. The presented extension is based on the procedure described in [47]. The usefulness of the result is at least twofold. It is used to establish the martingale property for certain processes in Chapter 2. In addition, it is used to formulate a general expression for the simple return process with any Lévy and semi Markov jump choices. We consider the following Lévy-type SDE:

$$dx_t = x_{t^-} dL_t^{\theta}, \quad x(0) = x_0, \tag{1.3.1}$$

where,

$$dL^{\theta}_t = \mu(\theta_t)dt + \sigma(\theta_t)dB_t + \int_{|z|>1} H(z,\theta_t)\psi(\theta_t, dz, dt)$$
$$+ \int_{|z|\le 1} G(z,\theta_t)\bar{\psi}(\theta_t, dz, dt), \tag{1.3.2}$$

$\theta$ is the semi Markov process defined in Section 1.2; $\psi$, $\nu$ $G$ and $H$ are in Definition 1.2.4. Following the procedure described in Chapter 2, [47], we break down (1.3.1) into the following four types of simplified SDEs.

$$\begin{cases} dx^1_t = x^1_t \mu(\theta_t)dt \\ dx^2_t = x^2_{t-}\sigma(\theta_t)dB_t \\ dx^3_t = x^3_{t-}\int_{|z|>1} H(z,\theta_t)\psi(\theta_t, dz, dt) \\ dx^4_t = x^4_{t-}\int_{|z|\le 1} G(z,\theta_t)\bar{\psi}(\theta_t, dz, dt). \end{cases} \tag{1.3.3}$$

Imitating the procedure in [47], the closed form solution processes of

$$dx^1_t = x^1_{t-}\mu(\theta_t)dt \text{ and } dx^2_t = \sigma(\theta_t)x^2_t dB_t,$$

are

$$x^1_t = \big[\exp\big(\int_0^t \mu(\theta_s)ds\big)\big]c_1 \text{ and } x^2_t = \exp\left[-\frac{1}{2}\int_0^t \sigma^2(\theta_s)ds + \int_0^t \sigma(\theta_s)dB_s\right]c_2, \tag{1.3.4}$$

respectively; $c_1$ and $c_2$ are arbitrary constants. We next consider the third type of SDE in (1.3.3):

$$dx^3_t = x^3_{t-}\int_{|z|>1} H(z,\theta_t)\psi(\theta_t, dz, dt). \tag{1.3.5}$$

We seek a solution of a form:

$$x^3_t = \exp\left[\int_0^t \int_{|z|>1} f_4(z,\theta_s)\psi(\theta_s, dz, ds)\right]c_3, \tag{1.3.6}$$

where $f_4$ is an unknown smooth function to be determined, and $c_3$ is a real random variable. The Ito integral for pure jump processes (1.3.6) yields:

$$\begin{aligned} x^3_{t+\Delta t} - x^3_t &= \sum_{t\le s\le t+\Delta t}\big(x^3_s - x^3_{s-}\big) \\ &= \int_t^{t+\Delta t}\int_{|z|>1}\big(x^3_{s-}e^{f_4(z,\theta_{s-})} - x^3_{s-}\big)\psi(\theta_{s-}, dz, ds) \\ &= \int_t^{t+\Delta t} x^3_{s-}\int_{|z|>1}\big(e^{f_4(z,\theta_{s-})} - 1\big)\psi(\theta_{s-}, dz, ds) \end{aligned} \tag{1.3.7}$$

As $\Delta t$ becomes very small, (1.3.7) reduces to:

$$dx_t^3 = x_{t^-}^1 \int_{|z|>1} \big(e^{f_4(z,\theta_t)} - 1\big)\psi(\theta_t, dz, dt). \tag{1.3.8}$$

Since $x^3$ is solution of stochastic differential equation (1.3.5), we repeat the procedure described in [47] and obtain:

$$\exp\big(f_4(z,\theta_t)\big) - 1 = H(z,\theta_t). \tag{1.3.9}$$

Hence,

$$f_4(z,\theta_t) = ln(1 + H(z,\theta_t)). \tag{1.3.10}$$

Therefore, the general solution of (1.3.5) is represented by

$$x_t^3 = \exp\{\big[\int_0^t \int_{|z|>1} ln(1 + H(z,\theta_s))\psi(\theta_s, dz, ds)\big]\}c_3. \tag{1.3.11}$$

$x^3$ is almost surely finite. Finally, we find a solution of the following stochastic differential equation:

$$dx_t^4 = x_{t^-}^4 \int_{|z|\le 1} G(z,\theta_t)\bar{\psi}(\theta_t, dz, dt). \tag{1.3.12}$$

We seek a solution process of(1.3.12) in the following form:

$$x_t^4 = \exp\{\Big[\int_0^t \int_{|z|\le 1} f_5(z,\theta_s)\bar{\psi}(\theta_{s^-}, dx, ds) \tag{1.3.13}$$

$$+ \int_0^t \int_{|z|\le 1} f_6(z,\theta_{s^-})\nu(\theta_{s^-}, dz)ds\Big]\}c_4, \tag{1.3.14}$$

where $f_5$ and $f_6$ are unknown smooth functions to be determined, and $c_4$ is a real valued random variable. $x^4$ in (1.3.12) is an exponential function semi martingale of the form $v = \int_0^t \int_{|z|\le 1} f_5(z,\theta_s)\bar{\psi}(\theta_s, dx, ds) + \int_0^t \int_{|z|\le 1} f_6(z,\theta_s)\nu(\theta_s, dz)ds$. Applying the Ito formula for discontinuous semi martingales, [1], we have

$$\begin{aligned} dx_t^4 =& \frac{\partial x_t^4}{\partial v} dv^c + \frac{1}{2}\frac{\partial^2 x_t^4}{\partial v^2} d(v^c)d(v^c) + \Big(\Delta x_t^4 - \frac{\partial x^4}{\partial L}\Delta L\Big) \\ =& x_{t^-}^4 \int_{|z|\le 1} \big(f_6(z,\theta_{t^-}) + \exp\left(f_5(z,\theta_{t^-})\right) - 1 - f_5(z,\theta_{t^-})\big)\nu(\theta_{t^-}, dz)dt \\ &+ x_{t^-}^4 \int_{|z|\le 1} \big(\exp\left(f_5(z,\theta_{t^-})\right) - 1\big)\bar{\psi}(\theta_{t^-}, dz, dt). \end{aligned} \tag{1.3.15}$$

Again, following the procedure for finding solution processes in [47], we get:

$$\begin{cases} f_6(z,\theta_t) + \exp\left(f_5(z,\theta_t)\right) - 1 - f_5(z,\theta_t) = 0 \\ \exp\left(f_5(z,\theta_t)\right) - 1 = G(z,\theta_t). \end{cases} \tag{1.3.16}$$

Hence,

$$\begin{cases} f_5(z,\theta_t) = ln(1+G(z,\theta_t)) \\ f_6(z,\theta_t) = ln(1+G(z,\theta_t)) - G(z,\theta_t). \end{cases} \tag{1.3.17}$$

Therefore,

$$x_t^4 = \exp\{\Big[\int_0^t \int_{|z|\le 1} \big[\ln(1+G(z,\theta_{s^-})) - G(z,\theta_{s^-})\big]\nu(\theta_{s^-},dz)ds \tag{1.3.18}$$

$$+ \int_0^t \int_{|z|\le 1} \ln(1+G(z,\theta_{s^-}))\bar{\psi}(\theta_{s^-},dz,ds)\Big]\}c_4. \tag{1.3.19}$$

The product of $x^1, x^2, x^3$ and $x^4$ in (1.3.4), (1.3.4), (1.3.11) and (1.3.19), respectively, yields the solution of initial value problem (1.3.1):

$$\begin{aligned} x_t =& x_0 \exp\{\Big[\int_0^t \Big[\mu(\theta_{s^-}) - \frac{1}{2}\sigma^2(\theta_{s^-}) + \int_{|z|\le 1} \big[\ln(1+G(z,\theta_{s^-})) - G(z,\theta_{s^-})\big]\nu(\theta_{s^-},dz)\Big]ds \\ &+ \int_0^t \sigma(\theta_{s^-})dB_s + \int_0^t \int_{|z|\le 1} \ln(1+G(z,\theta_{s^-}))\bar{\psi}(\theta_{s^-},dz,ds) \\ &+ \int_0^t \int_{|z|>1} \ln(1+H(z,\theta_{s^-}))\psi(\theta_{s^-},dz,ds)\Big]\}. \end{aligned} \tag{1.3.20}$$

In the following, we present a few versions of (1.3.20).

REMARK 1.3.1 We note that adding and subtracting

$$\int_0^t \int_{|z|\le 1} G(z,\theta_{s^-})\bar{\psi}(\theta_{s^-},dz,ds) \text{ and } \int_0^t \int_{|z|>1} H(z,\theta_{s^-})\psi(\theta_{s^-},dz,ds),$$

(1.3.20) reduces to:

$$\begin{aligned} x_t = x_0 \exp\{\Big[&\int_0^t \mu(\theta_{s^-})ds + \int_0^t \sigma(\theta_{s^-})dB_s + \int_0^t \int_{|z|\le 1} G(z,\theta_{s^-})\bar{\psi}(\theta_{s^-},dz,ds) \\ &+ \int_0^t \int_{|z|>1} H(z,\theta_{s^-})\psi(\theta_{s^-},dz,ds) - \frac{1}{2}\int_0^t \sigma^2(\theta_{s^-})ds \\ &+ \int_0^t \int_{|z|\le 1} \big[\ln\big(1+G(z,\theta_{s^-})\big) - G(z,\theta_{s^-})\big]\nu(\theta_{s^-},dz)ds \\ &+ \int_0^t \int_{|z|\le 1} \big[\ln\big(1+G(z,\theta_{s^-})\big) - G(z,\theta_{s^-})\big]\bar{\psi}(\theta_{s^-},dz) \\ &+ \int_0^t \int_{|z|>1} \big[\ln\big(1+H(z,\theta_{s^-})\big) - H(z,\theta_{s^-})\big]\psi(\theta_{s^-},dz,ds)\Big]\} \end{aligned} \tag{1.3.21}$$

Moreover, adding and subtracting $\int_0^t \int_{|z|\leq 1} G(z,\theta_{s^-})\bar{\psi}(\theta_{s^-},dz,ds)$,

$\int_0^t \int_{|z|>1} H(z,\theta_{s^-})\bar{\psi}(\theta_{s^-},dz,ds)$, and $\int_0^t \int_{|z|>1} H(z,\theta_{s^-})\nu(\theta_{s^-},dz)ds$,

(1.3.20) becomes:

$$\begin{aligned}
x_t = x_0 \exp\{ \Big[ & \int_0^t \mu(\theta_{s^-})ds + \int_0^t \sigma(\theta_{s^-})dB_s + \int_0^t \int_{|z|\leq 1} G(z,\theta_{s^-})\bar{\psi}(\theta_{s^-},dz,ds) \\
& + \int_0^t \int_{|z|>1} H(z,\theta_{s^-})\bar{\psi}(\theta_{s^-},dz,ds) \\
& + \int_0^t \int_{|z|>1} \ln\big(H(z,\theta_{s^-})+1\big)\nu(\theta_{s^-},dz)ds - \frac{1}{2}\int_0^t \sigma^2(\theta_{s^-})ds \\
& + \int_0^t \int_{|z|\leq 1} \big[\ln(1+G(z,\theta_{s^-})) - G(z,\theta_{s^-})\big]\bar{\psi}(\theta_{s^-},dz,ds) \\
& + \int_0^t \int_{|z|\leq 1} \big[\ln(1+G(z,\theta_{s^-})) - G(z,\theta_{s^-})\big]\nu(\theta_{s^-},dz)ds \\
& + \int_0^t \int_{|z|>1} \big[\ln(1+H(z,\theta_{s^-})) - H(z,\theta_{s^-})\big]\bar{\psi}(\theta_{s^-},dz,ds)\Big]\}. \qquad (1.3.22)
\end{aligned}$$

In the following remark, we take a look at a few particular cases of interest which will be used, subsequently.

REMARK 1.3.2 If $H(z,\theta_s)$, $G(z,\theta_s)$ and $L_s^\theta$ in (1.3.2) are replaced by $e^{H(z,\theta_s)}-1$, $e^{G(z,\theta_s)}-1$ and

$$\begin{aligned}
dL_s^\theta = & \mu(\theta_s)ds + \sigma(\theta_s)dB_s + \int_{|z|\leq 1} \big[e^{G(z,\theta_s)}-1\big]\bar{\psi}(\theta_s,dz,ds) \\
& + \int_{|z|>1} \big[e^{H(z,\theta_s)}-1\big]\psi(\theta_s,dz,ds), \qquad (1.3.23)
\end{aligned}$$

respectively, then the solution of the IVP (1.3.1) in (1.3.20), (1.3.21) and (1.3.22) reduce to:

$$\begin{aligned}
x_t = x_0 \exp\{ \Big[ & \int_0^t \Big[\mu(\theta_{s^-}) - \frac{1}{2}\sigma^2(\theta_{s^-}) \\
& + \int_{|z|\leq 1} \big[G(z,\theta_{s^-}) + 1 - e^{G(z,\theta_{s^-})}\big]\nu(\theta_{s^-},dz)\Big]ds \\
& + \int_0^t \sigma(\theta_{s^-})dB_s + \int_0^t \int_{|z|\leq 1} G(z,\theta_{s^-})\bar{\psi}(\theta_{s^-},dz,ds) \\
& + \int_0^t \int_{|z|>1} H(z,\theta_{s^-})\psi(\theta_{s^-},dz,ds)\Big]\}, \qquad (1.3.24)
\end{aligned}$$

$$x_t = x_0 \exp\{ \Big[ \int_0^t \mu(\theta_{s^-})ds + \int_0^t \sigma(\theta_{s^-})dB_s$$

$$
\begin{aligned}
&+\int_0^t\int_{|z|\le 1}\big[e^{G(z,\theta_{s^-})}-1\big]\bar\psi(\theta_{s^-},dz,ds)\\
&+\int_0^t\int_{|z|>1}\big[e^{H(z,\theta_{s^-})}-1\big]\psi(\theta_{s^-},dz,ds)-\frac{1}{2}\int_0^t\sigma^2(\theta_{s^-})ds\\
&+\int_0^t\int_{|z|\le 1}\big[G(z,\theta_{s^-})-e^{G(z,\theta_{s^-})}+1\big]\nu(\theta_{s^-},dz)ds\\
&+\int_0^t\int_{|z|\le 1}\big[G(z,\theta_{s^-})-e^{G(z,\theta_{s^-})}+1\big]\bar\psi(\theta_{s^-},dz,ds)\\
&+\int_0^t\int_{|z|>1}\big[H(z,\theta_{s^-})-e^{H(z,\theta_{s^-})}+1\big]\psi(\theta_{s^-},dz,ds)\Big]\}, \qquad (1.3.25)
\end{aligned}
$$

$$
\begin{aligned}
x_t = x_0\exp\{\Big[&\int_0^t\mu(\theta_{s^-})ds+\int_0^t\sigma(\theta_{s^-})dB_s\\
&+\int_0^t\int_{|z|\le 1}\big[e^{G(z,\theta_{s^-})-1}\big]\bar\psi(\theta_{s^-},dz,ds)\\
&+\int_0^t\int_{|z|>1}\big[e^{H(z,\theta_{s^-})}-1\big]\bar\psi(\theta_{s^-},dz,ds)\\
&+\int_0^t\int_{|z|>1}H(z,\theta_{s^-})\nu(\theta_{s^-},dz)ds-\frac{1}{2}\int_0^t\sigma^2(\theta_{s^-})ds\\
&+\int_0^t\int_{|z|\le 1}\big[G(z,\theta_{s^-})-e^{G(z,\theta_{s^-})}+1\big]\bar\psi(\theta_{s^-},dz,ds)\\
&+\int_0^t\int_{|z|\le 1}\big[G(z,\theta_{s^-})-e^{G(z,\theta_{s^-})}+1\big]\nu(\theta_{s^-},dz)ds\\
&+\int_0^t\int_{|z|>1}\big[H(z,\theta_{s^-})-e^{H(z,\theta_{s^-})}+1\big]\bar\psi(\theta_{s^-},dz,ds)\Big]\}. \qquad (1.3.26)
\end{aligned}
$$

In addition, if $\mu(\theta_s)$ in (1.3.2) is replaced by $\Big[\mu(\theta_s)+\frac{1}{2}\sigma^2(\theta_s)+\int_{|z|\le 1}\big[e^{G(z,\theta_s)}-1-G(z,\theta_s)\big]\nu(\theta_s,dz)\Big]$, then (1.3.24), (1.3.25) and (1.3.26), respectively, reduce to:

$$
\begin{aligned}
x_t &= x_0\exp\{\Big[\int_0^t\mu(\theta_{s^-})ds+\int_0^t\sigma(\theta_{s^-})dB_s\\
&\qquad+\int_0^t\int_{|z|\le 1}G(\theta_{s^-},z)\bar\psi(\theta_{s^-},dz,ds)\\
&\qquad+\int_0^t\int_{|z|>1}H(\theta_{s^-},z)\psi(\theta_{s^-},dz,ds)\Big]\}\\
&= x_0\exp\{[L_t^\theta]\}, \qquad (1.3.27)
\end{aligned}
$$

$$
\begin{aligned}
x_t = x_0\exp\{\Big[&\int_0^t\mu(\theta_{s^-})ds+\int_0^t\sigma(\theta_{s^-})dB_s\\
&+\int_0^t\int_{|z|\le 1}\big[e^{G(z,\theta_{s^-})}-1\big]\bar\psi(\theta_{s^-},dz,ds)
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^t\int_{|z|>1}\big[e^{H(z,\theta_{s^-})}-1\big]\psi(\theta_{s^-},dz,ds)\\
&+\int_0^t\int_{|z|\le 1}\big[G(z,\theta_{s^-})-e^{G(z,\theta_{s^-})}+1\big]\bar{\psi}(\theta_{s^-},dz)\\
&+\int_0^t\int_{|z|>1}\big[H(z,\theta_{s^-})-e^{H(z,\theta_{s^-})}+1\big]\psi(\theta_{s^-},dz,ds)\Big]\},
\end{aligned}
$$

$$
\begin{aligned}
x_t = x_0\exp\{\Big[&\int_0^t\mu(\theta_{s^-})ds+\int_0^t\sigma(\theta_{s^-})dB_s\\
&+\int_0^t\int_{|z|\le 1}\big[e^{G(z,\theta_{s^-})}-1\big]\bar{\psi}(\theta_{s^-},dz,ds)\\
&+\int_0^t\int_{|z|>1}\big[e^{H(z,\theta_{s^-})}-1\big]\bar{\psi}(\theta_{s^-},dz,ds)\\
&+\int_0^t\int_{|z|>1}\big[e^{H(z,\theta_{s^-})}-1\big]\nu(\theta_{s^-},dz)ds\\
&+\int_0^t\int_{|z|\le 1}\big[G(z,\theta_{s^-})-e^{G(z,\theta_{s^-})}+1\big]\psi(\theta_{s^-},dz,ds)\\
&+\int_0^t\int_{|z|>1}\big[H(z,\theta_{s^-})-e^{H(z,\theta_{s^-})}+1\big]\bar{\psi}(\theta_{s^-},dz,ds)\Big]\},
\end{aligned}
\tag{1.3.28}
$$

where $L^\theta$ is defined in (1.3.2). Moreover, if $\mu(\theta_{t^-})$ in (1.3.1) is replaced by $\Big[\mu(\theta_{s^-})+\frac{1}{2}\sigma^2(\theta_{s^-})+\int_{|z|\le 1}\big[e^{G(z,\theta_{s^-})}-1-G(z,\theta_{s^-})\big]\nu(\theta_{s^-},dz)+\int_{|z|>1}H(z,\theta_{s^-})\nu(\theta_{s^-},dz)\Big]$, then (1.3.26) reduces to;

$$
\begin{aligned}
x_t = x_0\exp\{\Big[&\int_0^t\mu(\theta_{s^-})ds+\int_0^t\sigma(\theta_{s^-})dB_s\\
&+\int_0^t\int_{|z|\le 1}\big[e^{G(z,\theta_{s^-})-1}\big]\bar{\psi}(\theta_{s^-},dz,ds)\\
&+\int_0^t\int_{|z|>1}\big[e^{H(z,\theta_{s^-})}-1\big]\bar{\psi}(\theta_{s^-},dz,ds)\\
&+\int_0^t\int_{|z|\le 1}\big[G(z,\theta_{s^-})-e^{G(z,\theta_{s^-})}+1\big]\bar{\psi}(\theta_{s^-},dz,ds)\\
&+\int_0^t\int_{|z|>1}\big[H(z,\theta_{s^-})-e^{H(z,\theta_{s^-})}+1\big]\bar{\psi}(\theta_{s^-},dz,ds)\Big]\}.
\end{aligned}
\tag{1.3.29}
$$

## 1.4 Ito Differential Formula

In this section, we define the asset price model and we derive the infinitesimal generator of the quadruplet $(t,y,\theta,x)$. We denote $L_t^\theta$ the Lévy process with Lévy triplet $\big(\mu(\theta_t),\sigma(\theta_t),\nu(\theta_t,dz)\big)$ defined (1.3.2). Following the argument used in [47], we formulate a linear stochastic hybrid dynamic model for stock price process under structural perturbations of semi Markov and Lévy processes.

DEFINITION 1.4.1 *A linear stochastic hybrid dynamic model under semi Markov and Lévy structural perturbations is defined as follows:*

$$\begin{cases} dx(t) = x(t^-)dL_t^{\theta_n} \, , \, x(T_n) = x_n \, , \, t \in [T_n, T_{n+1}) \\ x_n = \beta_n x(T_n^-, T_{n-1}, x_{n-1}) \, , \, x(0) = x_0 \, , \, n \in I(1,\infty) = \mathbb{N}, \end{cases} \tag{1.4.1}$$

*where* $\{T_n\}_{n=1}^{\infty}$ *is an increasing sequence of jump/regime switching times of the semi Markov process* $\theta$ *with* $T_0 = 0$ *introduced in Definition 1.2.1 ; for* $n \in I(0,\infty) = \{0,1,2,3,...\}$*,* $\beta_n$ *denotes the discrete time state jump process caused by the semi Markov process from state* $\theta_{n-1}$ *at* $T_{n-1}$ *to* $\theta_n$ *at* $T_n$*; it is denoted* $\beta_n = \beta_{\theta_{n-1},\theta_n}$*, highlighting the assumption that semi Markov jump distributions depend only on the previous and current market states. The density function of* $\beta_{i,j}$ *is* $b(.|i,j)$ *and* $L_t^{\theta}$ *is defined in* (1.3.2)*.*

REMARK 1.4.1 A few observations about the model in the context of Remark 1.3.2 are in order. The solution of (3.2.4) can be described by the following discrete time iterative process [47]:

$$\begin{cases} x(t, T_n, x_n) = x_n \exp\left[ \int_{T_n}^{t} dL_s^{\theta_n} \right], t \in [T_n, T_{n+1}) \\ x_n = \beta_n x(T_n^-, T_{n-1}, x_{n-1}), \end{cases} \tag{1.4.2}$$

where $L_t^{\theta}$ is defined as the exponent of the solution process of (1.3.1) as expressed in (1.3.27). The semi Markov process decomposes both the time and state domains causing structural changes in the stock price process, while the Lévy process directly decomposes the state domain of definition of the stochastic dynamic model.

In the following we are exhibiting three particular cases of the model developed in (1.4.1), which generalize systematically Black Scholes model, Merton Jump diffusion model and the Normal Inverse Gaussian model. We will subsequently use these generalized versions of these models to exhibit the role and scope of the model in (1.4.1). In all three instances of (1.4.1), we assume $G(z,\theta_t) = H(z,\theta_t) = z, \forall t \in [0,T]$. In addition, these models are referred to as: The semi Markov Black Scholes (SMBS),the semi Markov Merton Jump Diffusion (SMMJD), the semi Markov Normal Inverse Gaussian (SMNIG).

ILLUSTRATION 1.4.1 We are ready to formulate an illustration regarding the models SMBS, SMMJD and SNNIG as special cases of (1.4.1).

Semi Markov version of Black Scholes (SMBS) model: we assume the Itó Levy process $L$ has no Levy jumps, in other terms, the Poisson random measure $\psi_{bs}$ has intensity measure $\nu_{bs} = 0$. The Semi Markov version Black Scholes (SMBS) model in the context of (1.4.1) reduces to:

$$\begin{cases} dx(t) = x(t^-)\big[\mu(\theta_t)dt + \sigma(\theta_t)dB_t\big] \,,\, x(T_n) = x_n \,,\, t \in [T_n, T_{n+1}) \\ x_n = \beta_n x(T_n^-, T_{n-1}, x_{n-1}) \,,\, x(0) = x_0 \,,\, n \in I(1,\infty) = \mathbb{N}, \end{cases} \tag{1.4.3}$$

Semi Markov version of Merton Jump Diffusion (SMMJD) model: we assume the Itó Levy process $L$ has jumps of normal size with mean and standard deviation $\sigma_{mjd}$ and $\mu_{mjd}$, arriving at a finite Poisson rate $\lambda_{mjd}$. We also assume the corresponding Poisson random measure denoted $\psi_{mjd}$ has intensity measure $\nu_{mjd}(\theta_t) = \frac{\lambda_{mjd}(\theta_t)}{\sigma_{mjd}(\theta_t)\sqrt{2\pi}} \exp\big[-\frac{(z-\mu_{mjd}(\theta_t))^2}{2\sigma^2_{mjd}(\theta_t)}\big]$. The semi Markov version of Merton Jump Diffusion (SMMJD) model in the context of (1.4.1) reduces to:

$$\begin{cases} dx(t) = x(t^-)\big[\mu(\theta_t)dt + \sigma(\theta_t)dB_t + \int_{z\in\mathbb{R}} z\psi_{mjd}(\theta_n, dz, dt)\big] \,,\, x(T_n) = x_n \,,\, t \in [T_n, T_{n+1}) \\ x_n = \beta_n x(T_n^-, T_{n-1}, x_{n-1}) \,,\, x(0) = x_0 \,,\, n \in I(1,\infty) = \mathbb{N}, \end{cases} \tag{1.4.4}$$

Semi Markov version of Normal Inverse Gaussian (SMNIG) model: here we assume the Itó Levy process $L$ has no diffusion component ($\sigma(\theta_s) = 0$), with Levy measure

$$\nu_{mjd}(\theta_s) = \frac{\alpha_{nig}(\theta_t)\delta_{nig}(\theta_t)}{\pi x} K_1(\alpha_{nig}(\theta_t)|x|) \exp\big[(\beta_{nig}(\theta_t)x)\big],$$

where $K_1$ is the modified bessel function of the third kind. The corresponding random Poisson measure is denoted $\psi_{nig}$. The semi Markov version of Normal Inverse Gaussian (SMNIG) model, in the context of (1.4.1) reduces to:

$$\begin{cases} dx(t) = x(t^-)\big[\mu(\theta_t)dt + \sigma(\theta_t)dB_t + \int_{|z|>1} z\psi_{nig}(\theta_n, dz, dt) \\ + \int_{|z|\le 1} z\psi_{nig}(\theta_n, dz, dt)\big] \,,\, x(T_n) = x_n \,,\, t \in [T_n, T_{n+1}) \\ x_n = \beta_n x(T_n^-, T_{n-1}, x_{n-1}) \,,\, x(0) = x_0 \,,\, n \in I(1,\infty) = \mathbb{N}, \end{cases} \tag{1.4.5}$$

REMARK 1.4.2 From (1.3.27) and (1.4.1), the size of the jump in log price at time $T_n$ is $\ln(\beta_n)$. The density function of $\ln(\beta_n)$ is described by:

$$\bar{b}(z|\theta_{n-1}, \theta_n) = b(e^z|\theta_{n-1}, \theta_n)e^z, \tag{1.4.6}$$

where $b(.|\theta_{n-1}, \theta_n)$ denotes the density of $\beta_n$ and $e$ is the Naperian base. We further note that the discrete time dynamic system in (1.4.2) is an intervention process. A feature of interest of this model is its potential to capture, simultaneously, three important stylized facts. The volatility clustering exhibited in log return time series, the slowly decaying autocorrelation of square returns and the observed correlation between log returns and volatility [10, 16]. As the market switches from one state to another, the diffusion rate changes while the asset price is subjected to a jump. Thus the diffusion rate and the price jumps are modulated by the process $\theta$.

For the development of an infinitesimal generator, in the following, we define a point process encoding both the regime switches and the jumps of $x$ at regime switches. At each regime change, we note that the jump in log price is $\ln(\beta_n)$. We define $E^2 = \{(i,j), (i,j) \in E \times E, i \neq j\}$ and the power set of $E^2$, $\mathcal{P}(E^2)$. $\mathbb{B}(\mathbb{R})$ is the Borel sigma algebra of the real line. We are ready to define the aforementioned point process.

DEFINITION 1.4.2 *$\beta_n$ and $\theta_n$ are introduced in Definition 1.4.1. Let $N(t, A, B)$ be a stochastic process defined on $[0, T] \times \mathbb{B}(\mathbb{R}) \times P(E^2)$ into $[0, \infty)$ as:*

$$N(t, A, B) = \sum_{n \geq 1} 1_{\left(t \geq T_n, \ln(\beta_n) \in A, (\theta_{n-1}, \theta_n) \in B\right)} \tag{1.4.7}$$

*and $N(t, A, B)$ stands for the number of regime switches in B with corresponding log price jumps $\ln(\beta_n) \in A$ by time t.*

REMARK 1.4.3 We observe that:

$$N(t, A, B) = \sum_{(i,j) \in B} N(t, A, \{(i,j)\}), \tag{1.4.8}$$

where $N(t, A, \{(i,j)\})$ counts the number of regime switches from $i$ to $j$ with corresponding log price jump $\ln(\beta_n) \in A$.

In the following Lemma, we derive the predictable compensator process for $N(t, A, \{(i,j)\})$.

LEMMA 1.4.1 *Let $N(t, A, \{(i,j)\})$ be the point process introduced in Definition 1.4.2. Then*

$$N(t \wedge T_n, A, \{(i,j)\}) - \gamma(t \wedge T_n, A, \{(i,j)\}) \tag{1.4.9}$$

*is a martingale with respect to the filtration* $(\bar{\mathbb{H}}_t)_{t\geq 0}, \forall n \in I(1,\infty)$*, where:*

$$\gamma(t, A, \{(i,j)\}) = \int_0^t \int_{z\in A} \bar{b}(z|i,j)\lambda_{i,j}(y_s)dzds, \tag{1.4.10}$$

*and* $\lambda_{i,j}$ *are defined in* (1.2.9)*.*

*Proof.* From [8, 63], it is enough to prove that $N(t\wedge T_n, A, \{(i,j)\}) - \gamma(t \wedge T_n, A, \{(i,j)\})$ is an $\left(\bar{\mathbb{H}}_s\right)_{s\geq 0}$-martingale. For any $0 \leq s \leq t$ and for each $n \in I(1,\infty)$, it satisfies:

$$E\Big[\big[N(t\wedge T_n, A, \{(i,j)\}) - \gamma(t\wedge T_n, A, \{(i,j)\})\big] \\ -\big[N(s\wedge T_n, A, \{(i,j)\}) - \gamma(s\wedge T_n, A, \{(i,j)\})\big]\big|\bar{\mathbb{H}}_s\Big] = 0 \tag{1.4.11}$$

and if and only if

$$E\Big(N(t\wedge T_n, A, \{(i,j)\}) - N(s\wedge T_n, A, \{(i,j)\})\big|\bar{\mathbb{H}}_s\Big) \\ = E\Big(\gamma(t\wedge T_n, A, \{(i,j)\}) - \gamma(s\wedge T_n, A, \{(i,j)\})\big|\bar{\mathbb{H}}_s\Big). \tag{1.4.12}$$

We prove that (1.4.12) holds. We first prove that (1.4.12) holds when the jump process $N$ is stopped at $T_1$. We then prove by the Principle of Mathematical Induction that (1.4.12) is true when $N$ is stopped at time $T_n$. From Definition 1.4.2, (1.2.5) and for $0 \leq s \leq t$, we have

$$\begin{aligned}
&E\big(N(t\wedge T_1, A, \{(i,j)\}) - N(s\wedge T_1, A, \{(i,j)\})\big|\bar{\mathbb{H}}_s\big) \\
&= \begin{cases} E\Big(1_{(T_1\leq t, \ln(\beta_1)\in A, \theta_1=j, \theta_0=i)} - 1_{(T_1\leq s, \ln(\beta_1)\in A, \theta_1=j,\theta_0=i)}\big|\theta_0, T_1 > s\Big), \text{ for } T_1 > s \\ 0, \text{ for } T_1 \leq s, \end{cases} \\
&=1_{(T_1>s)}E\Big(1_{(T_1\leq t, \ln(\beta_1)\in A, \theta_1=j, \theta_0=i)} - 1_{(T_1\leq s, \ln(\beta_1)\in A, \theta_1=j,\theta_0=i)}\big|\theta_0, T_1 > s\Big) \\
&=1_{(T_1>s)}E\big(1_{(s\leq T_1\leq t, \ln(\beta_1)\in A, \theta_1=j, \theta_0=i)}|\theta_0, T_1 > s\big) \\
&=1_{(T_1>s)}1_{(\theta_0=i)}\frac{P(s\leq T_1\leq t, \ln(\beta_1)\in A, \theta_1=j|\theta_0=i)}{P(T_1>s|\theta_0=i)} \\
&=1_{(T_1>s)}1_{(\theta_0=i)}p_{ij}\frac{P(s\leq T_1, \ln(\beta_1)\in A|\theta_1=j, \theta_0=i)}{S(s|i)} \\
&\quad - 1_{(T_1>s)}1_{(\theta_0=i)}p_{ij}\frac{P(t\leq T_1, \ln(\beta_1)\in A|\theta_1=j, \theta_0=i)}{S(s|i)} \\
&=1_{(T_1>s)}1_{(\theta_0=i)}P(\ln(\beta_1)\in A|\ln(\beta_1)i,j)p_{ij}\frac{P(s\leq T_1|\theta_0=i, \theta_1=j)}{S(s|i)} \\
&\quad - 1_{(T_1>s)}1_{(\theta_0=i)}P(\ln(\beta_1)\in A|\ln(\beta_1)i,j)p_{ij}\frac{P(t\leq T_1|\theta_0=i, \theta_1=j)}{S(s|i)} \\
&=1_{(T_1>s)}1_{(\theta_0=i)}p_{ij}P(\ln(\beta_1)\in A|\theta_0=i, \theta_1=j)\frac{-\Delta S(t|i,j)}{S(s|i)},
\end{aligned} \tag{1.4.13}$$

where $\Delta S(t|i,j) = S(t|\theta_0 = i, \theta_1 = j) - S(s|\theta_0 = i, \theta_1 = j)$, with $S(.|i,j)$ denoting the conditional survival distribution of sojourn time when the process switches from $i$ to $j$. From (1.2.8), (1.2.9) and (1.4.6), we have

$$\begin{aligned} &E\big[N(t\wedge T_1, A, \{i,j\}) - N(s\wedge T_1, A, \{i,j\})\big] \\ &= \int_s^t \int_{z\in A} 1_{(T_1>s)} 1_{(\theta_0=i)} \bar{b}(z|i,j)\lambda_{i,j}(y_u)dudz. \end{aligned} \tag{1.4.14}$$

On the other hand, from (1.4.10) and (1.2.8), we obtain:

$$\begin{aligned}
&E\big[\gamma(t\wedge T_1, A, \{(i,j)\}) - \gamma(s\wedge T_1, A, \{(i,j)\})|\bar{\mathbb{H}}_s\big] \\
=&1_{(\theta_0=i)}1_{T_1>s}E\Big[\int_{z\in A}\int_0^{t\wedge T_1} \bar{b}(z|i,j)\lambda_{i,j}(y_u)dudz \\
&-\int_{z\in A}\int_0^{s\wedge T_1} \bar{b}(z|i,j)\lambda_{i,j}(y_u)dudz\big|\bar{\mathbb{H}}_s\Big] \\
=&1_{T_1>s}1_{(\theta_0=i)}E\Big[\int_{T_1\wedge s}^{T_1\wedge t} P(\ln(\beta_1)\in A|i,j)\lambda_{i,j}(y_u)du\big|T_1>s, \theta_0=i\Big], (\text{ Fubini's theorem}) \\
=&1_{T_1>s}1_{(\theta_0=i)}\int_s^\infty \int_{T_1\wedge s}^{T_1\wedge t} P(\ln(\beta_1)\in A|i,j)\lambda_{i,j}(y_u)du\frac{-dS(r|\theta_0=i)}{S(s|\theta_0=i)} \\
=&1_{T_1>s}1_{(\theta_0=i)}\Big[\int_s^t \int_{r\wedge s}^{r\wedge t} P(\ln(\beta_1)\in A|i,j)\lambda_{i,j}(y_u)du\frac{-dS(r|\theta_0=i)}{S(s|\theta_0=i)} \\
&+\int_t^\infty \int_{r\wedge s}^{r\wedge t} P(\ln(\beta_1)\in A|i,j)\lambda_{i,j}(y_u)du\frac{-dS(r|\theta_0)}{S(s|\theta_0=i)}\Big] \\
=&1_{T_1>s}1_{(\theta_0=i)}\Big[-\frac{1}{S(s|i,j)}\int_s^t\int_s^r P(\ln(\beta_1)\in A|i,j)\lambda_{i,j}(y_u)dudS(r|\theta_0) \\
&+\int_t^\infty\big[\int_s^t P(\ln(\beta_1)\in A|i,j)\lambda_{i,j}(y_u)du\big]\frac{-dS(r|\theta_0=i)}{S(s|\theta_0=i)}\Big] \\
=&1_{T_1>s}1_{(\theta_0=i)}\Big[-\frac{1}{S(s|\theta_0=i)}\int_s^t\int_s^r P(\ln(\beta_1)\in A|i,j)\lambda_{i,j}(y_u)dS(r|\theta_0)du \\
&+\int_s^t\big[\underbrace{\int_t^\infty \frac{-dS(r|\theta_0=i)}{S(s|\theta_0=i)}}\big]P(\ln(\beta_1)\in A|i,j)\lambda_{i,j}(y_u)du\Big] \\
=&1_{T_1>s}1_{(\theta_0=i)}\Big[-\frac{1}{S(s|\theta_0=i)}\int_s^t P(\ln(\beta_1)\in A|i,j)\lambda_{i,j}(y_u) \\
&\times\big[\int_u^t dS(r|\theta_0=i)\big]du \\
&+\frac{S(t|\theta_0=i)}{S(s|\theta_0=i)}\int_{[s,t]} P(\ln(\beta_1)\in A|i,j)\lambda_{i,j}(u)du\Big] \\
=&1_{T_1>s}1_{(\theta_0=i)}P(\ln(\beta_1)\in A|i,j)\Big[\frac{1}{S(s|\theta_0=i)}\int_s^t \lambda_{i,j}(y_u)\big[S(u|\theta_0=i)
\end{aligned}$$

$$
\begin{aligned}
&- S(t|\theta_0 = i)\big]du + \frac{S(t|\theta_0 = i)}{S(s|\theta_0 = i)} \int_s^t \lambda_{i,j}(y_u)du\Big] \\
=&1_{T_1>s}1_{(\theta_0=i)} \frac{P(\ln(\beta_1) \in A|i,j)}{S(s|\theta_0 = i)} \int_s^t \lambda_{i,j}(u)S(u|\theta_0 = i)du \\
=&1_{T_1>s}1_{(\theta_0=i)}p_{ij} \frac{P(\ln(\beta_1) \in A\big|i,j)}{S(s|\theta_0 = i)} \int_s^t S(u|\theta_0 = i)\frac{-dS(u|i,j)}{S(u|\theta_0 = i)} \\
=&1_{T_1>s}1_{(\theta_0=i)}p_{ij}P(\ln(\beta_1) \in A|i,j)\frac{S(s|i,j) - S(t|i,j)}{S(s|i)}.
\end{aligned} \tag{1.4.15}
$$

From (1.4.13) and (1.4.15), we get

$$
\begin{aligned}
&E\bigg(\gamma(t \wedge T_1, A, \{(i,j)\}) - \gamma(s \wedge T_1, A, \{(i,j)\})\big|\bar{\mathbb{H}}_s\bigg) \\
&= 1_{T_1>s}1_{(\theta_0=i)}p_{ij}P(\ln(\beta_1) \in A|i,j)\frac{S(s|i,j) - S(t|i,j)}{S(s|\theta_0)} \\
&= E\bigg(N(t \wedge T_1, A, \{(i,j)\}) - N(s \wedge T_1, A, \{(i,j)\})\big|\bar{\mathbb{H}}_s\bigg).
\end{aligned}
$$

This establishes (1.4.12). Hence the stopped point process $N(t \wedge T_1, A, \{(i,j)\})$ has predictable compensator $\gamma(t\wedge T_1, A\times\{(i,j)\})$ defined in (1.4.10). Assuming that (1.4.11) is valid for some $k \in I(1,\infty)$, and repeating the above argument, we verify the induction assumption. By the principle of mathematical induction, we conclude that $N((t \wedge T_k, t \wedge T_{k+1}], A, \{(i,j)\}) - \gamma(t \wedge T_k, t \wedge T_{k+1}], A, \{(i,j)\})$ is an $(\bar{\mathbb{H}}_t)_{t>0}$-martingale. □

Prior to turning our attention to the infinitesimal generator, we first establish Itó differential formula for (1.4.1).

THEOREM 1.4.1 (Ito Differential Formula) *Let* $V \in \mathcal{C}\Big[\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}, \mathbb{R}\Big]$ *be continuously differentiable in the first and second variables and twice continuously differentiable function in the fourth variable. Let $x$, $y$, $N$ and $\gamma$ be stochastic processes defined in* (1.4.1), (1.2.14), (1.4.7) *and* (1.4.10)*, respectively. Moreover, processes $N$ and $\psi$ do not jump simultaneously $P-$almost surely. Then*

$$
\begin{aligned}
&dV(s, y_{s^-}, \theta_{s^-}, x_{s^-}) \\
&=\big(\mathcal{L}V\big)(s, y_{s^-}, \theta_{s^-}, x_{s^-})ds + \sigma(\theta_{s^-})x_{s^-}\frac{\partial V}{\partial x}dB_s \\
&\quad + \int_{|z|\le 1} \Big[V(s, y_s, \theta_s, x_{s^-} + x_{s^-}G(z,\theta_s)) - V(y_s, \theta_s, x_{s^-})\Big]\bar{\psi}(\theta_s, dz, ds) \\
&\quad + \int_{|z|>1} \big[V(s, y_s, \theta_s, x_{s^-} + x_{s^-}H(z,\theta_s)) - V(s, y_s, \theta_s, x_{s^-})\big]\bar{\psi}(\theta_s, dz, ds)
\end{aligned}
$$

$$+\int_{z\in\mathbb{R}}\sum_{j\in E\backslash\{\theta_{s^-}\}}\big\{V(s,y_s,j,x_{s^-}e^z)$$

$$-V(s,y_{s^-},\theta_{s^-},x_{s^-})\big\}\tilde{N}(ds,dz,\{(\theta_{s^-},j)\}), \tag{1.4.16}$$

*for $\theta_{s^-}\in E$ where:*

$$\begin{aligned}
&\mathcal{L}V(s,y_{s^-},\theta_{s^-},x_{s^-})\\
=&\frac{\partial V}{\partial s}+\frac{\partial V}{\partial y}+\mu(\theta_{s^-})x_{s^-}\frac{\partial V}{\partial x}+\frac{1}{2}\sigma^2(\theta_{s^-})x_{s^-}^2\frac{\partial^2 V}{\partial x^2}\\
&+\int_{|z|\le 1}\Big[V(s,y_{s^-},\theta_{s^-},x_{s^-}+x_{s^-}G(z,\theta_{s^-}))\\
&-V(s,y_{s^-},\theta_{s^-},x_{s^-})-G(z,\theta_{s^-})x_{s^-}\frac{\partial V}{\partial x}\Big]\nu(\theta_{s^-},dz)\\
&+\int_{|z|>1}\Big[V(s,y_{s^-},\theta_{s^-},x_{s^-}+x_{s^-}H(z,\theta_{s^-}))\\
&-V(s,y_{s^-},\theta_{s^-},x_{s^-})\Big]\nu(\theta_{s^-},dz)\\
&+\int_{z\in\mathbb{R}}\sum_{j\neq\theta_{s^-}}\lambda_{\theta_{s^-},j}(y_{s^-})\Big[V(s,y_{s^-},j,x_{s^-}e^z)\\
&-V(s,y_{s^-},\theta_{s^-},x_{s^-})\Big]\bar{b}(z|\theta_{s^-},j)dz,
\end{aligned} \tag{1.4.17}$$

*$\theta_{s^-}\in E$ and $\tilde{N}=N-\gamma$.*

*Proof.* Let V be defined as in the theorem. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of semi Markov jump times and $T_0=0$. For $t\in\mathbb{R}^+$, we can find an interval $[T_n,T_{n+1}]$ such that $T_n\le t<t+\Delta t\le T_{n+1}$ for some $n\in\mathbb{N}$. Let $\{J_j^n\}_{j=0}^{k_n}\subset[T_n,T_{n+1}]$ and $J_0^n=T_n$ be a finite sequence of jump times due to the Lévy jump process for $k_n\in\mathbb{N}$. We further note that the interval can be rewritten as:

$$[T_n,T_{n+1}]=[T_n,T_{n+1}^-]\cup[T_{n+1}^-,T_{n+1}] \tag{1.4.18}$$

We observe that $[J_j^n,J_{j+1}^{n-}]\cap[J_{j+1}^{n-},J_{j+1}^n]=\emptyset$. In addition,

$$[T_n,T_{n+1}^-]=\bigcup_{j=0}^{k_n}\Big([J_j^n,J_{j+1}^{n-}]\cup[J_{j+1}^{n-},J_{j+1}^n]\Big) \tag{1.4.19}$$

It is known that the state dynamic process operating under the above stated conditions decomposes into three parts, namely, the continuous time, the Lévy jump time and the semi Markov jump time. In fact, the solution process of (1.4.1)/(1.3.1) can be rewritten as:

$$x_t=x_t^c+x_t^d+x_t^s, \tag{1.4.20}$$

where $x_t^c$, $x_t^d$ and $x_t^s$ are due to the presence of continuous process, Lévy process and semi Markov process, respectively. We further observe that for $s \in [T_n, T_{n+1}]$, we have: $s = s^- + (s - s^-) = s^- + \Delta s$, where $\Delta s = s - s^-$, $s^- \neq s$. From Definitions 1.2.3, 1.4.1, we note that $y_s = y_{s^-}$ and $\theta_s = \theta_{s^-}$ for $s \in [T_n, T_{n+1}^-]$ and for $s = T_{n+1}$, $s \neq s^-$, $y_{s^-} \neq y_s$ and $\theta_{T_{n+1}} \neq \theta_{s^-}$. Moreover, there is a $j \in I(1, k_n - 1)$ such that $s \in [J_j^n, J_{j+1}^n] \cup [J_{k_n}^n, T_{n+1}^-]$. We choose $\Delta s$ so that $s + \Delta s \in [J_j^n, J_{j+1}^n]$. For these choices of $s$ and $s + \Delta s$, we have:

$$\begin{cases} y_{s+\Delta s} = y_{s^-} + \Delta s \\ \theta_{s+\Delta s} = \theta_{s^-} \\ x_{s+\Delta s} = x_{s^-} + \Delta x_s^s, \end{cases} \tag{1.4.21}$$

Furthermore,

$$\Delta x_s = \begin{cases} \Delta x_s^c, \text{ if } s \in [J_j^n, J_{j+1}^n) \cup [J_{k_n}^n, T_{n+1}), \text{ for } j \in I(0, k_n - 1) \text{ and } n \in I(0, \infty) \\ \Delta x_s^d, \text{ if } s = J_{j+1}^n, \text{ for } j \in I(1, k_n - 1) \\ \Delta x_s^s, \text{ if } s = T_{n+1} \end{cases} \tag{1.4.22}$$

(1.4.22) implies that for $s, \Delta s \in [J_j^n, J_{j+1}^n)$, the change in state dynamic process is in the absence of the influence of Lévy jump process. On the other hand, for $s = J_j^n$ for each $j \in I(1, k_n)$, the dynamic process is interrupted by the presence of Lévy jumps. Finally, if s=$T_{n+1}$, then the dynamic system undergoes a structural change. Here the structural change is under the influence of the semi Markov process. Therefore, there is no contribution of the continuous time dynamic process.
Based on the nature of the dynamic process operating under continuous time process, semi Markov process and Lévy process, we compute the change in the auxiliary function V as:

$$V(s + \Delta s, y_{s+\Delta s}, \theta_{s+\Delta s}, x_{s+\Delta s}) - V(s, y_s, \theta_s, x_s) \tag{1.4.23}$$

in the context of state dynamic model (1.3.1).
The computation of change in (1.4.23) depends on the computation of changes over the time domain of decomposition of $[T_n, T_{n+1}]$ for $n \in I(0, \infty)$. For computation on $\cup_{j=0}^{k_n-1}[J_j^n, J_{j+1}^{n-}] \bigcup [J_{k_n}^n, T_{n+1}]$, we utilize the generalized mean value theorem. For this purpose, we pick $s, s+\Delta s \in [J_j^{n^-}, J_{j+1}^n) \subset [T_n, T_n^-]$. From (1.4.21), the computation of (1.4.23) on the time domain $\cup_{j=0}^{k_n}[J_j^n, J_{j+1}^{n-}] \bigcup [J_{k_n}^n, T_{n+1}]$

regarding the continuous part of state dynamic (1.4.1)/(1.3.1) is as follows: The decomposition of three subsets of time domain $[T_n, T_{n+1}]$, namely, $\cup_{j=0}^{k_n}[J_j^n, J_{j+1}^{n-}]\bigcup[J_{k_n}^n, T_{n+1}^-]$, or $\cup_{j=1}^{k_n}[J_{j+1}^{n-}, J_{j+1}^n]$, or $[T_{n+}^-, T_{n+1}]$ for $n, k_n \in I(0,\infty)$.

$$
\begin{aligned}
&V(s+\Delta s, y_{s+\Delta s}, \theta_{s+\Delta s}, x_{s+\Delta s}) - V(s, y_s, \theta_s, x_s)\\
&=\int_0^1 \Big[\frac{\partial V}{\partial s}(s+\eta\Delta s, y_s+\eta\Delta s, \theta_s, x_s+\eta\Delta x_s)\Delta s\\
&\quad+\frac{\partial V}{\partial y}(s+\eta\Delta s, y_s+\eta\Delta y_s, \theta_s, x_s+\eta\Delta x_s)\Delta y_s\\
&\quad+\frac{\partial V}{\partial x}(s+\eta\Delta s, y_s+\eta\Delta y_s, \theta_s, x_s+\eta\Delta x_s)\Delta x_s\Big]d\eta\\
&=\frac{\partial V}{\partial s}(s, y_s, \theta_s, x_s)\Delta s+\frac{\partial V}{\partial y}(s, y_s, \theta_s, x_s)\Delta y_s\\
&\quad+\frac{\partial V}{\partial x}(s, y_s, \theta_s, x_s)\Delta x_s\\
&\quad+\int_0^1 \Big[\frac{\partial V}{\partial x}(s+\eta\Delta s, y_s+\eta\Delta s, \theta_s, x_s+\eta\Delta x_s)\\
&\quad-\frac{\partial V}{\partial x}(s, y_s, \theta_s, x_s)\Big]\Delta x_s d\eta+\varepsilon_{s,y}(\Delta s),
\end{aligned}
\tag{1.4.24}
$$

where,

$$
\begin{aligned}
&\varepsilon_{s,y}(\Delta s)\\
&=\int_0^1 \Big[\frac{\partial V}{\partial s}(s+\eta\Delta s, y_s+\eta\Delta y_s, \theta_s, x_s+\eta\Delta x_s)\\
&\quad-\frac{\partial V}{\partial s}(s, y_s, \theta_s, x_s)\Big]\Delta s d\eta\\
&\quad+\int_0^1 \Big[\frac{\partial V}{\partial y}(s+\eta\Delta s, y_s\\
&\quad+\eta\Delta y_s, \theta_s, x_s+\eta\Delta x_s)-\frac{\partial V}{\partial y}(s, y_s, \theta_s, x_s)\Big]\Delta y_s d\eta.
\end{aligned}
\tag{1.4.25}
$$

We again apply the generalized mean value theorem to the integrand in (1.4.24), and we obtain:

$$
\begin{aligned}
&\frac{\partial V}{\partial x}(s+\eta\Delta s, y_s+\eta\Delta y_s, \theta_s, x_s+\eta\Delta x_s)\\
&\quad-\frac{\partial V}{\partial x}(s+\eta\Delta s, y_s+\eta\Delta y_s, \theta_s, x_s)\\
&=\frac{\partial^2 V}{\partial x^2}(s, y_s, \theta_s, x_s)\eta\Delta x_s\\
&\quad+\int_0^1 \big[\frac{\partial^2 V}{\partial x^2}(s+\Delta s, y_s+\eta\Delta y_s, \theta_s, x_s+\epsilon\eta\Delta x_s)\\
&\quad-\frac{\partial^2 V}{\partial x^2}(s, y_s, \theta_s, x_s)\big]\Delta x_s\eta d\epsilon.
\end{aligned}
\tag{1.4.26}
$$

From (1.4.26), the fourth term in (1.4.24) reduces to:

$$\int_0^1 \Big[\frac{\partial V}{\partial x}(s+\eta\Delta s, y_s+\eta\Delta y_s, \theta_s, x_s+\eta\Delta x_s) - \frac{\partial V}{\partial x}(s, y_s, \theta_s, x_s)\Big]\Delta x_s d\eta$$
$$= \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(s, y_s, \theta_s, x_s)(\Delta x_s)^2 + \varepsilon_x(\Delta s), \tag{1.4.27}$$

where,

$$\begin{aligned}\varepsilon_x(\Delta s) = &\int_0^1\int_0^1 \big[\frac{\partial^2 V}{\partial x^2}(s+\Delta s, y_s+\eta\Delta y_s, \theta_s, x_s \\ &+ \epsilon\eta\Delta x_s) - \frac{\partial^2 V}{\partial x^2}(s, y_s, \theta_s, x_s)\big](\Delta x_s)^2\eta d\eta d\epsilon \\ &+ \int_0^1 \big[\frac{\partial V}{\partial x}(s+\eta\Delta s, y_s+\eta\Delta y_s, \theta_s, x_s) \\ &- \frac{\partial V}{\partial x}(s, y_s, \theta_s, x_s)\big]d\eta\Delta x_s.\end{aligned}$$

From (1.4.24) and (1.4.27), we have:

$$\begin{aligned}&V(s+\Delta s, y_s+\Delta y_s, \theta_{s+\Delta s}, x_{s+\Delta s}) - V(s, y_s, \theta_s, x_s) \\ =&\frac{\partial V}{\partial s}(s, y_s, \theta_s, x_s)\Delta s + \frac{\partial V}{\partial y}(s, y_s, \theta_s, x_s)\Delta y_s \\ &+ \frac{\partial V}{\partial x}(s, y_s, \theta_s, x_s)\Delta x_s \\ &+ \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(s, y_s, \theta_s, x_s)(\Delta x_s)^2 + \varepsilon(\Delta s),\end{aligned} \tag{1.4.28}$$

where $\varepsilon(\Delta s) = \varepsilon_{s,y}(\Delta s) + \varepsilon_x(\Delta s)$. The expressions in (1.4.24), (1.4.26) and (1.4.27) are valid for all $s \in [J_j^n, J_{j+1}^{n-}] \cup [J_{k_n}^n, T_{n+1}^-]$ and for all $j \in I(1, k_n - 1)$ and hence they are valid on the intervals $[T_n, T_{n+1})$ for $n, k_n \in I(0, \infty)$.

Using Lévy integrals and a single jump value, we compute (1.4.23) under the influence of Lévy jump process. For this case, we first compute $V(J_{j+1}^n) - V(J_{j+1}^{n-})$, where $V(J_{j+1}^n) = V(s \wedge J_{j+1}^n, y_{s\wedge J_{j+1}^n}, \theta_{s\wedge J_{j+1}^n}, x_{s\wedge J_{j+1}^{n-}} + x_{s\wedge J_{j+1}^{n-}} G(z, \theta_{s\wedge J_{j+1}^n}))$ and $V(J_{j+1}^{n-}) = V(s, y_{s^-}, \theta_{s^-}, x_{s^-})$.
We set and compute:

$$\begin{aligned}&V(s\wedge J_{j+1}^n) - V(s\wedge J_{j+1}^{n-}) \\ =&\big[V(s\wedge J_{j+1}^n, y_{s\wedge J_{j+1}^n}, \theta_{s\wedge J_{j+1}^n}, x_{s\wedge J_{j+1}^{n-}} + x_{s\wedge J_{j+1}^{n-}} G(z, \theta_{s\wedge J_{j+1}^n})) \\ &- V(s, y_{s^-}, \theta_{s^-}, x_{s^-})\big]\psi(\theta_{s^-}, \Delta z, \Delta s) \\ &+ \big[V(s\wedge J_{j+1}^n, y_{s\wedge J_{j+1}^n}, \theta_{s\wedge J_{j+1}^n}, x_{s\wedge J_{j+1}^{n-}} + x_{s\wedge J_{j+1}^{n-}} H(z, \theta_{s\wedge J_{j+1}^n})) \\ &- V(s, y_{s^-}, \theta_{s^-}, x_{s^-})\big]\psi(\theta_{s^-}, \Delta z, \Delta s).\end{aligned} \tag{1.4.29}$$

From (2), for any $s \in [J_{j+1}^{n-}, J_{j+1}^{n}]$, $j \in I(0, k_n - 1)$ and $n, k_n \in I(0, \infty)$ we have

$$
\begin{aligned}
&V(s \wedge J_{j+1}^{n}) - V(s \wedge J_{j+1}^{n-})\\
&= \int_{s}^{s+\Delta s} \int_{|z| \leq 1} \big[V(s, y_s, \theta_s, x_{s^-} + x_{s^-} G(z, \theta_s))\\
&\quad - V(s, y_s, \theta_s, x_{s^-})\big] \psi(\theta_s, dz, ds)\\
&\quad + \int_{s}^{s+\Delta s} \int_{|z| > 1} \big[V(s, y_s, \theta_s, x_{s^-} + x_{s^-} H(z, \theta_s))\\
&\quad - V(s, y_s, \theta_s, x_{s^-})\big] \psi(\theta_s, dz, ds).
\end{aligned} \tag{1.4.30}
$$

The expression in (1.4.30) is over a subinterval $\cup_{j=0}^{k_n - 1}[J_{j+1}^{n-}, J_{j+1}^{n}]$ of $[T_n, T_{n+1}]$. Finally, for $s \in [T_{n+1}^{-}, T_{n+1}]$, and imitating the above argument, we compute (1.4.23) under the presence of semi Markov jump as follows:

$$
\begin{aligned}
&V\big(s \wedge T_{n+1}, y_{s \wedge T_{n+1}}, \theta_{s \wedge T_{n+1}}, x_{s \wedge T_{n+1}^{-}} + \Delta x_{s \wedge T_{n+1}^{-}}\big)\\
&\quad - V\big(s \wedge T_{n+1}^{-}, y_{s \wedge T_{n+1}^{-}}, \theta_{s \wedge T_{n+1}^{-}}, x_{s \wedge T_{n+1}^{-}}\big)\\
&= V(s \wedge T_{n+1}, y_{s \wedge T_{n+1}^{-} + \Delta s}, \theta_{s \wedge T_{n+1}^{-} + \Delta s}, x_{s \wedge T_{n+1}^{-}} e^{z}) - V(s, y_{s^-}, \theta_{s^-}, x_{s^-})\\
&= \int_{s}^{s+\Delta s} \int_{z \in \mathbb{R}} \big[V(u, y_{u^-}, \theta_{u^-}, x_{u^-} e^{z})\\
&\quad - V(u, y_{u^-}, \theta_{u^-}, x_{u^-})\big] N(ds, dz, \{\theta_{T_{n+1}^{-}}, \theta_{T_{n+1}}\}),
\end{aligned} \tag{1.4.31}
$$

hence, adding and subtracting:

$$
\begin{aligned}
&\int_{s}^{s+\Delta s} \int_{z \in \mathbb{R}} \big[V(u, y_{u^-}, \theta_{u^-}, x_{u^-} e^{z})\\
&\quad - V(u, y_{u^-}, \theta_{u^-}, x_{u^-})\big] \gamma(ds, dz, \{\theta_{u^-}, \theta_u\}),
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&V\big(s \wedge T_{n+1}, y_{s \wedge T_{n+1}}, \theta_{s \wedge T_{n+1}}, x_{s \wedge T_{n+1}^{-}} + \Delta x_{s \wedge T_{n+1}^{-}}\big)\\
&\quad - V\big(s \wedge T_{n+1}^{-}, y_{s \wedge T_{n+1}^{-}}, \theta_{s \wedge T_{n+1}^{-}}, x_{s \wedge T_{n+1}^{-}}\big)\\
&= \int_{s}^{s+\Delta s} \int_{z \in \mathbb{R}} \big[V(u, y_{u^-}, \theta_{u^-}, x_{u^-} e^{z})\\
&\quad - V(u, y_{u^-}, \theta_{u^-}, x_{u^-})\big] \gamma(du, dz, \{\theta_{u^-}, \theta_u\})\\
&\quad + \int_{s}^{s+\Delta s} \int_{z \in \mathbb{R}} \big[V(u, y_{u^-}, \theta_{u^-}, x_{u^-} e^{z})
\end{aligned}
$$

$$- V(u, y_{u^-}, \theta_{u^-}, x_{u^-})\big]\tilde{N}(du, dz, \{\theta_{u^-}, \theta_u\}). \tag{1.4.32}$$

This expression is on $[T_{n+1}^-, T_{n+1}]$ for $n \in I(0, \infty)$. From (1.4.28), (1.4.30) and (1.4.32), (1.4.23) reduces to:

$$\begin{aligned}
&V(s+\Delta s, y_{s+\Delta s}, \theta_{s+\Delta s}, x_{s+\Delta s}) - V(s, y_s, \theta_s, x_s) \\
&= \frac{\partial V}{\partial s}(s, y_s, \theta_s, x_s)\Delta s + \frac{\partial V}{\partial y}(s, y_s, \theta_s, x_s)\Delta y_s \\
&\quad + \frac{\partial V}{\partial x}(s, y_s, \theta_s, x_s)\Delta x_s \\
&\quad + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(s, y_s, \theta_s, x_s)(\Delta x_s)^2 \\
&\quad + \int_s^{s+\Delta s}\int_{|z|\le 1}\big[V(u, y_u, \theta_u, x_{u^-} + x_{u^-}G(z, \theta_{u^-})) \\
&\quad - V(u, y_{u^-}, \theta_{u^-}, x_{u^-})\big]\psi(\theta_u, dz, du) \\
&\quad + \int_s^{s+\Delta s}\int_{|z|>1}\big[V(u, y_u, \theta_u, x_{u^-} + x_{u^-}H(z, \theta_{u^-})) \\
&\quad - V(u, y_{u^-}, \theta_{u^-}, x_{u^-})\big]\psi(\theta_u, dz, du) \\
&\quad + \int_s^{s+\Delta s}\int_{z\in\mathbb{R}}\sum_{\theta_u \in E\setminus\{\theta_{u^-}\}}\big[V(u, y_{u^-}, \theta_{u^-}, x_{u^-}e^z) \\
&\quad - V(u, y_{u^-}, \theta_{u^-}, x_{u^-})\big]\gamma(ds, dz, \{\theta_{u^-}, \theta_u\}) \\
&\quad + \int_s^{s+\Delta s}\int_{z\in\mathbb{R}}\big[V(u, y_{u^-}, \theta_{u^-}, x_{u^-}e^z) \\
&\quad - V(u, y_{u^-}, \theta_{u^-}, x_{u^-})\big]\tilde{N}(ds, dz, \{\theta_{u^-}, \theta_u\}) + \varepsilon(\Delta s).
\end{aligned} \tag{1.4.33}$$

For small $\Delta s$, applying the concepts of stochastic differentials [1], adding and subtracting:

$$\begin{aligned}
&\int_s^{s+\Delta s}\int_{|z|>1}\big[V(u, y_u, \theta_u, x_{u^-} + x_{u^-}H(z, \theta_{u^-})) \\
&\qquad - V(u, y_{u^-}, \theta_{u^-}, x_{u^-})\big]\nu(\theta_{u^-}, dz)du
\end{aligned}$$

and

$$\begin{aligned}
&\int_s^{s+\Delta s}\int_{|z|\le 1}\big[V(u, y_u, \theta_u, x_{u^-} + x_{u^-}G(z, \theta_{u^-})) \\
&\qquad - V(u, y_{u^-}, \theta_{u^-}, x_{u^-})\big]\nu(\theta_{u^-}, dz)du,
\end{aligned}$$

(1.4.33) reduces to:

$$
\begin{aligned}
&dV(s, y_{s^-}, \theta_{s^-}, x_{s^-}) \\
&=\mathcal{L}V(s, y_{s^-}, \theta_{s^-}, x_{s^-})ds + \sigma(\theta_{s^-})x_{s^-}\frac{\partial V}{\partial x}(s, y_s, \theta_s, x_s)dB_s \\
&\quad + \int_{|z|>1} \big[V(s, y_s, \theta_s, x_{s^-} + x_{s^-}H(z, \theta_{s^-})x_{s^-}) \\
&\quad - V(s, y_{s^-}, \theta_{s^-}, x_{s^-})\big]\bar{\psi}(\theta_s, dz, ds) \\
&\quad + \int_{|z|\leq 1} \big[V(s, y_s, \theta_s, x_{s^-} + x_{s^-}G(z, \theta_{s^-})x_{s^-}) \\
&\quad - V(s, y_{s^-}, \theta_{s^-}, x_{s^-})\big]\bar{\psi}(\theta_s, dz, ds) \\
&\quad + \int_{z\in\mathbb{R}} \big[V(s, y_{s^-}, \theta_{s^-}, x_{s^-}e^z) \\
&\quad - V(s, y_{s^-}, \theta_{s^-}, x_{s^-})\big]\tilde{N}(ds, dz \times \{\theta_{s^-}, \theta_s\}) \qquad (1.4.34)
\end{aligned}
$$

This establishes Ito differential formula (1.4.16) for Lévy type stochastic differential equation under semi Markovian structural perturbations. Here $\mathcal{L}$ in (1.4.17) is the linear differential operator relative to (1.4.1). □

In the following, based on Theorem 1.4.1, we present a concept of infinitesimal generator and a few results as special cases.

DEFINITION 1.4.3 *For the function $V$ defined in Theorem 1.4.1 and using* (1.4.33)*, an infinitesimal generator of* (3.2.4) *is defined by:*

$$
\begin{aligned}
\lim_{\Delta t\to 0}\Big[\frac{1}{\Delta t}E\big[&V(t+\Delta t, y_{t+\Delta t}, \theta_{t+\Delta t}, x_{t+\Delta t}) \\
&- V(t, y_t, \theta_t, x_t)\big|y_t = y_{t^-}, \theta_t = \theta_{t^-}, x_t = x_{t^-}\big]\Big] \\
&= \mathcal{A}V(t, y_{t^-}, \theta_{t^-}, x_{t^-}), \text{ for } \theta_{t^-} \in E, \qquad (1.4.35)
\end{aligned}
$$

*Moreover, a one parameter family of semi-group is generated by*

$$
\frac{\partial V}{\partial t}(t, y_{t^-}, \theta_{t^-}, x_{t^-}) = \mathcal{A}V(t, y_{t^-}, \theta_{t^-}, x_{t^-}), \qquad (1.4.36)
$$

*where $\mathcal{A} = \mathcal{L}$ in* (1.4.16) *and $\frac{\partial V}{\partial t}(t, y_t, \theta_t, x_t)$ is the conditional partial derivative defined by the left-hand side of* (1.4.35).

We present special cases of the developed infinitesimal generator in Definition 1.4.3.

REMARK 1.4.4 From Remark 1.3.2, the infinitesimal generator $\mathcal{A}$ defined in (1.4.35) extends the earlier work in a systematic way. In fact, this generator includes the infinitesimal generator influenced by finite state Markov chain [16, 26]. Moreover, it also includes the generator influenced by a finite state semi Markov process [30, 42, 65]. If H and G are replaced by $e^G - 1$ and $e^H - 1$, then $\mathcal{A}$ in (1.4.35) in the context of (1.4.17) is

$$
\begin{aligned}
&\mathcal{A}V(s, y_{s^-}, \theta_{s^-}, \theta_s, x_{s^-}) \\
&= \frac{\partial V}{\partial s} + \frac{\partial V}{\partial y} + \mu(\theta_{s^-})x_{s^-}\frac{\partial V}{\partial x} + \frac{1}{2}x_{s^-}^2\sigma^2(\theta_{s^-})\frac{\partial^2 V}{\partial x^2} \\
&\quad + \int_{|z|\leq 1}\Big[V(s, y_{s^-}, \theta_{s^-}, \theta_s, x_{s^-} + x_{s^-}[e^{G(z,\theta_{s^-})} - 1]) \\
&\quad - V(s, y_{s^-}, \theta_{s^-}, \theta_s, x_{s^-}) - x_{s^-}[e^{G(z,\theta_{s^-})} - 1]\frac{\partial V}{\partial x}\Big]\nu(\theta_{s^-}, dz) \\
&\quad + \int_{|z|>1}\Big[V(s, y_{s^-}, \theta_{s^-}, \theta_s, x_{s^-} + x_{s^-}[e^{H(z,\theta_{s^-})} - 1]) \\
&\quad - V(s, y_{s^-}, \theta_{s^-}, \theta_s, x_{s^-})\Big]\nu(\theta_{s^-}, dz) \\
&\quad + \int_{z\in\mathbb{R}}\sum_{\theta_s\in E, \theta_s\neq\theta_{s^-}}\lambda_{\theta_{s^-},\theta_s}(y_s)\big[V(s, y_{s^-}, \theta_s, x_{s^-}e^z) \\
&\quad - V(s, y_{s^-}, \theta_{s^-}, x_{s^-})\big]\bar{b}(z|\theta_{s^-}, \theta_s)dz
\end{aligned}
$$

A few notes on the nature of the infinitesimal operator.

REMARK 1.4.5 We further remark that the infinitesimal generator defined in (1.4.35) can be rewritten in a $m \times m$ matrix form. In fact the partial differential equations in (1.4.36) are a system of partial differential equations of dimension m. More precisely, (1.4.36) is a linear system of partial differential equations with variable coefficients.

REMARK 1.4.6 For $V(t, y_t, \theta_t, x_t) = x_t$, the conclusion of Theorem 1.4.1 reduces to

$$
\begin{aligned}
dx_t = &\mathcal{L}V(t, y_{t^-}, \theta_{t^-}, x_{t^-})dt + \sigma(\theta_t^-)dB_t + \int_{|z|\leq 1} x_{t^-}G(z, \theta_{t^-})\bar{\psi}(\theta_{t^-}, dz, dt) \\
&+ \int_{|z|>1} x_{t^-}H(z, \theta_{t^-})\bar{\psi}(\theta_{t^-}, dz, dt) \\
&+ \int_{z\in\mathbb{R}}\sum_{j\in E\setminus\{\theta_{t^-}\}}\big[x_{t^-}(e^z - 1)\bar{N}(dt, dz, \{(\theta_{t^-}, j)\})\big], \qquad (1.4.37)
\end{aligned}
$$

where

$$\begin{aligned}
&\mathcal{L}V(t, y_{t-}, \theta_{t-}, x_{t-}) \\
=&x_{t-}\Big[\mu(\theta_{t-}) \\
&+\int_{|z|>1} H(z, \theta_{t-})\nu(\theta_{t-}, dz) \\
&+\int_{|z|>1} \sum_{j\in E\setminus\{\theta_{t-}\}} \big[(e^z-1)\lambda_{\theta_{t-},j}(y_{t-})\big]\bar{b}(z|\theta_{t-}, j)dz\Big].
\end{aligned} \tag{1.4.38}$$

$$dx_t = x_{t-}dM_t^\theta, \tag{1.4.39}$$

with

$$\begin{aligned}
dM_t^\theta =&\mu(\theta_{t-})dt + \int_{|z|\le 1} G(z, \theta_{t-})\bar{\psi}(\theta_{t-}, dz, dt) \\
&+\int_{|z|>1} H(z, \theta_{t-})\bar{\psi}(\theta_{t-}, dz, dt) \\
&+\int_{z\in\mathbb{R}} \sum_{j\in E\setminus\{\theta_{t-}\}} \big[(e^z-1)\bar{N}(dt, dz, \{(\theta_{t-}, j)\})\big] \\
&+\int_{|z|>1} \sum_{j\in E\setminus\{\theta_{t-}\}} \big[(e^z-1)\lambda_{\theta_{t-},j}(y_{t-})\big]\bar{b}(z|\theta_{t-}, j)dzdt \\
&+\int_{|z|>1} H(\theta_{t-}, z)\nu(\theta_{t-}, dz)dt \\
=&dL_t^\theta + \int_{z\in\mathbb{R}} \sum_{j\in E\setminus\{\theta_{t-}\}} \big[(e^z-1)N(dt, dz, \{(\theta_{t-}, j)\})\big],
\end{aligned} \tag{1.4.40}$$

where $L^\theta$ is defined in (1.3.2). Furthermore, we note that the solution process determined by (1.4.39) has another solution representation of (1.4.1) in the framework of Remark 1.3.2. In fact, the closed form solution representation of (1.4.39) is as follows:

$$\begin{aligned}
x_t = x_0\exp\Big\{\Big[&\int_0^t\int_{z\in\mathbb{R}} \sum_{j\in E\setminus\{\theta_{s-}\}} \big[zN(ds, dz, \{(\theta_{s-}, j)\})\big] + \int_0^t \mu(\theta_{s-})ds \\
&-\frac{1}{2}\int_0^t \sigma^2(\theta_{s-})ds + \int_0^t\int_{|z|\le 1} \big[\ln(1+G(z,\theta_{s-})) - G(z,\theta_{s-})\big]\nu(\theta_{s-}, dz)ds \\
&+\int_0^t \sigma(\theta_{s-})dB_s + \int_0^t\int_{|z|\le 1} \ln(1+G(z,\theta_{s-}))\bar{\psi}(\theta_{s-}, dz, ds) \\
&+\int_0^t\int_{|z|>1} \ln(1+H(z,\theta_{s-}))\psi(\theta_{s-}, dz, ds)\Big]\Big\}.
\end{aligned} \tag{1.4.41}$$

We exhibit three particular cases of the infinitesimal generator.

ILLUSTRATION 1.4.2 We exhibit the infinitesimal generators corresponding to SMBS, SMMJD and SMNIG models developed as Illustration 1.4.1.

The SMBS model: The infinitesimal generator of the SMBS defined in Illustration 1.4.1 reduces to:

$$\begin{aligned}\mathcal{A}V(s, y_{s^-}, \theta_{s^-}, \theta_s, x_{s^-}) &= \frac{\partial V}{\partial s} + \frac{\partial V}{\partial y} + \mu(\theta_{s^-})x_{s^-}\frac{\partial V}{\partial x} + \frac{1}{2}x_{s^-}^2\sigma^2(\theta_{s^-})\frac{\partial^2 V}{\partial x^2} \\ &+ \int_{z\in\mathbb{R}} \sum_{\theta_s\in E, \theta_s\neq\theta_{s^-}} \lambda_{\theta_{s^-},\theta_s}(y_s)\big[V(s, y_{s^-}, \theta_s, x_{s^-}e^z) - V(s, y_{s^-}, \theta_{s^-}, x_{s^-})\big]\bar{b}(z|\theta_{s^-}, \theta_s)dz\end{aligned} \tag{1.4.42}$$

The SMMJD model: The infinitesimal generator of the SMMJD defined in Illustration 1.4.1 reduces to:

$$\begin{aligned}\mathcal{A}V(s, y_{s^-}, \theta_{s^-}, \theta_s, x_{s^-}) &= \frac{\partial V}{\partial s} + \frac{\partial V}{\partial y} + \mu(\theta_{s^-})x_{s^-}\frac{\partial V}{\partial x} + \frac{1}{2}x_{s^-}^2\sigma^2(\theta_{s^-})\frac{\partial^2 V}{\partial x^2} \\ &+ \int_{z\in\mathbb{R}} \Big[V(s, y_{s^-}, \theta_{s^-}, \theta_s, x_{s^-} + x_{s^-}[e^z - 1]) - V(s, y_{s^-}, \theta_{s^-}, \theta_s, x_{s^-})\Big]\nu_{mjd}(\theta_{s^-}, dz) \\ &+ \int_{z\in\mathbb{R}} \sum_{\theta_s\in E, \theta_s\neq\theta_{s^-}} \lambda_{\theta_{s^-},\theta_s}(y_s)\big[V(s, y_{s^-}, \theta_s, x_{s^-}e^z) - V(s, y_{s^-}, \theta_{s^-}, x_{s^-})\big]\bar{b}(z|\theta_{s^-}, \theta_s)dz\end{aligned} \tag{1.4.43}$$

The SMNIG model: The infinitesimal generator defined in Illustration 1.4.1 reduces to:

$$\begin{aligned}\mathcal{A}V(s, y_{s^-}, \theta_{s^-}, \theta_s, x_{s^-}) &= \frac{\partial V}{\partial s} + \frac{\partial V}{\partial y} + \mu(\theta_{s^-})x_{s^-}\frac{\partial V}{\partial x} \\ &+ \int_{|z|\leq 1} \Big[V(s, y_{s^-}, \theta_{s^-}, \theta_s, x_{s^-} + x_{s^-}[e^z - 1]) - V(s, y_{s^-}, \theta_{s^-}, \theta_s, x_{s^-}) - x_{s^-}[e^z - 1]\frac{\partial V}{\partial x}\Big]\nu_{nig}(\theta_{s^-}, dz) \\ &+ \int_{|z|>1} \Big[V(s, y_{s^-}, \theta_{s^-}, \theta_s, x_{s^-} + x_{s^-}[e^z - 1]) - V(s, y_{s^-}, \theta_{s^-}, \theta_s, x_{s^-})\Big]\nu_{nig}(\theta_{s^-}, dz) \\ &+ \int_{z\in\mathbb{R}} \sum_{\theta_s\in E, \theta_s\neq\theta_{s^-}} \lambda_{\theta_{s^-},\theta_s}(y_s)\big[V(s, y_{s^-}, \theta_s, x_{s^-}e^z) - V(s, y_{s^-}, \theta_{s^-}, x_{s^-})\big]\bar{b}(z|\theta_{s^-}, \theta_s)dz\end{aligned} \tag{1.4.44}$$

In the following section, we utilize the infinitesimal generator of the exponential semi Markov Lévy switching process to find a closed form expression of the characteristic function of the $\ln(x_t)$ with $x_t$ solution of (1.4.1).

## 1.5 Characteristic function

In this section, we derive a closed form expression for the conditional characteristic function

$$\Psi(u,t,y,j,x) = E\Big[e^{iu\ln(x_t)}\Big|y_0 = y, \theta_0 = j, x_0 = x\Big] \tag{1.5.1}$$

of the log price process,

$$\ln\big(x_t\big) = \sum_{p=1}^{n(t)} \ln(\beta_p) + L_t^\theta, \tag{1.5.2}$$

where $\beta_n$, $L^\theta$ in Definition 1.4.1 and $x_t$ is the closed form solution process of (3.2.4) in the context of (1.3.27).

LEMMA 1.5.1 *Let $L_t^\theta$, $x$, $y$ and $\gamma$ be defined in* (1.3.2)*,* (1.4.1)*,* (1.2.14) *and* (1.4.10)*, respectively. A closed form expression for the conditional characteristic vector function of* $\ln(x)$ *is,*

$$\Psi(u,t,y,x) = \exp\left[iu\ln(x)\right]\exp\left[\int_y^{t+y} M(u,s)ds\right].\mathbf{1}, \tag{1.5.3}$$

*where $i = \sqrt{-1}$; $\Psi(u,t,y,x)$ is an m-dimensional column vector with k-th component $\Psi(u,t,y,k,x)$, for $k \in E$; $\mathbf{1}$ is $m \times 1$ vector with components ones, and $M(u,y) = \big(M_{pq}(u,y)\big)_{m\times m}$ is an $m \times m$ matrix defined by:*

$$M_{q,p}(u,y) = \begin{cases} iu\mu(q) - \frac{1}{2}\sigma^2(q)u^2 + \int_{|z|\le 1}\big[e^{iuG(z,q)} - 1 - iuG(z,q)\big]\nu(q,dz) \\ + \int_{|z|>1}\big[e^{iuH(z,q)} - 1\big]\nu(q,dz) + \lambda_{q,q}(y), \ \textit{if } p = q \\ \lambda_{q,p}(y)\int_{z\in\mathbb{R}} e^{iuz}\bar{b}(z|q,p)dz, \ \textit{otherwise}, \end{cases}$$

*and it is assumed to satisfy the Lie bracket-type condition:*

$$\big[M(u,y_1), M(u,y_2)\big] = 0 \quad, \forall y_1, y_2 \in \mathbb{R}^+ \tag{1.5.4}$$

*Proof.* From (1.5.3), first we observe that

$$\Psi(u,t,y,x) = \exp\left[iuL_t^\theta\right]\exp\Big[\int_y^{t+y} M(u,s)ds\Big].\mathbf{1} \tag{1.5.5}$$

We note that $\Psi(u,t,y,\theta_t,x)$ possesses all smoothness properties of $V$ defined in Theorem 1.4.1. Therefore, following the argument used in the proof of Theorem 1.4.1, Definition 1.4.3 and Remark 1.4.5, we conclude that $\Psi$ is in the domain of the infinitesimal generator of the process

$(y_t, \theta_t, iuL_t^{\theta})_{t\in[0,T]}$. Moreover, it satisfies the following system of linear partial differential equation

$$\frac{\partial \Psi(u,t,y,k,x)}{\partial t} = \mathcal{A}\Psi(u,t,y,k,x), \text{ for } k \in E, \tag{1.5.6}$$

where $\mathcal{A}$ is the operator defined in Definition 1.4.3. From Remark 1.4.4 with $\mu(\theta_{s^-})$ replaced by

$$\mu(\theta_{s^-}) + \frac{1}{2}\sigma(\theta_{s^-}) + \int_{|z|\le 1}\Big[e^{G(z,\theta_{s^-})} - G(Z,\theta_{s^-}) - 1\Big]\nu(\theta_{s^-}, dz)$$

and for $\Psi$ defined in (1.5.1), we have:

$$\begin{aligned}
&\mathcal{A}\Psi(u,s,y_{s^-},\theta_{s^-},x)\\
&=\frac{\partial \Psi(u,s,y,\theta_{s^-},x)}{\partial s} + \frac{\partial \Psi(u,s,y,\theta_{s^-},x)}{\partial y}\\
&\quad + iu\big[\mu(\theta_s) + \frac{1}{2}\sigma^2(\theta_s)\big]x_{s^-}\frac{\partial \Psi(u,s,y,\theta_{s^-},x)}{\partial x}\\
&\quad + \int_{|z|\le 1}\Big[\Psi(u,s,y,\theta_{s^-},x + x[e^{iuG(z,\theta_{s^-})} - 1])\\
&\quad - \Psi(u,s,y,\theta_{s^-},x) - iuxG(z,\theta_{s^-})\frac{\partial \Psi(u,s,y,\theta_{s^-},x)}{\partial x}\Big]\nu(\theta_{s^-}, dz)\\
&\quad + \int_{|z|>1}\Big[\Psi(u,s,y,\theta_{s^-},x + x[e^{iuH(z,\theta_{s^-})} - 1])\\
&\quad - \Psi(u,s,y,\theta_{s^-},x)\Big]\nu(\theta_{s^-}, dz) - \frac{1}{2}x_s^2u^2\sigma^2(\theta_{s^-})\frac{\partial^2 \Psi(u,s,y,\theta_{s^-},x)}{\partial x^2}\\
&\quad + \int_{z\in\mathbb{R}}\sum_{\theta_s\in E\setminus\{\theta_{s^-}\}}\Psi(u,s,y,\theta_s,xe^{iuz})\bar{b}(z|\theta_{s^-},\theta_s)\lambda_{\theta_{s^-},\theta_s}(y)dz\\
&\quad - \Psi(u,s,y,\theta_{s^-},x)\lambda_{\theta_{s^-},\theta_{s^-}}(y) \text{ for } \theta_{s^-}\in E.
\end{aligned} \tag{1.5.7}$$

Now, we assume that $\Psi(u,t,y,k,x) = \exp\Big[iu\ln(x)\Big]h(u,t,y,k)$, where $h(u,t,y,k)$ is the k-th component of an unknown m-dimensional vector function $\mathbf{h}(u,t,y) = \big[h(u,t,y,1),...,h(u,t,y,m)\big]^{\top}$. From this, (1.5.6) reduces to the following system of partial differential equations:

$$\begin{aligned}
&\frac{\partial h(u,t,y,k)}{\partial t}\\
&\qquad=\frac{\partial h(u,t,y,k)}{\partial y} + h(u,t,y,k)\Big[iu\big[\mu(k) + \frac{1}{2}\sigma^2(k)\big]\\
&\qquad\quad + \frac{1}{2}\sigma^2(k)[-iu-u^2] + \int_{|z|\le 1}\big[e^{iuG(z,k)} - 1 - iuG(z,k)\big]\nu(k,dz)\\
&\qquad\quad + \int_{|z|>1}\big[e^{iuH(z,k)} - 1\big]\nu(k,dz) + \lambda_{k,k}(y)\Big]\\
&\qquad\quad + \int_{z\in\mathbb{R}}\sum_{j\neq k}\lambda_{k,j}(y)h(u,t,y,j)(y)e^{iuz}\bar{b}(z|k,j)dz
\end{aligned}$$

$$
\begin{aligned}
=&\frac{\partial h(u,t,y,k)}{\partial y}+h(u,t,y,k)\Big[iu\mu(k)-\frac{1}{2}\sigma^2(k)u^2\\
&+\int_{|z|\le 1}\big[e^{iuG(z,k)}-1-iuG(z,k)\big]\nu(k,dz)\\
&+\int_{|z|>1}\big[e^{iuH(z,k)}-1\big]\nu(k,dz)+\lambda_{k,k}(y)\Big]\\
&+\int_{z\in\mathbb{R}}\sum_{j\neq k}\lambda_{k,j}(y)h(u,t,y,j)(y)e^{iuz}\bar{b}(z|k,j)dz.
\end{aligned}
\tag{1.5.8}
$$

As stated in Remark 1.4.5, the coefficients of $h$ are defined by the elements of $\mathcal{A}$ associated with $\Psi(u,t,y,x)$, in particular, the $m\times m$ matrix $M(u,y)=\big(M_{k,j}(u,y)\big)_{m\times m}$ defined in (1.5.4). From the definition of $\mathbf{h}(u,t,y)$, (1.5.8) reduces to:

$$
\frac{\partial \mathbf{h}(u,t,y)}{\partial t}=\frac{\partial \mathbf{h}(u,t,y)}{\partial y}+M(u,y)\mathbf{h}(u,t,y),\quad \mathbf{h}(u,0,y)=\mathbf{1}=\underbrace{(1,...,1)^\top}_{m\ ones}. \tag{1.5.9}
$$

Using the method of characteristics, the system of partial differential equations (1.5.9) can be solved. In this case, the characteristic curves are determined by $\frac{dy}{dt}=\pm 1$. Solving these differential equations, we obtain

$$
\eta=t-y \text{ and } \zeta=t+y. \tag{1.5.10}
$$

We use the above change of variable to define the transforms $\tilde{h}$ and $\tilde{M}$ from $h$ and $M$, respectively, as functions of $(\eta,\zeta)$:

$$
\begin{cases}
\tilde{\mathbf{h}}(u,\eta,\zeta)=\mathbf{h}(u,\frac{\eta+\zeta}{2},\frac{-\eta+\zeta}{2})\\
\tilde{M}(u,\eta,\zeta)=M(u,\frac{-\eta+\zeta}{2}).
\end{cases}
\tag{1.5.11}
$$

From (1.5.11), the initial value problem (1.5.9) reduces to the ODE:

$$
\frac{\partial \tilde{\mathbf{h}}(u,\eta,\zeta)}{\partial \eta}=\frac{1}{2}\tilde{M}(u,\eta,\zeta)\tilde{\mathbf{h}}(u,\eta,\zeta),\qquad \tilde{\mathbf{h}}(u,-y,y)=\mathbf{1}. \tag{1.5.12}
$$

Under condition (1.5.4), the general solution of the linear homogeneous ODE with time varying coefficients is [49]:

$$
\tilde{\mathbf{h}}(u,\eta,\zeta)=\exp\Big[\frac{1}{2}\int_0^\eta \tilde{M}(u,\kappa,\zeta)d\kappa\Big].g(\zeta), \tag{1.5.13}
$$

where g is an arbitrary m-dimensional vector function. Using the initial condition in (1.5.12), g is determined by

$$
g(\zeta)=\exp\Big[\frac{1}{2}\int_{-\zeta}^0 \tilde{M}(u,\kappa,\zeta)d\kappa\Big]\mathbf{1},\forall\zeta\in[0,T].
$$

This together with (1.5.13), yields the solution of the initial value problem (1.5.12) as:

$$\tilde{\mathbf{h}}(u,\eta,\zeta) = \exp\Big[\frac{1}{2}\int_{-\zeta}^{\eta}\tilde{M}(u,\kappa,\zeta)d\kappa\Big]\mathbf{1}.$$

Using the inverse of the transformation defined in (1.5.10), the solution of the original initial value problem (1.5.9) becomes:

$$\begin{aligned}\mathbf{h}(u,t,y) &= \exp\Big[\frac{1}{2}\int_{-t-y}^{t-y} M(u, \frac{-\kappa+t+y}{2})d\kappa\Big]\mathbf{1}\\ &= \exp\Big[\int_{y}^{t+y} M(u,s)ds\Big]\mathbf{1}.\end{aligned}$$

This establishes the conditional characteristic function for the log prices described by (1.4.1). □

REMARK 1.5.1 We note that the closed form exponential expression (1.5.3) holds only under condition (1.5.4). This is due to the fact that system of ode in (1.5.12) has time varying coefficients. Assuming the matrix $(M_{i,j})_{m\times m}$ has continuous entries with respect to $y$, the system of differential equations with time varying coefficients (1.5.12) has a fundamental matrix $\Phi(u,t,y)$. Therefore, the characteristic function in (1.5.1) is described by

$$\Psi(u,t,y,x) = \exp\left[iu\ln(x)\right]\left[\Phi(u,t,y)\Phi(u,0,y)^{-1}\right]\mathbf{1}. \tag{1.5.14}$$

The characteristic function $\Psi(u,t,y,x)$ has a closed form expression if $\Phi$ has a closed form expression. Theorem 1.5.1 corresponds to the particular case where $\Psi(u,t,y,x) = \exp\Big[\int_y^{t+y} M(u,s)ds\Big]$.

We next provide three particular instances of the matrix $M$ in (1.5.4), found inside the expression of the closed form characteristic function in (1.5.3).

ILLUSTRATION 1.5.1 We exhibit three instances of the matrix derived in (1.5.4).

The SMBS model: In the particular case of the SMBS model, we have

$$M_{q,p}(u,y) = \begin{cases} iu\mu(q) - \frac{1}{2}\sigma^2(q)u^2 + \lambda_{q,q}(y), \text{ if } p=q \\ \lambda_{q,p}(y)\int_{z\in\mathbb{R}} e^{iuz}\bar{b}(z|q,p)dz, \text{ otherwise,}\end{cases} \tag{1.5.15}$$

The SMMJD model: In the particular case of the SMMJD model, we have

$$M_{q,p}(u,y) = \begin{cases} iu\big[\mu(q) - \int_{|z|\le 1} z\nu_{mjd}(q,dz)\big] - \frac{1}{2}\sigma^2(q)u^2 + \int_{z\in\mathbb{R}}\left[e^{iuz}-1\right]\nu_{mjd}(q,dz) + \lambda_{q,q}(y), \\ \text{if } p=q \\ \lambda_{q,p}(y)\int_{z\in\mathbb{R}} e^{iuz}\bar{b}(z|q,p)dz, \text{ otherwise,}\end{cases} \tag{1.5.16}$$

The SMNIG model: In the particular case of the SMNIG, we have

$$M_{q,p}(u,y) = \begin{cases} iu\mu(q) - \frac{1}{2}\sigma^2(q)u^2 + \int_{z\in\mathbb{R}} \left[e^{iuz} - 1 - iuz1_{|z|\le 1}\right]\nu_{nig}(q,dz) + \lambda_{q,q}(y), \text{ if } p = q \\ \lambda_{q,p}(y)\int_{z\in\mathbb{R}} e^{iuz}\bar{b}(z|q,p)dz, \text{ otherwise,} \end{cases} \tag{1.5.17}$$

In the following, we present a consequence of Lemma 1.5.1 that extends the characteristic function of sojourn time of finite state Markov processes [25, 53].

COROLLARY 1.5.1.1 *We denote $O_s^t(k)$ the time spent by the semi Markov process $(\theta_t)_{t\in[0,T]}$ in its state k in the time interval $[s,t]$,$\forall s,t \in [0,T]$, with $s \le t$, and $O_s^t = \left[O_s^t(1), O_s^t(2), ..., O_s^t(m)\right]^\top$ denotes the m-dimensional occupation time vector of $\theta$. If $a = (a_1, a_2, ..., a_m)^\top$ is an $m \times 1$ vector of constant real numbers, then,*

$$E\left[e^{iu\langle J_s^t, a\rangle}\middle| y_s = y, \theta_s = k\right] = \left\langle \left[\exp\left[\int_y^{t+y} M(u,s)ds\right]\right].e_k, \mathbf{1}\right\rangle,$$

*where,*

$$M_{p,q}(u,y) = \begin{cases} iua_q + \lambda_{q,q}(y) & \text{, If } p = q \\ \lambda_{p,q}(y) & \text{, otherwise} \end{cases} \tag{1.5.18}$$

*Proof.* Let $(a_t)_{t\in[0,T]}$ denote a stochastic process with $a_t = a_j$ whenever $\theta_{n(t)} = j$.

$$\begin{aligned} \exp\left[iu\langle O_s^t, a\rangle\right] &= \exp\left[iu\sum_{k\in E} a_k O_s^t(k)\right] \text{ , for } i = \sqrt{-1} \\ &= \exp\left[iu\int_s^t a_{v^-}dv\right] \text{ , ( as } a \text{ is piecewise constant)} \\ &= \exp\left[iu\int_s^t a_{v^-}dv\right] \\ &= e^{iu\int_s^t dL_v^\theta}, \text{ with } L_v^\theta = a_v, \forall v \in [0,T]. \end{aligned}$$

The characteristic function of the semi Markov occupation times in the time interval $[s,t]$ becomes:

$$\begin{aligned} &E\left[e^{iu\langle J_s^t, a\rangle}\middle| y_s = y, \theta_s = k\right] \\ &= E\left[e^{iu\int_s^t dL_v^\theta}\middle| y_s = y, \theta_s = k\right] \\ &= E\left[e^{iuL_{t-s}^\theta}\middle| y_0 = y, \theta_0 = k\right], \end{aligned}$$

since the couple $(\theta, y)$ is homogeneous. Applying Lemma 1.5.1 with $\beta_n = 1\ \forall n \in I(0,\infty)$ proves the result. Corollary 1.5.1.1 is a direct extension of the results in [9, 36, 53]. ☐

### 1.6 Conclusion

We derived Itó formula for an asset price driven by a Levy process in a market with semi Markov regimes. However, applying such a model to the pricing of derivative products will present quite a few challenges. In Itó formula, the term due to semi Markov regime changes in the market depends on the backward recurrence time, while its counterpart in markets with Markov regimes is merely constant. This support the economic interpretation that regimes in semi Markov markets have a time dependent instantaneous propensity of switching to another state. In other terms, the instantaneous probability of changing state conditional on the current one depends on the "age" of the current state. If one considers the financial downturns seen in the market since the great depression, one might find sensible to assume that in the early age a financial crisis there is a decent chance that the market regime improves as it hasn't yet gangrened the financial system and can still be contained and fixed. However, as the issue lasts longer, it becomes more difficult to contain it as more sectors are affected. Markov market regimes assume that no matter the age of the market regime, the instantaneous conditional probability of switching to any particular regime remains constant.

In addition to the supplementary flexibility, the new model offers a simple but reasonable low hanging solution to the infinite market regime problem. Although the semi Markov market asset model defined in this chapter is assumed to have a finite number of regimes, each regime can have infinitely many sub-regimes. Indeed, let us assume that the market follows a three state model where the first state corresponds to a bearish market, the second state corresponds to a normal market and the last corresponds to a bullish market. Each observed bearish(respectively bullish, normal) market are not identical in semi Markov markets. A given bearish (respectively bullish, normal) market with an increasing conditional propensity of turning bullish (respectively bearish, normal) is not to be equalled to a bearish market with a decreasing conditional propensity of turning bullish. Although both are bearish markets, one has reasons to be more optimistic in the former than the latter as the signs point respectively to an imminent change of regime and a long bearish market. In other terms, semi Markov markets distinguish between a bearish market trending in the right direction and a bearish market trending in the wrong direction. Such differences are retrieved from the functional form of the conditional intensity matrix of the semi Markov process, which can be chosen in infinitely many ways hence supporting the the notion that although market regimes can be grouped in finitely many categories, market states are actually infinite.

The derivation of the characteristic function of the log price also allows us to envision the calibration of the semi Markov parameters using the Fourier transform.

# Chapter 2

# Risk neutral option price formula under semi Markov regimes

## 2.1 Introduction

In this Chapter, we introduce the conditional minimum equivalent martingale measuring the semi Markov jump risk and the Lévy risk. In addition, we also develop an unconditional minimum entropy martingale measure and the Esscher transform [64] measuring all three risks, namely, Lévy risk, semi Markov jump risk and regime switching risk. In Section 2.2, we exhibit a general change of measure and two risk neutral measures of interest, namely, the minimum entropy martingale measures and the pricing kernel discussed in [64]. The latter accounts for the regime risk, the jump risk and the Lévy risk. Section 2.3 is devoted to the presentation of a couple of option price formulas. The first is an application of the well known Fourier transform method developed in [11]. The second formula is a slight modification of the integral formula developed in [30].

## 2.2 Change of measure and Pricing Kernels

We first utilize the closed form solution representation of (1.3.1) to shed more light on the martingale property of the solution process of the Lévy type stochastic linear differential equations. For this purpose, let $x_t$ be the solution process of (1.3.1) and assume that it is a $\left(\mathbb{H}_t \vee \bar{\mathbb{L}}_t\right)-$martingale, that is, for $s \leq t$, $E\left[x_t - x_s \middle| \mathbb{H}_s \vee \bar{\mathbb{L}}_s\right] = 0$. This is represented by the following illustrations.

ILLUSTRATION 2.2.1 *(a) From* $E\left[x_t - x_s \middle| \mathbb{H}_s \vee \bar{\mathbb{L}}_s\right] = 0$ *and* (1.3.27)*, it is obvious that the solution* *process $x_t$ of* (1.3.1)

$$\begin{aligned}x_t =&x_0 \exp\{\Big[\int_0^t \mu(\theta_{s^-})ds + \int_0^t \sigma(\theta_{s^-})dB_s \\ &+ \int_0^t \int_{|z|\leq 1} G(z,\theta_{s^-})\bar{\psi}(\theta_{s^-},dz,ds) \\ &+ \int_0^t \int_{|z|>1} H(z,\theta_{s^-})\bar{\psi}(\theta_{s^-},dz,ds)\Big]\} \qquad (2.2.1)\end{aligned}$$

*is a martingale if and only if*

$$\begin{aligned}&\mu(\theta_{t^-}) + \frac{1}{2}\sigma^2(\theta_{t^-}) + \int_{|z|\leq 1}\Big[e^{G(z,\theta_{t^-})} - G(z,\theta_{t^-}) - 1\Big]\nu(\theta_{t^-},dz) \\ &+ \int_{|z|>1}\Big[e^{H(z,\theta_{t^-})} - 1\Big]\nu(\theta_{t^-},dz) = 0, \forall \theta_{t^-} \in E. \qquad (2.2.2)\end{aligned}$$

(b) *Furthermore if $L_t^\theta$ in* (1.3.2) *is replaced by $M_t^\theta$:*

$$\begin{aligned}dM_t^\theta =&\sigma(\theta_{t^-})dB_t + \int_{|z|\leq 1} G(z,\theta_{t^-})\bar{\psi}(\theta_{t^-},dz,dt) \\ &+ \int_{|z|>1} H(z,\theta_{t^-})\bar{\psi}(\theta_{t^-},dz,dt), \qquad (2.2.3)\end{aligned}$$

*then the solution process of* (1.3.1) *in* (1.3.20) *is indeed a martingale and is represented by:*

$$\begin{aligned}x_t =&x_0 \exp\{\Big[-\frac{1}{2}\int_0^t \sigma^2(\theta_{s^-})ds + \int_0^t \sigma(\theta_{s^-})dB_s \\ &+ \int_0^t \int_{|z|>1} \big[\ln\big(H(z,\theta_{s^-})+1\big) - H(z,\theta_{s^-})\big]\nu(\theta_{s^-},dz)ds \\ &+ \int_0^t \int_{|z|\leq 1} \ln\left(1+G(z,\theta_{s^-})\right)\bar{\psi}(\theta_{s^-},dz,ds) \\ &+ \int_0^t \int_{|z|\leq 1} \big[\ln\left(1+G(z,\theta_{s^-})\right) - G(z,\theta_{s^-})\big]\nu(\theta_{s^-},dz)ds \\ &+ \int_0^t \int_{|z|>1} \ln\left(1+H(z,\theta_{s^-})\right)\bar{\psi}(\theta_{s^-},dz,ds)\Big]\}. \qquad (2.2.4)\end{aligned}$$

(c) *Replacing $H(z,\theta_s)$, $G(z,\theta_s)$ and $L_s^\theta$ in* (1.3.2) *by $e^{H(z,\theta_s)} - 1$, $e^{G(z,\theta_s)} - 1$ and $M^\theta$ in* (2.2.3), *respectively, the solution of the (IVP)* (1.3.1) *in* (1.3.22) *is a martingale if and only if:*

$$\begin{aligned}
x_t =& x_0 \exp\{\Big[\int_0^t \sigma(\theta_{s^-})dB_s + \int_0^t \int_{|z|>1} H(z,\theta_{s^-})\nu(\theta_{s^-},dz)ds \\
&- \frac{1}{2}\int_0^t \sigma^2(\theta_{s^-})ds + \int_0^t \int_{|z|\le 1} \big[G(z,\theta_{s^-}) - e^{G(z,\theta_{s^-})} + 1\big]\nu(\theta_{s^-},dz)ds \\
&+ \int_0^t \int_{|z|\le 1} G(z,\theta_{s^-})\bar{\psi}(\theta_{s^-},dz,ds) \\
&+ \int_0^t \int_{|z|>1} H(z,\theta_{s^-})\bar{\psi}(\theta_{s^-},dz,ds)\Big]\}.
\end{aligned} \tag{2.2.5}$$

*(d)* $V(t, y_{t^-}, \theta_{t^-}, x_{t^-})$ *in* (1.4.16) *is a martingale if and only if* $\mathcal{L}V(t, y_{t^-}, \theta_{t^-}, x_{t^-})$ *is identically equal to zero. In particular, from* (1.4.37)*, the solution process of* (1.4.1) *is a local martingale if and only if* $\mathcal{L}V(t, y_{t^-}, \theta_{t^-}, x_{t^-})$ *in* (1.4.38) *is identically zero that is* $\mu(\theta_{t^-}) + \int_{|z|>1} H(z,\theta_{t^-})\nu(\theta_{t^-},dz) + \int_{|z|>1}\sum_{j\in E\backslash\{\theta_{t^-}\}}\big[(e^z-1)\lambda_{\theta_{t^-},j}(y_{t^-})\big]\bar{b}(z|\theta_{t^-},j)dz = 0$

We introduce and recall a few notations necessary for presenting the next lemma.

REMARK 2.2.1 We denote $\Phi_t$ a positive $\big(P,(\mathbb{H}_t \vee \bar{\mathbb{L}}_t)_{t\in[0,T]}\big)-$ martingale process with initial value $\Phi_0 = 1$. In fact for $x_0 = 1$, any one of the solution processes in Illustration 2.2.1 part (a), I can be represented by $\Phi_t$, that is the fundamental solution process of linear Lévy-type stochastic differential equations. Moreover, $\Phi_t$ is called a density process of a probability measure $\bar{P}$ with respect to a given probability measure $P$.

Based on a Girsanov theorem for Lévy [44] and point [8] processes, we present a Girsanov-type theorem for stochastic hybrid process described by (1.4.1). We highlight the effects of change of measures on both time and state domains of decomposition with respect to $(L_t^\theta)_{t\in[0,T]}$, $(\beta_n)_{n\ge 0}$ and $(\theta_t)_{t\in[0,T]}$. $(T_n)_{n\ge 0}$ are the jump times in Definition 1.2.1.

LEMMA 2.2.1 (Girsanov-type Theorem) *Let* $\eta$ *and* $Y$ *be piecewise deterministic stochastic processes defined on* $[0,T]\times\mathbb{R}$ *and* $[0,T]\times\mathbb{R}\times\mathbb{R}$ *into* $\mathbb{R}$*, respectively.* $\xi = \big(\xi_{i,j}(s,z)\big)_{m\times m}$ *is a* $\mathbb{R}^{m\times m}-$*valued and* $\bar{\mathbb{H}}_t-$*predictable process defined on* $[0,T]\times\mathbb{R}$ *into* $\mathbb{R}$*. Let us consider the process* $M_t^\theta$ *defined by:*

$$dM_t^\theta = \int_{z\in\mathbb{R}} \sum_{j\in E\backslash\{\theta_{t^-}\}} (e^z - 1)N(dt,dz,\{(\theta_{t^-},j)\}) + dL_t^\theta, \tag{2.2.6}$$

*where* $L_t^\theta$ *is defined in* (1.3.2)*. Furthermore, we make the following assumptions:*

**H1** *$\xi = (\xi_{i,j}(s,z))_{m\times m}$, $\big(\lambda_{i,j}(y_s)\big)_{m\times m}$ defined in* (1.2.9)*, $\eta$ and $Y$ satisfy the following conditions:*

$$\begin{cases} \int_0^t \int_{z\in\mathbb{R}} \xi_{i,j}(z,s)\lambda_{i,j}(y_{s^-})\bar{b}(z|i,j)dzds < \infty \\ Y \geq 0 \\ \int_0^t \eta(s,\theta_{s^-})\mu(\theta_{s^-})ds < \infty \\ \int_{z\in\mathbb{R}} \big[G(z,\theta_{s^-})1_{|z|\leq 1} \\ +H(z,\theta_{s^-})1_{|z|>1}\big]\big[Y(\theta_{s^-},z,s)-1\big]\nu(\theta_{s^-},dz) < \infty \\ \int_{z\in\mathbb{R}} \big[Y(\theta_{s^-},z,s)-1\big]^2\nu(\theta_{s^-},dz) < \infty \end{cases} \tag{2.2.7}$$

**H2** *let $Z_t$ be the solution process of the following linear SDE,*

$$\begin{aligned} dZ_t =& Z_{t^-}\Big[\eta(t,\theta_t)\sigma(\theta_t)dB_s + \int_{z\in\mathbb{R}} \big(Y(\theta_t,z,t)-1\big)\bar{\psi}(\theta_t,dz,dt) \\ &+ \sum_{(i,j)\in E^2}\int_{z\in\mathbb{R}} \big[\xi_{i,j}(t,z)-1\big]\bar{N}(dt,dz,\{(i,j)\})\Big] \; Z_0 = 1, \end{aligned} \tag{2.2.8}$$

*where, $\bar{N} = N - \gamma$ defined in* (1.4.17) *and $Z_t$ has the closed form representation:*

$$\begin{aligned} Z_t = \exp\{\Big[ &-\int_0^t \frac{1}{2}\eta(s,\theta_{s^-})^2\sigma(\theta_{s^-})^2 ds + \int_0^t \eta(s,\theta_{s^-})\sigma(\theta_{s^-})dB_s \\ &+ \int_0^t\int_{z\in\mathbb{R}} \big(Y(\theta_{s^-},z,s)-1\big)\bar{\psi}(\theta_{s^-},dz,ds) \\ &+ \int_0^t\int_{z\in\mathbb{R}} \big[\ln\big(Y(\theta_{s^-},z,s)\big) - \big(Y(\theta_{s^-},z,s)-1\big)\big]\psi(\theta_{s^-},dz,ds)\Big]\} \\ &\times \prod_{(i,j)\in E^2} \exp\{\Big[\int_0^t (1-\xi_{i,j}(s,z))\bar{b}(z|i,j)\lambda_{i,j}(y_{s^-})dzds \\ &+ \int_0^t\int_{z\in\mathbb{R}} \ln\big(\xi_{i,j}(s,z)\big)N(ds,dz,\{(i,j)\})\Big]\}. \end{aligned} \tag{2.2.9}$$

*Therefore, from Remark 2.2.1 and under a local equivalent probability measure $\bar{P}$ with density process $Z_t$ with respect to $P$, the following hold:*

1. *$B_t^{\bar{P}} = -\int_0^t \eta(s,\theta_{s^-})\sigma(\theta_{s^-})ds + B_t$ is a Brownian motion for each $\theta_{t^-} \in E$,*
2. *$\nu^{\bar{P}}(\theta_{t^-},) = Y(\theta_{t^-},t,)\nu(\theta_{t^-},)$ $P-$almost surely,*
3. *$\gamma^{\bar{P}}(dz,\{(i,j)\}) = \xi_{i,j}(t,z)\bar{b}(z|i,j)\lambda_{i,j}(y_t)dz$ $P-$almost surely,*

*4. $M_t^\theta$ defined in* (2.2.6) *can be expressed as follows:*

$$\begin{aligned} dM_t^\theta =& \Big[\mu(\theta_{t^-}) + \sigma^2(\theta_{t^-})\eta(t,\theta_{t^-}) \\ &+ \int_{|z|\le 1} G(z,\theta_{t^-})(Y(\theta_{t^-},z,t)-1)\nu(\theta_{t^-},dz) \\ &+ \int_{|z|>1} H(z,\theta_{t^-})Y(\theta_{t^-},z,t)\nu(\theta_{t^-},dz) \\ &+ \sum_{j\in E\setminus\{\theta_{t^-}\}} \int_{z\in\mathbb{R}} [e^z-1]\gamma^{\bar{P}}(dz,\{\theta_{t^-},j\})\Big]dt + \sigma(\theta_{t^-})dB^{\bar{P}} \\ &+ \int_{|z|\le 1} G(z,\theta_{t^-})\bar{\psi}^{\bar{P}}(\theta_{t^-},dz,dt) + \int_{|z|>1} H(z,\theta_{t^-})\bar{\psi}^{\bar{P}}(\theta_{t^-},dz,dt) \\ &+ \sum_{j\in E\setminus\{\theta_{t^-}\}} \int_{z\in\mathbb{R}} [e^z-1]\Big[N(dt,dz,\{\theta_{t^-},j\}) \\ &- \gamma^{\bar{P}}(dz,\{\theta_{t^-},j\})dt\Big]. \end{aligned} \tag{2.2.10}$$

*Proof.* From (2.2.8), we note that $E\big[Z_t - Z_s\big|\bar{\mathbb{H}}_s \vee \mathbb{L}_s\big] = 0, \forall s,t \in [0,T]$ with $s \le t$. Hence, $Z_t$ is a local martingale. From the initial condition $Z_0 = 1$ in (2.2.8) and Illustration 2.2.1, $Z_t$ is a density process of $\bar{P}$. Moreover, $\bar{P}(A) = \int_A Z_t(w)dP(w)$, for $A \in \bar{\mathbb{H}}_t \vee \mathbb{L}_t$. Consequently, $\bar{P}$ is a local equivalent probability measure with density $Z_t$ relative to $P$. From the definition of $B^{\bar{P}}$ in 1, it is obvious that it is a Brownian motion with mean $-\int_0^t \eta(s,\theta_{s^-})\sigma(\theta_{s^-})ds$ and variance $t$. It remains to show that $B^{\bar{P}}$ is a local martingale with respect to $\bar{P}$. For this purpose, we use (2.2.8) and apply Ito formula for the product $ZB^{\bar{P}}$ [47] and we have,

$$\begin{aligned} d(Z_tB_t^{\bar{P}}) =& Z_t dB^{\bar{P}} + B_t^{\bar{P}} dZ_t + dZ_t dB_t^{\bar{P}} \\ =& Z_t dB + B_t^{\bar{P}} dZ_t \\ =& Z_t\big[1 + \eta(t,\theta_{t^-})\sigma(\theta_{t^-})B_t^{\bar{P}}\big]dB_t \\ &+ Z_tB_t^{\bar{P}}\Big[\int_{z\in\mathbb{R}} \big(Y(\theta_t,z,t)-1\big)\bar{\psi}(\theta_t,dz,dt) \\ &+ \sum_{(i,j)\in E^2} \int_{z\in\mathbb{R}} \big[\xi_{i,j}(t,z)-1\big]\bar{N}(dt,dz,\{(i,j)\})\Big]. \end{aligned}$$

From this, we conclude that $B^{\bar{P}}$ is a $\bar{P}-$ continuous local martingale with quadratic variation $t$. From Lévy characterization of Brownian motions, $B^{\bar{P}}$ is a $\bar{P}-$standard Brownian motion. This establishes 1. We now prove that $\nu^{\bar{P}}(\theta_{t^-},dz) = Y(\theta_{t^-},t,z)\nu(\theta_{t^-},dz)$ is the $\bar{P}-$Lévy measure of

$\psi(\theta_{t^-},,)$. Knowing that $\bar{P}$ and $P$ are equivalent, following the argument[1], we define the conditional characteristic function for the Poisson process $\psi(\theta_{t^-},,)$ relative to the probability measure $\bar{P}$ as follows:

$$
\begin{aligned}
&E^{\bar{P}}\Big[\exp\Big[iu\int_0^t\int_{z\in\mathbb{R}}\psi(\theta_{s^-},dz,ds)\Big]\big|\mathbb{H}_T\Big]\\
&=\exp\Big[\int_0^t\int_{z\in\mathbb{R}}\Big[\big(e^{iu}-1\big)\Big]\nu\bar{P}(\theta_s,dz)\Big],
\end{aligned}
\tag{2.2.11}
$$

where $\nu\bar{P}$ is an intensity measure of $\psi$ with respect to $\bar{P}$. Using the closed form expression of the density process (2.2.9), the characteristic function in (2.2.11) is also computed as follows;

$$
\begin{aligned}
&E^{\bar{P}}\Big[\exp\{\Big[iu\int_0^t\int_{z\in\mathbb{R}}\psi(\theta_{s^-},ds,dz)\Big]\}\big|\mathbb{H}_T\Big]\\
=&E\Big[Z_t\exp\{\Big[iu\int_0^t\int_{z\in\mathbb{R}}\psi(\theta_{s^-},dz,ds)\Big]\}\big|\mathbb{H}_T\Big]\\
=&E\Big[\exp\{\Big[\int_0^t\int_{z\in\mathbb{R}}\big(Y(\theta_{s^-},z,s)-1\big)\bar{\psi}(\theta_{s^-},dz,ds)\\
&+\int_0^t\int_{z\in\mathbb{R}}\big[\ln\big(Y(\theta_{s^-},z,s)\big)-Y(\theta_{s^-},z,s)\\
&+1+iu\big]\psi(\theta_{s^-},dz,ds)\Big]\}\Big|\mathbb{H}_T\Big]\\
=&\exp\{\Big[\int_0^t\int_{z\in\mathbb{R}}\big(1-Y(\theta_{s^-},z,s)\big)\nu(\theta_{s^-},dz)ds\Big]\}\\
&\times E\Big[\exp\{\Big[\int_0^t\int_{z\in\mathbb{R}}\big(Y(\theta_{s^-},z,s)-1\big)\psi(\theta_{s^-},dz,ds)\\
&+\int_{z\in\mathbb{R}}\big[\ln\big(Y(\theta_{s^-},z,s)\big)-\big(Y(\theta_{s^-},z,s)-1\big)+iu\big]\psi(\theta_s,dz,ds)\Big]\}\Big|\mathbb{H}_T\Big]\\
=&\exp\{\Big[-\int_0^t\int_{z\in\mathbb{R}}\big(Y(\theta_{s^-},z,s)-1\big)\nu(\theta_{s^-},dz)ds\Big]\}\\
&\times E\Bigg[\exp\{\Big[\int_0^t\int_{z\in\mathbb{R}}\big[\ln\big(Y(\theta_{s^-},z,s)\big)+iu\big]\psi(\theta_{s^-},dz,ds)\Big]\}\big|\mathbb{H}_T\Bigg]
\end{aligned}
\tag{2.2.12}
$$

We note that $\int_0^t\int_{z\in\mathbb{R}}\big[\ln\big(Y(\theta_{s^-},z,s)\big)+iu\big]\psi(\theta_{s^-},dz,ds)$ is a compound Poisson process. From [1], (2.2.12) becomes,

$$
\begin{aligned}
&E^{\bar{P}}\Big[\exp\{\Big[iu\int_0^t\int_{z\in\mathbb{R}}\psi(\theta_{s^-},ds,dz)\Big]\}\big|\mathbb{H}_T\Big]\\
&=\exp\{\Big[-\int_0^t\int_{z\in\mathbb{R}}\big(Y(\theta_{s^-},z,s)-1\big)\nu(\theta_{s^-},dz)ds\Big]\}\\
&=\exp\{\Big[\int_0^t\int_{z\in\mathbb{R}}\big[e^{\ln(Y(\theta_s,z,s))+iu}-1\big]\nu(\theta_s,dz)ds\Big]\}
\end{aligned}
$$

$$= \exp\{\Big[\int_0^t \int_{z\in\mathbb{R}} \big[Y(\theta_{s^-}, z, s)e^{iu} - Y(\theta_{s^-}, z, s)\big]\nu(\theta_{s^-}, dz)\Big]\}$$
$$= \exp\{\Big[\int_0^t \int_{z\in\mathbb{R}} \Big[\big(e^{iu} - 1\big)\Big]Y(\theta_{s^-}, z, s)\nu(\theta_{s^-}, dz)\Big]\}, \tag{2.2.13}$$

From (2.2.11) and (2.2.13), it is obvious that the intensity of Lévy jump poisson measure $\psi$, with respect to $\bar{P}$, is $\bar{\nu} = Y\nu\ P-$almost surely. Based on the proof of 2, the proof of 3 can be reformulated, analogously. The verification of (2.2.10) follows from algebraic computations. □

REMARK 2.2.2 We recall that under the historical probability measure P, $(p_{i,j})_{(i,j)\in E^2}$ in (1.2.5), $F(|i,j)$ in Lemma 1.2.1 and $\bar{b}(|i,j)$ in (1.4.6) are the transition probability matrix of the embedded Markov chain, the sojourn time distribution and the log jump density, respectively. We denote $(p^{\bar{P}}_{i,j})_{(i,j)\in E^2}$, $F^{\bar{P}}(|i,j)$ and $\bar{b}^{\bar{P}}(|i,j)$ the transition probability matrix, the conditional cumulative distribution of sojourn times and the density of the log of jump due to the semi Markov process from state $i$ to state $j$ at jump time $T_{n-1}$, under the probability measure $\bar{P}$. Using these notions and part 3 of Lemma 2.2.1, we have

$$\bar{b}^{\bar{P}}(z|i,j)\lambda^{\bar{P}}_{i,j}(y_s) = \xi_{i,j}(s,z)\bar{b}(z|i,j)\lambda_{i,j}(y_s),$$
$$\text{with } \lambda^{\bar{P}}_{i,j}(y_s) = p^{\bar{P}}_{i,j}\frac{f^{\bar{P}}(y_s|i,j)}{1-\sum_{k\neq i} p^{\bar{P}}_{i,j}F^{\bar{P}}(y_s|i,k)}. \tag{2.2.14}$$

We further remark that $\bar{P}$ is a risk neutral measure, if the process $L^{\theta}_t - \int_0^t r(s)ds$ is a local martingale with respect to $\bar{P}$, whenever the drift coefficient satisfies the condition:

$$\mu(\theta_{t^-}) - r(t) + \sigma^2(\theta_{t^-})\eta(t,\theta_{t^-}) + \int_{|z|\leq 1} G(z,\theta_{t^-})\big(Y(\theta_{t^-},z,t) - 1\big)\nu(\theta_{t^-},dz)$$
$$+ \int_{|z|>1} H(z,\theta_{t^-})Y(\theta_{t^-},z,t)\nu(\theta_{t^-},dz)$$
$$+ \sum_{j\in E\setminus\{\theta_{t^-}\}} \int_{z\in\mathbb{R}} [e^z - 1]\gamma^{\bar{P}}(dz, \{(\theta_{t^-}, j)\}) = 0. \tag{2.2.15}$$

Given the 2-variate process $(\eta(t,\theta_t), Y(\theta_t,z,t))$ in (2.2.8), one can freely choose $\xi$. Hence, for each choice of $\xi$, one gets a distinct risk neutral measure. Furthermore, by the application of the first and the second fundamental theorem of asset pricing [4], the market under consideration is arbitrage free and incomplete.

Following arguments in [26, 51, 53], we define two particular equivalent martingale measures, namely the conditional and the unconditional minimum entropy martingale measure, respectively.

### 2.2.1 Conditional minimum entropy martingale measure(CMEMM)

We define the conditional minimum entropy martingale measure pricing Lévy and semi Markov jump risks. In the absence of risk associated with regime changes, a pricing kernel is computed through a random Esscher transform. Without loss in generality, we assume that investors always know past and future market regimes. Based on the idea in [52], we define the process $R^{\theta}$ as follows;

$$R_t^{\theta} = \int_0^t \int_{z\in\mathbb{R}} \sum_{j\in E\setminus\{\theta_{s-}\}} \left[(e^z-1)N(ds,dz,\{(\theta_{s-},j)\})\right] + L_t^{\theta}, \tag{2.2.16}$$

where $L_t^{\theta}$, $n(t)$ and $\beta_k$ are defined in (1.3.2),(1.2.3) and (1.4.1), respectively. Picking a locally bounded process $(\alpha_t)_{t\in[0,T]}$, we modify the process defined in (2.2.16) as:

$$\begin{aligned}\int_0^t \alpha_s dR_s^{\theta} = & \int_0^t \int_{z\in\mathbb{R}} \sum_{j\in E\setminus\{\theta_{s-}\}} \alpha_{s^-}\left[(e^z-1)N(ds,dz,\{(\theta_{s-},j)\})\right] \\ & + \int_0^t \alpha_{s^-} dL_s^{\theta}. \end{aligned} \tag{2.2.17}$$

In the following, we utilize the modified process (2.2.17) to formulate a dynamic process for the asset process.

DEFINITION 2.2.1 *Let $\alpha$ be a locally bounded process. We assume that* $E\left[e^{\int_0^t \alpha_{s^-} dR_s^{\theta}} \middle| \mathbb{H}_T\right] < \infty, \forall t \in [0,T]$. *We define the stochastic processes $Z^{\alpha}$ and $k(s,z,ds,dz)$ as follows;*

$$Z_t^{\alpha} = \frac{e^{\int_0^t \alpha_{s^-} dR_s^{\theta}}}{E\left[e^{\int_0^t \alpha_{s^-} dR_s^{\theta}} \middle| \mathbb{H}_T\right]}, \forall t \geq 0 \tag{2.2.18}$$

*and*

$$k(s,z,ds,dz) = \sum_{j\in E\setminus\{\theta_{s-}\}} \alpha_{s^-}(e^z-1)N(ds,dz,\{(\theta_{s-},j)\}), \forall s \geq 0, z \in \mathbb{R}. \tag{2.2.19}$$

*The stochastic process defined in* (2.2.18) *is called an Esscher transformation with Esscher parameters* $(\alpha_s)_{s\in[0,T]}$.

We first establish preliminary results useful for finding a necessary and sufficient condition under which the probability measure $P^{\alpha}$ with density relative to $P$ defined by the Esscher transform in (2.2.18) is an equivalent martingale measure relative to the asset price process $x_t$ described by (1.4.1).

LEMMA 2.2.2 *Under Definition 1.4.1, Remark 1.3.2 and the Esscher parameter $(\alpha_s)_{s\in[0,T]}$ in Definition 2.2.1, a stochastic process $x_t^\alpha$ exists and satisfies the following properties.*

*1.*

$$\begin{aligned} & E\left[x_t^\alpha \middle| \mathbb{H}_T\right] \\ & = \prod_{i=0}^{n(t)} E\left[\exp\left[\alpha_i \int_{T_i}^{T_{i+1}} \int_{z\in\mathbb{R}} k(s,z,ds,dz)\right] \middle| \theta_i, \theta_{i+1}\right] e^{\int_{T_i}^{T_{i+1}} f_i(s)ds}, \end{aligned} \tag{2.2.20}$$

*where,*

$$\begin{aligned} f_i(s) = & \alpha(i)\mu(\theta_i) + \frac{1}{2}\sigma^2(\theta_i)\alpha(i) \\ & + \int_{|z|\leq 1} \left[e^{\alpha(i)G(z,\theta_i)} - 1 - \alpha(i)G(z,\theta_i)\right]\nu(\theta_i, dz) \\ & + \int_{|z|>1} \left[e^{\alpha(i)H(z,\theta_i)} - 1\right]\nu(\theta_i, dz), \end{aligned} \tag{2.2.21}$$

*for $i \in I(0,\infty)$, $s \in [0,T]$;*

*2.*

$$\frac{x_t^\alpha}{E\left[x_t^\alpha \middle| \mathbb{H}_T\right]} = \prod_{i=0}^{n(t)} \frac{\exp\left[\alpha_i \int_{T_i}^{T_{i+1}} \int_{z\in\mathbb{R}} k(s,z,ds,dz)\right] e^{\int_{T_i}^{T_{i+1}} \alpha_{s^-} dM_s^{\theta_i}}}{E\left[\exp\left[\alpha_i \int_{T_i}^{T_{i+1}} \int_{z\in\mathbb{R}} k(s,z,ds,dz)\right] \middle| \theta_i, \theta_{i+1}\right]}, \tag{2.2.22}$$

*where $\alpha_{s^-} dM_s^{\theta_i} = \alpha_{s^-} dL_s^{\theta_i} - f_i(s)ds$;*

*3.* $E\left[\frac{x_t^\alpha}{E(x_t^\alpha|\mathbb{H}_T)} \middle| \mathbb{H}_T\right] = 1$;

*4.* $Z_t^\alpha = \frac{x_t^\alpha}{E\left[x^\alpha \middle| \mathbb{H}_T\right]}$ *is a $\left(P, \mathbb{H}_T \vee \bar{\mathbb{L}}\right)-$local martingale;*

*5. If $P^\alpha$ is a risk neutral measure with respect to $Z_t^\alpha$, then under $P^\alpha$ we have:*

   *(a)* $B_t^{P^\alpha} = B_t - \int_0^t \alpha_{s^-}\sigma(\theta_{s^-})ds$, *is a $P^\alpha-$standard Brownian motion process,*

   *(b)* $\nu^{P^\alpha}(\theta_{s^-}, dz) = e^{\left[H(z,\theta_{s^-})1_{(|z|>1)} + G(z,\theta_{s^-})1_{|z|\leq 1}\right]}\nu(\theta_{s^-}, dz)$, *is a $P^\alpha-$predictable compensator of the Poisson random measure $\psi(j,)$ for all $j \in E$,*

   *(c) the density of the n-th jump coefficient $\beta_n$ is* $\frac{\exp\left[\int_{T_n}^{T_{n+1}} \int_{z\in\mathbb{R}} k(s,z,ds,dz)\right]}{E\left[\exp\left[\int_{T_n}^{T_{n+1}} \int_{z\in\mathbb{R}} k(s,z,ds,dz)\right] |\theta_n, \theta_{n+1}\right]}$.

*Proof.* From Definition 1.4.1, $0 = T_0 \leq T_1 \leq T_2 \leq ... \leq T_{n-1}$ are the regime switching times caused by the semi Markov process prior to $t$. For notational convenience, we denote $\theta_{-1} = \theta_0$. Under the assumption of the Lemma, the solution process of (1.4.1) in the context of (1.4.2) and the simple return process (2.2.17) exists and it is represented as

$$x_t^\alpha = \prod_{i=0}^{n(t)} \exp\Big[\int_{T_i}^{T_{i+1}} \int_{z\in\mathbb{R}} k(s,z,ds,dz)\Big] e^{\int_{T_i}^{T_{i+1}} \alpha_{s-} dL_s^{\theta_i}},$$

with $\beta_0 = x_0 = 1$. For $t \in [T_n, T_{n+1}]$, from the independence of Lévy and semi Markov processes, we have:

$$E\Big[x_t^\alpha|\mathbb{H}_T\Big] = \prod_{i=0}^{n-1}\Bigg[E\Big[\exp\Big[\int_{T_i}^{T_{i+1}}\int_{z\in\mathbb{R}} k(s,z,ds,dz)\Big]\Big|\theta_{i-1},\theta_i\Big] \times E\Big[e^{\int_{T_i}^{T_{i+1}} \alpha_{s-} dL_s^i} e^{\int_{T_i}^{t} \alpha_{s-} dL_s^{\theta_i}}|\mathbb{H}_T\Big]\Bigg]. \tag{2.2.23}$$

This, together with an application of the Lévy Kintchine formula [55] yields,

$$\begin{aligned} E\Big[x_t^\alpha\Big|\mathbb{H}_T\Big] = \prod_{i=0}^{n(t)} E\Big[\exp\{\Big[\int_{T_i}^{T_{i+1}}\int_{z\in\mathbb{R}} k(s,z,ds,dz)\Big]\}\Big|\theta_{i-1},\theta_i\Big] \\ \times\Bigg[\exp\{\int_{T_i}^{T_{i+1}}\Big[\alpha_{s-}\mu(\theta_i) + \frac{1}{2}\sigma^2(\theta_i)\alpha_{s-}^2 \\ + \int_{|z|\leq 1}\Big[e^{\alpha_{s-}G(z,\theta_i)} - 1 - \alpha_{s-}G(z,\theta_i)\Big]\nu(\theta_i,dz) \\ + \int_{|z|>1}\Big[e^{\alpha_{s-}H(z,\theta_i)} - 1\Big]\nu(\theta_i,dz)\Big]ds\}\Bigg] \end{aligned}$$

This completes the proof of 1. For the proof of 2, we consider

$$\frac{x_t^\alpha}{E\big[x^\alpha\big|\mathbb{H}_T\big]}. \tag{2.2.24}$$

From (2.2.21), (2.2.24), we obtain

$$\begin{aligned} &\frac{x_t^\alpha}{E\big[x^\alpha\big|\mathbb{H}_T\big]} \\ &= \frac{\prod_{i=0}^{n(t)} \exp\Big[\int_{T_i}^{T_{i+1}}\int_{z\in\mathbb{R}} k(s,z,ds,dz)\Big] \exp\Big[\int_{T_i}^{T_{i+1}} \alpha_{s-} dL_s^{\theta_i}\Big]}{\prod_{i=0}^{n(t)} E\Big[\exp\Big[\int_{T_i}^{T_{i+1}}\int_{z\in\mathbb{R}} k(s,z,ds,dz)\Big]\Big|\theta_j,\theta_{j+1}\Big]\exp\Big[\int_{T_j}^{T_{j+1}} f_j(s)ds\Big]} \\ &= \prod_{i=0}^{n(t)} \frac{\exp\Big[\int_{T_i}^{T_{i+1}}\int_{z\in\mathbb{R}} k(s,z,ds,dz)\Big]\exp\Big[\int_{T_i}^{T_{i+1}}\big[\alpha_{s-}dL_s^{\theta_j} - f_i(s)ds\big]\Big]}{E\Big[\exp\Big[\int_{T_i}^{T_{i+1}}\int_{z\in\mathbb{R}} k(s,z,ds,dz)\Big]\Big|\theta_{j-1},\theta_j\Big]}. \end{aligned} \tag{2.2.25}$$

From (2.2.3), (2.2.17), (2.2.21) and (2.2.25), we observe that $\alpha_{s^-} dL_s^{\theta_j} - f_j(s)ds$ has a form similar to (2.2.3), that is

$$\alpha_{s^-} dL_s^{\theta_j} - f_j(s)ds = \alpha_{s^-} dM_t^{\theta_j}, \tag{2.2.26}$$

with coefficients $G$ and $H$ replaced by $e^G - 1$ and $e^H - 1$, respectively, hence establishing 2. Using 1, (2.2.25) and (2.2.26), we further remark that,

$$\begin{aligned}
&E\Big[\frac{x_t^\alpha}{E\big[x_t^\alpha\big|\mathbb{H}_T\big]}\Big|\mathbb{H}_T\Big] \\
&= E\Big[\prod_{i=0}^{n(t)} \frac{\exp\{\big[\int_{T_i}^{T_{i+1}}\int_{z\in\mathbb{R}} k(s,z,ds,dz)\big]\}\exp\{\big[\int_{T_i}^{T_{i+1}} \alpha_{s^-} dM_s^{\theta_i}\big]\}}{E\Big[\exp\{\big[\int_{T_i}^{T_{i+1}}\int_{z\in\mathbb{R}} k(s,z,ds,dz)\big]\}\Big|\theta_i,\theta_{i+1}\Big]}\Big|\mathbb{H}_T\Big] \\
&= \prod_{i=0}^{n(t)} \frac{E\Big[\exp\{\big[\int_{T_i}^{T_{i+1}}\int_{z\in\mathbb{R}} k(s,z,ds,dz)\big]\}\Big|\theta_{i-1},\theta_i\Big]}{E\Big[E\Big[\exp\{\big[\int_{T_i}^{T_{i+1}}\int_{z\in\mathbb{R}} k(s,z,ds,dz)\big]\}\Big|\theta_i,\theta_{i+1}\Big]\Big|\theta_{i-1},\theta_i\Big]} \\
&\times E\Big[\exp\{\Big[\int_{T_i}^{T_{i+1}} \big[\alpha_{s^-} dL_s^{\theta_i} - f_i(s)ds\big]\Big]\}\Big|\mathbb{H}_T\Big] \\
&= \prod_{i=0}^{n(t)} 1 = 1, \text{for } t\in[0,T],
\end{aligned}$$

which establishes 3. For the proof of 4 we consider

$$\frac{\frac{x_t^\alpha}{E\left[x_s^\alpha\middle|\mathbb{H}_T\right]}}{\frac{x_s^\alpha}{E\left[x_s^\alpha\middle|\mathbb{H}_T\right]}} = \prod_{i=n(s)+1}^{n(t)} \frac{\exp\Big[\alpha_i\int_{T_i}^{T_{i+1}}\int_{z\in\mathbb{R}} k(s,z,ds,dz)\Big]}{E\Big[\exp\Big[\int_{T_i}^{T_{i+1}}\int_{z\in\mathbb{R}} k(s,z,ds,dz)\Big]\Big|\theta_{i-1},\theta_i\Big]} \times \exp\Big[\int_{T_i}^{T_{i+1}}\big[\alpha_{s^-} dM_s^{\theta_i}\big]\Big].$$

The conditional expectation with respect to $\mathbb{H}_T \vee \bar{\mathbb{L}}_s$ yields

$$E\Big[\frac{x_t^\alpha}{E\big[x_t^\alpha\big|\mathbb{H}_T\big]}\Big|\mathbb{H}_T \vee \bar{\mathbb{L}}_s\Big] = \frac{x_s^\alpha}{E\big[x_s^\alpha\big|\mathbb{H}_T\big]}.$$

This proves 4. Moreover, from 1, 4 and (2.2.22), $Z^\alpha$ is a probability density process of a probability measure $P^\alpha$ with respect to $P$. The proof of statements in 5a and 5b of 5 follow by imitating the proofs of (1) and (2) of Lemma 2.2.1. We only establish 5c. For $B \subset \mathbb{B}_k$ and $t \in [T_k, T_{k+1})$. In fact

$$
\begin{aligned}
E^{P^\alpha}\big[1_B\big] &= E\big[1_B Z_t^{P^\alpha}\big]\\
=&E\Big[E\big[1_B Z_t^{P^\alpha}|\mathbb{H}_T\big]\Big],\\
=&E\left[E\left[1_B\prod_{i=0}^{n(t)}\frac{\exp\Big[\int_{T_i}^{T_{i+1}}\int_{z\in\mathbb{R}}k(s,z,ds,dz)\Big]}{E\Big[\exp\Big[\int_{T_i}^{T_{i+1}}\int_{z\in\mathbb{R}}k(s,z,ds,dz)\Big]\Big|\theta_{i-1},\theta_i\Big]}\bigg|\mathbb{H}_T\right]\right]\\
=&E\left[1_B\frac{\exp\Big[\int_{T_k}^{T_{k+1}}\int_{z\in\mathbb{R}}k(s,z,ds,dz)\Big]}{E\Big[\exp\Big[\int_{T_k}^{T_{k+1}}\int_{z\in\mathbb{R}}k(s,z,ds,dz)\Big]\Big|\theta_{k-1},\theta_k\Big]}\right]\\
&\times\prod_{i=1,i\neq k}^{n(t)}E\left[\frac{\exp\Big[\int_{T_i}^{T_{i+1}}\int_{z\in\mathbb{R}}k(s,z,ds,dz)\Big]}{E\Big[\exp\Big[\int_{T_i}^{T_{i+1}}\int_{z\in\mathbb{R}}k(s,z,ds,dz)\Big]\Big|\theta_{i-1},\theta_i\Big]}\bigg|\theta_{i-1},\theta_i\right]\\
=&E\left[1_B\frac{\exp\Big[\int_{T_k}^{T_{k+1}}\int_{z\in\mathbb{R}}k(s,z,ds,dz)\Big]}{E\Big[\exp\Big[\int_{T_k}^{T_{k+1}}\int_{z\in\mathbb{R}}k(s,z,ds,dz)\Big]\Big|\theta_{k-1},\theta_k\Big]}\right].
\end{aligned}
$$

Hence, $\forall B\in\mathbb{B}_k$, $E^{P^\alpha}\big[1_B\big]=E\left[1_B\frac{\exp\Big[\int_{T_i}^{T_{i+1}}\int_{z\in\mathbb{R}}k(s,z,ds,dz)\Big]}{E\Big[\exp\Big[\int_{T_i}^{T_{i+1}}\int_{z\in\mathbb{R}}k(s,z,ds,dz)\Big]\Big|\theta_{k-1},\theta_k\Big]}\right]$. From Radon Nikodym theorem [44], the density of $\beta_k$ under $P^\alpha$ is $\frac{\exp\Big[\int_{T_k}^{T_{k+1}}\int_{z\in\mathbb{R}}k(s,z,ds,dz)\Big]}{E\Big[\exp\Big[\int_{T_k}^{T_{k+1}}\int_{z\in\mathbb{R}}k(s,z,ds,dz)\Big]\Big|\theta_{k-1},\theta_k\Big]}$. This completes the proof of the Lemma. □

In the following lemma, we provide a sufficient condition for the price process to be a $\big(P^\alpha,(\mathbb{H}_T\vee\bar{\mathbb{L}}_t)_{t\in[0,T]}\big)$-martingale. The result obtained will be used to derived the martingale condition on the discounted price process.

LEMMA 2.2.3 *In addition to assumptions of Lemma 2.2.2, we assume that* $\int_{|z|>1}\big(H(z,\theta_s)+1\big)e^{\alpha(j)H(z,\theta_s)}\nu(j,dz)<\infty,\forall j\in E$. *Then the following results hold:*

1. $x$ *in* (1.4.1) *is a* $\big(P^\alpha,(\mathbb{H}_T\vee\bar{\mathbb{L}}_t)_{t\in[0,T]}\big)$*-martingale measure provided that:*

$$
\mu(\theta_n)+\alpha_t\sigma^2(\theta_n)+\int_{|z|\le 1}G(z,\theta_n)\big[e^{\alpha_t G(z,\theta_n)}-1\big]\nu(\theta_n,dz)
$$
$$
+\int_{|z|>1}H(z,\theta_n)e^{\alpha_t H(z,\theta_n)}\nu(\theta_n,dz)=0, \qquad (2.2.27)
$$

$$
E_{P^\alpha}\big[\beta_n\big|\theta_{n-1},\theta_n\big]=1,\forall t\in(T_n,T_{n+1}),\forall n\in I(0,\infty) \qquad (2.2.28)
$$

2. *the discounted price process* $\tilde{x}_t = e^{\int_0^t r_s ds} x_t$, *is a* $\big(P^{\alpha}, (\mathbb{H}_T \vee \bar{\mathbb{L}}_t)_{t\in[0,T]}\big)$*-martingale if:*

$$\begin{cases} \mu(\theta_n) + \alpha_t\sigma^2(\theta_n) + \int_{|z|\le 1}\big[G(z,\theta_n)e^{\alpha_t G(z,\theta_n)} - G(z,\theta_n)\big]\nu(\theta_n, dz) \\ + \int_{|z|>1}\big[e^{\alpha_t H(z,\theta_n)} - 1\big]\nu(\theta_n, dz) = r_t, \\ E_{P^{\alpha}}\big[\beta_n \big| \theta_{n-1}, \theta_n\big] = 1, \forall t \in (T_n, T_{n+1}), \forall n \in I(0,\infty); \end{cases} \tag{2.2.29}$$

3. *Let* $\alpha^{\star}$ *and* $P^{\alpha^{\star}}$ *be a solution process of equation* (2.2.27) *and the probability measure associated with the density process* $Z^{\alpha^{\star}}$*, respectively. Under* $P^{\alpha^{\star}}$*, the process* $R^{\theta}_t$ *in* (2.2.16) *could be expressed as follows:*

$$\begin{aligned} dR^{\theta}_t = &\int_{z\in\mathbb{R}} \sum_{j\in E\setminus\{\theta_{t^-}\}} \alpha_{t^-}(e^z - 1)N(dt, dz, \{(\theta_{t^-}, j)\}) + r_{t^-}dt + \sigma(\theta_{t^-})dB^{P^{\alpha^{\star}}} \\ &+ \int_{|z|\le 1} G(z,\theta_{t^-})\big[\psi(\theta_{t^-}, dt, dz) - \nu^{P^{\alpha^{\star}}}(\theta_{t^-}, dz)dt\big] \\ &+ \int_{|z|>1} H(z,\theta_{t^-})\big[\psi(\theta_{t^-}, dt, dz) - \nu^{P^{\alpha^{\star}}}(\theta_{t^-}, dz)dt\big], \end{aligned}$$

*with*

$$E_{P^{\alpha^{\star}}}\big[\beta_n \big| \theta_{n-1}, \theta_n\big] = 1.$$

*Proof.* From Radon Nicodym theorem, $x_t$ is a $\big(P^{\alpha}, (\mathbb{H}_T \vee \bar{\mathbb{L}}_t)_{t\in[0,T]}\big)$-martingale if and only if $x_t Z^{\alpha}_t$ is a $\big(P, (\mathbb{H}_T \vee \bar{\mathbb{L}}_t)_{t\in[0,T]}\big)$-martingale. From (1.3.20) and (1.4.1):

$$\begin{aligned} &x_t Z^{\alpha}_t \\ &= x_s Z^{\alpha}_s \prod_{i=n(s)+1}^{n(t)} \left[ \frac{\beta_i \exp\Big[\int_{T_i}^{T_{i+1}}\int_{z\in\mathbb{R}} k(s,z,ds,dz)\Big] e^{\int_{T_i}^{T_{i+1}}\big[\alpha_{s^-}dM^{\theta_i}_s + d\bar{L}^{\theta_i}_s\big]}}{E\bigg[\exp\Big[\int_{T_i}^{T_{i+1}}\int_{z\in\mathbb{R}} k(s,z,ds,dz)\Big]\bigg|\theta_{i-1},\theta_i\big|\theta_{i-1},\theta_i\bigg]} \right] \end{aligned} \tag{2.2.30}$$

with $\bar{L}^{\theta}_t$ defined as follows:

$$\begin{aligned} d\bar{L}^{\theta}_s = &\Big[\mu(\theta_{s^-}) - \frac{1}{2}\sigma^2(\theta_{s^-}) + \int_{|z|\le 1}\big[\ln(1 + G(z,\theta_{s^-})) - G(z,\theta_{s^-})\big]\nu(\theta_{s^-}, dz)\Big]ds \\ &+ \sigma(\theta_{s^-})B_s + \int_{|z|\le 1}\ln(1 + G(z,\theta_{s^-}))\bar{\psi}(\theta_{s^-}, dz, ds) \\ &+ \int_{|z|>1}\ln(1 + H(z,\theta_{s^-}))\psi(\theta_{s^-}, dz, ds) \end{aligned} \tag{2.2.31}$$

From (2.2.3), (2.2.30), and (2.2.31) we have:

$$
\begin{aligned}
&\alpha_{s^-} dM_s^{\theta_i} + d\bar{L}_s^{\theta} \\
&=\Big[\mu(\theta_{s^-}) - \frac{1}{2}\alpha_{s^-}^2\sigma^2(\theta_{s^-}) - \frac{1}{2}\sigma^2(\theta_{s^-})ds \\
&\quad - \int_{|z|>1} \big[e^{\alpha_{s^-}H(z,\theta_{s^-})} - 1\big]\nu(\theta_{s^-}, dz) \\
&\quad + \int_{|z|\leq 1} \big[\alpha_{s^-}G(z,\theta_{s^-}) - e^{\alpha_{s^-}G(z,\theta_{s^-})} + 1 \\
&\quad + \ln\left(G(z,\theta_{s^-}) + 1\right) - G(z,\theta_{s^-})\big]\nu(\theta_{s^-}, dz)ds\Big] + (\alpha_{s^-} + 1)\sigma(\theta_{s^-})dB_s \\
&\quad + \int_{|z|\leq 1} \Big[\alpha_{s^-}G(z,\theta_{s^-}) + \ln\left(G(z,\theta_{s^-}) + 1\right)\Big]\bar{\psi}(\theta_{s^-}, dz, ds) \\
&\quad + \int_{|z|>1} \Big[\alpha_{s^-}H(z,\theta_{s^-}) + \ln\left(H(z,\theta_{s^-}) + 1\right)\Big]\psi(\theta_{s^-}, dz, ds)
\end{aligned}
\tag{2.2.32}
$$

From (1.3.20), $d\big[e^{\int_0^t d(\alpha_{s^-}M_s^{\theta} + \bar{L}_s^{\theta})}\big] = e^{\int_0^t d(\alpha_{s^-}M_s^{\theta} + \bar{L}_s^{\theta})} dL_t^{\star}$ with:

$$
\begin{aligned}
dL_s^{\star} =& \Big[\mu(\theta_{s^-}) + \alpha_{s^-}\sigma^2(\theta_{s^-}) \\
&+ \int_{|z|\leq 1} \big[G(z,\theta_{s^-})e^{\alpha_{s^-}G(z,\theta_{s^-})} - G(z,\theta_{s^-})\big]\nu(\theta_{s^-}, dz)ds \\
&- \int_{|z|>1} H(z,\theta_{s^-})e^{\alpha_{s^-}H(z,\theta_{s^-})}\nu(\theta_{s^-}, dz)\Big]ds + \sigma(\theta_{s^-})(\alpha_{s^-} + 1)dB_s \\
&+ \int_{|z|\leq 1} \Big[(G(z,\theta_{s^-}) + 1)e^{\alpha_{s^-}G(z,\theta_{s^-})} - 1\Big]\bar{\psi}(\theta_{s^-}, dz, ds) \\
&+ \int_{|z|>1} \big[(H(z,\theta_{s^-}) + 1)e^{\alpha_{s^-}H(z,\theta_{s^-})} - 1\big]\bar{\psi}(\theta_{s^-}, dz, ds)
\end{aligned}
\tag{2.2.33}
$$

We now derive conditions under which $x_t Z_t^{\alpha}$, is a $(P, (\mathbb{H}_T \vee \bar{\mathbb{L}}_t)_{t\in[0,T]})$-martingale process. $x_t Z_t^{\alpha}$ is a $(P, (\mathbb{H}_T \vee \bar{\mathbb{L}}_t)_{t\in[0,T]})$-martingale process if and only if:

$$
E\Big[x_t Z_t^{\alpha}\big|\mathbb{H}_T \vee \bar{\mathbb{L}}_s\Big] = x_s Z_s^{\alpha}, \forall s, t \in [0, T] \tag{2.2.34}
$$

Applying Lemma 1.4.1 to $V(s, y_s, \theta_s, Z_s x_s) = x_t Z_t$ and replacing $G$, $H$, $\sigma$, $\mu$ and $\beta_i$ by,

$$
\begin{aligned}
&(G(z,\theta_s) + 1)e^{\alpha_s G(z,\theta_s)} - 1 \\
&,(H(z,\theta_s) + 1)e^{\alpha_s H(z,\theta_s)} - 1, \\
&(\alpha_s + 1)\sigma_s, \\
&\mu(\theta_s) + \alpha_s\sigma^2(\theta_s) + \int_{|z|\leq 1} \big[G(z,\theta_s)e^{\alpha_s G(z,\theta_s)} - G(z,\theta_s)\big]\nu(\theta_s, dz)
\end{aligned}
$$

$$- \int_{|z|>1} \left[e^{\alpha_s H(\theta_s)} - 1\right] \nu(\theta_s, dz)$$

and

$$\frac{\beta_i \exp\left[\int_{T_i}^{T_{i+1}} \int_{z\in\mathbb{R}} k(s,z,ds,dz)\right]}{E\left[\exp\left[\int_{T_i}^{T_{i+1}} \int_{z\in\mathbb{R}} k(s,z,ds,dz)\right]\Big|\theta_{i-1},\theta_i\right]},$$

respectively, we obtain;

$$\begin{aligned}
&x_t Z_t^\alpha - x_s Z_s^\alpha \\
&= \int_s^t x_{u^-} Z_{u^-}^\alpha \Big[\mu(\theta_{u^-}) + \alpha_{u^-}\sigma^2(\theta_{u^-}) \\
&\quad + \int_{|z|\le 1} \left[G(z,\theta_{u^-}) e^{\alpha_{u^-} G(z,\theta_{u^-})} - G(z,\theta_{u^-})\right] \nu(\theta_{u^-}, dz) \\
&\quad - \int_{|z|>1} \left[e^{\alpha_{u^-} H(z,\theta_{u^-})} - 1\right] \nu(\theta_{u^-}, dz) \\
&\quad + \int_{|z|>1} x_{u^-} Z_{u^-}^\alpha \left[(H(u,\theta_{u^-})+1) e^{\alpha_{u^-} H(z,\theta_{u^-})} - 1\right] \nu(\theta_{u^-}, dz)\Big] du \\
&\quad + \int_s^t \int_{z\in\mathbb{R}} \sum_{j\in E\setminus\{\theta_{u^-}\}} x_{u^-} Z_{u^-}^\alpha (e^z - 1) N(du, dz, \{\theta_{u^-}, j\}) \\
&\quad + \underbrace{\text{sum of martingale terms}}.
\end{aligned}$$

Taking the conditional expectation, we obtain;

$$\begin{aligned}
&E\left[x_t Z_t^\alpha - x_s Z_s^\alpha \middle| \mathbb{H}_T \vee \bar{\mathbb{L}}_s\right] \\
&= \int_s^t E\left[x_{u^-} Z_{u^-}^\alpha | \mathbb{H}_T \vee \bar{\mathbb{L}}_s\right] \Big[\mu(\theta_{u^-}) + \alpha_{u^-}\sigma^2(\theta_{u^-}) \\
&\quad + \int_{|z|\le 1} \left[G(z,\theta_{u^-}) e^{\alpha_{u^-} G(z,\theta_{u^-})} - G(z,\theta_{u^-})\right] \nu(\theta_{u^-}, dz) \\
&\quad - \int_{|z|>1} H(z,\theta_{u^-}) e^{\alpha_{u^-} H(z,\theta_{u^-})} \nu(\theta_{u^-}, dz)\Big] du \\
&\quad + E\Big[\int_s^t \int_{z\in\mathbb{R}} \sum_{j\in E\setminus\{\theta_{u^-}\}} \alpha_{u^-} x_{u^-} Z_{u^-}^\alpha (e^z - 1) N(du, dz, \{(\theta_{u^-}, j)\}) \Big| \mathbb{H}_T \vee \mathbb{L}_s\Big] \\
&= 0, \forall s, t \in [0, T],
\end{aligned} \tag{2.2.35}$$

for any $s, t$ and for small $\Delta s$ $s, t = s + \Delta s \in (T_n, T_{n+1})$ for some $n \in I(1, \infty)$. This together with (2.2.35) yields;

$$\mu(\theta_s) + \alpha_s \sigma^2(\theta_s) + \int_{|z|\le 1} \left[G(z,\theta_s) e^{\alpha_s G(z,\theta_s)} - G(z,\theta_s)\right] \nu(\theta_s, dz)$$

$$+\int_{|z|>1} H(s,\theta_s)e^{\alpha_s H(z,\theta_s)}\nu(\theta_s,dz)\Delta s = 0, \forall s \in (T_n, T_{n+1}), n \in I(1,\infty). \tag{2.2.36}$$

Lastly, we assume $[s,t[= [T_n, T_{n+1}[$. When $\Delta s$ is small. There is one regime change $[s,t[$ at $t = T_n$. Using (2.2.36), (2.2.35) becomes,

$$\begin{aligned}
&E\Big[\int_{T_n}^{T_{n+1}}\int_{z\in\mathbb{R}}\sum_{j\in E\setminus\{\theta_{u^-}\}}\alpha_{u^-}\big[x_{u^-}Z^{\alpha}_{u^-}\big]\Big|\mathbb{H}_T\vee\mathbb{L}_{T_n}\Big] = 0, \forall n \in I(0,\infty)\\
&Z_{T_n}x_nE\Big[\frac{\beta_n e^{\alpha_n(\beta_n-1)}}{E\big[e^{\alpha_n(\beta_n-1)}\big|\theta_{n-1},\theta_n\big]}\Big|\theta_{n-1},\theta_n\Big|\mathbb{H}_T\vee\bar{\mathbb{L}}_{T_n}\Big] - 1 = 0, \forall n \in I(0,\infty)\\
&E\Big[\frac{\beta_n e^{\alpha_n(\beta_n-1)}}{E\big[e^{\alpha_n(\beta_n-1)}\big|\theta_{n-1},\theta_n\big]}\Big|\theta_{n-1},\theta_n\Big] - 1 = 0, \forall n \in I(0,\infty),\\
&E\Big[\frac{\beta_n \exp\Big[\int_{T_n}^{T_{n+1}}\int_{z\in\mathbb{R}}k(s,z,ds,dz)\Big]}{E\big[\exp\Big[\int_{T_n}^{T_{n+1}}\int_{z\in\mathbb{R}}k(s,z,ds,dz)\big|\theta_{n-1},\theta_n\Big]\big]}\Big|\theta_{n-1},\theta_n\Big] - 1 = 0, \forall n \in I(0,\infty),\\
&E_{P^{\alpha}}\big[\beta_n\big|\theta_{n-1},\theta_n\big] - 1 = 0, \forall n \in I(0,\infty).
\end{aligned} \tag{2.2.37}$$

This completes the proof of part 1. Part 2 is a direct consequence of part 1 whenever $\mu(\theta_{s^-})$ is replaced by $\mu(\theta_{s^-})-r_s$. For the proof of part 3, we use (2.2.27) to derive the risk neutral dynamic of the process $R^{\theta}$ defined in (2.2.16). We denote $B^{P^{\alpha^\star}}$ and $\nu^{P^{\alpha\star}}$ the standard Brownian motion and the intensity process of the Poisson process $\psi$ under the probability measure $P^{\alpha^\star}$, respectively. From Lemma 2.2.2 part (5a), solving for $B$ in $B_t^{P^{\alpha^\star}} = B_t - \int_0^t \alpha^\star_{s^-}\sigma(\theta_{s^-})ds$, adding and subtracting $\nu^{P^{\alpha^\star}}$ inside the Poisson integrals, we obtain:

$$\begin{aligned}
dR_t^{\theta} =& \int_{z\in\mathbb{R}}\sum_{j\in E\setminus\{\theta_{t^-}\}}(e^z-1)N(ds,dz,\{(\theta_{t^-},j)\}) + \mu(\theta_{t^-})dt + \sigma(\theta_{t^-})dB_t\\
&+\int_{|z|\le 1}G(z,\theta_{t^-})\bar{\psi}(\theta_{t^-},dz,dt) + \int_{|z|>1}H(z,\theta_{t^-})\psi(\theta_{t^-},dz,dt)\\
=& \int_{z\in\mathbb{R}}\sum_{j\in E\setminus\{\theta_{t^-}\}}(e^z-1)N(dt,dz,\{(\theta_{t^-},j)\}) + \Big[\mu(\theta_{t^-}) + \sigma^2(\theta_{t^-})\alpha^\star_{t^-}\\
&+\int_{|z|\le 1}G(z,\theta_{t^-})\big[\nu^{P^{\alpha^\star}}(\theta_{t^-},dz) - \nu(\theta_{t^-},dz)\big]\\
&+\int_{|z|>1}H(z,\theta_{t^-})\nu^{P^{\alpha^\star}}(\theta_{t^-},dz)\Big]dt\\
&+\sigma(\theta_{t^-})dB^{P^{\alpha^\star}} + \int_{|z|\le 1}G(z,\theta_{t^-})\big[\psi(\theta_{t^-},dt,dz) - \nu^{P^{\alpha^\star}}(\theta_{t^-},dz)dt\big]\\
&+\int_{|z|>1}H(z,\theta_t)\big[\psi(\theta_t,dt,dz) - \nu^{P^{\alpha^\star}}(\theta_t,dz)dt\big], \forall t \in [T_n, T_{n+1}]
\end{aligned} \tag{2.2.38}$$

From Lemma 2.2.2 part (5a), one gets $\nu^{P^\alpha}(j,dz) = e^{\left[H(z,j)1_{(|z|>1)}+G(z,j)1_{|z|\leq 1}\right]}\nu(j,dz)$. Hence,

$$\begin{aligned}
dR_t^\theta = & \int_{z\in\mathbb{R}} \sum_{j\in E\setminus\{\theta_{t^-}\}} (e^z-1)N(dt,dz,\{(\theta_{t^-},j)\}) + \Big[\mu(\theta_{t^-})dt + \sigma^2(\theta_{t^-})\alpha^\star_{t^-} \\
& + \int_{|z|\leq 1} G(z,\theta_{t^-})\big[e^{\alpha^\star_{\theta_{t^-}} G(z,\theta_{t^-})} - 1\big]\nu(\theta_{t^-},dz)dt + \sigma(\theta_{t^-})dB^{P^{\alpha^\star}} \\
& + \int_{|z|>1} H(z,\theta_{t^-})e^{\alpha^\star_{\theta_{t^-}} H(z,\theta_{t^-})}\nu(\theta_{t^-},dz)dt\Big] \\
& + \int_{|z|\leq 1} G(z,\theta_{t^-})\big[\psi(\theta_{t^-},dt,dz) - \nu^{\alpha^\star}(\theta_{t^-},dz)dt\big] \\
& + \int_{|z|>1} H(z,\theta_t)\big[\psi(\theta_t,dt,dz) - \nu^{P^{\alpha^\star}}(\theta_t,dz)dt\big] \qquad (2.2.39)
\end{aligned}$$

$\alpha^\star$ satisfies the condition 1. Therefore, (2.2.39) becomes;

$$\begin{aligned}
dR_t^\alpha = & \int_{z\in\mathbb{R}} \sum_{j\in E\setminus\{\theta_{t^-}\}} (e^z-1)N(ds,dz,\{(\theta_{t^-},j)\}) + \sigma(\theta_{t^-})dB^{P^{\alpha^\star}} \\
& + \int_{|z|\leq 1} G(z,\theta_{t^-})\big[\psi(\theta_{t^-},dt,dz) - \nu^{P^{\alpha^\star}}(\theta_{t^-},dz)dt\big] \\
& + \int_{|z|>1} H(z,\theta_{t^-})\big[\psi(\theta_{t^-},dt,dz) - \nu^{P^{\alpha^\star}}(\theta_{t^-},dz)dt\big], \forall t\in[T_n,T_{n+1}], \qquad (2.2.40)
\end{aligned}$$

with

$$E_{P^\alpha}\big[\beta_n \big| \theta_{n-1},\theta_n\big] = 1,$$

which proves 3. This establishes the lemma. □

In the next remark, we introduce a particular case of $R_t^\theta$ corresponding to the simple return process [52] and we present a few properties of conditional entropies [67].

REMARK 2.2.3 Let $P_1$ and $P_2$ be two absolutely continuous probability measures relative to $P$. We recall three important properties of conditional entropies [51]:

1. $\mathcal{H}^{\mathbb{H}_T}_{\mathbb{H}_T\vee\mathbb{L}_t}(P_1|P) \geq 0$;
2. $\mathcal{H}^{\mathbb{H}_T}_{\mathbb{G}}(P_1|P) \leq \mathcal{H}^{\mathbb{H}_T}_{\mathbb{K}}(P_1|P)$, if $K\subset \mathbb{H}_T\vee\mathbb{L}_T$;
3. If $P_1$ is a $\big(P,(\mathbb{H}_T\vee\bar{\mathbb{L}}_t)_{t\in[0,T]}\big)-$absolutely continuous martingale measure, and $P_2$ is a probability measure equivalent to $P$ such that $\ln\big(\frac{dP_2}{dP}\big)$ is integrable with respect to $P_1$, then $\mathcal{H}^{\mathbb{H}_T}_{\mathbb{H}_T\vee\mathbb{L}_T}(P_1|P) \geq E_{P_1}\big[\ln(dP_2|dP)|\mathbb{H}_T\big]$.

We now state and prove the conditional minimum entropy property of the martingale measure $P^{\alpha^\star}$ when $R^\theta_t$ is the simple return process of $x_t$ in Remark 2.2.3.

LEMMA 2.2.4 *Let* $Q \in \mathcal{M}(\tilde{x}, P) = \{Q \ll P : \tilde{x}$ *is a* $\big(Q, (\mathbb{H}_T \vee \bar{\mathbb{L}}_t)_{t\in(0,T)}\big) -$ *Local martingale*$\}$. *Let* $P^{\alpha^\star}$ *be defined as in Definition 2.2.1 with* $\alpha^\star$ *solution process of* (2.2.29). *Then the following inequality holds:*

$$\mathcal{H}^{\mathbb{H}_T}_{\mathbb{H}_T\vee\bar{\mathbb{L}}_T}(Q|P) \geq \mathcal{H}^{\mathbb{H}_T}_{\mathbb{H}_T\vee\bar{\mathbb{L}}_T}(P^{\alpha^\star}|P), \forall Q \in \mathcal{M}(\tilde{x}, P).$$

*Where* $\mathcal{H}^{\mathbb{H}_T}_{\bar{\mathbb{L}}_T\vee\mathbb{H}_T}$ *is the conditional entropy defined in [26].*

*Proof.* We prove the lemma in two steps. The first step consists in minimizing the conditional relative entropy of any probability measure $Q$ in the set $\mathcal{M}(\tilde{x}, P)$. From (2.2.18) and Remark 2.2.3, one notes that $Z^\alpha_t$ can also be expressed as follows;

$$Z^\alpha_t = \frac{e^{\int_0^t \alpha_{s-} d\tilde{R}^\theta_s}}{E\Big[e^{\int_0^t \alpha_{s-} d\tilde{R}^\theta_s}\big|\mathbb{H}_T\Big]}, \forall t \geq 0. \tag{2.2.41}$$

By definition of an absolutely continuous local martingale measure, the discounted stock price $\tilde{x} = e^{-\int_0^t r_{s-} ds} x_t$ is a $(Q, \mathbb{H}_T \vee \bar{\mathbb{L}}_t)_{t\in[0,T]}$-local martingale process. The simple return processes $R_t$ and $\tilde{R}_t$ are associated with the price process $x_t$ and the discounted price process $\tilde{x}_t$ with respect to(3.2.4), defined by $dR^\theta_t = \frac{dx_t}{x_{t-}}$ and $d\tilde{R}^\theta_t = \frac{d\tilde{x}_t}{\tilde{x}_{t-}}$, respectively. $Q \in \mathcal{M}(\tilde{x}, P)$ implies that $\tilde{x}$ is a $(P, \mathbb{H}_T \vee \bar{\mathbb{L}}_t)-$martingale. Hence, $\tilde{R}^\theta_t$ is a $Q-$local martingale process. Furthermore, $\int_0^t \alpha_{s-} d\tilde{R}^\theta_s$ is a local $Q-$martingale. As a $Q-$local martingale, $\int_0^t \alpha^\star_{s-} d\tilde{R}^\theta$ is therefore integrable with respect to $Q$. From (2.2.41) and (2.2.21) we have,

$$\ln\big(Z^{\alpha^\star}_t\big) = \int_0^t \alpha^\star_{s-} d\tilde{R}^\theta_s - \int_0^t g(s) ds, \tag{2.2.42}$$

where,

$$g(t) = \ln\Big(E\big[e^{\int_0^t \alpha^\star_{s-} d\tilde{R}_s}|\mathbb{H}_T\big]\Big). \tag{2.2.43}$$

From (2.2.42), $\ln\big(Z^{\alpha^\star}_t\big)$ is integrable with respect to $Q$ as a sum of two integrable terms. Let $(t_n)_{n\in I(0,\infty)}$ be a local sequence of increasing stopping times with $\lim_{n\to\infty} t_n = T$, associated with the local martingale $\int_0^t \alpha_{s-} d\tilde{R}^\theta_s$. By definition of local sequences, the process $\int_0^{t_n\wedge t} \alpha_{s=} d\tilde{R}^\theta_s$

is a $Q-$martingale. Hence, for any $t \in [0,T]$, we have;

$$\begin{aligned} \mathcal{H}^{\mathbb{H}_T}_{\mathbb{H}_T \vee \bar{\mathbb{L}}_T}(Q|P) &\geq \mathcal{H}^{\mathbb{H}_T}_{\mathbb{H}_T \vee \bar{\mathbb{L}}_{t\wedge t_n}}(Q|P), (\text{ remark 2.2.3}) \\ &\geq E_Q\Big[\ln\big(\frac{dP^{\alpha^\star}}{dP}\big|_{\mathbb{H}_T \vee \bar{\mathbb{L}}_{t\wedge t_n}}\big)\big|\mathbb{H}_T\Big], (\text{ remark 2.2.3}). \end{aligned} \tag{2.2.44}$$

From (2.2.42) we have

$$\begin{aligned} & E_Q\Big[\ln\big(\frac{dP^{\alpha^\star}}{dP}\big|_{\mathbb{H}_T \vee \bar{\mathbb{L}}_{t\wedge t_n}}\big)\big|\mathbb{H}_T\Big] \\ &= E_Q\Big[\ln\big(Z^{\alpha^\star}_{t\wedge t_n}\big)\Big|\mathbb{H}_T\Big] \\ &= E_Q\big[\int_0^{t\wedge t_n} \alpha_{s^-} d\tilde{R}_s\big|\mathbb{H}_T\big] + E_Q\big[g(t\wedge t_n)|\mathbb{H}_T\big] \\ &= E_Q\big(\int_0^{t\wedge t_n} \alpha_s d\tilde{R}_s\big|\mathbb{H}_T \vee \bar{\mathbb{L}}_0\big) + E_Q\big[g(t\wedge t_n)|\mathbb{H}_T\big] \\ &= E_Q\big[g(t\wedge t_n)\big|\mathbb{H}_T\big], \end{aligned}$$

since $\int_0^{t\wedge t_n} \alpha_{s^-} d\tilde{R}(s)$ is a $\big(\mathbb{H}_T \vee \bar{\mathbb{L}}_t\big)_{t\in[0,T]}-$martingale. We note that, $\Big|E_Q\big[g(t\wedge t_n)\big|\mathbb{H}_T\big]\Big| \leq E_Q\big[|g(T)|\big|\mathbb{H}_T\big] = g(T), \forall t \in [0,T]$, since from (2.2.43), $g$ is $\mathbb{H}_T-$measurable. Hence, by the Dominated Convergence theorem we have,

$$\lim_{n\to\infty} E_Q\big[g(t\wedge t_n)\big|\mathbb{H}_T\big] = E_Q\big[g(T)\big|\mathbb{H}_T\big] = |g(T)|. \tag{2.2.45}$$

Taking the limit in (2.2.44), we obtain;

$$\mathcal{H}^{\mathbb{H}_T}_{\mathbb{H}_T \vee \bar{\mathbb{L}}_T}(Q|P) \geq E_Q\big[g(T)\big|\mathbb{H}_T\big] = g(T). \tag{2.2.46}$$

The second step of the proof consists in showing that the conditional relative entropy of the random Esscher transform achieves the minimum value in (2.2.46). Using (2.2.41) and the $P^{\alpha^\star}-$martingale property of $\tilde{R}_t$ the relative entropy of $P^{\alpha^\star}$ is computed as follows;

$$\begin{aligned} & \mathcal{H}^{\mathbb{H}_T}_{\mathbb{H}_T \vee \bar{\mathbb{L}}_T}(P^{\alpha^\star}|P) \\ &= E_{P^{\alpha^\star}}\Big[\ln\big(\frac{dP^{\alpha^\star}}{dP}\big)\big|\mathbb{H}_T\Big] \\ &= E_{P^{\alpha^\star}}\Big[\int_0^T \alpha_{s^-} d\tilde{R}_s\big|\mathbb{H}_T \vee \bar{\mathbb{L}}_0\Big] + E_{P^{\alpha^\star}}\big[g(T)\big|\mathbb{H}_T\big], (\text{ from (2.2.41)}) \\ &= E_{P^{\alpha^\star}}\big[g(T)\big|\mathbb{H}_T\big] = g(T), (\tilde{R}^\theta_t \text{ is a } P^{\alpha^\star} \text{ a martingale}). \end{aligned}$$

From (2.2.46), the lemma follows. □

### 2.2.2 Unconditional minimum entropy martingale measure (UMEMM)

We will define an equivalent martingale probability measure and we will establish that it has the unconditional minimum entropy martingale measure property. $(\lambda_{i,j}(t))_{m\times m}$ is the intensity matrix of the semi Markov process $\theta$ from (1.2.9) and $N$ is the point process defined in Definition 1.4.2.

DEFINITION 2.2.2 *Let $Q$ be a local absolutely continuous probability measure with respect to the historical probability measure $P$ on the filtered measurable space $\Big(\Omega, \mathbb{H}_T \vee \bar{\mathbb{L}}_T, \big(\mathbb{H}_t \vee \bar{\mathbb{L}}_t\big)_{t\in[0,T]}\Big)$; $\hat{Q}$ and $\hat{P}$, denote the regular versions of the conditional probabilites $P(|\mathbb{H}_T)$ and $Q(|\mathbb{H}_T)$ over $\mathbb{H}_T \vee \bar{\mathbb{L}}_T$. $Z_t^Q$ denotes a local martingale process with initial value 1, representing the density process of $Q$ with respect to $P$. $(\xi_{i,j}(t))_{m\times m}$ denotes a matrix entries with predictable processes. Moreover, the rows add up to 0 and satisfy $\sum_{(i,j)\in E^2}\int_0^t \big|\xi_{i,j}(s)\lambda_{i,j}(s)\big|ds < \infty$. $N$ and $\gamma$ are the processes defined in Definition 1.4.2 and Lemma 1.4.1, respectively.*

We first recall a decomposition theorem [53] and we establish a Girsanov-type lemma necessary in the proof of the UMEMM property.

LEMMA 2.2.5 *Let $Q$, $P$, $Z_t^Q$, $\hat{Q}(w,)$, $(\xi_{i,j})_{m\times m}$, $N$, $\bar{N}$ and $\hat{P}(w,)$ be processes and probability measures defined in Definitions 2.2.2, 1.4.2 and Lemma 1.4.1. The following claims hold.*

1. *there exist two density processes $Z_t^L$ and $Z_T^H$ such that;*

$$Z_t^Q = Z_t^L \times Z_T^H \tag{2.2.47}$$

   *with:*

$$\frac{d\hat{Q}}{d\hat{P}}\Big|_{\mathbb{L}_t\vee\mathbb{H}_T} = Z_t^L \text{ and } \frac{dQ}{dP}\Big|_{\mathbb{H}_T} = Z_T^H.$$

2. *If $\big(\lambda_{i,j}(t)\big)_{m\times m}$ is the matrix with conditional intensity of the semi Markov process $\theta$ in (1.2.9) and $\lambda_{i,j}(t) \neq 0, \forall t \in [0,T]$, then the following claims are equivalent;*

   *(a)*

$$\frac{dQ}{dP}\big|_{\mathbb{H}_t} = Z_t^H,$$

   *where $Z^H$ solves the SDE:*

$$dZ_t^H = Z_{t^-}^H \sum_{(i,j)\in E^2}\Big[-1 + \frac{\xi_{i,j}(t^-)}{\lambda_{i,j}(t^-)}\Big]\bar{N}(dt,\mathbb{R},\{(i,j)\}),\ Z_0^H = 1.$$

*(b) Under probability measure Q, with density process $Z_t^H$, the point process $M$ has conditional intensities matrix $\big(\xi_{i,j}(t)\big)_{m\times m}$.*

*Proof.* The proof of 1 follows closely [53]. As for 2, we note that 2a ⇒ 2b follows from the proof of Lemma 2.2.1. We now aim at proving that 2b ⇒ 2a. From Definition 2.2.2 and 2b, $Z_t^H$ and $N(,\mathbb{R},\{(i,j)\}) - \gamma(\mathbb{R},\{(i,j)\})$ are $\big(P,(\mathbb{H}_t)_{t\in[0,T]}\big)-$martingale processes. From the martingale representation property of $\bar{N}(t,\mathbb{R},\{(i,j)\}) = N(t,\mathbb{R},\{(i,j)\}) - \gamma(\mathbb{R},\{(i,j)\}) = N(t,\mathbb{R},\{(i,j)\}) - \lambda_{i,j}(t)$, there exists an $m\times m$ matrix of $\mathbb{H}_t-$predictable processes $(s_t^{i,j})_{m\times m}$ such that:

$$dZ_t^H = \sum_{(i,j)\in E^2} s_{t^-}^{i,j}\bar{N}(dt,\mathbb{R},\{(i,j)\}).$$

As $Z_t^H > 0$ $P$-almost surely, there exists an $m\times m$ matrix of predictable processes $\tilde{s}_t^{i,j}$ satisfying $s_t^{i,j} = Z_t^H\tilde{s}_t^{i,j}$. Hence,

$$dZ_t^H = Z_{t^-}^H\sum_{(i,j)\in E^2}\tilde{s}_{t^-}^{i,j}\bar{N}(dt,\mathbb{R},\{(i,j)\}).$$

From Lemma 2.2.1, the matrix of conditional $Q-$intensities of $N(,\mathbb{R},\{(i,j)\})$ is $\lambda_{i,j}(t)(1+\tilde{s}_t^{i,j})$. One needs to prove that the conditional intensity of $N(,\mathbb{R},\{(i,j)\})$ with respect to $Q$ is $\xi_{i,j}(t),\forall i,j\in I(1,m)$. Hence, equating both matrices and solving for $\tilde{s}^{i,j}$ yields;

$$\lambda_{i,j}(t)(1+\tilde{s}_t^{i,j}) = \xi_{i,j}(t), \forall t\in[0,T],(i,j)\in E^2,$$
$$\text{and hence, } \tilde{s}_t^{i,j} = \Big[-1+\frac{\xi_{i,j}(t)}{\lambda_{i,j}(t)}\Big], \forall t\in[0,T],(i,j)\in E^2.$$

Therefore, $Z^H$ is solution process of the SDE,

$$dZ_t^H = Z_{t^-}^H\sum_{(i,j)\in E^2}\Big[-1+\frac{\xi_{i,j}(t^-)}{\lambda_{i,j}(t^-)}\Big]\big[N(dt,\mathbb{R},\{(i,j)\}) - \xi_{i,j}(t^-)dt\big]\ Z_0 = 1.$$

From Lemma 2.2.1, the intensity matrix of the semi Markov process $\theta$ under the probability measure $Q$ is $(\xi_{i,j}(t))_{m\times m}$. This completes the proof of 2 and thence the lemma. □

We define a density process which we prove is the unconditional minimum entropy martingale measure.

DEFINITION 2.2.3 *Let $P^{\alpha^\star,\xi}$ be a risk neutral measure with density $Z_t = Z_t^{\alpha^\star} \times Z_T^{\xi}$, where $Z^{\alpha^\star}$ is introduced in Definition 2.2.1, with $R^\theta$ the simple return process of defined in* (2.2.16). *$\alpha^\star$ is the solution process of* (2.2.29) *and $Z_t^\xi$ is solution of the SDE $dZ_t^\xi = Z_{t^-}^\xi \sum_{(i,j)\in E^2} \tilde{s}_t^{i,j} d\bar{M}_t^{i,j}$.*

$$\begin{aligned}\frac{dP^{(\alpha^\star,\xi)}}{dP}\Big|_{\mathbb{H}_T\vee\bar{\mathbb{L}}_T} \\ =&\frac{e^{\int_0^T \alpha^\star_{s-}\,dR_s}}{E\big(e^{\int_0^T \alpha^\star_{s-}\,dR_s}\big|\mathbb{H}_T\big)}\prod_{(i,j)\in E^2}\exp\{\Big[\int_0^T(1-\xi_{i,j}(s^-))\lambda_{i,j}(s^-)ds \\ &+\int_0^T \ln\big(\xi_{i,j}(s^-)\big)N(ds,\mathbb{R},\{(i,j)\})\Big]\}.\end{aligned}$$

*We also define a functional $F$ as follows;*

$$\begin{aligned}F\big((\xi_{i,j})\big) =&E\Bigg[g(T)+\sum_{(i,j)\in E^2}\Big[\int_0^T(1-\xi_{i,j}(s^-))\lambda_{i,j}(s^-)ds \\ &+\int_0^T \ln(\xi_{i,j}(s^-))N(ds,\mathbb{R},\{(i,j)\})\Big] \\ &\prod_{(i,j)\in E^2}\exp\{\Big[\int_0^T(1-\xi_{i,j}(s^-))\lambda_{i,j}(s^-)ds \\ &+\int_0^T \ln(\xi_{i,j}(s^-))N(ds,\mathbb{R},\{(i,j)\})\Big]\}\Bigg],\end{aligned}$$

*where $g$ is defined in* (2.2.43).

We will next show that under a particular choice of $\xi$, $P^{\alpha^\star,\xi}$ has the unconditional minimum entropy martingale measure property.

LEMMA 2.2.6 *We denote $P^{\alpha^\star,\bar{\xi}}$ and $F$ the risk neutral measure and the functional from Definition 2.2.3, respectively. If $Q$ is a $\big(\mathbb{H}_t \vee \bar{\mathbb{L}}_t\big)_{t\in[0,T]}$ risk neutral measure and $(\bar{\xi}_t^{i,j})_{m\times m}$ minimizes the functional $F$, then the following holds:*

$$\mathcal{H}_{\mathbb{H}_T\vee\bar{\mathbb{L}}_T}\big(Q\big|P\big) \geq \mathcal{H}_{\mathbb{H}_T\vee\bar{\mathbb{L}}_T}\big(P^{(\alpha^\star,\bar{\xi})}\big|P\big).$$

*Proof.* Let $Q$ be a risk neutral measure. By definition of risk neutral measures, $Q$ is locally absolutely continuous with respect to $P$. From Lemma 2.2.5, there exists a process $Z_t^L$ and a process $Z_t^H$ such that $\frac{dQ}{dP}\Big|_{\mathbb{H}_T\vee\bar{\mathbb{L}}_t} = Z_t^L \times Z_T^H$. From Lemma 2.2.5, we have:

$$Z_t^H = \prod_{(i,j)\in E^2}\exp\Big[\int_0^t(1-\xi_{i,j}(s^-))\lambda_{i,j}(y_{s^-})ds+\int_0^T \ln(\xi(s^-))N(ds,\mathbb{R},\{(i,j)\})\Big],$$

for some $m \times m$ matrix of $\mathbb{H}_t$-predictable processes $\xi_{i,j}(s)$ as in Definition 2.2.2.

$$\begin{aligned}
&\mathcal{H}_{\mathbb{H}_T \vee \bar{\mathbb{L}}_T}(Q|P) \\
&\quad = E\Big[\ln(\frac{dQ}{dP})\frac{dQ}{dP}\Big] \\
&\quad = E\Big[Z_T^L Z_T^H(\ln(Z_T^L)) + Z_T^L Z_T^H \ln(Z_T^H)\Big] \\
&\quad = E\Big[E\big[Z_T^L Z_T^H \ln(Z_T^L)\big|\mathbb{H}_T\big] + E\big[Z_T^L Z_T^H \ln(Z_T^H)|\mathbb{H}_T\big]\Big] \\
&\quad = E\Big[Z_T^H E\big[Z_T^L \ln(Z_T^L)\big|\mathbb{H}_T\big] + Z_T^H \ln(Z_T^H)\Big] \\
&\quad = E\Big[Z_T^H \mathcal{H}^{\mathbb{H}_T}_{\mathbb{H}_T \vee \bar{\mathbb{L}}_T}(\hat{Q}|\hat{P}) + Z_T^H \ln(Z_T^H)\Big] \\
&\quad \geq E\Big[Z_T^H \mathcal{H}^{\mathbb{H}_T}_{\mathbb{H}_T \vee \bar{\mathbb{L}}_T}(\bar{P}^{\alpha}|\hat{P}) + Z_T^H \ln(Z_T^H)\Big] \\
&\quad = E\Bigg[g(T) + \sum_{(i,j)\in E^2}\Big[\int_0^T (1-\xi_{i,j}(s^-))\lambda_{i,j}(y_{s^-})ds \\
&\qquad + \int_0^T \ln(\xi_{i,j}(s^-))N(ds, \mathbb{R}, \{(i,j)\})\Big] \\
&\qquad \times \prod_{(i,j)\in E^2} \exp\{\Big[\int_0^T (1-\xi_{i,j}(s^-))\lambda_{i,j}(s^-)ds \\
&\qquad + \int_0^T \ln(\xi_{i,j}(s^-))N(ds, \mathbb{R}, \{(i,j)\})\Big]\}\Bigg] \\
&\quad = F(\xi_{i,j}) \\
&\quad \geq F(\bar{\xi}_{i,j}), (\text{ definition of } \bar{\xi}) \\
&\quad = E\Big[\ln(\frac{d\bar{P}^{\alpha^\star,\bar{\xi}}}{dP})\frac{d\bar{P}^{\alpha^\star,\bar{\xi}}}{dP}\Big] \\
&\quad = \mathcal{H}_{\mathbb{H}_T \vee \bar{\mathbb{L}}_T}(\bar{P}^{\alpha^\star,\bar{\xi}}|P),
\end{aligned}$$

which proves the result. □

### 2.2.3 Siu and Yang Kernel pricing all risks

Let $\alpha$ be a piecewise constant stochastic process. We define the density process $Z_t^\alpha$ of a probability measure, $P^\alpha$, on the filtration $(\mathbb{H}_t \vee \bar{\mathbb{L}}_t)_{t\in[0,T]}$, with an Esscher transform with parameter $\alpha$. The pricing kernel discussed here is based on the work in [64], in the context of a Markov switching asset price process.

DEFINITION 2.2.4 *Let $Z^\alpha$ be the following stochastic process;*

$$\frac{d\bar{P}^\alpha}{dP}\Big|_{\mathbb{H}_t\vee\bar{\mathbb{L}_t}} = Z_t^\alpha = \begin{cases} E\left[\frac{e^{-\int_0^T \alpha(s)dR_s^\theta}}{E\left[e^{-\int_0^T \alpha_{s-}dR_s^\theta}|\theta_0,y_0\right]}\Big|\mathbb{H}_t\vee\bar{\mathbb{L}}_t\right] & \text{if } \forall t\in(0,T] \\ 1 & \text{if } t=0, \end{cases}$$

*where, $R_t^\theta$ is the log price process in* (2.2.16)*, induced by $x$ from* (1.4.1) *in the context of solution* (1.4.2).

We note that from Lemma 1.5.1, one can retrieve any particular scalar conditional characteristic function from the vector characteristic function as follows: $\Psi(u,t,y,j,x) = \exp\left[iu\ln(x)\right]\Big\langle \exp\Big(\int_y^{t+y} M(u,s)ds\Big)$ where $e_{\theta_t} = (1_{\theta_t=1}, 1_{\theta_t=2}, ..., 1_{\theta_t=m})^\top$. $\alpha_t$ is the Esscher parameter process associated with the probability measure $\bar{P}^\alpha$.

LEMMA 2.2.7 *Let $Z_t^\alpha$ be the process in Definition 2.2.4. $Z_t^\alpha$ is an almost surely positive martingale with unitary expectation.*

*Proof.* We first prove that $Z_t^\alpha$ is a martingale. Let $0\le s\le t$

$$\begin{aligned} E\big[Z_t^\alpha\big|\mathbb{H}_s\vee\bar{\mathbb{L}}_s\big] &= E\Bigg[E\Big[\frac{e^{-\int_0^T \alpha_{s-}dR_s^\theta}}{E\Big[e^{-\int_0^T \alpha_{s-}dR_s^\theta}|y_0,\theta_0,L_0\Big]}\Big|\mathbb{H}_t\vee\bar{\mathbb{L}}_t\Big]\Big|\mathbb{H}_s\vee\bar{\mathbb{L}}_s\Bigg] \\ &= E\Big[\frac{e^{-\int_0^T \alpha_{s-}dR_s^\theta}}{E\Big[e^{-\int_0^T \alpha_{s-}dR_s^\theta}|y_0,\theta_0,L_0\Big]}\Big|\mathbb{H}_s\vee\bar{\mathbb{L}}_s\Big], (\mathbb{H}_s\vee\bar{\mathbb{L}}_s\subset\mathbb{H}_t\vee\bar{\mathbb{L}}_t) \\ &= Z_s^\alpha. \end{aligned}$$

Therefore, $Z_t^\alpha$ is a martingale. It follows that $Z_t^\alpha$ has unitary expectation,

$$E(Z_t^\alpha) = E\big[E(Z_t^\alpha)\big|\mathbb{H}_0\vee\bar{\mathbb{L}}_0\big] = E(Z_0^\alpha) = 1.$$

Noting that $Z_t^\alpha$ is an almost surely positive process by construction, the lemma follows. □

From the preceding lemma, $Z_t^\alpha$ is a density process. Hence, The Esscher transform in (2.2.4) defines a probability measure $\bar{P}^\alpha$ equivalent to $P$. It remains to show that $\bar{P}^\alpha$ is a martingale measure under a certain condition specified in the next Lemma.

LEMMA 2.2.8 *Let $Z_t^\alpha$ be from Definition 2.2.4 and $(x_t)_{t\in[0,T]}$ as defined in* (1.4.1)*. $M(u,y)$ and $\bar{M}(u,y)$ are defined in* (1.5.4) *with modified log price process defined by $dR_t^\theta = \alpha_{t-}\ln\big(\beta_{n(t)}\big) +$*

*$\alpha_{t^-} dL^\theta_t$ and $dR^\theta_t = \alpha_{t^-}\ln\big(\beta_{n(t)}\big) - r_{t^-}dt + (\alpha_{t^-}+1)dL^\theta_t$, respectively, with $L^\theta$ defined in* (1.3.2). *$\tilde{x}_t = e^{-\int_0^t r_{s^-}ds}x_t$ is a $\big(\bar{P}^\alpha, (\mathbb{H}_t \vee \bar{\mathbb{L}}_t)_{t\in[0,T]}\big)$-martingale if and only if:*

$$\Big\langle \exp\{\big[\int_y^{t+y}\bar{M}(-i,s)ds\big]\}.e_{\theta_0}, \mathbf{1}\Big\rangle \\ - \Big\langle \exp\{\big[\int_y^{t+y}M(-i,s)ds\big]\}^\top.e_{\theta_0}, \mathbf{1}\Big\rangle = 0 \quad \forall t\in[0,T], \theta_u \in E,$$

*respectively, where,*

$$\bar{M}_{p,q}(-i,y) \\ = \begin{cases} -r(q) + (\alpha_q+1)\mu(q) + \frac{1}{2}(\alpha_q+1)^2\sigma^2(q) + \int_{|z|>1}\big[e^{(\alpha_q+1)G(z,q)}-1\big]\nu(q,dz) \\ + \int_{|z|\le 1}\big[e^{(\alpha_q+1)G(z,q)} - 1 - (\alpha_q+1)G(z,q)\big]\nu(q,dz) + \lambda_{q,q}(y) \textit{ If } p=q \\ \lambda_{q,p}(y)\int_{z\in\mathbb{R}} e^{(\alpha_q+1)z}\bar{b}(z|q,p)dz \textit{ Otherwise }, \end{cases}$$

*and*

$$M_{p,q}(-i,y) \\ = \begin{cases} \alpha_q\mu(q) + \frac{1}{2}\alpha_q^2\sigma^2(q) + \int_{|z|\le 1}\big[e^{\alpha_q G(z,q)} - 1 - \alpha_q G(z,q)\big]\nu(q,dz) \\ + \int_{|z|>1}\big[e^{\alpha_q G(z,q)}-1\big]\nu(q,dz) + \lambda_{q,q}(y) \textit{ If } p=q \\ \lambda_{q,p}(y)\int_{z\in\mathbb{R}} e^{\alpha_q z}\bar{b}(z|q,p)dz \textit{ Otherwise }. \end{cases}$$

*Proof.* Let $0 \le u \le t$. From [53, 64] and by the abstract Bayes rule [44], we have;

$$E\big(Z^\alpha_t e^{-\int_0^t r_{s^-}ds}x_t \big| \mathbb{H}_u \vee \bar{\mathbb{L}}_u\big) \tag{2.2.48}$$

$$= e^{-\int_0^u r_{s^-}ds}x_u \frac{E\big[e^{-\int_u^t r_{s^-}ds + \int_u^t \big(\alpha(s^-)+1\big)dR^\theta_s}\big|\mathbb{H}_u \vee \bar{\mathbb{L}}_u\big]}{E\big[e^{\int_u^t \alpha_{s^-}dR^\alpha_s}\big|\mathbb{H}_u \vee \bar{\mathbb{L}}_u\big]} \tag{2.2.49}$$

Hence, $e^{-\int_0^t r_{s^-}ds}x_t$ is a $\big(\bar{P}^\alpha, (\mathbb{H}_t \vee \bar{\mathbb{L}}_t)_{t\in[0,T]}\big)$-martingale if and only if

$$\frac{E\big[e^{-\int_u^t r_{s^-}ds + \int_0^t \big(\alpha_{s^-}+1\big)dR^\alpha_s}\big|\mathbb{H}_u \vee \bar{\mathbb{L}}_u\big]}{E\big[e^{\int_0^t \alpha_{s^-}dR^\alpha_s}\big|\mathbb{H}_u \vee \bar{\mathbb{L}}_u\big]} = 1 \quad \forall u,t \in [0,T]. \tag{2.2.50}$$

From Lemma 1.5.1 applied to $dR^\theta_t = \alpha_{t^-}\ln\big(\beta_{n(t)}\big) + \alpha_{t^-}dL^\theta_t$ and $dR^\theta_t = \alpha_{t^-}\ln\big(\beta_{n(t)}\big) - r_{t^-}dt + (\alpha_{t^-}+1)dL^\theta_t$, respectively and on account of the Markov property and the homogeneity of the process $(\theta, y)$, the numerator and the denominator of (2.2.50) becomes,

$$E\big[e^{-\int_u^t r_{s^-}ds+\int_0^t \big[\alpha(s^-)+1\big]dL_s^\alpha}\big|\mathbb{H}_u \vee \bar{\mathbb{L}}_u\big] = \Big\langle \exp\big(\int_y^{y+t-u} \bar{M}(-i,s)ds\big).e_{\theta_0}, \mathbf{1}\Big\rangle$$

and

$$E\big[e^{\int_0^t \alpha(s^-)dL_s^\alpha}\big|\mathbb{H}_u \vee \bar{\mathbb{L}}_u\big] = \Big\langle \exp\big(\int_y^{y+t-u} M(-i,s)ds\big).e_{\theta_0}, \mathbf{1}\Big\rangle,$$

respectively, where,

$$\bar{M}_{p,q}(-i,y) = \begin{cases} -r(q) + (\alpha_q+1)\mu(q) + \frac{1}{2}(\alpha_q+1)^2\sigma^2(q) + \int_{|z|>1}\big[e^{(\alpha_q+1)G(z,q)}-1\big]\nu(q,dz) \\ +\int_{|z|\le 1}\big[e^{(\alpha_q+1)G(z,q)}-1-(\alpha_q+1)G(z,q)\big]\nu(q,dz) + \lambda_{q,q}(y) \text{ If } p=q \\ \lambda_{q,p}(y)\int_{z\in\mathbb{R}} e^{(\alpha_q+1)z}\bar{b}(z|q,p)dz \text{ Otherwise }, \end{cases}$$

and

$$M_{p,q}(-i,y) = \begin{cases} \alpha_q\mu(q) + \frac{1}{2}\alpha_q^2\sigma^2(q) + \int_{|z|\le 1}\big[e^{\alpha_q G(z,q)}-1-\alpha_q G(z,q)\big]\nu(q,dz) \\ +\int_{|z|>1}\big[e^{\alpha_q G(z,q)}-1\big]\nu(q,dz) + \lambda_{q,q}(y) \text{ If } p=q \\ \lambda_{q,p}(y)\int_{z\in\mathbb{R}} e^{\alpha_q z}\bar{b}(z|q,p)dz \text{ Otherwise }. \end{cases}$$

Hence, (2.2.50) becomes:

$$\Big\langle \exp\{\big[\int_y^{y+t-u}\bar{M}(-i,s)ds\big]\}.e_{\theta_0}, \mathbf{1}\Big\rangle - \Big\langle \exp\{\big[\int_y^{y+t-u} M(-i,s)ds\big]\}.e_{\theta_0}, \mathbf{1}\Big\rangle = 0,\ \forall u,t\in[0,T],\ \forall\theta_u\in E, \tag{2.2.51}$$

which completes the proof of the lemma. □

## 2.3 Option Pricing Formulas

In this section, we price a European style call option within the risk neutral pricing theory [62]. We denote $Q$ an equivalent martingale measure of the historical probability measure $P$, relative to the price process $x$ in (1.4.1). We derive a PIDE extending the PDE in [6] satisfied by European call prices. We also describe how two existing pricing methods blend seamlessly in the context of this paper.

DEFINITION 2.3.1 *Let $S$ be a function in $L^2(\Omega, Q)$ defined on $\mathbb{R}^+ \times \mathbb{R}^+$ into $\mathbb{R}$ representing the payoff of a contingent claim; $Q$ is a risk neutral probability measure of the price process $x$ defined by* (3.2.4) *with respect to the historical probability measure $P$; $K$ is a nonnegative real number denoting the strike price of a European type option contract with maturity $T$; $x_T$ denotes the asset price value at maturity; $C$ is the $Q-$risk neutral option price function defined on $[0, T] \times \mathbb{R}^+ \times \mathbb{R}^+ \times [0, T] \times E \times \mathbb{R}^+$ into $\mathbb{R}^+$ and $V$ denotes the discounted option price process defined by $V(t, T, K, y_t, \theta_t, x_t) = e^{-\int_0^t r_{s-} ds} C(t, T, K, y_t, \theta_t, x_t)$.*

LEMMA 2.3.1 *Let $S$ be a random variable representing the payoff of a general European style contingent claim with maturity T and strike price $K$ in Definition 2.3.1; let $Q$ be the risk neutral measure defined in Definition 2.3.1 and $C$ is the $Q-$risk neutral option price of a contingent claim. Then, the $Q-$risk neutral option price $C$ of a European contingent claim with maturity $T$, strike price $K$ and payoff $S$ can be expressed;*

$$C(t, T, K, y_t, \theta_t, x_t) = E_Q\big(e^{-\int_t^T r_s ds} S(x_T, K) | y_t, \theta_t, x_t\big). \tag{2.3.1}$$

*Proof.* From [62], the $Q-$risk neutral option price $C$ at time $t$ is given by;

$$C(t, T, K, y_t, \theta_t, x_t) = E_Q\big[e^{-\int_t^T r_s ds} S(x_T, K) | \mathbb{H}_t \vee \bar{\mathbb{L}}_t\big].$$

We note from Lemma 1.2.3, that the triplet $(y, \theta, x)$ is Markovian, hence

$$C(t, T, K, y_t, \theta_t, x_t) = E_Q\big[e^{-\int_t^T r_s ds} S(x_T, K) | y_t, \theta_t, x_t\big],$$

which proves the result. $\square$

A partial integro differential equation(PIDE) satisfied by a European style contingent claim with maturity $T$ and payoff $H$ is presented in the next Lemma.

LEMMA 2.3.2 *Let $Q$, $C$ and $V$ be the risk neutral measure, the $Q-$risk neutral option price function and the discounted option price process defined in 2.3.1, respectively. Then $V$ satisfies the following* *system of PIDE;*

$$
\begin{aligned}
&\frac{\partial V}{\partial s}+\frac{\partial V}{\partial y}+\mu(j)x_{s^-}\frac{\partial V}{\partial x}+\frac{1}{2}\sigma^2(j)x_s^2\frac{\partial^2 V}{\partial x^2}\\
&+\int_{|z|\le 1}\Big[V(s,T,K,y_s,\theta_s,x_{s^-}+x_{s^-}G(z,j))-V(s,T,K,y_s,j,x_{s^-})\\
&-G(z,j)x_{s^-}\frac{\partial V}{\partial x}\Big]\nu(j,dz)\\
&+\int_{|z|>1}\Big[V(s,T,K,y_s,\theta_s,x_{s^-}+x_{s^-}H(z,\theta_s))\\
&-V(s,T,K,y_s,\theta_s,x_{s^-})\Big]\nu(\theta_s,dz)\\
&+\int_{z\in\mathbb{R}}\sum_{j\neq i}\lambda_{i,j}(y_s)V(s,T,K,y_s,j,x_{s^-}e^z)\bar{b}(z|i,j)dz\\
&+V(s,T,K,y_{s^-},i,x_{s^-})\lambda_{j,j}(y)=0,
\end{aligned}
$$

*with terminal condition,*

$$V(T,T,K,y_T,\theta_T=j,x_T)=e^{-\int_0^T r(\theta_s)ds}S(x_T,K),\ \textit{for } j\in E.$$

*Proof.* From (2.3.1), the discounted price process could be expressed as follows:

$$
\begin{aligned}
V(t,T,K,y,j,x)&=e^{-\int_0^t r(\theta_s)ds}C(t,T,K,y,j,x)\\
&=E^Q\big(e^{-\int_{[0,T]} r(\theta_s)ds}S(x_T,K)|y_t,\theta_t,x_t\big) \qquad (2.3.2)
\end{aligned}
$$

V is a $\big(Q,\bar{\mathbb{L}}_t\vee\mathbb{H}_t)\big)$-Martingale since it is a $Q$-conditional expectation. We use the law of iterated expectation and $u\le t$ to prove it as follows:

$$
\begin{aligned}
&E\big(V(t,T,K,y,j,x)|\bar{\mathbb{L}}_u\vee\mathbb{H}_u\big)\\
&=E\big(e^{-\int_0^t r(\theta_s)ds}C(t,T,K,y,j,x)|\bar{\mathbb{L}}_u\vee\mathbb{H}_u\big)\\
&=E\Big[e^{-\int_0^t r(\theta_s)ds}E\big(e^{-\int_t^T r(\theta_s)ds}S(x,K)|\bar{\mathbb{L}}_t\vee\mathbb{H}_t\big)|\bar{\mathbb{L}}_u\vee\mathbb{H}_u\Big]\\
&=E\Big[E\big(e^{-\int_0^T r(\theta_s)ds}S(x,K)|\bar{\mathbb{L}}_t\vee\mathbb{H}_t\big)|\bar{\mathbb{L}}_u\vee\mathbb{H}_u\Big]\\
&=E\Big[e^{-\int_0^T r(\theta_s)ds}S(x,K)|\bar{\mathbb{L}}_u\vee\mathbb{H}_u\Big]\\
&=e^{-\int_0^u r(\theta_s)ds}E\Big[e^{-\int_u^T r(\theta_s)ds}S(x,K)|\bar{\mathbb{L}}_u\vee\mathbb{H}_u\Big]\\
&=V(t,T,K,y,j,x)
\end{aligned}
$$

From Ito differential formula in Lemma 1.4.1, we have:

$$dV(t,T,K,y_t,\theta_t,x(t)) = \mathcal{A}V(t,T,K,y_{t^-},\theta_{t^-},x(t^-))dt + \underbrace{\text{Martingale Terms}}$$

As $V$ is a martingale, the first term vanishes and (1.4.17), the following PIDE is obtained:

$$\begin{aligned}
&\frac{\partial V}{\partial s} + \frac{\partial V}{\partial y} + \mu(\theta_{s^-})x_{s^-}\frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2(\theta_{s^-})x_s^2\frac{\partial^2 V}{\partial x^2}\\
&+ \int_{|z|\le 1}\Big[V(s,T,K,y_s,\theta_s,x_{s^-} + x_{s^-}G(z,\theta_s)) - V(s,T,K,y_s,\theta_s,x_{s^-})\\
&- G(z,\theta_s)x_{s^-}\frac{\partial V}{\partial x}\Big]\nu(\theta_s,dz)\\
&+ \int_{|z|>1}\Big[V(s,T,K,y_s,\theta_s,x_{s^-} + x_{s^-}H(z,\theta_s)) - V(s,T,K,y_s,\theta_s,x_{s^-})\Big]\nu(\theta_s,dz)\\
&+ \int_{z\in\mathbb{R}}\sum_{j\neq i}\lambda_{i,j}(y_s)V(s,T,K,y_s,j,x_{s^-}e^z)\bar{b}(z|i,j)dz\\
&- V(s,T,K,y_{s^-},i,x_{s^-})\lambda_{\theta_{s^-},j}(y_s) = 0, \forall t\in[0,T].
\end{aligned}$$

Hence, the proof is complete. □

DEFINITION 2.3.2 *Let $\tilde{C}$ be a continuous function on $[0,T]\times\mathbb{R}^+\times\mathbb{R}\times[0,T]\times E\times\mathbb{R}^+$ into $\mathbb{R}^+$ representing the modified European call option price; let $\Upsilon$ be the characteristic function of $\tilde{C}$ with respect to its third variable, and $k$ denotes the logarithm of the positive real number $K$ in Definition 2.3.1.*

REMARK 2.3.1 Assuming a deterministic interest rate $r$, a closed form formula for the Fourier transform of a modified vanilla European call option price is known [11]. Let us denote $C$, $\tilde{C}$ and $\eta$ the European call price, the modified European call price of Carr and Madan type and a positive real number, respectively, for the payoff function of a European call option $S(x_T,K) = (x_T - K)^+ = (e^{\ln(x_T)} - e^k)^+$, with $k = \ln(K)$. Further assume that

$$\int_0^\infty |\tilde{C}(t,T,k,j,y)|dk < \infty, \forall j\in E.$$

Then the modified European call price defined by Carr and Madan is expressed as follows;

$$\tilde{C}(t,T,k,y_t,\theta_t,x(t)) = e^{\eta}E_Q\big(e^{-\int_t^T r(t)ds}(e^{\ln(x_T)} - e^k)^+\big|y_t,\theta_t,x_t\big) \tag{2.3.3}$$

In the following lemma, we recall the characteristic function of the modified European Carr and Madan type call option [11].

LEMMA 2.3.3 *If* $\Upsilon$ *and* $\Psi(u,t,y_0,\theta_0,x(0))$ *are the Fourier transform of* $\tilde{C}$ *defined in Definition 2.3.2 and the conditional characteristic function of the log price process defined in Lemma 1.5.1, respectively, then we have;*

$$\begin{aligned}&\Upsilon(T,w,y_0,\theta_0=j,x_0)\\ =&\frac{e^{-\int_0^T r_{s-}ds}}{(\eta+iw)(1+\eta+iw)}\Psi\big(wk-ix_0(1+\eta),T,y_0,\theta_0=j,x_0\big).\end{aligned}\tag{2.3.4}$$

*Proof.* The proof can be found in [11]. □

REMARK 2.3.2 We note that in the case of a regime switching interest rate [53] uses the Carr and Madan type transformation to obtain the characteristic function of European call option prices. The formula in [11] is based on the characteristic function of occupation times which is known in closed form when market states are described by a Markov Chain. We have derived the characteristic function of the occupation times in Corollary 1.5.1.1 which allows us to extend the results in [25, 53] when market states are described by a semi Markov process.

In the context of a price process driven by the Brownian motion [30], an integral option price formula is obtained. In the following result, we present a similar pricing formula [30] in the context of (1.4.1), where we assume that $f_s^j$ is the density of the increment of the log price process in an interval of length $s$, whenever the semi Markov process is in state $j$ for any $j\in I(1,m)=E$.

LEMMA 2.3.4 *An integral option pricing formula in the context of model* (1.4.1) *is represented by the following formula:*

$$\begin{aligned}&C(t,y_t,\theta_t,x_t)=P(t,y_t,\theta_t)C^{\theta_t}(t,T,K,x_t)\\ &+Q(t,y_t,\theta_t)\int_0^{T-t}e^{r(\theta_t)u}p(t,y_t,\theta_t)\Big[\int_0^{\infty}\tilde{C}(t+u,0,j,x_t)du\Big]dx,\end{aligned}\tag{2.3.5}$$

*with* $P(t,y_t,\theta_t)=\frac{1-F(y_t+T-t|\theta_t)}{1-F(y_t|\theta_t)}$,$1-P(t,y_t,\theta_t)=Q(t,y_t,\theta_t)$*,* $p(t,y_t,\theta_t)=\frac{f(y_t+T-t|\theta_t)}{1-F(y_t|\theta_t)}$ *and* $\tilde{C}(t+u,y_t,j,x_t)=\sum_{\theta_{t+u}=j,j\neq\theta_{t-}}C(t+u,y_t,j,x_t)f_u^j(\ln(x/S_t))$*, where* $F(.|\theta_{t-})$ *and* $f(.|\theta_{t-})$ *are defined in Remark 1.2.1.* $C^{\theta_t}$ *is the Black Scholes option price when the market is in state* $\theta_t$ *and* $C(t,y_t,\theta_t,x_t)$ *is short hand notation for* $C(t,T,K,y_t,\theta_t,x_t)$*.*

*Proof.* The lemma follows by imitating the proof of Theorem 3.1 of [30]. Let $V(t)=v(t,y_t,\theta_t,x_t)$ defined as in Lemma 1.4.1, using the risk neutral pricing formula, the tower Law of expectations, the identity $1 = 1_{T_{n(t)+1}\leq 1} + 1_{T_{n(t)+1}>1}$ and the notations:

$$E\Big[V(t)|s, y_s, \theta_s, x_s\Big] = E_s[V(t)],$$
$$E\Big[V(t)|s, y_s, \theta_s, x_s, T_{n(t)+1} < T\Big] = E_s^{\leq T}[V(t)], t$$
$$E\Big[V(t)|s, y_s, \theta_s, x_s, T_{n(t)+1} > T\Big] = E_s^{>T}[V(t)]$$
$$E\Big[V(t)|s, u, y_s, y_u, \theta_s, \theta_u, x_s, x_u\Big] = E_{s,u}[V(t)]$$
$$E\Big[V(t)|s, u, y_s, y_u, \theta_s, \theta_u, x_s, x_u, T_{n(t)+1} > T\Big] = E_{s,u}^{>T}[V(t)]$$
$$E\Big[V(t)|s, s+u, y_s, y_{s+u}, \theta_s, \theta_{s+u}, x_s, x_{s+u}, \tau_{n(s)} = y_t + u\Big] = E_{s,u}^{\tau=T}[V(t)],$$

$\forall s, t, u \in \mathbb{R}_+$ and $s < t$, we obtain:

$$\begin{aligned}
C(t,&y_t, \theta_t, x_t)\\
=&E_{t^-}\big[e^{\int_t^T r(\theta_s)ds}(x_T - K)^+\big],\\
=&E_{t^-}\Big[E\big[e^{\int_t^T r(\theta_s)ds}(x_T - K)^+\big|y_t, \theta_t, x_t, T_{n(t)+1}\big]\Big],\\
=&E_{t^-}\Big[1_{(T_{n(t)+1})>T}E\big[e^{\int_t^T r(\theta_{s^-})ds}(x_T - K)^+\big|y_t, \theta_t, x_t, T_{n(t)+1}\big]\Big]\\
&+ E\Big[1_{T_{n(t)+1}\leq T}E\big[e^{\int_t^T r(\theta_{s^-})ds}(x_T - K)^+\big|y_t, \theta_t, x_t, T_{n(t)+1}\big]\Big],\\
=&P\big(T_{n(t)+1} > T|y_{t^-}, \theta_{t^-}, x_{t^-}\big)E_{t^-}^{>T}\Big[E\big[C(t, y_{t^-}, \theta_{t^-}, x_{t^-})\big|T_{n(t)+1}\big]\Big]\\
&+ P\big((T_{n(t)+1} \leq T)|y_{t^-}, \theta_{t^-}, x_{t^-}\big)E_{t^-}^{\leq T}\Big[E\big(C(t, y_{t^-}, \theta_{t^-}, x_{t^-})\big|, T_{n(t)+1}\big]\Big]\\
=&P\big(T_{n(t)+1} > T|y_{t^-}, \theta_{t^-}, x_{t^-}\big)E\big[C(t, y_{t^-}, \theta_{t^-}, x_{t^-})\big|T_{n(t)+1} > T\big]\\
&+ P\big(T_{n(t)+1} \leq T|y_{t^-}, \theta_{t^-}, x_{t^-}\big)E\big[C(t, y_{t^-}, \theta_{t^-}, x_{t^-})\big|T_{n(t)+1} \leq T\big]\\
=&P\big(T_{n(t)+1} > T|y_{t^-}, \theta_{t^-}, x_{t^-}\big)C^{\theta_t}(t, x_t) + P\big(T_{n(t)+1} > T|y_{t^-}, \theta_{t^-}, x_{t^-}\big)\\
&\times E_{t^-}^{\leq T}\big[E\big(C(t, y_{t^-}, \theta_{t^-}, x_{t^-})\big|\tau_{n(t)} = y_t + u\big]\big]\\
=&P(t, y_t, \theta_t)C^{\theta_t}(t, x_t) + Q(t, y_t, \theta + t)\int_0^{T-t} p(T - u, y_t, \theta_t)\\
&\times E\big[C(t, y_{t^-}, \theta_{t^-}, x_{t^-})\big|\tau_{n(t)} = y_t + u\big]du\\
=&P(t, y_t, \theta_t)C^{\theta_t}(t, x_t) + Q(t, y_t, \theta_t)\int_0^{T-t} p(t, y_t, \theta_t)E_{t+u,t}\Big[\\
&e^{r(\theta_t)u}E\big[e^{\int_{t+u}^T r(\theta_{s^-})ds}(x_T - K)^+\big|y_{t^-}, \theta_{t^-}, x_{t^-}, \tau_{n(t)} = y_t + u\big]\Big]\\
=&P(t, y_t, \theta_t)C^{\theta_t}(t, x_t) + Q(t, y_t, \theta_t)\int_0^{T-t} E_t\Big[e^{r(\theta_t)u}p(T - u, y_t, \theta_t)
\end{aligned}$$

$$
\begin{aligned}
&\times E_{t,u}^{\tau=y_t+u}\Big[e^{\int_{t+u}^{T} r(\theta_{s^-})ds}(x_T-K)^+\Big]\Big]du\\
=&P(t,y_t,\theta_t)C^{\theta_t}(t,x_t)+Q(t,y_t,\theta_t)\int_0^{T-t} e^{r(\theta_t)u}p(T-u,y_t,\theta_t)du\\
&\times \int_0^{\infty}\sum_{\substack{\theta_{t+u}=j\\ j\neq\theta_t}} E\big[C(t+u,y_{t+u}=0,\theta_{t+u},x_{t^-})\big|\tau_{n(t)}=y_t+u\big]\\
&\times f_u^j(\ln(x/S_t))dudx\\
=&P(t,y_t,\theta_t)C^{\theta_t}(t,x_t)+Q(t,y_t,\theta_t)\int_0^{T-t} e^{r(\theta_t)u}p(T-u,y_t,\theta_t)\\
&\times \int_0^{\infty}\tilde{C}(t+u,0,j,x_t)dudx.
\end{aligned}
$$

□

## 2.4 Conclusion

Option pricing in a semi Markov switching regime and performed through risk neutral pricing raised the issue of choice of the martingale measure. Existence of uniqueness of the equivalent martingale measure in the well known case of Black Scholes model has been well documented. However, the completeness of the market is in general invalidated when additional sources of randomness are introduced. Semi Markov regimes do not escape the rule and render the market incomplete. The issue now is to pick the ”best” equivalent martingale measure to price derivative with. Such a choice is generally arduous as one needs to first describe an acceptable definition of ”best”. In the current literature, two preeminent versions of equivalent martingale measure are the Minimum martingale measure(MMM) and the minimum entropy martingale measure(MEMM). In this chapter, we applied a version of Girsanov theorem to present a general equivalent martingale measure and we exhibited one non optimal and two optimal martingale measures producing sensible derivative prices. We also presented the conditional minimal entropy martingale measure as it is closest to the historical probability measure with respect to Kulback Leibler distance. Carr and Madan pricing algorithm have consequently been used along with the Fourier transform derived last chapter. A semi closed pricing formula of the pricing formula is obtained. The new feature of the pricing formula is its conditional intensity and the backward recurrence time. One is interested on how much impact the semi Markov parameters have on the option prices and how much difference there is between Markov regime prices and semi Markov regime prices.

# Chapter 3

# Simulation of option prices and calibration of option prices parameters under semi-Markov and Levy process structural perturbations

## 3.1 Introduction

Stochastic hybrid models have been used in financial modeling by quite a few authors among which, [2, 16, 36, 45, 48, 63]. This is in response to the well documented limitations of the seminal Black Scholes stock price model [6]. The non normality of log returns is exhibited by a pronounced skewness and fat tails along with non constant implied volatility, therefore contradicting modeling assumptions underlying Black Scholes model [6]. Moreover, smiles, smirks and skew empirically observed in the option market are unexplained by the Black Scholes model. Heavy tailed and asymmetric distributions have been successfully applied as a remedy to the log return distribution misfits. However, the skew, smile and smirk are reproduced by exponential Levy models for asset prices with relative success for short to medium maturity [66]. A consensual agreement is that volatility is not constant as assumed by [6]. Furthermore, there is strong empirical evidence supporting stochastic volatility. Stochastic volatility and local volatility models have provided a better explanation for many stylized facts of the derivative market and log return times series. However, stochastic regime switching models with random volatility switching from one state to another provide economically interpretable alternative to stochastic volatility and local volatility models. Regime switching models have been first used in [38] in the context of time series in a two-state market regime. Since then, a slew of regime switching stock price models have ensued [2, 10, 14, 34, 43, 54]. However, most of the models developed are assumed to have Markov states. The convenience of Markov market states stems from the constant conditional intensity matrix of Markov processes which proves to be unrealistic for a market often undergoing structural changes. Indeed, under the assumption of constant conditional intensities matrix, the market has the same propensity of switching regime at any given time, regardless of occurring changes. We note that application of Markov regime switching models to financial derivatives is still a work in progress, namely, Markov regime switching exponential Levy models for asset prices with applications in credit risk are used in [36] and an option pricing method under Markov regime switching exponential jump diffusion [18]. On the other hand, semi Markov regime switching models are a relatively unexplored topic [2, 30, 41]. Moreover, simulation methods for option prices from [30, 41] along with continuous and discrete time MCMC calibration method formulated by [41] for semi Markov Black Scholes models of asset prices are developed. Both calibration methods rely on normal likelihood simulation and aren't extensible to other switching exponential Levy models. This is because they either do not have a closed form density function or their known density does not have easy-to-simulate-from conjugate priors. This issue is solved by [2], where a closed form expression for the characteristic function of log asset prices is developed. This paved the way for calibration and simulation of option prices induced by an arbitrary exponential Levy price process with closed form log price characteristic function.

In this paper, we explore four problems of interest: estimation of historical parameters of a semi Markov switching asset price model via LLGMM approach first developed by [56], estimation of the effects of the semi Markov sojourn distribution parameters on option prices, application of [11] and the Fourier space time stepping algorithm of [43] to semi Markov modulated stock price processes and comparison of Markov modulated and semi Markov modulated stock price models.

The paper is organized as follows: in Section 3.2, we define the model along with related filtrations. We use the LLGMM method of [56] to estimate the historical parameters of the model illustrated by three case studies in Section 3.3. Section 4 highlights the effects of risk neutral semi Markov parameters on option prices and volatility surfaces via simulations based on the Carr and Madan method. We also show that we can use the Fourier time stepping method of [60] to price American options and exotic options. Both algorithms are shown to blend naturally in the semi Markovian regime model due to the piecewise constant assumption imposed on the conditional intensity matrix. Section 3.4 ends with calibrations of Heston model, Markov and semi Markov regime switching Black Scholes models to a couple of option data, and we compare the fit of all models through the residual mean square error risk function. Section 3.5 concludes our work with a summary and a few problems encountered along the way, which haven't yet found a satisfying resolution.

## 3.2 Preliminary notations and definitions of the model

Let $T > 0$ and $T^\star > 0$ be the maturity date of an option contract and the time horizon of the market, respectively. We assume that the market is subjected to regime/state structural changes. It is assumed that the market structural states are governed by a semi Markov process $\theta_t$. $(\theta_n, T_n)$ is the corresponding Markov renewal process, where $T_n$ and $\theta_n = \theta_{T_n}$ are the time and the state of the process at the n-th regime change. We assume that the structural state domain $E$ of $(\theta_t)_{t>0}$ is finite, and $m = n(E)$. We also denote $\tau_n = T_{n+1} - T_n$ the sojourn time of the semi Markov process. Let $(\beta_n)_{n\geq 0}$ be a sequence of real nonnegative independent random variables. We assume that the jump in price only depends on the past and current transition states of the semi Markov process, namely, $\beta_n = \beta_{\theta_{n-1},\theta_n}$ with density $g(|\theta_{n-1}, \theta_n)$. $n(t) = \max_n\{n \in I(0, \infty), T_n \leq t\}$ denotes last regime change prior to or at time t. Let $\psi(\theta_{t^-}, dz, ds)$ and $\nu(\theta_{s^-}, dz)ds$ be a Poisson random measure and its intensity measure, respectively. We denote $\bar{\psi}(\theta_{s^-}, dz, ds) = \psi(\theta_{s^-}, dz, ds) - \nu(\theta_{s^-}, dz)ds$, as compensated measure of $\psi(\theta_{s^-}, dz, ds)$. $G$ and $H$ are smooth functions defined from $\mathbb{R} \times E$ into $\mathbb{R}$ satisfying

$$\int_{z\in\mathbb{R}} \Big[\big(1 + H^2(z, j)\big)1_{|z|>1} + 1_{|z|\leq 1}G^2(z, j)\Big]\nu(j, dz) < \infty, \forall j \in E. \tag{3.2.1}$$

Condition (3.2.1) ensures that $H$ and $G$ have slow growth enough to allow existence and finiteness of average transformed small and big jumps. It also ensures that the average big jump is finite, which will ensure existence of certain expected values for some versions of $H$ and $G$. Such features will be necessary in the remainder of the article as we will apply the isometry property of martingales Let $L_t^\theta$ and $\tilde{L}_t^\theta$ be stochastic processes defined by:

$$\begin{aligned} L_t^\theta = &\int_0^t \mu(\theta_{s^-}, s)ds + \int_0^t \sigma(\theta_{s^-}, s)dB_s + \int_0^t \int_{|z|\leq 1} G(z, \theta_{s^-})\bar{\psi}(\theta_{s^-}, dz, ds) \\ &+ \int_0^t \int_{|z|>1} H(z, \theta_{s^-})\psi(\theta_{s^-}, dz, ds), \end{aligned} \tag{3.2.2}$$

and

$$\begin{aligned} d\tilde{L}_t^\theta = &\int_0^t \Big[\int_{|z|\leq 1} (e^{G(z,\theta_{s^-})} - 1 - G(z, \theta_{s^-}))\nu(\theta_{s^-}, dz) + \mu(\theta_{s^-}, s) + \frac{1}{2}\sigma^2(\theta_{s^-}, s)\Big]ds \\ &+ \int_0^t \sigma(\theta_{s^-}, s)dB_s \\ &+ \int_0^t \int_{|z|\leq 1} (e^{G(z,\theta_{s^-})} - 1)\bar{\psi}(\theta_{s^-}, dz, ds) + \int_0^t \int_{|z|>1} (e^{H(z,\theta_{s^-})} - 1)\psi(\theta_{s^-}, dz, ds), \end{aligned} \tag{3.2.3}$$

respectively. The asset price process $\big(x(t)\big)_{t\in[0,T]}$ is described by the solution of a following Levy-type stochastic differential equation developed by [2, 47]:

$$\begin{cases} dx(t) = x(t^-)d\tilde{L}_t^{\theta_n}, x(T_n) = x_n, t \in [T_n, T_{n+1}), \\ x_n = \beta_n x(T_n^-, T_{n-1}, x_{n-1}), x(0) = x_0, \forall n \in I(0,\infty). \end{cases} \tag{3.2.4}$$

The solution process $x$ defined on each interval $[T_n, T_{n+1})$ takes the following form found by [2, 47]:

$$\begin{cases} x(t) = x_n \exp\Big[\int_{T_n}^t dL_s^{\theta_n}\Big], t \in [T_n, T_{n+1}) \\ x_n = \beta_n x(T_n^-, T_{n-1}, x_{n-1}), \forall n \in I(0,\infty). \end{cases} \tag{3.2.5}$$

Let $\big(\Omega, \mathbb{F}\big)$ be the reference measurable space. $(\mathbb{H}_t)_{t\in[0,T]}$, $(\mathbb{L}_t)_{t\in[0,T]}$ and $\mathbb{B}_n$ are filtration generated by the semi Markov process $\theta_t$, the Levy processes $L_s^j$, $s \in [0,t]$, $\forall j \in E = \{1,2,3,...,m\}$ and the discrete sequence $\beta_n$, respectively. We also denote $\bar{\mathbb{L}}_t = \mathbb{L}_t \vee \mathbb{B}_{n(t)}$, $\bar{\mathbb{G}}_t = \mathbb{H}_T \vee \bar{\mathbb{L}}_t$ and $\mathbb{G}_t = \mathbb{H}_t \vee \bar{\mathbb{L}}_t, \forall t \in [0, T^\star]$. Let $P$ and $Q$ be the historical probability and an equivalent martingale measures as found by [2], associated with the price process $\big(x(t)\big)_{t>0}$ defined on the measurable space $\big(\Omega, \mathbb{F}\big)$, respectively.

## 3.3 Parameter estimation via LLGMM.

### 3.3.1 Estimating equations

We recall the definition of the infinitesimal generator developed by [2].

DEFINITION 3.3.1 *Let $\mathcal{L}$ and $V$ represent the infinitesimal generator of the price process $x(t)$ solution of the SDE* (3.2.4) *and a function such that $V \in \mathcal{C}\Big[\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}, \mathbb{R}\Big]$, with $V$ continuously differentiable in the first and second variables and twice continuously differentiable function in the fourth variable. Let $s \in [T_n, T_{n+1})$ with $\theta_{T_n} = j$. We have,*

$$\begin{aligned} \mathcal{L}V(s, y_{s^-}, \theta_{s^-}, x_{s^-}) &= \frac{\partial V}{\partial s} + \frac{\partial V}{\partial y} + \big[\mu(\theta_{s^-}, s) + \frac{1}{2}\sigma^2(\theta_{s^-}, s)\big]x_{s^-}\frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2(\theta_{s^-}, s)x_{s^-}^2\frac{\partial^2 V}{\partial x^2} \\ &+ \int_{|z|\le 1}\Big[V(s, y_{s^-}, \theta_{s^-}, x_{s^-}e^{G(z,\theta_{s^-})}) - V(s, y_{s^-}, \theta_{s^-}, x_{s^-}) - G(z,\theta_{s^-})x_{s^-}\frac{\partial V}{\partial x}\Big]\nu(\theta_{s^-}, dz) \\ &+ \int_{|z|>1}\Big[V(s, y_{s^-}, \theta_{s^-}, x_{s^-}e^{H(z,\theta_{s^-})}) - V(s, y_{s^-}, \theta_{s^-}, x_{s^-})\Big]\nu(\theta_{s^-}, dz) \\ &+ \int_{z\in\mathbb{R}}\sum_{j\neq\theta_{s^-}}\lambda_{\theta_{s^-},j}(y_{s^-})\Big[V(s, y_{s^-}, j, x_{s^-}e^z) - V(s, y_{s^-}, \theta_{s^-}, x_{s^-})\Big]\bar{b}(z|\theta_{s^-}, j)dz. \end{aligned} \tag{3.3.1}$$

We establish two difference equations that are needed in the LLGMM estimation method.

LEMMA 3.3.1 *Let* $V \in \mathcal{C}\Big[\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}, \mathbb{R}\Big]$ *be continuously differentiable in the first and second variables and twice continuously differentiable function in the fourth variable.* $P^n_{M^n} = \{t_k\}_{k=0}^{M_n}, T_n = t_0 < t_1 < ... < t_{M_n} = T^-_{n+1}$ *is a partition of the time interval* $[T_n, T_{n+1})$*, where* $\theta_n = j$*. The conditional expectation and variance of* $V$*, associated with a discretized scheme of the transformed stochastic differential equation:*

$$\begin{aligned} dV(s, y_s, \theta_s, x_s) &= \mathcal{L}V(s, y_{s^-}, \theta_{s^-}, x_{s^-}) + \sigma(\theta_{s^-}, s)x_{s^-}\frac{\partial V}{\partial x}dB_s \\ &+ \int_{|z|\le 1}\Big[V(s, y_{s^-}, \theta_{s^-}, x_{s^-}e^{G(z,j)}) - V(s, y_{s^-}, \theta_{s^-}, x_{s^-})\Big]\bar{\psi}(j, dz, ds) \\ &+ \int_{|z|>1}\Big[V(s, y_{s^-}, \theta_{s^-}, x_{s^-}e^{H(z,j)}) - V(s, y_{s^-}, \theta_{s^-}, x_{s^-})\Big]\bar{\psi}(j, dz, ds), \end{aligned} \tag{3.3.2}$$

*are:*

$$E\big[\Delta V(t_{k+1}, y_{t_{k+1}}, j, x_{t_{k+1}})|\mathbb{G}_{t_k}\big] = \mathcal{L}V(t_k, y_{t_k}, j, x_{t_k})\Delta t_{k+1} \tag{3.3.3}$$

$$\begin{aligned} &E\Big[\Delta V(t_{k+1}, y_{t_{k+1}}, j, x_{t_{k+1}}) - E\big[\Delta V(t_{k+1}, y_{t_{k+1}}, j, x_{t_{k+1}})|\mathbb{G}_{t_k}\big]\Big|\mathbb{G}_{t_k}\Big]^2 \\ &= \begin{cases} \Big[x_{t_k}\text{-}\frac{\partial V}{\partial x}\sigma(j, t_k{}^-)\Big]^2\Delta t_{k+1} + \int_{|z|\le 1}\big[V(t_k, y_{t_k}, j, x_{t_k}e^{G(z,j)}) - V(t_k, y_{t_k}, j, x_{t_k})\big]^2\nu(j, dz)\Delta t_{k+1} \\ + \int_{|z|>1}\big[V(t_k, y_{t_k}, j, x_{t_k}e^{H(z,j)}) - V(t_k, y_{t_k}, j, x_{t_k})\big]^2\nu(j, dz)\Delta t_{k+1}. \end{cases} \end{aligned} \tag{3.3.4}$$

*Proof.* We apply Euler-Maruyama discretization process as formulated by [45], to the transformed Levy-type stochastic differential equation (3.3.2) and obtain:

$$\begin{aligned} \Delta V(t_{k+1}, y_{t_{k+1}}, j, x_{t_{k+1}}) &= \mathcal{L}V(t_k, y_{t_k}, j, x_{t_k})\Delta t_k + \sigma(j, t_k)x_{t_k}\frac{\partial V}{\partial x}\Delta B_{t_{k+1}} \\ &+ \int_{|z|\le 1}\Big[V(t_k, y_{t_k}, j, x_{t_k}e^{G(z,j)}) - V(t_k, y_{t_k}, j, x_{t_k})\Big]\bar{\psi}(j, dz, \Delta t_k) \\ &+ \int_{|z|>1}\Big[V(t_k, y_{t_k}, j, x_{t_k}e^{H(z,j)}) - V(t_k, y_{t_k}, j, x_{t_k})\Big]\bar{\psi}(j, dz, \Delta t_k) \text{ at } t_{k+1} \in P^n_{M^n}. \end{aligned} \tag{3.3.5}$$

Now we apply the conditional mean to the numerical scheme (3.3.5) and have:

$$E\big[\Delta V(t_{k+1}, y_{t_{k+1}}, j, x_{t_{k+1}})|\mathbb{G}_{t_k}\big] = \mathcal{L}V(t_k, y_{t_k}, j, x_{t_k})\Delta t_{k+1}, \tag{3.3.6}$$

$$
\begin{aligned}
&E\Big(\Delta V(t_{k+1},y_{t_{k+1}},j,x_{t_{k+1}})-E\big(\Delta V(t_{k+1},y_{t_{k+1}},j,x_{t_{k+1}})|\mathbb{G}_{t_k}\big)\Big|\mathbb{G}_{t_k}\Big)^2=E\Big[\Big[x_{t_k}\frac{\partial V}{\partial x}\sigma(j,t_k)\Delta B_{t_{k+1}}\Big]^2\Big|\mathbb{G}_{t_k}\Big]\\
&+E\Big[\Big[\int_{|z|\le 1}\big[V(t_k,y_{t_k},j,x_{t_k}e^{G(z,j)})-V(t_k,y_{t_k},j,x_{t_k})\big]\bar{\psi}(j,dz,\Delta t_{k+1})\Big]^2\Big|\mathbb{G}_{t_k}\Big]\\
&+E\Big[\Big[\int_{|z|>1}\big[V(t_k,y_{t_k},j,x_{t_k}e^{H(z,j)})-V(t_k,y_{t_k},j,x_{t_k})\big]\bar{\psi}(j,dz,\Delta t_{k+1})\Big]^2\Big|\mathbb{G}_{t_k}\Big]\\
&+2E\Big[\Big[x_{t_k}\frac{\partial V}{\partial x}\sigma(j,t_k)\Delta B_{t_{k+1}}\int_{|z|\le 1}\big[V(t_k,y_{t_k},j,x_{t_k}e^{G(z,j)})-V(t_k,y_{t_k},j,x_{t_k})\big]\bar{\psi}(j,dz,\Delta t_{k+1})\Big]\Big|\mathbb{G}_{t_k}\Big]\\
&+2E\Big[\Big[x_{t_k}\frac{\partial V}{\partial x}\sigma(j,t_k)\Delta B_{t_{k+1}}\int_{|z|>1}\big[V(t_k,y_{t_k},j,x_{t_k}e^{H(z,j)})-V(t_k,y_{t_k},j,x_{t_k})\big]\bar{\psi}(j,dz,\Delta t_{k+1})\Big]\Big|\mathbb{G}_{t_k}\Big].
\end{aligned}
\tag{3.3.7}
$$

$\bar{\psi}$ and $B$ are independent martingales. Hence, the products involving both have zero expectations. We also note that products involving the compensated Poisson measure $\bar{\psi}$ for large and small jumps vanish as they never jump, simultaneously. From Ito isometry, (3.3.7) becomes:

$$
\begin{aligned}
&E\Big(\Delta V(t_{k+1},y_{t_{k+1}},j,x_{t_{k+1}})-E\big(\Delta V(t_{k+1},y_{t_{k+1}},j,x_{t_{k+1}})|\mathbb{G}_{t_k}\big)\Big|\mathbb{G}_{t_k}\Big)^2=\Big[x_{t_k}\frac{\partial V}{\partial x}\sigma(j,t_k)\Big]^2\Delta t_{k+1}\\
&+\int_{|z|\le 1}\big[V(t_k,y_{t_k},j,x_{t_k}e^{G(z,j)})-V(t_k,y_{t_k},j,x_{t_k})\big]^2\nu(j,dz)\Delta t_{k+1}\\
&+\int_{|z|>1}\big[V(t_k,y_{t_k},j,x_{t_k}e^{H(z,j)})-V(t_k,y_{t_k},j,x_{t_k})\big]^2\nu(j,dz)\Delta t_{k+1}
\end{aligned}
\tag{3.3.8}
$$

This establishes the results. □

The following remark describes the jump integral estimation problem.

REMARK 3.3.1 (3.3.3) and (3.3.4) form the building blocks of the estimation procedure that is utilized to estimate the drift and diffusion coefficients. It is therefore possible that due to roundoff, discretization and computational errors, $\hat{\sigma}^2$ have negative values. Hence, it is critically important to chose an efficient numerical estimation methods of the Levy integrals. We chose to estimate Levy integrals via Monte Carlo integration method. We first note that compound Poisson processes have independent and identically distributed (iid) jump sizes. Hence, jumps sizes of Levy integrals are iid. We can apply the following monte carlo estimation scheme defined in [61]:

$$
\int_{z\in\mathbb{R}} g(z)\nu(j,dz)=E^{\nu(j,)}\big[g(z)\big]\approx\frac{1}{n(j)}\sum_{k=1}^{n(j)}g(z_k),
\tag{3.3.9}
$$

where $g$ is a $\nu(j,)-$integrable real valued function, and $(z_i)_{i=1}^{n(j)}$ is an iid sample of Levy jump sizes when the market is in state $\theta_t=j$. $n(j)$ denotes the number of Levy jump corresponding to the $j-th$ regime.

In the following Lemma, we present a particular case of interest along with an explicit formula for parameter estimates and a recursive formula for price simulation updates.

LEMMA 3.3.2 *(i) If $H(z,j) = G(z,j) = z, \forall z \in \mathbb{R}, \forall j \in E$ and $V(t, y_t, j, x(t)) = \ln(x(t))$, then the transformed stochastic Levy type differential equation, and the conditional expectations of Euler-Maruyama type discretization scheme in Lemma 3.3.1 reduce to:*

$$\begin{aligned} dV(s, y_s, \theta_s, x_s) &= \mathcal{L}V(s, y_{s^-}, \theta_{s^-}, x_{s^-}) + \sigma(\theta_{s^-}, s) x_{s^-} \frac{\partial V}{\partial x} dB_s \\ &+ \int_{|z|\leq 1} \Big[ V(s, y_{s^-}, \theta_{s^-}, x_{s^-} e^{G(z,j)}) - V(s, y_{s^-}, \theta_{s^-}, x_{s^-}) \Big] \bar{\psi}(j, dz, ds) \\ &+ \int_{|z|>1} \Big[ V(s, y_{s^-}, \theta_{s^-}, x_{s^-} e^{H(z,j)}) - V(s, y_{s^-}, \theta_{s^-}, x_{s^-}) \Big] \bar{\psi}(j, dz, ds), \end{aligned} \tag{3.3.10}$$

$$E\Big[\Delta \ln(x_{t_{k+1}}) | \mathcal{G}_{t_k}\Big] = \Big[ \int_{|z|>1} z\nu(j, dz) + \mu(j, t_k) \Big] \Delta t_{k+1}, \tag{3.3.11}$$

$$E\Bigg[\Big(\Delta \ln(x_{t_{k+1}}) - E(\Delta \ln(x_{t_{k+1}}) | \mathcal{G}_{t_k})\Big) \Bigg| \mathbb{G}_{t_k}\Bigg]^2 = \sigma^2(j, t_k) \Delta t_{k+1} + \int_{z\in\mathbb{R}} z^2 \nu(j, dz) \Delta t_{k+1}. \tag{3.3.12}$$

*(ii) At time $t_k$, we consider the subpartition $P^{n,k}_{M_n,m_k} = \{t_{k-m_k}, t_{k-m_k+1}, ..., t_{k-1}\}$ of $P^n_{M_n}$ consisting of the past $m_k$ consecutive data values of the price process $x_t$. We assume $\Delta t_k = \Delta t$, $\mu(j,t) = \mu(j)$ and $\sigma(j,t) = \sigma(j)$. We denote $\hat{\mu}^j_{t_k,m_k}$ and $\hat{\sigma}^j_{t_k,m_k}$ the estimates of $\mu(j)$ and $\sigma(j)$ relative to the subpartition $P^{n,k}_{M_n,m_k}$, respectively. Explicit formulas for $\hat{\mu}^j_{t_k,m_k}$ and $\hat{\sigma}^j_{t_k,m_k}$ can be expressed as follows:*

$$\hat{\mu}_{t_k,m_k} = \frac{1}{m_k \Delta t} \sum_{i=k-m_k}^{k-1} E(\Delta \ln(x_{t_i}) | \mathbb{G}_{t_{i-1}}) - \frac{1}{n(j)} \sum_{k=1}^{n(j)} z_k 1_{|z_k|>1} \tag{3.3.13}$$

$$(\hat{\sigma}^j_{t_k,m_k})^2 = \frac{1}{m_k - 1} \sum_{i=k-m_k}^{k-1} E\big(\Delta \ln(x_{t_i}) - E(\Delta \ln(x_{t_i}) | \mathbb{G}_{t_{i-1}}) \big| \mathbb{G}_{t_{i-1}}\big)^2 - \frac{m_k}{m_k - 1} \sum_{i=n-m_k}^{k-1} z_i \Delta t_k.. \tag{3.3.14}$$

*(iii) We denote $\hat{x}_{t_k} = E\big[x_{t_k} \big| \mathbb{G}_{t_{k-1}}\big]$ the estimated conditional mean asset price. The following recurrence relation holds:*

$$\hat{x}_{t_{k+1}} = \hat{x}_{t_k} \exp\Big[\hat{\mu}^j_{t_k,m_k} \Delta t_{k+1} + \frac{1}{2} (\hat{\sigma}^j_{t_k,m_k})^2 \Delta t_{k+1} + \frac{1}{n(j)} \sum_{l=1}^{n(j)} \big[e^{z_l} - 1 - z_l 1_{|z_l|\leq 1}\big]\Big]. \tag{3.3.15}$$

*Proof.* Under the assumption $H(z,j) = G(z,j) = z, \forall z \in \mathbb{R}, \forall j \in E$, and using $V(t, y_t, j, x(t)) = \ln(x(t))$, and applying Lemma 3.3.1, (3.3.3) and (3.3.4) reduce to (3.3.11) and (3.3.12), respectively. (ii) is a direct consequence of part (i). Summing up (3.3.13) and (3.3.14) over the subpartition $P^{n,k}_{M_n,m_k}$, we obtain:

$$\sum_{i=k-m_k}^{k-1} E\Big[\Delta \ln(x_{t_k})|\mathcal{G}_{t_{k-1}}\Big] = m_k\Big[\frac{1}{n(j)}\sum_{k=1}^{n(j)} z_k 1_{|z_k|>1} + m_k\mu(j)\Big]\Delta t \tag{3.3.16}$$

$$\sum_{i=k-m_k}^{k-1} E\Big[\Big(\Delta \ln(x_{t_k}) - E(\Delta \ln(x_{t_k})|\mathcal{G}_{t_{k-1}})\Big)\Big|\mathbb{G}_{t_{k-1}}\Big]^2 = m_k\sigma^2(j)\Delta t + m_k\int_{z\in\mathbb{R}} z^2\nu(j,dz)\Delta t. \tag{3.3.17}$$

Solutions of algebraic equations in (3.3.16) and (3.3.17) establish (3.3.13) and (3.3.14), respectively. For (iii), we consider $n \in I(1,M)$ and $t_{n-1}, t_n$, points of the partition $P_M$ such that $T_{k-1} < t_{n-1} < t_n < T_k$ where $\theta_{T_{k-1}} = j$ for some $k \in I(1,\infty)$. By Levy Kintchine formula [55], we have:

$$\begin{aligned}
E\big[x_{t_n}|\mathcal{G}_{t_{n-1}}\big] &= E\Big[x_{t_{n-1}}\exp\big[\int_{t_{n-1}}^{t_n} dL_s^j\big]\Big|\mathcal{G}_{t_{n-1}}\Big] \\
&= E\Big[x_{t_{n-1}}\exp\big[\mu(j)\Delta t + \sigma(j)\Delta B_n + \int_{t_{n-1}}^{t_n}\int_{|z|>1} z\psi(j,dz,ds) + \int_{t_{n-1}}^{t_n}\int_{|z|\le 1} z\bar{\psi}(j,dz,dt)\big]\Big|\mathcal{G}_{t_{n-1}}\Big] \\
&= x_{t_{n-1}}\exp\Big[\big[\mu(j) + \frac{1}{2}\sigma^2(j) + \int_{z\in\mathbb{R}}[e^z - 1 - z1_{|z|\le 1}]\nu(j,dz)\big]\Delta t\Big].
\end{aligned}$$

Hence, at each time step $t_n$, the simulated conditional mean observation is computed recursively as follows:

$$\hat{x}_{t_n} = \hat{x}_{t_{n-1}}\exp\Big[\hat{\mu}(j)\Delta t + \frac{1}{2}\hat{\sigma}^2(j)\Delta t + \frac{\Delta t}{n(j)}\sum_{k=1}^{n(j)}\big[e^{z_k} - 1 - z_k 1_{|z_k|<1}\big]\Big], \tag{3.3.18}$$

where $n(j)$ is the size of the data when the market is in regime j, hence proving the lemma. □

### 3.3.2 Parameter estimation for three real data.

We assume the regime switching times observable. Although semi Markov jumps $\beta$, are not expected to stand out by their size, large jumps have empirically been associated with local structural changes through clustering [10] and will therefore be chosen as jump times. These jump times could be used to estimate the sojourn time parameters of the semi Markov process, however, we focus on estimating the price jump distribution parameters and the parameters of the Levy distributions in between jumps. The IBM, Bank of America Corporation and BNP Paris Bas data were collected over a period of 20 years and six months (daily except for the weekends and market holidays) from January 2nd 1994 to July 11th 2014. Structural changes were identified in each data set and the corresponding jumps were considered semi Markov market price jumps. Based on Lemma 3.3.2, we estimate the parameters of model (3.2.4) between jumps using the LLGMM algorithm developed in [56]. The fit of the LLGMM is presented by the first columns of Figures 1, 2 and 3. Another feature of the LLGMM is the more obvious randomness in the volatility as opposed to that of Garch(1,1). The same conclusion is reached in the context of semi Markov exponential Levy asset prices as shown in the second columns of Figures 1, 2 and 3. As noted in Remark 3.3.1, jumps are independent and so are log jumps. Estimation of the parameters of the distribution of $\beta$ could be performed by maximum likelihood.

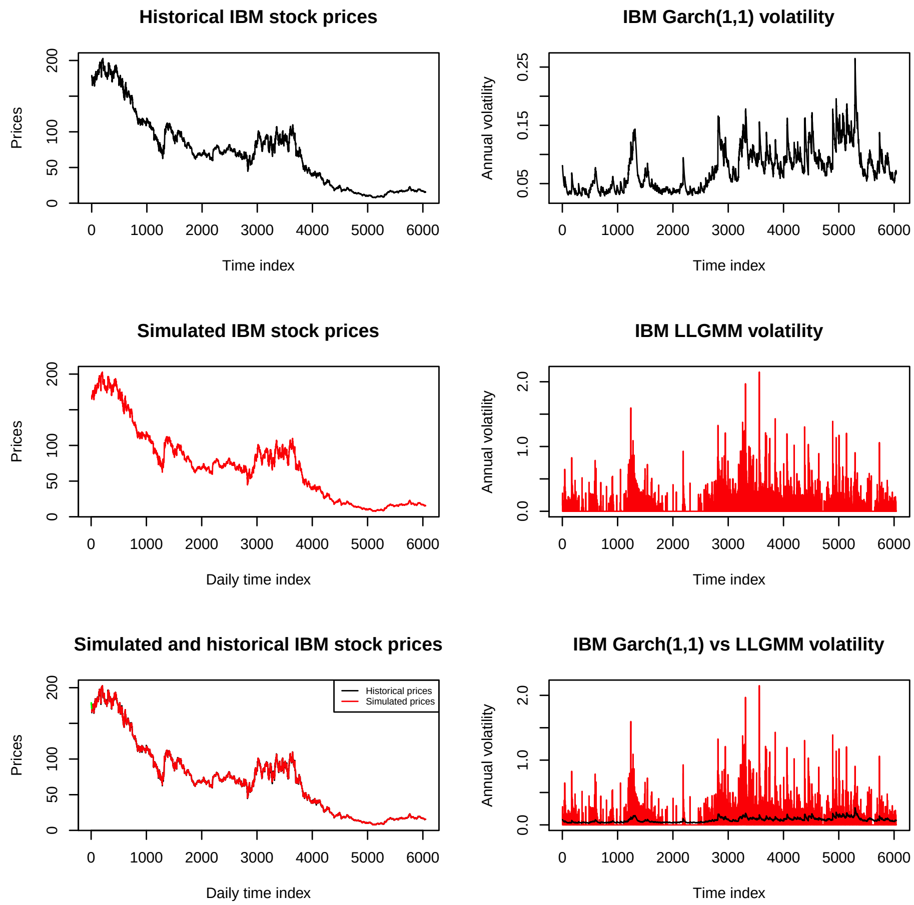


**Figure 1.**: The first column exhibits the fit of the LLGMM simulated IBM stock prices against historical prices. The second column illustrates a comparison of LLGMM and Garch(1,1) annual volatilities.

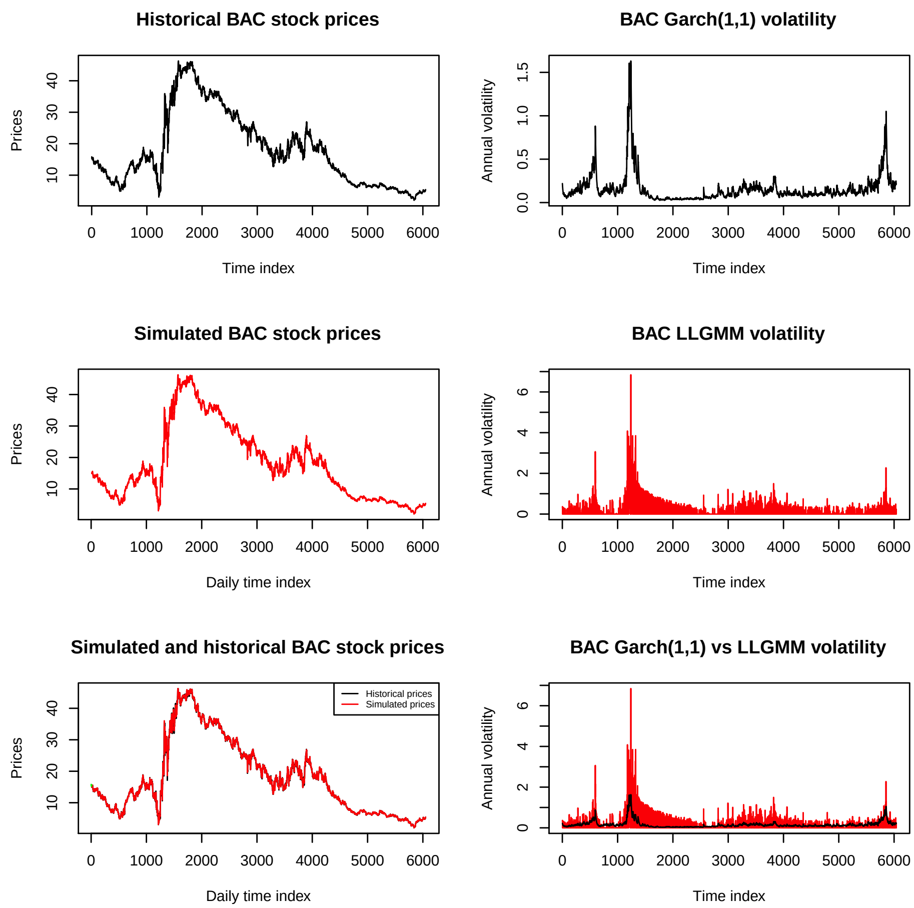


**Figure 2.**: The first column shows the fit for the LLGMM simulated BAC stock prices against historical prices. The second column exhibits the performance of the LLGMM and Garch(1,1) annual volatilities.

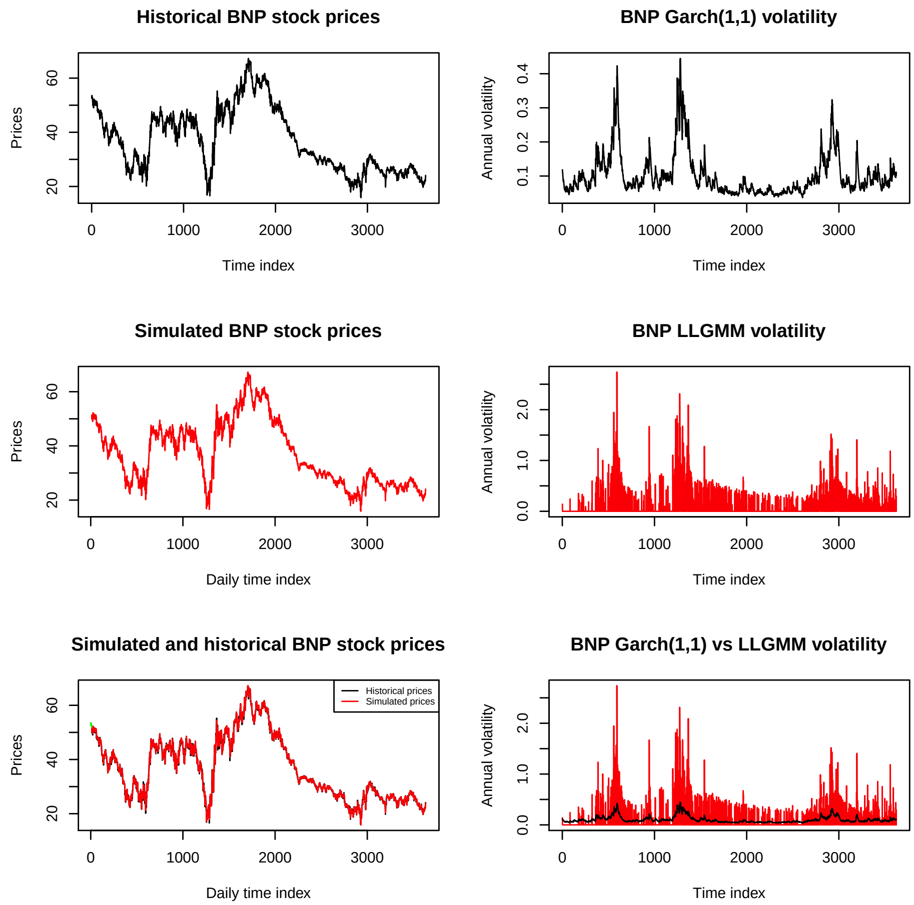


**Figure 3.**: The first column exhibits the fit of the LLGMM stock of Bank National de Paris (BNP) against historical prices. The second column illustrates a comparison of LLGMM and Garch(1,1) annual volatilities.

**Table 1**: IBM: Overview of a few LLGMM parameter estimates and simulated values

| Index: $t_k$ | Data: $y_{t_k}$ | Estimate: $\hat{y}_{\hat{m}_k}$ | Error: $\|y_{t_k} - \hat{y}_{\hat{m}_k}\|$ | Volatility: $s_{\hat{m}_k}$ | Drift: $\hat{\mu}(j)$ | Sample: $\hat{m}_k$: |
|---|---|---|---|---|---|---|
| 11 | 167.73 | 169.19 | 0.70 | 0.000033 | -0.013 | 2 |
| 12 | 169.72 | 168.82 | 0.90 | 0.000404 | -0.002 | 2 |
| 13 | 164.90 | 166.98 | 2.08 | 0.000434 | -0.011 | 3 |
| 14 | 165.45 | 165.71 | 0.27 | 0.000257 | -0.008 | 5 |
| 15 | 167.19 | 166.87 | 0.32 | 0.000026 | 0.007 | 2 |
| 16 | 169.02 | 168.68 | 0.35 | 0.000000 | 0.011 | 2 |
| 17 | 169.35 | 168.97 | 0.38 | 0.000240 | 0.002 | 6 |
| 18 | 169.55 | 169.77 | 0.22 | 0.000029 | 0.005 | 3 |
| 19 | 168.03 | 169.12 | 1.08 | 0.000052 | -0.004 | 2 |
| 6043 | 15.73 | 15.60 | 0.12 | 0.000259 | 0.009 | 2 |
| 6044 | 15.59 | 15.59 | 0.00 | 0.000170 | -0.001 | 296 |
| 6045 | 15.65 | 15.64 | 0.01 | 0.000156 | 0.003 | 4 |
| 6046 | 15.80 | 15.75 | 0.06 | 0.000019 | 0.007 | 2 |
| 6047 | 15.71 | 15.72 | 0.01 | 0.000116 | -0.002 | 100 |
| 6048 | 15.75 | 15.75 | 0.00 | 0.000136 | 0.002 | 14 |
| 6049 | 15.57 | 15.67 | 0.10 | 0.000049 | -0.005 | 3 |
| 6050 | 15.43 | 15.51 | 0.08 | 0.000003 | -0.010 | 2 |

**Table 2**: BAC: Overview of a few LLGMM parameter estimates and simulated values.

| Index: $t_k$ | Data: $y_{t_k}$ | Estimate: $\hat{y}_{\hat{m}_k}$ | Error:$\lvert y_{t_k} - \hat{y}_{\hat{m}_k}\rvert$ | Volatility:$s_{\hat{m}_k}$ | Drift: $\hat{\mu}(j)$ | Sample: $\hat{m}_k$: |
|---|---|---|---|---|---|---|
| 11 | 14.98 | 15.35 | 0.22 | 0.000427 | -0.018 | 2 |
| 12 | 15.04 | 15.13 | 0.09 | 0.000684 | -0.015 | 2 |
| 13 | 14.98 | 14.97 | 0.01 | 0.000380 | -0.011 | 3 |
| 14 | 15.05 | 14.99 | 0.06 | 0.000023 | 0.002 | 3 |
| 15 | 15.05 | 15.03 | 0.02 | 0.000011 | 0.002 | 2 |
| 16 | 15.35 | 15.18 | 0.17 | 0.000202 | 0.010 | 2 |
| 17 | 15.37 | 15.34 | 0.03 | 0.000177 | 0.011 | 2 |
| 18 | 15.35 | 15.35 | 0.01 | 0.000127 | 0.000 | 15 |
| 19 | 15.23 | 15.27 | 0.05 | 0.000025 | -0.005 | 2 |
| 6043 | 5.10 | 5.04 | 0.05 | 0.000000 | 0.029 | 2 |
| 6044 | 5.14 | 5.14 | 0.00 | 0.000207 | 0.019 | 2 |
| 6045 | 5.24 | 5.24 | 0.00 | 0.000104 | 0.019 | 3 |
| 6046 | 5.24 | 5.24 | 0.00 | 0.000650 | 0.000 | 4819 |
| 6047 | 5.21 | 5.23 | 0.02 | 0.000015 | -0.003 | 2 |
| 6048 | 5.33 | 5.31 | 0.02 | 0.000194 | 0.015 | 7 |
| 6049 | 5.44 | 5.42 | 0.02 | 0.000000 | 0.022 | 2 |
| 6050 | 5.40 | 5.42 | 0.02 | 0.000324 | -0.001 | 2156 |

**Table 3**: BNP: Overview of a few LLGMM parameter estimates and simulated values.

| Index: $t_k$ | Data: $y_{t_k}$ | Estimate: $\hat{y}_{\hat{m}_k}$ | Error: $\lvert y_{t_k} - \hat{y}_{\hat{m}_k} \rvert$ | Volatility: $s_{\hat{m}_k}$ | Drift: $\hat{\mu}(j)$ | Sample: $\hat{m}_k$ |
|---|---|---|---|---|---|---|
| 11 | 51.40 | 51.60 | 0.09 | 0.000078 | -0.008 | 3 |
| 12 | 50.89 | 51.01 | 0.12 | 0.000014 | -0.012 | 3 |
| 13 | 49.89 | 50.28 | 0.39 | 0.000049 | -0.015 | 2 |
| 14 | 51.00 | 50.36 | 0.64 | 0.000883 | 0.001 | 2 |
| 15 | 50.22 | 50.22 | 0.00 | 0.000155 | -0.003 | 14 |
| 16 | 49.99 | 49.99 | 0.00 | 0.000111 | -0.005 | 14 |
| 17 | 50.16 | 50.08 | 0.08 | 0.000252 | 0.001 | 4 |
| 18 | 50.38 | 50.30 | 0.08 | 0.000000 | 0.004 | 2 |
| 19 | 50.49 | 50.49 | 0.00 | 0.000001 | 0.003 | 3 |
| 3626 | 21.93 | 22.05 | 0.11 | 0.000202 | -0.008 | 2 |
| 3627 | 21.73 | 21.87 | 0.14 | 0.000102 | -0.008 | 3 |
| 3628 | 22.56 | 22.19 | 0.37 | 0.001114 | 0.014 | 2 |
| 3629 | 22.42 | 22.36 | 0.06 | 0.000696 | 0.007 | 3 |
| 3630 | 22.14 | 22.16 | 0.01 | 0.000017 | -0.009 | 2 |
| 3631 | 22.92 | 22.46 | 0.46 | 0.000697 | 0.013 | 4 |
| 3632 | 23.43 | 23.11 | 0.32 | 0.000082 | 0.028 | 2 |
| 3633 | 24.10 | 23.78 | 0.32 | 0.000041 | 0.028 | 3 |

## 3.4 Option pricing

We develop option pricing based on Carr and Madan method and the FST method. We first recall the following definitions from [2] useful in the remainder of this section:

DEFINITION 3.4.1 *Let $S$ denotes a continuous function defined on $\mathbb{R}^+ \times \mathbb{R}^+$ into $\mathbb{R}$ representing the payoff of a contingent claim; $Q$ is a risk neutral probability measure of the price process $x$ defined by* (3.2.4) *with respect to the historical probability measure $P$; $K$ is a nonnegative real number denoting the strike price of a derivative contract with maturity $T$; $x_T$ denotes the asset price value at maturity; $C$ is the $Q-$risk neutral option price function defined on $[0,T]\times\mathbb{R}^+\times\mathbb{R}^+\times[0,T]\times E\times\mathbb{R}^+$ into $\mathbb{R}^+$ and $V$ denotes the discounted option price process defined by $V(t,T,K,y_t,\theta_t,x_t) = e^{-\int_0^t r_{s-}ds}C(t,T,K,y_t,\theta_t,x_t)$. The Fourier transform and the inverse Fourier transform of an integrable function $f$, are interchangeably denoted $\mathcal{F}(f)$ or $\hat{f}$ and $\mathcal{F}^{-1}(f)$ or $\check{f}$, respectively. Let $N(t,A,B)$ be a stochastic process defined on $[0,T]\times\mathbb{B}(\mathbb{R})\times P(E^2)$ into $[0,\infty)$ as:*

$$N(t,A,B) = \sum_{n\geq 1} 1_{\left(t\geq T_n, \ln(\beta_n)\in A, (\theta_{n-1},\theta_n)\in B\right)} \tag{3.4.1}$$

*and $N(t,A,B)$ stands for the number of regime switches in $B$ with corresponding log price jumps $\ln(\beta_n) \in A$ by time $t$. The compensators $\gamma(t,A,\{(i,j)\}) = \int_0^t\int_{z\in A}\bar{b}(z|i,j)\lambda_{i,j}(y_s)dzds$ of $N(t,A,\{(i,j)\})$ are derived in [2].*

We model asset prices with the semi Markov switching exponential Levy process in (3.2.5), where $L^\theta$ defined in (3.2.2) is based on $H(z,j) = G(z,j) = z$. Options are priced based on the risk neutral theory. The martingale probability measure chosen for pricing purpose is the conditional minimum entropy martingale measure (CMEMM) $P^{\alpha^\star}$ with density process expressed in [2] as the following Esscher transform:

$$\frac{d\bar{P}^{\alpha^\star}}{dP}\Big|_{\mathbb{H}_T\vee\mathbb{L}_t} = \left[\prod_{i=1}^{n}\frac{e^{\alpha_i^\star\beta_i}}{E\left(e^{\alpha_i^\star\beta_i}\big|\theta_i,\theta_{i-1}\right)}\right]\frac{e^{\int_0^t\alpha_s^\star dL_s}}{E\left(e^{\int_0^t\alpha_s^\star dL_s}\big|\mathbb{H}_T\right)}, \forall t\in[T_n,T_{n+1}).$$

From [2], for $\forall t \in [T_n, T_{n+1}), \forall n \in I(1,\infty)$, the risk neutral conditions satisfied by the Esscher parameter process $(\alpha_t^\star)_{t\in[0,T]}$, are as follows:

$$\begin{cases} \mu(\theta_n) + \alpha_t^\star\sigma^2(\theta_n) + \int_{|z|\leq 1}\left[G(z,\theta_n)e^{\alpha_t^\star G(z,\theta_n)} - G(z,\theta_n)\right]\nu(\theta_n,dz) \\ + \int_{|z|>1}H(z,\theta_n)\left(e^{\alpha_t^\star H(z,\theta_n)}-1\right)\nu(\theta_n,dz) = r(t), \forall t\in(T_n,T_{n+1}) \\ \left[E^{P^{\alpha^\star}}[\beta_n|\theta_n,\theta_{n-1}]-1\right] = \left[E\left[\frac{\beta_n e^{\alpha_t^\star\beta_n}}{E\left[e^{\alpha_t^\star\beta_n}|\theta_n,\theta_{n-1}\right]}\big|\theta_n,\theta_{n-1}\right]-1\right] = 0 \text{ if } t = T_n. \end{cases} \tag{3.4.2}$$

The risk neutral pricing formula for European call options is described by:

$$\begin{aligned} C(t,T,K,y_t,j,x_t) &= E^{\bar{P}^{\alpha}}\big(e^{-\int_t^T r_s ds}(x_T-K)^+|\mathbb{H}_t \vee \bar{\mathbb{L}}_t\big) \\ &= E^{\bar{P}^{\alpha}}\big(e^{-\int_t^T r_s ds}(e^{\log(x_T)}-e^k)^+|y_t,\theta_t=j,x_t\big) \\ &= E^{\bar{P}^{\alpha}}\big(e^{k-\int_t^T r_s ds}(e^{k\left(\frac{\log(x_T)}{k}-1\right)}-1)^+|y_t,\theta_t=j,x_t\big), \end{aligned} \tag{3.4.3}$$

where $k=\ln K$. Given the known closed form expression of the characteristic function of log prices [2], one can apply Carr and Madan formula [11]:

$$\Upsilon(t,T,u,y_0,j,x_0) = \frac{e^{-\int_t^T r_s ds}}{(\alpha+iu)(1+\alpha+iu)}\Psi\big(u-i(1+\alpha),t,y_0,j,x_0\big), \tag{3.4.4}$$

where $\Upsilon$ is the characteristic function of the modified option price

$$\tilde{C}(t,T,k,y_t,j,x_t) = e^{\alpha k}E^{\bar{P}^{\alpha}}\Big[e^{-\int_t^T r_s ds}(e^{\ln(x_T)}-e^k)^+|y_t,\theta_t=j,x_t\Big]$$

$\Psi$ is the characteristics function of the log prices developed in [2]:

$$\begin{aligned} \Psi(t,T,u,x,j,y) =& E\big[e^{iu\ln(x_t)}\big|\theta_0=j,y_0=y,x_0=x\big] \\ &= \exp(iu\ln(x_t))\big\langle \exp\big(\int_y^{t+y} M(u,\eta)d\eta\big)e_j,\mathbf{1}\big\rangle, \end{aligned} \tag{3.4.5}$$

and $M$ is an $m\times m$ matrix function with components $M_{qp}$ are defined by:

$$M_{qp}(u,y) = \begin{cases} iu\mu(q)-\frac{1}{2}\sigma^2(q)u^2+\int_{|z|\le 1}\big[e^{iuG(z,q)}-1-iuG(z,q)\big]\nu(q,dz) \\ +\int_{|z|>1}\big[e^{iuH(z,q)}-1\big]\nu(q,dz)+\lambda_{q,q}(y),\ \text{if } p=q \\ \lambda_{q,p}(y)\int_{z\in\mathbb{R}} e^{iuz}\bar{b}(z|q,p)dz,\ \text{otherwise.} \end{cases} \tag{3.4.6}$$

$M$ is assumed to satisfies the Lie bracket condition [49]

$$[M(u,t_1),M(u,t_2)] = 0, \forall t_1,t_2\in\mathbb{R}_+. \tag{3.4.7}$$

$\mu$ satisfies the martingale condition in (3.4.2). We next explore the effects of parameters on option prices and implied volatilities. By developing a closed form solution to a PIDE which will allow us to apply the FST algorithm, hence paving the way for the pricing of exotic and American options [60] in the context of asset price model (3.2.4). The option price is the inverse Fourier transform of $\Upsilon$:

$$C(t,T,k,y_t,j,x_t) = \frac{e^{-\alpha k}}{2\pi}\int_{\mathbb{R}} e^{-iuk}\Upsilon(t,T,u,y,j,x)du. \tag{3.4.8}$$

The Fourier time stepping method from [60], is an option pricing method of vanilla and exotic option contracts based on the inverse Fourier transform. The FST has been used to price options when the market is subjected to Markov regime changes, [43, 53, 60]. We investigate an application of the FST method to semi Markov regimes with jumps at regime changes. We assume in this subsection that the asset price process in (3.2.5) is defined under a $\big(P, \mathbb{H} \vee \bar{\mathbb{L}}\big)-$ equivalent martingale measure $Q$.

LEMMA 3.4.1 *Let $S$ be a random variable representing the payoff of a general European style contingent claim with maturity $T$ and strike price $K$ in Definition 3.4.1; let $Q$ and $\mathcal{L}$ be the risk neutral measure and the infinitesimal generator defined in Definitions 3.4.1 and 3.3.1. Let $C$ be the $Q-$risk neutral option price of a contingent claim.*

*(i) Then, for $\theta_{s^-} = j$, the $Q-$risk neutral option price $C$ of a European contingent claim with maturity $T$, strike price $K$ and payoff $S$ satisfies the following PIDE:*

$$\begin{cases} \mathcal{L}V(s,T,K,y_{s^-},j,x_{s^-}) = 0 \\ V(T,T,K,y_T,\theta_T,x_T) = e^{\int_0^T r_u du} S(x_T,K) \end{cases} \tag{3.4.9}$$

*For any $j \in I(1,m)$, and $V(s,T,K,y_{s^-},j,x_{s^-}) = e^{-\int_0^s r_u du} C(s,T,K,y_{s^-},j,x_{s^-})$.*

*(ii) The vector solution $\mathbf{C}(s,T,K,y,x) = \big(C(s,T,K,y,1,x), C(s,T,K,y,2,x), ..., C(s,T,K,y,m,x)\big)$ has Fourier Transform (with respect to $\ln(x)$):*

$$\hat{\mathbf{C}}(s,T,K,y,w) = \Big[\hat{C}(s,T,K,y,1,w), \hat{C}(s,T,K,y,2,w), ..., \hat{C}(s,Y,K,y,m,w)\Big] = \exp\Big[\int_y^{T-t+y} M(u,s)ds\Big].\big[\hat{S}(w)\mathbf{1}\big], \tag{3.4.10}$$

*and the Fourier transform of option prices in individual regimes are:*

$$\hat{C}(s,T,K,y,j,w) = \big\langle \exp\Big[\int_y^{T-t+y} M(u,s)ds\Big].\big[\hat{S}(w).e_j\big], \mathbf{1}\big\rangle. \tag{3.4.11}$$

*Proof.* (i) is a direct consequence of the PIDE derived in [2] with $\mu$, $H$ and $G$ replaced by $\int_{|z|\leq 1}\big[e^{G(z,j)} - 1 - G(z,j)\big]\nu(j,dz) + \frac{1}{2}\sigma^2(j)$, $e^{G(z,j)} - 1$ and $e^{H(z,j)} - 1$. We prove (ii) using properties of the Fourier transform. Using the change of variable $\bar{x} = \ln(x)$, assuming $S, C \in \mathbb{L}^1(\Omega, \bar{\mathbb{G}}, Q)$ with respect to the first and the sixth variables, respectively, and using the properties $\mathcal{F}(\frac{\partial^n}{\partial \bar{x}^n}C)(t,T,K,y,j,w) = (iw)^n\hat{C}(t,T,K,y,j,w)$, $\mathcal{F}\big(C(t,T,K,y,j,xe^z)\big)(w) = e^{iwz}\hat{C}(t,T,K,y,j,w)$, along with linearity of the Fourier transform lead to:

$$\begin{cases} \frac{\partial \hat{C}}{\partial s} + \frac{\partial \hat{C}}{\partial y} + iw\mu(j) + \frac{1}{2}\sigma^2(j) + \Big[\int_{|z|\leq 1}\big[e^{iwG(z,j)} - 1 - iwG(z,j)\big]\nu(j,dz) \\ + \int_{|z|>1}\big[e^{iwH(z,j)} - 1\big]\nu(j,dz) \\ +\lambda_{j,j}(y_{s^-}) - r(s)\Big]\hat{C}(s,T,K,y_{s^-},j,w) \\ + \sum_{i\neq j}\Big[\big[\int_{z\in\mathbb{R}}\lambda_{j,i}(y_s)e^{iwz}\bar{b}(z|j,i)dz\big]\hat{C}(s,T,K,y_{s^-},i,w)\Big] = 0, \\ \hat{C}(T,T,K,y_T,\theta_T,x_T) = \hat{S}(w,K) \end{cases} \tag{3.4.12}$$

In vector form, (3.4.12) becomes:

$$\begin{cases} \frac{\partial \hat{\mathbf{C}}}{\partial s} + \frac{\partial \hat{\mathbf{C}}}{\partial y} + M(w,y)\hat{\mathbf{C}}(s,y) = 0 \\ \hat{\mathbf{C}}(T,T,K,y_T,w) = \hat{S}(w,K).\mathbf{1} \end{cases}$$

With the matrix $\big(M(w,y)\big)_{m\times m}$ defined by its elements:

$$M_{p,q}(w,y) = \begin{cases} iw\mu(p) - \frac{1}{2}w^2\sigma^2(p) + \int_{|z|\leq 1}\big[e^{iwG(z,p)} - 1 - iwe^{G(z,p)}\big]\nu(p,dz) & \\ + \int_{|z|>1}\big[e^{iwH(z,p)} - 1\big]\nu(p,dz) + \lambda_{p,p}(y) - r(p) & \text{If } p = q \\ \lambda_{q,p}(y)\int_{z\in\mathbb{R}} e^{iwz}\bar{b}(z|q,p)dz & \text{Otherwise,} \end{cases} \tag{3.4.13}$$

which proves (ii). Such a system of PIDE does not in general admit classical solutions as many payoff functions or derivative instruments are continuous but not differentiable. The solutions considered for this type of PIDE are weak solutions in the sense of viscosity, which are proven to exist in [53] in the Markov regime switching case. We assume in our case that the conditional intensity matrix is a piecewise constant function of the backward recurrence time. Hence, PIDE (3.4.9) has a unique viscosity solution and is solved in [2] yielding the general solution:

$$\hat{\mathbf{C}}(s,T,K,y,w) = \exp\Big[\int_y^{T-t+y} M(w,s)ds\Big].\big[\hat{S}(w,K)\mathbf{1}\big], \tag{3.4.14}$$

hence,

$$\hat{C}(s,T,K,y,j,w) = \big\langle \exp\Big[\int_y^{T-t+y} M(w,s)ds\Big].\big[\hat{S}(w,K).\mathbf{1}\big], e_j\big\rangle. \tag{3.4.15}$$

Which proves (iii). □

REMARK 3.4.1 We note that for any $t_2 > t_1$ we have:

$$\begin{cases} \hat{\mathbf{C}}(t_1, T, K, y, w) = \exp\Big[\int_y^{t_2-t_1+y} M(w,s)ds\Big].\hat{\mathbf{C}}(t_2, T, K, y_2, w) \\ \hat{C}(t_1, T, K, y, j, w) = \big\langle \exp\Big[\int_y^{t_2-t_1+y} M(w,s)ds\Big].\hat{\mathbf{C}}(t_2, T, K, y_2, w), e_j\big\rangle. \end{cases} \tag{3.4.16}$$

REMARK 3.4.2 We recall the discretization necessary for implementing the FST algorithm, [60]. First partition the time space $[0, T]$ in $N$ subintervals with $P^{t,n} = (t_n)_{n=0}^N$. Discretization of the log stock price space $(-\infty, \infty)$ is done by approximating the log price domain by a bounded domain $[x_{min}, x_{max}]$ and set $P^{x,M} = (x_i)_{i=0}^M$, $x_i = x_{min} + i\Delta x$ where $\Delta x = \frac{x_{min}-x_{max}}{M}$. As noted by [60], it is sometimes preferred to discretize either $\ln(x/x_0)$ or $\ln(x/K)$ with $K$ the strike price of the option contract depending on whether the pricing is needed around the strike price or not. The frequency domain $[0, w_{max}]$ is partitioned with $P^{w,N} = (w_i)_{i=0}^{(N+1)/2}$ ,$w_i = x_{min} + i\Delta w = 2w_{max}/N$ and $w_{max} = \frac{1}{2\Delta x}$, the Niquisdt critical frequency.

Next lemma describes the basic difference between option prices in semi Markov regimes and Markov regimes and shows that it boils down mainly to the difference in integrated conditional intensities.

LEMMA 3.4.2 *Let $M$, $M^m$, $C^m, \Upsilon^m$ and $\Psi^m$ be the matrix defined by* (3.4.6)*, the matrix defined by* (3.4.6) *when $(\theta_t)_{t\in[0,T]}$ is a Markov process with generator matrix components denoted $(\lambda_{i,j}^m)_{1\le i,j\le m}$, Carr and Madan option price via FFT, the characteristic function of option prices and the characteristic function of spot prices in Markov switching market regimes.*

$$\begin{aligned} &\frac{2\pi}{e^{-\alpha k}}\Big(C(t, T, k, y_t, j, x_t) - C^m(t, T, k, j, x_t)\Big) \\ &= \int_{\mathbb{R}} \frac{\exp\Big[(iu(\ln(x_t) - uk)) - \int_t^T r_s ds\Big]}{(\alpha + iu)(1 + \alpha + iu)} \sum_{n=1}^{\infty} \Big\langle \Big[\Big(\int_y^{t+y} M(u - i(1+\alpha), \eta)d\eta\Big)^n - \Big(tM^m(u - i(1+\alpha))\Big)^n\Big] \end{aligned} \tag{3.4.17}$$

*Proof.*

$$\begin{aligned} &\frac{2\pi}{e^{-\alpha k}}\big(C(t, T, k, y_t, j, x_t) - C^m(t, T, k, j, x_t)\big) \\ &= \frac{e^{-\alpha k}}{2\pi}\int_{\mathbb{R}} e^{-iuk}\Upsilon(t, T, u, y, j, x)du - \frac{e^{-\alpha k}}{2\pi}\int_{\mathbb{R}} e^{-iuk}\Upsilon^m(t, T, u, j, x)du, \\ &= \int_{\mathbb{R}} e^{-iuk}\big(\Upsilon(t, T, u, y, j, x) - \Upsilon^m(t, T, u, j, x)\big)du, \text{ from (3.4.8)}, \end{aligned}$$

$$= \int_{\mathbb{R}} \frac{e^{-iuk-\int_t^T r_s ds}}{(\alpha+iu)(1+\alpha+iu)}\Big[\Psi\big(u-i(1+\alpha),t,y_0,j,x_0\big)-\Psi^m\big(u-i(1+\alpha),t,j,x_0\big)\Big]du, \text{ from (3.4.4)}$$

$$= \int_{\mathbb{R}} \frac{\exp\Big[(iu(\ln(x_t)-uk))-\int_t^T r_s ds\Big]}{(\alpha+iu)(1+\alpha+iu)}\Big\langle\big[\exp\big(\int_y^{t+y} M(u-i(1+\alpha),\eta)d\eta\big)$$

$$-\exp\big(tM^m(u-i(1+\alpha))\big)\big]e_j,\mathbf{1}\Big\rangle du, \text{ from (3.4.5)}$$

$$= \int_{\mathbb{R}} \frac{\exp\Big[(iu(\ln(x_t)-uk))-\int_t^T r_s ds\Big]}{(\alpha+iu)(1+\alpha+iu)}\sum_{n=1}^{\infty}\Big\langle\Big[\Big(\int_y^{t+y} M(u-i(1+\alpha),\eta)d\eta\Big)^n$$

$$-\Big(tM^m(u-i(1+\alpha))\Big)^n\Big]e_j,\mathbf{1}\Big\rangle du$$

□

### 3.4.1 Effect of parameters on option prices.

Simulation of option prices is performed by computing the inverse Fourier transform of (3.4.4). We use Simpson rule of integration, with upper limit of integration in $w$ being $a$. Moreover, the frequency space is divided into N subintervals of equal lengths; the log strike k ranges from $-b$ to $b$ divided into N subintervals of equal lengths. Inverting the Fourier Transform of $\tilde{C}$ could be done quite efficiently by FFT or even by FRFT as suggested in [15]. Let's use the notation: $w_j = (j-1)\eta$ with $\eta = \frac{a}{N}$, $k_u = -b + \lambda(u-1)$ with $\lambda = \frac{2b}{N}$. To match the Discrete Fourier Transform with the FFT requires one to impose the condition $\lambda\eta = \frac{2\pi}{N}$:

$$C(t,T,k_u,y_t,j,x_t) \approx \mathcal{R}_e\Big[\frac{e^{-\alpha k_u}}{\pi}\sum_{j=1}^{N} e^{-i\frac{2\pi}{N}(j-1)(u-1)}\Upsilon(u_j,y_t,j,x_t)e^{ibu_j}\frac{\eta}{3}\big[3+(-1)^j-\delta_j\big]\Big] \tag{3.4.18}$$

The presented condition generates a tradeoff between precision of the integral approximation and step size of the log strike partitions. However the fractional Fourier transform (FRFT) allows to use independent log strikes step size and integration grid precision [15]. Under the risk neutral measure, we consider a couple of semi Markov spot price log-jump densities $\bar{b}_{i,j}$, when the market switches from regime $i$ to $j$, and we denote them $\kappa_{i,j}$ with:

$$\kappa_{i,j} = \begin{cases} 1 - \varepsilon_{i,j} \text{ with probability } p_{i,j} \\ 1 \text{ with probability } 1 - 2p_{i,j} \\ 1 + \varepsilon_{i,j} \text{ with probability } p_{i,j} \end{cases} \tag{3.4.19}$$

$$\text{with} \tag{3.4.20}$$

$$\varepsilon_{i,j} > 0, p_{i,j} \in [0, \frac{1}{2}], \forall i, j \in I(1, m). \tag{3.4.21}$$

We consider the following partition $0 = a_0 < a_1 < ... < a_{M-1} = T^\star$ of $[0, T^\star]$ and for convenience we denote $a_M = \infty$. We assume that the conditional intensity of the semi Markov process $\theta_t$ is a piecewise constant approximation of Weibull intensities. This is guided by a couple of motivations: first its flexibility and then the necessity of Lie bracket condition (3.4.7). On the one hand, Weibull intensities are quite flexible as they can simulate increasing, constant and decreasing rates. On the other hand, the Lie bracket condition is satisfied piecewise, since the $\lambda_{i,j}$ are piecewise constant. The piecewise conditional intensity approximation of Weibull intensities relative to the partition $(a)_{k=0}^{M-1}$ are defined:

$$\begin{aligned} \lambda_{i,j}(y_s) &= \begin{cases} p_{i,j} \sum_{k=0}^{M-1} \frac{\vartheta_i}{\varsigma_i} \big(\frac{a_k^\star}{\varsigma_i}\big)^{\vartheta_i - 1} 1_{[a_k, a_{k+1})}(y_s) \text{ if } i \neq j \\ -\sum_{j=1, j\neq i}^{m} \lambda_{i,j}(y_s) \text{ otherwise} \end{cases} \\ &= \begin{cases} \alpha_{i,j} \sum_{k=0}^{M-1} \big(a_k^\star\big)^{\vartheta_i - 1} 1_{[a_k, a_{k+1})}(y_s) \text{ if } i \neq j \\ -\sum_{j=1, j\neq i}^{m} \lambda_{i,j}(y_s) \text{ otherwise ,} \end{cases} \end{aligned} \tag{3.4.22}$$

$\forall s \in [0, T]$, with $\alpha_{i,j} = p_{i,j} \frac{\vartheta_i}{\varsigma_i^{\vartheta_i}}$ and $a_k^\star = \frac{a_k + a_{k+1}}{2}, \forall k \in I(0, M-2)$. Three notable sets of parameters are absent from most option price formulas whenever the market is subjected to Markov regime changes: the backward recurrence time, the semi Markov sojourn time distribution, and the price jumps associated with regime changes, respectively. We will examine the added flexibility of stock price models under semi Markov regimes due to the extra parameters and the impact of each of the first two parameters on option prices and implied volatilities. We first make a couple of observations necessary to shed more light on the simulation results. If the intensity function is a continuous function of the backward recurrence time, from the derivative of the matrix exponential under the Lie bracket condition [49], from (3.4.4) and (3.4.8) we have:

$$\frac{\partial C}{\partial y}(t,T,k,y,j,x) = \frac{e^{-\alpha k}}{2\pi}\int_{\mathbb{R}} e^{-iuk}\frac{\partial \Upsilon}{\partial y}(t,T,u,y,j,x)du$$

$$= \frac{e^{-\alpha k}}{2\pi}\int_{u\in\mathbb{R}}\Big\langle\Big[M(u-i(1+\eta),y+t) \tag{3.4.23}$$

$$-M(u-i(1+\eta),y)\Big]\exp\Big(\int_y^{t+y}M(u,\eta)d\eta\Big)e_j, e^{iu(\ln(x)-k)}\mathbf{1}\Big\rangle du,$$

(3.4.24)

where the difference of matrices in (3.4.24) is performed componentwise and yields,

$$M_{pq}(u-i(1+\eta),y+t) - M_{pq}(u-i(1+\eta),y) \tag{3.4.25}$$

$$=\begin{cases}\lambda_{p,p}(y+t)-\lambda_{p,p}(y) \text{ if p=q}\\ \big[\lambda_{p,q}(y+t)-\lambda_{p,q}(y)\big]E_{\kappa_{p,q}}\Big[e^{i\ln(\beta_{p,q})\big[u-i(1+\eta)\big]}\Big],\end{cases} \tag{3.4.26}$$

where $E_{\kappa_{p,q}}$ is the expected value with respect to $k_{p,q}$. (3.4.25) shows an interesting feature of option prices in semi Markov market regimes. The derivative with respect to the backward recurrence time of option prices First, if the rate matrix $(\lambda_{i,j}(y))_{m\times m}$ is monotonic componentwise, the backward recurrence time effect on option prices is most significant for long range maturity options. Assuming the intensity functions $\lambda_{i,j}$ have finite limits when the backward recurrence time grows to infinity, we have

$$\lim_{y\to\infty}\frac{\partial C}{\partial y}(t,T,k,y,x) = 0, \forall i,j\in E. \tag{3.4.27}$$

Hence, the semi Markov conditional intensity matrix is asymptotically constant, which implies that asymptotically, semi Markov market regime prices are identical to Markov market regime option prices. Examining Figures 6, 7 and 8 shows that irrespective of the model, option prices from prices processes in Markov market regimes can be sandwiched between semi Markov market regimes with shape parameter $\vartheta$ above and below 1. More importantly, the specific observation that prices from semi Markov market regimes with $\vartheta_i < 1$ and $\vartheta_i > 1$ are higher and lower than Markov prices, respectively is consistent with the underlying mathematical and economic theory. Indeed, our price model accounts for two sources of risk, the Levy and the semi Markov switching risks.

The former is hedged against by the conditional minimum entropy martingale measure as the Levy jump process is turned into a martingale, and the latter isn't considered hedgeable and directly affects option prices. Hence, the higher the regime switching risk the larger the option price. From (3.4.22), choosing $\vartheta_i < 1$ (resp. $\vartheta_i > 1$) implies a decreasing (resp. increasing) transition rate, which translates in a decreasing (resp. increasing) regime switching risk. Simulations for Figures 6 and 7 are performed with $y = .1$ year and $\vartheta_i < 1$, hence, $\lambda_{i,j}(y)$ is largest for $\vartheta_i < 1$, which justifies why prices for $\vartheta_i < 1$ lie above prices corresponding to $\vartheta_i = 1$ and $\vartheta_i > 1$, respectively. Simulation of Figure 8 was performed with $y = 1.1$ year. It shows that option prices are higher for larger shape parameters. This is in agreement with (3.4.22) as it shows that $\lambda$ is higher for higher values of $\vartheta_i$. One of the most documented shortcomings of Levy models for price processes is their inability to capture long term implied volatility smiles, [50]. Markov switching Levy price models succeed in slowing the dampening of the implied volatility smiles through the conditional intensity rate matrix, [50]. In our context, the conditional rate matrix is time dependent and could be affected by $y$, $\alpha$ and $\vartheta$, hence offering more control than Markov market regime models over long term smiles. Figure 15 shows that, irrespective of the Levy process used, long term smiles an smirks which often vanish at $T = 1$ year in Markov regimes [7], are still persistent at $T = 2$ years when market regimes are semi Markovian. In addition, Figures 16, 17 and 18 show that the backward recurrence time, the shape and the scale parameters do have a prominent effect on the implied volatility surface. The three rows of Figure 9 display three different effects of the backward recurrence time on the difference in prices between all market regimes, based on three different values of the shape parameter $\vartheta$. The first and the third row were simulated based upon $\vartheta_i < 1, \forall i \in E$ and $\vartheta_i > 1, \forall i \in E$, respectively. Option prices are decreasing in the first row and increasing in the third row in all market regimes. However, in each regime, prices decrease or increase at different rates, hence affecting the price differences between market regime prices. One therefore observe either an exacerbation or a reduction of differences in regimes as evidenced in the first and third row. The second row corresponds to $\vartheta_i = 1, \forall i \in E$ and shows no change in option prices as the price model reduces to Markov market regime price model which is independent of $y$. Indeed, from (3.4.22), when $\vartheta_i = 1, \forall i \in E$, $\lambda_{i,j}$ will reduce to a mere constant and will therefore be free of $y$. Similar observations are made in Figures 10 and 11. One also note from Figure 12, 13 and 14 that $y$, $\alpha$ and $\vartheta$, respectively, affect in-the money, at-the-money and out-of-the-money options prices and leave deep in-the-money (low call strikes relative to the spot price)and out-of-the-money (high call strikes relative to the spot price) option prices relatively unchanged. This stems from (3.4.3), where the payoff vanishes or grows substantially causing the option price to vanish or grow as well irrespective of the market regime when the log strike $k$ grows or decreases relative to the spot price respectively.

### 3.4.2 FST pricing of vanilla and exotic option contracts

We first look into the pricing of two vanilla option contracts: single asset European option contracts and single asset American option contracts. We recall that American option contracts can be exercised any time before expiration of the contract, unlike European option contracts which are settled at maturity. It has been shown [28] that it is not optimal to exercise a non-dividend-paying American option contract before maturity. Hence, American and European call options contracts have the same price provided that the underlying asset does not pay dividends, [32]. One will therefore be concerned only with pricing and comparing prices induced by [11], and the FST numerical methods, [60], respectively. The FST pricing of European option contracts requires one time step despite the assumption of time dependent conditional intensity matrix. It is based on (3.4.16) applied at $(t, y, x)$ as follows:

$$\mathbf{C}(t, T, K, y, x) = \mathcal{F}^{-1}\Big[\exp\Big[\int_y^{T-t+y} M(w, s)ds\Big].\hat{S}(w, K).\mathbf{1}\Big].$$

Simulation parameters used for pricing purpose are as follows: $m = 3$ market states, interest rate $r = .05$, spot price $S = 100$, $\sigma = (.3, .5, .7)$, $\alpha = (-3, 2, 1; 2, -6, 4; .5, 1, -1.5)$, $\vartheta = (5, 3, .3)$, $\epsilon = (0, .2, .1; .4, 0, .1; .1, .3, 0)$ and the jump and drop probabilities $p = (0, .2, .1; .25, 0, .3; .1, 0, 0)$. We notice from Figure 4 that SMBS call option prices obtained from FST and Carr and Madan are identical up to the third decimal as is the case with call option in Markov regime markets [53]. However, the error plot shows that Carr and Madan prices are consistently slightly larger than FST prices. As for the pricing of American options in semi Markov markets, (3.4.16) allows us to use the FST method, [60], thus far applied to Markov regime markets. The FST is applied based on the discretization scheme presented in Remark 3.4.2. American put option prices are larger than their payoff, as they can be either exercised or held at each time step. The option holder always chooses the alternative netting the larger benefit. Such a maximum condition is enforced in the design of the pricing algorithm as follows:

$$\mathbf{C}^{\star}(t_n, T, K, y, x_n) = \mathcal{F}^{-1}\Big[\exp\Big[\int_{y}^{y+\Delta t} M(w,s)ds\Big].\hat{\mathbf{C}}(t_{n+1}, T, K, y, w)\Big],$$

which is the holding price of the option at time $t_n$, while the price of the option at the same time is

$$\mathbf{C}(t_n, T, K, y, x_n) = \max\big(\mathbf{C}^{\star}(t_n, T, K, y, x_n), \mathbf{C}(T, T, K, y, x_n)\big),$$

where the maximum is applied component wise. We simulate prices of European style Digital and

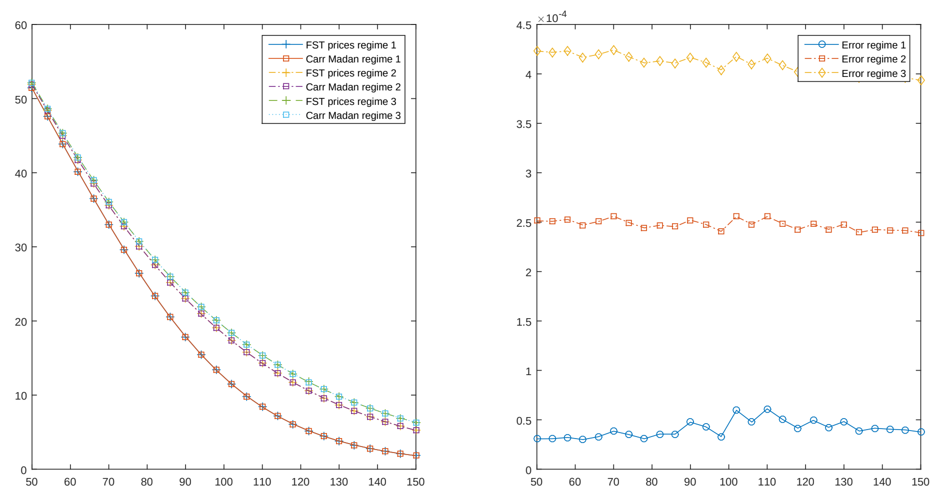


**Figure 4.**: The left hand side plot presents comparisons of Semi Markov Back Scholes FST and Carr and Madan prices of European option in every state of the market while the right hand side plot exhibits error differences in each regimes.

asset-or-nothing exotic option prices. We recall that at expiry, digital call option contracts pay $\$1$, if the spot price is at least at large as the strike and nothing otherwise. Digital put options however, pay nothing if the spot price is larger than the strike and $\$1$ otherwise. Asset-or-nothing option contracts are similar to digital option contracts with the only difference that they pay the asset price worth or nothing. Their respective payoff functions can be expressed as follows:

$$S(x_T, k) = \begin{cases} 1_{(x_T \geq K)} \text{ for digital calls,} \\ 1_{(x_T < K)} \text{ for digital puts,} \\ x_T 1_{(x_T \geq K)} \text{ for asset-or-nothing calls,} \\ x_T 1_{(x_T < K)} \text{ for asset-or-nothing puts.} \end{cases}$$

The pricing of this style of path-independent exotic option contracts in semi Markov regime switching could be done using the FST method with one single time step. The first row of Figure shows that the effect of the backward recurrence time on European vanilla observed in the preceding section carries over to American and exotic option contracts.

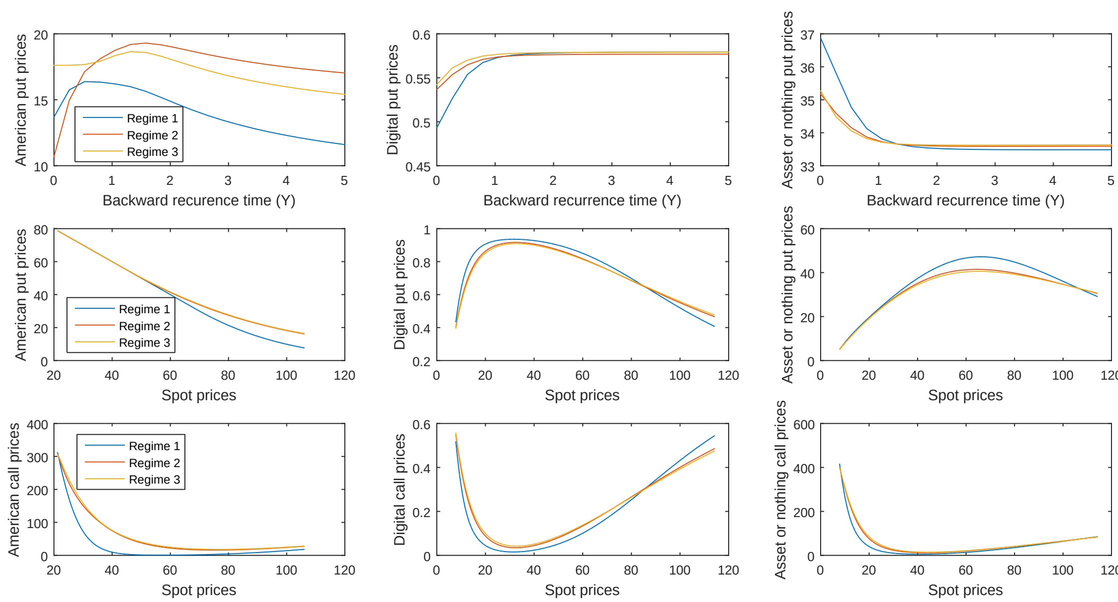


**Figure 5.**: The first row captures the effect of backward recurrence on American style put, digital put and asset-or-nothing put option prices while the second and third rows present all three put and call option prices at each market state.

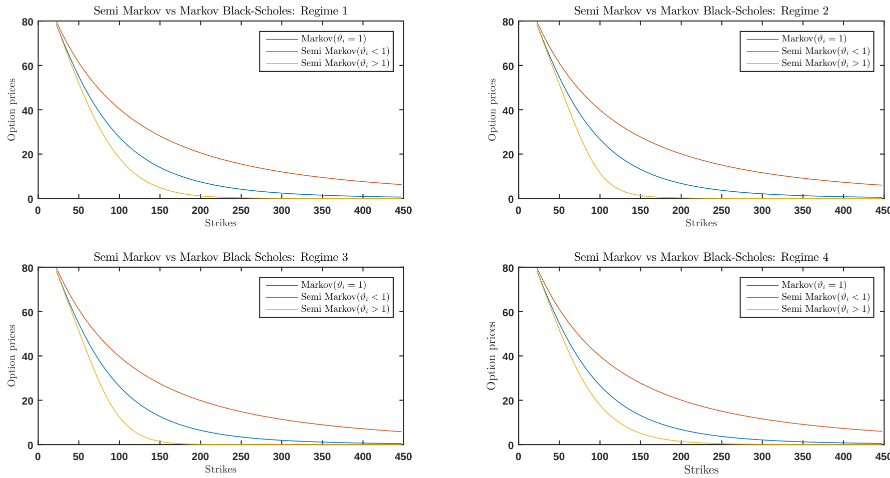


**Figure 6.**: Comparison of option prices in a Markov switching Black Scholes model and a semi Markov switching Black Scholes (SMBS) model.

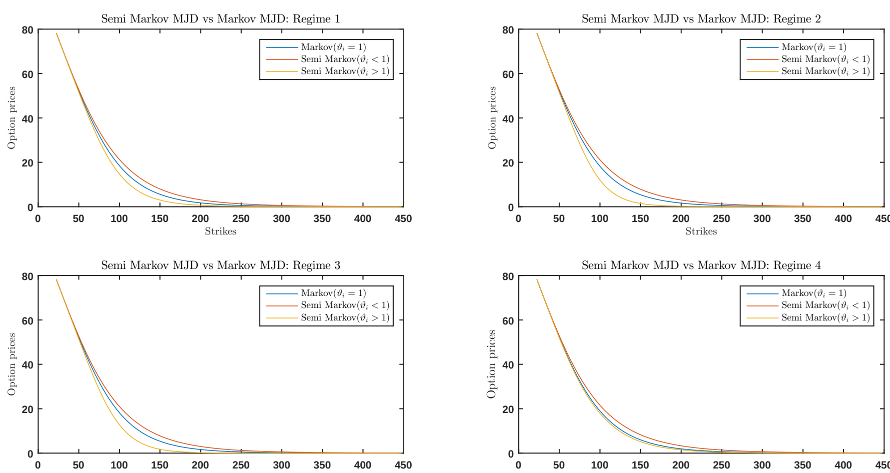


**Figure 7.**: Comparison of option prices in a Markov switching Merton Jump diffusion model and a semi Markov switching Merton Jump diffusion (SMMJD) model.

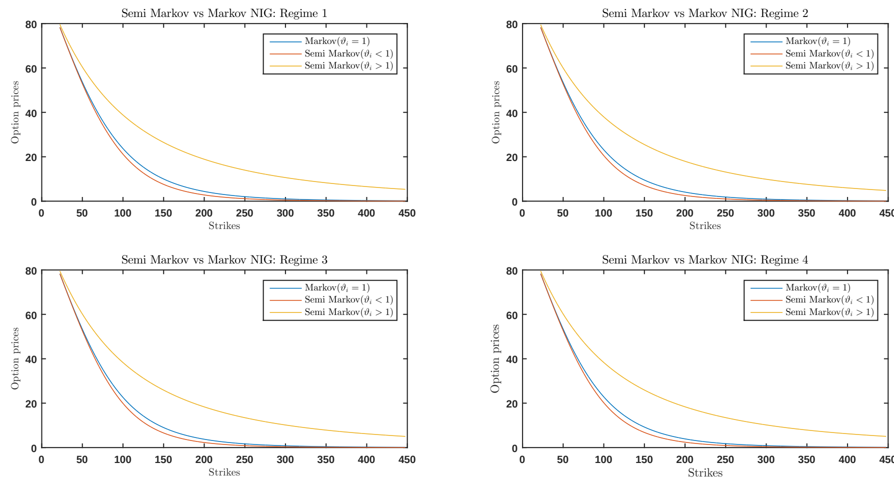


**Figure 8.**: Comparison of option prices in a Markov switching Normal Inverse Gamma model and a semi Markov switching Normal Inverse Gamma (SMNIG) model.

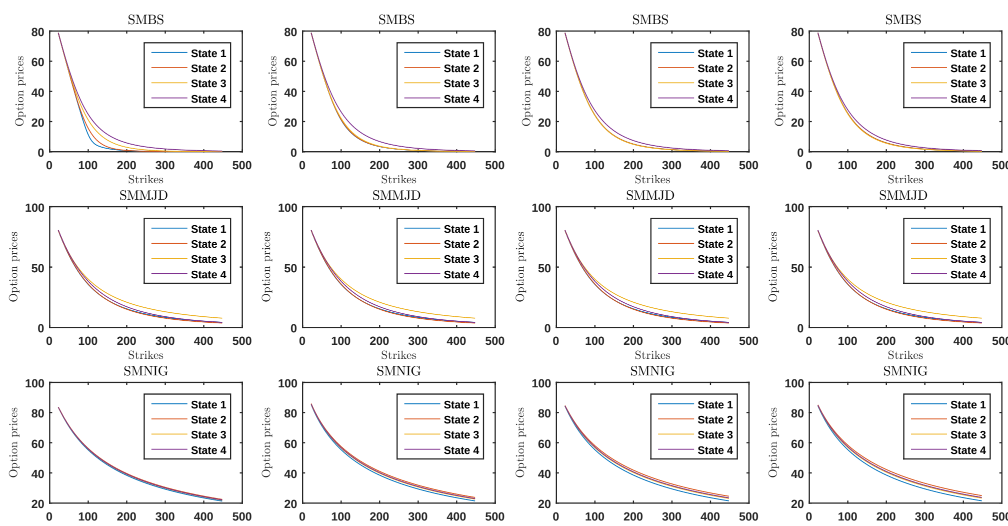


**Figure 9.**: Effects of the backward recurrence time on option price. The first, second and third rows are simulated with $\vartheta_{i,j} < 1, \forall i, j \in E$, $\vartheta_{i,j} > 1, \forall i, j \in E$ and $\vartheta_{i,j} = 1, \forall i, j \in E$, respectively.

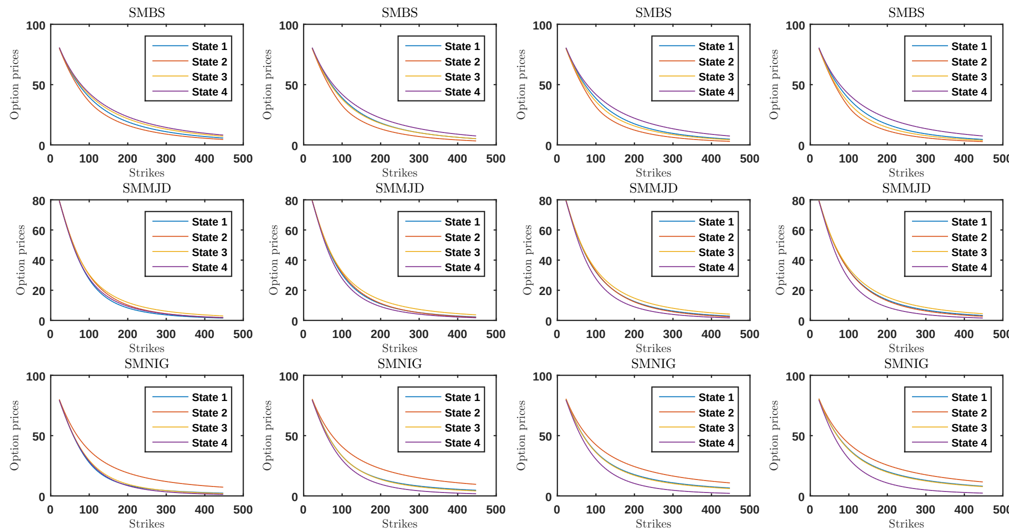


**Figure 10.**: Effects of the scale parameter $\alpha$ on option prices $C$.

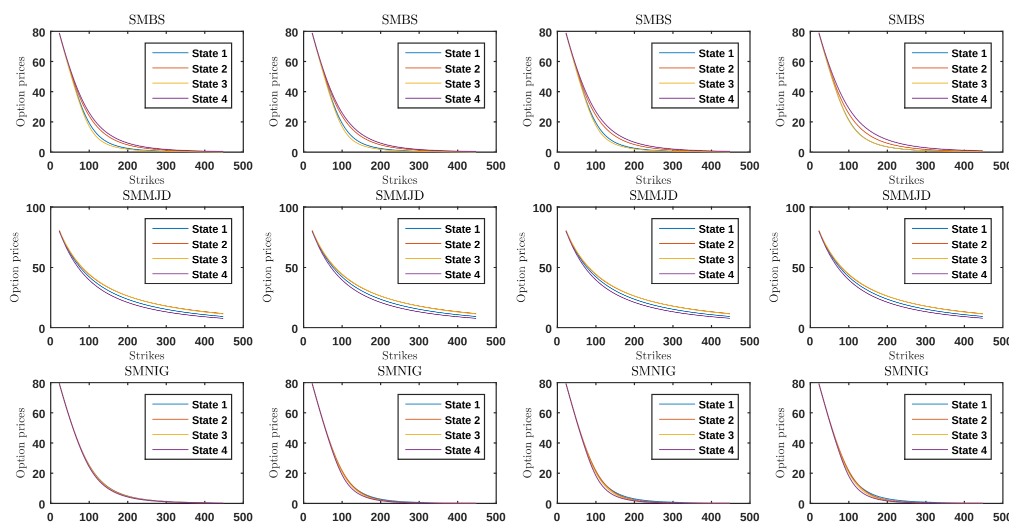


**Figure 11.**: The effects of the shape parameter on option prices. The shape parameter vector in the simulation is $\zeta_k \vartheta$ where $\zeta_k \in \{.25, .5, .75, 1\}$.

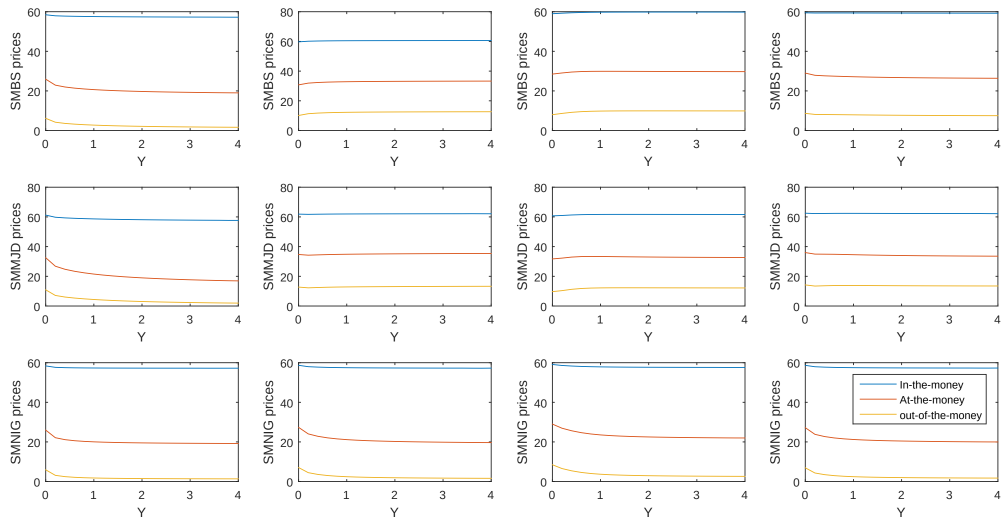


**Figure 12.**: The effects of the backward recurrence time $y_t$ on option prices $C$ from the standpoint of the strike price $K$ of the option and the model used (SMBS, SMMJD or SMNIG).

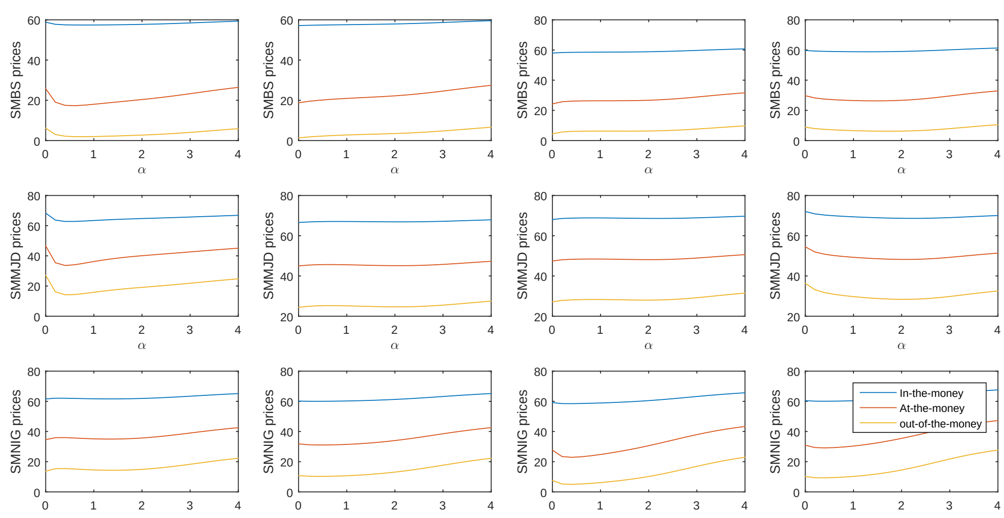


**Figure 13.**: The effects of the scale parameters $\alpha$ on option prices $C$ is noticeable for all three models regardless of the option moneyness.

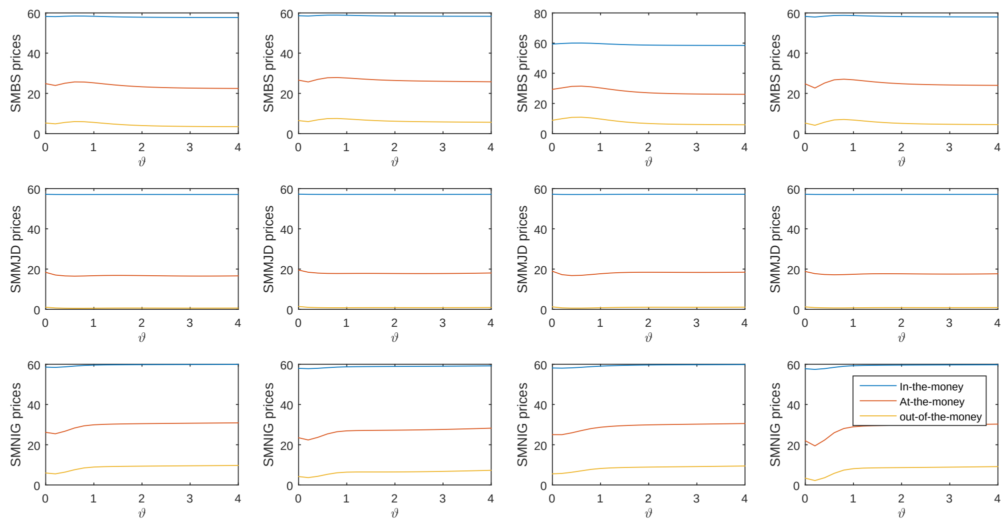


**Figure 14.**: The shape parameter vector used is $(1/i, (2.1 - B(i))/1, .9^{B}(i), B(i)/2)$, where $i \in I(1, 10)$ and $B(i) = .2(i - 1)$. Effects of the shape parameters $\vartheta$ on option prices $C$ are noticeable for all three models and regardless of the option moneyness.

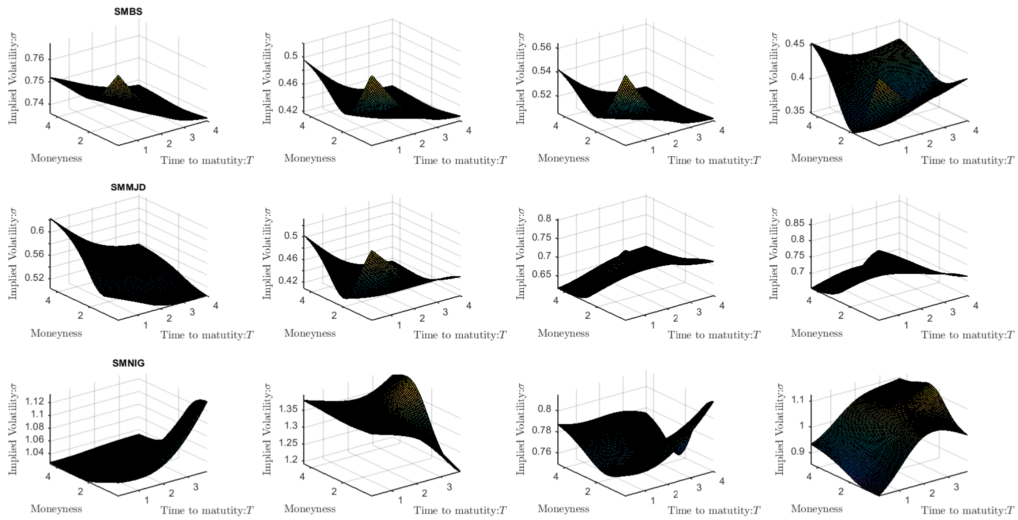


**Figure 15.**: Implied volatility surfaces induced by option prices generated by SMBS (first row), SMMJD (second row) and SMNIG (third row). Column 1-4 correspond to market regimes 1-4, respectively.

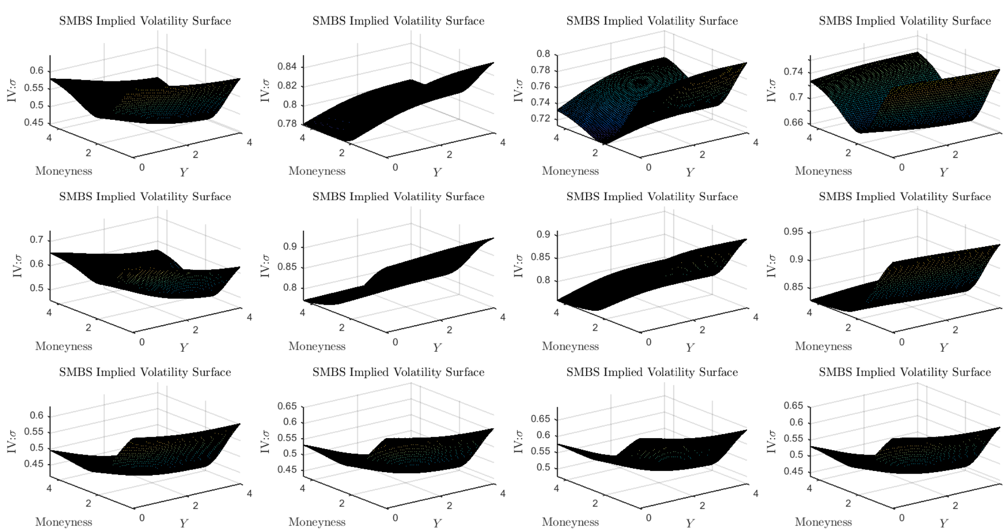


**Figure 16.**: Effects of the backward recurrence time $y_t$ on the implied volatility.

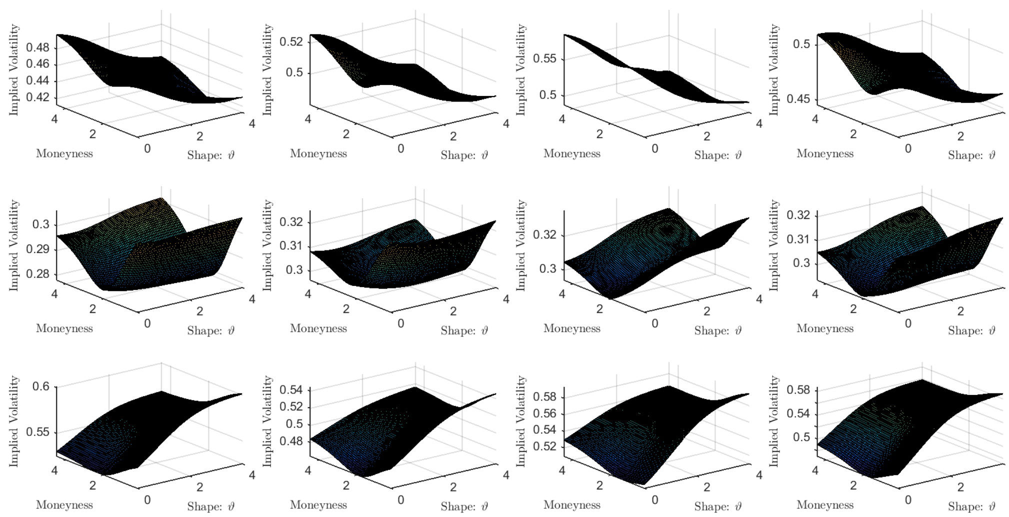


**Figure 17.**: Effects of the shape parameter $\vartheta$ on the implied volatility.

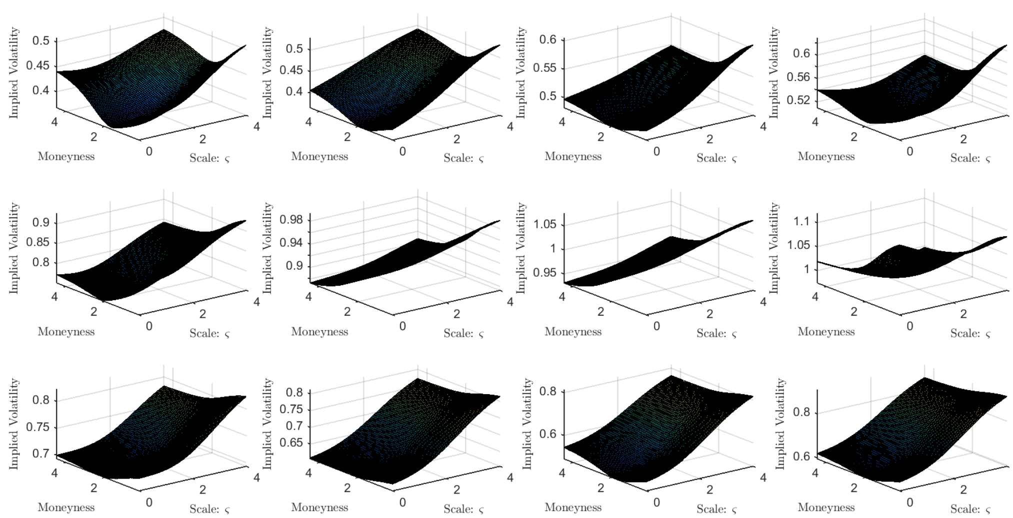


**Figure 18.**: Effects of the scale parameter $\alpha$ on the implied volatility.

### 3.4.3 Calibration

In this section, we estimate the risk neutral parameters inducing option prices closest to observed market option prices in the least square sense. Here the CMEMM is considered to be the risk neutral measure. Hence, the martingale condition (3.4.2) holds. The sojourn time distribution of the semi Markov process $(\theta_s)_{s\in[0,T]}$ is assumed to be piecewise exponential with intensity function defined in (3.4.22). The jump distribution at a regime switch is defined in (1.4.1). Option prices induced by Levy processes are well known to fit market prices better [66], than BS induced option prices. It is therefore more appropriate to isolate the effects of market regime by calibrating regime switching BS parameters to market option prices. On the other hand, in Subsection 3.4.1, we have concluded that the effects of the backward recurrence and sojourn time distribution parameters increase as time to expiry goes up. Hence, our choice of the time to expiry $T = 1.2$ years. We will use $4$ data sets to calibrate semi Markov regime switching Black Scholes models. We will show that in our framework, calibration results provide a fit at least as good as Markov switching models with the added advantage of more insight into the dynamic of market regimes. Data of interest are in-the-money and at-the-money European call option contract quotes on the Dow Jones Industrial Average Index (DJX) and the NASDAQ index (NDX), both collected March 2008 and March 2015, respectively. Data is presented in Table 3.4.3. We note that in-crisis DJX quotes of 2008 have been used in [22] to calibrate standard exponential Levy processes and in [53] to calibrate Markov regime switching exponential Levy processes whereas post crisis data have been retrieved from the website *www.optionseducation.org*. The sum of squares (SS) and root mean square error (RMSE) are reported in Tables 5 and 6. Markov switching models are known to improve the fit of exponential Levy models discussed in [16, 25, 54]. It appears from Figures 19 and 20 that SMBS fits the market data at least as well as Markov switching BS and from 19 and 20 visibly better. Such a feature is hardly unexpected as the theoretical set up developed in [2] and the associated estimation techniques parallel and extend the results of [16, 26, 34, 53]; showing that Markov BS model estimation methods are nested inside SMBS. Furthermore, the parameter estimates of the sojourn time distribution of the semi Markov process shed an additional ray of light on the market regimes behavior. Although risk neutral parameters are a reflection of market makers perception of the future, one can still glean a decent insight on the market behavior through calibrated parameters. One notices that calibrated 2008 DJX Markov regimes have two very similar regimes with nearly identically low volatilities ($17.5\%$ and $17.6\%$), which is a reflection of the market widespread panic observed at the end of the year, hence suggesting that Markov regime models support a one state market. However, semi Markov parameters calibrated to the same data rather contend that $\alpha_2$ and $\vartheta_2$ are much higher than $\alpha_1$ and $\vartheta_1$, hence showing that the market will spend much more time in regime 2, the regime with the highest probability of price jumps or drops (regime 1, $\hat{p} = .258$), which is in line with the sell-off observed throughout 2008 when the Dow Jones Industrial Average dropped by nearly $20\%$ from June 2007 to June 2008. Post crisis Markov and semi Markov market state parameter estimates in Table 5 present the same conundrum as in-crisis parameter estimates. Indeed, Markov market model parameters describe a market with volatility non reflective of the easing of the mood observed in the market. In fact, Markov market regime model supports evidence that volatility is higher in the post crisis market and the regime risks are similar. Another lingering effect of the financial crisis that has not been captured by Markov regime models is the remnant and even mounting fear of a market crash or correction which became even more acute since the DJIA and NASDAQ have reached all time intraday highs May 19 2015 and April 23 2015, respectively. Both market features are captured by semi Markov parameters which provide a more intuitive interpretation of future behavior of the market regimes and crash fears. In fact, a look at the last line of Table 5 shows that volatility has decreased ($.143 < .173$ and $.044 < .141$) while most of the remaining fear in the market is centered around unexpected crashes ($\hat{\epsilon}_1 = 18\%, \hat{p}_1 = .477$). When the market is in state 1 seldom does it switch to state 2 as $\vartheta_1 < \vartheta_2$ and $\alpha_1 < \alpha_2$. However, the switching rate from state 2 to state 1 grows as $\vartheta_2$ is bigger than 1. Hence, the market is expected to have short stays in state 2 which has low volatility ($.044 < .144$) and low probability of drop or jump ($.028 < .477$) and longer stays in the first state. This is also in line with the notion that the 2008 financial crisis has lingering effects and market makers expect significant market corrections and are incline to over-reacting to new information. Similar observations are made from Table 6.

**Table 4**: European call option quotes written on the DJIA and NASDAQ during and after the financial crisis. Deep out-of-the-money options have been weeded out as they are of value close to zero. The spot prices are as follows: Dow Jones Industrial average, $122 and $180 for the 2008 and the 2015 data sets. NASA Index $1775 and $4323

| 2008 DJX quotes | | 2015 DJX quotes | | 2008 NDX quotes | | 2015 NDX quotes | |
|---|---|---|---|---|---|---|---|
| Strikes | Call prices | Strikes | Call prices | Strikes | Call prices | Strikes | Call prices |
| 98 | 24.43 | 50 | 129.85 | 1400 | 334.95 | 4050 | 339 |
| 99 | 23.40 | 55 | 124.85 | 1425 | 311.55 | 4075 | 319.1 |
| 100 | 22.50 | 60 | 119.875 | 1450 | 288.35 | 4100 | 299.35 |
| 101 | 21.55 | 65 | 114.9 | 1475 | 265.5 | 4125 | 280.15 |
| 102 | 20.63 | 70 | 109.9 | 1500 | 242.6 | 4150 | 261.3 |
| 103 | 19.68 | 75 | 104.925 | 1525 | 220.5 | 4175 | 242.95 |
| 104 | 18.75 | 80 | 99.95 | 1550 | 198.95 | 4200 | 224.9 |
| 105 | 17.83 | 85 | 94.975 | 1575 | 178.45 | 4210 | 217.9 |
| 106 | 16.90 | 90 | 89.95 | 1600 | 158.55 | 4220 | 210.85 |
| 107 | 15.98 | 95 | 85.05 | 1625 | 139.60 | 4225 | 207.4 |
| 108 | 15.10 | 100 | 80.1 | 1650 | 121.5 | 4230 | 204 |
| 109 | 14.23 | 105 | 75.15 | 1675 | 104.45 | 4240 | 197.1 |
| 110 | 13.33 | 110 | 70.225 | 1700 | 88.45 | 4250 | 190.4 |
| 111 | 12.45 | 115 | 65.325 | 1725 | 73.8 | 4260 | 183.7 |
| 112 | 11.63 | 120 | 60.425 | 1750 | 60.4 | 4270 | 177.7 |
| 113 | 10.78 | 125 | 55.575 | 1775 | 48.45 | 4275 | 174.4 |
| 114 | 9.95 | 130 | 50.75 | 1800 | 38.05 | 4280 | 171.2 |
| 115 | 9.18 | 135 | 45.95 | 1825 | 29.2 | 4290 | 164.75 |
| 116 | 8.40 | 140 | 41.25 | 1850 | 21.65 | 4300 | 158.6 |
| 117 | 7.68 | 145 | 36.55 | 1875 | 15.65 | 4310 | 152.35 |
| 118 | 6.93 | 150 | 32.025 | 1900 | 10.95 | 4320 | 146.25 |
| 119 | 6.23 | 155 | 27.475 | 1925 | 7.45 | 4325 | 143.2 |
| 120 | 5.58 | 160 | 23.125 | ... | ... | 4330 | 139.7 |
| 121 | 4.95 | 165 | 18.925 | ... | ... | 4340 | 134.45 |
| 122 | 4.35 | 170 | 15.1 | ... | ... | 4350 | 128.65 |
| 123 | 3.80 | 175 | 11.375 | ... | ... | 4360 | 123.05 |
| 124 | 3.25 | 180 | 8.075 | ... | ... | 4370 | 117.35 |
| 125 | 2.74 | 185 | 5.275 | ... | ... | 4375 | 114.35 |
| 126 | 2.28 | ... | ... | ... | ... | 4380 | 111.95 |
| 127 | 1.90 | ... | ... | ... | ... | 4390 | 106.65 |
| 128 | 1.52 | ... | ... | ... | ... | 4400 | 101.1 |

**Table 5**: This Table reports calibration results of model parameters using option contracts on the Dow Jones Industrial Average(DJX). We assume the market has two regimes ie $E = \{1, 2\}$ and at inception of the contract, the market has been in its current state for $y = 1.2$yrs. Only call options with maturity $T = 47$ days are used for illustration.

| Market type | Model Type | $\hat{\alpha}_1$ | $\hat{\alpha}_2$ | $\hat{\vartheta}_1$ | $\hat{\vartheta}_2$ | $\hat{\epsilon}_1$ | $\hat{\epsilon}_2$ | $\hat{p}_1$ | $\hat{p}_2$ | $\hat{\sigma}_1$ | $\hat{\sigma}_2$ | SS | RMSE |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| In-crisis Market | Markov BS | 1.735 | 20.525 | 1 | 1 | 0 | 0 | 0 | 0 | .175 | .176 | 3.73 | .2454 |
| | SMBS | .501 | 10.957 | 4.038 | 12.724 | .038 | .016 | .258 | .08 | .173 | .141 | 3.72 | .2449 |
| Post crisis Market | Markov BS | 3.54 | 20.025 | 1 | 1 | 0 | 0 | 0 | 0 | .25 | .224 | 24 | .65 |
| | SMBS | 1.061 | 9.651 | 2.058 | 6.794 | .186 | .143 | .477 | .028 | .143 | .044 | 21 | .61 |

**Table 6**: This table reports calibration results of model parameters using option contracts on the NASDAQ (NDX). We assume that the market has two regimes ie $E = \{1, 2\}$ and at inception of the contract, the market has been in its current state for $y = 1.2$ yrs. Only call options with maturity $T = 47$ days are used for illustration.

| Market type | Model Type | $\hat{\alpha}_1$ | $\hat{\alpha}_2$ | $\hat{\vartheta}_1$ | $\hat{\vartheta}_2$ | $\hat{\epsilon}_1$ | $\hat{\epsilon}_2$ | $\hat{p}_1$ | $\hat{p}_2$ | $\hat{\sigma}_1$ | $\hat{\sigma}_2$ | SS | RMSE |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| In-crisis Market | Markov BS | 5.8 | 58.5 | 1 | 1 | 0 | 0 | 0 | 0 | .206 | .268 | 400 | 3.02 |
| | SMBS | 14.402 | 18.389 | 24.306 | 7.166 | .012 | .017 | .096 | .102 | .003 | .196 | 384 | 2.95 |
| Post crisis Market | Markov BS | 11.68 | 44.493 | 1 | 1 | 0 | 0 | 0 | 0 | .172 | .159 | 993 | 4 |
| | SMBS | 10.32 | 45.85 | 3.25 | 12.73 | .062 | .015 | .194 | .038 | .047 | .018 | 384 | 2.49 |

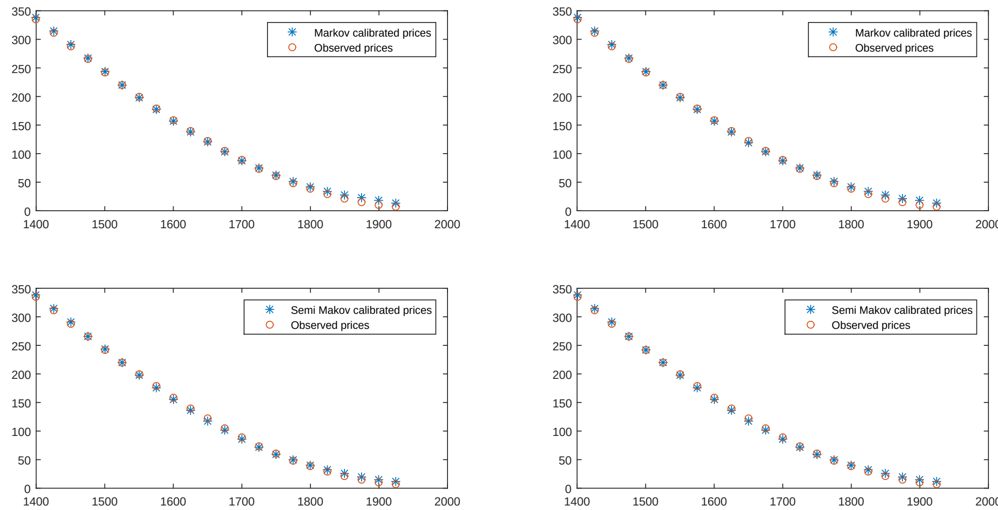


**Figure 19.**: Calibration results of the Markov and semi Markov regime switching models to NDX observed prices in 2008 with the financial crisis in full swing.

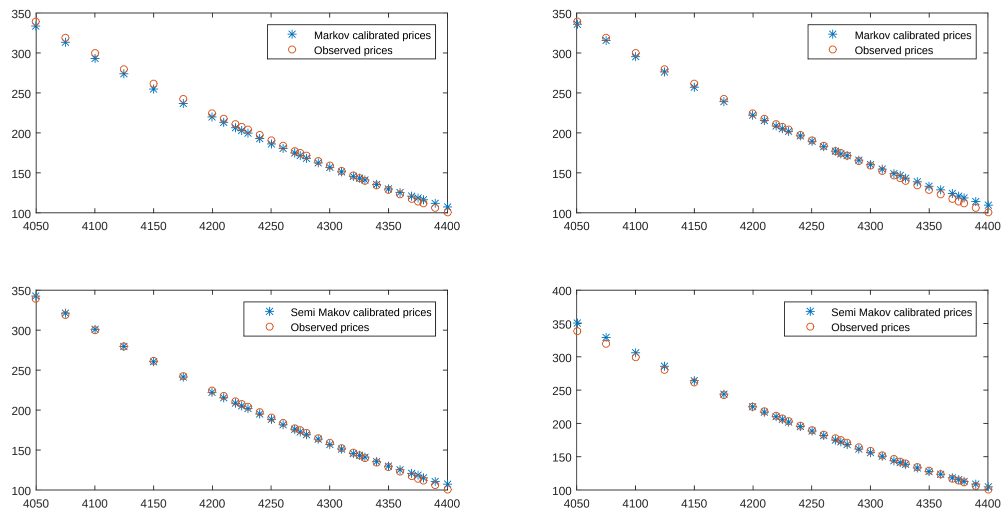


**Figure 20.**: Calibration results of the Markov and semi Markov regime switching models to NDX observed prices in 2015 post financial crisis.

### 3.4.4 Comparison with Heston model

In this section, we veer away from sequential calibration of option prices for each market maturity as performed in last section. We rather perform a full implied volatility surface calibration which we compare with the implied volatility surface generated by Markov regime switching Black Scholes and Heston models. The parameters of interest in Heston model as in most literature are $V_0$ the initial spot price volatility, $\kappa$ the speed of mean reversion of the spot price volatility, $\theta$ the long term spot price volatility and $\rho$ the correlation of the Brownian motions driving the spot price and the Brownian motion driving the mean reverting volatility of the spot price. We recall that Heston volatility and Black Scholes models are two amongst the most used market models by practitioners in the financial market. Two of the main attractions of both models is their relative tractability as far as option pricing and the clarity of the economic interpretation of their calibrated parameters as well as their calibration performances relative to more complex models (Levy models for instance). We look into a full implied volatility surface of the semi Markov Black Scholes model, the Markov Black Scholes model and Heston model for a data set of option prices on the NASDAQ index quoted under the handle NDX and obtained from the website optioneducation.org. A summary of the calibration process is given on Table 7. It appears from rows 1, 2 and 4 of Table 7 that the semi Markov Black Scholes regime switching model fits a volatility surface as well and even slightly better than the Heston model (RMSE are 3.8 vs 4.6) and even better than Markov regime Black Scholes models (RMSE 3.8 vs 7.1). A closer look at the reason why the semi Markov Black Scholes outperforms Heston volatility surface is because of its ability to reproduce more accurately short term option prices as evidenced by Figure 21. Given that the basic Black Scholes model only supports flat volatility surfaces, one can attribute the added flexibility of the volatility surface generated by the semi Markov Black Scholes model to its switching nature which is in turn modeled as a semi Markov process. One can note the negligible contribution of the extra jump component at regime switches as the optimum parameters of the calibration algorithm leaves all four jump parameters equal to 0 and hence with no effect of option prices. Despite these encouraging model fit diagnostic of the semi Markov regime switching model, one cannot be oblivious to the lack of efficiency of its calibration algorithm compared to Heston and the Markov regime Black Scholes. Much research as been devoted to successfully improving on the efficiency of the Heston calibration model leading to efficient algorithm while the relative novelty of Semi Markov regime switching models in the derivative market is relatively unexplored, hence offering a decent research avenue. All algorithms used in this sections were implemented in MATLAB version R2016a and the global optimization tool was extensively used to avoid the additional bias of initial guesses in the comparison. The genetic and the simulated annealing algorithms were used in particular to obtain an initial guess and find an optimum solution respectively. [h]

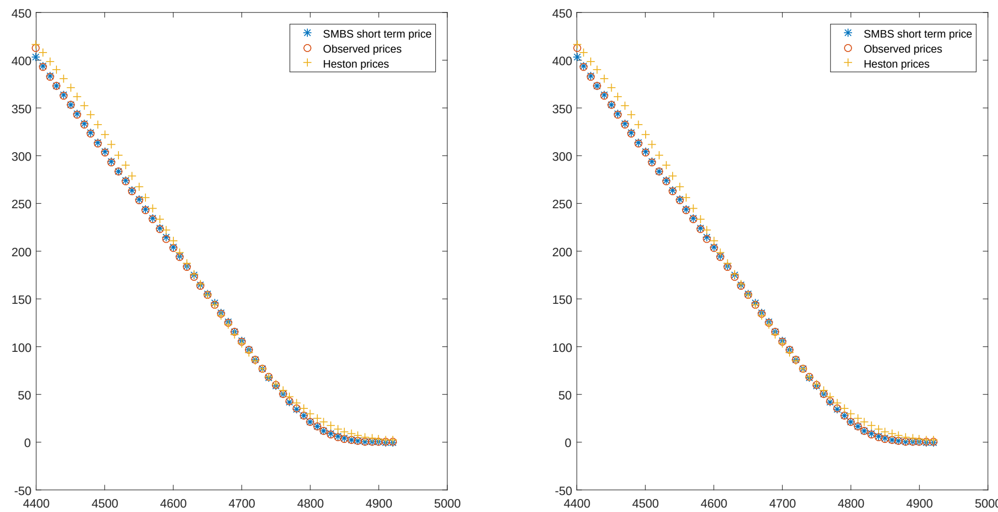


**Figure 21.**: Comparison of short term NDX prices observed(August 2016), generated from the SMBS and Heston models .

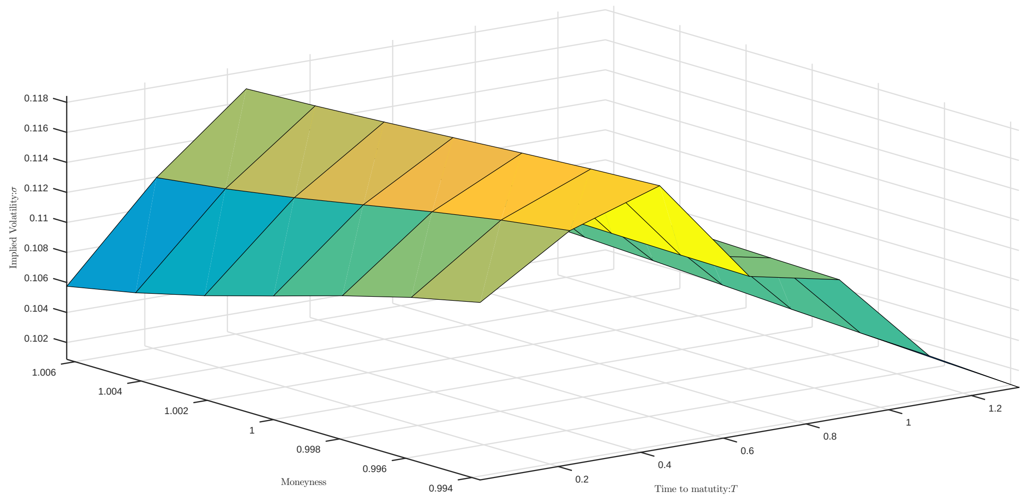


**Figure 22.**: Implied volatility surface of NDX observed prices with the semi Markov regime switching Black Scholes model in August 2016 as the market still recovers from the 2008 financial crisis assuming the market in regime 2.

**Table 7**: This table reports calibration results of model parameters using option contracts on the NASDAQ (NDX). We assume that the market has two regimes ie $E = \{1, 2\}$ and at inception of the contract, the market has been in its current state for $y = 1.2$yrs. An option price data across a spectrum of maturity times and an option data for very short term volatility are used for calibration.

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Markov BS | IVS | $\hat{\alpha}_1 = .313$ | $\hat{\alpha}_1 = 14.304$ | $\hat{\vartheta}_1 = 1$ | $\hat{\vartheta}_2 = 1$ | $\hat{\epsilon}_1 = 0$ | $\hat{\epsilon}_2 = 0$ | $\hat{p}_1 = 0$ | $\hat{p}_2 = 0$ | $\hat{\sigma}_1 = .107$ | $\hat{\sigma}_2 = .122$ | $SS = 4933$ | RMSE=7.1 | CPU time=5hours |
| SMBS Model | IVS | $\hat{\alpha}_1 = 39.716$ | $\hat{\alpha}_1 = 8.699$ | $\hat{\vartheta}_1 = 4.7$ | $\hat{\vartheta}_2 = 8.058$ | $\hat{\epsilon}_1 = 0$ | $\hat{\epsilon}_2 = 0$ | $\hat{p}_1 = 0$ | $\hat{p}_2 = 0$ | $\hat{\sigma}_1 = .09$ | $\hat{\sigma}_2 = .172$ | SS=712 | RMSE=3.8 | CPU time=22hours |
| | Short Maturity | $\hat{\alpha}_1 = 3.789$ | $\hat{\alpha}_1 = 23.899$ | $\hat{\vartheta}_1 = 54.82$ | $\hat{\vartheta}_2.353 =$ | $\hat{\epsilon}_1 = .001 =$ | $\hat{\epsilon}_2 = .019$ | $\hat{p}_1 = .153$ | $\hat{p}_2 = .003$ | $\hat{\sigma}_1 = 7.215$ | $\hat{\sigma}_2 = .053$ | SS=244 | RMSE=2.1 | CPU time=4hours |
| Heston Model | IVS | $V_0 = .017$ | $\kappa = .12$ | $\theta = .126$ | $\sigma = .5$ | $\rho = .207$ | | | | | | SS=1010 | RMSE=4.54 | CPU time=2hours |
| | Short Maturity | $V_0 = .009$ | $\kappa = 2.438$ | $\theta = .139$ | $\sigma.245$ | $\rho = .88$ | | | | | | SS=6104 | RMSE=10.7 | CPU time=.5hours |

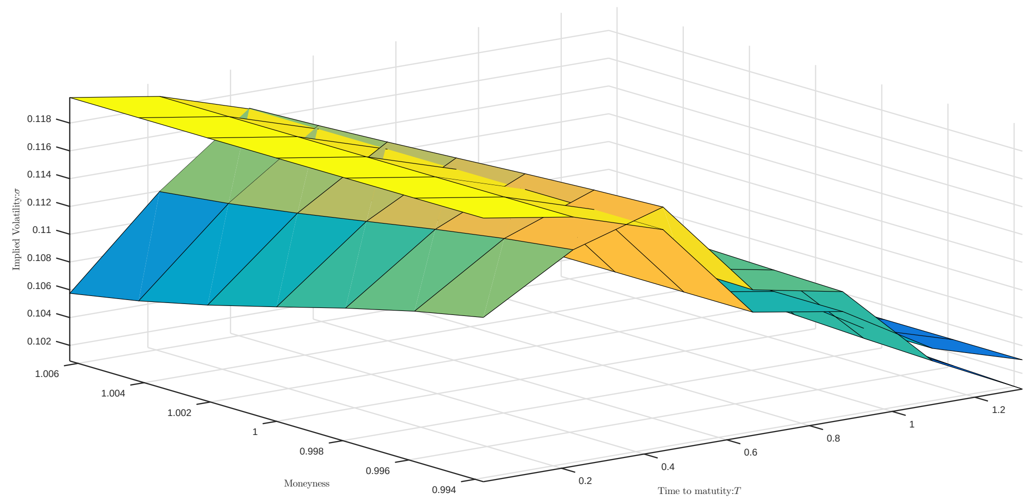


**Figure 23.**: Implied volatility surface of NDX observed prices with the semi Markov regime switching Black Scholes model in August 2016 as the market still recovers from the 2008 financial crisis assuming the market in regime 1 along with market volatility prices. The market implied volatility surface is blue-green for short maturities and the SMBS implied volatility surface is yellow for short maturities.

### 3.5 Conclusion

In this chapter, we have extended the ubiquitous Markov market regime model to a semi Markov market regime model in the context of option pricing. It allows a more accurate description of the risk neutral market regime dynamics as it assumes a time dependent conditional intensity of state changes. The main drawback is the increased complexity of the partial differential equation satisfied by the option price which translated in longer CPU times of calibration algorithms. We considered semi Markov processes with piecewise constant conditional intensity matrices, which allowed us to use Carr and Madan and the Fourier time stepping methods for option simulations and calibrations. An analysis of the semi Markov parameters effects on option prices shows that semi Markov parameters influence option prices to a visible extend, hence legitimizing the use of semi Markov regimes in derivative pricing. We performed a fit comparison of models with semi Markov and Markov markets regimes and showed that Black Scholes model under semi Markov regimes shows a slight improvement in sequential calibration (for each maturity) over Markov regime switching models and a substantial improvement in the calibration of full implied volatility surface over both Markov regime switching and Heston models. As previously mentioned, the runtime of the calibration algorithm is slower than Heston and Markov regime models, and every regime switching model (Markov or semi Markov) induces an incomplete market. Incompleteness of the market renders the risk neutral pricing argument more complex as there exist more than one risk neutral measures. This brings up the issue of choice of the risk neutral measure. In this article we used the minimal entropy martingale measure which is heuristically the risk neutral measure closest to the historical measure probability measure in the sense of Kulback Leibler distance, which minimizes the distance between the risk neutral and the historical view of the market. Another useful risk neutral measure is the minimal martingale measure which allows the best (with respect to certain risk functions) replication of option contracts for portfolio risk hedging purpose. Future research may look into the development of a minimal martingale measure for semi Markov regime switching models along with improving the calibration algorithm for a more economical CPU time in semi Markov regime switching models.

# Chapter 4

## Non linear multidimensional Levy-Itó semi Markov regime switching models

### 4.1 Introduction

We introduce a family of semi Markov regime switching multidimensional non linear models extending the commodity model developed in [56, 57] and the most common stochastic volatility models and local volatility models, namely, Heston-type models, [20, 40, 58], Constant Elasticity of Volatility (CEV) type models, Garch models and Cox-Ingerson-Ross (CIR) models [19, 23, 29] among others. Stochastic volatility models are ubiquitous in financial modeling, as they are a significant upgrade over Black Scholes model [6] both in derivative pricing and in asset return prediction. In derivative pricing, stochastic volatility model provide an implied volatility surface in compliance with many empirical features of the market such as smiles and smirks. As for historical parameter estimation, Stochastic volatility models provide a log return distribution exhibiting many empirically observed features (skewness, fatness of tails and high peak). Despite the abundant literature of Heston models and its well documented reproduction of many stylized empirical facts, a semi Markov switching market has the potential of adding to the already well documented flexibility of the model. Semi Markov regime switching market represent a non obvious generalization of the more common Markov regime switching models. Such models have been studied recently by [2, 3] who finds a Fourier methods through a characteristic function formula, for pricing derivatives in Levy driven financial markets.

The families of models introduced also allow the modeling of multi-asset baskets. Financial portfolios are often divided in sectors (technology, energy, commodity and so on) which are assumed to show significant intra-sector correlation and little to no inter-sector correlation. Hence prediction of one specific asset price could be improved when accounting for prices of assets in the same group. [56, 57] first developed a calibration technique for such a class of model, namely the LLGMM method, accounting for a unique layer of interaction between asset prices through their diffusion coefficients. We extend such a model to semi Markov markets and provide in the same token an opportunity to assess the unpredictable jumps effects on the calibration results. The rest of the chapter is organized as follows: Section 2 is an introduction to the general multidimensional non linear model. Section 3 is the special case of a coupled multidimensional stochastic differential equation model describing commodity prices and an extension of Heston model to account for semi Markov regime changes. Section 3 is a presentation of numerical results and simulations.

### 4.2 LLGMM preliminary set up

Let $\big(\Omega, \mathbb{F}\big)$ and $m$ be the reference measurable space and a whole number, respectively. We denote $T^*$, $T$ and $\psi$ the market time horizon and the maturity time of some derivative contract, and a Poisson random measure, respectively. $(\theta_t)_{t\in[0,T^*]}$ is a semi Markov process with state space $E = \{1, 2, ..., m\}$, state switching times $(T_k)_{k\in\mathbb{N}}$ and $(\boldsymbol{\beta}_k)_{k\in\mathbb{N}}$ is a discrete sequence of non negative vector valued numbers with $\boldsymbol{\beta}_k = \big(\beta_k^1, \beta_k^2, ..., \beta_k^n\big)$. We consider $n, q, l \in \mathbb{N}^\star$ and $\boldsymbol{x}$ an n-dimensional vector stochastic process, $\boldsymbol{\mu} \in \mathcal{C}\big[\mathbb{R}_+ \times \mathbb{R}_+ \times E \times \mathbb{R}^n, \mathbb{R}^n\big]$, $\boldsymbol{\sigma} \in \mathcal{C}\big[\mathbb{R}_+ \times \mathbb{R}_+ \times E \times \mathbb{R}^n, \mathbb{R}^n \times \mathbb{R}^l\big]$, $\boldsymbol{B}$ is an $l$ dimensional vector of Brownian motions, $\boldsymbol{G}, \mathbf{H} \in \mathcal{C}\big[\mathbb{R}_+ \times \mathbb{R}_+ \times E \times \mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n\big]$ and $R \in \mathcal{B}\big[\mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n\big]$ a bounded function.

$\boldsymbol{\mu}, \boldsymbol{\sigma}, \boldsymbol{G}$ and $\boldsymbol{H}$ are assumed smooth enough to ensure existence of a solution of the following system of stochastic partial differential equations:

$$\begin{aligned} d\boldsymbol{x}_t &= \boldsymbol{\mu}(t, y_{t^-}, \theta_k, \mathbf{x}_{t^-})dt + \boldsymbol{\sigma}(t, y_{t^-}, \theta_k, \boldsymbol{x}_{t^-})d\mathbf{B}_t \\ &+ \int_{|z|<1} \boldsymbol{G}(t, y_{t^-}, \theta_k, \boldsymbol{x}_{t^-}, z)\bar{\psi}(\theta_k, dt, dz) \\ &+ \int_{|z|>1} \boldsymbol{H}(t, y_{t^-}, \theta_k, \boldsymbol{x}_t, z)\psi(\theta_k, dt, dz), \end{aligned} \tag{4.2.1}$$

$$\boldsymbol{x}_k = \boldsymbol{\beta}_k \boldsymbol{x}_k^- \text{ , } \forall k \in I(1,\infty) = \mathbb{N}, \text{ where } \boldsymbol{x}_{T_k} = \boldsymbol{x}_k, \forall k \in I(1,\infty) = \mathbb{N}, \forall t \in [T_k, T_{k+1}) \tag{4.2.2}$$

$$\ln(\boldsymbol{\beta}_k) \sim \bar{\boldsymbol{b}}\boldsymbol{(}|\boldsymbol{\theta_{k-1}}, \boldsymbol{\theta_k}\boldsymbol{)} \tag{4.2.3}$$

$(\mathbb{H}_t)_{t\in[0,T]}$, $(\mathbb{L}_t)_{t\in[0,T]}$ and $\mathbb{B}_n$ are filtrations generated by the semi Markov process $\theta_t$, Levy processes $L_s^j$, $s \in [0,t]$, $\forall j \in E = \{1, 2, 3, ..., m\}$ and the discrete vector sequence $(\boldsymbol{\beta}_k)_{k\in\mathbb{N}}$, respectively. We also denote $\bar{\mathbb{L}}_t = \mathbb{L}_t \vee \mathbb{B}_{n(t)}$, $\bar{\mathbb{G}}_t = \mathbb{H}_T \vee \bar{\mathbb{L}}_t$ and $\mathbb{G}_t = \mathbb{H}_t \vee \bar{\mathbb{L}}_t, \forall t \in [0, T^\star]$. Let $P$ and $Q$ be the historical probability and an equivalent martingale measures [2], respectively, associated with the price process $\big(\boldsymbol{x}(t)\big)_{t>0}$ defined on the reference space $\big(\Omega, \mathbb{F}\big)$. We present an extension of Itó's lemma in [2], for a function $V \in \mathcal{C}^{1,1,0,2}\big[\mathbb{R}^+ \times \mathbb{R}^+ \times E \times \mathbb{R}^n, \mathbb{R}^q\big]$. $\frac{\partial V}{\partial \boldsymbol{x}}$ denotes the $q \times n$ first derivative matrix of $V$ and $\frac{\partial^2 V}{\partial x^2} = \big(\frac{\partial^2 V_k}{\partial x_i \partial x_j}\big)_{n\times n\times q}$ represents a tensor of rank 3. We use the following notation: $\boldsymbol{G} = (G_1, ..., G_q)^T$, $\boldsymbol{H} = (H_1, ..., H_q)^T$ and $Tr\Big[a^T \frac{\partial^2 V}{\partial \boldsymbol{x}^2} a\Big] = \Big(Tr\big(a^T \frac{\partial^2 V_1}{\partial \boldsymbol{x}^2} a\big), ..., Tr\big(a^T \frac{\partial^2 V_q}{\partial \boldsymbol{x}^2} a\big)\Big)^T$, $\forall a \in \mathbb{R}^n \times \mathbb{R}^p$, $\forall p \in \mathbb{N}^*$. We define the following set $E^\star = E^2 - \{(i,i), i \in E\}$. We denote $N(t, A, B)$ a stochastic process on $[0,T] \times \mathbb{B}(\mathbb{R}) \times P(E^\star)$ into $[0, \infty)$ as follows:

$$N(t, A, B) = \sum_{n\geq 1} 1_{\big(t\geq T_n, \ln(\boldsymbol{\beta}_n)\in A, (\theta_{n-1},\theta_n)\in B\big)}. \tag{4.2.4}$$

$N(t, A, B)$ is the number of regime switches in subset $B$ of $E^\star$ with corresponding log price jumps $\ln(\beta_n) \in A \subset \mathbb{R}$ by time $t$. The compensator $\gamma(t, A, B) = \sum_{(i,j)\in B} \int_0^t \int_{z\in A} \bar{\boldsymbol{b}}(z|i,j)\lambda_{i,j}(y_s) dz ds$ of $N(t, A, B)$ and Itó formula for the function $V$ are derived in [2]:

$$\begin{aligned}
& dV(s, y_s, \theta_s, \boldsymbol{x}_s) = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial \boldsymbol{x}} d\boldsymbol{x}_s + \frac{1}{2}(d\boldsymbol{x}_s^c)^T \frac{\partial^2 V}{\partial \boldsymbol{x}^2} d\boldsymbol{x}_s^c \\
& + \int_{|z|\leq 1} \Big[V(s, y_{s^-}, \theta_{s^-}, \boldsymbol{x}_{s^-} + \boldsymbol{G}(s, y_{s^-}, \theta_{s^-}, \boldsymbol{x}_{s^-}, z)) - V(s, y_{s^-}, \theta_{s^-}, \boldsymbol{x}_{s^-}) \\
& - \frac{\partial V}{\partial \boldsymbol{x}} \boldsymbol{G}(s, y_{s^-}, \theta_{s^-}, \boldsymbol{x}_{s^-}, z)\Big] \bar{\psi}(\theta_{s^-}, dz, ds) \\
& + \int_{|z|>1} \Big[V(s, y_{s^-}, \theta_{s^-}, \boldsymbol{x}_{s^-} + \boldsymbol{H}(s, y_{s^-}, \theta_{s^-}, \boldsymbol{x}_{s^-}, z)) - V(s, y_{s^-}, \theta_{s^-}, \boldsymbol{x}_{s^-}) \\
& - \frac{\partial V}{\partial \boldsymbol{x}} \boldsymbol{H}(s, y_{s^-}, \theta_{s^-}, \boldsymbol{x}_{s^-}, z)\Big] \psi(\theta_{s^-}, dz, ds) \\
& \int_{|z|\leq 1} \Big[V(s, y_{s^-}, \theta_{s^-}, \boldsymbol{x}_{s^-} + \boldsymbol{G}(s, y_{s^-}, \theta_{s^-}, \boldsymbol{x}_{s^-}, z)) - V(s, y_{s^-}, \theta_{s^-}, \boldsymbol{x}_{s^-}) \\
& - \frac{\partial V}{\partial \boldsymbol{x}} \boldsymbol{G}(s, y_{s^-}, \theta_{s^-}, \boldsymbol{x}_{s^-}, z)\Big] \nu(\theta_{s^-}, dz) ds \\
& + \sum_{j=1, j\neq\theta_{s^-}}^{m} \int_{z\in\mathbb{R}} \Big[V(s, y_s, j, \boldsymbol{x}_{s^-} e^z) - V(s, y_{s^-}, \theta_{s^-}, \boldsymbol{x}_{s^-})\Big] N(dz, ds, \{(\theta_{s^-}, j)\}) \qquad (4.2.5) \\
& = \frac{\partial V}{\partial s} ds + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial \boldsymbol{x}} \boldsymbol{\mu}(t, \theta_k, y_{s^-}, \boldsymbol{x}_{s^-}) ds + Tr\Big[\frac{1}{2} \boldsymbol{\sigma}^T(s, y_{s^-}, \theta_{s^-}, \boldsymbol{x}_{s^-}) \frac{\partial^2 V}{\partial \boldsymbol{x}^2} \boldsymbol{\sigma}(s, y_{s^-}, \theta_{s^-}, \boldsymbol{x}_{s^-})\Big] ds \\
& + \frac{\partial V}{\partial \boldsymbol{x}} \boldsymbol{\sigma}(s, \theta_k, y_{s^-}, \boldsymbol{x}_{s^-}) d\boldsymbol{B}_s \\
& + \int_{|z|\leq 1} \Big[V(s, y_{s^-}, \theta_{s^-}, \boldsymbol{x}_{s^-} + \boldsymbol{x}_{s^-} \boldsymbol{G}(s, y_{s^-}, \theta_{s^-}, \boldsymbol{x}_{s^-}, z)) - V(s, y_{s^-}, \theta_{s^-}, \boldsymbol{x}_{s^-})\Big] \bar{\psi}(\theta_{s^-}, dz, ds) \\
& + \int_{|z|>1} \Big[V(s, y_{s^-}, \theta_{s^-}, \boldsymbol{x}_{s^-} + \boldsymbol{H}(s, y_{s^-}, \theta_{s^-}, \boldsymbol{x}_{s^-}, z)) - V(s, y_{s^-}, \theta_{s^-}, \boldsymbol{x}_{s^-})\Big] \psi(\theta_{s^-}, dz, ds)
\end{aligned}$$

$$\int_{|z|\le 1}\Big[V(s,y_{s^-},\theta_{s^-},\boldsymbol{x}_{s^-}+\boldsymbol{x}_{s^-}\boldsymbol{G}(s,y_{s^-},\theta_{s^-},\boldsymbol{x}_{s^-},z))-V(s,y_{s^-},\theta_{s^-},\boldsymbol{x}_{s^-})$$

$$-\frac{\partial V}{\partial x}\boldsymbol{G}(s,y_{s^-},\theta_{s^-},\boldsymbol{x}_{s^-},z)\Big]\nu(\theta_{s^-},dz)ds$$

$$+\sum_{j=1,j\neq\theta_{s^-}}^{m}\int_{z\in\mathbb{R}}\Big[V(s,y_s,j,\boldsymbol{x}_{s^-}e^z)-V(s,y_{s^-},\theta_{s^-},\boldsymbol{x}_{s^-})\Big]N(dz,ds,\{(\theta_{s^-},j)\}) \tag{4.2.6}$$

$$=\mathcal{L}V(s,\theta_s,y_s,\boldsymbol{x}_s)ds+\frac{\partial V}{\partial \boldsymbol{x}}\boldsymbol{\sigma}(s,\theta_k,y_{s^-},\boldsymbol{x}_{s^-})d\boldsymbol{B}_s$$

$$+\int_{|z|\le 1}\Big[V(s,y_{s^-},\theta_{s^-},\boldsymbol{x}_{s^-}+\boldsymbol{G}(s,y_{s^-},\theta_{s^-},\boldsymbol{x}_{s^-},z))-V(s,y_{s^-},\theta_{s^-},\boldsymbol{x}_{s^-})\Big]\bar{\psi}(\theta_{s^-},dz,ds)$$

$$\int_{|z|>1}\Big[V(s,y_{s^-},\theta_{s^-},\boldsymbol{x}_{s^-}+\boldsymbol{H}(s,y_{s^-},\theta_{s^-},\boldsymbol{x}_{s^-},z))-V(s,y_{s^-},\theta_{s^-},\boldsymbol{x}_{s^-})\Big]\bar{\psi}(\theta_{s^-},dz,ds)$$

$$+\sum_{j=1,j\neq\theta_{s^-}}^{m}\int_{z\in\mathbb{R}}\Big[V(s,y_s,j,\boldsymbol{x}_{s^-}e^z)-V(s,y_{s^-},\theta_{s^-},\boldsymbol{x}_{s^-})\Big]\bar{N}(dz,ds,\{(\theta_{s^-},j)\}), \tag{4.2.7}$$

where

$$\mathcal{L}V(s,\theta_{s^-},y_{s^-},\boldsymbol{x}_{s^-})=\frac{\partial V}{\partial s}+\frac{\partial V}{\partial y}+\frac{\partial V}{\partial \boldsymbol{x}}\boldsymbol{\mu}(t,\theta_k,y_{s^-},\boldsymbol{x}_{s^-})$$

$$+\frac{1}{2}Tr\Big[\boldsymbol{\sigma}^T(s,y_{s^-},\theta_{s^-},\boldsymbol{x}_{s^-})\frac{\partial^2 V}{\partial \boldsymbol{x}^2}\boldsymbol{\sigma}(s,y_{s^-},\theta_{s^-},\boldsymbol{x}_{s^-})\Big]$$

$$+\int_{|z|\le 1}\Big[V(s,y_{s^-},\theta_{s^-},\boldsymbol{x}_{s^-}+\boldsymbol{G}(s,y_{s^-},\theta_{s^-},\boldsymbol{x}_{s^-},z))-V(s,y_{s^-},\theta_{s^-},\boldsymbol{x}_{s^-}) \tag{4.2.8}$$

$$-\frac{\partial V}{\partial \boldsymbol{x}}\boldsymbol{G}(s,y_{s^-},\theta_{s^-},\boldsymbol{x}_{s^-},z)\Big]\nu(\theta_{s^-},dz)$$

$$+\int_{|z|>1}\Big[V(s,y_{s^-},\theta_{s^-},\boldsymbol{x}_{s^-}+\boldsymbol{H}(s,y_{s^-},\theta_{s^-},\boldsymbol{x}_{s^-},z))-V(s,y_{s^-},\theta_{s^-},\boldsymbol{x}_{s^-})\Big]\nu(\theta_{s^-},dz)$$

$$+\int_{z\in\mathbb{R}}\sum_{j\in E-\{\theta_{s^-}\}}\Big[V(s,y_s,j,\boldsymbol{x}_{s^-}e^z)-V(s,y_{s^-},\theta_{s^-},\boldsymbol{x}_{s^-})\Big]\gamma(dz,ds,\{(\theta_{s^-},j)\}) \tag{4.2.9}$$

We denote $P^n_{M^n}=\{t_k\}_{k=0}^{M_n}, 0=t_0<t_1<...<t_{M_n}=T$ a partition of the time interval $[0,T]$. Using the notations $V_k=V(t_k,\theta_{t_k},y_{t_k},\boldsymbol{x}_{t_k})$, $G_k=G(t_k,y_{t_k},\theta_{t_k},\boldsymbol{x}_{t_k},z)$, $H_k=H(t_k,y_{t_k},\theta_{t_k},\boldsymbol{x}_{t_k},z)$, $V^G_{k-1}(z)=V(t_{k-1},y_{t_{k-1}},\theta_{t_{k-1}},x_{t_{k-1}}+\boldsymbol{G}(t_{k-1},y_{t_{k-1}},\theta_{t_{k-1}},\boldsymbol{x}_{t_{k-1}},z))$ and $V^H_{k-1}(z)=V(t_{k-1},y_{t_{k-1}},\theta_{t_{k-1}},x_{t_{k-1}}+\boldsymbol{H}(t_{k-1},y_{t_{k-1}},y_{t_{k-1}},\boldsymbol{x}_{t_{k-1}},z))$, the first and second moments are presented below:

$$E\big[\Delta V_k\big|\mathbb{G}_{t_{k-1}}\big]=\mathcal{L}V_{k-1}\Delta t_k$$

$$E\Bigg[\Big[\Delta V_k-E\big[\Delta V_k\big|\mathbb{G}_{t_{k-1}}\big]\Big]\Big[\Delta V_k-E\big[\Delta V_k\big|\mathbb{G}_{t_{k-1}}\big]\Big]^T\Bigg|\mathbb{G}_{t_{k-1}}\Bigg] \tag{4.2.10}$$

$$=\frac{\partial V}{\partial \boldsymbol{x}}\boldsymbol{\sigma}(s,y_{s^-},\theta_{s^-},\boldsymbol{x}_{s^-})\boldsymbol{\sigma}^T(s,y_{s^-},\theta_{s^-},\boldsymbol{x}_{s^-})\frac{\partial V}{\partial \boldsymbol{x}}^T\Delta t_{k-1}$$

$$+\int_{|z|\leq 1}\left[V^G_{k-1}(z)-V_{k-1}\right]\left[V^G_{k-1}(z)-V_{k-1}\right]^T\nu(\theta_{t_{k-1}},dz)\Delta t_k$$
$$+\int_{|z|>1}\left[V^H_{k-1}(z)-V_{k-1}\right]\left[V^H_{k-1}(z)-V_{k-1}\right]^T\nu(\theta_{t_{k-1}},dz)\Delta t_k$$
$$+\sum_{j\in E-\{\theta_{t_{k-1}}\}}\int_{z\in\mathbb{R}}\left[V(t_{k-1},y_{t_{k-1}},j,\boldsymbol{x}_{t_{k-1}}e^z)-V_{k-1}\right]\left[V(t_{k-1},y_{t_{k-1}},j,\boldsymbol{x}_{t_{k-1}}e^z)-V_{k-1}\right]^T$$
$$\times\gamma(dz,\Delta t_k,\{(\theta_{t_{k-1}},j)\}) \quad (4.2.11)$$

## 4.3 Illustrations

In the following, we examine two particular cases of the model in (4.2.1).

### 4.3.1 Modeling of a basket of interdependent assets

We extend the system of interconnected commodity price process network in [56] by considering Levy jumps representing shocks specific to each member of the network. We assume that each asset in the network is affected by independent unpredictable shocks/informations. If the network considered is a network of financial assets, unpredictable shocks may originate from sudden information affecting investors views of the particular financial sector of interest or other unpredictable change affecting parameters playing a preeminent role on the asset price. Unlike [56, 57], we assume two layers of interactions of asset prices in the network. We assume asset price interactions in the diffusion coefficient and in the price jumps. Similarly to [56, 57], the i-th asset price impacts the j-th asset price through cross diffusion coefficients of order (i,j). A second layer of interaction introduced in this paper is achieved by assuming that a jump of the i-th asset price impacts the j-th asset price through an appropriately modeled cross dependence parameter of order (i,j). Let us assume that the network entails $n\geq 1$ assets. We model these shocks with a family of Lévy-Itó processes with Poisson measures $\boldsymbol{\psi}^m=(\psi^m_1,...\psi^m_n)^T$ having intensity processes $\boldsymbol{\nu}^m=(\nu^m_1,...\nu^m_n)^T$ and $\boldsymbol{\psi}^p=(\psi^p_1,...\psi^p_n)^T$ having intensity processes $\boldsymbol{\nu}^p=(\nu^p_1,...\nu^p_n)^T$. We also denote $\bar{\boldsymbol{\psi}}^m=(\bar{\psi}^m_1,...\bar{\psi}^m_n)^T=\boldsymbol{\psi}^m=(\psi^m_1,...\psi^m_n)^T-\boldsymbol{\nu}^m=(\nu^m_1,...\nu^m_n)^T$ and $\boldsymbol{\psi}^p=(\psi^p_1,...\psi^p_n)^T$ with intensity processes $\boldsymbol{\nu}^p=(\nu^p_1,...\nu^p_n)^T$. We assume that the vector mean process $\boldsymbol{m}^m=(m^m_1,...m^m_n)$ and vector price process $\boldsymbol{p}^m=(p^m_1,...p^m_n)$ potentially react differently to market shocks through Poisson integrands $(G^m_i)^n_{i=1},(H^m_i)^n_{i=1},(G^p_i)^n_{i=1}$ and $(H^p_i)^n_{i=1}$. We introduce the following matrices: $\boldsymbol{G}^m=(G^m_j)_{j\leq n},\boldsymbol{G}^p=(G^p_j)_{j\leq n},\boldsymbol{H}^m=(H^m_j)_{j\leq n},\boldsymbol{G}^p=(H^p_j)_{j\leq n},\boldsymbol{W}=$

$(W_{i,j})_{i,j\leq n}$ and $and\boldsymbol{Z} = (Z_{i,j})_{i,j\leq n}$ are real numbers $\forall i,j \in I(1,n)$. $W_{i,j}$ and $Z_{i,j}$ are independent Brownian motion processes. A coupled system of Levy type stochastic differential equations under regime changes and subjected to structural perturbations can be expressed as follows:

$$\begin{cases} dm_j = (u_j - m_j)\Big[\sum_{l=1}^{n}\kappa_{lj}(t^-)m_l(t^-)dt + \delta_{jj}dW_{j,j}(t) + \sum_{l=1,l\neq j}^{n}\delta_{lj}m_l(t^-)dW_{l,j}(t)\Big] \\ + \int_{|z|<1} G_l^m(t,\boldsymbol{m}_{t^-},z)\bar{\psi}_l^m(dt,dz) + \int_{|z|>1} H_l^m(t,\boldsymbol{m}_t,z)\psi_l^m(dt,dz), t\in[T_k,T_{k+1}), m_j(t_0) = m_{j,0}, \\ m_j(T_k) = m_j^k \text{ and } m_j^k = \pi_j^k m(T_k^-,T_{k-1},\boldsymbol{m}), \forall k \in I(1,\infty), \\ dp_j = p_j\Big[\big[\gamma_{jj}(m_j - p_j) + \beta_j + \sum_{l=1,l\neq j}^{n}\gamma_{lj}(t^-)p_l(t^-)\big]dt + \sigma_{jj}dZ_{j,j}(t) + \sum_{l=1,l\neq j}^{n}\sigma_{lj}p_l(t^-)dZ_{l,j}(t)\Big] \\ + \int_{|z|<1} G_l^p(t,\boldsymbol{p}_{t^-},z)\bar{\psi}_l^p(dt,dz) + \int_{|z|>1} H_l^p(t,\boldsymbol{p}_t,z)\psi_l^p(dt,dz), t\in[T_k,T_{k+1}), p_j(t_0) = p_{j,0}, \\ p_j(T_k) = p_j^k \text{ and } p_j^k = \omega_j^k p(T_k^-,T_{k-1},\boldsymbol{p},\boldsymbol{m}), \forall k \in I(1,\infty). \end{cases} \tag{4.3.1}$$

We define the following matrices:

$$\boldsymbol{\kappa} = \begin{pmatrix} \kappa_{11} & \dots & \kappa_{1n} \\ \vdots & \dots & \vdots \\ \kappa_{n1} & \dots & \kappa_{nn} \end{pmatrix}, \boldsymbol{\gamma} = \begin{pmatrix} -\gamma_{11} & \dots & \gamma_{1n} \\ \vdots & \dots & \vdots \\ \gamma_{n1} & \dots & -\gamma_{nn} \end{pmatrix},$$

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_{11} & \dots & \sigma_{1n}p_1 \\ \vdots & \dots & \vdots \\ \sigma_{n1}p_n & \dots & \sigma_{nn} \end{pmatrix}, \boldsymbol{\Upsilon} = \begin{pmatrix} \delta_{11} & \dots & \delta_{1n}m_1 \\ \vdots & \dots & \vdots \\ \delta_{n1}m_n & \dots & \delta_{nn} \end{pmatrix},$$

$$\boldsymbol{W} = \begin{pmatrix} W_{11} & \dots & W_{1n} \\ \vdots & \dots & \vdots \\ W_{n1} & \dots & W_{nn} \end{pmatrix}, \boldsymbol{Z} = \begin{pmatrix} Z_{11} & \dots & Z_{1n} \\ \vdots & \dots & \vdots \\ Z_{n1} & \dots & Z_{nn} \end{pmatrix}$$

This coupled system in matrix form becomes:

$$\begin{aligned} d\boldsymbol{m}_t &= \boldsymbol{a}(t,y_t,\theta_j,\boldsymbol{m})dt + \Upsilon(t,y_t,\theta_j,\boldsymbol{m})d\boldsymbol{W}(t) + \int_{|z|<1}\boldsymbol{G}^m(t,y_t,\theta_j,\boldsymbol{m},z)\bar{\psi}_m(\theta_j,dt,dz) \\ &+ \int_{|z|>1}\boldsymbol{H}^m(t,y_t,\theta_t,\boldsymbol{m},z)\psi(\theta_j,dt,dz), \text{ where } \boldsymbol{m}(t_0) = \boldsymbol{m}_0, \text{ if } t\in[T_j,T_{j+1}) \\ d\boldsymbol{p}_t &= \boldsymbol{b}(t,y,\theta_j,\boldsymbol{p},\boldsymbol{m})dt + \boldsymbol{\Sigma}(t,y,\theta_j,\boldsymbol{p})d\boldsymbol{Z}(t) + \int_{|z|<1}\boldsymbol{G}^p(t,y,\theta_j,\boldsymbol{p},z)\bar{\psi}_p(\theta_j,dt,dz) \\ &+ \int_{|z|>1}\boldsymbol{H}^p(t,y_t,\theta_j,\boldsymbol{p},z)\psi(\theta_j,dt,dz), \text{ where } \boldsymbol{p}(t_0) = \boldsymbol{p}_0, \text{ if } t\in[T_j,T_{j+1}) \end{aligned} \tag{4.3.2}$$

$$\boldsymbol{m}^i = \Pi^i\boldsymbol{m}^{i-1}(T_i^-,T_{i-1},\boldsymbol{m}^{i-1}), \tag{4.3.3}$$

$$\boldsymbol{p}^i = \Omega^i \boldsymbol{p}^{i-1}(T_i^-, T_{i-1}, \boldsymbol{p}^{i-1}), \tag{4.3.4}$$

with

$$\Pi^i = diag(\pi_1, ..., \pi_n), \text{ where } \ln(\pi_k) \sim g(|\theta_i, \theta_{i-1}), \forall k \in I(1,n), i \in I(1,\infty), \tag{4.3.5}$$

$$\Omega^i = diag(\omega_1, ...\omega_n), \text{ where } \ln(\omega_k) \sim h(|\theta_i, \theta_{i-1}), \forall k \in I(1,n), i \in I(1,\infty), \tag{4.3.6}$$

where $\boldsymbol{a}(t, y, \theta_j, \boldsymbol{m})$ and $\boldsymbol{b}(t, y, \theta_j, \boldsymbol{p}, \boldsymbol{m})$ are $n$ dimensional vectors with $\boldsymbol{a}(t, y, \theta_j, \boldsymbol{p}, \boldsymbol{m}) = \boldsymbol{\kappa}^T(t^-)\boldsymbol{p}(t^-)$ and $\boldsymbol{b}(t, y, \theta_j, \boldsymbol{p}, \boldsymbol{m}) = -\boldsymbol{\gamma} diag(m_1, ..., m_n) + \boldsymbol{\beta} + \boldsymbol{\gamma}\boldsymbol{p}$; We consider the particular case of (4.3.1) where, $H_l^p(t, y_t, \theta_t, \boldsymbol{p}_t, z) = G_l^p(t, y_t, \theta_t, \boldsymbol{p}_t, z) = z$ and $H_l^m(t, y_t, \theta_t, \boldsymbol{m}_t, z) = G_l^m(t, y_t, \theta_t, \boldsymbol{p}_t, z) = z$. We also consider the following families of Lyapunov functions: $V^{j,q}(t, y_t, \theta_t, \boldsymbol{m}) = m_j^q$ and $V^{j,q}(t, y_t, \theta_t, \boldsymbol{p}) = p_j^q \; \forall j \in I(1,n), q \in I(1,\infty)$. Using the multivariate real valued version of Itó formula in (4.2.8) and (4.2.10) applied to $U$, we obtain, respectively:

$$\begin{aligned}
d(m_j^q(t)) &= q m_j^{q-1}(t^-)(u_j - m_j(t^-))\Big[\sum_{l=1}^n \kappa_{j,l} m_l(t^-)\Big]dt \\
&+ \frac{1}{2} q(q-1) m_j^{q-2}(t^-)(u_j - m_j(t^-))^2 \Big[\sum_{l=1}^n \delta_{j,l}^2 m_l^2(t^-)\Big]dt \\
&+ \int_{|z|\le 1} \Big[\big[m_j(t^-) + m_j(t^-) G_l^m(t^-)\big]^q - m_j^q(t^-)\Big] \bar{\psi}_j^m(\theta_{t^-}, dz) \\
&+ \int_{|z|>1} \Big[\big[m_j^q(t^-) + m_j(t^-) H_j^m(t^-)\big]^q - m_j^q(t^)\Big] \bar{\psi}_j^m(\theta_{t^-}, dz) \\
&\int_{|z|\le 1} \Big[\big[m_j(t^-) + m_j(t^-) G_l^m(t^-)\big]^q - m_j^q(t^-) - q m_j^q(t^-) G_m^j\Big] \nu_j^m(\theta_{t^-}, dz) \\
&+ \int_{|z|>1} \Big[\big[m_j^q(t^-) + m_j(t^-) H_j^l(t^-)\big]^q - m_j^q(t^)\Big] \nu_j^m(\theta_{t^-}, dz) \\
&+ q m^{q-1}(t^-)(u_j - m_j(t^-))\big[\sum_{l=1}^n \delta_{j,l} m_l(t^-) dW_{j,l}(t)\big], \forall t \in [T_s, T_{s+1}], s \in I(1,\infty),
\end{aligned} \tag{4.3.7}$$

$$\begin{aligned}
d(p_j^q(t)) &= q p_j^{q-1}(t^-) p_j(t^-)\big[\gamma_{j,j}(m_j(t^-) - p_j(t^-)) + \beta_j + \sum_{l=1}^n \gamma_{j,l} p_l(t)\big]dt \\
&+ \frac{1}{2} q(q-1) p_j^{q-2}(t^-) p_j^2(t^-)\Big[\sum_{l=1}^n \sigma_{j,l}^2 p_l^2(t^-)\Big] \\
&+ \int_{|z|\le 1} \Big[\big[p_j(t^-) + p_j(t^-) G_j^p(t^-)\big]^q - p_j^q(t^-)\Big] \bar{\psi}_j^p(\theta_{t^-}, dz) \\
&+ \int_{|z|>1} \Big[\big[p_j(t^-) + p_j(t^-) H_j^p(t^-)\big]^q - p_j^q(t^-)\Big] \bar{\psi}_p^j(\theta_{t^-}, dz) \\
&\int_{|z|\le 1} \Big[\big[p_j(t^-) + p_j(t^-) G_j^p\big]^q - p_j^q(t^-) - q p^q(t^-) G_p^j\Big] \nu_j^p(\theta_{t^-}, dz)
\end{aligned}$$

$$+\int_{|z|>1}\Big[\big[p_j(t^-)+p_j(t^-)H_j^p\big]^q-p_j^q(t^-)\Big]\nu_p^j(\theta_{t^-},dz)$$

$$qp^{q-1}(t^-)p_j(t^-)\big[\sum_{l=1}^{n}\sigma_{j,l}p_l(t^-)dZ_{j,l}(t^-)\big],\forall t\in[T_s,T_{s+1}],s\in I(1,\infty). \tag{4.3.8}$$

Applying Euler Maruyama discretization scheme at $t_k$ leads to:

$$E\big[\Delta m_j^q(t_k)\big|\mathbb{G}_{t_{k-1}}\big]=qm_j^{q-1}(t_{k-1})(u_j-m_j(t_{k-1}))\Big[\sum_{l=1}^{n}\kappa_{j,l}m_l(t^-))\Big]\Delta t_k$$
$$+\frac{1}{2}q(q-1)m_j^{q-2}(t_{k-1})(u_j-m_j(t_{k-1}))^2\Big[\sum_{l=1}^{2}\delta_{j,l}^2(t_{k-1})\Big]$$
$$+\int_{|z|\leq 1}\Big[\big[m_j(t_{k-1})+m_j(t^-)G_j^m(t_{k-1})\big]^q-m_j^q(t_{k-1})-qm_j^q(t_{k-1})G_j^m(t_{k-1})\Big]\nu_j^m(\theta_{t_{k-1}},dz)$$
$$+\int_{|z|>1}\Big[\big[m_j(t_{k-1})+m_j(t^-)H_j^m(t_{k-1})\big]^q-m_j^q(t_{k-1})\Big]\nu_j^m(\theta_{s^-},dz), \tag{4.3.9}$$

$$E\Big[\big[\Delta m_j^q(t_k)-E\big[\Delta m_j^q(t_k)\big|\mathbb{G}_{t_{k-1}}\big]\big]^2\Big|\mathbb{G}_{t_{k-1}}\Big]$$
$$=(u_j-m_j(t_{k-1}))^2\Bigg[q^2m_j^{2(q-1)}(t_{k-1})\Big[\sum_{l=1}^{n}\delta_{j,l}^2p_l(t_{k-1})\Big]\Delta t_k$$
$$+\int_{|z|\leq 1}\Big[\big[m_j(t_{k-1})+m_j(t^-)G_m^j(t_{k-1})\big]^q-m_j^q(t_{k-1})\Big]^2\nu_j^m(\theta_{t_{k-1}},dz)\Delta t_k$$
$$+\int_{|z|>1}\Big[\big[m_j(t_{k-1})+m_j(t^-)H_m^j(t_{k-1})\big]^q-m_j^q(t_{k-1})\Big]^2\nu_m^j(\theta_{t_{k-1}},dz)\Delta t_k\Big]\Bigg], \tag{4.3.10}$$

and

$$E\big[\Delta p_j^q(t_k)\big|\mathbb{G}_{t_{k-1}}\big]=qp_j^{q-1}(t_{k-1})p_j(t_{k-1})\big[\gamma_{j,j}(m_j(t^-)-p_j(t^-))+\beta_j+\sum_{l=1}^{n}\gamma_{j,l}m_l(t)\big]\Delta t_k$$
$$+\frac{1}{2}q(q-1)p_j^{q-2}(t_k)p_j^2(t_{k-1})\Big[\sum_{l=1}^{n}\sigma_{j,l}^2p_l^2(t_{k-1})\Big]$$
$$+\int_{|z|\leq 1}\Big[\big[p_j(t_{k-1})+p_j(t_{k-1})G_j^p(t_{k-1})\big]^q-p_j^q(t_{k-1})-qp_j^q(t_{k-1})G_j^p(t_{k-1})\Big]\nu_j^p(\theta_{s^-},dz)$$
$$+\int_{|z|>1}\Big[\big[p_j(t_{k-1})+p_j(t_{k-1})H_j^p(t_k)\big]^q-p_j^q(t_{k-1})\Big]\nu_p^l(\theta_{s^-},dz), \tag{4.3.11}$$

$$E\Big[\big[\Delta p_j^q(t_k)-E\big[\Delta p_j^q(t_k)\big|\mathbb{G}_{t_{k-1}}\big]\big]^2\Big|\mathbb{G}_{t_{k-1}}\Big]=q^2p_j^{2q(t_{k-1})}\Big[\sum_{l=1}^{n}\sigma_{j,l}^2(t_{k-1})\Big]\Delta t_k$$
$$+\int_{|z|\leq 1}\Big[\big[p_j(t_{k-1})+p_j(t_{k-1})G_l^p(t_{k-1})\big]^q-p_j^q(t_{k-1})\Big]^2\nu_p^j(\theta_{t_{k-1}},dz)\Delta t_k$$
$$+\int_{|z|>1}\Big[\big[p_j(t_{k-1})+p_j(t_{k-1})H_j^p(t_{k-1})\big]^q-p_j^q(t_{k-1})\Big]^2\nu_p^j(\theta_{t_{k-1}},dz)\Delta t_k. \tag{4.3.12}$$

We note that Levy jumps sizes of a fixed price process(resp: fixed mean process) are independent and identically distributed for a given market regime. The Euler Maruyama discretization in 4.3.9 and (4.3.11) involve first and second moments of Levy jump sizes which we will estimate using Monte Carlo integration. We note from model definition in 4.3.1 that the contribution of small and big Levy jumps in $dp_j(t)$(resp: $m_j(t)$), the price (resp: the mean) change in asset $j$ at time $t$ are respectively $p_(t^-)\int_{|z|<1} G_l^p(t, \boldsymbol{p}_{t^-}, z)$ and $p_j(t^-)\int_{|z|>1} H_l^p(t, \boldsymbol{p}_t, z)$ (resp: $m_(t^-)\int_{|z|<1} G_l^m(t, \boldsymbol{m}_{t^-}, z)$ and $m_j(t^-)\int_{|z|>1} H_l^m(t, \boldsymbol{m}_t, z)$). Hence, the Levy Jumps are estimated from the price and mean process return time series in the following manner (see [61]): $\int_s^t \int_{z\in\mathbb{R}} f(\eta, z)\nu_l^m(i, dz)d\eta \approx \frac{1}{n^l(i,m,[s,t])}\sum_{\zeta_i^{m_l}\in[s,t]}^{n(m_l,[s,t])} f(\zeta_i^{m_l}, z_{\zeta_i^{m_l}}), \forall t, s \in [0, T]$ with $\theta_\eta = i, \forall \eta \in [s, t]$. $(\zeta_i^{m_l})_{i\in\mathcal{N}^\star}$ is the sequence of Levy time jumps of the mean price process $m_l$. $n(m_l, [s, t])$ denotes the total number of jumps of the $l - th$ mean asset price process while in regime $i$ in the time interval $[s, t]$ and $\left(z_{\zeta_i^{m_l}}\right)_{i\in\mathcal{N}^\star}$ represents the corresponding sequence of Levy jump sizes. We assume that the function $f$ is integrable with respect to the product measure $\nu_l^m(i, )dt$ in the domain $\mathbb{R} \times [0, T]$. Throughout the discretization process, the four functions used inside the Levy integral are $f(\eta, z) = 1_{(|z|\leq 1)}, f(\eta, z) = 1_{(|z|\leq 1)}z^2, f(\eta, z) = 1_{(|z|>1)}z$ and $f(\eta, z) = 1_{(|z|>1)}z^2$. For convenience of notation, from here on, for each of the preceding functions we denote the Monte Carlo estimates by $M_1^s(m_l), M_2^s(m_l), M_1^b(m_l)$ and $M_2^b(m_l)$, respectively. Approximations of the exponential in the Levy integrand and estimation of the Levy integrals are performed by Taylor expansions and Monte Carlo sums respectively. These lead to the following:

$$E\big[\Delta m_j^q(t_k)\big|\mathbb{G}_{t_{k-1}}\big] = qm_j^{q-1}(t_{k-1})(u_j - m_j(t_{k-1}))\Big[\sum_{l=1}^{n}\kappa_{j,l}m_l(t^-))\Big]\Delta t_k$$
$$+ \frac{1}{2}q(q-1)m_j^{q-2}(t_{k-1})(u_j - m_j(t_{k-1}))^2\Big[\sum_{l=1}^{2}\delta_{j,l}^2(t_{k-1})\Big] + qm_j^q(t^-)M_j^1(bm), \tag{4.3.13}$$
$$E\Big[\big[\Delta m_j^q(t_k) - E\big[\Delta m_j^q(t_k)\big|\mathbb{G}_{t_{k-1}}\big]\big]^2\Big|\mathbb{G}_{t_{k-1}}\Big]$$
$$= (u_j - m_j(t_{k-1}))^2\Big[q^2 m_j^{2(q-1)}(t_{k-1})\Big[\sum_{l=1}^{n}\delta_{j,l}^2 m_l^2(t_{k-1})\Big]\Delta t_k\Big]$$
$$+ q^2 m_j^{2q}(t_k)M_j^2(sm) + q^2 m_j^{2q}(t_k)M_j^2(bm), \tag{4.3.14}$$

and

$$E\big[\Delta p_j^q(t_k)\big|\mathbb{G}_{t_{k-1}}\big] = qp_j^{q-1}(t_{k-1})p_j(t_{k-1})\big[\gamma_{j,j}(m_j(t^-) - p_j(t^-)) + \beta_j + \sum_{l=1}^{n}\gamma_{j,l}m_l(t)\big]\Delta t_k$$

$$+\frac{1}{2}q(q-1)p_j^{q-2}(t_k)p_j^2(t_{k-1})\Big[\sum_{l=1}^{n}\sigma_{j,l}^2p_l^2(t_{k-1})\Big]+qp_j^q(t_{k-1})M_j^1(bp), \tag{4.3.15}$$

$$E\Big[\big[\Delta p_j^q(t_k)-E\big[\Delta p_j^q(t_k)\big|\mathbb{G}_{t_{k-1}}\big]\big]^2\Big|\mathbb{G}_{t_{k-1}}\Big]=q^2p_j^{2q}(t_{k-1})\Big[\sum_{l=1}^{n}\sigma_{j,l}^2(t_{k-1})p_j^2(t_{k-1})\Big]\Delta t_k$$

$$+q^2p_j^{2q}(t_{k-1})M_j^2(sp)\Delta t_k+q^2p_j^{2q}(t_{k-1})M_j^2(bp)\Delta t_k. \tag{4.3.16}$$

We note that the particular case $q=1$ in (4.3.9) and (4.3.11) yields the following systems:

$$E\big[\Delta m_j(t_k)\big|\mathbb{G}_{t_{k-1}}\big]=(u_j-m_j(t_{k-1}))\Big[\sum_{l=1}^{n}\kappa_{j,l}m_l(t)\Big)\Big]\Delta t_k$$

$$+m_j(t_{k-1})M_j^1(bm), \tag{4.3.17}$$

$$E\Big[\big[\Delta m_j(t_k)-E\big[\Delta m_j(t_k)\big|\mathbb{G}_{t_{k-1}}\big]\big]^2\Big|\mathbb{G}_{t_{k-1}}\Big]=(u_j-m_j(t_{k-1}))^2\sum_{l=1}\delta_{j,l}^2m_l^2(t_{k-1})\Delta t_k$$

$$+m_j^2(t_{k-1})[M_j^2(sm)\Delta t_k+M_j^2(bm)\Delta t_k], \tag{4.3.18}$$

and

$$E\big[\Delta p_j(t_k)\big|\mathbb{G}_{t_{k-1}}\big]=p_j(t_{k-1})\big[\gamma_{j,j}(m_j(t^-)-p_j(t^-))+\beta_j+\sum_{l=1,l\neq j}\gamma_{j,l}p_l(t)\big]\Delta t_k$$

$$+p_j(t_{k-1})M_j^2(bp), \tag{4.3.19}$$

$$E\Big[\big[\Delta p_j(t_k)-E\big[\Delta p_j(t_k)\big|\mathbb{G}_{t_{k-1}}\big]\big]^2\Big|\mathbb{G}_{t_{k-1}}\Big]=p_j^2(t_{k-1})\Big[\Big[\sum_{l=1}^{n}\sigma_{j,l}^2p_l^2(t_{k-1})\Big]\Delta t_k$$

$$+M_2^s(p_j)\Delta t_{k-1}+M_2^b(p_j)\Delta t_{k-1}\Big]. \tag{4.3.20}$$

When $q=1$, from (4.3.13) we have

$$(u_j-m_j(t_{k-1}))^2\Big[\sum_{l=1}\delta_{j,l}^2m_l^2(t_{k-1})\Big]=E\Big[\big[\Delta m_j(t_k)-E\big[\Delta m_j(t_k)\big|\mathbb{G}_{t_{k-1}}\big]\big]^2\Big|\mathbb{G}_{t_{k-1}}\Big]$$

$$-m_j^2(t_k)M_j^2(sm)-m_j^2(t_k)M_j^2(bm). \tag{4.3.21}$$

Using (4.3.21) in the first moment equation of (4.3.13) yields:

$$E\big[\Delta m_j^q(t_k)\big|\mathbb{G}_{t_{k-1}}\big]=qm_j^{q-1}(t_{k-1})(u_j-m_j(t_{k-1}))\Big[\sum_{l=1}^{n}\kappa_{j,l}m_l(t^-)\Big)\Big]\Delta t_k$$

$$+\frac{1}{2}q(q-1)m_j^{q-2}(t_{k-1})\Big[E\Big[\big[\Delta m_j(t_k)-E\big[\Delta m_j(t_k)\big|\mathbb{G}_{t_{k-1}}\big]\big]^2\Big|\mathbb{G}_{t_{k-1}}\Big]-m_j^2(t_k)M_j^2(sm)$$

$$-m_j^2(t_k)M_j^2(bm)\Big]+qm_j^q(t^-)M_j^1(bm), \tag{4.3.22}$$

hence,

$$E\big[\Delta m_j^q(t_k)\big|\mathbb{G}_{t_{k-1}}\big] - \frac{1}{2}q(q-1)m_j^{q-2}(t_{k-1})E\Big[\big[\Delta m_j(t_k) - E\big[\Delta m_j(t_k)\big|\mathbb{G}_{t_{k-1}}\big]\big]^2\Big|\mathbb{G}_{t_{k-1}}\Big]$$
$$+\frac{1}{2}q(q-1)m_j^q(t_k)M_j^2(sm) + \frac{1}{2}q(q-1)m_j^q(t_k)M_j^2(bm) - qm_j^q(t^-)M_j^1(bm)$$
$$= \kappa_{jj}\Big[qu_jm_j^q(t_{k-1}) - qm_j^{q+1}(t_{k-1})\Big] + \kappa_{jl}\Big[qu_jm_j^{q-1}(t_{k-1})m_l(t_{k-1}) - qm_j^q(t_{k-1})m_l(t_{k-1})\Big]. \tag{4.3.23}$$

Writing the estimating equations at time $t_k$ yields:

$$\sum_{k=s-n_s}^{s-1} E\big[\Delta m_j^q(t_k)\big|\mathbb{G}_{t_{k-1}}\big] - \frac{1}{2}q(q-1)\sum_{k=s-n_s}^{s-1} m_j^{q-2}(t_{k-1})E\Big[\big[\Delta m_j(t_k) - E\big[\Delta m_j(t_k)\big|\mathbb{G}_{t_{k-1}}\big]\big]^2\Big|\mathbb{G}_{t_{k-1}}\Big]$$
$$+\frac{1}{2}q(q-1)M_j^2(sm)\sum_{k=s-n_s}^{s-1} m_j^q(t_k) + \frac{1}{2}q(q-1)M_j^2(bm)\sum_{k=s-n_s}^{s-1} m_j^q(t_k) - q\sum_{k=s-n_s}^{s-1} m_j^q(t^-)M_j^1(bm)$$
$$= \kappa_{jj}\Big[qu_j\sum_{k=s-n_s}^{s-1} m_j^q(t_{k-1}) - q\sum_{k=s-n_s}^{s-1} m_j^{q+1}(t_{k-1})\Big] + \kappa_{jl}\Big[qu_j\sum_{k=s-n_s}^{s-1} m_j^{q-1}(t_{k-1})m_l(t_{k-1})$$
$$-q\sum_{k=s-n_s}^{s-1} m_j^q(t_{k-1})m_l(t_{k-1})\Big] \tag{4.3.24}$$

Let us set:

$$A_{j,m}^q = q\sum_{k=s-n_s}^{s-1} m_j^q(t_{k-1})(u_j - m_j(t_{k-1})) = a_1^q u + a_2^q$$
$$B_{j,m}^q = q\sum_{k=s-n_s}^{s-1} m_j^{q-1}(t_{k-1})(u_j - m_j(t_{k-1}))m_l(t_{k-1}) = b_1^q u + b_2^q$$
$$C_{j,m}^q = \sum_{k=s-n_s}^{s-1} E\big[\Delta m_j^q(t_k)\big|\mathbb{G}_{t_{k-1}}\big]$$
$$-\frac{1}{2}q(q-1)\sum_{k=s-n_s}^{s-1} m_j^{q-2}(t_{k-1})E\Big[\big[\Delta m_j(t_k) - E\big[\Delta m_j(t_k)\big|\mathbb{G}_{t_{k-1}}\big]\big]^2\Big|\mathbb{G}_{t_{k-1}}\Big]$$
$$+\frac{1}{2}q(q-1)M_j^2(sm)\sum_{k=s-n_s}^{s-1} m_j^q(t_k) + \frac{1}{2}q(q-1)M_j^2(bm)\sum_{k=s-n_s}^{s-1} m_j^q(t_k) - q\sum_{k=s-n_s}^{s-1} m_j^q(t^-)M_j^1(bm)$$
$$a_1^q = q\sum_{k=s-n_s}^{s-1} m_j^q(t_{k-1}), a_2^q = q\sum_{k=s-n_s}^{s-1} m_j^{q+1}(t_{k-1})$$
$$b_1^q = q\sum_{k=s-n_s}^{s-1} m_l^{q-1}(t_{k-1})m_l(t_{k-1}), b_2^q = q\sum_{k=s-n_s}^{s-1} m_j^q(t_{k-1})m_l(t_{k-1}).$$

For $l \in \{1,2\}$ with, $l \neq j$ we seek to estimate the parameters $u_j, \kappa_{lj}^m$ and $\kappa_{lj}^m$. The deterministic interaction coefficient parameters $\kappa_{lj}$ are estimated using the first moment equation in (4.3.24), for any three real values $q_1, q_2$ and $q_3$ of the parameter $q$.

$$A_{j,m}^{q_1}\kappa_{jj} + B_{j,m}^{q_1}\kappa_{lj} = C_{j,m}^{q_1}$$
$$A_{j,m}^{q_2}\kappa_{jj} + A_{j,m}^{q_2}\kappa_{lj} = C_{j,m}^{q_2},$$
$$A_{j,m}^{q_3}\kappa_{jj} + A_{j,m}^{q_3}\kappa_{lj} = C_{j,m}^{q_3},$$

yielding the solutions:

$$\begin{aligned}
&u_j^2\big[a_1^{q_3}b_1^{q_2}c^{q_1} - a_1^{q_3}b_1^{q_1}c^{q_2} + a_1^{q_1}b_1^{q_3}c^{q_2} - a_1^{q_2}b_1^{q_3}c^{q_1} - a_1^{q_1}b_1^{q_2}c^{q_3} + a_1^{q_2}b_1^{q_1}c^{q_3}\big]\\
&+ u_j\big[a_1^{q_3}b_2^{q_2}c^{q_1} + a_2^{q_3}b_1^{q_2}c^{q_1} - a_1^{q_3}b_2^{q_2}c^{q_2} - a_2^{q_3}b_1^{q_1}c^{q_2} + a_1^{q_1}b_2^{q_3}c^{q_2} + a_2^{q_1}b_1^{q_3}c^{q_2} - a_2^{q_2}b_1^{q_3}c^{q_1} - a_1^{q_2}b_2^{q_3}c^{q_1}\\
&- a_1^{q_1}b_2^{q_2}c^{q_3} - a_2^{q_1}b_1^{q_2}c^{q_3} + a_1^{q_2}b_2^{q_1}c^{q_1} + a_1^{q_1}b_2^{q_2}c^{q_3}\big]\\
&+ \big[a_2^{q_3}b_2^{q_2}c^{q_1} - a_2^{q_3}b_2^{q_1}c^{q_2} + a_2^{q_1}b_2^{q_3}c^{q_2} - a_2^{q_2}b_2^{q_3}c^{q_1} - a_2^{q_1}b_2^{q_2}c^{q_3} + a_2^{q_2}b_2^{q_1}c^{q_3}\big] = 0\\
&\kappa_{jj} = \frac{B_{j,m}^{q_2}C_{j,m}^{q_1} - C_{j,m}^{q_2}B_{j,m}^{q_1}}{B_{j,m}^{q_2}A_{j,m}^{q_1} - A_{j,m}^{q_2}B_{j,m}^{q_1}}\\
&\kappa_{lj} = \frac{C_{j,m}^{q_2}A_{j,m}^{q_1} - A_{j,m}^{q_2}C_{j,m}^{q_1}}{B_{j,m}^{q_2}A_{j,m}^{q_1} - A_{j,m}^{q_2}B_{j,m}^{q_1}}.
\end{aligned} \tag{4.3.25}$$

We estimate $\delta_{jl}, l \in \{1,2\}, l \neq j$ the continuous random interaction coefficients of the mean processes $(m_l(t))_{t\in[0,T]}, l \in \{1,2\}$ associated with the Brownian motion. We use the second order moment equation in (4.3.14)

$$\begin{aligned}
&\sum_{k=s-n_s}^{s-1} E\Big[\big[\Delta m_j^q(t_k) - E\big[\Delta m_j^q(t_k)\big|\mathbb{G}_{t_{k-1}}\big]\big]^2\Big|\mathbb{G}_{t_{k-1}}\Big] - q^2M_j^2(sm)\sum_{k=s-n_s}^{s-1} m_j^{2q}(t_k)\\
&- q^2M_j^2(bm)\sum_{k=s-n_s}^{s-1} m_j^{2q}(t_k)\\
&= \delta_{jj}^2\sum_{k=s-n_s}^{s-1}(u_j - m_j(t_{k-1}))^2q^2m_j^{2q}(t_{k-1}) + \delta_{jl}^2\sum_{k=s-n_s}^{s-1}(u_j - m_j(t_{k-1}))^2q^2m_j^{2(q-1)}(t_{k-1})m_l(t_{k-1})
\end{aligned} \tag{4.3.26}$$

applied to any two distinct values of $q$, $q_1$ and $q_2$ as follows:

$$A_{j,m}^{q_1,j}\delta_{jj}^2 + B_{j,m}^{q_1,l}\delta_{lj}^2 = C_{j,m}^1 \tag{4.3.27}$$

$$A_{j,m}^{q_2,j}\delta_{jj}^2 + B_{j,m}^{q_2,l}\delta_{lj}^2 = C_{j,m}^2, \tag{4.3.28}$$

where the coefficients are defined as follows:

$$A^{q}_{j,m} = \sum_{k=s-n_s}^{s-1} (u_j - m_j(t_{k-1}))^2 q^2 m_j^{2q}(t_{k-1})\Delta t_k$$

$$B^{q}_{j,m} = \sum_{k=s-n_s}^{s-1} (u_j - m_j(t_{k-1}))^2 q^2 m_j^{2(q-1)}(t_{k-1}) m_l^2(t_{k-1})\Delta t_k$$

$$C^{q}_{j,m} = \sum_{k=s-n_s}^{s-1} E\Big[\big[\Delta m_j^q(t_k) - E\big[\Delta m_j^q(t_k)\big|\mathbb{G}_{t_{k-1}}\big]\big]^2\Big|\mathbb{G}_{t_{k-1}}\Big] - q^2 M_j^2(sm)\sum_{k=s-n_s}^{s-1} m_j^{2q}(t_k)$$

$$- q^2 M_j^2(bm)\sum_{k=s-n_s}^{s-1} m_j^{2q}(t_k)$$

this, therefore yields the solutions:

$$\delta^2_{jj} = \frac{C^{q_1,l}_{j,m} B^{q_2,l}_{j,m} - B^{q_1,l}_{j,m} C^{q_2,l}_{j,m}}{A^{q_1,j}_{j,m} B^{q_2,l}_{j,m} - A^{q_2,j}_{j,m} B^{q_1,l}_{j,m}} \tag{4.3.29}$$

$$\delta^2_{lj} = \frac{A^{q_1,j}_{j,m} C^{q_2,l}_{j,m} - C^{q_1,l}_{j,m} A^{q_2,j}_{j,m}}{A^{q_2,j}_{j,m} B^{q_2,l}_{j,m} - A^{q_2,j}_{j,m} B^{q_1,l}_{j,m}}. \tag{4.3.30}$$

We turn our attention to estimating the parameters of the price processes $p_j, j \in \{1,2\}$. Equation of (4.3.16) has parameters $\sigma^2_{lj}, l \in \{1,2\}$. From

$$\sum_{k=s-n_s}^{s-1} E\Big[\big[\Delta p_j^q(t_k) - E\big[\Delta p_j^q(t_k)\big|\mathbb{G}_{t_{k-1}}\big]\big]^2\Big|\mathbb{G}_{t_{k-1}}\Big] - q^2 \sum_{k=s-n_s}^{s-1} p_j^{2q}(t_{k-1}) M_j^2(sp)\Delta t_k \tag{4.3.31}$$

$$- q^2 \sum_{k=s-n_s}^{s-1} p_j^{2q}(t_{k-1}) M_j^2(bp)\Delta t_k \tag{4.3.32}$$

$$= q^2 \sum_{k=s-n_s}^{s-1} p_j^{2q}(t_{k-1})\Big[\sum_{l=1}^{n} \sigma^2_{j,l}(t_{k-1}) p_l^2(t_{k-1})\Big]\Delta t_k \tag{4.3.33}$$

The parameters $\sigma^2_{lj}, l \in \{1,2\}$ are estimated through the general equation,

$$E_{q,j}\sigma^2_{jj} + F_{q,l}\sigma^2_{lj} = D_{q,l},$$

where,

$$E_{q,j} = q^2 \sum_{k=s-n_s}^{s-1} p_j^{2q+2}(t_{k-1})$$

$$F_{q,l} = q^2 \sum_{k=s-n_s}^{s-1} p_j^{2q}(t_{k-1}) p_l^{2q}(t_{k-1})$$

$$
D_{q,l} = \sum_{k=s-n_s}^{s-1} E\Big[\big[\Delta p_j^q(t_k) - E\big[\Delta p_j^q(t_k)\big|\mathbb{G}_{t_{k-1}}\big]\big]^2\Big|\mathbb{G}_{t_{k-1}}\Big] - q^2 \sum_{k=s-n_s}^{s-1} p_j^{2q}(t_{k-1}) M_j^2(sp)\Delta t_k
$$

$$
- q^2 \sum_{k=s-n_s}^{s-1} p_j^{2q}(t_{k-1}) M_j^2(bp)\Delta t_k
$$

For two distinct values $q_1$ and $q_2$ of $q$ we form the following system:

$$
\begin{aligned}
E_{q_1,j}\sigma_{jj}^2 + F_{q_1,l}\sigma_{lj}^2 &= D_{q_1,l} \\
E_{q_2,j}\sigma_{jj}^2 + F_{q_2,l}\sigma_{lj}^2 &= D_{q_2,l},
\end{aligned}
$$

with solutions;

$$
\sigma_{jj}^2 = \frac{F_{q_2,l}D_{q_1,l} - D_{q_2,l}F_{q_1,l}}{E_{q_1,j}F_{q_2,l} - E_{q_2,j}F_{q_1,l}}, \tag{4.3.34}
$$

$$
\sigma_{lj}^2 = -\frac{D_{q_2,l}E_{q_1,j} - D_{q_1,l}E_{q_2,j}}{E_{q_1,j}F_{q_2,l} - E_{q_2,j}F_{q_1,l}}. \tag{4.3.35}
$$

The last parameters of the price processes are the deterministic interaction coefficient parameters $\gamma_{jj}, \beta_j$ and $\gamma_{lj}, l \neq j$. In order to estimate them, we assume the parameters $\sigma_{jl}, l \in \{1,2\}$ known from the estimation equations (4.3.34). We use the first moment equation in (4.3.15).

$$
\sum_{k=s-n_s}^{s-1} E\big[\Delta p_j^q(t_k)\big|\mathbb{G}_{t_{k-1}}\big] - \frac{1}{2}q(q-1)\sum_{k=s-n_s}^{s-1} p_j^q(t_k)\Big[\sigma_{j,l}^2 + \sigma_{j,l}^2 p_l^2(t_{k-1})\Big] - q\sum_{k=s-n_s}^{s-1} p_j^q(t_{k-1})M_j^1(bp)
$$

$$
= q\sum_{k=s-n_s}^{s-1} p_j^q(t_k)\big[\gamma_{jj}(m_j(t_{k-1}) - p_j(t_{k-1})) + \beta_j + \sum_{l=1,l\neq j}^{n} \gamma_{jl}p_l(t_{k-1})\big]\Delta t_k \tag{4.3.36}
$$

Equation (4.3.36) could also be written

$$
K_{q,j}\gamma_{jj} + Q_q\beta_j + O_{q,l}\gamma_{lj} = S_{q,l},
$$

where the equation coefficients are expressed as follows,

$$
S_{q,l} = \sum_{k=s-n_s}^{s-1} E\big[\Delta p_j^q(t_k)\big|\mathbb{G}_{t_{k-1}}\big] - \frac{1}{2}q(q-1)\sum_{k=s-n_s}^{s-1} p_j^q(t_k)\Big[\sum_{l=1}^{n}\sigma_{j,l}^2 p_l^2(t_{k-1})\Big]
$$

$$
- q\sum_{k=s-n_s}^{s-1} p_j^q(t_{k-1})M_j^1(bp)
$$

$$
O_{q,l} = q\sum_{k=s-n_s}^{s-1} p_j^q(t_k)
$$

$$Q_q = q \sum_{k=s-n_s}^{s-1} p_j^q(t_k)$$
$$K_{q,j} = q \sum_{k=s-n_s}^{s-1} p_j^q(t_k)(m_j(t_{k-1}) - p_j(t_{k-1}))$$

For three distinct values $q_1, q_2$ and $q_3$ of $q$, we form the system:

$$K_{q_1,j}\gamma_{jj} + Q_{q_1}\beta_j + O_{q_1,l}\gamma_{lj} = S_{q_1,l},$$
$$K_{q_2,j}\gamma_{jj} + Q_{q_2}\beta_j + Oq_2, l\gamma_{lj} = S_{q_2,l},$$
$$K_{q_3,j}\gamma_{jj} + Q_{q_3}\beta_j + O_{q_3,l}\gamma_{lj} = S_{q_3,l}.$$

Solutions are expressed in closed form through Cramer rule

$$\gamma_{jj} = \frac{\begin{vmatrix} S_{q_1,l} & Q_{q_1} & O_{q_1,l} \\ S_{q_2,l} & Q_{q_2} & O_{q_2,l} \\ S_{q_3,l} & Q_{q_3} & O_{q_3,l} \end{vmatrix}}{\begin{vmatrix} K_{q_1,j} & Q_{q_1} & O_{q_1,l} \\ K_{q_2,j} & Q_{q_2} & O_{q_2,l} \\ K_{q_3,j} & Q_{q_3} & O_{q_3,l} \end{vmatrix}}$$

$$\beta_j = \frac{\begin{vmatrix} K_{q_1,j} & S_{q_1,l} & O_{q_1,l} \\ K_{q_2,j} & S_{q_2,l} & O_{q_2,l} \\ K_{q_3,j} & S_{q_3,l} & O_{q_3,l} \end{vmatrix}}{\begin{vmatrix} K_{q_1,j} & Q_{q_1} & O_{q_1,l} \\ K_{q_2,j} & Q_{q_2} & O_{q_2,l} \\ K_{q_3,j} & Q_{q_3} & O_{q_3,l} \end{vmatrix}}$$

$$\gamma_{lj} = \frac{\begin{vmatrix} K_{q_1,j} & Q_{q_1} & S_{q_1,l} \\ K_{q_2,j} & Q_{q_2} & S_{q_2,l} \\ K_{q_3,j} & Q_{q_3} & S_{q_3,l} \end{vmatrix}}{\begin{vmatrix} K_{q_1,j} & Q_{q_1} & O_{q_1,l} \\ K_{q_2,j} & Q_{q_2} & O_{q_2,l} \\ K_{q_3,j} & Q_{q_3} & O_{q_3,l} \end{vmatrix}}.$$

### 4.3.2 Modeling of asset prices with a regime switching Heston model

This illustration extends [27] and presents how vanilla option pricing could be performed using a model with stochastic volatility in a market with semi Markov regimes. However, the results and proofs in this second illustration of the general model of coupled stochastic dynamics will rely on a different but equivalent notation for market regimes. We first introduce the new notation for the semi Markov process and a few necessary results.

DEFINITION 4.3.1 *1. The semi Markov process $(\theta_t)_{t\in[0,T]}$ is now represented by a vector process denoted $(\Theta_t)_{t\in[0,T]}$ with $\Theta_t = (1_{(\theta_t=1)}, 1_{(\theta_t=2)}, ..., 1_{(\theta_t=m)})$,*

2. *$\Lambda(y_t) = (\lambda_{ij}(y_t))_{i,j\leq m}$ denotes the conditional matrix of intensities of the semi Markov process $(\Theta_t)_{t\in[0,T]}$,*
3. *the notation $\Lambda^T$ denotes the matrix transposed of $\Lambda$,*
4. *we denote the states of the semi Markov process $(\Theta_t)_{t\in[0,T]}$, $e_1, ...e_m$, where $e_i = \underbrace{(0, ...0, 1, 0, ...0)}_{i-th}$ and $\boldsymbol{I}$ represents the identity matrix.*

We establish a martingale theorem for the vector valued semi Markov process as a particular case of [8].

LEMMA 4.3.1 *Let $\Theta_t = (1_{(\theta_t=1)}, 1_{(\theta_t=2)}, ..., 1_{(\theta_t=m)})$ and $\Lambda(t) = (\lambda_{ij}(y_t))_{i,j\leq m}$ as in Definition 4.3.1, be the vector carrying instantaneous states of the semi Markov process $\Theta_t$ and the conditional intensity matrix of $\Theta_t$. There exists an $\mathbb{R}^m$ vector valued martingale process $M_t$, right continuous with left limits such that:*

$$\Theta_t = \Theta_0 + \int_{t_0}^{t} \Lambda^{'}(y_{u^-})\Theta_u^- du + M_t. \tag{4.3.37}$$

*Proof.* From Definition 4.3.1 and the intensity theorem for general marked point processes in [8], the conditional intensity process $\lambda_i$ of $1_{(\theta_t=i)}$ could be expressed as follows:

$$\begin{aligned}
\lambda_i(t) =& \sum_{n\geq 0} p_{\theta_n i} \frac{f_{\theta_n i}(t-T_n)dt}{S_{\Theta_n}(t-T_n)} 1_{(T_n\leq t<T_{n+1})} 1_{(T_n<\infty)} \\
=& \sum_{j\in E}\sum_{n\geq 0} p_{ji} \frac{f_{ji}(t-T_n)dt}{S_{\Theta_n}(t-T_n)} 1_{T_n\leq t<T_{n+1}} 1_{T_n<\infty} \\
=& \sum_{j\in E} \lambda_{j,i}(t-T_n) 1_{\theta_t=j}, \text{ where, } \lambda_{j,i} = \sum_{n\geq 0} p_{ji} \frac{f_{ji}(t-T_n)dt}{S_{\Theta_n}(t-T_n)} 1_{T_n\leq t<T_{n+1}} 1_{T_n<\infty}.
\end{aligned}$$

Hence from [8], there exists real valued cádlág martingale processes $M^i(t), \forall i = 1, ..., m$ such that

$$M^i(t) = 1_{(\theta_t=i)} - \int_0^t \lambda_i(y_{s^-})ds.$$

We set $M(t) = (M^1(t), ..., M^m(t))$ and rewrite

$$d(1_{(\theta_t=i)}) = \sum_{j\in E} \lambda_{j,i}(y_{t^-})1_{(\theta_{t^-}=j)}dt + dM^i(t)$$

$$d(1_{(\theta_t=i)}) = \big(\lambda_{1,i}(y_{t^-}), \lambda_{2,i}(y_{t^-}), ..., \lambda_{m,i}(y_{t^-})\big).\Theta_t^- dt + dM^i(t)$$

hence:

$$d\Theta_t = \Lambda(y_{t^-})'\Theta_t^- dt + dM(t),$$

hence proving a martingale decomposition of the vector process $(\Theta_t)_{t\in[0,T]}$. □

We consider the particular case where the asset price and its volatility follow the risk neutral Heston Model:

$$dx_t = \big(r(t,\Theta_t,y_t) - .5\sigma(t,\Theta_t,y_t)\big)dt + x_t\sqrt{\sigma(t,\Theta_t,y_t)}dW_t^1 \tag{4.3.38}$$

$$d\sigma(t,\Theta_t,y_t) = a(t,\Theta_t,y_t)\big(b(t,\Theta_t,y_t) - \sigma(t,\Theta_t,y_t)\big)dt + \upsilon(t,\Theta_t,y_t)\sqrt{\sigma(t,\Theta_t,y_t)}dW_t^2 \tag{4.3.39}$$

$$\text{with: } dW_t^1 dW_t^2 = \rho_t dt, \text{ and } \sigma(0,\Theta_0,y_0) = \sigma_0 > 0, \tag{4.3.40}$$

where $r$ is the risk free interest rate, $x$ is the log asset price model, $\sigma$ is the asset volatility, $a$ is the speed of the mean reversion of the asset volatility, $b$ is the long term asset volatility and $\upsilon$ determines the variance of the volatility which is referred to as the volatility of volatility.
We make the following simplifying assumptions on the model parameters:

1. A sufficient condition for the volatility remains non negative we assume: $2a(t,\Theta_t,y_t)b(t,\theta_t,y_t) > \upsilon^2(t,\Theta_t,y_t), \forall t \in [0,T]$,

2. The only parameter subjected to mean reversion is the long term volatility: $r(t,\Theta_t,y_t) = r(t)$, $\rho(t,\Theta_t,y_t) = \rho(t)$, $\upsilon(t,\Theta_t,y_t) = \upsilon(t)$

3. the speed of mean reversion of the volatility, the interest rate and the volatility of volatility are assumed to be positive and constant: $a(t) = a > 0$,$r(t) = r > 0$,$\upsilon(t) = \upsilon > 0$.

4. The correlation between asset price and volatility is assume to be constant: $\rho(t) = \rho$.

The main goal is to derive a general formula for the vanilla European call option prices both from the perspective of Carr and Madan's algorithm and as an extension of the formula derived in [**?** ]. Both problems are simplified if one can derive an expression for the following conditional expectation:

$$E\big(e^{iwx_u}\big|\mathbb{F}_t\big), \text{for a fixed } u \geq t.$$

We first present a useful Lemma providing a general conditional expectation formula needed in this illustration.

LEMMA 4.3.2 *We denote* $\boldsymbol{I}$, $\mathbf{1} = \underbrace{(1, ..., 1)}_{m}^{T}$, $f$, $\Lambda(s) = \big(\lambda_{i,j}(s)\big)_{1\leq i,j\leq m}$ *and* $\Phi$ *the* $m \times m$ *identity matrix, an* $m \times 1$ *vector of real numbers, a real valued process continuous in its second and third variables, the* $m \times m$ *real valued conditional intensity matrix of the semi Markov process* $\Theta$ *and the* $m \times m$ *matrix of real numbers satisfying the linear matrix partial differential equation with terminal condition:*

$$\frac{\partial \Phi}{\partial t} + \frac{\partial \Phi}{\partial y} + \boldsymbol{A}(t, y_t)\Phi(T, t, y_t) = 0, \text{ with } \Phi(T, T, y_T) = \boldsymbol{I} \tag{4.3.41}$$

*where:*

$$\boldsymbol{A}(T, t, y_t) = \Lambda^{'}(y_t) - diag\big(\boldsymbol{f}(t, y_t)\big),$$

$$\boldsymbol{f}(t, y) = \big(f(t, y_t, e_1), f(t, y_t, e_2), ..., f(t, y_t, e_m)\big).$$

*We assume that the following conditions are satisfied:*

$$E\big[e^{-\int_0^T f(u, y_{u^-}, \Theta_{u^-})du}\big|\mathbb{H}_t \vee \mathbb{L}_T\big] < \infty, \forall t \in [0, T] \tag{4.3.42}$$

$$\int_0^T \lambda_{i,j}(s)ds < \infty \tag{4.3.43}$$

*We define a real valued function* $F$ *and a vector valued function* $\boldsymbol{F}$ *as follows:*

$$F(T, t, y_t, \Theta_t) = E\big(e^{-\int_t^T f(u, y_{u^-}, \Theta_{u^-})du}\big|\mathbb{H}_t \vee \mathbb{L}_T\big), \tag{4.3.44}$$

$$\boldsymbol{F}(T, t, y_t) = \big(F(T, t, y_t, e_1), F(T, t, y_t, e_2), ..., F(T, t, y_t, e_m)\big), . \tag{4.3.45}$$

*Let* $K$ *be a* $m \times m$ *real matrix function.* $M$ *is said to satisfy the bracket condition if* $\forall t_1, t_1 \in \mathbb{R}^+$, $\big[K(t_1), K(t_2)\big] = 0$*, where* $[,]$ *denotes Lie matrix bracket.*

1. *The vector valued function* $\boldsymbol{F}$ *and the real valued function* $F$ *can be respectively expressed as follows:*

$$\boldsymbol{F}(T,t,y_t) = \Phi(T,t,y_t)\mathbf{1}, \tag{4.3.46}$$

$$F(T,t,y_t,\Theta_t) = \big\langle \Phi(T,t,y_t)\mathbf{1}, \Theta_t \big\rangle = \big\langle \Phi(T,t,y_t)\Theta_t, \mathbf{1} \big\rangle, \tag{4.3.47}$$

2. *If* $\boldsymbol{A}$ *satisfies the bracket condition, a closed form expression for the fundamental matrix of* (4.3.41) *could be expressed as follows:*

$$\Phi(T,t,y_t) = \exp\Big[\int_y^{T-t+y} \boldsymbol{A}(v-y+t,v)dv\Big], \tag{4.3.48}$$

*Proof.* We define the following product of processes,

$$\begin{aligned} P_t &= e^{-\int_0^t f(u,y_{u^-},\Theta_{u^-})du} F(T,t,y_t,\theta_t) && (4.3.49)\\ &= e^{-\int_0^t f(u,y_{u^-},\Theta_{u^-})du} E\Big[e^{-\int_t^T f(u,y_{u^-},\Theta_{u^-})du}\Big|\mathbb{H}_t \vee \mathbb{L}_T\Big] && (4.3.50)\\ &= E\Big[e^{-\int_0^T f(u,y_{u^-},\Theta_{u^-})du}\Big|\mathbb{H}_t \vee \mathbb{L}_T\Big]. && (4.3.51) \end{aligned}$$

In addition to being integrable as per assumption (4.3.42), the process $\big((P)_{t\in[0,T]}\big)$ satisfies the following properties:

$$\begin{aligned} E\big[P_t\big|\mathbb{H}_s \vee \mathbb{L}_T\big] &= E\Big[E\big[e^{-\int_0^T f(u,y_{u^-},\Theta_{u^-})du}\Big|\mathbb{H}_t \vee \mathbb{L}_T\big]\big|\mathbb{H}_s \vee \mathbb{L}_T\Big]\\ &= E\Big[e^{-\int_0^T f(u,y_{u^-},\Theta_{u^-})du}\big|\mathbb{H}_s \vee \mathbb{L}_T\Big]\\ &= P_s, \forall s < t \text{ with } s,t \in \mathbb{R}. \end{aligned}$$

Hence, $P_t$ has the martingale property with respect to the filtration $(\mathbb{H}_t \vee \mathbb{L}_T)_{t\in[0,T]}$. Writing Itó differential formula for $P_t$ from [1] yields:

$$\begin{aligned} & d\Big[e^{-\int_0^t f(u,y_{u^-},\Theta_{u^-})du} F(T,t,y_t,\Theta_t)\Big] = -f(t^-,y_{t^-},\Theta_{t^-}) e^{\int_0^t f(u,y_{u^-},\Theta_{u^-})du} F(T,t,y_t,\Theta_t)dt\\ & + e^{-\int_0^t f(u,y_{u^-},\Theta_{u^-})du} dF(T,t,y_t,\Theta_t)dt\\ & - f(t^-,\Theta t^-,y_{t^-}) e^{-\int_0^t f(u,\Theta_{u^-})du} F(T,t,y_t,\Theta_t)dt\\ & = -f(t^-,y_{t^-},\Theta_{t^-}) e^{-\int_0^t f(u,\Theta_{u^-})du} F(T,t,y_t,\Theta_t)dt + e^{-\int_0^t f(u,\Theta_{u^-})du} dF(T,t,y_t,\Theta_t) \end{aligned}$$

Applying a particular case of Itó rule for semi Markov regime switching processes developed in [2] yields:

$$dF(T,t,y_t,\theta_t) = \frac{\partial F}{\partial t}dt + \frac{\partial V}{\partial y}dt + \Big\langle \boldsymbol{F}(T,s,y_{s^-}), d\Theta_s \Big\rangle$$

Hence,

$$d\Big[e^{-\int_0^t f(u,\Theta_{u^-})du}F(T,t,y_t,\Theta_t)\Big] = -f(t^-,\Theta_{t^-},y_{t^-})e^{-\int_0^t f(u,\Theta_{u^-})du}F(T,t,y_t,\Theta_t)dt$$

$$+ e^{\int_0^t f(u,\Theta_{u^-})du}\Big[\frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial y}dt + \Big\langle \boldsymbol{F}(T,t,y_{t^-}), d\Theta_s\Big\rangle\Big] \tag{4.3.52}$$

$$= -f(t^-,\Theta_{t^-},y_{t^-})e^{-\int_0^t f(u,\Theta_{u^-})du}F(T,t,y_t,\Theta_t)dt$$

$$+ e^{-\int_0^t f(u,\Theta_{u^-})du}\Big[\frac{\partial F}{\partial t} + \frac{\partial F}{\partial y} + \Big\langle \boldsymbol{F}(T,t,y_{t^-}), \Lambda(y_{t^-})^T\Theta_{t^-}\Big\rangle\Big]dt$$

$$+ e^{-\int_0^t f(u,\Theta_{u^-})du}\Big\langle \boldsymbol{F}(T,t,y_{t^-}), dM_{t^-})\Big\rangle \tag{4.3.53}$$

$$= -e^{-\int_0^t f(u,\Theta_{u^-})du}\langle diag\big(\boldsymbol{f}(t^-,y_{t^-})\big)\boldsymbol{F}(T,t,y_t),\Theta_{t^-}\rangle dt$$

$$+ e^{-\int_0^t f(u,\Theta_{u^-})du}\Big[\langle\frac{\partial \boldsymbol{F}}{\partial t},\Theta_{t^-}\rangle dt + \langle\frac{\partial \boldsymbol{F}}{\partial y},\Theta_{t^-}\rangle dt + \Big\langle \Lambda(y_{t^-})\boldsymbol{F}(T,t,y_{t^-}),\Theta_{t^-}\Big\rangle\Big]dt$$

$$+ e^{-\int_0^t f(u,\Theta_{u^-})du}\Big\langle \boldsymbol{F}(T,t,y_{t^-}), dM_{t^-})\Big\rangle. \tag{4.3.54}$$

As $e^{\int_0^t f(u,y_{u^-},\Theta_{u^-})du}F(T,t,y_t,\Theta_t)$ is a martingale process, the bounded variation term of Itó formula in (4.3.52) is identically zero. It reads:

$$\frac{\partial \boldsymbol{F}}{\partial t} + \frac{\partial \boldsymbol{F}}{\partial y} + \boldsymbol{A}(t^-,y_{t^-})\boldsymbol{F}(T,t,y_t) = 0, \text{ with } \boldsymbol{A}(t,y_t) = -diag\big(\boldsymbol{f}(t,y_t)\big) + \Lambda^{'}(y_t). \tag{4.3.55}$$

Assuming matrix $\boldsymbol{A}$ has continuous components with respect to both $t$ and $y$ there exists $\Phi$ a fundamental solution of (4.3.55), ie $\Phi$ satisfies the matrix ODE

$$\frac{\partial \Phi}{\partial t} + \frac{\partial \Phi}{\partial y} + \boldsymbol{A}(t^-,y_{t^-})\Phi(T,t,y_t) = 0, \text{ with } \boldsymbol{A}(t,y_t) = -diag\big(\boldsymbol{f}(t,y_t)\big) + \Lambda^{'}(y_t). \tag{4.3.56}$$

Hence, the solution of the ODE (4.3.55) with terminal condition $\boldsymbol{F}(T,T,y_T) = \mathbf{1}$ is:

$$\boldsymbol{F}(T,t,y_t) = \Phi(T,t,y_t)\mathbf{1},$$

where $\Phi$ is solution of the matrix partial differential equation (4.3.56), with terminal condition $\Phi(T,T,y_T) = I$. This therefore proves (4.3.46). We establish (4.3.47) as follows:

$$\begin{aligned} F(T,t,y_t,\Theta_t) &= E\big(e^{\int_t^T f(u,y_{u^-},\Theta_{u^-})du}\big|\mathbb{H}_t \vee \mathbb{L}_T\big) \\ &= \big\langle \boldsymbol{F}(T,t,y_t),\Theta_t\big\rangle, \text{ From (4.3.45)} \\ &= \big\langle \Phi(T,t,y_t)\mathbf{1},\Theta_t\big\rangle \text{ Since the } e_j, j=1,...,m \text{ form an orthonormal basis} \\ &= \big\langle \Phi(T,t,y_t)\Theta_t,\mathbf{1}\big\rangle. \end{aligned}$$

The proof of part (2) proceeds from the result in part (1). Indeed, assuming the bracket condition is satisfied one can derive a closed form expression for the solution of (4.3.55). We first use the method of characteristic to solve the system of PDEs (4.3.55). We consider the variable transforms:

$$\eta = t - y \text{ and } \zeta = t + y. \tag{4.3.57}$$

Based on (4.3.57), we define the transforms $\tilde{\boldsymbol{F}}$ and $\tilde{\boldsymbol{A}}$ from $\boldsymbol{F}$ and $\boldsymbol{A}$, respectively, as functions of $(\eta, \zeta)$:

$$\begin{cases} \tilde{\boldsymbol{F}}(T, \eta, \zeta) = \boldsymbol{F}(T, \frac{\eta+\zeta}{2}, \frac{-\eta+\zeta}{2}) \\ \tilde{\boldsymbol{A}}(\eta, \zeta) = \boldsymbol{A}(\frac{\eta+\zeta}{2}, \frac{-\eta+\zeta}{2}). \end{cases} \tag{4.3.58}$$

Simple algebra shows that the system of PDEs (4.3.55) becomes :

$$\frac{\partial \tilde{\boldsymbol{F}}(T, \eta, \zeta)}{\partial \zeta} = -\frac{1}{2} \tilde{\boldsymbol{A}}(\eta, \zeta) \tilde{\boldsymbol{F}}(T, \eta, \zeta) \quad . \tag{4.3.59}$$

Assuming continuity of the components of the matrix $\tilde{\boldsymbol{A}}$, the ODE (4.3.59) has general solution

$$\tilde{\boldsymbol{F}}(T, \eta, \zeta) = \exp\Big[ -\frac{1}{2} \int_0^{\zeta} \tilde{\boldsymbol{A}}(\eta, s) ds \Big] . c(\eta), \tag{4.3.60}$$

where $c$ is a vector function of $\eta$ only. Assuming the terminal condition $\boldsymbol{F}(T, T, y) = \mathbf{1}$, the function $c$ becomes

$$c(T - y) = \exp\Big[ \frac{1}{2} \int_0^{T+y} \tilde{\boldsymbol{A}}(T - y, s) ds \Big],$$

which leads to ,

$$c(\eta) = \exp\Big[ \frac{1}{2} \int_0^{2T-\eta} \tilde{\boldsymbol{A}}(\eta, s) ds \Big].$$

Hence, the solution of the system of PDEs (4.3.55) becomes

$$\begin{aligned} \boldsymbol{F}(T, t, y) &= \exp\Big[ \frac{1}{2} \int_{t+y}^{2T-t+y} \boldsymbol{A}(T, \frac{t-y+s}{2}, \frac{y-t+s}{2}) ds \Big] . \mathbf{1} \\ &= \exp\Big[ \int_y^{T-t+y} \boldsymbol{A}(v - y + t, v) dv \Big] . \mathbf{1}, \end{aligned}$$

where $v = \frac{y-t+s}{2}$. Hence, one can verify that the fundamental matrix of the matrix system is:

$$\Phi(T, t, y_t) = \exp\Big[ \int_y^{T-t+y} \boldsymbol{A}(v - y + t, v) dv \Big]. \tag{4.3.61}$$

□

We note that the semi Markov process $\theta_t$ paired with its corresponding backward recurrence time $y_t$ form a Markov process. This important fact allows to claim that the preceding conditional characteristic function is a function of only the current values of the variables. Next Lemma review the characteristic function formula of the log price in the context of Heston model with no market regime.

LEMMA 4.3.3 *If a log asset price* $(x_t)_{t\in[0,T]}$ *and its volatility process* $(\sigma_t)_{t\in[0,T]}$ *follow the dynamic of the model in* (4.3.38) *and* (4.3.39)*, with one single market regime (that is no regime change), the characteristic function of the log price is expressed as follows:*

$$E\big(e^{iwx_u}\big|\mathbb{F}_t\big) = e^{A_t+B_t\sigma_t+iwx_t}, \forall u \geq t \tag{4.3.62}$$

$$A_t = irw(u-t) + a\int_t^u B_s b_s ds \tag{4.3.63}$$

$$B_t = \frac{a - i\rho\upsilon w + \eta}{\upsilon^2}\Big(\frac{1-e^{\eta(u-t)}}{1-\gamma e^{\eta(u-t)}}\Big) \tag{4.3.64}$$

*where,*

$$\eta = \sqrt{(a - i\rho\upsilon w + \eta)^2 + \upsilon^2 w(w+i)} \tag{4.3.65}$$

$$\gamma = \frac{a - i\rho\upsilon w + \eta}{a - i\rho\upsilon w + \eta}, \tag{4.3.66}$$

*where,* $i = \sqrt{-1}$.

*Proof.* The Markov property of the pair $(\Theta_t, y_t)$ implies that the quadruplet $(\Theta_t, y_t, x_t, \sigma_t)$ is Markovian as well. Therefore, we can use the notation:

$$h(u, \theta_t, y_t, \sigma_t, x_t) = E\big(e^{ibx_u}\big|\mathbb{F}_t\big), \text{for a fixed } u > t.$$

We derive the system of partial differential equations satisfied by $h$ when the market has one single state $\theta_t = j, \forall t \in [0,T]$. From 4.2.8 Itó Lemma applied to $h$ yields:

$$\begin{aligned} dh(t,j,\sigma_t,x_t) = &\frac{\partial h(t,j,\sigma_t,x_t)}{\partial t}dt + \frac{\partial h(t,j,\sigma_t,x_t)}{\partial x}dx_t + \frac{\partial h(t,j,\sigma_t,x_t)}{\partial \sigma}d\sigma_t \\ &+ \frac{1}{2}\frac{\partial^2 h(t,j,\sigma_t,x_t)}{\partial x^2}dx_t dx_t + \frac{1}{2}\frac{\partial^2 h(t,j,\sigma_t,x_t)}{\partial \sigma^2}d\sigma_t d\sigma_t + \frac{1}{2}\frac{\partial^2 h(t,j,\sigma_t,x_t)}{\partial \sigma\partial x}dx_t d\sigma_t \end{aligned}$$

From (4.3.38), we have:

$$\begin{aligned}
dh(t,j,\sigma_t,x_t) = {} & \frac{\partial h(t,j,\sigma_t,x_t)}{\partial t}dt + (r(t,j) - \frac{1}{2}\sqrt{\sigma(t,j)})\frac{\partial h(t,j,\sigma_t,x_t)}{\partial x}dt \\
& + a(t,j)(b(t,j) - \sigma(t,j)\frac{\partial h(t,j,\sigma_t,x_t)}{\partial \sigma}dt + \frac{1}{2}\sigma_t\frac{\partial^2 h(t,j,\sigma_t,x_t)}{\partial x^2}dt \\
& + \frac{1}{2}\upsilon^2(t,j)\sigma(t,j)\frac{\partial^2 h(t,j,\sigma_t,x_t)}{\partial \sigma^2}dt + \frac{1}{2}\upsilon(t,j)\sigma(t,j)\frac{\partial^2 h(t,j,\sigma_t,x_t)}{\partial \sigma \partial x}dt \\
& + \sqrt{\sigma(t,j)}\frac{\partial h(t,j\sigma_t,x_t)}{\partial x}dW_t^1 + \upsilon(t,j)\sqrt{\sigma(t,j)}\frac{\partial h(t,j,\sigma_t,x_t)}{\partial \sigma}dW_t^1. \quad (4.3.67)
\end{aligned}$$

As $h$ is defined through the means of conditional expectations, it is easy to prove that h is therefore a martingale process with respect to $(\mathbb{P},(\mathbb{F}_t)_{t\in[0,T]})$. Therefore, the bounded variation terms of the right hand side of equation (4.3.67) reduce to 0.

$$\begin{aligned}
& \frac{\partial h(t,j,\sigma_t,x_t)}{\partial t} + (r(t,j) - \frac{1}{2}\sqrt{\sigma(t,j)})\frac{\partial h(t,j,\sigma_t,x_t)}{\partial x} \\
& + a(t,j)(b(t,j) - \sigma(t,j))\frac{\partial h(t,j,\sigma_t,x_t)}{\partial \sigma} + \frac{1}{2}\sigma_t\frac{\partial^2 h(t,j,\sigma_t,x_t)}{\partial x^2} \\
& + \frac{1}{2}\upsilon^2(t,j)\sigma(t,j)\frac{\partial^2 h(t,j,\sigma_t,x_t)}{\partial \sigma^2} + \frac{1}{2}\rho\upsilon(t,j)\sigma(t,j)\frac{\partial^2 h(t,j,\sigma_t,x_t)}{\partial \sigma \partial x} \\
& = 0 \quad (4.3.68)
\end{aligned}$$

with boundary condition :

$$h(u,j,y_0,\sigma_0,x_0) = e^{iwx_0}. \quad (4.3.69)$$

From [27], we assume that the the characteristic function is of the form:

$$h(t,j) = e^{(A_t + B_t\sigma_t + iwx_t)}.$$

We apply substitution in (4.3.68) and from [27, 40] the following system of ODE is obtained:

$$irw + a(t,j)b(t,j)B_t + \dot{A}_t = 0 \quad (4.3.70)$$

$$-.5w^2 + iw\rho\upsilon B_t + .5\upsilon^2 B_t^2 - .5w - aB_t + \dot{B}_t = 0 \quad (4.3.71)$$

Solutions of such a coupled system are found in [40] and [27].

$$B_t = \frac{a - i\rho\upsilon w + \eta}{\upsilon^2}\Big(\frac{1 - e^{\eta(u-t)}}{1 - \gamma e^{\eta(u-t)}}\Big) \quad (4.3.72)$$

$$A_t = irw(u-t) + a\int_t^u B_s b_s ds \quad (4.3.73)$$

where,

$$\eta = \sqrt{(a - i\rho\upsilon w + \eta)^2 + \upsilon^2 w(w+i)} \text{ and } \gamma = \frac{a - i\rho\upsilon w + \eta}{a - i\rho\upsilon w - \eta} \quad (4.3.74)$$

□

We systematically extend the main result derived in [27] by considering semi Markov regimes. We find a similar but more general formula for vanilla call prices.

LEMMA 4.3.4 *Let $\boldsymbol{M}(t, y_t, u), u \geq t$, $\Theta_t$, $K$ and $\Lambda$ be an $m \times m$ real valued matrix function, a semi Markov process, the strike price of an option contract and the conditional intensity matrix of $\Theta_t$. We assume that $\boldsymbol{M}$ is solution of the matrix PDE:*

$$\frac{\partial \boldsymbol{M}}{\partial t} + \frac{\partial \boldsymbol{M}}{\partial y} + \boldsymbol{A}(t, y)\boldsymbol{M}(t, y, u) = 0, \text{ with } \boldsymbol{M}(u, y_u, u) = \boldsymbol{I}. \tag{4.3.75}$$

1. *The Vanilla European Heston call price from Carr and Madan algorithm is given by the semi analytic formula*

$$C(0, \Theta_0, y_0, x_0, \upsilon_0) = \frac{e^{-\alpha k}}{\pi} \mathcal{R}e\Big( \int_0^t e^{i\phi k} \frac{e^{-rT}\psi(\phi - (1 + \alpha))}{\alpha^2 + \alpha - v^2 + i(2\alpha + 1)\phi} \Big), \tag{4.3.76}$$

   *with $k = log(K)$ and $\alpha$ the Carr and Madan parameter. $\psi$ is the characteristic function of the log asset price $x$, given by the following expression:*

$$\psi(u, \Theta_0, y_0, x_0, \upsilon_0, w) = E(e^{i\phi x_u}|\mathbb{F}_0) = e^{ir\phi u + B_0\sigma_0 + iwx_0} \big\langle \Phi(u, 0, y)\Theta_0, \mathbf{1} \big\rangle, \tag{4.3.77}$$

   *where $\Phi$ satisfies the equation (4.3.75) with $\boldsymbol{A}(t, y) = -diag\big(\boldsymbol{f}(0, y_0)\big) + \Lambda^T(y_0)$ and with $f(t, y_t) = aB_t b(t, y_t)$.*

2. *A semi closed expression for vanilla option prices with Heston model in a market with semi Markov regimes is as follows:*

$$\begin{aligned} C(u, \Theta_0, y_0, x_0, \upsilon_0) =& e^{-ru}\Big(\frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathcal{R}e\Big[\frac{e^{iwru + iwx_0 + B_0\sigma_0}\big\langle \Phi(u, 0, y_0)\Theta_0, \mathbf{1}\big\rangle}{iw}\Big]dw\Big) \\ &+ Ke^{-ru}\Big(\frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathcal{R}e\Big[\frac{e^{-iwlog(K) + irwu + ibx_0 + \bar{B}_0\sigma_0 + iwx_0}\big\langle \bar{\Phi}(u, 0, y_0)\Theta_0, \mathbf{1}\big\rangle}{iw}\Big]dw\Big), \end{aligned} \tag{4.3.78}$$

   *where $\Phi$ (respectively, $\bar{\Phi}$) are solutions of the system of matrix partial differential equations* (4.3.75) *when $\boldsymbol{A}(t, y_t) = -diag\big(\boldsymbol{f}(t, y_t)\big) + \Lambda^T(y_t)$ with $f(t, y_t) = aB_t b(t, y_t)$ (respectively,*

*$f(t,y_t)=a\bar{B}_t b(t,y_t)$) with $A_t$,$B_t$,$\eta_t$ and $\gamma_t$ defined as in Lemma 4.3.3 for any $t>0$.*

$$\bar{B}_t=\frac{a-\rho\upsilon-i\rho\upsilon w+\bar{\eta}}{\upsilon^2}\Big(\frac{1-e^{\bar{\eta}(u-t)}}{1-\bar{\gamma}e^{\bar{\eta}(u-t)}}\Big),$$

*where,*

$$\bar{\eta}=\sqrt{(a-\rho\upsilon-i\rho\upsilon w+\bar{\eta})^2+\upsilon^2 w(w+i)}\text{ and }\bar{\gamma}=\frac{a-\rho\upsilon-i\rho\upsilon w+\bar{\eta}}{a-\rho\upsilon-i\rho\upsilon b-\bar{\eta}}$$

$$\bar{A}_t=irw(u-t)+a\int_t^u\bar{B}_s b_s ds.$$

*Proof.* We note that (4.3.76) is a well known formula derived in [11]. The critical issue is to prove (4.3.77). The first part of the lemma boils down to deriving an expression for the characteristic function of the log asset price of Heston model in a regime switching market.

$$\begin{aligned}E\big[e^{iwx_t}|\mathbb{F}_0\big]&=E\Big[E\big[e^{iwx_u}|\mathbb{H}_T\vee\mathbb{L}_0\big]|\mathbb{F}_0\Big]\\&=e^{iwru+iwx_u+B_0\sigma_0}E\big[e^{a\int_0^u B_s b(s,\theta_s^-,y_{s^-})ds}|\mathbb{F}_0\big]\\&=e^{ibru+ibx_0+B_0\sigma_0}\big\langle\Phi(u,0,y_0)\Theta_0,\mathbf{1}\big\rangle,\text{ From Lemma }4.3.2,\text{ with }\Phi\text{ satisfying (4.3.75)}\end{aligned}$$

where, $f(t,y_t)=aB_t b(t,y_t)$. This proves the first part of the lemma.

In order to complete the proof of the second part of the lemma, we recall the risk neutral pricing formula for a vanilla call option with initial cost $C$, strike price $K$ and in the context of a regime switching Heston driven market:

$$\begin{aligned}C(u,\Theta_0,y_0,x_0,\upsilon_0)&=E\Big[e^{-ru}(e^{x^u}-K)^+|\mathbb{F}_0\Big]\\&=e^{-ru}E\Big[e^{x^u}1_{x_u\geq\log K}|\mathbb{F}_0\Big]-Ke^{-ru}E\Big[x_u\geq\log(K)|\mathbb{F}_0\Big]\\&=e^{-ru}E\Big[e^{x^u}1_{x_u\geq\log K}|\mathbb{F}_0\Big]-Ke^{-ru}\mathbb{P}\big(x_u\geq\log K|\mathbb{F}_0\big)\end{aligned}$$

The second term of the last equation has been expressed as a conditional survival probability. We will express the first term in similar fashion. We first define $\bar{\mathbb{P}}$ for any $A\in\mathbb{F}_u$ as follows:

$$\begin{aligned}\bar{\mathbb{P}}(A)&=\frac{1}{e^{x_0}}E\big[e^{-ru+x_u}1_A|\mathbb{F}_0\big]\\&=\frac{1}{e^{x_0}}E\big[e^{-.5\int_0^u\sigma_s ds+\int_0^u\sqrt{\sigma_s}dB_s}1_A|\mathbb{F}_0\big],\text{ from (4.3.38).}\end{aligned}$$

It is easy to prove via Itó formula that the process

$$L_t=e^{-.5\int_0^u\sigma_s ds+\int_0^u\sqrt{\sigma_s}dB_s},\tag{4.3.79}$$

which satisfies the differential form of Itó lemma

$$dL_t = L_t \int_0^u \sqrt{\sigma_s} dB_s, \tag{4.3.80}$$

is a martingale process [44] and consequently is an appropriate density process. Hence, from Girsanov theorem [1], $\bar{\mathbb{P}}$ is a probability measure absolutely continuous with respect to the conditional probability measure $\mathbb{P}\big(|\mathbb{F}_0\big)$. The pricing formula can therefore be rewritten:

$$C(u, \Theta_0, y_0, x_0, \upsilon_0) = e^{-ru}\bar{\mathbb{P}}\big(1_{x_u \geq \log K}\big) - Ke^{-ru}\mathbb{P}\big(x_u \geq \log K|\mathbb{F}_0\big) \tag{4.3.81}$$

We first derive an expression for the second survival probability of (4.3.81). Under the probability measure $\mathbb{P}$, the model follows the dynamic in (4.3.38). Hence,

$$E\big[e^{ibx_u}|\mathbb{H}_T \vee \mathbb{L}_0\big] = e^{A_0 + B_0\sigma_0 + ibx_0}$$

where,

$$B_t = \frac{a - i\rho\upsilon b + \eta}{\upsilon^2}\Big(\frac{1 - e^{\eta(u-t)}}{1 - \gamma e^{\eta(u-t)}}\Big)$$

where,

$$\eta = \sqrt{(a - i\rho\upsilon b + \eta)^2 + \upsilon^2 b(b+i)} \text{ and } \gamma = \frac{a - i\rho\upsilon b + \eta}{a - i\rho\upsilon b - \eta}$$

$$A_t = irb(u-t) + a\int_t^u B_s b_s ds,$$

therefore from Lemma 4.3.3, the characteristic function has the form,

$$E\big[e^{iwx_u}|\mathbb{H}_T \vee \mathbb{L}_0\big] = e^{iwru + iwx_0 + B_0\sigma_0 + a\int_t^u B_s b(s,\theta_s^-, y_{s^-})ds}$$

The only regime switching term involved is $b$. Hence,

$$\begin{aligned} E\big[e^{iwx_t}|\mathbb{F}_0\big] &= E\Big[E\big[e^{iwx_u}|\mathbb{H}_T \vee \mathbb{L}_0\big]|\mathbb{F}_0\Big] \\ &= e^{iwru + iwx_u + B_0\sigma_0} E\big[e^{a\int_0^u B_s b(s,\theta_s^-, y_{s^-})ds}|\mathbb{F}_0\big] \end{aligned}$$

The characteristic function problem now boils down to deriving an expression for $F(u, \theta_0, y_0) = E\big[e^{a\int_0^u B_s b(s,\theta_s^-, y_{s^-})ds}|\mathbb{F}_0\big]$. From Lemma (4.3.2) we have:

$$F(u, \theta_t, y_t) = \big\langle \Phi(u,t,y_t)\Theta_t, \mathbf{1}\big\rangle, \text{ with } \Phi \text{ solution of the PDE:}$$

$$\frac{\partial \Phi}{\partial t} + \frac{\partial \Phi}{\partial y} + \boldsymbol{A}(t, y_t)\Phi(u,t,y_t) = 0, \text{ with } \Phi(u,u,y_u) = \boldsymbol{I}, \forall u \geq t \geq 0$$

where:

$$\boldsymbol{A}(t, y_t) = \Lambda^T(y_t) - aB_t diag\big(\boldsymbol{b}(t, y_t)\big).$$

Therefore the characteristic function and the survival probability sought are:

$$\begin{aligned}E\big[e^{ibx_u}|\mathbb{H}_T\vee\mathbb{L}_0\big] &= e^{ibru+ibx_0+B_0\sigma_0}E\big[e^{a\int_0^u B_s b(s,\theta_s^-,y_{s^-})ds}|\mathbb{F}_0\big]\\ &= e^{ibru+ibx_0+B_0\sigma_0}\big\langle\Phi(u,0,y_0)\Theta_0,\mathbf{1}\big\rangle\\ \mathbb{P}\big(x_u\geq\log K|\mathbb{F}_0\big) &= \frac{1}{2}+\frac{1}{\pi}\int_0^\infty\mathcal{R}\Big[\frac{e^{ibru+ibx_0+B_0\sigma_0}\big\langle\Phi(u,0,y_0)\Theta_0,\mathbf{1}\big\rangle}{iw}\Big]dw.\end{aligned}$$

We turn our attention to deriving a semi closed expression for the first term of (4.3.81). We note that from Girsanov theorem [44], under the probability $\bar{\mathbb{P}}$, the standard Brownian motion in (4.3.38) become

$$d\bar{B}_t = dB_t - \sqrt{\sigma_t}dt.$$

The new dynamic of the Heston model under the probability measure $\bar{\mathbb{P}}$ is as follows:

$$\begin{aligned}dx_t &= (r+.5\sigma_t)dt+\sqrt{\sigma_t}d\bar{B}_t\\ d\sigma_t &= \big(a_tb_t-(a_t-\rho\upsilon_t)\sigma_t\big)+\upsilon_t\sqrt{\sigma_t}dW_t.\end{aligned}$$

In a derivation similar to that of the first characteristic function and default probability, we obtain:

$$\begin{aligned}E\big[e^{ibx_u}|\mathbb{F}_0\big] &= e^{-iwlog(k)+irwu+ibx_0+\bar{B}_0\sigma_0+iwx_0}\big\langle\bar{\Phi}(u,0,y_0)\Theta_0,\mathbf{1}\big\rangle\\ \bar{\mathbb{P}}\big(x_u\geq\log K|\mathbb{F}_0\big) &= \frac{1}{2}+\frac{1}{\pi}\int_0^\infty\mathcal{R}\Big[\frac{e^{-iwlog(k)+irwu+ibx_0+\bar{B}_0v_0+iwx_0}\big\langle\bar{\Phi}(u,0,y_0)\Theta_0,\mathbf{1}\big\rangle}{iw}\Big]dw\end{aligned}$$

$$\frac{\partial\bar{\Phi}}{\partial t}+\frac{\partial\bar{\Phi}}{\partial y}+\boldsymbol{A}(t,y_t)\bar{\Phi}(u,t,y_t)=0,\ \text{with } \bar{\Phi}(u,u,y_u)=\boldsymbol{I}$$

where:

$$\begin{aligned}\boldsymbol{A}(t,y_t) &= \Lambda^T(y_t)-a\bar{B}_t diag\big(\boldsymbol{b}(t,y_t)\big),\\ \bar{B}_t &= \frac{a-\rho\upsilon-i\rho\upsilon w+\bar{\eta}}{\upsilon^2}\Big(\frac{1-e^{\bar{\eta}(u-t)}}{1-\bar{\gamma}e^{\bar{\eta}(u-t)}}\Big),\end{aligned}$$

where,

$$\begin{aligned}\bar{\eta} &= \sqrt{(a-\rho\upsilon-i\rho\upsilon w+\bar{\eta})^2+\upsilon^2w(w+i)}\ \text{and}\ \bar{\gamma}=\frac{a-\rho\upsilon-i\rho\upsilon w+\bar{\eta}}{a-\rho\upsilon-i\rho\upsilon b-\bar{\eta}}\\ \bar{A}_t &= irw(u-t)+a\int_t^u\bar{B}_sb_sds.\end{aligned}$$

Hence, (4.3.81) yields the result to be proved. □

### 4.4 Conclusion

This chapter aimed at calibrating the historical parameters and the risk neutral parameters of two nonlinear semi Markov regime switching coupled system of stochastic differential equations, respectively representing a basket of commodity prices and the risk neutral dynamic of a stock price. In the former case, we obtained closed form parameter estimates and in the latter case we obtained a couple of semi closed form formulas for European call option prices, hence proving the tractability of both model when the market is assumed to follow a semi Markov dynamic.

### 4.5 *

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# Chapter 5

## Conclusion and future Research

Semi Markov regime switching financial models prove to be a generalization of Markov switching processes preserving the economic interpretability of the regime switching parameters. One of the downside of semi Markov switching models is the challenging nature of solving the system of partial integro differential equations (PIDE) of call option prices. The PIDE is usually solved using finite difference methods in a rectangular domain. However, in semi Markov models, the backward recurrence is a variable in the derived pricing PIDE. Given that the current time is always at least equal to the backward recurrence time, the rectangular structure of the variables involved in the PIDE does not hold anymore. Which result in a more challenging finite difference scheme. Our approach for solving the pricing problem throughout this work is based on the characteristic function formula of the log asset price. The tractability of the formula obtained depends highly on the choice of the intensity matrix. For computational simplicity and tractability of the results in this work, we opted to work with piecewise constant Weibull intensities as they are flexible enough to encompass upward trending, downward trending and constant instantaneous switching propensities. The comparative results obtained showed that calibrating Markov and semi Markov regime models to observed data, lead to substantial model fit improvement and preserved the ease of interpretation of parameters.

However we are still looking into possible improvements of the model. In order to benefit from the full scale semi Markov regime models, one will need to drop the piecewise constant assumption and pick a smoother model for conditional intensity. This of course will leave us with only the finite difference method for option pricing.

Another application of interest is in fixed income derivatives and in structured finance. Credit default swap, collateralized debt obligation are interest rate sensitive products more likely to be affected by regime changes than options. Although in this work we have shown that the effect of regimes could be substantial, there are reasons to believe that fixed income and structured finance products are more sensitive to regime changes and therefore semi Markov switching models would provide a substantially better avenue for modeling asset dynamics.

Finally, we are working at comparing what was referred to in this work as the pseudo-infinite number of regimes of semi Markov Markets, with the explicit infinite number of regimes in hidden Markov model with Dirichlet priors. The purpose of such a comparison would be to determine if there is a parallel between the estimated regimes in both models and to determine if one has a more realistic interpretation.

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