3-23-2017

Linear Extremal Problems in the Hardy Space H^p for $0 < p < 1$

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by

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A thesis submitted in partial fulfillment
of the requirements for the degree of
Mathematics M.A.
Department of Mathematics & Statistics
College of Arts and Sciences

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Date of Approval:
March 21, 2017

Keywords: Hardy Spaces, Extremal Problems, Analytic Functions, Numerical Analysis,

Dedication

I would like to dedicate this to my wife Corey, and my parents and grandparents:

Corey, you have always supported me completely and without your patience and love my long journey of education would not have been possible. I wish you a successful trek as you begin your graduate journey.

To my family, from a young age, everyone of you have pushed my intellectual pursuits. You have constantly encouraged me to be the best version of myself and to learn everything I can. My dreams are built on this, and I wouldn't be the man I am today without you all.

Corey, Mom, Dad, Grammy, and Grandma, I love you all dearly, thank you for everything!

Acknowledgments

Dr. Catherine Bénéteau has been an excellent mentor throughout my graduate career. As her student, she pushed me to work harder than I ever had before and ever since the day I entered the M.A. program at USF, she has been a constant source of advice and guidance. Without her help this thesis would have never happened and I am in debt to all she has done for me. In general, the math faculty at USF have all had a great impact on my life. Working for this department as a T.A. has been the most fulfilling work of my life so far and I am so thankful to have been given the opportunity to work with these wonderful people.

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Abstract

In this thesis, we consider linear extremal problems in the H^p spaces. For many of these extremal problems, a unique solution can be guaranteed. We will examine some of the classical examples of extremal problems in these spaces. With this framework in place we will then consider a particular problem which does not always have a unique solution.

Chapter 1
Introduction to H^p Spaces

In this thesis, we will be examining a linear extremal problem in the Hardy Spaces (H^p), which are families of spaces of analytic functions. In Chapter 1 we provide necessary background into the Hardy Spaces, along with some minor historical context. In Chapter 2 we examine the properties of H^p functions. Having explored the necessary background of H^P Spaces, we look at some classic extremal problems for those spaces in Chapter 3. In Chapter 4, we consider a particular extremal problem in H^p for $0 < p < 1$, which does not have a unique solution. In Chapter 5, we discuss an algorithm written to provide numerical analysis of the extremal problem, which supports the lack of uniqueness of the extremal problem under consideration. We conclude this chapter with future work.

1.1 Background

Let us begin with some background:

DEFINITION 1.1.1 *The Lebesgue Space (L^p): Let (X, μ) be a measure space. The $L^p(X)$ space for a given $0 < p < \infty$, is the set of complex measurable functions f (on X), such that the following L^p norm is is finite*

$$\|f\|_p := \left\{ \int_X |f|^p d\mu \right\}^{\frac{1}{p}} < \infty. \tag{1.1}$$

For the special case of $p = \infty$ the Lebesgue space is defined in the following way:

DEFINITION 1.1.2 *The Lebesgue Space (L^∞): Let (X, μ) be a measure space. The $L^\infty(X)$ space is the set of complex measurable functions f (on X), such that the essential supremum of $|f|$ is finite.*

We define the L^∞-norm of the function f, as $\|f\|_\infty =$ esssup$|f| = \inf\{C \geq 0 : |f(x)| \leq C$ for almost every $x\}$.

The definition of L^p for $1 \leq p < \infty$ uses the more general Lebesgue integral (as opposed to the Riemann integral), which was introduced in 1904 by Henri Lebesgue [6]. The L^p space is a complete normed linear space, in other words, for any $1 \leq p \leq \infty$, we say that $L^p(X)$ is a Banach space. (In 1920, Stefan Banach initially formalized this space in his thesis [6], in which he laid out the axioms that form its basis.)

The Hardy Spaces can be viewed as subspaces of L^p, which we will discuss later, but involve analytic functions defined on the open unit disk $\mathbb{D}$.

DEFINITION 1.1.3 *Analytic Function: An analytic function is a complex valued function of a complex variable which can be represented locally by a convergent power series expansion.*

DEFINITION 1.1.4 *The Hardy Space (H^p): For functions analytic in the disk and $0 < p < \infty$, if*

$$\|f\|_p := \lim_{r \to 1^-} M_p(r, f) = \lim_{r \to 1^-} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{i\theta}\right) \right|^p d\Theta \right)^{\frac{1}{p}} < \infty, \tag{1.2}$$

then the function is said to be an element of H^p. For the special case $p = \infty$, we require that $M_\infty(r, f) = \max\limits_{0 \leq \theta \leq 2\pi} \left| f\left(re^{i\theta}\right) \right| < \infty$ and we write $f \epsilon H^\infty$.

Note that the H^p Spaces form a decreasing sequence of spaces, i.e. if $p < q$ then $H^q \subset H^p$.

Note that any complex function which is analytic in the closed disk $(|z| \leq 1)$ is bounded, and hence will be in H^p for all values of $p > 0$. Therefore, we can see that polynomial, $\sin(z)$, $\cos(z)$, e^z, and rational functions with poles outside of the closed disk are all functions of H_p for any $p > 0$.

However, when a function has singularities on the unit circle, things are not so clear. For example, the function $\frac{1}{1-z}$ has a singularity at $z = 1$.

PROPOSITION 1 The function $f(z) = (1 - z)^{-1}$ is in H^p for every $0 < p < 1$, but is not in H^1, and thus not in any H^p space for any $p \geq 1$.

Note: The solution to this problem is well known and illustrates the need for clever estimates in order to determine the convergence of integrals.

Proof.

Part 1 Let us first show that $f \in H^p$ for $0 < p < 1$. That means we need to determine the convergence of the following integral limit:

$$\lim_{r\to 1^-} \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{1}{1-re^{i\theta}} \right|^p d\theta. \tag{1.3}$$

It is not hard to see that the above equals $\lim\limits_{r\to 1^-} \frac{1}{2\pi} \int\limits_0^{2\pi} \frac{1d\theta}{(\sqrt{1+r^2-2r\cos\theta})^p}$.

Notice that: $1-2r\cos(\theta)+r^2 = (1-r)^2+4r\sin^2\left(\frac{\theta}{2}\right)$. For $0 \le x \le \frac{\pi}{2}$ we have that $\sin x \ge \frac{2}{\pi}x$

$$\Rightarrow (1-r)^2 + 4r\sin^2\left(\frac{\theta}{2}\right) \ge (1-r)^2 + \frac{4r}{\pi^2}\theta^2$$

$$\Rightarrow \frac{1}{2\pi}\int_0^{\frac{\pi}{2}} \frac{1d\theta}{(1+r^2-2r\cos\theta)^{\frac{p}{2}}} \le \frac{1}{2\pi}\int_0^{\frac{\pi}{2}} \frac{1d\theta}{\left((1-r)^2+\left(\frac{4r\theta^2}{\pi^2}\right)\right)^{\frac{p}{2}}} \le \frac{1}{2\pi}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\left(\frac{4r\theta^2}{\pi^2}\right)^{\frac{p}{2}}}$$

$$= \frac{1}{2\pi}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\left(\frac{2\theta}{\pi}r^{\frac{1}{2}}\right)^p} = \frac{\pi^{p-1}}{2^{p+1}\cdot r^{\frac{p}{2}}}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\theta^p}.$$

Notice that $\int\limits_0^{\frac{\pi}{2}} \theta^{-p} d\theta$ diverges for $p = 1$ but converges for $p < 1$.

Since,

$$\frac{1}{2\pi}\int_0^{\frac{\pi}{2}} \frac{1d\theta}{(1+r^2-2r\cos\theta)^p} = \frac{1}{2\pi}\int_{\frac{3\pi}{2}}^{2\pi} \frac{1d\theta}{(1+r^2-2r\cos\theta)^p}$$

the previous statement gives the convergence for both of these integrals when $p < 1$.

Now, we need only show that $\int\limits_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(\frac{1d\theta}{\sqrt{1+r^2-2r\cos\theta}}\right)^p$ converges, but this is obvious for r close to 1.

So, we have that

$$\lim_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} \frac{1d\theta}{(\sqrt{1+r^2-2r\cos\theta})^p} < \infty \tag{1.4}$$

for $p < 1$.

Next we must show that (1.3) diverges for $p = 1$.

In this case,

$$\lim_{r\to 1^-} \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{1}{1-re^{i\theta}}\right|^p d\theta = \int_o^{2\pi} \frac{1d\theta}{\left(\sqrt{2+2\cos\theta}\right)^p} = \int_0^{2\pi} \frac{1d\theta}{2\sin\left(\frac{\theta}{2}\right)} = \int_0^{\pi} \frac{dx}{\sin(x)} \tag{1.5}$$

Since $\sin(x) < x$ for $0 < x < \frac{\pi}{2}$, then we have that $\int\limits_0^{\frac{\pi}{2}} \frac{dx}{\sin(x)} > \int\limits_0^{\frac{\pi}{2}} \frac{dx}{x}$ and since $\int\limits_0^{\frac{\pi}{2}} \frac{dx}{x}$ diverges which implies that $\lim\limits_{r\to 1} \frac{1}{2\pi} \int\limits_0^{2\pi} \left| \frac{1}{1-e^{i\theta}} \right|^p d\theta = \infty$ and therefore $(1-z)^{-1} \notin H^1$.

Notice that since $\frac{1}{z-1} \notin H^1$, it is not in any H^p space for $p \geq 1$.

□

An important preliminary result in H_p spaces began with Hardy's Convexity Theorem, in 1915, which states that $M_p(r, f)$ is a non-decreasing function of r and is logarithmically convex [6].

PROPOSITION 2 (Hardy's Convexity Theorem) For $f \in H^p$, $M_p(r, f)$ is a non-decreasing function of r and is logarithmically convex.

Below we outline a (known) alternative approach to the classical proof using Littlewood's Subordination Theorem. First a definition:

DEFINITION 1.1.5 *A function $f(z)$ analytic in $|z| < 1$ is said to be subordinate to an analytic function $F(z)$ ($f \prec F$) if $f(z) = F(w(z))$ for some function $w(z)$ analytic in $|z| < 1$, satisfying $|w(z)| \leq |z|$. This is notated $f \prec F$.*

Littlewood's Subordination Theorem states,

THEOREM 1.1 *Let $f(z)$ and $F(z)$ be analytic in $|z| < 1$, and suppose $f \prec F$. Then $M_p(r, f) \leq M_p(r, F)$, $0 < p \leq \infty$.*

Proof. Let $F(z)$ be analytic in $|z| < 1$. Let $0 < k < 1$ and define $f(z) = F(kz)$. Then it is obvious that $f(z)$ is analytic in $|z| < 1$ and $f \prec F$.
It is also clear that $\frac{1}{2\pi} \int\limits_0^{2\pi} \left|F\left(kre^{i\theta}\right)\right|^p d\theta \leq \frac{1}{2\pi} \int\limits_0^{2\pi} \left|F\left(re^{i\theta}\right)\right|^p d\theta$.
Now by the Littlewood Subordination Theorem, we have that
$\frac{1}{2\pi} \int\limits_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^p d\theta \leq \frac{1}{2\pi} \int\limits_0^{2\pi} \left|F\left(re^{i\theta}\right)\right|^p d\theta$ and therefore $\frac{1}{2\pi} \int\limits_0^{2\pi} \left|F\left(kre^{i\theta}\right)\right|^p d\theta \leq \frac{1}{2\pi} \int\limits_0^{2\pi} \left|F\left(re^{i\theta}\right)\right|^p d\theta$.
Since $0 < kr < r \leq 1$ we have that $M_p(q, F) \leq M_p(r, F)$ for any $q \leq r$.
This shows that $M_p(r, F)$ is a nondecreasing function of r.

□

Although the seeds had been planted by Hardy years before, it wasn't until 1923 that the Riesz brothers coined the term H_δ (later changed to H^p). Furthermore, the H^p Space wasn't shown to be a Banach Space (for $p \geq 1$) until Taylor defined the H^p norm near the end of the 1940's, over 20 years after the corresponding result for L^p Spaces (See [6] p. 473 and 474).

In fact, it is Hardy's Convexity theorem that allowed Taylor to define this norm, since, if $f \in H^p$, then $\lim\limits_{r \to 1^-} M_p(r, f) = M_p(1, f)$. Therefore, we can define a norm on H^p by: $\|f\|_p = \lim\limits_{r \to 1^-} M_p(r, f)$.

On the other hand, in the case of $p < 1$, things are slightly less simple. In 1952 Arthur E. Livingston showed that for $0 < p < 1$, the H^p space is not normable. His argument hinged on "a theorem of Kolmogoroff, [that] a linear topological space has an equivalent normed topology if and only if the space contains a bounded open convex set." Livingston then shows that for " $0 < p < 1$, [H^p] contains no convex neighborhood of the origin; this is clearly sufficient to show that H^p $(0 < p < 1)$ contains no bounded open convex set, and hence is not normable." (See [5].)

1.2 H^p as a subspace of L^p

By considering boundary values of functions in H^p we can consider it as a subspace of L^p. But in order to do this we must first consider a key result.

THEOREM 1.2 (Fatou's Theorem [2]) *If $f \in H^p$, i.e., if $\|f\|_p < \infty$ then $\lim\limits_{r \to 1^-} f\left(re^{i\theta}\right)$ exists almost everywhere and we can call this limit $f(e^{i\theta})$, additionally,* $\lim\limits_{r \to 1^-} \left(\frac{1}{2\pi}\int\limits_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^p\right)^{\frac{1}{p}} = \left(\frac{1}{2\pi}\int\limits_0^{2\pi}\left|f\left(e^{i\theta}\right)\right|^p\right)^{\frac{1}{p}}$.

Therefore, we have that if $f \in H^p$, then

$$\|f\|_{H^p} = \left(\frac{1}{2\pi}\int\limits_0^{2\pi}\left|f\left(e^{i\theta}\right)\right|^p\right)^{\frac{1}{p}}. \tag{1.6}$$

Let $\pi = \{z \in \mathbb{C} : |z| = 1\}$ and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Then by applying Fatou's Theorem, we see that the H^p and L^p norms for any function in H^p are identical. Of course, if a sequence of functions f_n converge to f in H^p, then $\|f\|_{H^p} < \infty$. Therefore, $H^p(\mathbb{D})$ is a closed subspace of $L^p(\pi)$. Although not true in general, when X is a compact space, $L^p(X)$ has the property that $L^p(X) \supset L^q(X)$ if $0 < p < q \leq \infty$. Since π is a compact space this property holds for H^p spaces as well.

In other words, if we consider the set $\mathscr{H}^p$ (of boundary functions for H^p), then $\mathscr{H}^p$ is a vector subspace of L^p. In fact, for $0 < p < \infty$, $\mathscr{H}^p$ is the L^p closure of the set of polynomials in $e^{i\theta}$. This means that in a, (perhaps limited) way, we can apply many results for L^p spaces to their corresponding H^p space, by analyzing the boundary functions for that particular class.

In the following chapter, we will examine the properties of H^p functions more closely. In doing this, we see that functions f that are elements of H^p possess a particular structure that we will analyze.

Chapter 2

Properties of H^p Functions

2.1 Basic Results

Up to this point, we have examined the general properties of the H_p Space. Now we would like to look more closely at the structure of functions in H_p. We would like to know what kinds of properties a Hardy Space function has. We will see that the functions in H_1 can be characterized as Poisson, Poisson-Stieltjes, Cauchy, and Cauchy-Stieltjes Integrals, and satisfy a Canonical Factorization Theorem. We will first need some definitions:

DEFINITION 2.1.1 *A function f is called harmonic if it is twice continuously differentiable, maps from an open subset of $\mathbb{C}$ to $\mathbb{R}$, and satisfies the Laplace equation:*

$$\triangle f := \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0, \tag{2.1}$$

where $z = x + iy$.

DEFINITION 2.1.2 *Poisson Integral: The Poisson integral of a function u defined on the unit circle is:*

$$\frac{1}{2\pi}\int_0^{2\pi} P\left(r, \theta - t\right) u\left(e^{it}\right) dt \tag{2.2}$$

where

$$P\left(r, \theta\right) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}. \tag{2.3}$$

The space h^p is defined similarly to H^p, but for harmonic functions. If u is a real valued harmonic function and if $M_p\left(r, u\right)$ is bounded, we say that $u \in h^p$. It is important to note that all such h^1 functions can be written as Poisson integrals of their boundary functions. This can be combined with another useful fact: Every analytic function $f(z)$ is in H^p if and only if both its real and imaginary parts are in h_p. Therefore, all functions in H^1 can be recovered from their corresponding boundary functions (See [1] p. 2). We will explore this characterization further with Theorem 2.1 which is presented in [1] (p.41), though it is left unproven in this source. We need a few more definitions before examining and proving this theorem.

DEFINITION 2.1.3 *Cauchy Integral: The Cauchy integral of a complex valued integrable function ϕ is:*

$$F(z) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{e^{it}\phi(e^{it})}{e^{it}-z}dt. \tag{2.4}$$

DEFINITION 2.1.4 *Cauchy-Stieltjes Integral: The Cauchy-Stieltjes integral with respect to a complex-valued function $\mu(t)$ of bounded variation is:*

$$F(z) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{e^{it}}{e^{it}-z}d\mu(t). \tag{2.5}$$

DEFINITION 2.1.5 *Poisson-Stieltjes Integral: The Poisson-Stieltjes integral with respect to a complex-valued function $\mu(t)$ which is of bounded variation on $[0, 2\pi]$ is:*

$$\frac{1}{2\pi}\int_0^{2\pi} P\left(r, \theta - t\right) d\mu(t) \tag{2.6}$$

where

$$P\left(r, \theta\right) = \frac{1-r^2}{1-2r\cos\theta + r^2}. \tag{2.7}$$

THEOREM 2.1 *If a function $f(z)$ analytic in $|z| < 1$ can be represented in any one of the following four ways:*

i as a Cauchy-Stieltjes integral with $F(z) \equiv 0$ in $|z| < 1$;

ii as a Cauchy integral with $F(z) \equiv 0$ in $|z| < 1$;

iii as a Poisson-Stieltjes integral;

iv as a Poisson integral;

then it can be represented in each of the other three ways. The class of functions so representable is H^1.

The following three theorems are necessary in the proof of this theorem:

THEOREM 2.2 *(See [1] p.34) A function $f(z)$ analytic in $|z| < 1$ is representable as the Poisson integral of a function $\varphi \in L^1$ if and only if $f \in H^1$. In this case $\varphi(t) = f(e^{it})$ almost everywhere.*

THEOREM 2.3 *(See [1] p.40) For a complex-valued function $\mu(t)$ of bounded variation, the following three statements are equivalent:*

i $\int\limits_0^{2\pi} e^{int} d\mu\left(t\right) = 0$, *n=1,2,3,...*

ii The Cauchy-Stieltjes integral $F\left(z\right) = \frac{1}{2\pi}\int\limits_0^{2\pi} \frac{e^{it}d\mu(t)}{e^{it}-z}$ *vanishes identically in* $|z| < 1$.

iii The Poisson-Stieltjes integral $f\left(z\right) = \frac{1}{2\pi}\int\limits_0^{2\pi} P\left(r, \theta - t\right) d\mu\left(t\right)$ *is analytic in* $|z| < 1$.

THEOREM 2.4 (The F. and M. Riesz Theorem [7]) *Let $\mu(t)$ be a normalized complex-valued function function of bounded variation on [0,2π], with the property $\int\limits_0^{2\pi} e^{int} d\mu\left(t\right) = 0$n n=1,2,3,... Then $\mu(t)$ is absolutely continuous with respect to arclength measure dt.*

(For more on these theorems see [1].)

Assuming theorems 2.2, 2.3, and 2.4 we now return to the matter of proving Theorem 2.1.

Proof. Theorem 2.1

I (i) $\Rightarrow$ (iii) This is the stated result of Theorem 2.3.

II (i) $\Rightarrow$ (ii) Thm. 2.3 gives the result that $\int\limits_0^{2\pi} e^{int} d\mu\left(t\right) = 0$, n=1,2,3,...

This implies $\mu(t)$ is absolutely continuous by The F. and M. Riesz Theorem

Therefore, $d\mu(t) = \varphi(t)dt$ for some integrable function $\varphi(t)$.

This proves (ii).

III (ii) $\Rightarrow$ (i) This is due to the fact that the Cauchy-Stieltjes integral is more general than the Cauchy integral.

IV (iv) $\Rightarrow$ (iii) This is due to the fact that the Poisson-Stieltjes integral is more general than the Poisson integral.

V Theorem 2.3 gives the result that $\int\limits_0^{2\pi} e^{int} d\mu\left(t\right) = 0$, n=1,2,3,...

This implies $\mu(t)$ is absolutely continuous by the F. and M. Riesz Theorem.

Therefore, $d\mu(t) = \varphi(t)dt$ for some integrable function $\varphi(t)$.

This proves (iv).

All other implications follow from combining the above results.

Finally, we have that the functions which are representable in these ways is the space H^1, as the direct result of Thm. 2.2.

□

2.2 H^p Factorization

Next we explore a critical result, the ability to break down any non-zero H^p function uniquely into 3 factors whose properties fully determine the behavior of the function. These factors are the Blaschke Product, a Singular Inner Function and an outer function for the class H^p. This Canonical Factorization allows one to reduce analyzing the original function down to these constituent parts. We consider now, the properties of these factors:

DEFINITION 2.2.1 *Blaschke Product: A function of the following form,*

$$B\left(z\right) = z^m \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \bar{a}_n\, z} \tag{2.8}$$

where $0 < |a_n| < 1$ *and* $\sum_{n=1}^{\infty} \left(1 - |a_n|\right) < \infty$*, is called a Blaschke Product.*

This Blaschke Product has the property that $|B(e^{i\theta})| = 1$ almost everywhere, i.e., it has modulus 1 a.e. on the unit circle.

DEFINITION 2.2.2 *Inner Function: An inner function* $I : \mathbb{D} \to \mathbb{C}$ *is a function such that* $|I(z)| \leq 1$ *for* $|z| < 1$*, and* $|I(e^{i\theta})| = 1$ *almost everywhere.*

It can be shown that every inner function can be factored into a Blaschke Product and a Singular Inner Function, which we will now define:

DEFINITION 2.2.3 *Singular Inner Function: Let a function have the following form,*

$$S(z) = Ce^{-\int\limits_{o}^{2\pi} \frac{e^{it}+z}{e^{it}-z} d\mu(t)} \tag{2.9}$$

where μ is a positive Borel measure for which $\mu(\pi) < \infty$ and that is singular with respect to the Lebesgue measure. Additionally, C is a constant of modulus 1. Any function which satisfies these conditions is called a Singular Inner Function.

Note that $|S(e^{i\theta})| = 1$ almost everywhere. Now we look at the last remaining factor in the Canonical Factorization of H^p:

DEFINITION 2.2.4 *Outer Function for the Class H^p: A function with the following form,*

$$F(z) = e^{i\gamma} e^{\frac{1}{2\pi} \int\limits_{0}^{2\pi} \frac{e^{it}+z}{e^{it}-z} \log\Psi(t) dt} \tag{2.10}$$

where $\gamma \in \mathbb{R}$, $\psi(t) \geq 0$, $\log\Psi(t) \in L^1$ and $\Psi(t) \in L^p$ is called an outer function for the class H^p.

Now we present the "Canonical Factorization Theorem", due to Smirnov, in its full form:

THEOREM 2.5 (The Canonical Factorization Theorem [8]) *Every function $f(z) \not\equiv 0$ of class H^p $(p > 0)$ has a unique factorization of the form $f(z) = B(z)S(z)F(z)$, where $B(z)$ is a Blaschke product, $S(z)$ is a singular inner function, and $F(z)$ is an outer function for the class of H^p. Conversely, every such product $B(z)S(z)F(z)$, with $F \in H^p$, belongs to H^p.*

The Blaschke factor serves an interesting role in this factorization. $B(z)$ contains all of the zeros for $f(z)$ which ensures that the remaining factors for $f(z)$ are nonvanishing functions. Additionally, since both $\left|B\left(e^{i\theta}\right)\right| = 1$ and $\left|S\left(e^{i\theta}\right)\right| = 1$ almost everywhere we get the result that $F(z)$ has the same H^p norm as $f(z)$, i.e., $\|F\|_p = \|f\|_p$. We also have that $|F(z)| \geq |f(z)|$ in $|z| < 1$.

The following proposition illustrates the use of this key theorem in identifying outer functions:

PROPOSITION 3 If $f(z)$ is analytic and $Re\{f(z)\} > 0$ in $|z| < 1$, then f is an outer function.

Note: Theorem 3.16 of [1] states that if f(z) is analytic and schlicht in $|z| < 1$, then $f \in H^p$ for all $p < \frac{1}{2}$. (A schlicht function f is an analytic function if it does not take any value twice, i.e., it is one-to-one.)

Proof. Since f is analytic and $Re\{f(z)\} > 0$ in $|z| < 1$ then we have that $f \in H^p$ for all $p < 1$. (Thm. 3.2 of [1]).

Since f is strictly positive in $|z| < 1$, we see that $\frac{1}{f}$ is analytic and one-to-one in $|z| < 1$. This implies that $\frac{1}{f} \in H^p$ for $p < \frac{1}{2}$, by Thm. 3.16 of [1], and therefore $S(z) \equiv 1$ (by the Canonical Factorization Theorem), since both f and $\frac{1}{f}$ have canonical factorizations with singular inner functions that are less than or equal to one.

Finally, this gives that $f(z) = B(z)F(z)$, but since f has no zeros this implies $B(z) = 1$ and therefore $f(z) = F(z)$ where F(z) is the outer function given by the canonical representation.

□

In fact, the Hardy Spaces are all subsets of the Nevanlinna class N, and the factorization applies in general to Nevanlinna functions as well. Every function of class N can be expressed in the form $f(z) = B(z)\frac{S_1(z)}{S_2(z)}F(z)$. The canonical factorization theorem gives us a powerful tool in analyzing the properties of H^p functions.

2.3 Additional Useful Facts

Many useful facts regarding H^p spaces are simple in their statements(though perhaps not in their derivation).

First a definition:

DEFINITION 2.3.1 *A function g(z) is said to have a harmonic majorant in D if there is a function U(z) harmonic in D such that $g(z) \leq U(z)$ for any $z \in D$.*

Given the preceding definition we can make use of the following theorem "If f(z) is analytic in $|z| < 1$, then $f \in H^p (0 < p < \infty)$ if and only if $|f(z)|^p$ has a harmonic majorant in $|z| < 1$."

Another theorem states that every analytic function f(z) with positive real part in $|z| < 1$ is of class H^p for all $p < 1$, which follows from the previous proposition and the fact that every outer function for H^p is an element of H^p.

The preceding theorems provide two tools for determining whether a function is in H^p for $p < 1$. As the calculations for checking a function directly are often tedious or require clever tricks, theorems of this nature are important tools that can substantially reduce the burden of work when analyzing a particular function.

Next we will examine some general results for extremal problems in the H^p Spaces. We will see that for certain extremal problems the solutions are unique, and additionally, we will see the forms of those unique solutions.

Chapter 3

Extremal Problems

We now turn to a discussion of linear extremal problems in Hardy Space. The Canonical Factorization Theorem, provides a key tool for analyzing these problems. Let us begin by examining how linear functionals on H^p Spaces can be represented.

The well known Riesz-Representation Theorem has a close analogue in the theory of H^p spaces. We follow the presentation in [1] (p.113) due to A.E. Taylor:

THEOREM 3.1 *For $1 < p < \infty$, the bounded linear functionals ϕ on H^p can be expressed in the form*

$$\phi(f) = \frac{1}{2\pi}\int_0^{2\pi} f\left(e^{i\theta}\right) k\left(e^{-i\theta}\right) d\theta \tag{3.1}$$

where $k(e^{i\theta}) \in H^q$, $(\frac{1}{p} + \frac{1}{q} = 1)$ [$k(z)$ is called the kernel of ϕ.]

The argument used to construct this functional representation breaks down in the case that $p < 1$, because the space H^p is not normable when $p < 1$. In order to find a representation for the case that $p < 1$, first, we define $A(\mathbb{D})$ (called the disk algebra) as, the class of functions which are analytic in $|z| < 1$ and continuous in $|z| \leq 1$.

THEOREM 3.2 *(See [1] p.115.) To each bounded linear functional ϕ on H^p, $0 < p < 1$, there corresponds a unique function $g \in A$ such that*

$$\phi(f) = \lim_{r\to 1^-} \frac{1}{2\pi}\int_0^{2\pi} f\left(re^{i\theta}\right) g\left(e^{-i\theta}\right) d\theta, f \in H^p. \tag{3.2}$$

We will deal initially with the case that $1 \leq p < \infty$, and examine the extremal problem of finding:

$$\|\phi\| = \sup_{f\in H^p, \|f\|_p \leq 1} |\phi\left(f\right)|. \tag{3.3}$$

These functionals are subject to a powerful relation called the duality relation, which, for our current choice of p values, takes the form:

$$\sup_{f\in H^p,\|f\|_p\le 1}\frac{1}{2\pi}\left|\int\limits_{|z|=1} f(z)k(z)dz\right| = \min_{g\in H^q}\|k-g\|_q. \tag{3.4}$$

As the linear functional is identified uniquely by its kernel $k(e^{i\theta})$, we can frame this duality relation in the following way: Finding the norm of a functional on H^p is equivalent to finding the function in H^q that best approximates the kernel (which is a function in L^p, where $\frac{1}{p}+\frac{1}{q}=1$). When $1 < p \le \infty$ the supremum of (12) is attained, and the minimum of (13) is also attained. Therefore, in these cases a solution to the duality problem is guaranteed. However, in the case that $p = 1$ a solution to (12) is guaranteed only in the case that the kernel ($k(e^{it})$) is a continuous function.

The following theorem summarizes the result:

THEOREM 3.3 (The Duality Relation) *(See [6] p.132) For each p $(1 \le p \le \infty)$ and for each function $k(e^{i\theta}) \in L^q$ ($\frac{1}{p}+\frac{1}{q}=1$) with $k \notin H^q$, the duality relation*

$$\sup_{f\in H^p,\|f\|_p\le 1}|\phi(f)| = \inf_{g\in H^q}\|k-g\|_q \tag{3.5}$$

holds, where $\phi(f)$ is defined by

$$\phi(f) = \frac{1}{2\pi i}\int\limits_{|z|=1} f(z)k(z)dz. \tag{3.6}$$

If $p > 1$ then there exists a unique extremal function f for which $\phi(f) > 0$. If $p = 1$ and $k(e^{i\theta})$ is continuous, at least one such extremal function exists. If $p > 1$ ($q < \infty$), the dual extremal problem has a unique solution. If $p = 1$ ($q = \infty$), the dual extremal problem has at least one solution; it is unique if an extremal function exists.

Now we consider a particular extremal problem, the Carathéodory-Féjer problem.

The Carathéodory-Féjer Problem: Let $\beta_1, ..., \beta_m$ be points in the unit disk and let $c_1, ..., c_m$ be complex numbers. Among the functions $f \in H^p$ satisfying the interpolation conditions

$$f(\beta_j) = c_j, j = 1, ..., m, \tag{3.7}$$

we must find one for which the infimum

$$\inf \|f\|_p = \inf \left\{ \int_0^{2\pi} \left| f\left(e^{i\theta}\right) \right|^p d\theta \right\}^{\frac{1}{p}} \tag{3.8}$$

is attained. In doing so, we shall consider functionals of the form: $\phi_j(f) = f(\beta_j)$, for $j = 1, 2, ..., m$, where β_j are fixed points in the unit disk.

S. Ya. Khavinson ([4]) shows that this problem has a unique solution f for $p \geq 1$ and that the following equality holds:

$$\min \|f\|_q = \max_{g \in H^1_\gamma} \left| \sum_{j=1}^{m} c_j \phi_j(g) \right| 2\pi \tag{3.9}$$

where H^1_γ is the space of functions f whose product with $\gamma = \Pi_1^m |\zeta - \beta_j|$ is in H^1.

The form of this unique solution is:

$$f^*(z) = C \prod_{j=1}^{k} \frac{z - \alpha_j}{1 - \bar{\alpha}_j z} \prod_{j=1}^{k} \left(1 - \bar{\alpha}_j z\right)^{\frac{2}{q}} \prod_{j=1}^{k} (1 - \beta_j z)^{-\frac{2}{q}} \tag{3.10}$$

S. Ya Khavinson proves that although the methods of proof are necessarily different for the case $p < 1$, nonetheless, the solution for the extremal problem (3.8) still has the same form. This is achieved by associating the extremal function for $0 < p < 1$ with a function which is extremal for some $p' > p$, where $p' > 1$ [4].

Now we consider another linear extremal problem solved by Macintyre, Rogosinski and S. Ya Khavinson in 1950.

Theorem: Suppose, given $a_0, a_1, ..., a_n \in \mathbb{C}$, f is the unique extremal function that solves

$$\inf\{\|f\|_p : f(0) = a_0, f'(0) = a_1, ..., f^{(n)}(0) = a_n\}. \tag{3.11}$$

Then f has the form:

$$f(z) = C \prod_{|a_j|<1} \frac{z - \alpha_j}{1 - \bar{\alpha}_j z} \prod_{j=1}^{n} \left(1 - \bar{\alpha}_j\, z\right)^{\frac{2}{p}} \tag{3.12}$$

where C is a constant, $|a_j| \leq 1$, for $j = 1, ..., n$, and the first product is taken over some or all of the a_j that are contained inside the unit disk.

The following lemma of Kabaila (See [3]) provides the key to showing that the extremal solution has the same form for the case $p < 1$.

Lemma: Given $a_0, a_1, ..., a_n \in \mathbb{C}$, suppose f^* is extremal for the problem

$$\inf\{\|f\|_p : f(0) = a_0, f'(0) = a_1, ..., f^{(n)}(0) = a_n\}. \tag{3.13}$$

Let $B(z)$ be the Blaschke factor of f^*. Then, for any fixed $p' > p$, the function

$$g^*(z) = \left(\frac{f^*(z)}{B(z)}\right)^{\frac{p}{p'}} B(z) \tag{3.14}$$

is extremal for the problem

$$\inf\{\|g\|_{p'} : g(0) = c_0, g'(0) = c_1, ..., g^{(n)}(0) = c_n\}, \tag{3.15}$$

where the constants C_j, $j = 0, ..., n$ are the first $n+1$ Taylor coefficients of the function g^*.

Since g^* solves (3.15) for $p' \geq 1$, by the previous theorem we have that

$$g^*(z) = C \prod_{|a_j|<1} \frac{z - \alpha_j}{1 - \bar{\alpha}_j z} \prod_{j=1}^{n} \left(1 - \bar{\alpha_j}\ z\right)^{\frac{2}{p'}}. \tag{3.16}$$

Therefore, f^* has the form

$$f^*(z) = C^{\frac{p}{p'}} \prod_{|a_j|<1} \frac{z - \alpha_j}{1 - \bar{\alpha}_j z} \prod_{j=1}^{n} \left(1 - \bar{\alpha_j}\ z\right)^{\frac{2}{p}}. \tag{3.17}$$

We summarize these results in the following theorem,

THEOREM 3.4 *Let $p > 0$. Fix $a_0, a_1, ..., a_n \in \mathbb{C}$, and consider the extremal problem of finding*

$$inf\{\|f\|_p : f(0) = a_0, f'(0) = a_1, ..., f^{(n)}(0) = a_n\}. \tag{3.18}$$

The extremal function f^* always exists and has the form

$$f(z) = C \prod_{|\alpha_j|<1} \frac{z - \alpha_j}{1 - \bar{\alpha}_j z} \prod_{j=1}^{n} \left(1 - \bar{\alpha_j}\ z\right)^{\frac{2}{p}} \tag{3.19}$$

where $|\alpha_j| \leq 1$, C is a constant, and the first product is taken over some or all of the α_j that are contained inside the unit disk. If $p \geq 1$, the extremal function is unique. If $p = \infty$, $\frac{2}{p}$ should be interpreted as being 0.

The problem of uniqueness is a key difference for the case $p < 1$. In this case the extremal problem does not necessarily have a unique solution. This is shown by Kabaila when he considered the following, more simple, "model" problem:

LEMMA 3.1 *Fix $0 < p < 1$ and $a > 0$. For the problem of finding the extremal functions for*

$$\inf\{\|f\|_p : f(0) = 1, f'(0) = a\}, \tag{3.20}$$

we have that the extremal function has 1 of 2 forms:

$$f_1^*(z) = C\frac{z-\alpha}{1-\bar{\alpha}\, z}\left(1-\bar{\alpha}\, z\right)^{\frac{2}{p}}, \tag{3.21}$$

where $|\alpha| < 1$, or

$$f_2^*(z) = C\left(1-\bar{\beta}\, z\right)^{\frac{2}{p}} \tag{3.22}$$

where $|\beta| \leq 1$. We call the first function an extremal of type I and the second an extremal function of type II.
We conclude that for each $0 < p < 1$, there exists a unique value $a > 0$ such that

$$\inf\{\|f\|_p : f(0) = 1, f'(0) = a\} \tag{3.23}$$

has exactly two extremal solutions, one having a zero and the other being non-vanishing.

Now we will turn to the main extremal problem of this thesis. We will see that the solutions to the "model" problem are the key to analyzing our next extremal problem.

Chapter 4

An Extremal Problem With Non-Unique Solutions

We consider the following extremal problem: for a fixed $0 < p < 1$ and $0 < c < 1$, to find

$$\sup\left\{Re(f'(0)) : f(0) = c, \|f\|_p \leq 1\right\}. \tag{4.1}$$

It may seem initially that this problem is distinctly different from the extremal problems we've just considered. In fact, this is not the case, as the following lemma will show. But first, we note that if f is extremal, so is the function $f(e^{it}z)$, therefore, without loss of generality we can consider f to be normalized so that $f'(0) > 0$.

LEMMA 4.1 *If* $0 < p < 1$ *and* $0 < c < 1$ *and if* f *is extremal for the problem*

$$m := sup\{Re(f'(0)) : f(0) = c, \|f\|_p \leq 1\}, \tag{4.2}$$

then f is also extremal for the problem

$$\inf\{\|f\|_p : f(0) = c, f'(0) = m\} \tag{4.3}$$

Proof. Suppose f is extremal for (4.2). Then $f(0) = c$ and $f'(0) = m$, and $\|f\|_p \leq 1$. Now suppose there were a function g such that $g(0) = c$ and and $g'(0) = m$ but $\|g\|_p < \|f\|_p \leq 1$. Pick $\epsilon > 0$ small enough such that

$$\|g(z) + \epsilon z\|_p^p \leq \|g\|_p^p + \|\epsilon z\|_p^p = \|g\|_p^p + \epsilon^p < 1 \tag{4.4}$$

Then the function $g_\epsilon(z) := g(z) + \epsilon z$ satisfies $g(0) = c$ and $\|g_\epsilon\|_p \leq 1$, but $g'_\epsilon(0) = m + \epsilon > m$ which is a contradiction. Therefore, f is also extremal for (4.3). □

We have seen in lemma 3.1 that this problem has two potential forms for its solutions, and we will see that for each $0 < p < 1$ the extremal solution is dependent upon the interval in which c lies. These two potential forms are as follows:

$$f_1(z) = A\frac{z-\alpha}{1-\bar{\alpha}\, z}\left(1-\bar{\alpha}\, z\right)^{\frac{2}{p}}, \tag{4.5}$$

for $|\alpha| < 1$ and $A > 0$, or

$$f_2(z) = B\left(1-\bar{\beta}\, z\right)^{\frac{2}{p}}, \tag{4.6}$$

where $|\beta| \leq 1$.

In order to determine which form will be extremal and for what values of p and c, we will need to make use of the given constraints. First we note that if f is a solution to the problem, then $1 = \|f\|_p^p$ (Otherwise, one can construct a function $f(z) + \epsilon$ and argue as in the previous lemma.) This leads to the following result for each potential equation:

$$1 = \|f_1\|_p^p = A^p\|\left(1-\bar{\alpha}\, z\right)^{\frac{2}{p}}\|_p^p = A^p\|1-\bar{\alpha}\, z\|_2^2 = A^p\left(1+|\alpha|^2\right), \text{ and} \tag{4.7}$$

$$1 = \|f_2\|_p^p = B^p(1+|\beta|^2). \tag{4.8}$$

The second equality in (4.7) follows because the Blaschke product $\frac{z-\alpha}{1-\bar{\alpha}z}$ has modulus 1 on the unit circle.

We will now examine each of the functions more closely, beginning with $f_1(z)$. In order to determine where $f_1(z)$ is the extremal solution we will need to derive some useful facts first.
Since $0 < c < 1$, $f_1(0) = -A\alpha = c$ and $|\alpha| < 1$, we see that α is real and $-1 < \alpha < 0$. Using $\alpha = -\frac{c}{A}$ we can rewrite the constraint from (4.7) as follows:

$$A^p\left(1+\frac{c^2}{A^2}\right) = 1. \tag{4.9}$$

The fact that $-A\alpha = c$ and $|\alpha| < 1$ imply that $A > c$. Additionally, using constraint (4.9) we see that if $A \leq 2^{-\frac{1}{p}}$ then $A^p\left(1+|\alpha|^2\right) \leq 2^{-1}\left(1+|\alpha|^2\right) \leq \frac{1}{2}\left(1+|\alpha|^2\right) < 1$ which is a contradiction, and therefore, $A > 2^{-\frac{1}{p}}$.

Using these constraints, we will see the form of $f'(z)$ at the point $z = 0$.

LEMMA 4.2 *For* $0 < p < 1$ *and* $0 < c < 1$ *the derivative of* $f_1(z)$ *at zero is:*

$$f_1'(0) = A + \frac{c^2}{A}\left(\frac{2-p}{p}\right). \tag{4.10}$$

Proof. Using the fact that α is real valued and rewriting $f_1(z)$ we have that
$f_1(z) = A(z-\alpha)(1-\alpha z)^{\frac{2-p}{p}}$. Calculating the derivative, we find that
$f_1'(z) = A[(1-\alpha z)^{\frac{2-p}{p}} + (z-\alpha)(-\alpha)(\frac{2-p}{p})(1-\alpha z)^{\frac{2-2p}{p}}]$.
Taking $z = 0$, $f_1'(0) = A[1+\alpha^2(\frac{2-p}{p})]$. Using the constraint that $\alpha = \frac{-c}{A}$, we get the desired result.

□

Now we see that multiplying each side of (4.9) by A^2 results in the equation:

$$A^{(p+2)} + A^p c^2 - A^2 = 0 \tag{4.11}$$

Applying these facts leads to the following lemma:

Lemma

i *Let* $0 < p < 1$. *If* $0 < c \le 2^{-\frac{1}{p}}$, *then there exists a unique solution x in the interval* $(2^{-\frac{1}{p}}, 1)$ *to the equation*

$$x^{p+2} + x^p c^2 - x^2 = 0. \tag{4.12}$$

ii *If* $2^{-\frac{1}{p}} < c < c_p := \left(\frac{2-p}{2}\right)^{\frac{1}{p}} \sqrt{\frac{p}{2-p}}$, *then there are 2 solutions to the equation (4.12) in the interval* $(c, 1)$.

iii *If* $c = c_p$ *then there exists a unique solution to (4.12).*

iv *If* $c_p < c < 1$, *then there are no solutions to (4.12) in the interval (c,1).*

Proof. Let $0 < p < 1$ and $0 < c < 1$. Then there exists a solution x in $(0,1)$ to (4.12) if and only if $c^2 = x^{2-p} - x^2$. We wish to see in what intervals the equation $g(x) := x^{2-p} - x^2$ satisfies $g(x) = c^2$. Clearly g is continuous on $[0,1]$, and infinitely differentiable on $(0,1)$. Since $x^{2-p} > x^2$ on $(0,1)$ we see that $g(x) > 0$ for $x \in (0,1)$. Additionally, $g(0) = g(1) = 0$. By taking the derivative of $g(x)$ we get:

$$g'(x) = (2-p)\,x^{1-p} - 2x > 0. \tag{4.13}$$

It is easy to see that $g'(x)$ is continuous on $(0,1)$. Solving the equation $g'(x) = 0$ on $(0,1)$, for x, gives the solution $x = \left(\frac{2-p}{2}\right)^{\frac{1}{p}} =: a_p$. From these facts we can conclude that g is strictly increasing on $(0, a_p)$ and strictly decreasing on $(a_p, 1)$, therefore, g has its maximum at

$$c_p^2 := g(a_p) = \left(\frac{2-p}{2}\right)^{\frac{2}{p}} \left(\frac{p}{2-p}\right). \tag{4.14}$$

This implies that if $c_p < c < 1$, then there is no solution to (4.12), which proves (iv). If $c = c_p$, then a_p is the unique solution to (4.12), which proves (iii). This also shows that for $0 < c < c_p$, there are 2 solutions for (4.12) in $(0,1)$. In order to prove (i) and (ii) we must examine the equation more closely. If $\sqrt{g(x)} = x$ then $x^{2-p} - x^2 = x^2$, when restricted to (0,1), then we see that $x^{-p} = 2$ and therefore $x = 2^{-\frac{1}{p}}$. Now we want to show that $g(x) > x^2$ only when $0 < x < 2^{-\frac{1}{p}}$. To see this, assume that $x < 2^{-\frac{1}{p}}$, then $x^p < 2^{-1}$, which gives $x^{-p} > 2$, therefore $x^{2-p} > 2x^2$, $x^{2-p} - x^2 > x^2$ and finally $g(x) > x^2$. Similarly, if we assume $x > 2^{-\frac{1}{p}}$ then we will get that $g(x) < x^2$. Additionally, since $0 < p < 1$, $\frac{2-p}{2} > \frac{1}{2}$ and therefore $a_p > 2^{-\frac{1}{p}}$.

This leads to the conclusion that if $0 < c \leq 2^{-\frac{1}{p}}$, then there exists one solution to (4.12) in the interval $(0, 2^{-\frac{1}{p}}]$ and one in the interval $(2^{-\frac{1}{p}}, 1)$ which proves (i).

On the other hand if $2^{-\frac{1}{p}} < c < c_p$, then there are two solutions to (4.12) in the interval $(c, 1)$, which proves (ii). Now we have that every part of the lemma has been shown.

□

Now that we have examined $f_1(z)$ and its constraints more closely, we shall examine $f_2(z)$.

LEMMA 4.3 *For $0 < p < 1$ and $0 < c < 1$ the derivative of $f_2(z)$ at zero is:*

$$f_2'(0) = \frac{2c}{p}\sqrt{\frac{1}{c^p} - 1} \tag{4.15}$$

and if $0 < c < 2^{-\frac{1}{p}}$, then there is no solution of type II.

Proof. We already have that $f_2(z) = B(1 - \bar{\beta} z)^{\frac{2}{p}}$.

Since $f_2(0) = c$ it follows immediately that $B = c$ so that $f_2(z) = c\left(1 - \bar{\beta} z\right)^{\frac{2}{p}}$ and hence

$$f_2'(z) = c\frac{2}{p}\left(1 - \bar{\beta} z\right)^{\frac{2}{p}-1}\left(-\bar{\beta}\right). \tag{4.16}$$

Then we get that

$$f_2'(0) = -\frac{\bar{\beta}\, 2c}{p}. \tag{4.17}$$

By normalizing f_2 such that $f_2'(0) > 0$, we see that β is real and negative. If f_2 is a solution to (1) then we have that

$$1 = \|f_2\|_p^p = c^p(1+\beta^2). \tag{4.18}$$

Since $|\beta| \le 1$ we get that $c^p \ge \frac{1}{2}$. Therefore, we can conclude that if $0 < c < 2^{-\frac{1}{p}}$ then there is no solution of type f_2. On the other hand if $c \ge 2^{-\frac{1}{p}}$, then

$$\beta = -\sqrt{\frac{1}{c^p} - 1}, \tag{4.19}$$

which gives the desired result. □

Now some conclusions can be drawn about problem (4.1), and thus we formulate the following theorem:

THEOREM 4.1 *Let $0 < p < 1$. For each $0 < c < 1$, there exists a solution to the problem of finding*

$$\sup\left\{\left|f'(0)\right| : f(0) = c, \|f\|_p \le 1\right\}. \tag{4.20}$$

i If $0 < c < 2^{-\frac{1}{p}}$, then

$$f_1(z) = A\frac{(z-\alpha)}{\left(1-\bar{\alpha}\, z\right)}\left(1-\bar{\alpha}\, z\right)^{\frac{2}{p}}, \tag{4.21}$$

is extremal, where $\alpha = -\frac{c}{A}$ and A is the unique solution to

$$x^{p+2} + x^p c^2 - x^2 = 0 \tag{4.22}$$

in the interval $(2^{-\frac{1}{p}}, 1)$.

ii If $2^{-\frac{1}{p}} < c < c_p$ then the solution could be of either form $f_1(z)$ or $f_2(z)$. Additionally, for at least some fixed p, there exists some $2^{-\frac{1}{p}} < c < c_p$ such that $f_1(0) = f_2(0)$ and hence the extremal solution is not unique.

iii If $c_p < c < 1$ then f_2 is extremal.

iv If $c = 2^{-\frac{1}{p}}$ or $c = c_p$ then a solution exists, though the type has not been conclusively determined.

Proof. (i) This follows immediately from the previous conclusions that there is no solution of type f_2 for $0 < c < 2^{-\frac{1}{p}}$ and that the value for the constant A must be greater than $2^{-\frac{1}{p}}$.

(ii) We will consider the particular case where $p = \frac{1}{2}$. We will show 2 different values for c ($2^{-\frac{1}{p}} < c < c_p$) which result in f_1 being the extremal function for the first chosen value, c_1, while f_2 is the extremal function for the second value, c_2. This combined with a continuity argument proves the result. We will now examine our first choice of c.

For $c_1 = \frac{13}{50}$: We first note that since $p = \frac{1}{2}$ we have that $2^{-\frac{1}{p}} = \frac{1}{4}$ and $c_p = \frac{\sqrt{3}}{4}$ and therefore $c_1 \in (2^{-\frac{1}{p}}, c_p)$. 2 simple calculations show that for this value of c we get that $f_1'(0) = A + \frac{507}{2500A}$ and $f_2'(0) = \frac{52}{50}\sqrt{(\frac{50}{13})^{\frac{1}{2}} - 1}$. In order to solve $A^{(p+2)} + A^p c^2 - A^2 = 0$ for A, we have turned to a computer algebra system. In particular, we have used the Matlab solve function, which returns exact solutions to the equations. However, the expression of these solutions would be highly impractical to list here, so we have opted to express the following values rounded to 4 decimal places. As previously shown, there are 2 solutions to this equation in $(c, 1)$ and in this particular case we get that $A_1 \approx .2709$ and $A_2 \approx .8289$. When evaluating with A_1 we see that $f_1'(0) \approx 1.0196$ and with A_2 we see that $f_1'(0) \approx 1.0735$. From our previous result we have that $f_2'(0) \approx 1.0196$. Therefore f_1 is extremal with the constant A_2.

For $c_2 = \frac{8}{25}$: We note that $c_2 \in (2^{-\frac{1}{p}}, c_p)$. In this case 2 simple calculations show that that $f_1'(0) = A + \frac{192}{625A}$ and $f_2'(0) = \frac{32}{25}\sqrt{(\frac{25}{8})^{\frac{1}{2}} - 1}$. In this case our CAS returns $A_1 \approx .2709$ and $A_2 \approx .8289$. When evaluating with A_1 we see that $f_1'(0) \approx 1.1190$ and with A_2 we see that $f_1'(0) \approx 1.1200$. From our previous result we have that $f_2'(0) \approx 1.1216$. Therefore f_2 is extremal for this case.

Note that $f_1'(0)$ and $f_2'(0)$ are both continuous, as functions of c (for $0 < c < 1$), and therefore, their difference is also continuous as a function of c. We see that $[f_1'(0) - f_2'(0)] > 0$ for c_1, and that $[f_1'(0) - f_2'(0)] < 0$ for c_2. Since $[f_1'(0) - f_2'(0)]$ is continuous as a function of c for $(0 < c < 1)$ there must exist some c_0 such that for c_0, $[f_1'(0) - f_2'(0)] = 0$ (by the Intermediate Value Theorem) and therefore $f_1'(0) = f_2'(0)$ for c_0. Hence, the solution to the extremal problem is not unique.

[Although we haven't been able to find a proof for general choice of p and c here, it is probably true that for each value of $0 < p < 1$ there exists a c ($2^{-\frac{1}{p}} < c < c_p$) such that both functions are extremal.]

(iii) The case for $c_p < c < 1$ follows immediately from the result that there is no solution for the constant A of the first function f_1 when $c_p < c < 1$.

(iv) The fact that a solution exists at these points is trivial, since both potential extremal solutions are defined at these points. We have not been able to find a general proof for these particular cases, although, as we will discuss in Chapter 5, our numerical approach suggests each case has a unique extremal solution. [The results suggest that f_1 is extremal in the case that $c = 2^{-\frac{1}{p}}$ and that f_2 is extremal in the case that $c = c_p$.] We examine these situations further, in the next two lemmas. $\square$

LEMMA 4.4 *In the case that $c = 2^{-\frac{1}{p}}$, the problem of determining whether f_1 or f_2 is extremal, reduces to the problem of determining if the expression*

$$Ap2^{\frac{1-p}{p}} + \frac{2^{-\frac{1}{p}}}{A} - \frac{p \cdot 2^{\frac{-(p+1)}{p}}}{A} \tag{4.23}$$

is greater than 1 or less than 1, respectively.

Proof. When $c = 2^{-\frac{1}{p}}$ We can write:

$$f_1'(0) = A + \left(\frac{2^{\frac{-2}{p}}}{A}\right)\left(\frac{2-p}{p}\right) \tag{4.24}$$

and

$$f_2'(0) = \frac{2^{\frac{p-1}{p}}}{p}. \tag{4.25}$$

Taking the quotient of these derivatives gives

$$\frac{f_1'(0)}{f_2'(0)} = \frac{AP}{2^{\frac{p-1}{p}}} + \frac{2^{-\frac{2}{p}}p}{A2^{\frac{p-1}{p}}}\left(\frac{2-\rho}{p}\right) \tag{4.26}$$

$$= Ap2^{\frac{1-p}{p}} + \frac{2^{-\frac{(p+1)}{\rho}}(2-\rho)}{A} \tag{4.27}$$

$$= Ap2^{\frac{1-p}{p}} + \frac{2^{-\frac{1}{p}}}{A} - \frac{p \cdot 2^{\frac{-(p+1)}{p}}}{A}. \tag{4.28}$$

Since, both $f_1'(0)$ and $f_2'(0)$ are positive terms, we have that if $Ap2^{\frac{1-p}{p}} + \frac{2^{-\frac{1}{p}}}{A} - \frac{p \cdot 2^{\frac{-(p+1)}{p}}}{A} > 1$, then f_1 is extremal and if $Ap2^{\frac{1-p}{p}} + \frac{2^{-\frac{1}{p}}}{A} - \frac{p \cdot 2^{\frac{-(p+1)}{p}}}{A} < 1$ then f_2 is extremal. $\square$

LEMMA 4.5 *In the case that $c = c_p$, the problem of determining whether f_1 or f_2 is extremal reduces to the problem of showing that the function*

$$h(p) = \sqrt{2p^{\frac{-p-2}{2}}(2-p)^{\frac{p-4}{2}} - p^{-1}(2-p)^{-1}} \tag{4.29}$$

is either strictly less than or strictly greater than 1 for $0 < p < 1$ respectively. Equivalently, one can show that $h'(p) < 0$ or $h'(p) > 0$ for $0 < p < 1$.

Proof. In the case that $c = c_p := \left(\frac{2-p}{2}\right)^{\frac{1}{p}} \sqrt{\frac{p}{2-p}}$, we have already shown that $A = a_p := (\frac{2-p}{2})^{\frac{1}{p}}$. In this case we can rewrite both $f_1'(0)$ and $f_2'(0)$ in terms of p alone. We get that,

$$\begin{aligned} f_1'(0) &= A + \frac{c^2}{A}\left(\frac{2-p}{p}\right) \\ &= \left(\frac{2-p}{2}\right)^{\frac{1}{p}} + \left(\frac{2-p}{2}\right)^{\frac{1}{p}} \\ &= 2\left(\frac{2-p}{2}\right)^{\frac{1}{p}} \end{aligned}$$

and

$$\begin{aligned} f_2'(0) &= \frac{2c}{p}\sqrt{\frac{1}{c^p} - 1} \\ &= \frac{2}{p}\left(\frac{2-p}{2}\right)^{\frac{1}{p}} \sqrt{\frac{p}{2-p}}\sqrt{\left(\frac{2}{2-p}\right)\left(\frac{2-p}{p}\right)^{\frac{p}{2}} - 1} \\ &= \frac{2^{1-\frac{1}{p}}(2-p)^{\frac{1}{p}-\frac{1}{2}}}{p^{\frac{1}{2}}}\sqrt{2p^{-\frac{p}{2}}(2-p)^{\frac{p}{2}-1} - 1} \\ &= \frac{2^{\frac{p-1}{p}}(2-p)^{\frac{2-p}{2p}}}{p^{\frac{1}{2}}}\sqrt{2p^{-\frac{p}{2}}(2-p)^{\frac{p-2}{2}} - 1}. \end{aligned}$$

Now we take the quotient and define $h(p) = \frac{f_2'(0)}{f_1'(0)}$. We get that

$$h(p) = \frac{(2-p)^{-\frac{1}{2}}}{p^{\frac{1}{2}}}\sqrt{2p^{-\frac{p}{2}}(2-p)^{\frac{p-2}{2}} - 1} \tag{4.30}$$

$$= \sqrt{\frac{2p^{-\frac{p}{2}}(2-p)^{\frac{p-2}{2}} - 1}{p\,(2-p)}} \tag{4.31}$$

$$= \sqrt{2p^{\frac{-p-2}{2}}(2-p)^{\frac{p-4}{2}} - p^{-1}(2-p)^{-1}}. \tag{4.32}$$

Since, both $f_1'(0)$ and $f_2'(0)$ are positive, we see that if $h(p) < 1$ then f_1 is extremal and if $h(p) > 1$ then f_2 is extremal. This proves our first claim. Looking at (4.30) it is easy to see that the term inside of the radical is strictly greater than or equal to 0. Then, clearly we have that $h(p)$ is continuous for $0 < p \leq 1$. Furthermore, it is easy to see that $h(1) = 1$ and therefore, if $h'(p) < 0$ then as p decreases from 1, we get that $h(p) > 1$. Similarly, if $h'(p) > 0$, then as p decreases from 1, we get that $h(p) < 1$. Now our second claim has been shown.

□

We have now seen that the extremal problem (4.1) does not have a unique solution for all values of p and c, although we can determine that it has a unique solution for certain values of c. Furthermore, we have examined some conditions which would guarantee the uniqueness of solutions for the special cases $c = 2^{-\frac{1}{p}}$ and $c = c_p$. In the next chapter we will examine the use of computer programming and how it assisted in determining the non-uniqueness of solutions to the extremal problem.

Chapter 5
Numerical Analysis

5.1 Our Approach

Now we turn to a discussion of numerical analysis. In this research, we have seen that modern programming provides powerful tools that can be used in approaching rigorous mathematical problems. In the principle problem of this thesis, these tools were extremely helpful in directing the early efforts towards solving this problem. The numerical analysis was focused on the particular cases $c = 2^{-\frac{1}{p}}$, $c = c_p$, and the interval $c \in (2^{-\frac{1}{p}}, c_p)$. In fact, it was the use of this programming that led to the proof that our extremal problem did not have a unique solution. It is very important to note, in the programming examples that follow, we have used the vpasolve function in Matlab. This is because it runs much more quickly than the solve function, which was important for testing large numbers of values. However, vpasolve only provides numerical estimates, rather than exact values. As such, when constructing our argument for the main theorem, we instead used the solve function to calculate exact values for those two values of c.

Pseudocode

The program worked by testing individual values of p and c in the formulas for $f_1'(0)$ and $f_2'(0)$ with the goal of comparing their values to determine the extremal function.

The following is pseudocode that represents the MATLAB code used to analyze this extremal problem (following the pseudocode we provide a more in depth description of the algorithms steps):
For p from .01 till .98 do
for c from $2^{-\frac{1}{p}}$ till c_p do
Solve for constant A using $(A^{p+2}) + (A^p)(c^2) - (A^2) = 0$
Evaluate $f_1'(0)$ and $f_2'(0)$ using p, c, and A.
Check $max\{f_1'(0), f_2'(0)\}$

Output 1 if f_1 is extremal. Output 2 if f_2 is extremal.

After fixing an individual p value, the program would then start by testing a particular value for c. Using p and c the program solves $A^{p+2} + A^p c^2 - A^2 = 0$ for the constant A which is needed in f_1. Using the values for p,c, and A the program can now compute both $f_1'(0)$ and $f_2'(0)$.

After comparing the results for this case of p and c the program moves on to the next value of c. Once the program has checked the designated range of c values for a fixed p then a new loop starts for the next value of p. This continues until the sampled values for p have all been tested, at which time the results for the program are printed. The values of c are separated into the three cases mentioned above because the values $c = 2^{-\frac{1}{p}}$ and $c = c_p$ are interesting special cases and, in particular, the functions can be rewritten and simplified for these cases.

Interpretation of Results

For the case $c = 2^{-\frac{1}{p}}$ numerical analysis indicated that $f_1(z)$ function was extremal, while for the second case ($c = c_p$) the numerical analysis indicated that $f_2(z)$ was extremal for the problem.

For $c \in (2^{-\frac{1}{p}}, c_p)$ (with p fixed) the numerical analysis often indicated that the extremal function changed between functions 1 and 2. There was no clear pattern to the shift, however, this did indicated the likely possibility that there are values of p and c for which both functions are extremal, which would show that the problem does not always have a unique solution. Although, we were able to provide a proof that for at least one value of p there exists a c such that the solution was not unique, it would be interesting to prove this fact for all values of p.

5.2 The Code

The programs used were written in MATLAB R2015b. Although the pseudocode describes how the algorithm operated, we also provide the precise code. The code used for each of the 3 cases follows: (Its formatting has been adjusted slightly, so as to be more friendly to the LaTeX environment.)

Case $c = 2^{-\frac{1}{p}}$

for p=0.1:0.02:0.98

c1=$2^{(-1)/p}$;

syms x

A1=vpasolve($(x^{p+2}) + (x^p) * ((c1)^2) - (x^2) == 0$, x, $[c1, 1]$);

f1=vpa(A1+((($c1)^2$)/A1)*((2-p)/p));

f2=vpa((2*(c1))/p)*(((1/(($c1)^p$))-1)$^{(1/2)}$);

B=[f1,f2];

$[M, I] = max(B)$;

[p, c1, f1, f2, I]

end

Case $c = c_p$

for p=0.1:0.02:0.98

$c1 = ((((2-p)/2)^{(1/p)}) * ((p/(2-p))^{(1/2)}))$;

syms x

$A1 = ((2-p)/2)^{(1/p)}$;

$f1 = vpa(A1 + ((c1^2)/A1) * ((2-p)/p))$;

$f2 = vpa((2*c1)/p) * (((1/((c1)^p)) - 1)^{(1/2)})$;

B=[f1,f2];

[M,I]=max(B);

p; c1; M; I; A1 ;

[p, c1, f1, f2, I]

end

Case $c \in (2^{-\frac{1}{p}}, c_p)$

for $p = 0.1 : 0.02 : 0.98$

$c1 = round(2^{((-1)/p)}, 3);$

$c2 = floor(((((2-p)/2)^{(1/p)}) * ((p/(2-p))^{(1/2)})) * 1000)/1000;$

$c3 = ((2-p)/2)^{(1/p)};$

for $c = c1 : 0.005 : c2$

$F = [c1, c]$

$[N, J] = max(F)$

syms x

A1=vpasolve$((x^{(p+2)}) + (x^p) * (c^2) - (x^2) == 0, x, [N, c3]);$

A2=vpasolve$((x^{(p+2)}) + (x^p) * (c^2) - (x^2) == 0, x, [c3, 1])$;

f1=vpa$(A1 + ((c^2)/A1) * ((2 - p)/p))$;

f2=vpa$(A2 + ((c^2)/A2) * ((2 - p)/p))$;

f3=vpa$((2 * c)/p) * (((1/(c^p)) - 1)^{(1/2)})$;

$B = [f2, f3]$;

$[M, I] = max(B)$;

$[p, c, M, A1, A2, f2, f3, I]$

Result Samples

For the program $c = 2^{-\frac{1}{p}}$, the formatting is $[p, c, f]$ where p and c represent the values for each variable, and f indicates whether the first or second function is extremal. Thus, one sees that the following numerical results indicate that f_1 is extremal.

0.1000 0.0010 1.0000

0.1200 0.0031 1.0000

0.1400 0.0071 1.0000

0.1600 0.0131 1.0000

0.1800 0.0213 1.0000

0.2000 0.0313 1.0000

0.2200 0.0428 1.0000

0.2400 0.0557 1.0000

0.2600 0.0695 1.0000

0.2800 0.0841 1.0000

0.3000 0.0992 1.0000

0.3200 0.1146 1.0000

0.3400 0.1302 1.0000

0.3600 0.1458 1.0000

0.3800 0.1614 1.0000

0.4000 0.1768 1.0000

0.4200 0.1920 1.0000

0.4400 0.2069 1.0000

0.4600 0.2216 1.0000

0.4800 0.2360 1.0000

0.5000 0.2500 1.0000

0.5200 0.2637 1.0000

0.5400 0.2770 1.0000

0.5600 0.2900 1.0000

0.5800 0.3027 1.0000

0.6000 0.3150 1.0000

0.6200 0.3269 1.0000

0.6400 0.3386 1.0000

0.6600 0.3499 1.0000

0.6800 0.3608 1.0000

0.7000 0.3715 1.0000

0.7200 0.3819 1.0000

0.7400 0.3919 1.0000

0.7600 0.4017 1.0000

0.7800 0.4112 1.0000

0.8000 0.4204 1.0000

0.8200 0.4294 1.0000

0.8400 0.4382 1.0000

0.8600 0.4466 1.0000

0.8800 0.4549 1.0000

0.9000 0.4629 1.0000

For the program $c = c_p$, one sees that the following numerical results indicate f_2 is extremal for the problem.

0.1000 0.1374 2.0000

0.1200 0.1509 2.0000

0.1400 0.1634 2.0000

0.1600 0.1751 2.0000

0.1800 0.1862 2.0000

0.2000 0.1968 2.0000

0.2200 0.2070 2.0000

0.2400 0.2168 2.0000

0.2600 0.2263 2.0000

0.2800 0.2354 2.0000

0.3000 0.2444 2.0000

0.3200 0.2531 2.0000

0.3400 0.2616 2.0000

0.3600 0.2700 2.0000

0.3800 0.2782 2.0000

0.4000 0.2862 2.0000

0.4200 0.2941 2.0000

0.4400 0.3019 2.0000

0.4600 0.3096 2.0000

0.4800 0.3172 2.0000

0.5000 0.3248 2.0000

0.5200 0.3322 2.0000

0.5400 0.3396 2.0000

0.5600 0.3469 2.0000

0.5800 0.3541 2.0000

0.6000 0.3613 2.0000

0.6200 0.3684 2.0000

0.6400 0.3755 2.0000

0.6600 0.3826 2.0000

0.6800 0.3896 2.0000

0.7000 0.3966 2.0000

0.7200 0.4035 2.0000

0.7400 0.4105 2.0000

0.7600 0.4174 2.0000

0.7800 0.4243 2.0000

0.8000 0.4312 2.0000

0.8200 0.4380 2.0000

0.8400 0.4449 2.0000

0.8600 0.4518 2.0000

0.8800 0.4587 2.0000

0.9000 0.4655 2.0000

0.9200 0.4724 2.0000

0.9400 0.4793 2.0000

0.9600 0.4862 2.0000

0.9800 0.4931 2.0000

For the program $c \in (2^{-\frac{1}{p}}, c_p)$, we have only included a small but relevant portion of the results, as we had tested hundreds of combinations of p and c. Additionally the format we use here is slightly different as we are including more data. In this case, the format is [p, c, A1, A2, $f_1'(0)$ using A1, $f_2'(0)$ using A2, extremal function].

[1/2, 51/200, 1.07044, 0.26020, 0.83752, 1.070442312935, 1.00990, 1]

[1/2, 13/50, 1.07353, 0.27085, 0.82885, 1.07353, 1.01960, 1]

[1/2, 53/200, 1.07671, 0.28198, 0.81970, 1.07671, 1.02911, 1]

[1/2, 27/100, 1.08, 0.29366, 0.81, 1.08, 1.03843, 1]

[1/2, 11/40, 1.08338, 0.30595, 0.79967, 1.08338, 1.04755, 1]

[1/2, 7/25, 1.08687, 0.31894, 0.78864, 1.08687, 1.05650, 1]

[1/2, 57/200, 1.09048, 0.33276, 0.77679, 1.09048, 1.06525, 1]

[1/2, 29/100, 1.09422, 0.34754, 0.76397, 1.09422, 1.07383, 1]

[1/2, 59/200, 1.09808, 0.36349, 0.74997, 1.09808, 1.08222, 1]

[1/2, 3/10, 1.10209, 0.38092, 0.73448, 1.10209, 1.09044, 1]

[1/2, 61/200, 1.10625, 0.40028, 0.71707, 1.10625, 1.09848, 1]

[1/2, 31/100, 1.11060, 0.42235, 0.69694, 1.11060, 1.10635, 1]

[1/2, 63/200, 1.11516, 0.44865, 0.67258, 1.11516, 1.11404, 1]

[1/2, 8/25, 1.12156, 0.48316, 0.64, 1.12, 1.12156, 2]

5.3 Conclusion

In this thesis, we have examined the basics of the theory of H^p Spaces. Using this framework, we considered classic linear extremal problems on H^p. We have seen, that for the problems that were considered in Chapter 3, there exists a unique solution whenever $p > 1$. In Chapter 4, we have proven that the extremal problem of finding f such that

$$\sup\left\{\left|f'(0)\right| : f(0) = c, \|f\|_p \leq 1\right\} \tag{5.1}$$

for a fixed $0 < p < 1$ and $0 < c < 1$, does not always have a unique extremal solution. However, for any fixed p, we can place constraints on c which do in fact guarantee uniqueness.

We were able to come to strong conclusion regarding the cases that $0 < c < 2^{-\frac{1}{p}}$, $2^{-\frac{1}{p}} < c < c_p$, and $c_p < c < 1$. However, a definitive conclusion for the cases $c = 2^{-\frac{1}{p}}$ and $c = c_p$ eluded us. We did provide two lemmas which may guide future attempts at completing this aspect of the problem. Additionally, our numerical analysis strongly suggests that f_1 is extremal if $c = 2^{-\frac{1}{p}}$, while f_2 is extremal if $c = c_p$. As such, we would suggest that any future attempts be aimed at proving these statements conclusively.

In conclusion, although many extremal problems have unique solutions for $p > 1$, these same problems may not have unique solutions when $0 < p < 1$. This failure of uniqueness gives opportunities for interesting research into the solutions of such extremal problems.

References

[1] P. L. Duren, Theory of H^p Spaces, Academic Press (1970).

[2] P. Fatou, Séries trigonométriques et séries de Taylor, Acta Math. **30** (1906), 335-400.

[3] V. Kabaĭla, Some interpolation problems in the classes H_δ for $\delta < 1$, Studies of Current Problems of the Theory of Functions of a Complex Variable (Fourth All-Union Conf., Moscow, 1958; A. I. Markushevich, editor), Fizmatgiz, Moscow, 1961, pp. 180-187. (Russian)

[4] S. Ya. Khavinson, Two papers on extremal problems in complex analysis, American Mathematical Society (1986).

[5] Livingston, Arthur E. The space H^p, $0 < p < 1$, is not normable. Pacific J. Math. 3 (1953), no. 3, 613–616. http://projecteuclid.org/euclid.pjm/1103051332.

[6] A. Pietsch, History of Banach Spaces and Linear Operators, Birkhuser Boston (2007).

[7] F. and M. Riesz, Über die Randwerte einer analytischen Funktion, Quatrième Congrès des Math. Scand. Stockholm (1916), 27-44.

[8] V. I. Smirnov, Sur les valeurs limites des fonctions, régulières à l'intérieur d'un cercle. Journal de la Société Phys.-Math. de Lénningrade **2** (1929), no. 2, 22-37.

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