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On Spectral Properties of Single Layer Potentials

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by

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Dedication

To the memory of my beloved mother.

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Abstract

We show that the singular numbers of single layer potentials on smooth curves asymptotically behave like $\mathrm{O}(\frac{1}{n})$. For the curves with singularities, as long as they contain a smooth piece, the resulting single layer potentials are never trace-class. We provide upper bounds for the operator and the Hilbert-Schmidt norms of single layer potentials on smooth and chord-arc curves. Regarding the injectivity of single layer potentials on planar curves, we prove that among single layer potentials on dilations of a given curve, only one yields a non-injective single layer potential. A criterion for injectivity of single layer potentials on ellipses is given. We establish an isoperimetric inequality for Schatten $p-$norms of logarithmic potentials over quadrilaterals and its analogue for Newtonian potentials on parallelepipeds.

1 Introduction

The study of the interaction between the geometry of domains and differential operators defined on these domains has a long history. In 1966, Mark Kac posed a question regarding what nowadays is called "isospectral" domains. He showed that it is possible to gain information about the shape of domains by studying the eigenvalues of the Laplacian. He stated in his 1966 paper [33] that he had heard about the problem from S. Bochner. This problem had been studied before by H. Weyl and others [59]. Almost immediately after Kac's paper was published, John Milnor [42] by using a theorem of Ernest Witt, showed that there exists a pair of 16-dimensional tori that have the same eigenvalues for the Hodge-Laplace operator but have different shapes. The problem in two dimensions remained open until 1992 when C. Gordon, D. Webb and S. Wolpert[30] in their breakthrough article constructed isospectral planar regions for both Dirichlet and Neumann boundary conditions.

A similar problem for self-adjoint compact operators has been studied by many mathematicians. In particular, the spectral properties of integral operators induced by Cauchy, logarithmic and Newtonian kernels have been extensively studied by Arazy and Khavinson in [5], Anderson, Khavinson and Lomonosov in [4], and Dostanić in [19], [20] and [21].

This thesis deals with spectral behavior of single layer potentials and relations between the geometry of boundary curves and eigenvalues of single layer potentials defined on these curves. In the last chapter, we slightly diverge from this topic and discuss some results pertaining to the behavior of Newtonian potentials.

This thesis is divided into six chapters.

Chapter 2 discusses the injectivity of single layer potentials. We show that injective single layer potentials are "rare" to come by. Also we present a criterion for injectivity of single layer potentials defined on ellipses.

Chapter 3 deals with Schatten class membership of single layer potentials. The major result of this chapter states that singular values of single layer potentials over smooth curves asymptotically have the same behavior, namely behave like $O(\frac{1}{n})$.

Chapter 4 is dedicated to a free boundary problem for single layer potentials. In this chapter we show that single layers potentials with monomial, analytic, eigenfunctions occurs only for circles.

Chapter 5 deals with a conjecture regarding logarithmic potentials on polygons inspired by Ruzhanksy's work in [51], and is based on the paper [62]. Essentially we establish an isoperimetric inequality for Schatten $p-$norm of logarithmic potentials over rectangles. This an improvement of the result in [51] .

Chapter 6 summarizes our results and also outlines directions for further study.

1.1 Operator Theory Preliminaries

In this chapter we present selected results from operator theory that will be used in subsequent chapters. For a given Banach space X, by $\mathbf{B}(X)$ we denote the space of bounded operators from X into itself. An operator $T \in \mathbf{B}(X)$ is said to be *compact* if the image of the closed unit ball of X under T has compact closure.

Theorem 1.1.1 *The set of compact operators on X is a closed two-sided ideal in the algebra of $\boldsymbol{B}(X)$ with the norm topology.*

See [61, p.13] for a proof.

Theorem 1.1.2 *Suppose T is a compact operator on a Hilbert space $\mathcal{H}$. Then T is the norm limit of finite rank operators.*

Proof. Let B be the closed unit ball in $\mathcal{H}$. Since $T(B)$ is relatively compact, it is totally bounded. For a given $\epsilon > 0$, we can cover $T(B)$ by open balls of radius ϵ centered at points $y_1, \cdots, y_n$. Let P be the orthogonal projection to the finite-dimensional subspace F spanned by the y_k and define $T_\epsilon = PT$. Note that for any $y \in \mathcal{H}$ and for any y_k,

$$|P(y) - y_k| \leq |y - y_k|,$$

since $y = P(y) + y'$ with y' orthogonal to all y_k. For $x \in \mathcal{H}$ with $|x| \leq 1$, by construction there is y_k such that $|Tx - y_k| < \epsilon$. Then

$$|Tx - T_\epsilon x| \leq |Tx - y_k| + |T_\epsilon x - y_k| < 2\epsilon.$$

Thus, $T_\epsilon - T \to 0$ in operator norm as $\epsilon \to 0$.

■

Whether there exist Banach spaces with compact operators which are not norm-limits of finite-rank operators was an unsolved question for more than forty years; in the end Per Enflo (see [25]) gave a counter-example.

Definition 1.1.3 The adjoint T^* of a bounded operator $T \in \mathbf{B}(\mathcal{H})$ is the bounded operator $T^* \in \mathbf{B}(\mathcal{H})$ such that for all $x, y \in \mathcal{H}$

$$\langle Tx, y \rangle = \langle x, T^* y \rangle.$$

Existence and uniqueness of T^* follows from the Riesz representation theorem. An operator T is called self-adjoint (Hermitian) if $T = T^*$.

The following is well known.

Proposition 1.1.4 *The operator norm of a bounded self-adjoint operator T on a Hilbert space $\mathcal{H}$ can be written as*

$$\|T\|_{op} = \sup_{\|f\|=1} |\langle Tf, f \rangle|.$$

See [15, 2.13] for a proof.

The following result is due to Rayleigh and Ritz (see [26, 0.43]):

Theorem 1.1.5 *Suppose T is a compact self-adjoint operator on a Hilbert space $\mathcal{H}$. Then either $\|T\|_{op}$ or $-\|T\|_{op}$ is an eigenvalue for T.*

Proof. By the previous proposition, there exists a sequence $\{x_n\}$ with $\|x_n\| = 1$, such that $\langle Tx_n, x_n\rangle \to M$, where $M = \pm\|T\|_{op}$. Now note that

$$\begin{aligned}\|Tx_n - Mx_n\|^2 &= \|Tx_n\|^2 + M^2\|x_n\|^2 - 2M\langle Tx_n, x_n\rangle \\ &\leq M^2 + M^2 - 2M\langle Tx_n, x_n\rangle \to 0.\end{aligned}$$

By the compactness of T there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$, so that $Tx_{n_i} \to x$. Since $Tx_{n_i} - Mx_{n_i} \to 0$, then $Mx_{n_i} \to x \neq 0$, and $Tx = Mx$.

■

The following is due to D. Hilbert and E. Schimdt.

Theorem 1.1.6 *If T is a compact self-adjoint operator on a Hilbert space $\mathcal{H}$, then $\mathcal{H}$ has an orthonormal basis consisting of eigenvectors of T.*

Refer to [28, 5.1] for a proof.

Definition 1.1.7 Suppose (X, μ) is a measure space and μ is $\sigma-$finite. Let $K \in L^2(\mu \times \mu)$, and define the operator T_K on $L^2(X, \mu)$ by

$$T_K f(x) = \int K(x, y)f(y)d\mu(y).$$

T_K is said to be an integral operator on $L^2(X, \mu)$.

Proposition 1.1.8 *The operator T_K is compact on $L^2(X, \mu)$ and $\|T_K\|_{op} \leq \|K\|_2$.*

See [15, 4.7] for a proof.

Recall that an operator T on a Hilbert space $\mathcal{H}$ is called Hilbert-Schmidt if

$$\sum_{i \in I} \|Te_i\|^2 < \infty,$$

where $\{e_i : i \in I\}$ an orthonormal basis of $\mathcal{H}$. The set of all Hilbert-Schmidt operators on $\mathcal{H}$ is denoted by $\mathfrak{S}_2(\mathcal{H})$.
Every Hilbert-Schmidt operator on $L^2(X,\mu)$ is an integral operator.

Theorem 1.1.9 *Suppose that (X,μ) is a measure space, and let $K \in L^2(X \times X, \mu \times \mu)$. Then the corresponding integral operator T_K belongs to $\mathfrak{S}_2$, and*

$$\|T_K\|_2 = \|K\|_2 = \left(\int_X \int_X |K(x,y)|^2 d\mu(x) d\mu(y) \right)^{1/2}.$$

Conversely, if $T \in \mathfrak{S}_2\big(L^2(X,\mu)\big)$, then there exists a unique Hilbert-Schmidt kernel $K \in L^2(X \times X, \mu \times \mu)$ such that $T = T_K$.

See [61, §3] for a proof.

Theorem 1.1.10 (Generalized Young's Inequality) *Let (X,μ) be a $\sigma-$finite measure space, $1 \le p \le \infty$, and $C > 0$. Suppose K is a measurable function in $X \times X$ such that*

$$\sup_{x \in X} \int_X \big|K(x,y)\big| d\mu(y) \le C, \qquad \sup_{y \in X} \int_X \big|K(x,y)\big| d\mu(x) \le C.$$

Then $T_K f$ is well-defined almost everywhere for each $f \in L^p(X,\mu)$, and $\|T_K f\|_p \le C\|f\|_p$.

See [26, §0.10] for a proof.

1.2 Schatten $p-$Classes

Let $\mathcal{H}$ be a separable Hilbert space. Recall that the singular numbers of a compact operator $T \in \mathbf{B}(\mathcal{H})$ are the square roots of the eigenvalues of T^*T arranged in decreasing order, repeated according to multiplicity, and we denote them by $s_1(T) \ge s_2(T) \ge \cdots$. For $p > 1$, the operator T is said to belong to the Schatten $p-$class $\mathfrak{S}_p(\mathcal{H})$, if $s_n(T) \in \ell_p$ and is said to belong to the weak Schatten $p-$class $\mathfrak{S}_{p,\infty}(\mathcal{H})$ if

$s_n(T) = O(\frac{1}{n^{1/p}})$. For $1 \leq p < \infty$, the Schatten p -norm is defined by

$$\|T\|_p = \Big(\sum_{n=1}^{\infty} (s_n(T))^p \Big)^{1/p}.$$

Each Schatten $p-$class is a two-sided ideal in $\mathbf{B}(\mathcal{H})$. The class $\mathfrak{S}_p(\mathcal{H})$ is a Banach space for $p \in [1, \infty)$, and a Hilbert space for $p = 2$. For $p \in (0, 1)$ the quantity $\|.\|_p$ defines a quasi-norm. Clearly, $\mathfrak{S}_1$ and $\mathfrak{S}_2$ consist of trace-class and Hilbert-Schmidt operators, respectively. Schatten classes can be regarded as non-commutative analogous to the Lebesgue spaces (We refer the interested reader to [13] for a biography of Schatten).

Let us review the properties of the singular numbers and Schatten ideals.

Theorem 1.2.1 *Suppose A, B and C are compact operators on $\boldsymbol{B}(\mathcal{H})$. Then*
i) $s_n(A) = s_n(A^*)$;
ii) $s_n(cA) = |c| s_n(A)$ *for* $c \in \mathbb{C}$;
iii) $s_n(ABC) \leq \|A\| s_n(B) \|C\|$;
iv) $s_{m+n-1}(A + B) \leq s_m(A) + s_n(B)$;
v) $s_{m+n-1}(AB) \leq s_m(A) s_n(B)$;
vi) $\|A\|_{op} = s_1(A) \geq s_2(A) \geq \cdots \geq 0$;
vii) If rank $A < n$, *then* $s_n(A) = 0$.

We refer the reader to [46, §2] for a proof.

Theorem 1.2.2 *Suppose $T \in \boldsymbol{B}(\mathcal{H})$ is compact. Then*

$$s_n(T) = \inf\{\|T - T_n\|_{op} : T_n \in \boldsymbol{B}(\mathcal{H}) \text{ with } rank(T_n) < n\}.$$

See [46, §2] for a proof.

Theorem 1.2.3 *Suppose $U, V \in \boldsymbol{B}(\mathcal{H})$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then*
i)If $U \in \mathfrak{S}_p(\mathcal{H})$ *and* $V \in \mathfrak{S}_q(\mathcal{H})$, *then* $UV \in \mathfrak{S}_1(\mathcal{H})$;

ii)For $1 \le p_1 \le p_2 \le \infty$

$$\|U\|_1 \ge \|U\|_{p_1} \ge \|U\|_{p_2} \ge \|U\|_{op};$$

iii)$\|V\|_p = \{|\langle V, T\rangle| : \|T\|_q = 1\}$*, where* $\langle V, T\rangle = tr(V^*T)$.

See [10, I.8.7.3] for a proof.

1.3 Single Layer Potentials

Let Ω be a smoothly bounded domain in $\mathbb{R}^n$ with $n \ge 2$ and ds be $(n-1)-$dimensional Hausdorff measure on $\partial\Omega$. Suppose

$$k_n(x) = \begin{cases} \frac{1}{2\pi}\log|x| & n = 2 \\ \frac{1}{(n-2)\sigma_n}\frac{1}{\|x\|^{n-2}} & n \ge 3, \end{cases}$$

where $\sigma_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ denotes the surface area of $\mathbb{S}^{n-1}$.

The integral operator $\mathcal{S}_{\partial\Omega}$ on $L^2(\partial\Omega, ds)$, defined by

$$\mathcal{S}_{\partial\Omega}f(x) = \int_{\partial\Omega} f(y)k_n(x-y)ds_y,$$

is called the *single layer potential* on $\partial\Omega$.

The single layer potentials can be extended continuously to the whole complex plane (or $\mathbb{R}^n$, if $n \ge 3$), see [26, 3.25].

Theorem 1.3.1 *If* $f \in C(\partial\Omega, ds)$*, then* $\mathcal{S}_{\partial\Omega}f$ *extends to a continuous in* $\mathbb{R}^n$.

Proof. We need only show continuity on $\partial\Omega$. Given $x_0 \in \partial\Omega$ and $\delta > 0$, let $B_\delta = \{y \in \partial\Omega : |x_0 - y| < \delta\}$. Then

$$\begin{aligned} |\mathcal{S}_{\partial\Omega}f(x_0) - \mathcal{S}_{\partial\Omega}f(x)| \le & \int_{B_\delta} |f(y)k_n(x-y)|ds_y + \int_{B_\delta} |f(y)k_n(x_0-y)|ds_y \\ & + \int_{\partial\Omega\setminus B_\delta} |f(y)||k_n(x-y) - k_n(x_0-y)|ds_y. \end{aligned}$$

Since f is bounded and $k_n(x-y) = \mathrm{O}(\frac{1}{|x-y|^{n-2}})$ for $n \geq 3$ (or $\mathrm{O}(\log|x-y|)$ for $n=2$), an integration in polar coordinates shows that the first two terms on the right are $\mathrm{O}(\delta)$ (or $\mathrm{O}(\delta \log \delta)$). Given $\epsilon > 0$, then, we can make these terms less than $\frac{\epsilon}{3}$ by choosing δ small enough. If we now require that $|x - x_0| < \frac{\delta}{2}$, the integrand in the third term is bounded on $\partial\Omega \setminus B_\delta$ and tends uniformly to zero as $x \to x_0$, so by choosing $|x - x_0|$ small enough we can make the third term less than $\frac{\epsilon}{3}$.

■

The following is well known (see [26, 3.31]).

Proposition 1.3.2 *Suppose $n = 2$. If $f \in L^2(\partial\Omega, ds)$, the single layer potential with moment f is harmonic at infinity if and only in $\int_{\partial\Omega} f = 0$, in which case the potential vanishes at infinity.*

Proof. We have

$$\mathcal{S}_{\partial\Omega} f(z) = \int_{\partial\Omega} \Big(\log|z-w| - \log|z| \Big) f(w) ds_w + \log|z| \int_{\partial\Omega} f(w) ds_w.$$

Since $\log|z-w| - \log|z| \to 0$ uniformly for $w \in \partial\Omega$ as $|z| \to \infty$, the first term vanishes and $\lim_{|z|\to\infty} \dfrac{\mathcal{S}_{\partial\Omega} f(z)}{\log|z|} = \int_{\partial\Omega} f$. The result follows from the fact that a function on the complement of a bounded set in $\mathbb{R}^2$ is harmonic at infinity if and only if it is of order $\mathrm{o}(\log|z|)$ as $|z| \to \infty$ (see [26, 2.74]).

■

Definition 1.3.3 A function f from $S \subset \mathbb{R}^n \to \mathbb{R}^m$ is called Hölder continuous on S of order $\alpha > 0$, if there is a constant C such that $\|f(x) - f(y)\| \leq C\|x-y\|^\alpha$, for all $x, y \in S$. The class of Hölder continuous functions of order α on S is often denoted by $C^{0,\alpha}(S)$.

We conclude this chapter with a theorem of Sokhotski and Plemelj[1], and its analogue for single layer potentials.

[1]The theorem is named after Julian Karol Sokhotski, who proved it in 1868, and Josip Plemelj, who rediscovered it as a main ingredient of his solution of the Riemann-Hilbert problem in 1908.

Theorem 1.3.4 (Plemelj-Sokhotski) *Suppose C is a smooth curve and $f \in C^{0,\alpha}$ with $0 < \alpha \leq 1$. Let $F(z) = \frac{1}{2\pi i}\, P.V. \int_C \frac{f(\zeta)}{\zeta - z} d\zeta$. Then*

$$F^+(z) = -\frac{1}{2} f(z) + \frac{1}{2\pi i}\, P.V. \int_C \frac{f(\zeta)}{\zeta - z} d\zeta,$$

and

$$F^-(z) = \frac{1}{2} f(z) + \frac{1}{2\pi i}\, P.V. \int_C \frac{f(\zeta)}{\zeta - z} d\zeta,$$

where $F^+(z)$ and $F^-(z)$ are the limits approaching $z \in C$ from the exterior and interior of C respectively.

See [12, 2.5.1] for a proof.

Suppose $g \in C^1(\Omega) \cap C(\overline{\Omega})$. Let $\partial_{n^+} g$ and $\partial_{n^-} g$ denote the exterior and interior normal derivatives of g, namely $\partial_n^- g(x) = \lim_{t<0, t\to 0} \mathbf{n}(x) \cdot \nabla g(x + t\mathbf{n}(x))$ and $\partial_n^- g(x) = \lim_{t>0, t\to 0} \mathbf{n}(x) \cdot \nabla g(x + t\mathbf{n}(x))$, where $\mathbf{n}(x)$ is the unit normal to the curve at point x. The single layer potentials satisfy the following "jump" property:

Theorem 1.3.5 *Suppose $f \in C(\partial\Omega)$, then*

$$\partial_n^- \mathcal{S}_{\partial\Omega} f - \partial_n^+ \mathcal{S}_{\partial\Omega} f = f.$$

See [26, 3.29] for a proof.

2 Some Operator Theoretic Aspects of Single Layer Potentials

In the present chapter we investigate certain spectral properties of single layer potentials. In the first section we provide upper bounds for the operator and Hilbert-Schmidt norms of single layer potentials. The second section is devoted to Schatten class membership of single layers. In the next two sections, we present some results on injectivity of single layer potentials. In section five, the nodal sets are discussed. We conclude this section by some results concerning real analyticity of eigenfunctions of single layer potentials.

2.1 Upper Bounds for Chord-Arc Curves

Definition 2.1.1 A Jordan curve Γ is called *chord-arc* or *Lavrentiev*[1] if and only if Γ is rectifiable and there exists a constant $C > 0$ such that for all $z, w \in \Gamma$

$$C\sigma(z,w) \leq |z-w|,$$

where $\sigma(z,w)$ denotes the length of the shorter arc on Γ joining z to w. Examples of chord arc curves are smooth curves, Lipschitz curves and curves with corners

Presumably, the following is well known. For the convenience of the reader, we give a proof.

Proposition 2.1.2 *Suppose* Γ *is a chord-arc curve. Then* $\mathcal{S}_\Gamma \in \mathfrak{S}_p$ *for* $p \geq 2$.

[1]This notion was first introduced by Mikhail A. Lavrentiev in 1936. See, for instance [27, p. 246].

Proof. Let Γ be a chord-arc curve with chord-arc constant C. For simplicity, let us assume $\operatorname{diam}(\Gamma) \le 1$ and put $|\Gamma| = \ell$.

$$\begin{aligned}
\|\mathcal{S}_\Gamma\|_2^2 &= \int_\Gamma\int_\Gamma \log^2|z-w|ds_z ds_w \le \int_\Gamma\int_\Gamma \log^2 C\sigma(z,w)ds_z ds_w \\
&= \int_\Gamma\int_\Gamma \log^2 C ds_z ds_w + 2\log C\int_\Gamma\int_\Gamma \log\sigma(z,w)ds_z ds_w \\
&+ \int_\Gamma\int_\Gamma \log^2\sigma(z,w)ds_z ds_w \\
&= \Big(\frac{\ell\log C}{2\pi}\Big)^2 + \frac{\log C}{2\pi^2}\int_0^\ell\int_0^\ell \log\Big(\min\{|x-y|, \ell-|x-y|\}\Big)dxdy \\
&+ \int_0^\ell\int_0^\ell \log^2\Big(\min\{|x-y|, \ell-|x-y|\}\Big)dxdy \\
&= \Big(\frac{\ell\log C}{2\pi}\Big)^2 + \frac{\ell^2\log C}{\pi^2}\Big(\log\frac{\ell}{2}-1\Big) + \frac{\ell^2}{2\pi^2}\Big(\log^2\frac{\ell}{2} - 2\log\frac{\ell}{2}+2\Big) \\
&= \frac{\ell^2}{4\pi^2}\left(2\log^2\frac{\ell}{2} + 4\log\frac{\ell}{2}(\log C - 1) + (\log^2 C - 4\log C + 4)\right).
\end{aligned}$$

Thus,

$$\|\mathcal{S}_\Gamma\|_2 \le \frac{\ell}{2\pi}\left(2\log^2\frac{\ell}{2} + 4\log\frac{\ell}{2}(\log C - 1) + (\log^2 C - 4\log C + 4)\right)^{1/2}.$$

■

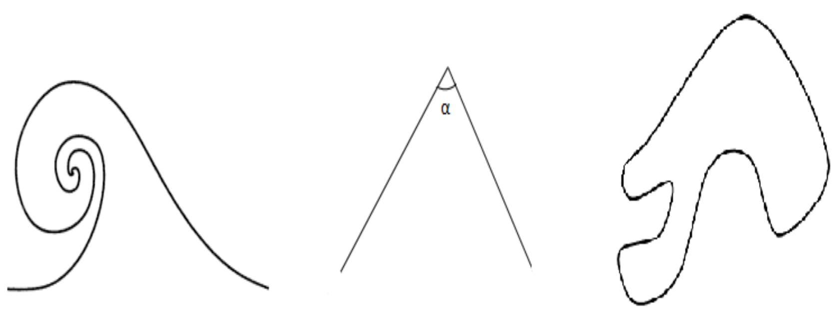

Figure 2.1: Examples of chord-arc curves.

2.2 Curves with Algebraic Cusps

In this section we shall study the spectral behavior of single layer potentials over curves with certain specific cusps. Recall that a *cusp* is a singular point on a planar curve where two branches have a common tangent. A cusp point of an algebraic curve is called an *algebraic cusp*. A plane algebraic curve $F(x, y)$ with an algebraic cusp at the origin, after a suitable linear transformation of the coordinates, can be represented by $F(x, y) = y^2 + x^3 + ax^2y + bxy^2 + cy^3$+higher order terms (see [57]).

Plane curves might have non-algebraic cusps. For instance $C = \{(x, e^{-1/x}) : x \in [0, 1]\} \cup \{(x, -e^{-1/x}) : x \in [0, 1]\}$ has a non-algebraic cusp at the origin.

Proposition 2.2.1 *Suppose* Γ *is a Jordan curve with an algebraic cusp. Then* $\mathcal{S}_\Gamma \in \mathfrak{S}_p$*, for* $p \geq 2$*.*

Proof.

Without loss of generality, we may assume that the cusp is at the origin and $\Gamma_+ \cup \Gamma_- \subset \Gamma$ for positive integers $m < n$, where $\Gamma_+ := \{t + it^{\frac{m}{n}} : 0 \leq t \leq \frac{1}{2}\}$ and $\Gamma_- := \{-s + is^{\frac{m}{n}} : 0 \leq s \leq \frac{1}{2}\}$. For $z_1 \in \Gamma_+$ and $z_2 \in \Gamma_-$,

$$0 \geq \log|z_1 - z_2| = \frac{1}{2}\log\left((t+s)^2 + (t^{\frac{m}{n}} - s^{\frac{m}{n}})^2\right) \geq \log(t+s).$$

Assume z and w lie on the sub-arc $\gamma_\epsilon := \{t+it^{\frac{m}{n}} : 0 \leq t \leq \epsilon\} \cup \{-s+is^{\frac{m}{n}} : 0 \leq s \leq \epsilon\}$

with $\epsilon < 1/2$. Choose $\alpha \in (1-\frac{m}{n}, 1)$ so that $-\log x \leq \frac{1}{x^{\alpha/2}}$ for all $x \in (0, 1/2)$. Then

$$\begin{aligned}
&\int_{\gamma_\epsilon}\int_{\gamma_\epsilon} \log^2|z-w| ds_z ds_w \\
&\leq \int_0^\epsilon\int_0^\epsilon \log^2(t+s)\sqrt{1+(\frac{m}{n}t^{\frac{m}{n}-1})^2}\sqrt{1+(\frac{m}{n}s^{\frac{m}{n}-1})^2}dtds \\
&\leq 2(\frac{m}{n})^2\int_0^\epsilon\int_0^\epsilon \frac{\log^2(t+s)}{(st)^{\frac{m}{n}-1}}dtds \\
&\leq 2(\frac{m}{n})^2\int_0^\epsilon\int_0^\epsilon \frac{\log^2(t+s)}{(st)^{\frac{m}{n}-1}}dtds \\
&\leq 2(\frac{m}{n})^2\int_0^\epsilon\int_0^\epsilon \frac{1}{(t+s)^\alpha(st)^{\frac{m}{n}-1}}dtds \\
&\leq 2(\frac{m}{n})^2\int_0^\epsilon\int_0^\epsilon \frac{1}{(st)^{\alpha+\frac{m}{n}-1}}dtds = 2(\frac{m}{n})^2\frac{1}{(1-\theta)^2}\epsilon^{2(1-\theta)} \quad (\text{since } t+s \geq ts),
\end{aligned}$$

where $\theta = \alpha + \frac{m}{n} - 1$.

■

In a similar fashion, one can show that if the underlying curve contains finitely many algebraic cusps, then the corresponding single layer potential has finite Hilbert-Schmidt norm.

Corollary 2.2.2 *If Γ is a Jordan curve with finitely many algebraic cusps, then $\mathcal{S}_\Gamma \in \mathfrak{S}_p$ for $p \geq 2$.*

Let us briefly discuss the compactness of single layer potentials in higher dimensions. The following is well known (see [26, §3]), but for the sake of completeness we give a proof.

Proposition 2.2.3 *Assume $\Omega \subset \mathbb{R}^n$, $n \geq 3$, is a smoothly bounded domain. Then $\mathcal{S}_{\partial\Omega}$ is a compact operator from $L^2(\partial\Omega, ds)$ into itself.*

Proof. For $\epsilon > 0$, let T_ϵ be the operator induced by the kernel

$$K_\epsilon(x,y) = \begin{cases} \frac{1}{|x-y|^{n-2}} & \text{if } |x-y| > \epsilon \\ 0 & \text{if } |x-y| \leq \epsilon. \end{cases}$$

The kernel K_ϵ is bounded on $\partial\Omega \times \partial\Omega$, so T_ϵ is compact. Let $B_\epsilon(x) = \{y : |x-y| < \epsilon\}$, and take $f \in L^2(\partial\Omega)$ with $\|f\|_2 = 1$,

$$\begin{aligned} \|\mathcal{S}_{\partial\Omega} f - T_\epsilon f\|_2^2 &= \int_{\partial\Omega} \left(\int_{\partial\Omega} \frac{f(y) ds_y}{\|x-y\|^{n-2}} - \int_{\partial\Omega} f(y) K_\epsilon(x,y) ds_y \right)^2 ds_x \\ &= \int_{\partial\Omega} \left(\int_{\partial\Omega \cap B_\epsilon(x)} \frac{f(y) ds_y}{\|x-y\|^{n-2}} \right)^2 ds_x. \end{aligned}$$

An integration in polar coordinates of the latter integral shows that $\|\mathcal{S}_{\partial\Omega} f - T_\epsilon f\|_2$ is of order $\mathrm{O}(\epsilon)$ as $\epsilon \to 0$. Therefore $\|T_\epsilon - \mathcal{S}_{\partial\Omega}\|_{op} \to 0$, when $\epsilon \to 0$. This completes the proof.

■

Assume that $\Omega \subset \mathbb{R}^n$, with $n \geq 3$, is a smoothly bounded domain and let δ be the diameter of $\partial\Omega$, then

$$\begin{aligned} \int_{\partial\Omega} \frac{ds_y}{\|x-y\|^{n-2}} &= \int\limits_{\partial\Omega \setminus B_\delta(x)} \frac{ds_y}{\|x-y\|^{n-2}} + \int\limits_{\partial\Omega \cap B_\delta(x)} \frac{ds_y}{\|x-y\|^{n-2}} \\ &\leq \frac{\text{surface area}(\partial\Omega)}{\delta^{n-2}} + A_0 \int_0^\delta \frac{r^{n-2} dr}{r^{n-2}} \quad \text{(using polar coordinates)} \\ &= B_0 \frac{1}{\delta^{n-2}} + A_0 \delta, \end{aligned}$$

where A_0 and B_0 depend on $\partial\Omega$. Similarly, we observe that $\int_{\partial\Omega} \int_{\partial\Omega} \frac{ds_x ds_y}{\|x-y\|^{2(n-2)}}$ converges if and only if $\int_0^\delta \frac{r^{n-2} dr}{r^{2(n-2)}}$ converges. But the latter integral converges if and only if $n < 3$. We therefore have the following proposition

Proposition 2.2.4 *Assume $\Omega \subset \mathbb{R}^n$, with $n \geq 3$, is a smoothly bounded domain. Then $\mathcal{S}_{\partial\Omega}$ is not Hilbert-Schmidt.*

2.3 Injectivity of Single Layer Potentials

A natural question regarding injectivity of single layer potentials is that how dilating a given smooth simple curve translates to injectivity of the corresponding single layer potential. In this section we discuss this matter. We show that among single layer potentials over dilations of a given curve, there is only one that is non-injective.

We first deal with a particular case, where the boundary curve is an ellipse. Before discussing the injectivity of single layer potentials on ellipses, let us recall the notion of the elliptic coordinate system.

Elliptic Coordinate System[2]: Let a be a positive constant. A point (μ, ϕ) with $\mu \geq 0$ and $0 \leq \phi \leq 2\pi$ in the elliptic coordinate system corresponds to $x = \frac{a}{2}\cosh\mu\cos\phi$ and $y = \frac{a}{2}\sinh\mu\sin\phi$ or equivalently $x + iy = \frac{a}{2}\cosh(\mu + i\phi)$ in the Cartesian coordinate system. Obviously $\mu =$constant represents an ellipse.

The arc-length element is given by

$$ds = \frac{a}{2}\tau d\phi,$$

where $\tau = \sqrt{\cosh^2\mu\sin^2\phi + \sinh^2\mu\cos\phi^2}$. Consider the points $z = (\mu, \phi)$ and $z_0 = (\mu_0, \phi_0)$. For the logarithmic distance between two points z and z_0, namely $\log|z - z_0|$,

[2]For more details, we refer the interested reader to [44, p. 1195].

we find that

$$\log|z-z_0| = \mu_0 + \log\frac{a}{4} \\ -2\sum_{n=1}^{\infty}\frac{e^{-n\mu_0}}{n}\Big(\cosh n\mu\cos n\phi_0\cos n\phi + \sinh n\mu\sin n\phi_0\sin n\phi\Big);\ \mu_0 > \mu,$$

$$(\dagger)\ \log|z-z_0| = \mu + \log\frac{a}{4} \\ -2\sum_{n=1}^{\infty}\frac{e^{-n\mu}}{n}\Big(\cosh n\mu_0\cos n\phi_0\cos n\phi + \sinh n\mu_0\sin n\phi_0\sin n\phi\Big);\ \mu_0 < \mu.$$

For the sake of completeness we verify the second equality, where $\mu_0 < \mu$, and the other one can be verified in the same manner. Let $w = \mu + i\phi$, $w_0 = \mu_0 + i\phi_0$, $z = \frac{a}{2}\cosh w$ and $z_0 = \frac{a}{2}\cosh w_0$,

$$\begin{aligned}
\log(z-z_0) &= \log\Big(\frac{a}{2}\frac{e^{w}+e^{-w}}{2}-\frac{a}{2}\frac{e^{w_0}+e^{-w_0}}{2}\Big) = \log\Big[\frac{a}{4}\big(e^{\frac{w+w_0}{2}}-e^{-\frac{w+w_0}{2}}\big)\big(e^{\frac{w-w_0}{2}}-e^{-\frac{w-w_0}{2}}\big)\Big]\\
&= \log\frac{a}{4}+\log\Big[e^{\frac{w+w_0}{2}}(1-e^{-(w+w_0)})e^{\frac{w-w_0}{2}}(1-e^{-w+w_0})\Big]\\
&= \log\frac{a}{4}+\log e^{\frac{w+w_0}{2}}+\log e^{\frac{w-w_0}{2}}+\log(1-e^{-(w+w_0)})+\ln(1-e^{-w+w_0})\\
&= \log\frac{a}{4}+w+\ln(1-e^{-(w+w_0)})+\ln(1-e^{-w+w_0})\\
&= \log\frac{a}{4}+w-\Big(\sum_{n=1}^{\infty}\frac{1}{n}e^{-n(w+w_0)}+\sum_{n=1}^{\infty}\frac{1}{n}e^{n(-w+w_0)}\Big)\\
&= \log\frac{a}{4}+w-\sum_{n=1}^{\infty}\frac{1}{n}e^{-nw}(e^{nw_0}+e^{-nw_0})\\
&= \log\frac{a}{4}+w-\sum_{n=1}^{\infty}\frac{2e^{-n(\mu+i\phi)}}{n}\cosh n(\mu_0+i\phi_0)\\
&= \log\frac{a}{4}+w-\sum_{n=1}^{\infty}\frac{2}{n}e^{-n\mu}(\cos n\phi-i\sin n\phi)(\cosh n\mu_0\cos n\phi_0+i\sinh n\mu_0\sin n\phi_0)\\
&= \log\frac{a}{4}+w-\sum_{n=1}^{\infty}\frac{2}{n}e^{-n\mu}(\cos n\phi\cosh n\mu_0\cos n\phi_0+\sin n\phi\sinh n\mu_0\sin n\phi_0)\\
&-i\sum_{n=1}^{\infty}\frac{2}{n}e^{-n\mu}(\cos n\phi\sinh n\mu_0\sin n\phi_0-\sin n\phi\cosh n\mu_0\cos n\phi_0).
\end{aligned}$$

Similarly, $\log(\bar{z}-\bar{z_0})$ can be represented as a series expansion. Consequently,

$$\begin{aligned}
\log|z-z_0| &= \frac{1}{2}\log(z-z_0)+\frac{1}{2}\log(\bar{z}-\bar{z_0}) = \frac{w+\bar{w}}{2}+\log a4\\
&-2\sum_{n=1}^{\infty}\frac{e^{-n\mu}}{n}\Big(\cosh n\mu_0\cos n\phi_0\cos n\phi+\sinh n\mu_0\sin n\phi_0\sin n\phi\Big)\\
&=\mu+\log\frac{a}{4}\\
&\quad-2\sum_{n=1}^{\infty}\frac{e^{-n\mu}}{n}\Big(\cosh n\mu_0\cos n\phi_0\cos n\phi+\sinh n\mu_0\sin n\phi_0\sin n\phi\Big).
\end{aligned}$$

This verifies (†).

Suppose a and μ are two positive numbers, and let E be the ellipse $(\frac{2x}{a\cosh\mu})^2+$

$(\frac{2y}{a\sinh\mu})^2 = 1$. Let $z_0 \in \text{int}(E)$ with $z_0 = (\mu_0, \phi_0)$. The single layer potential on E with moment f can be written as

$$\int_E f(w)\log|z_0 - w|\,ds_w = \frac{a}{4\pi}\int_0^{2\pi} f(\mu,\phi)\Big[\mu$$
$$+ \log\frac{a}{4} - \sum_{n=1}^{\infty}\frac{2e^{-n\mu_0}}{n}(\cosh n\mu_0\cos n\phi\cos n\phi_0 + \sinh n\mu_0\sin n\phi\sin n\phi_0)\Big]\tau(\mu,\phi)d\phi.$$

Our goal is to provide a simple geometric condition for the injectivity of the single layer potential defined on E. We shall show that $\mathcal{S}_E$ is not injective if $\mu = -\log\frac{a}{4}$, and in this case $\ker(\mathcal{S}_E) = \text{span}\{\frac{1}{\tau}\}$.
Take $z_0 \in \text{int}(E)$ with $z_0 = (\mu_0, \phi_0)$. Then

$$\int_0^{2\pi}\frac{-1}{\tau(\mu,\phi)}\sum_{n=1}^{\infty}\frac{e^{-n\mu_0}}{n}\Big(\cosh n\mu_0\cos n\phi\cos n\phi_0 + \sinh n\mu_0\sin n\phi\sin n\phi_0\Big)\tau(\mu,\phi)\,d\phi$$
$$= -\sum_{n=1}^{\infty}\frac{e^{-n\mu_0}}{n}\int_0^{2\pi}\Big(\cosh n\mu_0\cos n\phi\cos n\phi_0 + \sinh n\mu_0\sin n\phi\sin n\phi_0\Big)\,d\phi = 0.$$

For $z = (\mu,\phi) \in E$ by approaching from the interior, i.e., $z_0 \to z$ with $z_0 \in \text{int}(E)$, and using continuity of the single layer potential, it follows that

$$\mathcal{S}_E(\frac{1}{\tau}) = 0.$$

Thus, $\mathcal{S}_E$ is not injective. Moreover, since the kernel of the single layer potential is of dimension one, it is spanned by $\frac{1}{\tau}$. Let us summarize this result in the following proposition:

Proposition 2.3.1 *Let E be the ellipse $(\frac{2x}{a\cosh\mu})^2 + (\frac{2y}{a\sinh\mu})^2 = 1$. Then $\mathcal{S}_E$ is injective if and only if $\mu \neq -\log\frac{a}{4}$.*

Remark 2.3.2 *In the general case, $(\frac{x}{a_0})^2 + (\frac{y}{b_0})^2 = 1$ with $a_0 > b_0$, taking $a = \sqrt{a_0^2 - b_0^2}$ and $\mu = \coth^{-1}(\frac{a_0}{b_0})$, the previous proposition holds.*

2.4 Injectivity of Single Layer Potentials over Dilated Curves

A natural question regarding injectivity of single layer potentials is how dilating a given smooth simple curve affects the injectivity of the corresponding single layer potentials. In this section we shall discuss this matter and show that among all dilated curves, only one has non-injective single layer potential. We start with the following proposition.

Proposition 2.4.1 *Let D be a simply connected domain in the plane bounded by a C^1 curve. Then the kernel of $\mathcal{S}_{\partial D}$ is of dimension at most one.*

Proof. Set $\Gamma = \partial D$, and let $f \in \ker(\mathcal{S}_\Gamma)$. If $\displaystyle\int_\Gamma f\, ds = 0$, then by Proposition 1.3.2, the function $\mathcal{S}_\Gamma f$ must be harmonic at infinity with zero boundary data and so must be identically zero. Therefore by the Sokhotski-Plemelj theorem

$$f = \frac{\partial \mathcal{S}_\Gamma f}{\partial n_+} - \frac{\partial \mathcal{S}_\Gamma f}{\partial n_-} = 0 \quad \text{on } \Gamma.$$

If $\displaystyle\int_\Gamma f\, ds \neq 0$, take $g \in \ker(\mathcal{S}_\Gamma)$ with $\displaystyle\int_\Gamma g\, ds \neq 0$. Consider the function $h = \dfrac{1}{\int f\, ds} f - \dfrac{1}{\int g\, ds} g$. We have

$$\int_\Gamma h\, ds = \int_\Gamma \Big(\frac{1}{\int f\, ds} f - \frac{1}{\int g\, ds} g\Big)\, ds = 0 \text{ and } \mathcal{S}_\Gamma(h) = \mathcal{S}_\Gamma\Big(\frac{1}{\int f\, ds} f - \frac{1}{\int g\, ds} g\Big) = 0.$$

Then $h \equiv 0$, which implies $g = \left(\dfrac{\displaystyle\int g\, ds}{\displaystyle\int f\, ds}\right) f$.

■

Remark 2.4.2 *In the previous proposition if the single layer potential is replaced with the double layer potential, namely $Kf(z) = \displaystyle\int_\Gamma f(w) \frac{\partial}{\partial n_w} \log|z-w|\, ds_w$, then the similar result does not hold. Ebenfelt et al. in [24] showed that the only possibility for an eigenvalue of the double layer potential over a lemniscate to be of infinite multiplicity is if it is 0. For the case where the boundary curve is a circle centered at* *zero, the corresponding double layer potential has infinite dimensional null space (see [3] and [24]).*

Lemma 2.4.3 *Assume $\mathcal{S}_\Gamma$ is not injective and $\mathcal{S}_\Gamma(f) = \alpha$ for some $\alpha \in \mathbb{C}$ and $f \in L^2(\Gamma, ds)$. Then α must be zero and consequently f belongs to* $\ker(\mathcal{S}_\Gamma)$.

Proof. Since $\mathcal{S}_\Gamma$ is not injective, then by the previous lemma we can find a non-zero function g in $\ker(\mathcal{S}_\Gamma)$ such that $\int_\Gamma g(z)\, ds_z \neq 0$. From $\int_\Gamma f(w) \log|z-w| ds_w = \alpha$ it follows that

$$\begin{aligned}
\alpha g(z) =& g(z) \int_\Gamma f(w) \log|z-w| ds_w \\
&\Rightarrow \alpha \int_\Gamma g(z)\, ds_z = \Big(\int_\Gamma g(z)\, ds_z\Big) \int_\Gamma f(w) \log|z-w| ds_w \\
&\Rightarrow \alpha \int_\Gamma g(z)\, ds_z = \int_\Gamma f(w) \int_\Gamma g(z) \log|w-z|\, ds_z\, ds_w = 0.
\end{aligned}$$

Thus, $\alpha = 0$.

■

It turns out that for a given simple smooth curve Γ, among all the curves in the class $\{\lambda\Gamma |\, \lambda > 0\}$, there exists a parameter λ_0 for which $\mathcal{S}_{\lambda_0\Gamma}$ is not injective.

Proposition 2.4.4 *Let Γ be a smooth Jordan curve in the plane. Then among all operators in the class $\{\mathcal{S}_{t\Gamma} |\, t > 0\}$, there exists only one that is not injective.*

Proof. First let us recall a theorem of Fredholm [45, page 184] on integral equations of the first kind which states that equations of the type $\int_\Gamma f(w) \log|z-w| ds_w =$ constant have solutions. It follows from this theorem that either $\int_\Gamma f(w) \log|z-w| ds_w = 0$ or $\int_\Gamma f(w) \log|z-w| ds_w = 1$ has non-trivial solutions.

If $\int_\Gamma f(w) \log|z-w| ds_w = 0$ has a non-trivial solution, then $\mathcal{S}_\Gamma$ is not injective. Thus, $\mathcal{S}_{\lambda_0\Gamma}$ is not injective for $\lambda_0 = 1$. If $\int_\Gamma f(w) \log|z-w| ds_w = 0$ has no non-trivial solution, then $\int_\Gamma f(w) \log|z-w| ds_w = 1$ must have a solution. It follows that the average of f, i.e. $\beta = \int_\Gamma f(z) ds_z$, is non-zero, because if the single layer potential with the density f is constant and the average of f is zero, then $f \equiv 0$ (see [26, Lemma 3.32]). For a positive number λ, set $g(w) = f(\frac{w}{\lambda})$ where $w \in \lambda\Gamma$. Then

$$\begin{aligned}\int_{\lambda\Gamma} g(w) \log |w-t| ds_w = &\int_\Gamma g(\lambda\zeta) \log |\lambda z - \lambda\zeta| \lambda ds_\zeta \\ =&\lambda \log \lambda \int_\Gamma f(\zeta) ds_\zeta + \lambda \int_\Gamma f(\zeta) \log |z-\zeta| ds_\zeta \\ =&\lambda(\beta \log \lambda + 1).\end{aligned}$$

For $\lambda_0 = e^{-1/\beta}$ the latter quantity becomes zero, which implies that $\mathcal{S}_{\lambda_0\Gamma}$ is not injective.

Now we show that this parameter is unique. Let λ_0 be the parameter for which the single layer potential on $\lambda_0\Gamma$ is not injective. Suppose λ is a positive number not equal to λ_0 and $f \in \ker(\mathcal{S}_{\lambda\Gamma})$. Then

$$\begin{aligned}0 &= \int_{\lambda\Gamma} f(w) \log |w-t| \, ds_w \\ &= \int_{\lambda_0\Gamma} f(\frac{\lambda}{\lambda_0}\zeta) \log \left| \frac{\lambda}{\lambda_0}\zeta - \frac{\lambda}{\lambda_0} z \right| \frac{\lambda}{\lambda_0} ds_\zeta \\ &= \log \frac{\lambda}{\lambda_0} \int_{\lambda\Gamma} f(w) \, ds_w + \frac{\lambda}{\lambda_0} \int_{\lambda_0\Gamma} f(\frac{\lambda}{\lambda_0}\zeta) \log |\zeta - z| \, ds_\zeta.\end{aligned}$$

We deduce that $\int_{\lambda_0\Gamma} f(\frac{\lambda}{\lambda_0}\zeta) \log |\zeta - z| \, ds_\zeta = -\frac{\lambda_0}{\lambda} \log \frac{\lambda}{\lambda_0} \int_{\lambda\Gamma} f(w) \, ds_w$. It follows that $\int_{\lambda\Gamma} f(w) \, ds_w = 0$, and since $f \in \ker(\mathcal{S}_{\lambda\Gamma})$, then $f \equiv 0$.

■

Remark 2.4.5 *The phenomenon in the previous proposition is peculiar to curves in the plane and does not occur in higher dimensions. In dimensions higher than two, the geometry of the boundary is immaterial to the injectivity of its single layer potential. The single layer potentials over the boundary of smoothly bounded domains in $\mathbb{R}^n$, with $n \geq 3$, are always injective (see [56, Theorem 3.3]).*

2.5 Nodal Sets

We refer to the zero sets of eigenfunctions of single layer potentials as *nodal sets*. To be precise if f_λ is an eigenfunction for an eigenvalue λ, then the nodal set of f_λ is defined by

$$N(f_\lambda) = \{x \in \partial\Omega : f_\lambda(x) = 0\}.$$

Proposition 2.5.1 *Suppose f is a real-valued eigenfunction for the single layer potential on a smooth Jordan curve corresponding to the eigenvalue λ with empty nodal set. Then, any real-valued eigenfunction corresponding to an eigenvalue different from λ has a non-empty nodal set.*

Proof. Since $\mathcal{S}_{\partial\Omega}$ is self-adjoint, its eigenfunctions can be chosen to be real-valued. Let λ be an eigenvalue, and f_λ be a corresponding eigenfunction such that f_λ has no zeros on $\partial\Omega$. Without loss of generality we may assume that $f_\lambda > 0$. Now let μ be any other eigenvalue and f_μ be an eigenfunction associated with μ. Since $\int_{\partial\Omega} f_\mu(x)\, f_\lambda(x) ds_x = 0$, then f_μ should change sign. Thus, $N(f_\mu)$ must be non-empty.

■

Remark 2.5.2 *We observe that nodal sets of eigenfunctions for the single layer potential on the unit circle, namely $\{\sin nt\}_{n=1}^{\infty} \cup \{\cos nt\}_{n=1}^{\infty}$, are non-empty, except for the constant functions belonging to the nullspace.*

2.6 Real Analyticity of Eigenfunctions

In this section, we investigate the real analyticity of eigenfunctions of single layer potentials. Since single layer potentials are self-adjoint, their eigenfunctions can be chosen to be real-valued. These eigenfunctions can be viewed as functions defined on a segment when parametrized by the arc-length. We start by looking at the simplest case, where the single layer potential is defined on a segment.

Proposition 2.6.1 *Let $I = [0, 1]$ and $f \in L^2[0, 1]$, then $\|\mathcal{S}_I f\|_\infty \leq \frac{1}{\pi}\|f\|_2$.*

Proof. For $x \in (0,1)$,

$$\int_0^1 \log^2 |x-y| dy = \int_0^x \log^2 t dt + \int_0^{1-x} \log^2 t dt$$
$$= (1-x)x\log^2(1-x) - 2(1-x)\log(1-x) + 2(1-x)$$
$$+ (x\log^2 x - 2x\log x + 2x) \leq 4.$$

So

$$\|\mathcal{S}_I f\|_{op} = \frac{1}{2\pi} \sup_{x\in[0,1]} \Big| \int_0^1 f(y)\log|x-y|dy \Big| \leq \frac{1}{\pi}\|f\|_2.$$

■

Proposition 2.6.2 *The operator $\mathcal{S}_I$ maps $L^2[0,1]$ into $C^{0,\alpha}$, the space of Hölder continuous functions of order $\alpha \in (0, \frac{1}{2}]$.*

Proof. If $f \in L^2[0,1]$, then $\mathcal{S}_I f$ is differentiable and $(\mathcal{S}_I f)' \in L^2[0,1]$, i.e. $\mathcal{S}_I f$ belongs to the Sobolev space $W^{1,2}$ (see [54, Theorem 90]). For $x, y \in [0,1]$, with $x < y$,

$$|\mathcal{S}_I f(x) - \mathcal{S}_I f(y)| = \Big| \int_x^y (\mathcal{S}_I f)'(t)dt \Big| \leq \Big(\int_x^y dt \Big)^{1/2} \Big(\int_x^y |(\mathcal{S}_I f)'|^2 \Big)^{1/2}$$
$$\leq \|(\mathcal{S}_I f)'\|_2 \sqrt{y-x}.$$

Thus, $|\mathcal{S}_I f(x) - \mathcal{S}_I f(y)| \leq \|(\mathcal{S}_I f)'\|_2 |x-y|^\alpha$, for all $x, y \in [0,1]$ and $\alpha \in (0, \frac{1}{2}]$.

■

Proposition 2.6.3 *Eigenfunctions of the operator $\mathcal{S}_I$ can be extended analytically to the whole complex plane.*

Proof. Suppose (f, λ) is an eigenpair, i.e. $\lambda f(x) = \int_0^1 f(y)\log|x-y|dy$, for all $x \in [0,1]$. Then f is differentiable and $f' \in L^2[0,1]$. Moreover $\Big\| \frac{d}{dx} \int_0^1 f(y)\log|x-y|dy \Big\|_2 = \pi\|f\|_2$ (see [54, Thm. 90]).

Since $\lambda f'(x) = \frac{d}{dx}\int_0^1 f(y)\log|x-y|dy$, then

$$|\lambda|\|f'\|_2 = \Big\|\frac{d}{dx}\int_0^1 f(y)\log|x-y|dy\Big\|_2 = \pi\|f\|_2.$$

Differentiating once more, $\lambda f''(x) = \frac{d}{dx}\int_0^1 f'(y)\log|x-y|dy$, we find that

$$|\lambda|\|f''\|_2 = \Big\|\frac{d}{dx}\int_0^1 f'(y)\log|x-y|dy\Big\|_2 = \pi\|f'\|_2 = \frac{\pi^2}{|\lambda|}\|f\|_2.$$

So $\|f''\|_2 \leq (\frac{\pi}{|\lambda|})^2\|f\|_2$. Repeating the same argument successively, it follows that

$$\|f^{(n)}\|_2 \leq (\frac{\pi}{|\lambda|})^n\|f\|_2, \quad n = 1, 2, \cdots. \tag{2.6.1}$$

On the other hand, for $x \in (0,1)$,

$$|\lambda||f'(x)| = \Big|\int_0^1 f'(y)\log|x-y|dy\Big| \leq \|f'\|_2\Big(\int_0^1 \log^2|x-y|dy\Big)^{1/2} \leq 2\|f'\|_2,$$

where the last inequality follows from the fact that

$$\begin{aligned}&\int_0^1 \log^2|x-y|dy = \int_0^x \log^2 t dt + \int_0^{1-x}\log^2 t dt\\ &= (1-x)x\log^2(1-x) - 2(1-x)\log(1-x) + 2(1-x)\\ &+ (x\log^2 x - 2x\log x + 2x) \leq 4.\end{aligned}$$

This shows that $\|f'\|_\infty \leq \frac{2}{|\lambda|}\|f'\|_2$. Repeating, we get

$$\|f^{(n)}\|_\infty \leq \frac{2}{|\lambda|}\|f^{(n)}\|_2, \quad n = 1, 2, \cdots. \tag{2.6.2}$$

Combing (2.6.1) and (2.6.2), it follows that

$$\|f^{(n)}\|_\infty \leq \frac{2}{|\lambda|}(\frac{\pi}{|\lambda|})^n\|f\|_2 \quad n = 1, 2, \cdots.$$

As a result of the latter inequality, f can be extended to an analytic function on a domain containing I, but

$$\overline{\lim}\,\Big(\frac{\|f^{(n)}\|_\infty}{n!}\Big)^{1/n} \leq \frac{\pi}{|\lambda|}\,\overline{\lim}\,\frac{(\frac{2}{|\lambda|})^{1/n}}{n!^{1/n}} = 0.$$

Therefore f can be extended to an entire function.

■

We conclude this section with a similar result for single layer potentials over real analytic curves.

Theorem 2.6.4 *Assume Γ is a real analytic curve. Then eigenfunctions of $\mathcal{S}_\Gamma$ are real analytic.*

Proof. Suppose (f,λ) be an eigenpair, i.e. $\lambda f = \mathcal{S}_\Gamma f$ and f is real-valued. For simplicity let us assume that $|\Gamma| = \ell \leq 1$. Let $\psi : [0,\ell] \to \Gamma$ denote the arc-length parametrization of Γ. For $x, y \in [0,\ell]$,

$$\log|\psi(x) - \psi(y)| = \log|x-y| + \log|\phi(x,y)|,$$

where ϕ is smooth in $(0,\ell)\times(0,\ell)$, $|\phi| > 0$ and $\phi(x,y) = \phi(y,x)$ for $(x,y) \in [0,\ell]\times[0,\ell]$. Define $k(x,y) = \log|\phi(x,y)|$ on $[0,\ell]\times[0,\ell]$. Then, for $t \in (0,\ell)$,

$$\left(\int_0^\ell \Big|\frac{\partial^n k(t,y)}{\partial y^n}\Big|^2 dy\right)^{1/2} \leq \sqrt{\ell}\Big\|\frac{\partial^n k}{\partial x^n}\Big\|_\infty \leq \Big\|\frac{\partial^n k}{\partial x^n}\Big\|_\infty \leq c^{n+1} n!, \quad n = 1, 2, \cdots,$$

where $c > 0$ is a constant that depends on ϕ. Without loss of generality we may assume that $c \geq \pi$ and $|\lambda| \geq 1$. Set $F = f \circ \psi$, then

$$\begin{aligned}
\|F'\|_2 \leq |\lambda|\|F'\|_2 &= \Big\|\frac{d}{dx}\int_0^\ell F(y)\log|x-y|dy + \int_0^\ell F(y)\frac{\partial k(x,y)}{\partial x}dy\Big\|_2 \\
&\leq \Big\|\frac{d}{dx}\int_0^\ell F(y)\log|x-y|dy\Big\|_2 + \Big\|\int_0^\ell F(y)\frac{\partial k(x,y)}{\partial x}dy\Big\|_2 \\
&\leq \pi\|F\|_2 + c\|F\|_2 = 2c\|F\|_2.
\end{aligned}$$

Differentiating once more,

$$\begin{aligned}\|F''\|_2 \le |\lambda|\|F''\|_2 &= \Big\|\frac{d}{dx}\int_0^\ell F'(y)\log|x-y|dy + \int_0^\ell F(y)\frac{\partial^2 k(x,y)}{\partial y^2}dy\Big\|_2 \\ &\le \pi\|F'\|_2 + M^2\|F\|_2 \le M\|F'\|_2 + M^2\|F\|_2 \\ &\le 3!c^2\|F\|_2.\end{aligned}$$

Repeating the same argument successively, we find that

$$\|F^{(n)}\|_2 \le \Big(n!+(n-1)!+\cdots+1\Big)M^n\|F\|_2 \le (n+1)!M^n\|F\|_2, \quad n=1,2,\cdots. \quad (2.6.3)$$

Let $x \in [0,\ell]$ and $n \in \mathbb{N}$,

$$|F^{(n)}(x)| \le |\lambda||F^{(n)}(x)| = \Big|\int_0^\ell F^n(y)\log|\psi(x)-\psi(y)|dy\Big| \le \|F^{(n)}\|_2\Big(\int_0^\ell \log^2|\psi(x)-\psi(y)|dy\Big)^{1/2}.$$

If we let $\alpha = \max_{x\in[0,\ell]}\Big(\int_0^\ell \log^2|\psi(x)-\psi(y)|dy\Big)^{1/2}$, then

$$\|F^{(n)}\|_\infty \le \alpha\|F^{(n)}\|_2.$$

The latter inequality, together with (5.2.8), implies that

$$\|F^{(n)}\|_\infty \le \alpha(n+1)!M^n\|F\|_2.$$

So F is real-analytic on $(0,\ell)$.

■

3 Schatten Class Membership of Single Layer Potentials

Our focus in this chapter will be on the decay rate of eigenvalues of single layer potentials. We will derive a result for Schatten class membership of single layer potentials over smooth curves.

3.1 Positive Definiteness of Single Layer Potentials

In this section we review known results on positive and negative definiteness of several potential operators. Let us begin with Riesz potentials.

Suppose $\Omega \subset \mathbb{R}^n$ is a set of finite Lebesgue measure. The Riesz potential operators are defined by

$$(\mathcal{R}_{\alpha,\Omega} f)(x) = \int_{\Omega} f(y) k_{\alpha,n}(|x-y|)dy, \;\; f \in L^2(\Omega), \;\alpha \in (0,n),$$

where $k_{\alpha,n}(|x-y|) = c_{\alpha,n}|x-y|^{\alpha-n}$, and $c_{\alpha,n} = 2^{\alpha}\pi^{-\alpha/2}\frac{\Gamma(\alpha/2)}{\Gamma((n-\alpha)/2)}$.

The operator $R_{\alpha,\Omega}$ generalizes the Riemann-Liouville ones to several variables and the Newtonian potentials to fractional orders (see, for example [1, 1.2.2]). In particular if $\Omega \in \mathbb{R}^n$, with $n \geq 3$, then $\mathcal{R}_{2,\Omega}$ is precisely the Newtonian potential. The following is known:

Proposition 3.1.1 *The operator $\mathcal{R}_{\alpha,\Omega}$ is positive definite.*

See [50] for a proof.

It is noteworthy that the previous result can be extended to a larger class of potential operators. For $0 < \alpha < n$, the Riesz[1] energy of two signed measures μ and

[1] The Riesz potential is named after Marcel Riesz.

ν is defined by

$$\mathcal{E}_\alpha[\mu,\nu] = \iint \frac{1}{|x-y|^s}\mu(x)\,d\nu(y).$$

Note that the case $\mathcal{E}_\alpha[\mu,\nu] = +\infty$ might occur. Götz in [31, Corollary 3] generalized the previous theorem by proving the following:

Theorem 3.1.2 *Assume that ν is a signed measure with finite Riesz energy. Then the energy of ν is positive, unless it is the zero-measure.*

For the case of logarithmic potentials in the plane, we have the following known result from potential theory:

Theorem 3.1.3 *Suppose ν is a compactly supported signed measure in the plane and one of the following conditions is satisfied:*
(a)$\nu \perp 1$,
(b)$supp(\nu) \subset \mathbb{D}$;
then $\displaystyle\iint \ln \frac{1}{|x-y|} d\nu(x) d\nu(y) \geq 0.$

See [39, Theorem 1.16] for a proof.

Inclusion of the support of the measure in a disk of radius one is necessary for the previous theorem to hold. Let us illustrate this with a simple example.

Example 3.1.4 Let C_r be the boundary of the disk of radius $r > 0$, centered at the origin. Then

$$\begin{aligned}\int_{C_r}\int_{C_r} \ln\frac{1}{|z-w|}\,ds_z\,ds_w &= \int_{\mathbb{T}}\int_{\mathbb{T}} \ln\frac{1}{r|\zeta-\eta|}\,ds_\zeta\,ds_\eta \\ &= -\int_{\mathbb{T}}\int_{\mathbb{T}} \ln r\,ds_z\,ds_w - \int_{\mathbb{T}}\int_{\mathbb{T}} \ln|z-w|\,ds_z\,ds_w \\ &= -(2\pi)^2\log r.\end{aligned}$$

Obviously for any $r > 1$, the latter quantity becomes negative.

The following proposition immediately follows from Theorem 3.1.3.

Corollary 3.1.5 *Assume $\Gamma \subset \mathbb{D}$ is a smooth rectifiable curve. Then $\mathcal{S}_\Gamma$ is negative semi-definite on $L^2(\Gamma)$.*

Remark 3.1.6 *As we observed in example 3.1.4, in general, single layer potentials over planar curves need not be negative definite.*

For the higher dimensions, inclusion in the unit ball is not needed. The following proposition follows from 3.1.1.

Proposition 3.1.7 *Suppose* $\Omega \subset \mathbb{R}^n$ *with* $n \geq 3$ *is a smoothly bounded domain. Then* $\mathcal{S}_{\partial\Omega}$ *is positive definite.*

3.2 Effect of Dilation and Translation on Singular Numbers

In this section we establish an inequality which demonstrates a relation between the singular numbers of single layer potentials over the set of all dilations of a given curve. For higher dimensions, our result becomes trivial. Suppose Ω is a smoothly bounded domain in the plane. For brevity of notation, by Γ we denote the boundary of Ω.

Proposition 3.2.1 *Suppose* Ω *is a smoothly bounded domain in the plane and* $t > 0$. *Then* $\mathcal{S}_{t\Gamma} \in \mathfrak{S}_p$ *if and only if* $\mathcal{S}_\Gamma \in \mathfrak{S}_p$. *Moreover the following inequality holds;*

$$s_{n+2}(\mathcal{S}_\Gamma) \leq \frac{1}{t} s_{n+1}(\mathcal{S}_{t\Gamma}) \leq s_n(\mathcal{S}_\Gamma).$$

Proof. Let us introduce four auxiliary operators. Define $\mathcal{M}f(z) = f(tz)$ and $\mathcal{N}g(w) = g(\frac{w}{t})$ from $L^2(t\Gamma)$ into $L^2(\Gamma)$ and from $L^2(\Gamma)$ into $L^2(t\Gamma)$, respectively. Set $\mathcal{F}_1(f) = -\log t \int_\Gamma f ds$ and $\mathcal{F}_2(g) = \log t \int_{t\Gamma} g\, ds$ acting on $L^2(\Gamma)$ and $L^2(t\Gamma)$ respectively. Clearly $\text{rank}(\mathcal{F}_1) = \text{rank}(\mathcal{F}_2) = 1$. We have

$$\|\mathcal{M}f\|_2^2 = \int_\Gamma |\mathcal{M}f(z)|^2 ds(z) = \int_\Gamma |f(tz)|^2 ds(z) = \frac{1}{t}\int_{t\Gamma} |f(w)|^2 ds(w).$$

So $\|\mathcal{M}\|_{op} \leq \frac{1}{\sqrt{t}}$. Similarly, we can show that $\|\mathcal{N}\|_{op} \leq \sqrt{t}$. Now let us return to the proof of the theorem. Take $f \in L^2(t\Gamma, ds)$,

$$\begin{aligned}
(\mathcal{S}_{t\Gamma} f)(w) &= \int_{t\Gamma} f(\eta) \log |w-\eta| ds_\eta \\
&= t \int_{\Gamma} f(t\zeta) \log(t|\zeta - \frac{w}{t}|) ds_\zeta \\
&= t \log t \int_{\Gamma} f(t\zeta) ds(\zeta) + t \int_{\Gamma} f(t\zeta) \log |\zeta - \frac{w}{t}| ds_\zeta \\
&= (\mathcal{F}_2 f)(w) + t(\mathcal{S}_\Gamma \mathcal{M} f)(\frac{w}{t}) \\
&= (\mathcal{F}_2 f)(w) + t(\mathcal{N}\mathcal{S}_\Gamma \mathcal{M} f)(w). \qquad (3.2.1)
\end{aligned}$$

For $f \in L^2(\Gamma, ds)$,

$$\begin{aligned}
(\mathcal{S}_\Gamma f)(z) &= \int_{\Gamma} f(\zeta) \log |z-\zeta| ds_\zeta \\
&= \frac{1}{t} \int_{t\Gamma} f(\frac{\eta}{t}) \log(\frac{1}{t}|\eta - tz|) ds_\eta \\
&= -\frac{\log t}{t} \int_{t\Gamma} f(\frac{\eta}{t}) ds(\eta) + \frac{1}{t} \int_{t\Gamma} f(\frac{\eta}{t}) \log |\eta - tz| ds_\eta \\
&= (\mathcal{F}_1 f)(z) + \frac{1}{t}(\mathcal{S}_{t\Gamma} \mathcal{N} f)(tz) \\
&= (\mathcal{F}_1 f)(z) + \frac{1}{t}(\mathcal{M}\mathcal{S}_{t\Gamma} \mathcal{N} f)(z). \qquad (3.2.2)
\end{aligned}$$

So $\mathcal{S}_\Gamma = \mathcal{F}_1 + \frac{1}{t}\mathcal{M}\mathcal{S}_{t\Gamma}\mathcal{N}$ and $\mathcal{S}_{t\Gamma} = \mathcal{F}_2 + t\mathcal{N}\mathcal{S}_\Gamma\mathcal{M}$. From the elementary properties of singular numbers (see 1.2.1), together with the latter equalities for $\mathcal{S}_\Gamma$ and $\mathcal{S}_{t\Gamma}$, it follows that:

$$\begin{aligned}
s_{n+1}(\mathcal{S}_\Gamma) = s_{n+1}(\mathcal{F}_1 + \frac{1}{t}\mathcal{M}\mathcal{S}_{t\Gamma}\mathcal{N}) &\leq s_2(\mathcal{F}_1) + s_n(\frac{1}{t}\mathcal{M}\mathcal{S}_{t\Gamma}\mathcal{N}) \\
&\leq \frac{1}{t}\|\mathcal{M}\|_{op}\|\mathcal{N}\|_{op} s_n(\mathcal{S}_{t\Gamma}) \leq \frac{1}{t} s_n(\mathcal{S}_{t\Gamma}),
\end{aligned}$$

and similarly,

$$s_{n+1}(\mathcal{S}_{t\Gamma}) = s_{n+1}(\mathcal{F}_2 + t\mathcal{N}\mathcal{S}_\Gamma\mathcal{M}) \leq t s_n(\mathcal{S}_\Gamma)\|\mathcal{N}\|_{op}\|\mathcal{M}\|_{op} \leq t s_n(\mathcal{S}_\Gamma).$$

Therefore $s_{n+2}(\mathcal{S}_\Gamma) \le \frac{1}{t} s_{n+1}(\mathcal{S}_{t\Gamma}) \le s_n(\mathcal{S}_\Gamma)$, for $n = 1, 2, \cdots$. This implies $\mathcal{S}_{t\Gamma} \in \mathfrak{S}_p$ if and only if $\mathcal{S}_\Gamma \in \mathfrak{S}_p$.

■

Corollary 3.2.2 *Let* Γ *and* $t\Gamma$ *be as in above. Then*

$$\alpha_0 \le \frac{1}{t^p} \|\mathcal{S}_{t\Gamma}\|_p^p - \|\mathcal{S}_\Gamma\|_p^p \le \beta_0,$$

for any $p > 1$, *where constants* α_0 *and* β_0 *depend on* p *and* Γ.

Proof. For $p > 1$, by summing up the $p-$th powers of the terms of the last inequality in the proof of the previous proposition, we find that

$$\sum_{n=3}^{\infty} s_n^p(\mathcal{S}_\Gamma) \le \frac{1}{t^p} \sum_{n=2}^{\infty} s_n^p(\mathcal{S}_{t\Gamma}) \le \sum_{n=1}^{\infty} s_n^p(\mathcal{S}_\Gamma),$$

then

$$\|\mathcal{S}_\Gamma\|_p^p - s_1^p(\mathcal{S}_\Gamma) - s_2^p(\mathcal{S}_\Gamma) \le \frac{1}{t^p} \|\mathcal{S}_{t\Gamma}\|_p^p - \frac{1}{t^p} s_1^p(\mathcal{S}_{t\Gamma}) \le \|\mathcal{S}_\Gamma\|_p^p.$$

Therefore

$$\alpha_0 \le \frac{1}{t^p} \|\mathcal{S}_{t\Gamma}\|_p^p - \|\mathcal{S}_\Gamma\|_p^p \le \beta_0,$$

where $\alpha_0 = \frac{1}{t^p} s_1^p(\mathcal{S}_{t\Gamma}) - s_1^p(\mathcal{S}_\Gamma) - s_2^p(\mathcal{S}_\Gamma)$, and $\beta_0 = \frac{1}{t^p} s_1^p(\mathcal{S}_{t\Gamma})$.

■

Remark 3.2.3 *In higher dimensions, there is a simple relation between point spectra of single layer potentials over dilations of smooth surfaces. If* $f(x)$ *is an eigenfunction for* $\mathcal{S}_{\partial\Omega}$, *then so is* $F(y) := f(y/t)$ *for* $\mathcal{S}_{t\partial\Omega}$. *Conversely, if* $g(x)$ *is an eigenfunction of* $\mathcal{S}_{t\partial\Omega}$, *then so is* $G(y) := g(ty)$ *for* $\mathcal{S}_{\partial\Omega}$. *For eigenvalues we have that if* λ *is an eigenvalue of* $\mathcal{S}_{\partial\Omega}$, *then* $\frac{\lambda}{t^{n-2}}$ *is an eigenvalue of* $S_{t\partial\Omega}$. *Therefore the same holds in higher dimensions and we have the following conclusion:* $\mathcal{S}_{t\partial\Omega} \in \mathfrak{S}_p$ *if and only if* $\mathcal{S}_{\partial\Omega} \in \mathfrak{S}_p$.

We conclude this section with the following observation regarding translation of the underlying curve (or surfaces, for $n \geq 3$). We state and prove the result for $n = 2$, and the case $n \geq 3$ can be done in a similar fashion.

Proposition 3.2.4 *Let Ω be a simply connected domain in the plane and $a \in \mathbb{C}$. Then*

$$s_n(\mathcal{S}_{\partial\Omega+a}) = s_n(\mathcal{S}_{\partial\Omega}).$$

Proof. Put $\Gamma = \partial\Omega$ and define the embedding operators $\mathcal{I} : L^2(\Gamma) \to L^2(\Gamma + a)$, and $\mathcal{J} : L^2(\Gamma + a) \to L^2(\Gamma)$, by

$$(\mathcal{I}f)(z) = f(z - a),\ z \in \Gamma + a \textit{ and } (\mathcal{J}g)(w) = g(w + a),\ w \in \Gamma,$$

for $f \in L^2(\Gamma)$ and $g \in L^2(\Gamma + a)$. It is clear that $\|\mathcal{I}\| = \|\mathcal{J}\| = 1$. Moreover, $\mathcal{S}_\Gamma = \mathcal{I}\mathcal{S}_{\Gamma+a}\mathcal{J}$ and $\mathcal{S}_{\Gamma+a} = \mathcal{J}\mathcal{S}_\Gamma\mathcal{T}$. Thus,

$$s_n(\mathcal{S}_\Gamma) = s_n(\mathcal{I}\mathcal{S}_{\Gamma+a}\mathcal{J}) \leq \|\mathcal{I}\|\|\mathcal{J}\|s_n(\mathcal{S}_{\Gamma+a}) = s_n(\mathcal{S}_{\Gamma+a}),$$

and

$$s_n(\mathcal{S}_{\Gamma+a}) = s_n(\mathcal{J}\mathcal{S}_\Gamma\mathcal{I}) \leq \|\mathcal{I}\|\|\mathcal{J}\|s_n(\mathcal{S}_\Gamma) = s_n(\mathcal{S}_\Gamma),$$

for $n = 1, 2, \cdots$. Therefore, $s_n(\mathcal{S}_\Gamma) = s_n(\mathcal{S}_{\Gamma+a})$, for all $n = 1, 2, \cdots$.

■

3.3 Single Layer Potentials on Smooth Curves

For the decay rate of eigenvalues of integral operators, a general rule of thumb for integral operators is that " the smoother the kernel is, the faster the eigenvalues decay". In this section we shall show that the singular numbers of single layer potentials over smooth boundary curves have similar asymptotic behavior. We will show that as long as the curve Γ is smooth, the singular numbers of $\mathcal{S}_\Gamma$ are of order $\frac{1}{n}$. Let us begin with some elementary observations.

Assume Γ is a smooth Jordan curve of length ℓ, and let $\gamma : [0, \ell] \to \Gamma$ be the arc-length parametrization of Γ. Then

$$\gamma(s) - \gamma(t) = (s - t)\varphi(t, s), \quad \text{for } s, t \in [0, \ell],$$

where φ is a smooth function in $(0, \ell) \times (0, \ell)$. By smoothness of Γ, $|\varphi(s, t)| > 0$ for all $(s, t) \in (0, \ell) \times (0, \ell)$. Let us define the following operators:

$$\begin{aligned} A :& L^2(\Gamma) \to L^2[0, \ell], & B :& L^2[0, \ell] \to L^2(\Gamma), & V :& L^2[0, \ell] \to L^2[0, \ell] \\ (Af)(x) &= f(\gamma(x)), & \big(Bg\big)(\gamma(x)) &= g(\gamma^{-1}(x)), & (Vf)(x) &= \int_0^\ell g(y) \log |\varphi(x, y)| dy. \end{aligned}$$

Lemma 3.3.1 *The operators A, B and V are bounded. Moreover $A = B^{-1}$.*

Proof. The operator V, having continuous kernel, is a bounded (in fact it is Hilbert-Schmidt) on $L^2[0, \ell]$. Now we show that A is bounded. Let $f \in L^2(\Gamma)$, then

$$\|Af\|_2^2 = \int_0^\ell |(Af)(x)|^2 dx = \int_0^\ell |f(\phi(x))|^2 dx = \int_\Gamma |f(z)|^2 ds_z = \|f\|_{L^2(\Gamma)}^2.$$

Thus, A is bounded and $\|A\|_{op} = 1$. Similarly, one can show that B is bounded, and that $\|B\|_{op} = 1$. The last part of the statement is trivial.

■

Theorem 3.3.2 *Let Γ be a smooth Jordan curve in the plane. Then $\mathcal{S}_\Gamma \in \mathfrak{S}_p$ if and only if $p > 1$. Moreover $s_n(\mathcal{S}_\Gamma) = O(\frac{1}{n})$.*

Proof. Set $J = [0, \ell]$, and let $f \in L^2(\Gamma, ds)$. Take $x \in [0, \ell]$ and let $z = \gamma(x)$, then

$$\begin{aligned}
(\mathcal{S}_\Gamma f)(z) &= \int_\Gamma f(w) \log|w - z| ds_w \\
&= \int_0^\ell f(\gamma(y)) \log\big|\gamma(x) - \gamma(y)\big| dy \\
&= \int_0^\ell f(\gamma(y)) \log\big|(x - y)\phi(x, y)\big| dy \\
&= \int_0^\ell f(\gamma(y)) \log|x - y| dy + \int_0^\ell f(\gamma(y)) \log|\phi(x, y)| dy \\
&= \int_0^\ell (Af)(y) \log|x - y| dy + \int_0^\ell (Af)(y) \log|\phi(x, y)| dy \\
&= (\mathcal{S}_J Af)(x) + (VAf)(x) \\
&= (\mathcal{S}_J Af)\big(\gamma^{-1}(x)\big) + (VAf)\big(\gamma^{-1}(x)\big) \\
&= (B\mathcal{S}_J Af)(\gamma(x)) + (BVAf)(\gamma(x)) = (B\mathcal{S}_J Af)(z) + (BVAf)(z).
\end{aligned}$$

So

$$\mathcal{S}_\Gamma = B\mathcal{S}_J A + BVA.$$

The latter equality, together with the previous lemma, gives us

$$\mathcal{S}_J = B^{-1}\mathcal{S}_\Gamma A^{-1} - V = A\mathcal{S}_\Gamma B - V.$$

By the properties of the singular numbers, (see [29, p. 30]):

$$s_{2n}(\mathcal{S}_\Gamma) = s_{2n}(B\mathcal{S}_J A + BVA) \leq \|A\|\|B\| s_n(\mathcal{S}_J) + s_{n+1}(V) \leq s_n(\mathcal{S}_J) + s_{n+1}(V), \quad (1)$$

and

$$s_{2n-1}(\mathcal{S}_\Gamma) = s_{2n}(B\mathcal{S}_J A + BVA) \leq \|A\|\|B\| s_n(\mathcal{S}_J) + s_n(V) \leq s_n(\mathcal{S}_J) + s_n(V), \quad (2)$$

for $n = 1, 2, \cdots$. Similarly, for the singular numbers of $\mathcal{S}_J$, we find that

$$s_{2n}(\mathcal{S}_J) \leq s_n(\mathcal{S}_\Gamma) + s_{n+1}(V), \quad (3)$$

and

$$s_{2n-1}(\mathcal{S}_J) \leq s_n(\mathcal{S}_\Gamma) + s_n(V), \quad (4)$$

for $n = 1, 2, \cdots$. We note that the singular numbers of V decay faster than $\frac{1}{n}$. In fact, the singular numbers of operators with smooth kernels are of order $o(\frac{1}{n^k})$ for all positive integers k (see [29, p. 122] and [40, §30.5]). It follows from the inequalities (1)-(4) that $s_n(\mathcal{S}_\Gamma) = O(\frac{1}{n})$.

■

Smoothness of the boundary curve in the previous result can be slightly relaxed. First we will show that, if the boundary curve is piecewise smooth Jordan with finitely many algebraic cusps, the associated single layer operator is *not* trace-class.

Corollary 3.3.3 *Suppose* Γ *is a simple piecewise smooth rectifiable curve in the plane with finitely many algebraic cusps. Then* $\mathcal{S}_\Gamma \notin \mathfrak{S}_1$.

Proof. Choose a smooth sub-arc $\gamma \subset \Gamma$ and define the compact self-adjoint operator T_γ, acting on $L^2(\gamma, ds)$ by $(T_\gamma f)(z) = \int_\gamma f(w) \log|z-w|\, ds_w$. Let $\{\lambda_n\}_{n=1}^\infty$ be the set of singular numbers of T_γ. As we showed earlier, $\{\lambda_n\}_{n=1}^\infty \in \ell^p$ if and only if $p > 1$. Now let $\{f_n\}_{n=1}^\infty$ be an orthonormal basis, consisting of eigenfunctions of T_γ. For $n \in \mathbb{N}$ define e_n on Γ by $e_n = \chi_\gamma f_n$. For $m, n \in \mathbb{N}$, we find

$$\langle e_n, e_m \rangle = \int_\Gamma e_n \overline{e_m}\, ds = \int_\gamma e_n \overline{e_m} ds = \int_\gamma g_n \overline{g_m}\, ds = \delta_{m,n},$$

where $\delta_{m,n}$ is the Kronecker delta. Thus, $\{e_n\}_{n=1}^\infty$ form an orthonormal set in $L^2(\Gamma, ds)$

although they do not form a basis.

$$
\begin{aligned}
|\langle \mathcal{S}_\Gamma e_n, e_n\rangle| &= \Big| \int_\Gamma (\mathcal{S}_\Gamma e_n)(z)\overline{e_n(z)}\, ds_z| = \Big| \int_\gamma (\mathcal{S}_\Gamma e_n)(z)\overline{e_n(z)}\, ds_z \Big| \\
&= \Big| \int_\gamma \overline{e_n(z)} \Big(\int_\Gamma e_n(w) \log|z-w|\, ds_w \Big)\, ds_z \Big| \\
&= \Big| \int_\gamma \overline{e_n(z)} \Big(\int_\gamma e_n(w) \log|z-w|\, ds_w \Big)\, ds_z \Big| \\
&= \Big| \int_\gamma \overline{g_n(z)} \Big(\int_\gamma g_n(w) \log|z-w|\, ds_w \Big)\, ds_z \Big| \\
&= |\langle T_\gamma g_n, g_n\rangle| = \lambda_n.
\end{aligned}
$$

Therefore, $\sum_{n=1}^{\infty} |\langle \mathcal{S}_\Gamma e_n, e_n\rangle| = \sum_{n=1}^{\infty} \lambda_n = +\infty$, which implies $\mathcal{S}_\Gamma \notin \mathfrak{S}_1$. ■

We can generalize the previous proposition to obtain a relation between Schatten class membership of single layer potentials over curves and their sub-arcs. We state the following result without proof. The proof is similar to the one above.

Proposition 3.3.4 *Let $p > 1$ and suppose γ is a sub-arc of Γ. If $\mathcal{S}_\gamma \notin \mathfrak{S}_p$. Then $\mathcal{S}_\Gamma \notin \mathfrak{S}_p$.*

It is well known that, for the case where the boundary curve is the circle of radius one centered at the origin, then eigenfunctions are monomials. For the sake of completeness, let us show the simple relevant computations below. Let $n \in \mathbb{N}$ and $z \in \mathbb{D}$,

$$\begin{aligned}\mathcal{S}_{\mathbb{T}}(z^n) &= \frac{1}{2\pi}\int_{\mathbb{T}} \zeta^n \log|z-\zeta|\, ds_\zeta \\ &= \frac{1}{4\pi}\int_{\mathbb{T}} \zeta^n \Big[\log(1-\frac{z}{\zeta}) + \log(1-\overline{z}\zeta)\Big]\, ds_\zeta \\ &= \frac{1}{4\pi}\int_{\mathbb{T}} \zeta^n \Big[-\sum_{k=1}^{\infty}\frac{1}{k}(\frac{z}{\zeta})^k - \sum_{k=1}^{\infty}\frac{1}{k}(\overline{z}\zeta)^k\Big]\, ds_\zeta \\ &= \frac{-1}{4n\pi}\int_{\mathbb{T}} z^n\, ds_\zeta = \frac{-1}{2n}z^n.\end{aligned}$$

In a similar way, we can show that

$$\mathcal{S}_{\mathbb{T}}(z^n) = -\frac{1}{2|n|}z^n,$$

for $z \in \mathbb{D}$ and $n = -1, -2, \cdots$.

It follows from the mean value property of harmonic functions that $\mathcal{S}_{\mathbb{T}}(1) = 0$. Let $z \in \mathbb{T}$ and $\{\zeta_k\}$ be a sequence in $\mathbb{D}$ approaching z. For each positive integer k and non-zero integer n we have $\mathcal{S}_{\mathbb{T}}(\zeta_k^n) = -\frac{1}{2|n|}\zeta_k^n$. Letting k go to infinity, it follows from the continuity of single layer potentials that $\mathcal{S}_{\mathbb{T}}(z^n) = -\frac{1}{2|n|}z^n$. Therefore, for $z \in \mathbb{T}$ we have

$$\mathcal{S}_{\mathbb{T}}(z^n) = \begin{cases} \frac{-1}{2|n|} z^n, & n \in \mathbb{Z}^* \\ 0, & n = 0. \end{cases}$$

In particular, each eigenvalue $\frac{1}{2n}$ for $n = 1, 2, \cdots$, is of multiplicity two. Knowing the singular numbers, we immediately obtain the Hilbert-Schmidt norm as well as the operator norm:

$$\|\mathcal{S}_{\mathbb{T}}\|_{op} = s_1(\mathcal{S}_{\mathbb{T}}) = \frac{1}{2},$$

and

$$\|\mathcal{S}_{\mathbb{T}}\|_2^2 = (\frac{1}{2})^2 + (\frac{1}{2})^2 + \cdots + (\frac{1}{2n})^2 + (\frac{1}{2n})^2 + \cdots = 2\sum_{n=1}^{\infty} \frac{1}{4n^2} = \frac{\pi^2}{12}.$$

Let $r\mathbb{T}$ be the circle of radius $r > 0$, centered at the origin. Take $z \in \mathbb{C} \setminus r\mathbb{D}$, and $n \in \mathbb{N}$,

$$\begin{aligned}
\mathcal{S}_{r\mathbb{T}}(z^n) &= \frac{1}{2\pi} \int_{r\mathbb{T}} \zeta^n \log|z-\zeta| \, ds_\zeta \\
&= \frac{1}{2\pi} \int_{r\mathbb{T}} \zeta^n \Big(\log|z| + \log\Big|1 - \frac{\zeta}{z}\Big| \Big) \, ds_\zeta \\
&= \frac{1}{4\pi} \int_{r\mathbb{T}} \zeta^n \Big[\log(1 - \frac{\zeta}{z}) + \log(1 - \frac{\overline{\zeta}}{\overline{z}}) \Big] \, ds_\zeta \\
&= \frac{1}{4\pi} \int_{r\mathbb{T}} \zeta^n \Big[-\sum_{k=1}^{\infty} \frac{1}{k} (\frac{\zeta}{z})^k - \sum_{k=1}^{\infty} \frac{1}{k} (\frac{\overline{\zeta}}{\overline{z}})^k \Big] \, ds_\zeta \\
&= \frac{1}{4\pi} \int_{r\mathbb{T}} \zeta^n \Big[-\sum_{k=1}^{\infty} \frac{1}{k} (\frac{\zeta}{z})^k - \sum_{k=1}^{\infty} \frac{1}{k} (\frac{r^2}{\zeta\overline{z}})^k \Big] \, ds_\zeta \\
&= \frac{-r}{4n\pi} \int_{r\mathbb{T}} (\frac{r^2}{\overline{z}})^n \, ds_\zeta = \frac{-1}{2n} \Big(\frac{r^2}{|z|^2} \Big)^n r z^n.
\end{aligned}$$

Approaching the boundary of the unit disk from its exterior and using the continuity of single layer potentials implies that for $z \in t\mathbb{T}$,

$$\mathcal{S}_{r\mathbb{T}}(z^n) = -\frac{r}{2|n|} z^n, \quad n = \pm 1, \pm 2 \cdots.$$

Therefore, the eigenvalues are $-\frac{r}{2n}$ for $n = 1, 2, \cdots$. We note that $\mathcal{S}_{r\mathbb{T}}(1) = r\log r$. As a byproduct of our calculations, we infer that, $\mathcal{S}_{r\mathbb{T}}$ is injective if and only if $r \neq 1$.

In [36, Theorem 2.3] Ebenfelt et al. showed that under a mild smoothness assumption on the boundary curve, if the exterior domain is Smirnov (see [22]) and $f \equiv 1$ is an eigenfunction for the single layer potential, then the boundary curve must be a circle. This result can be interpreted as follows.

If there is a constant density of a mass distribution on the boundary curve $\partial\Omega$ such that the corresponding logarithmic potential remains constant in Ω, then $\partial\Omega$ must be a circle. This is two-dimensional analogue of the fundamental problem of electrostatics dealing with distribution of charge on the boundary of a conductor. As we saw earlier, monomials serve as eigenfunctions for the single layer potentials. Next we show that the converse is also true and that single layer potentials, to some extent, characterize circles. We will make use of the following results which are interesting in their own right (we refer the reader to [37]).

Theorem 4.0.5 *Let Γ be a rectifiable Jordan curve such that its exterior Ω_+ is Smirnov and $T(z)$ the tangent vector to Γ defined on Γ. Suppose that*

$$\overline{T(z)} = H(z), \quad a.e. \ on \ \Gamma$$

where $H(z)$ stands for the non-tangential boundary values of a bounded analytic function H in the exterior Ω_+ of Γ with $H(\infty)$. Then Γ must be a circle.

See [38] for a proof.

Theorem 4.0.6 *Suppose that Γ, Ω_+ and H satisfy conditions of the previous theorem, but assume H has a simple pole at a given finite point $z_0 \in \Omega_+$. Then Γ is a circle of the form*

$$\Gamma = \left\{ a\zeta + z_0 : \left|\zeta - \frac{p}{1-p^2}\right| = \frac{p^2}{1-p^2} \right\}$$

with some $a \in \mathbb{C}^*$ and $0 < p < 1$.

See [37] for a proof.

Proposition 4.0.7 *Assume Ω is a smoothly bounded domain in the plane. If $\mathcal{S}_\Gamma$ has a polynomial eigenfunction $p(z)$ with at most one simple zero in Ω_+, then Γ must be* *circle.*

Proof. Assume $\mathcal{S}_\Gamma(p) = \lambda p$, for some $\lambda \in \mathbb{R}$ and polynomial $p(z)$ such that $\#\{z \in \Omega_+ : p(z) = 0\} \leq 1$. Since $\lambda p(z) = \frac{1}{2\pi} \int_\Gamma p(\zeta) \log|z - \zeta|\, ds_\zeta$ for all $z \in \overline{\Omega}$, then

$$-4\pi\lambda p'(z) = \int_\Gamma \frac{p(\zeta)}{\zeta - z}\, ds_\zeta = \int_\Gamma \frac{p(\zeta)\overline{T(\zeta)}}{\zeta - z} d\zeta \quad \text{for } z \in \Omega,$$

where T denotes the unit tangent.

Define $F(z) = \int_\Gamma \frac{p(\zeta)\overline{T(\zeta)}}{\zeta - z} d\zeta$ on the exterior domain $\Omega_+ = \widehat{\mathbb{C}} \setminus \overline{\Omega}$. The function F is analytic in Ω_+ and $F(\infty) = 0$. By the Sokhotski-Plemelj jump theorem (see [?]) we find that the following holds almost everywhere on Γ:

$$2\pi i p(z)\overline{T(z)} = \lim_{\substack{w \to z,\\ w \in \Omega_+}} \int_\Gamma \frac{p(\zeta)\overline{T(\zeta)}}{\zeta - w} d\zeta - \lim_{\substack{t \to z,\\ t \in \Omega}} \int_\Gamma \frac{p(\zeta)\overline{T(\zeta)}}{\zeta - t} d\zeta = \lim_{\substack{w \to z,\\ w \in \Omega_+}} F(w) + 4\pi\lambda \lim_{\substack{t \to z,\\ t \in \Omega}} p'(t)$$

$$= \lim_{\substack{w \to z,\\ w \in \Omega_+}} F(w) + 4\pi\lambda \lim_{\substack{t \to z,\\ t \in \Omega}} p'(t) = \lim_{\substack{w \to z,\\ w \in \Omega_+}} \Big[F(w) + 4\pi\lambda p'(w)\Big].$$

This can be rewritten as

$$\overline{T(z)} = \lim_{\substack{w \to z,\\ w \in \Omega_+}} \frac{F(w)}{2\pi i p(w)} + \frac{2\lambda p'(w)}{i p(w)}, \quad \text{a.e. on } \Gamma.$$

The function $\Phi(w) = \frac{F(w)}{p(w)} + 2\lambda \frac{p'(w)}{p(w)}$ on Ω_+ has at most one finite simple pole in Ω_+ and $\Phi(\infty) = 0$. Since $\Phi \equiv \overline{T}$ a.e. on Γ, then by the previous theorem, Γ must be a circle.

■

The next corollary follows immediately.

Corollary 4.0.8 *Assume Ω contains the origin and $\mathcal{S}_{\partial\Omega}$ has a monomial eigenfunction of the form $p(z) = z^n$. Then, Ω must be a disk.*

We conclude this chapter with the following remark regarding the counterpart of the single layer operator in $L^2(\Omega, dA)$.

The logarithmic potential on Ω is defined by

$$(\mathcal{L}_\Omega f)(w) = \frac{1}{2\pi}\int_\Omega f(w)\log|z-w|\,dA_w, \quad f \in L^2(\Omega, dA).$$

We refer the interested readers to [5] for detailed discussions on the spectral properties of $\mathcal{L}_\Omega$. The operator $\mathcal{L}_\Omega$ is a compact Hilbert-Schmidt operator (see [5]). We shall show that unlike single layer potentials, $\mathcal{L}_\Omega$ does not possess polynomial eigenfunctions.

Proposition 4.0.9 *The operator $\mathcal{L}_\Omega$ has no polynomial eigenfunction of the form $p(z,\overline{z})$.*

Proof. Assume to the contrary that there exists $\lambda \in \mathbb{R}$ and a polynomial $p = p(z,\overline{z})$, not identically zero, such that $\mathcal{L}_\Omega p = \lambda p$. We can find a polynomial $q(z,\overline{z})$ so that $\Delta q = p$. Applying Green's formula,

$$\begin{aligned}\lambda p(z,\overline{z}) &= \frac{1}{2\pi}\int_\Omega p(w,\overline{w})\log|w-z|\,dA_w \\ &= \frac{1}{2\pi}\int_\Omega \Delta q(w,\overline{w})\log|w-z|\,dA_w \\ &= q(z,\overline{z}) + \int_{\partial\Omega}\Big(\frac{\partial q}{\partial n_\zeta}\log|\zeta-z| - q(\zeta,\overline{\zeta})\frac{\partial}{\partial n_\zeta}\log|\zeta-z|\Big)ds_\zeta,\end{aligned}$$

for $z \in \Omega$. Taking $\overline{\partial}_z$-derivatives we obtain

$$\lambda\frac{\partial p}{\partial \overline{z}} = \frac{\partial q}{\partial \overline{z}} + \frac{1}{2}\int_{\partial\Omega}\Big[\frac{\partial q}{\partial n_\zeta}\frac{1}{\overline{z}-\overline{\zeta}} - q\frac{\partial}{\partial n_\zeta}\big(\frac{1}{\overline{z}-\overline{\zeta}}\big)\Big]ds_\zeta.$$

By taking ∂_z-derivatives we obtain $4\lambda\Delta p = 4\Delta q$, but $p = \Delta q$, which implies $\lambda\Delta p = p$. Let k be the smallest positive integer for which $\Delta^k p$ vanishes, then it follows from $\lambda\Delta p = p$, that $0 = 4\lambda\Delta^k p = \lambda\Delta^{k-1}p \neq 0$, but this is a contradiction.

■

5 Two Isoperimetric Inequalities

This chapter is divided into two sections. In the first section, we briefly give some elementary results regarding single layer potentials on segments. In the second section, we give an account of isoperimetric inequalities related to logarithmic potentials. Our major results are isoperimetric inequalities for the Schatten norms of logarithmic and Newtonian potentials over rectangles and parallelepipeds respectively.

5.1 Single Layer Potentials on Segments

One of the most fundamental questions regarding spectra of single layer potentials is whether these operators over "different" curves can have the same spectra. In other words, are there "isospectral" boundary curves for single layer potentials? We shall provide an answer to this question for the simplest case where the single layer potential is defined on the space of square- integrable functions defined on a segment. We begin with some elementary observations.

Proposition 5.1.1 *Suppose I is a segment of length $\ell < 1$. Then*

$$\|\mathcal{S}_I\|_{op} \leq \frac{\ell}{2\pi}\Big(1 - \log\frac{\ell}{2}\Big).$$

Proof. By straightforward calculations we find that for $x \in [0, \ell]$,

$$\frac{1}{2\pi}\int_0^\ell \Big|\log|x-y|\Big|\,dy = \frac{1}{2\pi}\Big(-x\log x - (\ell - x)\log(\ell - x) + \ell\Big).$$

Then

$$\max_{0\le x\le \ell} \frac{1}{2\pi}\int_0^\ell \Big|\log|x-y|\Big|\,dy = \frac{1}{2\pi}\max_{0\le x\le \ell}\Big\{-x\log x-(\ell-x)\log(\ell-x)+\ell\Big\}$$
$$= \frac{1}{2\pi}(\ell-\ell\log\frac{\ell}{2}).$$

By Schur's test (see [61, 3.6]), $\|\mathcal{S}_I\|_{op} \le \frac{\ell}{2\pi}(1-\log\frac{\ell}{2})$.

■

Proposition 5.1.2 *If A and A' are two segments of equal length, then $\|\mathcal{S}_A\|_{op} = \|\mathcal{S}_A'\|_{op}$.*

Proof. For $a < b$ and $a' < b'$ with $b-a = b'-a'$, let $A = [a,b]$ and $A' = [a',b']$. Define the embedding operators $\mathcal{I} : L^2[a,b] \to L^2[a',b']$, and $\mathcal{J} : L^2[a',b'] \to L^2[a,b]$, by

$$(\mathcal{I}f)(x) = f(x+a-a'),\ \ \textit{and}\ (\mathcal{J}g)(x) = g(x+a'-a),$$

for $f \in L^2[a,b]$ and $g \in L^2[a',b']$. Obviously $\|\mathcal{I}\|_{op} = \|\mathcal{J}\|_{op} = 1$. Then it follows from $\mathcal{S}_{A'} = \mathcal{J}\mathcal{S}_A\mathcal{I}$ and $\mathcal{S}_A = \mathcal{I}\mathcal{S}_{A'}\mathcal{J}$ that

$$\|\mathcal{S}_A\|_{op} = \|\mathcal{S}_{A'}\|_{op}.$$

■

We can easily extend the previous proposition to all Schatten norms.

Corollary 5.1.3 *If A and A' are two segments of equal length, then $\|\mathcal{S}_A\|_p = \|\mathcal{S}_A'\|_p$, for all $p > 1$.*

Proof. Let $\mathcal{I}$ and $\mathcal{J}$ be as in above. Then

$$s_n(\mathcal{S}_A) = s_n(\mathcal{I}\mathcal{S}_{A'}\mathcal{J}) \le \|\mathcal{I}\|\|\mathcal{J}\|s_n(\mathcal{S}_{A'}) = s_n(\mathcal{S}_{A'}),$$

and

$$s_n(\mathcal{S}_{A'}) = s_n(\mathcal{J}\mathcal{S}_A\mathcal{I}) \le \|\mathcal{I}\|\|\mathcal{J}\|s_n(\mathcal{S}_A) = s_n(\mathcal{S}_A),$$

for $n = 1, 2, \cdots$. Thus, $s_n(\mathcal{S}_A) = s_n(\mathcal{S}_{A'})$, for $n = 1, 2, \cdots$, which implies $\|\mathcal{S}_A\|_p = \|\mathcal{S}_{A'}\|_p$, for all $p > 1$. The equality trivially holds.

■

Next we show that our previous result is not necessarily true for two intervals.

Proposition 5.1.4 *Let I be a segment. Then there exist two disjoint segments J and K with $|J| + |K| = |I|$ such that $\|\mathcal{S}_I\|_{op} < \|\mathcal{S}_{J\cup K}\|_{op}$.*

Proof. For $a > 0$, let $I = [0, a]$. Now take $a' \in (0, a)$ and choose a'' large enough so that

$$2\int_0^{a'}\int_{a''}^{a''+a-a'} \log|x-y|dxdy$$
$$> a\|\mathcal{S}_I\|_{op} + \left|\int_0^{a'}\int_0^{a'} \log|x-y|dxdy\right| + \left|\int_{a'}^{a}\int_{a'}^{a} \log|x-y|dxdy\right|.$$

Put $J = [0, a']$, $K = [a'', a''+a-a']$ and define $g(x) = \frac{1}{\sqrt{a}}$ on $J \cup K$ (clearly $\|g\|_2 = 1$).

$$\begin{aligned}
&|\langle \mathcal{S}_{J\cup K}(g), g\rangle| \\
&= \left|\int_0^{a'}\int_0^{a'} \frac{\log|x-y|}{a}dxdy + \int_{a''}^{a''+a-a'}\int_{a''}^{a''+a-a'} \frac{\log|x-y|}{a}dxdy\right. \\
&\left. + 2\int_0^{a'}\int_{a''}^{a''+a-a'} \frac{\log|x-y|}{a}dxdy\right| \\
&= \frac{1}{a}\left|\int_0^{a'}\int_0^{a'} \log|x-y|dxdy + \int_0^{a-a'}\int_0^{a-a'} \log|x-y|dxdy\right. \\
&\left. + 2\int_0^{a'}\int_{a''}^{a''+a-a'} \log|x-y|dxdy\right| \\
&\geq \frac{2}{a}\int_0^{a'}\int_{a''}^{a''+a-a'} \log|x-y|dxdy \\
&- \frac{1}{a}\left|\int_0^{a'}\int_0^{a'} \log|x-y|dxdy\right| - \frac{1}{a}\left|\int_0^{a-a'}\int_0^{a-a'} \log|x-y|dxdy\right| > \|\mathcal{S}_I\|_{op}.
\end{aligned}$$

Therefore, $\|\mathcal{S}_{J\cup K}\|_{op} > \|\mathcal{S}_I\|_{op}$.

■

Proposition 5.1.5 *Suppose I is a segment of length ℓ, then*

$$\|\mathcal{S}_I\|_2^2 = \frac{\ell^2}{2\pi^2}\Big(\log^2\ell - 3\log\ell + \frac{7}{2}\Big).$$

Proof. Without loss of generality we may assume $I = [0, \ell]$.

$$\begin{aligned}
\|\mathcal{S}_I\|_2^2 &= \frac{1}{4\pi^2}\int_0^\ell\int_0^\ell \log^2|x-y|dxdy = \frac{1}{4\pi^2}\int_0^\ell\Big(\int_0^y \log^2 t\,dt + \int_0^{\ell-y}\log^2 t\,dt\Big)dy \\
&= \frac{1}{4\pi^2}\int_0^\ell\Big[(y\log^2 y - 2y\log y + 2y) + \big((\ell-y)\log^2(\ell-y) \\
&- 2(\ell-y)\log(\ell-y) + 2(\ell-y)\big)\Big]dy \\
&= \frac{1}{2\pi^2}\int_0^\ell (x\log^2 x - 2x\log x + 2x)\,dx = \frac{\ell^2}{2\pi^2}\Big(\log^2\ell - 3\log\ell + \frac{7}{2}\Big).
\end{aligned}$$

■

For two segments I and J of equal length, we showed earlier that $s_n(\mathcal{S}_I) = s_n(\mathcal{S}_J)$, for $n = 1, 2, \cdots$. Now we show that the converse is also true.

Proposition 5.1.6 *Suppose I and J are two segments. Then $\|\mathcal{S}_I\|_2 = \|\mathcal{S}_J\|_2$ if and only if $|I| = |J|$.*

Proof. If I and J are two segments with $|I| = |J|$, then $\|\mathcal{S}_I\|_2 = \|\mathcal{S}_J\|_2$. Now assume I and J are two segments such that $\|\mathcal{S}_I\|_2 = \|\mathcal{S}_J\|_2$. Consider the function $f(x) = x^2(\log^2 x - 3\log x + \frac{7}{2})$ on $(0,\infty)$. Since $f'(x) = 2x\log^2 x + 4x\log x + 4x = 2x\Big((\log x + 1)^2 + 1\Big)$, then $f' > 0$ in $(0,\infty)$. So f is strictly increasing on $(0,\infty)$. We note that $f(x)$ is precisely equal to $\|\mathcal{S}_{[0,x]}\|_2^2$. Since $\|\mathcal{S}_I\|_2 = \|\mathcal{S}_J\|_2$, then by Proposition 5.1.5, $|I| = |J|$.

■

The following corollary follows directly from the previous proposition.

Corollary 5.1.7 *Suppose I and J are two segments. Then*

$$s_n(\mathcal{S}_I) = s_n(\mathcal{S}_J), \quad n = 1, 2, \cdots,$$

if and only if $|I| = |J|$.

Let us allow I to have finitely many disjoint intervals. A natural question to ask is whether it is possible to tell the defining segments apart by looking at the Hilbert-Schmidt norms of the single layer potentials for the general case where the underlying set is a union of finitely many disjoint segments. The following example shows this question has negative answer.

Example 5.1.8 Assume $I = [0, a] \cup [b, c]$ with $0 < a < b < c$, and let us compute the Hilbert-Schmidt norm of $\mathcal{S}_I$:

$$\|\mathcal{S}_I\|_2^2 = \frac{1}{4\pi^2}\int_I\int_I \log^2|x-y|dxdy = \frac{1}{4\pi^2}\int_I(\int_0^a \log^2|x-y|\,dx + \int_b^c \log^2|x-y|\,dx)dy$$
$$= \frac{1}{4\pi^2}\Bigg\{\int_0^a\int_0^a \log^2|x-y|\,dx + \int_b^c\int_b^c \log^2|x-y|\,dxdy$$
$$+ \int_0^a\int_b^c \log^2|x-y|\,dxdy + \int_b^c\int_0^a \log^2|x-y|\,dxdy\Bigg\}$$
$$= \frac{1}{4\pi^2}\Bigg\{\int_0^a\int_0^a \log^2|x-y|\,dxdy + \int_b^c\int_b^c \log^2|x-y|\,dxdy + 2\int_0^a\int_b^c \log^2|x-y|\,dxdy\Bigg\}$$
$$= \frac{1}{4\pi^2}\Bigg\{A(a) + A(c-b) + 2\int_0^a\int_b^c \log^2|x-y|\,dxdy\Bigg\}$$
$$= \frac{1}{4\pi^2}\Bigg\{A(a) + A(c-b) + 2\Bigg[c^2(\frac{1}{2}\log^2 c - \frac{1}{2}\log c - \frac{3}{4}) - b^2(\frac{1}{2}\log^2 b - \frac{1}{2}\log b - \frac{3}{4})$$
$$- (c-a)^2(\frac{1}{2}\log^2(c-a) - \frac{1}{2}\log(c-a) - \frac{3}{4})$$
$$+ (b-a)^2(\frac{1}{2}\log^2(b-a) - \frac{1}{2}\log(b-a) - \frac{3}{4})\Bigg] + 4a(c-b)\Bigg\},$$

where $A(x) = x^2(\log^2 x - 3\log x + \frac{7}{2})$ for $x > 0$. Denote the quantity in the latter equality by $f(a,b,c)$ and define $F(x,y) = f(x,y,1) - A(1-y+x)$ on the rectangle $[0,\frac{1}{4}] \times [\frac{1}{2},1]$. We observe that the zero set of F appears in $[0,\frac{1}{4}] \times [\frac{1}{2},1]$ (see Fig. 5.1). Thus, for some $0 < t < \frac{1}{4}$ and $\frac{1}{2} < s < 1$ we have that $F(t,s) = f(t,s,1) - A(1-s+t) = 0$. We note that $A(1-s+t) = \|\mathcal{S}_{[0,1-s+t]}\|_2^2$ and $f(t,s,1) = \|\mathcal{S}_{[0,t]\cup[s,1]}\|_2^2$. Thus,

$$\|\mathcal{S}_{[0,1-s+t]}\|_2^2 - \|\mathcal{S}_{[0,t]\cup[s,1]}\|_2^2 = f(t,s,1) - A(1-s+t) = 0.$$

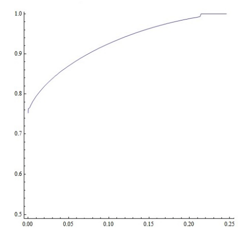

Figure 5.1: The graph of $f(x,y,1) - A(1-y+x) = 0$ for $(x,y) \in [0,\frac{1}{4}] \times [\frac{1}{2},1]$.

Corollary 5.1.9 *There exist segments I, J and K with $I \cap J = \emptyset$ and $|I \cup J| = |K|$ such that $\|\mathcal{S}_{I\cup J}\|_2 = \|\mathcal{S}_K\|_2$.*

Remark 5.1.10 *We observe that statements like the previous propositions do not hold for the case of simple closed curves with non-empty interior. For instance let $\mathbb{T} = \{z : |z| = 1\}$ and Γ be a square with sides of length $\frac{\pi}{2}$ then $|\mathbb{T}| = |\Gamma|$. As we showed earlier*

$$\|\mathcal{S}_{\mathbb{T}}\|_2^2 = 2\sum_{n=1}^{\infty} \frac{1}{4n^2} = \frac{\pi^2}{12} \approx 0.822.$$

On the other hand, for $\|\mathcal{S}_\Gamma\|_2^2$ we find that

$$\|\mathcal{S}_\Gamma\|_2^2 = \frac{1}{4\pi^2}\Big\{4\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \log^2|x-y|dxdy + 2\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \log^2(x^2+y^2)dxdy$$
$$+ 4\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \log^2(1+|x-y|^2)dxdy\Big\} \approx 0.774.$$

Thus, $\|\mathcal{S}_\Gamma\|_2 < \|\mathcal{S}_{\mathbb{T}}\|_2$, even though $|\mathbb{T}| = |\Gamma|$.

Proposition 5.1.11 *Suppose C_r and C_R are two circles of radii r and R respectively. Then $\|\mathcal{S}_{C_r}\|_2 = \|\mathcal{S}_{C_R}\|_2$ if and only if $r = R$.*

Proof. One finds by straightforward computation that

$$\|\mathcal{S}_{C_x}\|_2^2 = 4\pi^2 x^2 \log^2 2x + x^2 \int_0^{2\pi}\int_0^{2\pi} \log^2 \left| \sin(\frac{\phi-\theta}{2})\right| d\phi d\theta$$
$$+ 2x^2 \log 2x \int_0^{2\pi}\int_0^{2\pi} \log \left| \sin(\frac{\phi-\theta}{2})\right| d\phi d\theta,$$

for $x > 0$. Let $a = \int_0^{2\pi}\int_0^{2\pi} \log^2 \left| \sin(\frac{\phi-\theta}{2})\right| d\phi d\theta$ and $b = 2\int_0^{2\pi}\int_0^{2\pi} \log \left| \sin(\frac{\phi-\theta}{2})\right| d\phi d\theta$. Note that $a \approx 51.43$ and $b \approx -54.72$. The function $f(x) = 4\pi^2 x^2 \log^2 2x + ax^2 + bx^2 \log 2x$ is strictly increasing on $(0,\infty)$, therefore $\|\mathcal{S}_{C_R}\|_2 \geq \|\mathcal{S}_{C_r}\|_2$ if and only if $R \geq r$, and the equality is achieved if and only if $R = r$.

■

5.2 Isoperimetric Inequalities for Quadrilaterals and Parallelepipeds

In this section we establish isoperimetric inequalities for Schatten $p-$norms of single layer potentials as well as logarithmic potentials over rectangles and parallelepipeds. Our method is based on the Purkiss principle [58] in optimization which will be discussed later in this chapter and differs from the method used in [51], which is based on so-called Steiner symmetrization. On the other hand, a big disadvantage of our method is that it cannot be generalized to other polygons.

Proposition 5.2.1 *Let γ be a square. Then*

$$\|\mathcal{S}_\gamma\|_2 \leq \|\mathcal{S}_\Gamma\|_2$$

for any rectangle Γ with $|\Gamma| = |\gamma|$.

Proof. For the sake of simplicity let $|\gamma| = 2$. Suppose Γ is a rectangle of side length ℓ and $1-\ell$ where $\ell \in (0,1)$. Then

$$\begin{aligned}\|\mathcal{S}_\Gamma\|_2^2 &= \frac{1}{\pi}\int_0^\ell\int_0^\ell \log^2|x-y|dxdy + \frac{1}{\pi}\int_0^{1-\ell}\int_0^{1-\ell}\log^2|x-y|dxdy \\ &+ \frac{1}{4\pi}\int_0^\ell\int_0^\ell \log^2\left((1-\ell)^2+|x-y|^2\right)dxdy + \frac{1}{4\pi}\int_0^{1-\ell}\int_0^{1-\ell}\log^2\left(\ell^2+|x-y|^2\right)dxdy \\ &+ \frac{1}{\pi}\int_0^\ell\int_0^{1-\ell}\log^2(x^2+y^2)dxdy.\end{aligned}$$

Let $f(\ell)$ denote the right hand side of the above equality where $\ell \in (0,1)$. We observe that $f(x+\frac{1}{2}) = f(-x+\frac{1}{2})$ for $0 < x \le \frac{1}{2}$ and that f is decreasing on $(0, \frac{1}{2}]$.

Moreover by performing straightforward differentiation we find that $f'(\frac{1}{2}) = 0$ and f has only one critical point. Therefore, for $\ell = \frac{1}{2}$ the Hilbert-Schmidt norm is minimized which implies the corresponding curve for the minimizer is a square of side length $\frac{1}{2}$.

■

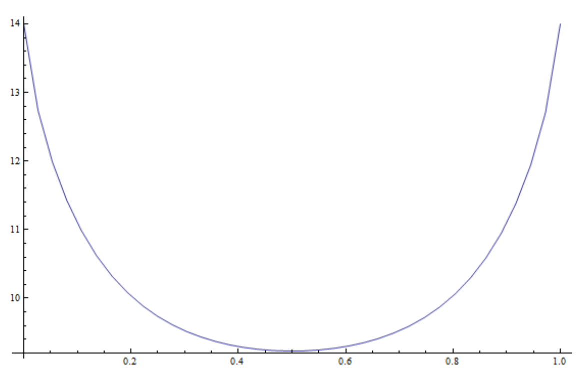

Figure 5.2: The graph of $f(x)$.

For a bounded domain $\Omega \subset \mathbb{R}^n$, with $n \ge 3$, the Newtonian potential on $L^2(\Omega, dV)$ is defined by

$$(\mathcal{N}_\Omega f)(X) = \frac{1}{\omega_n}\int_\Omega \frac{f(Y)}{\|X-Y\|^{n-2}}dV(Y), \;\; f \in L^2(\Omega, dV),$$

where ω_n is the surface area of the unit sphere $\mathbb{S}^{n-1}$. It is easy to show that both operators $\mathcal{N}_\Omega$ and $\mathcal{L}_\Omega$ are self-adjoint and compact. The logarithmic potentials are Hilbert-Schmidt (see [26, Chap. 3]). The necessary and sufficient condition for the Newtonian potentials to be Hilbert-Schmidt is that $n = 3$.

It is well known from potential theory that if plane domain Ω, is inside a disk of radius one, then $\mathcal{L}_\Omega$ is positive (see [39]). In general, $\mathcal{L}_\Omega$ need not be positive. Yet it has at most one negative eigenvalue (see Kac [33]), whereas the Newtonian potentials are positive (see, for instance [50, proposition 2.1]). The spectral properties of potential operators have been extensively studied (see [4], [5], [20], [33] and [55]). Recently Ruzhansky and Suragan in [51] established an isoperimetric inequality for the Schatten $p-$norms of logarithmic potentials over bounded domains of a given area. The result of Ruzhansky and Suragan coincides with the isoperimetric inequalities for the Laplace operator (see [18], [35], [48], and [60]), in the sense that among all domains of the same area, disks and only disks yield the maximum of the largest singular number. For the Laplace operator this fact is known as the Rayleigh-Faber-Krahn or Pólya inequality. In particular Ruzhansky and Suragan in [51] showed that the Schatten $p-$norm is maximized on the equilateral triangle centered at the origin among all triangles of a given area. Moreover, they conjectured the following:

Conjecture 5.2.2 *For any integer p, the regular $n-$gon is the maximizer of Schatten $p-$norms of the logarithmic potential over all convex $n-$gons.*

In the spirit of isoperimetric inequalities for the Laplacian (see [18] and [48]) and also the symmetric geometry of regular polygons, it seems fair to say that the Ruzhansky-Suragan conjecture should have an affirmative answer. We shall show that the conjecture holds for quadrilaterals. Our result agrees with Henrot's result in [32] for the Laplacian eigenvalue problem. To our knowledge, the answer to this question for the Laplacian over $n-$gons with $n \geq 5$, is unknown (see [32]). The two main isoperimetric inequalities appearing in [51] are as follows:

Theorem 5.2.3 *Let Δ be an equilateral triangle and let Ω be a bounded open triangle with $|\Omega| = |\Delta|$. Assume that the logarithmic potential operator is positive for Ω and*

Δ. *Then,*

$$\|\mathcal{L}_\Omega\|_p \le \|\mathcal{L}_\Delta\|_p,$$

for any integer $2 \le p \le \infty$, *where* $\|.\|_\infty$ *is the operator norm.*

Theorem 5.2.4 *Let* D *be a disk and let* Ω *be a bounded open domain with* $|\Omega| = |D|$. *Assume that the logarithmic potential operator is positive for* Ω *and* D. *Then,*

$$\|\mathcal{L}_\Omega\|_p \le \|\mathcal{L}_D\|_p,$$

for any integer $2 \le p \le \infty$, *where* $\|.\|_\infty$ *denotes the operator norm.*

In order to maximize the Hilbert-Schmidt norm of logarithmic potentials among all rectangles of a given area $A > 0$, it suffices to consider open rectangles of the form $\Omega_t = (0,t) \times (0, \frac{A}{t})$ with $t > 0$. Define $F(t) = \|\mathcal{L}_{\Omega_t}\|_2$, on $\mathbb{R}_+$. Then,

$$\begin{aligned} F(t) &= \frac{1}{2\pi}\left\{ \int_{\Omega_t}\int_{\Omega_t} \log^2|z-w|dA_z dA_w \right\}^{1/2} \\ &= \frac{1}{2\pi}\left\{ \int_0^{\frac{A}{t}}\int_0^t\int_0^{\frac{A}{t}}\int_0^t \log^2\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}dx_1dy_1dx_2dy_2 \right\}^{1/2}. \end{aligned}$$

Clearly F is a positive differentiable function on $(0,\infty)$ and it satisfies the functional equation $F(t) = F(\frac{A}{t})$. So $F'(t) = -\frac{A}{t^2}F'(\frac{A}{t})$ which implies $F'(\sqrt{A}) = 0$. It is known that for $A = 1$, the solutions to this functional equation are of the form $g(\log t)$, where g is an even differentiable function. This yields a representation for the Hilbert-Schmidt norm of the logarithmic potentials over rectangles of unit area. For some sample values, see the table below.

Computer calculations (using Wolfram Mathematica ®) shows that $F(1) = \|\mathcal{L}_{\Omega_1}\|_2 \approx 0.1624$ and $F(10) = \|\mathcal{L}_{\Omega_1}\|_2 \approx 0.2148$. Therefore, the Hilbert-Schmidt norm over a 10 by $\frac{1}{10}$ rectangle. For some sample values, see the table below.

We shall show that for any integer $p \ge 3$, among all the rectangular parallelepipeds of a given volume, the cube is a maximizer of the Schatten $p-$norm of the Newto-

a	1/10	1/4	1/2	1	4/3	3	10
$F(a)$	0.2148	0.1536	0.1545	0.1624	0.150	0.1502	0.2148

Table 5.1: Some numerical values of $f(a)$.

nian potential. Essentially, this problem can be viewed as an optimization problem for symmetric functions, subject to symmetric constraints. In general, there is no guarantee that symmetric functions subject to symmetric constraints, have diagonal extrema i.e., of the form $x_1 = x_2 = \cdots = x_n$, and one requires stronger assumptions. We shall appeal to the so-called Purkiss principle.

A function $f(x_1, \cdots, x_n)$ is said to be symmetric if for all σ belonging to the group of permutations on $\{x_1, \cdots, x_n\}$,

$$f(x_1, \cdots, x_n) = f\big(\sigma(x_1), \cdots, \sigma(x_n)\big).$$

Theorem 5.2.5 (The Purkiss Principle) *Let $f(x_1, \cdots, x_n)$ and $g(x_1, \cdots, x_n)$ be two symmetric functions with continuous second derivatives in a neighborhood of a point $P = (r, \cdots, r)$. On the set where $g = g(P)$, the function f will have a local minimum or maximum at P except in degenerate case, i.e. where $\nabla g \equiv 0$.*

For the sake of completeness we give a proof of this principle, but first we need the following lemmas (see [58] for details on the Purkiss principle and proofs):

Lemma 5.2.6 *Suppose that $f(x_1, \cdots, x_n)$ is a symmetric differentiable function. Then, at a point $x_1 = \cdots = x_n = r$, all the partial derivatives $\frac{\partial f}{\partial x_i}$ are equal.*

Lemma 5.2.7 *Suppose $f(x_1, \cdots, x_n)$ is a symmetric twice differentiable function. Then, at a point $x_1 = \cdots = x_n = r$ all the $\frac{\partial^2 f}{\partial x_i^2}$ are equal and all $\frac{\partial^2 f}{\partial x_i \partial x_j}$ for $i \neq j$ are equal.*

Lemma 5.2.8 *Suppose a quadratic form Q is given by*

$$Q(v_1, \cdots, v_n) = \sum_i b v_i v_i + \sum_{i \neq j} c v_i v_j.$$

Then for all v satisfying $\sum v_i = 0$, we have $Q(v) = (c - b) \sum v_i^2$.

Now let us come back to the proof of the Purkiss principle.

Proof. Assume $\nabla g(P) \neq 0$, we recall from Lemma 5.2.6 that $\nabla f(P)$ has the form $\lambda \nabla g(P)$. The second partial derivatives of f and g satisfy the equalities in Lemma 5.2.7 , and hence the terms $\frac{\partial^2 f}{\partial x_i \partial x_j}(P) - \lambda \frac{\partial^2 g}{\partial x_i \partial x_j}(P)$ also satisfy those equalities. Lemma 5.2.6 shows that the vectors v perpendicular to $\nabla g(P)$ are those with $\sum_i v_i = 0$. Lemma 5.2.8 shows that on those v, our quadratic form (if not identically zero) is positive or negative definite. The result follows from the Lagrange multiplier criterion.

■

We should point out that in the assumptions of the Purkiss principle, no specific relationship between f and g is assumed.

Proposition 5.2.9 *Let K be a cube and Ω be a rectangular parallelepiped in $\mathbb{R}^3$ with $|\Omega| = |K|$. Then*

$$\|\mathcal{N}_\Omega\|_p \leq \|\mathcal{N}_K\|_p$$

for any integer $3 \leq p < \infty$.

Proof. Without loss of generality we may confine ourselves to the case where the given area is equal to one. Let us define

$$N(x,y,z) = \int_0^x \int_0^y \int_0^z \int_0^x \int_0^y \int_0^z \int_0^x \int_0^y \int_0^z G(X_1, X_2, X_3) dV_{X_1} dV_{X_2} dV_{X_3},$$

where $G(X_1, X_2, X_3) = \frac{1}{\omega_3^3 \|X_1 - X_2\| \|X_1 - X_3\| \|X_2 - X_3\|}$ with $X_k = (x_k, y_k, z_k)$ for $k = 1, 2, 3$.

For $(x, y, z) \in \mathbb{R}^3_+$ the function N is precisely the Schatten $3-$norm, raised to the power three, of $\mathcal{N}$ over the rectangular parallelepiped $(0, x) \times (0, y) \times (0, z)$. The function N is non-negative, symmetric and has continuous second derivatives (by the fundamental theorem of calculus).

Now consider the function $g(x, y, z) = xyz$. The function g has non-vanishing gradient on $\{(a, b, c)|$ at most one of a, b, c is zero$\}$, is symmetric, and $g(1, 1, 1) = 1$.

By the Purkiss principle, N has a maximum or minimum at $(1,1,1)$. It is easy to show that $(1,1,1)$ cannot be a minimum. Therefore, $(1,1,1)$ is a maximizer.

This result can be easily generalized to all Schatten $p-$norms with integral $p \geq 3$. For that, we let

$$N_p(x,y,z) = \int_0^x \int_0^y \int_0^z \cdots \int_0^x \int_0^y \int_0^z G(X_1, X_2, \cdots, X_p) dV_{X_1} \cdots dV_{X_p},$$

where $G(X_1, \cdots, X_p) = \frac{1}{\omega_3^p \|X_1 - X_2\| \cdots \|X_p - X_1\|}$ with $X_k = (x_k, y_k, z_k)$ for $k = 1, 2, \cdots, p$. We note that $N_p(x,y,z) = \|\mathcal{N}\|_p^p$ over the rectangular parallelepiped $(0,x) \times (0,y) \times (0,z)$. It is obvious that N_p is symmetric and has continuous second derivatives. Therefore, by the Purkiss principle, $\mathcal{N}_p$ has a maximum at $(x,y,z) = (1,1,1)$.

■

Next, we state the logarithmic analogue of the previous result. The proof is similar and we leave it to the reader.

Theorem 5.2.10 *Let K be a square and Ω be a rectangle in $\mathbb{R}^2$ with $|\Omega| = |K|$. Assume that the logarithmic potential operator is positive for Ω and K. Then*

$$\|\mathcal{L}_\Omega\|_p \leq \|\mathcal{L}_K\|_p$$

for any integer $3 \leq p < \infty$.

Remark 5.2.11 *In the previous propositions, the only reason that we require p to be greater that two is to guarantee that the defining $p-$norms are C^2-differentiable. We suspect that the same result holds for $p = 2$.*

Remark 5.2.12 *In both Ruzhansky's work and in this chapter, only Schatten $p-$norms with integral p were taken into account. We are not aware whether Ruzhansky's results in [51] or our results will remain valid if one considers non-integral p. Imposing stronger assumptions, we shall obtain an inequality for all Schatten $p-$norms for real $p \geq 2$.*

We will state and prove a weak version of the Ruzhansky-Suragan inequality for all Schatten $p-$norms with real $p \geq 2$. First we need to recall the celebrated Weyl-Littlewood-Polya inequality which is also known as the Karamata's inequality (see [28, Lemma 3.4]).

Theorem 5.2.13 *Suppose f is a strictly convex function on a segment I and $x_1 \geq x_2 \geq \cdots \geq x_n \geq \cdots$ and $y_1 \geq y_2 \geq \cdots \geq y_n \geq \cdots$ are two sequences of numbers in I such that*

$$\sum_{k=1}^{m} x_k \geq \sum_{k=1}^{m} y_k \quad \textit{for } m = 1, 2, \cdots .$$

Then

$$\sum_{k=1}^{m} f(x_k) \geq \sum_{k=1}^{m} f(y_k) \quad \textit{for } m = 1, 2, \cdots .$$

The equalty

$$\sum_{k=1}^{m} f(x_k) = \sum_{k=1}^{m} f(y_k) \quad \textit{for } m = 1, 2, \cdots$$

holds if and only if $x_k = y_k$ for $k = 1, 2, \cdots$.

Let Ω_0 be a bounded domain. By $\mathcal{M}_{\Omega_0}$ we denote the class of all bounded domains Ω with the following property:

$$\sum_{n=1}^{m} s_n^2(\mathcal{L}_{\Omega}) \leq \sum_{n=1}^{m} s_n^2(\mathcal{L}_{\Omega_0}) \text{ for } m = 1, 2, \cdots .$$

Proposition 5.2.14 *Let Ω_0 be a bounded domain, then*

$$\|\mathcal{L}_{\Omega}\|_p \leq \|\mathcal{L}_{\Omega_0}\|_p$$

for any $\Omega \in \mathcal{M}_D$ and any real number $2 \leq p < \infty$.

Proof. Assume $p > 2$ is a fixed real number. Take $\Omega \in \mathcal{M}_{\Omega_0}$ and denote the singular numbers of $\mathcal{L}_{\Omega}$ and $\mathcal{L}_{\Omega_0}$ by $\{a_k\}_1^{\infty}$ and $\{b_k\}_1^{\infty}$ respectively. It follows from the definition of $\mathcal{M}_{\Omega_0}$ that

$$\sum_{k=1}^{m} a_k^2 \leq \sum_{k=1}^{m} b_k^2 \quad , m = 1, 2, \cdots .$$

The function F defined by $F(x) = x^{p/2}$ is strictly convex on $\mathbb{R}_+$. By Karamata's inequality

$$\sum_{k=1}^{\infty} F(a_k^2) \leq \sum_{k=1}^{\infty} F(b_k^2),$$

which implies

$$\sum_{k=1}^{\infty} a_k^p \leq \sum_{k=1}^{\infty} b_k^p.$$

The quantity on the left hand side is precisely $\|\mathcal{L}_\Omega\|_p^p$ and the one on the right hand side is $\|\mathcal{L}_{\Omega_0}\|_p^p$. The proof is complete.

■

Proposition 5.2.15 *Let Ω_0 be as above and $\Omega \in \mathcal{M}_{\Omega_0}$ so that*

$$\|\mathcal{L}_\Omega\|_p = \|\mathcal{L}_{\Omega_0}\|_p$$

for some number $p \in [2, \infty)$. Then Ω_0 and Ω are isospectral.

Proof. This proof follows immediately from the equality case of the Karamata's inequality.

■

6 Summary

6.1 Review of the Results

We showed that the singular numbers of single layer potentials on smooth curves asymptotically behave like $\frac{1}{n}$. For Jordan curves, as long as they contain a smooth piece, the resulting single layer potentials are never trace-class. We provide upper bounds for the operator and Hilbert-Schmidt norms of single layer potentials on smooth and chord-arc curves. Regarding the injectivity of single layer potentials on planar curves, we proved that among all operators on dilations of a curve, only one yields a non-injective single layer potential. A criterion for injectivity of single layer potentials over ellipses was given. We established isoperimetric inequalities for Schatten $p-$norms of logarithmic potentials over quadrilaterals, and its analogue for Newtonian potentials over parallelepipeds.

6.2 Open Problems

1. Ruzhanksy and Suragan in [51] provide an isoperimetric inequality for the logarithmic potentials defined on triangles, stating that of all triangles of the same area, the equilateral triangle is the maximizer of Schatten $p-$norms of the corresponding logarithmic potentials. We generalized their work to quadrilaterals, by showing that among all quadrilaterals of the same area, the square is the maximizer of Schatten $p-$norms of logarithmic potentials (see [62]). We suspect that the maximizer of Schatten $p-$norms of logarithmic potentials among all polygons must be the regular polygon. Neither Suragan's method, which is based upon Steiner symmetrization, nor our method, due to the complexity of constraints, are fruitful. In order to tackle this problem, a different machinery is needed. This problem remains open.

2. For the double layer potentials, it is known that the circle gives the maximum Hilbert-Schmidt norm among all the simple smooth curves of a fixed perimeter (see [43]). In the case of the single layer operator, the answer is not known. We can go further and pose the same question for all Schatten $p-$norms (see [27] and [46]).

3. Suppose the single layer operators associated with two "different" planar curves have identical spectrum. It is not known whether this property forces the curves to be the same. One potential area of future research is to investigate if there exist different curves (or surfaces) for which the corresponding single layer operators have the same spectra.

4. One might wonder if there are two "different" smooth Jordan curves having isospectral single layer potentials. For non-smooth curves, can we detect the corners or cusps of a given Jordan curve by studying the singular numbers of single layer potentials?

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