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Global Attractors and Random Attractors of Reaction-Diffusion Systems

Junyi Tu
University of South Florida, junyi@mail.usf.edu

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by

Junyi Tu

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Department of Mathematics & Statistics
College of Arts and Sciences
University of South Florida

Major Professor: Yuncheng You, Ph.D.
Mohamed Elhamdadi, Ph.D.
Sherwin Kouchekian, Ph.D.
Marcus McWaters, Ph.D.

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Dedication

To my family and teachers.

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Abstract

The dissertation studies about the existence of three different types of attractors of three multi-component reaction-diffusion systems. These reaction-diffusion systems play important role in both chemical kinetics and biological pattern formation in the fast-growing area of mathematical biology.

In Chapter 2, we prove the existence of a global attractor and an exponential attractor for the solution semiflow of a reaction-diffusion system called Boissonade equations in the L^2 phase space. We show that the global attractor is an (H, E) global attractor with the L^∞ and H^2 regularity and that the Hausdorff dimension and the fractal dimension of the global attractor are finite. The existence of exponential attractor is also shown. The upper-semicontinuity of the global attractors with respect to the reverse reaction rate coefficient is proved.

In Chapter 3, the existence of a pullback attractor for non-autonomous reversible Selkov equations in the product L^2 phase space is proved. The method of grouping and rescaling estimation is used to prove that the L^4-norm and L^6-norm of solution trajectories are asymptotical bounded. The new feature of pinpointing a middle time in the process turns out to be crucial to deal with the challenge in proving pullback asymptotic compactness of this typical non-autonomous reaction-diffusion system.

In Chapter 4, asymptotical dynamics of stochastic Brusselator equations with multiplicative noise is investigated. The existence of a random attractor is proved via the exponential transformation of Ornstein-Uhlenbeck process and some challenging estimates. The proof of pullback asymptotic compactness here is more rigorous through the bootstrap pullback estimation than a non-dynamical substitution of Brownian motion by its backward translation. It is also shown that the random attractor has the L^2 to H^1 attracting regularity by the flattening method.

Chapter 1
Introduction

Dynamical system is a ubiquitous and rapidly expanding area in mathematics since its inception by the founding work of Henri Pioncaré (1854-1912), Alexander Mikhailovich Lyapunov (1857-1918) and George David Birkhoff (1884-1944). The well known concepts in dynamical systems include stability, Lyapunov function, Lyapunov exponents and Birkhoff Ergodic Theorem, to name a few. Dynamical system in its infancy was called the qualitative theory of ordinary differential equations (ODEs). Along with the development of functional analysis and partial differential equations (PDEs), the well-posed problem of parabolic or hyperbolic partial differential equations formulated as abstract evolutionary equations generates a flow (or semiflow) of solution trajectories in Banach spaces. It then gives rise to the rich theory of an infinite dimensional dynamical system. The principal concept depicting the longtime dynamics of infinite dimensional dynamical system is global attractor of a semiflow.

The first construction of a global attractor for some dissipative PDEs was in the seminal work of Ladyzhenskaya [28] in 1972, then the theory of global attractors was developed by Foias and Temam [20], Babin and Vishik [3] and Hale [22]. Generally speaking, global attractor is a comparatively smaller and coherent subset in phase space and it captures all the important permanent structure of the concerned infinite dimensional dynamical systems, including all steady states, periodic orbits, homoclinic and heteroclinic orbits, and unstable manifolds. In particular, the fractal dimension of an attractor estimates the number of degrees of freedom for an infinite dimensional dynamical system. The rigorous definitions of global attractor and relevant concepts are given in Chapter 2.

Global attractor is an instrumental tool in the theory of infinite dimensional dynamical systems to characterize the long-term and asymptotical behavior of the underlying PDEs, delay differential equations or lattice equations, such as Navier-Stoks equations, reaction-diffusion systems and nonlinear wave equations. The existence theory of global attractors for a series of multi-component reaction-diffusion systems such as the Brusselator equations, Gray-Scott equations, Schnackenberg equations and Oregonator equations has been established recently by Y. You [42, 43, 45, 47].

The theory of non-autonomous dynamical systems attracts much attention as it synergizes the developments on time-dependent differential equations, control systems, random and stochastic differential equations. In the study of the asymptotical behavior of non-autonomous dynamical system, the concept of pullback attractor was introduced by H. Crauel, F. Flandoli [15] and B. Schmalfuß [35] to study the dynamics of certain stochastic differential equations within the framework of the random dynamical systems. The general theory of pullback attractor could be phrased in the language of cocycles, which will be defined in Chapter 4. The term pullback attractor became widely accepted after its use by P. Kloeden [26]. For more discussion about non-autonomous dynamical system, see the recent books by P. Kloeden and M. Rasmussen [27] and by Alexandre Carvalho, José A. Langa and James Robinson [11].

Global attractor of an autonomous dynamical system is invariant with respect to time translation. But for time-dependent systems, the starting time is as crucial as the elapsed time. Thus for non-autonomous dynamical systems we could generalize the concept in two directions. One is forward attraction, which is less applicable as shown in [11]. The other is pullback attraction, which means attracting by freezing final time while pulling back the initial time as early as possible. For more comparison of these concepts, readers are referred to [11].

Since Itô's stochastic integral was invented in 1944 [23] and stochastic analysis was developed in 1970s, it is natural to ask how to generate a random dynamical system from stochastic differential equations. The theory of random dynamical systems was developed mainly by Ludwig Arnold and his Bremen Group around 1980s. As L. Arnold pointed out in [1], one of the historical gates in the development of the theory of stochastic differential equations was the discovery that their solution is a cocycle over an ergodic dynamical system which models randomness, i.e., a random dynamical system. Since L. Arnord's classical book [2] appeared in 1998, there have been rapid progresses in various aspects of random dynamical system in the last two decades, see, e.g., [4, 5, 13, 25, 41, 48, 49]. One important and fruitful aspect is the theory and applications of random attractors generated by stochastic/random ordinary/partial differential equations.

In a nutshell, random dynamical system is defined in an environment described by another metric dynamical system modeling the noise, usually the white noise. The concept of random attractor was first introduced by H. Crauel and F. Flandoli [14], B. Schmalfuß [35] to study the asymptotic behaviors of Navier-Stokes equations with multiplicative and additive white noise. For various stochastic PDEs there have been a great deal of results addressing the random attractors, see, e.g., [5, 34, 41, 48, 49].

The main contributions of the dissertation can be summarized as follows. In Chapter 2, we analyses a new reaction-diffusion system called Boissonade equations from biology, and show the existence of a global attractor and an exponential attractor for the solution semiflow in the L^2 phase space. The challenge includes the proof of the H^2 regularity and the proof of the upper-semicontinuity of the global attractors with respect to the reverse reaction rate coefficient. These results could be useful in explaining the stability and robustness of the underlying models in the lab of biology.

In Chapter 3, the existence of the pullback attractor for non-autonomous reversible Selkov equations in the product L^2 phase space is proved. The method of grouping and re-scaling estimation is used to prove that the L^4-norm and L^6-norm of solution trajectories are asymptotical bounded. The new feature of pinpointing a middle time in the process turns out to be crucial to deal with the challenge in proving pullback asymptotic compactness of this typical non-autonomous reaction-diffusion system.

In Chapter 4, the asymptotical dynamics of stochastic Brusselator equations with multiplicative white noise is investigated. The existence of random attractor is proved via the exponential transformation of Ornstein-Uhlenbeck process and some challenging estimates. The proof of pullback asymptotic compactness here is more rigorous through the bootstrap pullback estimation than a non-dynamical substitution of Brownian motion by its backward translation. There are some novelties in this chapter such as:

First, it is new to use the exponential transformation of Ornstein-Uhlenbeck process to convert the stochastic terms to random coefficients for reaction-diffusion equations in the multiplicative white noise case.

Second, we use the rigorous pullback estimates other than the shortcut many authors did by the non-dynamical substitute of noise shift. The bootstrap method is applied in the light of the decomposition method used initially for deterministic Brusselator equations in [42].

Next we introduce some common notation in this dissertation.

Let $\Gamma \subset \mathbb{R}^n (n \leq 3)$ be a bounded domain with a locally Lipschitz continuous boundary. Define the product Hilbert spaces as follows,

$$H = L^2(\Gamma) \times L^2(\Gamma), \quad E = H_0^1(\Gamma) \times H_0^1(\Gamma), \quad \Pi = H_0^1(\Gamma) \cap H^2(\Gamma) \times H_0^1(\Gamma) \cap H^2(\Gamma).$$

These product spaces are the phase spaces of different regularity for the component functions $u(t, \cdot), v(t, \cdot)$. We denote the norm and inner-product of H or any component space $L^2(\Gamma)$ by $\| \cdot \|$ and $\langle \cdot, \cdot \rangle$, respectively. The norm of $L^p(\Gamma)$ or the product space $\mathbb{L}^p(\Gamma) = L^p(\Gamma) \times L^p(\Gamma)$ will be denoted by $\| \cdot \|_{L^p}$, if $p \neq 2$. By the Poincar$\acute{e}$ inequality and the homogeneous Dirichlet boundary condition, there is a constant $\gamma > 0$ such that

$$\|\nabla \xi\|^2 \geq \gamma \|\xi\|^2, \quad \text{for } \xi \in H_0^1(\Gamma) \text{ or } E, \tag{1.0.1}$$

and we take $\|\nabla \xi\|$ to be the equivalent norm $\|\xi\|_E$ or $\|\xi\|_{H_0^1(\Gamma)}$. We use $| \cdot |$ to denote the Lebesgue measure of sets as well as the absolute value and the vector norm in Euclidean spaces.

By the Lumer-Phillips theorem and the analytic semigroup generation theorem [37], the linear operator

$$A = \begin{pmatrix} d_1 \Delta & 0 \\ 0 & d_2 \Delta \end{pmatrix} : D(A)(= \Pi) \longrightarrow H \tag{1.0.2}$$

where d_1, d_2 are positive constants and Δ is the Laplacian operator, is the generator of an analytic and exponentially stable C_0-semigroup on the Hilbert space H, which will be denoted by $\{e^{At}\}_{t \geq 0}$.

Consider an *Initial Value Problem* for an abstract nonlinear evolutionary equation of the form

$$\begin{aligned} &\partial_t u = Au + F(u,t), \quad t > t_0, \\ &u(t_0) = u_0 \in H. \end{aligned} \tag{1.0.3}$$

on the Hilbert space H. Let $I = [t_0, t_0 + \tau)$ be a time interval on $\mathbb{R}$.

A pair (u, I) is said to be a *mild solution* of (1.0.3) in the space H on I, provided that $u \in C(I, H)$ and u is a solution of the integral equation

$$u(t) = e^{A(t-t_0)}u_0 + \int_{t_0}^{t} e^{A(t-s)}F(u(s), s)ds, \quad t \in I \tag{1.0.4}$$

A pair (u, I) is said to be a *strong solution* of (1.0.3) in the space H on I, provided that $u \in C(I, H)$, $u(t_0) = u_0$, u is strongly differentiable almost everywhere with $\partial_t u$ and Au in $L^1_{loc}(I, H)$, and u satisfies the differential equation

$$\partial_t u(t) = Au(t) + F(u(t), t), \quad \text{a.e.} \quad \text{in } H, \quad \text{on } I.$$

Chapter 2
Global attractor of Boissonade system

2.1 Introduction

Turing pattern is mathematically demonstrated by systems of partial differential equations called reaction diffusion systems [30, 39]. The chemical reaction kinetics is controlled by two antagonistic feedback loops, an activation process and an inhibitory process. The experimental evidence of Turing pattern was observed in the so-called chlorite-iodine-malonic acid (CIMA) reaction after almost 40 years since Turing's paper was published in 1952 [39]. In experimentation, it is a 3D reactor and the Turing structures are found to form one layer after the other, there is a single layer called "monolayer" beyond the pattern onset [17]. In order to clarify the relation between the genuine homogeneous 2D systems and the monolayers, V. Dufiet and J. Boissonade studied the selection of patterns close to onset for the same model in both geometries. They introduced a simple appropriate reaction-diffusion model that exhibits Turing pattern in [17]. In this chapter, we explore the long-time global dynamical behavior of this model, which we call the Boissonade equations,

$$\frac{\partial u}{\partial t} = d_1 \Delta u + u - \alpha v + \gamma uv - u^3, \tag{2.1.1}$$

$$\frac{\partial v}{\partial t} = d_2 \Delta v + u - \beta v, \tag{2.1.2}$$

for $(t, x) \in (0, \infty) \times \Gamma$, where $\Gamma \subset \mathbb{R}^n (n \leq 3)$ is a bounded domain with a locally Lipschitz continuous boundary, and the coefficients $d_i's, (i = 1, 2), \alpha, \beta, \gamma$ are positive constants. Given the homogeneous Dirichlet boundary condition

$$u(t, x) = v(t, x) = 0, \quad t > 0, \; x \in \partial\Gamma, \tag{2.1.3}$$

and an initial condition

$$u(0,x) = u_0(x),\ v(0,x) = v_0(x),\ x \in \Gamma, \tag{2.1.4}$$

we shall study the long-time and asymptotical dynamics of semiflow of the weak solutions to this initial-boundary value problem (2.1.1) – (2.1.4). Specifically, we prove the existence of a global attractor and show some important properties of it.

Note that the Boissonade system is different from the FitzHugh-Nagumo equations [12] in the quadratic term uv instead of u^2. This difference makes the nonlinear term of Boissanade system disqualify the dissipative condition, on the other hand it is also the main merit of our work.

First we formulate the Boissonade system into an evolutionary equation in the abstract functional space. By Sobolev embedding theorem, $H_0^1(\Gamma) \hookrightarrow L^6(\Gamma)$ is a continuous embedding for $n \le 3$. Invoking the generalized Hölder inequality, we have

$$\|uv\| \le |\Gamma|^{\frac{1}{6}} \|u\|_{L^6} \|v\|_{L^6}, \quad \|u^3\| \le \|u\|_{L^6}^3, \quad \text{for } u, v \in L^6(\Gamma).$$

From these facts we can verify that the nonlinear mapping

$$f(g) = \begin{pmatrix} u - \alpha v + \gamma uv - u^3 \\ u - \beta v \end{pmatrix} : E \longrightarrow H, \tag{2.1.5}$$

where $g = (u, v)$, is a locally Lipschitz continuous mapping defined on E. Thus the initial-boundary value problem (2.1.1)–(2.1.4) of this Boissonade system is formulated into an initial value problem in H:

$$\begin{aligned} &\frac{dg}{dt} = Ag + f(g), \quad t > 0. \\ &g(0) = g_0 = (u_0, v_0) \in H. \end{aligned} \tag{2.1.6}$$

We consider the weak solutions to the initial value problem (2.1.6), as defined below.

Definition 2.1.1. A function $g(t,x), (t,x) \in [0,\tau] \times \Gamma$, is called a weak solution to the IVP of the parabolic evolutionary equation (2.1.6), if the following two conditions are satisfied:

(i) $\frac{d}{dt}(g, \eta) = (Ag, \eta) + (f(g), \eta)$ is satisfied for a.e. $t \in [0,\tau]$ and any $\eta \in E$;

(ii) $g(t,\cdot) \in L^2(0,\tau;E) \cap C_w([0,\tau];H)$ such that $g(0) = g_0$,

where $C_w([0,\tau];H)$ denotes the space of weakly continuous functions on $[0,\tau]$ valued in H, and $(\cdot,\cdot)$ stands for the (E^*,E) dual product.

Next we recall some basic concepts and facts in the theory of infinite dimensional dynamical systems, cf. [12, 37, 38].

Definition 2.1.2 (Semiflow)**.** Let X be a Banach space, a semiflow in X is a family of maps $S(t) : t \geq 0$ such that the following holds:

1. $S(0) = Id$,

2. $S(t+s) = S(t)S(s)$, for all $t, s \in \mathbb{R}^+$,

3. $(t,x) \mapsto S(t)x$ is continuous with respect to t and x, where $t \geq 0, x \in X$.

Definition 2.1.3 (Absorbing Set)**.** Let $\{S(t)\}_{t\geq 0}$ be a semiflow on a Banach space X. A bounded subset B_0 of X is called an *absorbing set in* X if, for any bounded subset $B \subset X$, there is some finite time $t_0 \geq 0$ depending on B such that $S(t)B \subset B_0$ for all $t > t_0$.

Definition 2.1.4 (Asymptotic Compactness)**.** A semiflow $\{S(t)\}_{t\geq 0}$ on a Banach space X is called *asymptotically compact in* X if for any bounded sequences $\{x_n\}$ in X and $\{t_n\} \subset (0,\infty)$ with $t_n \to \infty$, there exist subsequences $\{x_{n_k}\}$ of $\{x_n\}$ and $\{t_{n_k}\}$ of $\{t_n\}$, such that $\lim_{k\to\infty} S(t_{n_k})x_{n_k}$ exists in X.

Definition 2.1.5 (Global Attractor)**.** Let $\{S(t)\}_{t\geq 0}$ be a semiflow on a Banach space X. A subset $\mathscr{A}$ of X is called a *global attractor in* X for this semiflow, if the following conditions are satisfied:

(i) $\mathscr{A}$ is a nonempty, compact, and invariant set in the sense that

$$S(t)\mathscr{A} = \mathscr{A} \quad \text{for any} \ \ t \geq 0.$$

(ii) $\mathscr{A}$ attracts any bounded set B of X in terms of the Hausdorff distance, i.e.

$$\text{dist}(S(t)B, \mathscr{A}) = \sup_{x\in B} \inf_{y\in\mathscr{A}} \|S(t)x - y\|_X \to 0, \quad \text{as} \ \ t \to \infty.$$

Definition 2.1.6 (($\mathbb{X}, \mathbb{Y}$) Global Attractor)**.** Let $\{\Sigma(t)\}_{t\geq 0}$ be a semiflow on a Banach space X and let Y be a densely and compactly imbedded subspace of X. A subset $\mathcal{A}$ of Y is called an (X, Y) global attractor for this semiflow if $\mathcal{A}$ has the following properties,

(i) $\mathcal{A}$ is a nonempty, compact, and invariant set in Y.

(ii) $\mathcal{A}$ attracts any bounded set $B \subset X$ with respect to the Y-norm, namely, there is a time $\tau = \tau(B)$ such that $\Sigma(t)B \subset Y$ for $t > \tau$ and $dist_Y(\Sigma(t)B, \mathcal{A}) \to 0$, as $t \to \infty$.

Definition 2.1.7 (Upper Semicontinuity)**.** Suppose that there is a family of semiflows $\{\{S_\lambda(t)\}_{t\geq 0}\}_{\lambda\in\Lambda}$ on a Banach space X , where $\Lambda \subset \mathbb{R}$ is an interval, and that there exists a global attractor $\mathscr{A}_\lambda$ in X for each semiflow$\{S_\lambda(t)\}_{t\geq 0}, \lambda \in \Lambda$. If $\lambda_0 \in \Lambda$ and

$$\text{dist}(\mathscr{A}_\lambda, \mathscr{A}_{\lambda_0}) \to 0, \quad \text{as } \lambda \to \lambda_0 \text{ in } \Lambda,$$

then we say that the family of global attractors $\{\mathcal{A}_\lambda\}_{\lambda\in\Lambda}$ is upper-semicontinuous at λ_0 , or that $\mathcal{A}_\lambda$ has the robustness at λ_0.

The following proposition states the basic result on the existence of a global attractor for a semiflow, cf. [33, 37, 38].

Proposition 2.1.8. Let $\{S(t)\}_{t\geq 0}$ be a semiflow on a Banach space X. If the following conditions are satisfied:

(i) $\{S(t)\}_{t\geq 0}$ has an absorbing set B_0 in X, and

(ii) $\{S(t)\}_{t\geq 0}$ is asymptotically compact in X,

then there exists a global attractor $\mathscr{A}$ in X for this semiflow, which is given by

$$\mathscr{A} = \omega(B_0) \stackrel{\text{def}}{=} \bigcap_{\tau\geq 0} \text{Cl}_X \bigcup_{t\geq\tau} (S(t)B_0).$$

Note that a global attractor $\mathscr{A}$ does not depends on the particular choice of absorbing set B_0.

First we state the main results of this chapter. We *emphasize* that these results are established unconditionally, neither assuming initial data or solutions are nonnegative, nor imposing any restriction on any positive parameters involved in the equations (2.1.1)–(2.1.2).

First Main Theorem. For any positive parameters $d_1, d_2, \alpha, \beta, \gamma$, there exists a global attractor $\mathscr{A}$ in the phase space H for the semiflow $\{S(t)\}_{t\geq 0}$ of the weak solutions generated by the Boissonade evolutionary equation (2.1.6). Moreover the following properties hold:

(i) $\mathcal{A}$ is a bounded set in $\Pi \cap \mathbb{L}^\infty(\Gamma)$, and it is an (H, E) global attractor.

(ii) $\mathcal{A}$ has a finite Hausdorff dimension and a finite fractal dimension.

(iii) There exists an exponential attractor for this semiflow $\{S(t)\}_{t\geq 0}$.

Second Main Theorem. Given any positive parameters d_1, d_2, α, β, the family of global attractors $\mathscr{A}_\gamma, \gamma \geq 0$, has the upper semi-continuity in E with respect to $\gamma \geq 0$ as it converges to zero, that is

$$\text{dist}_E(\mathscr{A}_\gamma, \mathscr{A}_0) \to 0, \quad \text{as} \quad \gamma \to 0^+.$$

The rest of the chapter is organized as follows. In Section 2 we shall prove the global existence of the weak solutions of the Boissonade evolutionary equation (2.1.6). In Section 3 we show the absorbing property. In Section 4 we prove the asymptotical compactness of this solution semiflow. In Section 5 we show the existence of a global attractor for this semiflow and its properties as being the (H, E) global attractor and the $\mathbb{L}^\infty$ and H^2 regularity. We also prove that the global attractor has finite Hausdorff dimension and fractal dimension. In Section 6 we show the existence of exponential attractor. In Section 7 and 8 we prove the upper-semicontinuity of the global attractors with respect to the reverse reaction rate coefficient as it tends to zero.

2.2 Global Existence of Weak Solutions

We write $u(t,x), v(t,x)$ simply as $u(t), v(t)$, or even as u, v, etc.

Lemma 2.2.1. For any given initial data $g_0 \in H$, there exists a unique local weak solution $g(t) = (u(t), v(t))$, $t \in [0, \tau]$ for some $\tau > 0$, of the IVP of the Boissonade evolutionary equation (2.1.6), which satisfies

$$g \in C([0,\tau]; H) \cap C^1((0,\tau); H) \cap L^2(0,\tau; E). \tag{2.2.1}$$

The proof of this lemma is by using Galerkin approximations and the Lions-Magenes type of compactness treatment involving the following *a priori* estimates for the weak solution. The process is standard and can be found in [46].

Lemma 2.2.2. For any initial data $g_0 = (u_0, v_0) \in H$, there exists a unique global weak solution $g(t) = (u(t), v(t))$, $t \in [0, \infty)$, of the IVP of the Boissonade evolutionary equation (2.1.6) and it becomes a strong solution on the time interval $(0, \infty)$.

Proof. Taking the inner products $\langle (2.1.1), u(t)\rangle$ and $\langle (2.1.2), cv(t)\rangle$ where c is a positive constant to be determined later, and summing them up, by Young's inequality we get

$$\begin{aligned}
&\frac{1}{2}\left(\frac{d}{dt}\|u\|^2 + c\frac{d}{dt}\|v\|^2\right) + d_1\|\nabla u\|^2 + cd_2\|\nabla v\|^2 \\
=&\int_\Gamma \left(u^2 + (c-\alpha)uv + \gamma u^2 v - u^4 - a\beta v^2\right) dx \\
\le&\int_\Gamma \left(u^2 + \frac{(c-\alpha)^2 u^2}{c\beta} + \frac{c\beta}{4}v^2 + \frac{u^4}{4} + \gamma^2 v^2 - u^4 - c\beta v^2\right) dx \\
=&\int_\Gamma \left((1 + \frac{(c-\alpha)^2}{c\beta})u^2 - \frac{3}{4}u^4 - (c\beta - \frac{c\beta}{4} - \gamma^2)v^2\right) dx \\
\le&(1 + \frac{(c-\alpha)^2}{c\beta})^2|\Gamma| - \int_\Gamma \frac{1}{2}u^4 dx - \frac{\gamma^2}{2}\int_\Gamma v^2 dx.
\end{aligned} \tag{2.2.2}$$

where we take $c = \frac{2\gamma^2}{\beta}$.

It follows that

$$\begin{aligned}
&\frac{1}{2}\left(\frac{d}{dt}\|u\|^2 + c\frac{d}{dt}\|v\|^2\right) + \frac{1}{2}\|u\|^2 + \frac{\gamma^2}{2}\|v\|^2 + d_1\|\nabla u\|^2 + cd_2\|\nabla v\|^2 \\
\le&\frac{1}{2}\left(\frac{d}{dt}\|u\|^2 + c\frac{d}{dt}\|v\|^2\right) + \int_\Gamma \frac{1}{2}u^4 dx + \frac{1}{8}|\Gamma| + \frac{\gamma^2}{2}\int_\Gamma v^2 dx + d_1\|\nabla u\|^2 + cd_2\|\nabla v\|^2 \\
\le&(1 + \frac{(c-\alpha)^2}{c\beta})^2|\Gamma| + \frac{1}{8}|\Gamma|.
\end{aligned} \tag{2.2.3}$$

Let $b_1 = \min\{1, \frac{\gamma^2}{c}\}$, $M_1 = 2(1 + \frac{(c-\alpha)^2}{c\beta})^2|\Gamma| + \frac{1}{4}|\Gamma|$, we end up with

$$\frac{d}{dt}\left(\|u\|^2 + c\|v\|^2\right) + b_1\left(\|u\|^2 + c\|v\|^2\right) \le M_1 \tag{2.2.4}$$

Thanks to the Gronwall's inequality, we have

$$\|u\|^2 + c\|v\|^2 \le e^{-b_1 t}(\|u_0\|^2 + c\|v_0\|^2) + \frac{M_1}{b_1} \tag{2.2.5}$$

Let $c_1 = \min\{1, c\}$, $c_2 = \max\{1, c\}$, we get

$$\|u\|^2 + \|v\|^2 \le \frac{c_2}{c_1} e^{-b_1 t}(\|u_0\|^2 + \|v_0\|^2) + \frac{M_1}{b_1 c_1} \tag{2.2.6}$$

From (2.2.6) we conclude that for any initial data $g_0 \in H$, the weak solution $g(t) = (u(t), v(t))$ is uniformly bounded in $[0, T_{max})$ if T_{max} is finite. Therefore, for any $g_0 \in H$, the weak solution $g(t)$ of (2.1.6) will never blow up in H at any finite time. Moreover, by the regularity of weak solution (2.2.1), any weak solution turns out to be a strong solution on the time interval $(0, \infty)$. The proof is completed. $\square$

By the global existence and uniqueness of the weak solutions and their continuous dependence on initial data, the family of all the global weak solutions $\{g(t; g_0) : t \ge 0, g_0 \in H\}$ defines a semiflow on H,

$$S(t) : g_0 \mapsto g(t; g_0), \quad g_0 \in H,\ t \ge 0,$$

which will be called the *Boissonade semiflow* associated with the Boissonade evolutionary equation (2.1.6).

Lemma 2.2.3. There exists a constant $K_1 > 0$, such that the set

$$B_0 = \left\{ g \in H : \|g\|^2 \le K_1 \right\} \tag{2.2.7}$$

is an absorbing set in H for the Boissonade semiflow $\{S(t)\}_{t \ge 0}$.

Proof. From (2.2.6) we obtain

$$\limsup_{t \to \infty} (\|u(t)\|^2 + \|v(t)\|^2) < K_1 = \frac{2M_1}{b_1 c_1}, \tag{2.2.8}$$

For any given bounded set $B = \{g \in H : \|g\| \le R\}$ in H, there exists a finite time $T_0 = \frac{1}{b_1} \ln \frac{b_1 c_2 R^2}{M_1}$ such that $\|u(t)\|^2 + \|v(t)\|^2 < K_1$ for any $g_0 \in B$ and all $t > T_0$. $\square$

Note that for any $t \ge T_0$, integration of (2.2.3) implies that

$$\begin{aligned} &\int_t^{t+1} 2(d_1 \|\nabla u(s)\|^2 ds + c d_2 \|\nabla v(s)\|^2) ds \\ &\le M_1 + \|u(t)\|^2 + c\|v(t)\|^2 \\ &\le M_1 + \max\{1, c\} K_1, \end{aligned} \tag{2.2.9}$$

which is useful later.

Next we show the absorbing properties of the (u, v) components of the Boissonade semiflow in the product Banach spaces $[L^{2p}(\Gamma)]^2$, for integers $p = 2, 3$.

Lemma 2.2.4. For $p = 2, 3$, there exists a positive constant K_p such that the absorbing inequality

$$\limsup_{t\to\infty} \|(u(t), v(t))\|_{L^{2p}}^{2p} < K_p \tag{2.2.10}$$

is satisfied by the (u, v) components of the Boissonade semiflow $\{S(t)\}_{t\geq 0}$ for any initial data $g_0 \in H$.

Proof. According to the solution property (2.2.1) with $T_{max} = \infty$ for all solutions, for any given initial status $g_0 \in H$ there exists a time $t_0 \in (0, 1)$ such that

$$S(t_0)g_0 \in E = [H_0^1(\Gamma)]^2 \hookrightarrow \mathbb{L}^6(\Gamma) \hookrightarrow \mathbb{L}^4(\Gamma). \tag{2.2.11}$$

Then the weak solution $g(t) = S(t)g_0$ becomes a strong solution on $[t_0, \infty)$ and satisfies

$$S(\cdot)g_0 \in C([t_0, \infty); E) \cap L^2(t_0, \infty; \Pi) \subset C([t_0, \infty); \mathbb{L}^6(\Gamma)) \subset C([t_0, \infty); \mathbb{L}^4(\Gamma)), \tag{2.2.12}$$

for $n \leq 3$. Based on this observation, without loss of generality, we can simply *assume* that $g_0 \in \mathbb{L}^6(\Gamma)$ for the purpose of studying the long-term dynamics. Then by the bootstrap argument, we can *assume* that $g_0 \in \Pi \subset \mathbb{L}^8(\Gamma)$ so that $S(t)g_0 \in \Pi \subset \mathbb{L}^8(\Gamma), t \geq 0$.

For $p = 2, 3$, we take the L^2 inner-product $\langle(2.1.1), u^{2p-1}\rangle$ and $\langle(2.1.2), v^{2p-1}\rangle$ and sum them up, by Young's inequality we obtain

$$
\begin{aligned}
&\frac{1}{2p}\frac{d}{dt}\left(\|u(t)\|_{L^{2p}}^{2p}+\|v(t)\|_{L^{2p}}^{2p}\right)\\
&+(2p-1)d_1\|u(t)^{2p-2}\nabla u(t)\|^2+(2p-1)d_2\|v(t)^{2p-2}\nabla v(t)\|^2\\
=&\int_\Gamma\left(u^{2p}+uv^{2p-1}-\alpha vu^{2p-1}+\gamma u^{2p}v-u^{2p+2}-\beta v^{2p}\right)dx\\
\le&\int_\Gamma\left(u^{2p}+\frac{1}{2p}(\frac{2(2p-1)}{p\beta})^{2p-1}u^{2p}+\frac{\beta}{4}v^{2p}+\frac{\beta}{4}v^{2p}+\frac{2p-1}{2p}(\frac{2}{p\beta})^{\frac{1}{2p-1}}\alpha^{\frac{2p}{2p-1}}u^{2p}\right)dx\\
&+\int_\Gamma\left(\frac{1}{p+1}(\frac{4p}{p+1})^p\gamma^{p+1}v^{p+1}+\frac{1}{4}u^{2p+2}-u^{2p+2}-\beta v^{2p}\right)dx\\
=&\int_\Gamma\left(A_1u^{2p}+A_2v^{p+1}-\frac{3}{4}u^{2p+2}-\frac{\beta}{2}v^{2p}\right)dx\\
\le&\int_\Gamma\left(\frac{1}{p+1}(\frac{4p}{p+1})^pA_1^{p+1}+\frac{1}{4}u^{2p+2}+\frac{\beta}{4}v^{2p}\right)dx\\
&+\int_\Gamma\left(\frac{p-1}{2p}(\frac{4(p+1)}{2p\beta})^{\frac{p+1}{p-1}}A_2^{\frac{2p}{p-1}}-\frac{3}{4}u^{2p+2}-\frac{\beta}{2}v^{2p}\right)dx\\
=&\left(\frac{1}{p+1}(\frac{4p}{p+1})^pA_1^{p+1}+\frac{p-1}{2p}(\frac{4(p+1)}{2p\beta})^{\frac{p+1}{p-1}}A_2^{\frac{2p}{p-1}}\right)|\Gamma|-\frac{1}{2}\int_\Gamma u^{2p+2}dx-\frac{\beta}{4}\int_\Gamma v^{2p}dx,
\end{aligned}
\tag{2.2.13}
$$

where

$$
A_1=1+\frac{1}{2p}(\frac{2(2p-1)}{p\beta})^{2p-1}+\frac{2p-1}{2p}(\frac{2}{p\beta})^{\frac{1}{2p-1}}\alpha^{\frac{2p}{2p-1}},
$$

$$
A_2=\frac{1}{p+1}(\frac{4p}{p+1})^p\gamma^{p+1}.
$$

It yields,

$$
\begin{aligned}
&\frac{1}{2p}\frac{d}{dt}\left(\|u(t)\|_{L^{2p}}^{2p}+\|v(t)\|_{L^{2p}}^{2p}\right)+\int_\Gamma u^{2p}dx+\frac{\beta}{4}\int_\Gamma v^{2p}dx\\
\le&\frac{1}{2p}\frac{d}{dt}\left(\|u(t)\|_{L^{2p}}^{2p}+\|v(t)\|_{L^{2p}}^{2p}\right)+\frac{1}{p+1}(\frac{2p}{p+1})^p|\Gamma|+\frac{1}{2}\int_\Gamma u^{2p+2}dx+\frac{\beta}{4}\int_\Gamma v^{2p}dx\\
\le&\left(\frac{1}{p+1}(\frac{4p}{p+1})^pA_1^{p+1}+\frac{p-1}{2p}(\frac{4(p+1)}{2p\beta})^{\frac{p+1}{p-1}}A_2^{\frac{2p}{p-1}}\right)|\Gamma|+\frac{1}{p+1}(\frac{2p}{p+1})^p|\Gamma|.
\end{aligned}
\tag{2.2.14}
$$

Putting $b_p=\min\{2p,\frac{p\beta}{2}\}$, and

$$
M_p=2p\left(\frac{1}{p+1}(\frac{4p}{p+1})^pA_1^{p+1}+\frac{p-1}{2p}(\frac{4(p+1)}{2p\beta})^{\frac{p+1}{p-1}}A_2^{\frac{2p}{p-1}}\right)|\Gamma|+(\frac{2p}{p+1})^{p+1}|\Gamma|,
$$

we get

$$\frac{d}{dt}\|u(t)\|_{L^{2p}}^{2p} + \frac{d}{dt}\|v(t)\|_{L^{2p}}^{2p} + b_p\|u(t)\|_{L^{2p}}^{2p} + b_p\|v(t)\|_{L^{2p}}^{2p} \leq M_p. \tag{2.2.15}$$

Applying the Gronwall inequality, we have

$$\|u(t)\|_{L^{2p}}^{2p} + \|v(t)\|_{L^{2p}}^{2p} \leq e^{-b_p t}(\|u_0\|_{L^{2p}}^{2p} + \|v_0\|_{L^{2p}}^{2p}) + M_p. \tag{2.2.16}$$

It follows that

$$\limsup_{t\to\infty} (\|u(t)\|_{L^{2p}}^{2p} + \|v(t)\|_{L^{2p}}^{2p}) < K_p = 2M_p. \tag{2.2.17}$$

The proof is completed. □

2.3 Asymptotic Compactness

In this section, we show that the Boissonade semiflow $\{S(t)\}_{t\geq 0}$ is asymptotically compact through the following lemma. We shall use the notation $\|(y_1, y_2)\|^2 = \|y_1\|^2 + \|y_2\|^2$ for conciseness.

Lemma 2.3.1. For any given initial data $g_0 \in B_0$, the (u, v) components of the solution trajectories $g(t) = S(t)g_0$ of the IVP (2.1.6) satisfy

$$\|\nabla(u(t), v(t))\|^2 \leq Q_1, \quad \text{for } \ t > T_1, \tag{2.3.1}$$

where $Q_1 > 0$ is a constant depending only on K_1 and $|\Gamma|$ but independent of initial data, and $T_1 > 0$ is finite and only depends on the absorbing ball B_0.

Proof. Taking the inner-products $\langle(2.1.2), -\Delta v(t)\rangle$ to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla v\|^2 + d_2\|\Delta v\|^2 \\
&= \int_\Gamma \left(-u\Delta v - \beta|\nabla v|^2\right)\, dx \\
&\leq \int_\Gamma \left(\frac{u^2}{2d_2} + \frac{d_2}{2}|\Delta v|^2\right)\, dx - \beta\|\nabla v\|^2.
\end{aligned}$$

It follows that

$$\frac{d}{dt}\|\nabla v\|^2 + d_2\|\Delta v\|^2 + 2\beta\|\nabla v\|^2 \leq \frac{\|u\|^2}{2d_2}.$$

Next, taking the inner-products $\langle(2.1.1), -\Delta u(t)\rangle$, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 + d_1\|\Delta u\|^2 \\
&= \int_\Gamma \left(|\nabla u|^2 + \alpha v\Delta u - \gamma uv\Delta u - 3u^2|\nabla u|^2\right)\, dx \\
&\leq \int_\Gamma \left(|\nabla u|^2 + \frac{\alpha^2 v^2}{d_1} + \frac{d_1}{4}|\Delta u|^2 + \frac{\gamma^2 u^2 v^2}{d_1} + \frac{d_1}{4}|\Delta u|^2\right)\, dx \\
&\leq \int_\Gamma \left(|\nabla u|^2 + \frac{\alpha^2 v^2}{d_1} + \frac{d_1}{4}|\Delta u|^2 + \frac{\gamma^2}{2d_1}\left(u^4 + v^4\right) + \frac{d_1}{4}|\Delta u|^2\right)\, dx.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\frac{d}{dt}\|\nabla u\|^2 + d_1\|\Delta u\|^2 \\
&\leq 2\|\nabla u\|^2 + \frac{2\alpha^2}{d_1}\|v\|^2 + \frac{\gamma^2}{d_1}\left(\|u\|_{L^4}^4 + \|v\|_{L^4}^4\right).
\end{aligned}
$$

Adding up the two components gives

$$
\begin{aligned}
&\frac{d}{dt}(\|\nabla u\|^2 + \|\nabla v\|^2) + d_1\|\Delta u\|^2 + d_2\|\Delta v\|^2 + 2\beta\|\nabla v\|^2 \\
&\leq 2\|\nabla u\|^2 + \frac{\|u\|^2}{2d_2} + \frac{2\alpha^2}{d_1}\|v\|^2 + \frac{\gamma^2}{d_1}\left(\|u\|_{L^4}^4 + \|v\|_{L^4}^4\right).
\end{aligned}
\tag{2.3.2}
$$

Note that we have taken $\|\nabla\varphi\|$ as the norm of E and there is a positive constant $\eta > 0$ associated with the Sobolev imbedding inequality

$$
\|\varphi\|_{L^4(\Gamma)} \leq \eta\|\varphi\|_E = \eta\|\nabla\varphi\|, \quad \text{for any } \varphi \in E. \tag{2.3.3}
$$

Since B_0 in (2.2.7) is an absorbing ball, there is a finite time $T_0 > 0$ depending only on B_0 such that $S(t)B_0 \subset B_0$ for all $t > T_0$. In other words, we have the finite time $T_0 > 0$ depending only on B_0 such that

$$
\|u(t)\|^2 + \|v(t)\|^2 \leq K_1, \quad \text{for any } t > T_0,\ g_0 \in B_0. \tag{2.3.4}
$$

Then (2.3.2) along with these facts shows that for any initial datum $g_0 \in B_0$ one has

$$
\begin{aligned}
&\frac{d}{dt}\|(\nabla u, \nabla v)\|^2 \\
&\leq \frac{\gamma^2}{d_1}\eta^4(\|\nabla u\|^4 + \|\nabla v\|^4) + 2\|\nabla u\|^2 + \max\{\frac{1}{2d_2}, \frac{2\alpha^2}{d_1}\}K_1 \\
&\leq \|(\nabla u, \nabla v)\|^2\left(2 + \frac{\gamma^2\eta^4}{d_1}\|(\nabla u, \nabla v)\|^2\right) + \max\{\frac{1}{2d_2}, \frac{2\alpha^2}{d_1}\}K_1,\ \ t > T_0.
\end{aligned}
\tag{2.3.5}
$$

The differential inequality (2.3.5) can be written as

$$\frac{d}{dt}\beta \leq \rho\,\beta + h, \quad \text{for } t > T_0,\ g_0 \in B_0, \tag{2.3.6}$$

where

$$\beta(t) = \|(\nabla u, \nabla v)\|^2, \quad \rho(t) = \left(2 + \frac{\gamma^2\eta^4}{d_1}\|(\nabla u, \nabla v)\|^2\right),$$

and

$$h(t) = \max\{\frac{1}{2d_2}, \frac{2\alpha^2}{d_1}\}K_1.$$

From (2.2.9) we see that, for any given initial data $g_0 = (u_0, v_0) \in B_0$,

$$\int_t^{t+1} \beta(s)\,ds \leq \frac{M_1 + \max\{1, c\}K_1}{2\min\{d_1, cd_2\}} \quad \text{for } t > T_0,\ g_0 \in B_0. \tag{2.3.7}$$

Hence,

$$\int_t^{t+1} \rho(s)\,ds \leq M \triangleq 2 + \frac{\gamma^2\eta^4}{d_1}\left(\frac{M_1 + \max\{1, c\}K_1}{2\min\{d_1, cd_2\}}\right). \tag{2.3.8}$$

Then we apply the uniform Gronwall inequality, cf. [37, 38], to (2.3.6) to get

$$\begin{aligned} &\|(\nabla u(t,\cdot), \nabla v(t,\cdot))\|^2 \\ &\leq \left(\frac{M_1 + \max\{1, c\}K_1}{2\min\{d_1, cd_2\}} + \max\{\frac{1}{2d_2}, \frac{2\alpha^2}{d_1}\}K_1\right)\exp M. \end{aligned} \tag{2.3.9}$$

for any $t > T_0 + 1, g_0 \in B_0$.

Thus we complete the proof by setting $T_1 = T_0 + 1$ and

$$Q_1 = \left(\frac{M_1 + \max\{1, c\}K_1}{2\min\{d_1, cd_2\}} + \max\{\frac{1}{2d_2}, \frac{2\alpha^2}{d_1}\}K_1\right)\exp M.$$

□

2.4 Global Attractor and Its Properties

In this section we reach the proof of the First Main Theorem on the existence of a global attractor, which will be denoted by $\mathscr{A}$, for the Boissonade semiflow $\{S(t)\}_{t\geq 0}$ and we shall show several properties of this global attractor $\mathscr{A}$, namely, the regularity of $\mathscr{A}$, the property of being an (H, E) global attractor, and the finite Hausdorff and fractal dimensionality.

Proof of the Existence of Global Attractor. In Lemma 2.2.3 we have shown that the Boissonade semiflow $\{S(t)\}_{t\geq 0}$ has an absorbing set B_0 in H. From Lemma 2.3.1 we have that

$$\|S(t)g_0\|_E^2 \leq Q_1, \quad \text{for } \ t > T_2 \ \text{ and for } \ g_0 \in B_0,$$

which implies that $\{S(t)B_0 : t > T_2\}$ is a bounded set in E and consequently a precompact set in H. Therefore, the Boissonade semiflow $\{S(t)\}_{t\geq 0}$ is asymptotically compact in H. Thus we can apply Proposition 2.1.8 to reach the conclusion that there exists a global attractor $\mathscr{A}$ in H for this Boissonade semiflow $\{S(t)\}_{t\geq 0}$. □

Next we show that the global attractor $\mathscr{A}$ of the Boissonade semiflow is an (H, E) global attractor with the regularity $\mathscr{A} \subset \Pi \cap \mathbb{L}^\infty(\Gamma)$. First we have

Lemma 2.4.1. Let $\{g_m\}$ be a sequence in E such that $\{g_m\}$ converges to $g_0 \in E$ weakly in E and $\{g_m\}$ converges to g_0 strongly in H, as $m \to \infty$. Then

$$\lim_{m\to\infty} S(t)g_m = S(t)g_0 \ \text{ strongly in } E,$$

and the convergence is uniform with respect to t in any given compact interval $[t_0, t_1] \subset (0, \infty)$.

The proof of this lemma is seen in [45, Lemma 10].

Theorem 2.4.2. The global attractor $\mathscr{A}$ in H for the Boissonade semiflow $\{S(t)\}_{t\geq 0}$ is an (H, E) global attractor and $\mathscr{A}$ is a bounded subset in $\mathbb{L}^\infty(\Gamma)$, i.e.

$$\|g\|_{\mathbb{L}^\infty} \leq C, \quad \text{for any} \quad g \in \mathscr{A}. \tag{2.4.1}$$

Proof. By Lemma 2.3.1, we can assert that there exists a bounded absorbing set $B_1 \subset E$ for the Boissonade semiflow $\{S(t)\}_{t\geq 0}$ on H and this absorbing is in the E-norm. Indeed,

$$B_1 = \{g \in E : \|g\|_E^2 = \|\nabla g\|^2 \leq Q_1\}.$$

Now we show that this semiflow $\{S(t)\}_{t\geq 0}$ is asymptotically compact with respect to the strong topology in E. For any time sequence $\{t_n\}, t_n \to \infty$, and any bounded sequence $\{g_n\} \subset E$, there exists a finite time $t_0 \geq 0$ such that $S(t)\{g_n\} \subset B_0$, for any $t > t_0$. Then for an arbitrarily given

$T > t_0 + T_1$, where T_1 is the time specified in Lemma 2.3.1, there is an integer $n_0 \geq 1$ such that $t_n > 2T$ for all $n > n_0$. According to Lemma 2.3.1 ,

$$\{S(t_n - T)g_n\}_{n>n_0} \text{ is a bounded set in } E.$$

Since E is a Hilbert space, there is an increasing sequence of integers $\{n_j\}_{j=1}^{\infty}$ with $n_1 > n_0$, such that

$$\lim_{j\to\infty} S(t_{n_j} - T)g_{n_j} = g^* \ \text{ weakly in } E.$$

By the compact imbedding $E \hookrightarrow H$, there is a further sebsequence of $\{n_j\}$, but relabeled as the same as $\{n_j\}$, such that

$$\lim_{j\to\infty} S(t_{n_j} - T)g_{n_j} = g^* \ \text{ strongly in } H.$$

Then by Lemma 2.4.1, we have the following convergence with respect to the E-norm,

$$\lim_{j\to\infty} S(t_{n_j})g_{n_j} = \lim_{j\to\infty} S(T)S(t_{n_j} - T)g_{n_j} = S(T)g^* \ \text{ strongly in } E.$$

This proves that $\{S(t)\}_{t\geq 0}$ is asymptotically compact in E.

Therefore, by Proposition 2.1.8, there exists a global attractor $\mathscr{A}_E$ for the semiflow $\{S(t)\}_{t\geq 0}$ in E. According to Definition 2.1.6 and the fact that B_1 attracts B_0 in the E-norm due to Lemma 2.3.1 , we see that this global attractor $\mathscr{A}_E$ is an (H, E) global attractor. Moreover, the invariance and the boundedness of $\mathscr{A}$ in H and in E imply that

$$\mathscr{A}_E \text{ attracts } \mathscr{A} \text{ in } E, \text{ so that } \mathscr{A} \subset \mathscr{A}_E;$$

$$\mathscr{A} \text{ attracts } \mathscr{A}_E \text{ in } H, \text{ so that } \mathscr{A}_E \subset \mathscr{A}.$$

Therefore, $\mathscr{A} = \mathscr{A}_E$ and we proved that the global attractor $\mathscr{A}$ in H is indeed an (H, E) global attractor for this Boissonade semiflow.

Next we show that $\mathscr{A}$ is a bounded subset in $\mathbb{L}^\infty(\Gamma)$. By the (L^p, L^∞) regularity of the analytic C_0-semigroup $\{e^{At}\}_{t\geq 0}$ stated in [37, Theorem 38.10], one has $e^{At} : \mathbb{L}^p(\Gamma) \longrightarrow \mathbb{L}^\infty(\Gamma)$ for $t > 0$, and there is a constant $C(p) > 0$ such that

$$\|e^{At}\|_{\mathcal{L}(\mathbb{L}^p, \mathbb{L}^\infty)} \leq C(p)\, t^{-\frac{n}{2p}}, \quad t > 0, \quad \text{where } n = \dim \Gamma. \tag{2.4.2}$$

By the variation-of-constant formula satisfied by the mild solutions (of course strong solutions), for any $g \in \mathscr{A}$ $(\subset E)$, we have

$$\begin{aligned} \|S(t)g\|_{L^\infty} &\le \|e^{At}\|_{\mathcal{L}(L^2,L^\infty)}\|g\| + \int_0^t \|e^{A(t-\sigma)}\|_{\mathcal{L}(L^2,L^\infty)}\|f(S(\sigma)g)\|\,d\sigma \\ &\le C(2)t^{-\frac{3}{4}}\|g\| + \int_0^t C(2)(t-\sigma)^{-\frac{3}{4}}L(Q_1)\|S(\sigma)g\|_E\,d\sigma, \quad t \ge 0, \end{aligned} \tag{2.4.3}$$

where $C(2)$ is in the sense of (2.4.2), and $L(Q_1)$ is the Lipschitz constant of the nonlinear map f restricted on the closed, bounded ball centered at the origin with radius Q_1 in E. By the invariance of the global attractor $\mathscr{A}$, we have

$$\{S(t)\mathscr{A} : t \ge 0\} \subset B_0\,(\subset H) \quad \text{and} \quad \{S(t)\mathscr{A} : t \ge 0\} \subset B_1\,(\subset E).$$

Then from (2.4.3) we get

$$\begin{aligned} \|S(t)g\|_{L^\infty} &\le C(2)K_1t^{-\frac{3}{4}} + \int_0^t C(2)L(Q_1)Q_1(t-\sigma)^{-\frac{3}{4}}\,d\sigma \\ &= C(2)[K_1t^{-\frac{3}{4}} + 4L(Q_1)Q_1t^{\frac{1}{4}}], \quad \text{for } t > 0. \end{aligned} \tag{2.4.4}$$

Specifically one can take $t = 1$ in (2.4.4) and use the invariance of $\mathscr{A}$ to obtain

$$\|g\|_{L^\infty} \le C(2)(K_1 + 4L(Q_1)Q_1), \quad \text{for any } g \in \mathscr{A}.$$

Thus the global attractor $\mathscr{A}$ is a bounded subset in $\mathbb{L}^\infty(\Gamma)$. $\square$

Now we show the global attractor $\mathscr{A}$ is bounded in $H^2(\Gamma) \times H^2(\Gamma)$.

Theorem 2.4.3. The global attractors $\mathscr{A}$ for the Boissonade semiflow $\{S(t)\}_{t\ge0}$ is bounded in $H^2(\Gamma) \times H^2(\Gamma)$.

Proof. Due to the homogeneous Dirichlet boundary condition and the fact that $C_0^\infty(\Gamma)$ is dense in $H^2(\Gamma)$ as well as

$$\|\nabla\varphi\|^2 = |\langle\varphi, \Delta\varphi\rangle| \le \frac{1}{2}(\|\Delta\varphi\|^2 + \|\varphi\|^2)$$

we have a constant $\varrho > 0$ such that

$$\|\varphi\|_{H^2}^2 \le \varrho(\|\Delta\varphi\|^2 + \|\varphi\|^2)$$

Since $\mathscr{A}$ is bounded in H, we only need to estimate $\|\Delta u\|^2$ and $\|\Delta v\|^2$. Below we write u_t to be the partial derivative with respect to time t of $u(t,x)$, etc. Taking inner-product $\langle(2.1.1), u_t\rangle$ and $\langle(2.1.1), v_t\rangle$ respectively and summing up, we get

$$\begin{aligned}
&\|u_t\|^2 + \|v_t\|^2 + \frac{1}{2}\frac{d}{dt}\left(d_1\|\nabla u\|^2 + d_2\|\nabla v\|^2\right)\\
=&\int_\Gamma \left(uu_t - \alpha u_t v + \gamma uvu_t - u^3 u_t + uv_t - \beta vv_t\right) dx\\
\leq&\int_\Gamma \left(C + \alpha C + \gamma C^2 + C^3\right)|u_t| dx + \int_\Gamma (C + \beta C)|v_t| dx\\
=&\frac{1}{2}\left(C + \alpha C + \gamma C^2 + C^3\right)^2 |\Omega| + \frac{1}{2}\|u_t\|^2 + \frac{1}{2}(C+\beta C)^2|\Omega| + \frac{1}{2}\|v_t\|^2,
\end{aligned} \tag{2.4.5}$$

where C comes from (2.4.1).

It follows that

$$\|u_t\|^2 + \|v_t\|^2 + \frac{d}{dt}\left(d_1\|\nabla u\|^2 + d_2\|\nabla v\|^2\right) \leq \left(C + \alpha C + \gamma C^2 + C^3\right)^2|\Omega| + (C+\beta C)^2|\Omega|. \tag{2.4.6}$$

Integrating (2.4.6) over time interval $[0,1]$, we have, for $(u,v) \in \mathscr{A}$,

$$\begin{aligned}
\int_0^1 (\|u_t\|^2 + \|v_t\|^2) ds \leq& d_1\|\nabla u(0)\|^2 + d_2\|\nabla v(0)\|^2\\
&+ \left(C + \alpha C + \gamma C^2 + C^3\right)^2|\Omega| + (C+\alpha C)^2|\Omega|\\
\leq& (d_1+d_2)Q_1 + \left(C + \alpha C + \gamma C^2 + C^3\right)^2|\Omega| + (C+\alpha C)^2|\Omega|.
\end{aligned} \tag{2.4.7}$$

Differentiate (2.1.1) and (2.1.2) to obtain

$$u_{tt} = d_1\Delta u_t + u_t - \alpha v_t + \gamma(u_t v + uv_t) - 3u^2 u_t, \tag{2.4.8}$$

$$v_{tt} = d_2\Delta v_t + u_t - \beta v_t. \tag{2.4.9}$$

Taking the inner products $\langle(2.4.8), t^2 u_t\rangle$ and $\langle(2.4.9), t^2 v_t\rangle$ and adding up, we have

$$\begin{aligned}
&- t\|u_t\|^2 - t\|v_t\|^2 + \frac{1}{2}\frac{d}{dt}\left(\|tu_t\|^2 + \|tv_t\|^2\right) + t^2 d_1\|\nabla u_t\|^2 + t^2 d_2\|\nabla v_t\|^2\\
=&\int_\Gamma t^2\left(u_t^2 - \alpha u_t v_t + \gamma u_t v u_t + \gamma uv_t u_t - 3u^2u_t^2 + u_t v_t - \beta v_t^2\right) dx\\
\leq&\int_\Gamma t^2\left(u_t^2 + \frac{\alpha}{2}(u_t^2+v_t^2) + C\gamma u_t^2 + \frac{C\gamma}{2}(u_t^2+v_t^2) + (u_t^2+v_t^2)\right) dx\\
=&t^2\left(2 + \frac{\alpha}{2} + \frac{3C\gamma}{2}\right)\|u_t\|^2 + t^2\left(1 + \frac{\alpha}{2} + \frac{C\gamma}{2}\right)\|v_t\|^2.
\end{aligned} \tag{2.4.10}$$

Integrating the above inequality on $[0,t]$, we have

$$\begin{aligned}&\frac{1}{2}\left(\|tu_t\|^2+\|tv_t\|^2\right)\leq\int_0^t s^2\left(2+\frac{\alpha}{2}+\frac{3C\gamma}{2}\right)\|u_t(s)^2\|ds\\&+\int_0^t s^2\left(1+\frac{\alpha}{2}+\frac{C\gamma}{2}\right)\|v_t(s)^2\|ds+\int_0^t s(\|u_t(s)\|^2+\|v_t(s)\|^2)ds.\end{aligned}\tag{2.4.11}$$

Putting $t=1$, we have

$$\begin{aligned}&\|u_t(1)\|^2+\|v_t(1)\|^2\\\leq&2\int_0^1\left(2+\frac{\alpha}{2}+\frac{3C\gamma}{2}\right)\|u_t(s)^2\|ds+2\int_0^t\left(1+\frac{\alpha}{2}+\frac{C\gamma}{2}\right)\|v_t(s)^2\|dx\\&+2\int_0^t(\|u_t(s)\|^2+\|v_t(s)\|^2)ds\\\leq&M_4.\end{aligned}\tag{2.4.12}$$

where

$$M_4=2\left(4+\alpha+2C\gamma\right)\left(\left(C+\alpha C+\gamma C^2+C^3\right)^2|\Omega|+\left(C+\alpha C\right)^2|\Omega|\right)$$

Next, due to the invariance of $\mathscr{A}$, for any $g\in\mathscr{A}$, there exists a $\tilde{g}\in\mathscr{A}$, such that $g=S(1)\tilde{g}$, thus from (2.1.1) and (2.1.2) we have

$$\begin{aligned}d_1\|\Delta u\|+d_2\|\Delta v\|\leq&\|u_t\|+\|v_t\|+\|u\|+\beta\|v\|+\|u\|+\alpha\|v\|+\gamma\|uv\|+\|u^3\|\\=&\|\tilde{u}_t(1)\|+\|\tilde{v}_t(1)\|+(2+\frac{\gamma}{2})\|u\|+(\alpha+\frac{\gamma}{2})\|v\|+\|u\|_{L^6}^3\\\leq&M_4^{\frac{1}{2}}+(2+\frac{\gamma}{2}+\alpha+\frac{\gamma}{2})K_1^{\frac{1}{2}}+K_3^{\frac{1}{2}}.\end{aligned}\tag{2.4.13}$$

Therefore $\mathscr{A}$ is bounded in $H^2(\Gamma)\times H^2(\Gamma)$. □

Next we consider the Hausdorff and fractal dimensions of the global attractor $\mathscr{A}$. The background concepts and results can be seen in [38, Chapter V]. Let $q_m=\limsup_{t\to\infty}q_m(t)$, where

$$q_m(t)=\sup_{g_0\in\mathscr{A}}\sup_{\substack{g_i\in H,\|g_i\|=1\\i=1,\cdots,m}}\left(\frac{1}{t}\int_0^t\operatorname{Tr}\left[\left(A+f'(S(\tau)g_0)\right)P_m(\tau)\right]\,d\tau\right),\tag{2.4.14}$$

in which $\operatorname{Tr}\left[(A+f'(S(\tau)g_0))\Gamma_m(\tau)\right]$ is the trace of the linear operator $(A+f'(S(\tau)g_0))P_m(\tau)$, $f'(g)$ is the Fréchet derivative of the Nemytskii map f in (2.1.6), and $P_m(t)$ stands for the orthogonal projection of the space H on the subspace spanned by $G_1(t),\cdots,G_m(t)$, with

$$G_i(t)=L(S(t),g_0)g_i,\quad i=1,\cdots,m.\tag{2.4.15}$$

Here $f'(S(\tau)g_0)$ is the Fréchet derivative of the map f defined by (2.1.5) at $S(\tau)g_0$, and $L(S(t), g_0)$ is the Fréchet derivative of the map $S(t)$ at g_0, with t fixed.

We use the following proposition [38, Chapter V] to show the finite upper bounds of the Hausdorff and fractal dimensions of this global attractor $\mathscr{A}$.

Proposition 2.4.4. If there is an integer m such that $q_m < 0$, then the Hausdorff dimension $d_H(\mathscr{A})$ and the fractal dimension $d_F(\mathscr{A})$ of $\mathscr{A}$ satisfy

$$d_H(\mathscr{A}) \leq m, \quad \text{and} \quad d_F(\mathscr{A}) \leq m \max_{1\leq j\leq m-1} \left(1 + \frac{(q_j)_+}{|q_m|}\right) \leq 2m. \tag{2.4.16}$$

It can be shown that for any given $t > 0$, $S(t)$ is Fréchet differentiable in H and uniformly Fréchet differentiable in $\mathscr{A}$. Its Fréchet derivative at g_0 is the bounded linear operator $L(S(t), g_0)$ given by

$$L(S(t), g_0)G_0 \stackrel{\text{def}}{=} G(t) = (U(t), V(t)),$$

for $G_0 = (U_0, V_0) \in H$, where $(U(t), V(t))$ is the weak solution of the following extended Boissonade variational equation

$$\begin{aligned}
&\frac{\partial U}{\partial t} = d_1 \Delta U + U - \alpha V + \gamma U v + \gamma u V - 3u^2 U, \\
&\frac{\partial V}{\partial t} = d_2 \Delta V - \beta V + U \\
&U(0) = U_0, \quad V(0) = V_0.
\end{aligned} \tag{2.4.17}$$

Here $g(t) = (u(t), v(t)) = S(t)g_0$ is the weak solution of (2.1.6) with the initial condition $g(0) = g_0$. The initial value problem (2.4.17) can be written as

$$\frac{dG}{dt} = (A + f'(S(t)g_0))G, \quad G(0) = G_0. \tag{2.4.18}$$

As we have shown, the invariance of $\mathscr{A}$ implies $\mathscr{A} \subset B_0 \cap B_1$, so that

$$\sup_{g_0 \in \mathscr{A}} \|S(t)g_0\|^2 \leq K_1 \quad \text{and} \quad \sup_{g_0 \in \mathscr{A}} \|S(t)g_0\|_E^2 \leq Q_1. \tag{2.4.19}$$

Theorem 2.4.5. The global attractors $\mathscr{A}$ for the Boissonade semiflow $\{S(t)\}_{t\geq 0}$ has a finite Hausdorff dimension and a finite fractal dimension.

Proof. By Proposition 2.4.4, we shall estimate Tr $[(A + f'(S(\tau)g_0))P_m(\tau)]$. At any given time $\tau > 0$, let $\{\zeta_j(\tau) : j = 1, \cdots, m\}$ be an H-orthonormal basis for the subspace

$$P_m(\tau)H = \text{Span}\,\{G_1(\tau), \cdots, G_,(\tau)\},$$

where $G_1(\tau), \cdots, G_m(\tau)$ satisfy (2.4.18) and, without loss of generality, assuming that the initial vectors $G_{1,0}, \cdots, G_{m,0}$ are linearly independent in H. Let $d_0 = \min\{d_1, d_2\}$. Denote the components of $\zeta_j(\tau)$ by $\zeta_j^i(\tau), i = 1, 2$. Then we have

$$\begin{aligned}\text{Tr}\,[(A + f'(S(\tau)g_0)P_m(\tau)] &= \sum_{j=1}^{m} \left(\langle A\zeta_j(\tau), \zeta_j(\tau)\rangle + \langle f'(S(\tau)g_0)\zeta_j(\tau), \zeta_j(\tau)\rangle\right) \\ &\le -d_0 \sum_{j=1}^{m} \|\nabla\zeta_j(\tau)\|^2 + J_1 + J_2 + J_3,\end{aligned} \tag{2.4.20}$$

where

$$\begin{aligned}J_1 = &\sum_{j=1}^{m} \int_\Gamma \left(|\zeta_j^1(\tau)|^2 + \zeta_j^1(\tau)\zeta_j^2(\tau) - \alpha\zeta_j^2(\tau)\zeta_j^1(\tau) - \beta|\zeta_j^2(\tau)|^2\right) dx, \\ &\le (2 + \alpha + \beta)m\end{aligned} \tag{2.4.21}$$

$$\begin{aligned}J_2 =& \sum_{j=1}^{m} \int_\Gamma \left(\gamma v(\tau)|\zeta_j^1(\tau)|^2 + \gamma u(\tau)\zeta_j^1(\tau)\zeta_j^2(\tau)\right) dx \\ \le& \sum_{j=1}^{m} \left(\|\gamma\|_{L^4}\|v(\tau)\|_{L^4}\|\zeta_j^1(\tau)\|_{L^4}\|\zeta_j^1(\tau)\|_{L^4} + \|\gamma\|_{L^4}\|u(\tau)\|_{L^4}\|\zeta_j^1(\tau)\|_{L^4}\|\zeta_j^2(\tau)\|_{L^4}\right), \\ J_3 =& \sum_{j=1}^{m} \int_\Gamma \left(-3u^2(\tau)|\zeta_j^1(\tau)|^2\right) dx \\ \le& \sum_{j=1}^{m} 3\left(\|u(\tau)\|_{L^4}\|u(\tau)\|_{L^4}\|\zeta_j^1(\tau)\|_{L^4}\|\zeta_j^1(\tau)\|_{L^4}\right).\end{aligned}$$

By the generalized Hölder inequality, the embedding $H_0^1(\Gamma) \hookrightarrow L^4(\Gamma)$ (for $n \le 3$) and (2.4.19), we get

$$
\begin{aligned}
& J_2 + J_3 \\
\le & \sum_{j=1}^{m} \left(\gamma|\Gamma|^{\frac{1}{4}} \|v(\tau)\|_{L^4} \|\zeta_j^1(\tau)\|_{L^4}^2 + \gamma|\Gamma|^{\frac{1}{4}} \|u(\tau)\|_{L^4} \|\zeta_j^1(\tau)\|_{L^4} \|\zeta_j^2(\tau)\|_{L^4} \right) \\
& + \sum_{j=1}^{m} \left(3\|u(\tau)\|_{L^4}^2 \|\zeta_j^1(\tau)\|_{L^4}^2 \right), \\
\le & \sum_{j=1}^{m} \left(2\gamma|\Gamma|^{\frac{1}{4}} \|S(\tau)g_0\|_{L^4} \|\zeta_j(\tau)\|_{L^4}^2 + 3\|S(\tau)g_0\|_{L^4}^2 \|\zeta_j(\tau)\|_{L^4}^2 \right), \\
\le & \left(2\gamma|\Gamma|^{\frac{1}{4}} \eta \|\nabla S(\tau)g_0\| + 3\eta^2 \|\nabla S(\tau)g_0\|^2 \right) \sum_{j=1}^{m} \|\zeta_j(\tau)\|_{L^4}^2 \\
\le & \, (2\gamma\eta|\Gamma|^{\frac{1}{4}} Q_1^{\frac{1}{2}} + 3\eta^2 Q_1) \sum_{j=1}^{m} \|\zeta_j(\tau)\|_{L^4}^2,
\end{aligned}
\tag{2.4.22}
$$

where η is the embedding coefficient given in (2.3.3). Now we invoke the Garliardo-Nirenberg interpolation inequality for Sobolev spaces [37, Theorem B.3],

$$
\|\zeta\|_{W^{k,p}} \le \tilde{C} \|\zeta\|_{W^{m,q}}^{\theta} \|\zeta\|_{L^r}^{1-\theta}, \quad \text{for } \zeta \in W^{m,q}(\Gamma), \tag{2.4.23}
$$

provided that $p, q, r \ge 1, 0 < \theta \le 1$, and

$$
k - \frac{n}{p} \le \theta \left(m - \frac{n}{q} \right) - (1-\theta)\frac{n}{r}, \quad \text{where } n = \dim \Gamma.
$$

Here with $W^{k,p}(\Gamma) = L^4(\Gamma), W^{m,q}(\Gamma) = H_0^1(\Gamma), L^r(\Gamma) = L^2(\Gamma)$, and $\theta = n/4 \le 3/4$, it follows from (2.4.23) that

$$
\|\zeta_j(\tau)\|_{L^4} \le \tilde{C} \|\nabla \zeta_j(\tau)\|^{\frac{n}{4}} \|\zeta_j(\tau)\|^{1-\frac{n}{4}} = \tilde{C} \|\nabla \zeta_j(\tau)\|^{\frac{n}{4}}, \quad j = 1, \cdots, m, \tag{2.4.24}
$$

since $\|\zeta_j(\tau)\| = 1$, where $\tilde{C}$ is a universal constant. Substitute (2.4.24) into (2.4.22) to obtain

$$
J_2 + J_3 \le (2\gamma\eta|\Gamma|^{\frac{1}{4}} Q_1^{\frac{1}{2}} + 3\eta^2 Q_1) \tilde{C}^2 \sum_{j=1}^{m} \|\nabla \zeta_j(\tau)\|^{\frac{n}{2}}. \tag{2.4.25}
$$

Substituting (2.4.21) and (2.4.25) into (2.4.20), we obtain

$$\begin{aligned}\text{Tr}\,[(A+f'(S(\tau)g_0)\Gamma_m(\tau)] \le & -d_0\sum_{j=1}^{m}\|\nabla\zeta_j(\tau)\|^2 \\ & + (2\gamma\eta|\Gamma|^{\frac{1}{4}}Q_1^{\frac{1}{2}}+3\eta^2Q_1)\tilde{C}^2\sum_{j=1}^{m}\|\nabla\zeta_j(\tau)\|^{\frac{n}{2}}+(2+\alpha+\beta)m.\end{aligned} \tag{2.4.26}$$

By Young's inequality, for $n\le 3$, we have

$$(2\gamma\eta|\Gamma|^{\frac{1}{4}}Q_1^{\frac{1}{2}}+3\eta^2Q_1)\tilde{C}^2\sum_{j=1}^{m}\|\nabla\zeta_j(\tau)\|^{\frac{n}{2}} \le \frac{d_0}{2}\sum_{j=1}^{m}\|\nabla\zeta_j(\tau)\|^2+m\,Q_3(n),$$

where $Q_3(n)$ is a universal positive constant depending only on $n=\dim(\Gamma)$. Hence,

$$\text{Tr}\,[(A+f'(S(\tau)g_0)\Gamma_m(\tau)] \le -\frac{d_0}{2}\sum_{j=1}^{m}\|\nabla\zeta_j(\tau)\|^2+m(Q_3(n)+b+k), \quad \tau>0,\ g_0\in\mathscr{A}.$$

According to the generalized Sobolev-Lieb-Thirring inequality [38, Appendix, Corollary 4.1], since $\{\zeta_1(\tau),\cdots,\zeta_m(\tau)\}$ is an orthonormal set in H, there exists a universal constant $Q^*>0$ only depending on the shape and dimension of Γ, such that

$$\sum_{j=1}^{m}\|\nabla\zeta_j(\tau)\|^2 \ge Q^*\frac{m^{1+\frac{2}{n}}}{|\Gamma|^{\frac{2}{n}}}.$$

Therefore,

$$\text{Tr}\,[(A+f'(S(\tau)g_0)\Gamma_m(\tau)] \le -\frac{d_0Q^*}{2|\Gamma|^{\frac{2}{n}}}m^{1+\frac{2}{n}}+m(Q_3(n)+b+k), \quad \tau>0,\ g_0\in\mathscr{A}. \tag{2.4.27}$$

Then we can conclude that

$$\begin{aligned} q_m &= \limsup_{t\to\infty}\, q_m(t) \\ &= \limsup_{t\to\infty}\sup_{g_0\in\mathscr{A}}\sup_{\substack{g_i\in H,\|g_i\|=1\\ i=1,\cdots,m}}\left(\frac{1}{t}\int_0^t \text{Tr}\,[(A+f'(S(\tau)g_0))\,\Gamma_m(\tau)]\,d\tau\right) \\ &\le -\frac{d_0Q^*}{2|\Gamma|^{\frac{2}{n}}}m^{1+\frac{2}{n}}+m(Q_3(n)+b+k)<0,\end{aligned} \tag{2.4.28}$$

if the integer m satisfies the following condition,

$$m-1\le\left(\frac{2(Q_3(n)+2+\alpha+\beta)}{d_0L_1}\right)^{n/2}|\Gamma|<m. \tag{2.4.29}$$

According to Proposition 2.4.4, we have proved that the Hausdorff dimension and the fractal dimension of the global attractor $\mathscr{A}$ are finite and their upper bounds are given by

$$d_H(\mathscr{A}) \leq m \quad \text{and} \quad d_F(\mathscr{A}) \leq 2m,$$

where m satisfies (2.4.29). □

2.5 Existence of An Exponential Attractor

In this section, we show the existence of an exponential attractor for the Boissonade semiflow $\{S(t)\}_{t\geq 0}$.

Definition 2.5.1. Let $\{S(t)\}_{t\geq 0}$ be a semiflow on a Banach space X. A subset $\mathscr{E}$ of X is called an *exponential attractor in* X for this semiflow, if the following conditions are satisfied:

- $\mathscr{E}$ is a nonempty, compact, and positively invariant set in the sense that

$$S(t)\mathscr{A} \subseteq \mathscr{A} \quad \text{for any } \; t \geq 0.$$

- $\mathscr{E}$ has a finite fractal dimension, and
- $\mathscr{E}$ attracts any bounded set B of X exponentially, in other words, there exist positive constants μ and $C(B)$ which depends on B, such that

$$\text{dist}(S(t)B, \mathscr{E}) \leq C(B)e^{-\mu t}, \quad \text{for } \; t \geq 0.$$

The basic theory and construction of exponential attractor was established by A.Eden, C.Foias, B.Nicolaenko and R.Temam in [18] for discrete and continuous semiflows on Hilbert spaces.

Another important concept in the area of infinite dimensional dynamical systems is the inertial manifold $\mathscr{M}$ defined as follows.

Definition 2.5.2. Let $\{S(t)\}_{t\geq 0}$ be a semiflow on a Banach space X. An inertial manifold $\mathscr{M}$ for this semiflow is a subset in X such that the following conditions are satisfied:

(i) $\mathscr{M}$ is a finite dimensional Lipschitz continuous manifold.

(ii) $\mathscr{M}$ is positively invariant under the semiflow S(t).

(iii) $\mathscr{M}$ attracts all the trajectories of the semiflow at a uniform exponential rate, i.e., there exists a positive constant ν such that for any bounded set B in X, there exist a constant $K(B)$ such that

$$\operatorname{dist}(S(t)B, \mathscr{M}) \leq K(B)e^{-\nu t}, \quad \text{for } t \geq 0.$$

For a continuous semiflow, if all the three objects(a global attractor $\mathscr{A}$, an exponential attractor $\mathscr{E}$, and an inertial manifold $\mathscr{M}$ of the same exponential attraction rate) exist, then the following inclusion relationship holds,

$$\mathscr{A} \subseteq \mathscr{E} \subseteq \mathscr{M}.$$

Here we prove the existence of an exponential attractor for the Boissonade semiflow by using the following lemma, which is a modified version of the result shown in [46, Lemma 6.3], whose proof was based on the squeezing property [18,29] and the constructive argument in [29, Theorem 4.5]. This lemma provides a way to directly check the sufficient conditions for the existence of an exponential attractor of a semiflow on a positively invariant cone in a Hilbert space. First we introduce the concept of squeezing property.

Definition 2.5.3. Let $\Psi : D(\Psi) \to \mathcal{H}$ be a nonnegative, self-adjoint, linear operator with a compact resolvent, let P_N be a spectral (orthogonal) projection with respect to Ψ, in other words, P_N maps the Hilbert space $\mathcal{H}$ onto the N-dimensional subspace $\mathcal{H}_N$ spanned by a set of the first N eigenvectors of the operator Ψ, we defined a cone

$$\mathscr{C}_{P_N} = \left\{ y \in \mathcal{H} : \|(I - P_N)(y)\|_{\mathcal{H}} \leq \|P_N(y)\|_{\mathcal{H}} \right\}.$$

A continuous mapping S_* satisfies the *discrete squeezing property* relative to a set $B \subset \mathcal{H}$ if there exist a constant $\kappa \in (0, 1/2)$ and a spectral projection P_N on $\mathcal{H}$ such that for any pair of points $y_0, z_0 \in B$, if

$$S_*(y_0) - S_*(z_0) \notin \mathscr{C}_{P_N},$$

then

$$\|S_*(y_0) - S_*(z_0)\|_{\mathcal{H}} \leq \kappa \|y_0 - z_0\|_{\mathcal{H}}.$$

The following lemma and its proof can be seen in [46, Lemma 6.3].

Lemma 2.5.4. Let $\mathcal{X}$ be a Hilbert space and $X \subset \mathcal{X}$ be an open cone with the vertex at the origin and X_c be the closure of X in $\mathcal{X}$. Consider an evolutionary equation

$$\frac{d\varphi}{dt} + \Psi\varphi = \Phi(\varphi), \quad t > 0, \tag{2.5.1}$$

where $\Psi : D(\Psi) \to \mathcal{X}$ is a nonnegative, self-adjoint, linear operator with compact resolvent, and $\Phi : \mathcal{Y} = D(\Psi^{1/2}) \to \mathcal{X}$ is a locally Lipschitz continuous mapping, where $\mathcal{Y}$ is a compactly imbedded subspace of $\mathcal{X}$. Suppose that the weak solution of (2.5.1) for each initial point $\varphi(0) = \varphi_0 \in X_c$ uniquely exists and is confined in X_c for all $t \geq 0$, which turn out to be a strong solution for $t > 0$. All these solutions in X_c form a semiflow denoted by $\{\Sigma(t)\}_{t\geq 0}$. Assume that the following conditions are satisfied:

(i) There exist a compact, positively invariant, absorbing set $\mathcal{B}_c$ in X_c with respect to the topology of $\mathcal{X}$.

(ii) There is a positive integer N such that the norm quotient $\Gamma(t)$ defined by

$$\Gamma(t) = \frac{\left\|\Psi^{1/2}\left(\varphi_1(t) - \varphi_2(t)\right)\right\|_{\mathcal{X}}^2}{\|\varphi_1(t) - \varphi_2(t)\|_{\mathcal{X}}^2} \tag{2.5.2}$$

for any distinct trajectories $\varphi_1(\cdot)$ and $\varphi_2(\cdot)$ starting from the set $\mathcal{B}_c \backslash \mathscr{C}_{P_N}$ satisfies

$$\frac{d\Gamma}{dt} \leq \rho\left(\mathcal{B}_c\right)\Gamma(t), \quad t > 0,$$

where $\rho\left(\mathcal{B}_c\right)$ is a positive constant only depending on $\mathcal{B}_c$.

(iii) For any given finite $T > 0$ and any given $\varphi \in \mathcal{B}_c$, $\Sigma(\cdot)\varphi : [0, T] \to \mathcal{B}_c$ is Hölder continuous with exponent $\theta = 1/2$ and the coefficient of Hölder continuity, $K_\theta(\varphi) : \mathcal{B}_c \to (0, \infty)$, is a bounded function.

(iv) For any $t \in [0, T]$ where $T > 0$ is arbitrarily given, $\Sigma(t)(\cdot) : \mathcal{B}_c \to \mathcal{B}_c$ is Lipschitz continuous and the Lipschitz constant $L(t) : [0, T] \to (0, \infty)$ is a bounded function.

Then there exists an exponential attractor $\mathscr{E}$ in X_c for this semiflow $\{\Sigma(t)\}_{t\geq 0}$.

Using this lemma, we can prove

Theorem 2.5.5. Given any positive parameters in the Boissonade system (2.1.1)–(2.1.2), there exists an exponential attractor $\mathscr{E}$ in H for the Boissonade semiflow $\{S(t)\}_{t\geq 0}$.

Proof. By Theorem 2.4.2, there exists an (H, E) global attractor $\mathscr{A}$ for the Boissonade semiflow $\{S(t)\}_{t\geq 0}$. Hence by Corollary 5.7 of [46], there exists a compact, positively invariant, absorbing set $\mathcal{B}_E$ in H, which is a bounded set in E for this semiflow.

Secondly, we prove that the second condition in Lemma 2.5.4 is satisfied by this Boissonade semiflow. Consider any two points $g_1(0), g_2(0) \in \mathcal{B}_E$ and let $g_i(t) = (u_i(t), v_i(t))$, $i = 1, 2$, be the corresponding solutions of (2.1.6) , respectively. Let $y(t) = g_1(t) - g_2(t)$, $t \geq 0$, where $g_1(0) \neq g_2(0)$. The associated norm quotient of $(g_1(t), g_2(t))$ is given by

$$\Gamma(t) = \frac{\left\|(-A)^{1/2} y(t)\right\|^2}{\|y(t)\|^2}, \quad t \geq 0.$$

For $t > 0$, we can calculate

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\Gamma(t) &= \frac{1}{\|y(t)\|^4}\left[\langle(-A)^{1/2}y(t), (-A)^{1/2}y_t\rangle\|y(t)\|^2 - \|(-A)^{1/2}y(t)\|^2\langle y(t), y_t\rangle\right] \\
&= \frac{1}{\|y(t)\|^2}\left[\langle(-A)y(t), y_t\rangle - \Gamma(t)\langle y(t), y_t\rangle\right] \\
&= \frac{1}{\|y(t)\|^2}\langle(-A)y(t) - \Gamma(t)y(t), Ay(t) + f(g_1(t)) - f(g_2(t))\rangle \\
&= \frac{1}{\|y(t)\|^2}\langle(-A)y(t) - \Gamma(t)y(t), Ay(t) + \Gamma(t)y(t) + f(g_1(t)) - f(g_2(t))\rangle \\
&= \frac{1}{\|y(t)\|^2}\left[-\|Ay(t) + \Gamma(t)y(t)\|^2 - \langle Ay(t) + \Gamma(t)y(t), f(g_1(t)) - f(g_2(t))\rangle\right] \\
&\leq \frac{1}{\|y(t)\|^2}\left[-\frac{1}{2}\|Ay(t) + \Gamma(t)y(t)\|^2 + \frac{1}{2}\left\|f(g_1(t)) - f(g_2(t))\right\|^2\right] \\
&\leq \frac{\left\|f(g_1(t)) - f(g_2(t))\right\|^2}{2\|y(t)\|^2}
\end{aligned} \tag{2.5.3}$$

where we used the identity $-\langle Ay(t) + \Gamma(t)y(t), \Gamma(t)y(t)\rangle = 0$. Note that the compact, positively invariant, H-absorbing set $\mathcal{B}_E$ described earlier in this proof is a bounded set in E . Hence there is a constant $R_1 > 0$ only depending on $\mathcal{B}_E$ such that

$$\|(u, v)\| \leq R_1, \quad \text{for any } (u, v) \in \mathcal{B}_E. \tag{2.5.4}$$

We have

$$
\begin{aligned}
&\|f(g_1(t)) - f(g_2(t))\| \\
&= \left\|(u_1 - u_2) - \alpha(v_1 - v_2) + \gamma(u_1 v_1 - u_2 v_2) - (u_1^3 - u_2^3)\right\| + \|(u_1 - u_2) - \beta(v_1 - v_2)\| \\
&\le 2\|u_1 - u_2\| + (\alpha + \beta)\|v_1 - v_2\| + \gamma\|u_1(v_1 - v_2)\| + \gamma\|v_2(u_1 - u_2)\| + 3R_1^2\|u_1 - u_2\| \\
&\le (2 + \alpha + \beta + 2\gamma R_1 + 3R_1^2)\|y(t)\| \\
&\le \lambda(2 + \alpha + \beta + 2\gamma R_1 + 3R_1^2)\|\nabla y(t)\|,
\end{aligned}
\tag{2.5.5}
$$

for $t > 0$. Let

$$N(R_1) = \lambda(2 + \alpha + \beta + 2\gamma R_1 + 3R_1^2). \tag{2.5.6}$$

In view of (1.0.2), from (2.5.3) and (2.5.5) it follows that

$$
\begin{aligned}
\frac{d}{dt}\Gamma(t) &\le \frac{\|f(g_1(t)) - f(g_2(t))\|}{\|y(t)\|^2} \le N^2(R_1)\frac{\|\nabla y(t)\|^2}{\|y(t)\|^2} \\
&\le \rho(\mathcal{B}_E)\frac{\|(-A)^{1/2} y(t)\|^2}{\|y(t)\|^2} = \rho(\mathcal{B}_E)\Gamma(t), \quad t > 0,
\end{aligned}
\tag{2.5.7}
$$

where

$$\rho(\mathcal{B}_E) = \frac{N^2(R_1)}{d_0}.$$

where $d_0 = \min\{d_1, d_2\}$. Thus the second condition in Lemma 2.5.4 is satisfied.

Thirdly we check the Hölder continuity of $S(\cdot)g : [0, T] \to \mathcal{B}_E$ for any given $g \in \mathcal{B}_E$ and any given compact interval $[0, T]$. Pick any $0 \le t_1 < t_2 \le T$, we have

$$
\begin{aligned}
\|S(t_2)g - S(t_1)g\| \le \left\|\left(e^{A(t_2 - t_1)} - I\right)e^{At_1}g\right\| &+ \int_{t_1}^{t_2} \left\|e^{A(t_2 - \sigma)} f(S(\sigma)g)\right\| d\sigma \\
&+ \int_0^{t_1} \left\|\left(e^{A(t_2 - t_1)} - I\right)e^{A(t_1 - \sigma)} f(S(\sigma)g)\right\| d\sigma.
\end{aligned}
\tag{2.5.8}
$$

Since $\mathcal{B}_E$ is positively invariant with respect to the Boissonade semiflow $\{S(t)\}_{t \ge 0}$ and $\mathcal{B}_E$ is bounded in E, there exists a constant $K_{\mathcal{B}_E} > 0$ such that for any $g \in \mathcal{B}_E$, we have

$$\|S(t)g\|_E \le K_{\mathcal{B}_E}, \quad t \ge 0.$$

Recall that $f : E \to H$ is locally Lipschitz continuous, there is a Lipschitz constant $L_{\mathcal{B}_E} > 0$ of f relative to this positively invariant set $\mathcal{B}_E$. Moreover, by [37, Theorem 37.5], for the analytic, contracting, linear semigroup $\{e^{At}\}_{t\geq 0}$, there exist positive constants N_0 and N_1 such that

$$\left\|e^{At}g - g\right\|_H \leq N_0\, t^{1/2}\|g\|_E, \quad \text{for } t \geq 0,\ g \in E,$$

and

$$\left\|e^{At}\right\|_{\mathcal{L}(H,E)} \leq N_1\, t^{-1/2}, \quad \text{for } t > 0.$$

It follows that

$$\left\|\left(e^{A(t_2-t_1)} - I\right) e^{At_1} g\right\| \leq N_0\,(t_2 - t_1)^{1/2}\, K_{\mathcal{B}_E}$$

and

$$\int_{t_1}^{t_2} \left\|e^{A(t_2-\sigma)} f(S(\sigma)g)\right\| d\sigma \leq \int_{t_1}^{t_2} \frac{N_1 L_{\mathcal{B}_E} K_{\mathcal{B}_E}}{\sqrt{t_2-\sigma}}\, d\sigma = 2K_{\mathcal{B}_E} L_{\mathcal{B}_E} N_1\,(t_2-t_1)^{1/2}.$$

Moreover,

$$\begin{aligned}\int_0^{t_1} \left\|\left(e^{A(t_2-t_1)} - I\right) e^{A(t_1-\sigma)} f(S(\sigma)g)\right\| d\sigma &\leq N_0\,(t_2-t_1)^{1/2} \int_0^{t_1} \frac{N_1 L_{\mathcal{B}_E} K_{\mathcal{B}_E}}{\sqrt{t_1-\sigma}}\, d\sigma \\ &= 2K_{\mathcal{B}_E} L_{\mathcal{B}_E} N_0 N_1 \sqrt{T}\,(t_2-t_1)^{1/2}.\end{aligned}$$

Substituting the above three inequalities into (2.5.8), we obtain

$$\|S(t_2)\,g - S(t_1)\,g\| \leq K_{\mathcal{B}_E}\left(N_0 + 2L_{\mathcal{B}_E} N_1 (1 + N_0\sqrt{T})\right)(t_2-t_1)^{1/2}, \tag{2.5.9}$$

for $0 \leq t_1 < t_2 \leq T$. Thus the third condition in Lemma 2.5.4 is satisfied. For the forth condition, we can use Theorem 47.8 (specifically (47.20) therein) in [37] to confirm the Lipschitz continuity of the mapping $S(t)(\cdot) : \mathcal{B}_E \to \mathcal{B}_E$ for any $t \in [0,T]$ where $T > 0$ is arbitrarily given. Finally, we can apply Lemma 2.5.4 to reach the conclusion that there exists an exponential attractor $\mathscr{E}$ in H for the Boissonade semiflow $\{S(t)\}_{t\geq 0}$. $\square$

2.6 Uniform Dissipativity and Uniform E-Bound of Global Attractors

In the section, we shall prove the upper semi-continuity(also called robustness) of the global attractors for the family of Biossonade semiflows with respect to γ as it converges to zero. Let $\{\{S_\gamma(t)\}_{t\geq 0}\}$ denote the weak solution semiflow of (2.1.6) with $\gamma > 0$ and $\{\{S_0(t)\}_{t\geq 0}\}$ denote the solution semiflow of (2.1.6) with $\gamma = 0$.

Definition 2.6.1. A family of semiflow $\{\{S_\lambda(t)\}_{t\geq 0}\}_{\lambda\in\Lambda}$ on a Banach space X is called *uniformly dissipative* at $\lambda_0 \in \Lambda$, where Λ is an interval of $\mathbb{R}$, if there is a neighborhood U of λ_0 in Λ and there is a bounded set $B \subset X$ such that B is an absorbing set for each semiflow $S_\lambda(t)$, $\lambda \in U$.

Lemma 2.6.2. The family of semiflow $\{\{S_\gamma(t)\}_{t\geq 0}\}_{\gamma\geq 0}$ on H is uniformly dissipative at $\gamma = 0$. Specifically, there exists a constant $K_H > 0$ such that the ball $B_H(0, K_H)$ in H is a common absorbing set for the semiflows $\{S_\gamma(t)\}_{t\geq 0}$ for all $\gamma \in [0, 1]$.

Proof. Taking the inner products $\langle(2.1.1), u(t)\rangle$ and $\langle(2.1.2), zv(t)\rangle$ where z is a positive adjusting parameter to be determined later, and summing them up, by Young's inequality and $\gamma \in [0, 1]$ we get

$$\begin{aligned}
&\frac{1}{2}\left(\frac{d}{dt}\|u\|^2 + z\frac{d}{dt}\|v\|^2\right) + d_1\|\nabla u\|^2 + zd_2\|\nabla v\|^2 \\
=&\int_\Gamma \left(u^2 + (z-\alpha)uv + \gamma u^2 v - u^4 - z\beta v^2\right) dx \\
\leq&\int_\Gamma \left(u^2 + \frac{(z-\alpha)^2 u^2}{z\beta} + \frac{z\beta}{4}v^2 + \frac{u^4}{4} + \gamma^2 v^2 - u^4 - z\beta v^2\right) dx \\
\leq&\int_\Gamma \left((1 + \frac{(z-\alpha)^2}{z\beta})u^2 - \frac{3}{4}u^4 - (\frac{3z\beta}{4} - 1)v^2\right) dx \\
\leq&(1 + \frac{(z-\alpha)^2}{z\beta})^2|\Gamma| - \frac{1}{2}\int_\Gamma u^4 dx - \frac{1}{4}\int_\Gamma v^2 dx \\
=&(1 + (\frac{1}{\beta} - \alpha)^2)^2|\Gamma| - \frac{1}{2}\int_\Gamma u^4 dx - \frac{1}{4}\int_\Gamma v^2 dx.
\end{aligned} \tag{2.6.1}$$

where the uniformly adjusting parameter $z = \frac{5}{3\beta}$.

It follows that

$$\begin{aligned}
&\frac{1}{2}\left(\frac{d}{dt}\|u\|^2 + \frac{5}{3\beta}\frac{d}{dt}\|v\|^2\right) + \frac{1}{2}\|u\|^2 + \frac{1}{4}\|v\|^2 + d_1\|\nabla u\|^2 + \frac{5d_2}{3\beta}\|\nabla v\|^2 \\
\leq&\frac{1}{2}\left(\frac{d}{dt}\|u\|^2 + \frac{5}{3\beta}\frac{d}{dt}\|v\|^2\right) + \int_\Gamma \frac{1}{2}u^4 dx + \frac{1}{8}|\Gamma| + \frac{1}{4}\int_\Gamma v^2 dx + d_1\|\nabla u\|^2 + \frac{5d_2}{3\beta}\|\nabla v\|^2 \\
\leq&(1 + (\frac{1}{\beta} - \alpha)^2)^2|\Gamma| + \frac{1}{8}|\Gamma|.
\end{aligned} \tag{2.6.2}$$

Putting $\tilde{b} = \min\{1, \frac{3\beta}{10}\}$, $\tilde{M} = 2(1 + (\frac{1}{\beta} - \alpha)^2)^2|\Gamma| + \frac{1}{4}|\Gamma|$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\|u\|^2 + \frac{5}{3\beta}\|v\|^2\right) + \tilde{b}(\|u\|^2 + \frac{5}{3\beta}\|v\|^2) \\
&\leq \frac{d}{dt}\left(\|u\|^2 + \frac{5}{3\beta}\|v\|^2\right) + \|u\|^2 + \frac{1}{2}\|v\|^2 \\
&\leq \tilde{M}.
\end{aligned}$$

Applying the Gronwall's inequality, we have

$$\|u\|^2 + \frac{5}{3\beta}\|v\|^2 \leq e^{-\tilde{b}t}(\|u_0\|^2 + \frac{5}{3\beta}\|v_0\|^2) + \frac{\tilde{M}}{\tilde{b}}. \tag{2.6.3}$$

which implies

$$\|u\|^2 + \|v\|^2 \leq \frac{\max\{1, \frac{5}{3\beta}\}}{\min\{1, \frac{5}{3\beta}\}} e^{-\tilde{b}t}(\|u_0\|^2 + \|v_0\|^2) + \frac{\tilde{M}}{\tilde{b}\min\{1, \frac{5}{3\beta}\}}. \tag{2.6.4}$$

From (2.6.4) we obtain

$$\limsup_{t\to\infty} (\|u(t)\|^2 + \|v(t)\|^2) \leq K_H^2 = \frac{2\tilde{M}}{\tilde{b}\min\{1, \frac{5}{3\beta}\}}, \tag{2.6.5}$$

Now K_H is a universal constant independent of both $\gamma \in [0,1]$ and initial data. In other words, for any given bounded set $B_H = \{g_0 : \|g_0\| \leq R\}$ in H, there exists a finite time $T_3 = \frac{1}{\tilde{b}} \ln \frac{\tilde{b}\max\{1, \frac{5}{3\beta}\}R^2}{\tilde{M}}$ such that $\|u(t)\|^2 + \|v(t)\|^2 \leq K_H^2$ for any $g_0 \in B_H$ and all $t \geq T_3$. Therefore, we have shown that the ball $B_H(0, K_H)$ is a common absorbing set for the semiflow $\{S_\gamma\}$ for each $\gamma \in [0,1]$.

□

Let

$$\mathcal{U} = \bigcup_{0\leq\gamma\leq 1} \mathscr{A}_\gamma.$$

The following corollary shows that the bundles of trajectories through $\mathcal{U} = \bigcup_{0\leq\gamma\leq 1} \mathscr{A}_\gamma$ under the semiflows $\{S_\gamma\}$ for any $\gamma \in [0,1]$ are uniformly bounded in H.

Corollary 2.6.1. *There exists a constant $M_H > 0$ such that*

$$\sup_{0\leq\gamma\leq 1} \sup_{t\geq 0} S_\gamma(t)\mathcal{U} \subset B_H(0, M_H),$$

where $B_H(0, M_H)$ is the closed ball in H centered at the origin with radius M_H.

Proof. From (2.6.4) we have $M_H^2 = \frac{\max\{1,\frac{5}{3\beta}\}}{\min\{1,\frac{5}{3\beta}\}} K_H^2 + \frac{2\tilde{M}}{\min\{1,\frac{5}{3\beta}\}}$. □

Next we show that the family of global attractors $\mathscr{A}_\gamma$ are bounded in E.

Lemma 2.6.3. There is a constant $K_E > 0$ such that

$$\mathcal{U} = \bigcup_{0\le\gamma\le 1} \mathscr{A}_\gamma \subset B_E(0, K_E),$$

where $B_E(0, K_E)$ is the closed ball in E centered at the origin with radius K_E.

Proof. For any $t \ge 0$, (2.2.3) implies that

$$\begin{aligned}\int_t^{t+1} 2(d_1\|\nabla u(s)\|^2 ds + \frac{5d_2}{3\beta}\|\nabla v(s)\|^2) ds &\le \tilde{M} + \|u(t)\|^2 + \frac{5}{3\beta}\|v(t)\|^2 \\ &\le \tilde{M} + \max\{1, \frac{5}{3\beta}\} K_H^2.\end{aligned} \tag{2.6.6}$$

Since any trajectories started from any $(u_0, v_0) \in \mathcal{U} \subset E$ are strong solution in E, we have $\|\nabla u(s)\|^2 + \|\nabla v(s)\|^2 \in C([t, t+1]; \mathbb{R})$, for any $t > 0$. By (2.6.6) and the Mean Value Theorem, for any given $t > 0$ there is a time $\tau \in [t, t+1]$ such that

$$\begin{aligned}\|S_\gamma(\tau)(u_0, v_0)\|_E^2 &= \|\nabla u(\tau)\|^2 + \|\nabla v(\tau)\|^2 \\ &\le \frac{\tilde{M} + \max\{1, \frac{5}{3\beta}\} K_H^2}{2\min\{d_1, \frac{5d_2}{3\beta}\}},\end{aligned} \tag{2.6.7}$$

for any $(u_0, v_0) \in \mathcal{U}$ and any $\lambda \in [0, 1]$.

By the invariance of every global attractor $\mathscr{A}_\gamma$ for $\gamma \in [0,1]$, namely, $S_\gamma(t)\mathscr{A}_\gamma = \mathscr{A}_\gamma$ for any $t > 0$, we have proved that

$$\mathscr{A}_\gamma \in B_E(0, K_E), \quad \text{for all } \gamma \in [0,1]$$

where $B_E(0, K_E)$ is the closed ball in E centered at the origin with radius

$$K_E = \left(\frac{\tilde{M} + \max\{1, \frac{5}{3\beta}\} K_H^2}{2\min\{d_1, \frac{5d_2}{3\beta}\}}, \right)^{\frac{1}{2}}.$$

So the union of $\mathscr{A}_\gamma$, where $\gamma \in [0,1]$, is bounded in $B_E(0, K_E)$. □

2.7 Upper Semi-continuity of Global Attractors

We have shown in Lemma2.6.3 that the union of the family of global attractors $\mathcal{U} = \bigcup_{0\leq\gamma\leq 1} \mathscr{A}_\gamma$ is bounded in E, but it does not automatically imply that the trajectories $S_\gamma(t)\mathcal{U}, t \geq 0$, are uniformly E-bounded with respect to all $\gamma \in [0,1]$, since the union set $\mathcal{U}$ is not an invariant set with respect to each particular semiflow $\{S_\gamma\}$for a given γ.

The following theorem shows that the bundles of trajectories through $\mathcal{U} = \bigcup_{0\leq\gamma\leq 1} \mathscr{A}_\gamma$ under the semiflows $\{S_\gamma\}$ for any $\gamma \in [0,1]$ are uniformly bounded in E.

Theorem 2.7.1. There exists a constant $M_E > 0$ such that

$$\sup_{0\leq\gamma\leq 1} \sup_{t\geq 0} S_\gamma(t)\mathcal{U} \subset B_E(0, M_E), \tag{2.7.1}$$

where $B_E(0, M_E)$ is the closed ball in E centered at the origin with radius M_E.

Proof. Taking the inner-products $\langle(2.1.2), -\Delta v(t)\rangle$ to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla v\|^2 + d_2\|\Delta v\|^2 \\
&= \int_\Gamma \left(-u\Delta v - \beta|\nabla v|^2\right)\, dx \\
&\leq \int_\Gamma \left(\frac{u^2}{2d_2} + \frac{d_2}{2}|\Delta v|^2\right)\, dx - \beta\|\nabla v\|^2.
\end{aligned}$$

It follows that

$$\frac{d}{dt}\|\nabla v\|^2 + d_2\|\Delta v\|^2 + 2\beta\|\nabla v\|^2 \leq \frac{\|u\|^2}{2d_2}$$

By the Gronwall inequality and uniform boundedness of $\mathcal{U}$ in H and E, we have

$$\|\nabla v\|^2 \leq e^{-2\beta t}\|\nabla v_0\|^2 + \frac{1}{2d_2}\left(\frac{c_2}{c_1}K_H^2 + \frac{2M_1}{b_1 c_1}\right) \leq D_1, \tag{2.7.2}$$

for any $t \geq 0, g_0 \in \mathcal{U}$ and $\gamma \in [0,1]$, where

$$D_1 = K_E^2 + \frac{1}{2d_2}\left(\frac{c_2}{c_1}K_H^2 + \frac{2M_1}{b_1 c_1}\right).$$

Next, take the inner-products $\langle(2.1.1), -\Delta u(t)\rangle$, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 + d_1\|\Delta u\|^2 \\
&= \int_\Gamma \left(|\nabla u|^2 + \alpha v \Delta u - \gamma u v \Delta u - 3u^2|\nabla u|^2\right) dx \\
&\le \int_\Gamma \left(|\nabla u|^2 + \frac{\alpha^2 v^2}{d_1} + \frac{d_1}{4}|\Delta u|^2 + \frac{\gamma^2 u^2 v^2}{d_1} + \frac{d_1}{4}|\Delta u|^2\right) dx \\
&\le \int_\Gamma \left(|\nabla u|^2 + \frac{\alpha^2 v^2}{d_1} + \frac{d_1}{4}|\Delta u|^2 + \frac{1}{2d_1}\left(u^4 + v^4\right) + \frac{d_1}{4}|\Delta u|^2\right) dx
\end{aligned}
$$

where note $\gamma \in [0, 1]$. It follows that

$$
\frac{d}{dt}\|\nabla u\|^2 + d_1\|\Delta u\|^2 \tag{2.7.3}
$$

$$
\le 2\|\nabla u\|^2 + \frac{2\alpha^2}{d_1}\|v\|^2 + \frac{1}{d_1}\left(\|u\|_{L^4}^4 + \|v\|_{L^4}^4\right) \tag{2.7.4}
$$

Now we use the Garliardo-Nirenberg interpolation inequality again [37, Theorem B.3],

$$
\|\zeta\|_{W^{k,p}} \le C\|\zeta\|_{W^{m,q}}^{\theta}\|\zeta\|_{L^r}^{1-\theta}, \quad \text{for } \zeta \in W^{m,q}(\Gamma), \tag{2.7.5}
$$

provided that $p, q, r \ge 1, 0 < \theta \le 1$, and

$$
k - \frac{n}{p} \le \theta\left(m - \frac{n}{q}\right) - (1-\theta)\frac{n}{r}, \quad \text{where } n = \dim\Gamma.
$$

Here with $W^{k,p}(\Gamma) = L^4(\Gamma), W^{m,q}(\Gamma) = H^2(\Gamma) \cap H_0^1(\Gamma), L^r(\Gamma) = L^2(\Gamma)$, $\theta = 3/8$ and $n \le 3$, we have

$$
\|\varphi\|_{L^4} \le C_2\|\varphi\|_{H^2\cap H_0^1}^{\frac{3}{8}}\|\varphi\|^{\frac{5}{8}} \tag{2.7.6}
$$

where $C_2 > 0$ is constant. Due to the Dirichlet boundary condition and the operator interpolation property, we can take the equivalent norm

$$
\|\phi\|_{H^2\cap H_0^1}^2 = \|\Delta\phi\|^2 + \|\nabla\phi\|^2 \quad \text{for } \phi \in H^2 \cap H_0^1.
$$

Thus we deduce that

$$
\begin{aligned}
\frac{1}{d_1}\|u\|_{L^4}^4 &\le d_1^{-1}C_2^4[\|u\|_{H^2\cap H_0^1}^{\frac{3}{8}}\|u\|^{\frac{5}{8}}]^4 \\
&= d_1^{-1}C_2^4\|u\|_{H^2\cap H_0^1}^{\frac{3}{2}}\|u\|^{\frac{5}{2}} \\
&\le \frac{3d_1}{4}\|u\|_{H^2\cap H_0^1}^2 + \frac{1}{4}d_1^{-7}C_2^{16}\|u\|^{10} \\
&\le \frac{3d_1}{4}(\|\Delta u\|^2 + \|\nabla u\|^2) + \frac{1}{4}d_1^{-7}C_2^{16}\|u\|^{10}
\end{aligned}
$$

From the embedding $H_0^1(\Gamma) \hookrightarrow L^4(\Gamma)$ (for $n \leq 3$), we get

$$\|v\|_{L^4}^4 \leq \eta^4 \|\nabla v\|^4, \quad \text{since} \quad v(t) \in E \quad t > 0.$$

Substitute the above two estimates into (2.7.3), we have

$$\begin{aligned}
&\frac{d}{dt}\|\nabla u\|^2 + d_1\|\Delta u\|^2 \\
&\leq 2\|\nabla u\|^2 + \frac{2\alpha^2}{d_1}\|v\|^2 + \frac{3d_1}{4}(\|\Delta u\|^2 + \|\nabla u\|^2) + \frac{1}{4}d_1^{-7}C_2^{16}\|u\|^{10} + \frac{\eta^4}{d_1}\|\nabla v\|^4 \\
&\leq \frac{8+3d_1}{4}\|\nabla u\|^2 + \frac{2\alpha^2}{d_1}M_H^2 + \frac{3d_1}{4}\|\Delta u\|^2 + \frac{1}{4}d_1^{-7}C_2^{16}M_H^5 + \frac{\eta^4}{d_1}D_1^2
\end{aligned}$$

To proceed, we note that

$$\|\nabla u\|^2 = \int_\Gamma |u\Delta u| dx \leq \int_\Gamma \left(\frac{1}{2(8+3d_1)}|\Delta u|^2 + \frac{(8+3d_1)}{2}u^2\right) dx$$

Hence we have

$$\begin{aligned}
&\frac{d}{dt}\|\nabla u\|^2 + \frac{d_1}{8}\|\Delta u\|^2 \\
&\leq \frac{(8+3d_1)}{2}M_H^2 + \frac{2\alpha^2}{d_1}M_H^2 + \frac{1}{4}d_1^{-7}C_2^{16}M_H^5 + \frac{\eta^4}{d_1}D_1^2
\end{aligned}$$

Thus we get

$$\frac{d}{dt}\|\nabla u\|^2 + \frac{d_1}{4}\|\nabla u\|^2 \leq \frac{d}{dt}\|\nabla u\|^2 + \frac{d_1}{8}(\|\Delta u\|^2 + \|u\|^2) \leq D_2, \quad t > 0, g_0 \in \mathfrak{U}, \gamma \in [0,1] \tag{2.7.7}$$

where D_2 is universal constant given by

$$D_2 = \frac{d_1}{8}M_H^2 + \frac{(8+3d_1)}{2}M_H^2 + \frac{2\alpha^2}{d_1}M_H^2 + \frac{1}{4}d_1^{-7}C_2^{16}M_H^5 + \frac{\eta^4}{d_1}D_1^2$$

Apply the Gronwall inequality to (2.7.7), we have

$$\|\nabla u\|^2 \leq e^{\frac{-d_1 t}{4}}\|\nabla u_0\|^2 + \frac{4D_2}{d_1} \leq K_E^2 + \frac{4D_2}{d_1} \tag{2.7.8}$$

for any $t \geq 0, g_0 \in \mathfrak{U}$ and $\gamma \in [0,1]$. Sum up (2.7.2) and (2.7.8), we conclude that

$$\|S_\gamma(t)g_0\|_E^2 \leq D_1 + K_E^2 + \frac{4D_2}{d_1}. \quad t > 0, g_0 \in \mathfrak{U}, \gamma \in [0,1] \tag{2.7.9}$$

Therefore, (2.7.1) is valid with $M_E = (D_1 + K_E^2 + \frac{4D_2}{d_1})^{1/2}$. The proof is complete. $\square$

Corollary 2.7.1. *The global attractor $\mathscr{A}_0$ for the Boissonade semiflow $\{S_0\}$ attracts the union set*

$$\mathcal{U} = \bigcup_{0 \le \gamma \le 1} \mathscr{A}_\gamma$$

with respect to E-norm.

Next we present the Gronwall-Henry inequality [37] in detail, which will be utilized to prove the uniform convergence theorem in the next step.

Lemma 2.7.2. Let $\psi(t)$ be a nonnegative function in $L^\infty_{loc}([0,T), R)$ and $\zeta(\cdot) \in L^1_{loc}([0,T))$ such that the inequality

$$\psi(t) \le \zeta(t) + \mu \int_0^t (t-s)^{r-1} \psi(s) ds, \; t \in (0,T)$$

where $0 < T \le \infty$ and $r > 0$ is satisfied. Then it holds that

$$\psi(t) \le \zeta(t) + \kappa \int_0^t \Phi(\kappa(t-s)) \psi(s) ds, \; t \in (0,T)$$

where $\kappa = (\mu \Gamma(r))^{\frac{1}{r}}$, $\Gamma(\cdot)$ is the Gamma function, and the function $\Phi(t)$ is given by

$$\Phi(t) = \sum_{n=1}^\infty \frac{1}{\Gamma(nr)} t^{nr-1}$$

Theorem 2.7.3. For any given $t \ge 0$, it holds that

$$\sup_{g_0 \in \mathcal{U}} \| S_\gamma(t) g_0 - S_0(t) g_0 \|_E \to 0, \quad \text{as} \quad \gamma \to 0^+ \tag{2.7.10}$$

where $\mathcal{U} = \bigcup_{0 \le \gamma \le 1} \mathscr{A}_\gamma$.

Proof. For any given $g_0 = (u_0, v_0) \in \mathcal{U} \subset B_H(0, M_H) \bigcap B_E(0, M_E)$, let $(u(t), v(t)) = S_\gamma(t)(u_0, v_0)$ and $(\tilde{u}(t), \tilde{v}(t)) = S_0(t)(u_0, v_0)$ be the two trajectories starting from the same initial point g_0 under the semiflows $\{S_\gamma\}$ and $\{S_0\}$, respectively. Define

$$w(t) = S_\gamma(t) g_0 - S_0(t) g_0, \quad t \ge 0,$$

and we have $w(0) = 0$. Since both $S_\gamma(t) g_0$ and $S_0(t) g_0$ are strong solutions in E, they are the mild solution, so that they satisfy the integral equation

$$w(t) = \int_0^t e^{A(t-\sigma)} [f_0(S_\gamma(\sigma) g_0) - f_0(S_0(\sigma) g_0)] d\sigma + \gamma \int_0^t e^{A(t-\sigma)} h(S_\gamma(\sigma) g_0) d\sigma, \quad t \ge 0. \tag{2.7.11}$$

where $e^{A}, t \geq 0$, is the C_0-semigroup generated by $A : D(A) \to H$,

$$f_0(u,v) = \begin{pmatrix} u - \alpha v - u^3 \\ \\ u - \beta v \end{pmatrix}, \text{and} \quad h(u,v) = \begin{pmatrix} uv \\ \\ 0 \end{pmatrix}. \tag{2.7.12}$$

Obviously $f_0(u,v) : E \to H$ is locally Lipschitz continuous. Thus there is a Lipschitz constant $L(M_E) > 0$ depending only on M_E given in (2.7.1), such that

$$\|f_0(g_1) - f_0(g_2)\| \leq L(M_E)\|g_1 - g_2\|_E, \quad \text{for any} \quad g_1, g_2 \in B_E(0, M_E).$$

By Theorem 2.7.1 we see that $\sup_{0\leq\gamma\leq 1}\sup_{t\geq 0} S_\gamma(t)\mathfrak{U} \subset B_E(0, M_E)$. Hence,

$$\begin{aligned} \|w(t)\|_E \leq & \int_0^t \|e^{A(t-\sigma)}\|_{\mathcal{L}(H,E)} L(M_E)\|S_\gamma(\sigma)g_0 - S_0(\sigma)g_0\|_E d\sigma \\ & + \gamma \int_0^t \|e^{A(t-\sigma)}\|_{\mathcal{L}(H,E)}\|h(S_\gamma(\sigma)g_0)\| d\sigma, \quad t \geq 0. \end{aligned} \tag{2.7.13}$$

where, we have, cf. [37, Theorem 38.10],

$$\|e^{A(t-\sigma)}\|_{\mathcal{L}(H,E)} \leq M_5\,(t-\sigma)^{-\frac{1}{2}}, \ \ t > \sigma \geq 0, \tag{2.7.14}$$

and

$$\|h(S_\gamma(\sigma)g_0)\| = \left(\int_\Omega u(\sigma,x)v(\sigma,x)dx\right)^{\frac{1}{2}} \leq \frac{1}{2}(\|u\|^2 + \|v\|^2) \leq \frac{1}{2}K_H, \ \ t \geq 0, \tag{2.7.15}$$

for any $g_0 = (u_0, v_0) \in \mathfrak{U}$. Substitute these two inequalities into (2.7.13) to obtain

$$\begin{aligned} \|w(t)\|_E \leq & \frac{\gamma}{2}M_5K_H \int_0^t (t-\sigma)^{-\frac{1}{2}} d\sigma + \int_0^t (t-\sigma)^{-\frac{1}{2}} L(M_E)\|w(\sigma)\|_E d\sigma \\ = & \gamma M_5 K_H t^{\frac{1}{2}} + L(M_E)\int_0^t (t-\sigma)^{-\frac{1}{2}}\|w(\sigma)\|_E d\sigma, \quad t \geq 0. \end{aligned} \tag{2.7.16}$$

To apply the Gronwal-Henry inequality (Lemma 2.7.2), put

$$\psi(t) = \|w(t)\|_E, \zeta(t) = \gamma M_5 K_H t^{\frac{1}{2}}, \mu = L(M_E), \text{and} \quad r = \frac{1}{2}.$$

it yields the estimate

$$\|S_\gamma(t)g_0 - S_0(t)g_0\|_E = \|w(t)\|_E \leq \gamma\left(\xi(t) + \varphi\int_0^t \Phi(\kappa(t-s))\xi(s)ds\right), \ t \in (0,T). \tag{2.7.17}$$

Here we have $\xi(t) = M_5 K_H t^{\frac{1}{2}}, \varphi = (L(M_E)\Gamma(\frac{1}{2}))^2 = L^2(M_E)\pi$, and

$$\Phi(t) = \sum_{n=1}^{\infty} \frac{1}{\Gamma(\frac{n}{2})} t^{\frac{n}{2}-1} = \frac{1}{\sqrt{\pi}} t^{\frac{-1}{2}} + 1 + \sum_{n=1}^{\infty} \frac{2}{n\Gamma(\frac{n}{2})} t^{\frac{n}{2}},$$

thus we conclude that

$$\int_0^t \Phi(\kappa(t-s))\xi(s)ds = M_5 K_H \int_0^t \Phi(\kappa(t-s)) s^{\frac{1}{2}} ds$$

is a nonnegative continuous function of $t \in [0, \infty)$.

The inequality (2.7.17) confirms that, for each fixed $t \geq 0$, the uniform convergence (2.7.10) holds:

$$\sup_{g_0 \in \mathfrak{U}} \|S_\gamma(t)g_0 - S_0(t)g_0\|_E \to 0, \quad \text{as} \quad \gamma \to 0^+.$$

The proof is completed. $\square$

Finally we come to the second main theorem.

Theorem 2.7.4. Given any positive parameters d_1, d_2, α, β, the family of global attractors $\mathscr{A}_\gamma, \gamma \geq 0$, has the upper semi-continuity in E with respect to $\gamma \geq 0$ as it converges to zero, that is

$$\text{dist}_E(\mathscr{A}_\gamma, \mathscr{A}_0) \to 0, \quad \text{as} \quad G \to 0^+.$$

Proof. Given an arbitrarily small $\epsilon > 0$, by Corollary 2.7.1 the global attractor $\mathscr{A}_0$ attracts the union set

$$\mathfrak{U} = \bigcup_{0 \leq \gamma \leq 1} \mathscr{A}_\gamma$$

with respect to the Hausdorff semi-distance in E-norm under the action of the semiflow $\{S_0\}$. In other words, there is a finite time $t_0 > 0$ such that

$$S_0(t_0)\mathfrak{U} \subset \mathscr{N}(\mathscr{A}_0, \frac{\varepsilon}{2}), \tag{2.7.18}$$

where $\mathscr{N}(\mathscr{A}_0, \frac{\varepsilon}{2})$ is the neighborhood of $\mathscr{A}_0$ in the space E. Then by Theorem 2.7.3, there exists a $\gamma_0 \in (0, 1]$ such that

$$\|S_\gamma(t)g_0 - S_0(t)g_0\|_E < \frac{\varepsilon}{2}, \quad \text{for any} \quad \gamma \in (0, \gamma_0). \tag{2.7.19}$$

Since every global attractor $\mathscr{A}_\gamma$ is an invariant set, from (2.7.18) and (2.7.19) one has

$$\mathscr{A}_\gamma = S_\gamma(t_0)\mathscr{A}_\gamma \subset S_\gamma(t_0)\mathfrak{U} \subset \mathscr{N}(S_0(t_0)\mathfrak{U}, \frac{\varepsilon}{2}) \subset \mathscr{N}(\mathscr{A}_0, \varepsilon),$$

for any $0 < \gamma < \gamma_0$. Therefore, the upper semi-continuity of this family of global attractors in the space E, which is

$$\text{dist}_E(\mathscr{A}_\gamma, \mathscr{A}_0) \to 0, \quad \text{as} \quad \gamma \to 0^+.$$

is proved.

$\square$

Chapter 3

Pullback attractor of non-autonomous Selkov system

3.1 Introduction

The Selkov system was originally a system of ODEs proposed by E.E. Selkov [36] as a simplified model of a biochemical process called *glycolysis*, through which living cells get energy from breaking down sugar. It is a prototype of cubic-autocatalytic chemical and biochemical reactions that include the chlorite-iodide-malonic-acid (CIMA) reaction. The reversible cubic-autocatalytic Selkov equations can be derived from the following schemes of reversible chemical reaction:

$$A \leftrightarrows^{k_1}_{k_{-1}} S, \quad S + 2P \leftrightarrows^{k_2}_{k_{-2}} 3P, \quad P \leftrightarrows^{k_3}_{k_{-3}} B \tag{3.1.1}$$

These reactions involve *non-zero reaction rates*, which implies “reversible”.

In this chapter, we are concerned with the following non-autonomous reversible Selkov system,

$$\frac{\partial u}{\partial t} = d_1 \Delta u + \rho - au + u^2 v - Gu^3 + h(t,x), \tag{3.1.2}$$

$$\frac{\partial v}{\partial t} = d_2 \Delta v + \beta - bv - u^2 v + Gu^3 + k(t,x), \tag{3.1.3}$$

for $(t,x) \in \mathbb{R} \times \Gamma$, where $\Gamma \subset \mathbb{R}^n (n \leq 3)$ is a bounded domain with a locally Lipschitz continuous boundary, and the coefficients $d_1, d_2, \rho, \beta, a, b$ are positive constants. $h(t,x), k(t,x)$ are time-dependent perturbation terms. We shall study the asymptotic dynamics of the weak solutions to this reaction diffusion system (3.1.2) – (3.1.3) coupled with the homogeneous Dirichlet boundary condition

$$u(t,x) = v(t,x) = 0, \quad t \in \mathbb{R},\ x \in \partial\Gamma, \tag{3.1.4}$$

and an initial condition at $\tau - t \in \mathbb{R}$,

$$u(\tau - t, x) = u_{\tau - t}(x),\ v(\tau - t, x) = v_{\tau - t}(x), \tag{3.1.5}$$

The existence and properties of global attractors for the autonomous reversible Selkov system was studied by Y. You [47]. The main difficulty of proving the absorbing property and asymptotic compactness lies in the signed quadratic $\pm uv$ and cubic terms $\pm Gu^3$. It is nontrivial to generalize the autonomous system to the non-autonomous case. The challenge includes carefully handling the initial time and pinpointing a middle time in the process in order to prove the pullback asymptotic compactness.

First we formulate the problem into mathematical setting. By Sobolev embedding theorem, $H_0^1(\Gamma) \hookrightarrow L^6(\Gamma)$ is a continuous embedding for $n \leq 3$. Via the generalized Hölder inequality, we have

$$\|u^2 v\| \leq \|u\|_{L^6}^2 \|v\|_{L^6}, \quad \|u^3\| \leq \|u\|_{L^6}^3, \quad \text{for } u, v \in L^6(\Gamma).$$

Hence the nonlinear mapping

$$f(g) = \begin{pmatrix} \rho - au + u^2 v - Gu^3 \\ \beta - bv - u^2 v + Gu^3 \end{pmatrix} : E \longrightarrow H, \tag{3.1.6}$$

where $g = (u, v)$, is a locally Lipschitz continuous mapping defined on E. Thus the initial-boundary value problem (3.1.2)–(3.1.5) of the non-autonomous Selkov system is formulated into an initial value problem:

$$\begin{aligned} \frac{dg}{dt} &= Ag + f(g) + P(t), \quad t \in \mathbb{R}. \\ g(\tau - t) &= g_{\tau - t} = (u_{\tau - t}, v_{\tau - t}) \in H, \end{aligned} \tag{3.1.7}$$

where $P(t) = (h(t), k(t))$. We have the standing assumptions for the non-autonomous perturbation terms:

1.
$$\int_{-\infty}^{\tau} e^{\gamma \xi} (\|h(\xi)\|^{2p} + \|k(\xi)\|^{2p}) d\xi < \infty, \quad \forall \, \tau \in \mathbb{R}, \tag{3.1.8}$$

2.
$$\sup_{t \in \mathbb{R}} \int_t^{t+1} (\|h(\xi)\|^2 + \|k(\xi)\|^2) d\xi = H < \infty, \tag{3.1.9}$$

where $p = 1, 2, 3$ and $\gamma = \min\{a, b\}$.

We point out that the second assumption is necessary to prove the asymptotic pullback compactness, which needs additional treatment than the reaction diffusion equation in [40]. The second assumption implies the first one for $p = 1$. It is not uncommon to impose the second assumption since it was assumed in proving the existence of uniform attractors [12].

The rest of the chapter is organized as follows. In Section 2 we shall prove the global existence of the weak solutions of the non-autonomous reversible Selkov equation (3.1.7). In Section 3 we show the pullback absorbing property of this non-autonomous reversible Selkov process. We show the pullback asymptotic compactness and the existence of the pullback attractor for this process in Section 4.

3.2 Preliminaries and Formulation

In this section we present some basic concepts in the theory of pullback attractors of non-autonomous dynamical systems. The readers are referred to [11] for a detailed introduction.

Definition 3.2.1. Let X be a Banach space, a process in X is a family of maps $S(t, s) : t \geq s$ such that the following holds:

1. $S(t, t) = I$, for all $t \in \mathbb{R}$,
2. $S(t, s) = S(t, \tau)S(\tau, s)$, for all $t \geq \tau \geq s$,
3. $(t, s, x) \mapsto S(t, s)x$ is continuous with respect to t, s, x, where $t \geq s, x \in X$.

In the sequel, we shall denote a process by $\{S(\tau, \tau - t)\}_{\tau\in\mathbb{R},t\geq 0}$ or simply by $\{S(\tau, \tau - t)\}_{t\geq 0}$ to highlight the "pullback" action. In other words, we freeze the final time at τ and pullback the initial time as early as possible by sending $t \to +\infty$. For simplicity, we write " the process $S(\cdot, \cdot)$" if the time parameters are not stressed.

Definition 3.2.2. Let $S(\cdot, \cdot)$ be a process on X. A bounded subset B_0 of X is called a *pullback absorbing set in* X at time τ if, for any bounded subset $B \subset X$, there is some finite time $t_0 \geq 0$

depending on B such that $S(\tau, \tau - t)B \subset B_0$ for all $t > t_0$. A family of bounded subsets $B(\cdot)$ pullback absorbs bounded sets if $B(\tau)$ pullback absorbs bounded set for any time $\tau \in \mathbb{R}$.

Definition 3.2.3. A process $\{S(\tau, \tau - t)\}_{t \geq 0}$ on a Banach space X is called *pullback asymptotically compact in* X if for any bounded sequences $\{x_n\}$ in X and $\{t_n\} \subset (0, \infty)$ with $t_n \to \infty$, there exist subsequences $\{x_{n_k}\}$ of $\{x_n\}$ and $\{t_{n_k}\}$ of $\{t_n\}$, such that $\lim_{k \to \infty} S(\tau, \tau - t_{n_k})x_{n_k}$ exists in X.

Definition 3.2.4. Let $S(\cdot, \cdot)$ be a process on a Banach space X. A family $\{\mathscr{A}(\tau) : \tau \in \mathbb{R}\}$ in X is called *a pullback attractor in* X for this process, if the following conditions are satisfied:

(i) $\mathscr{A}(\tau)$ is nonempty, compact for each $\tau \in \mathbb{R}$,

(ii) $\mathscr{A}(\cdot)$ is invariant in the sense that

$$S(\tau, s)\mathscr{A}(s) = \mathscr{A}(\tau) \quad \text{for any } \ \tau \geq s.$$

(iii) For each fixed $\tau \in \mathbb{R}$, $\mathscr{A}(\tau)$ pullback attracts any bounded set B of X in terms of the Hausdorff semi-distance, i.e.,

$$\text{dist}(S(\tau, s)B, \mathscr{A}(\tau)) = \sup_{x \in B} \inf_{y \in \mathscr{A}(\tau)} \|S(\tau, s)x - y\|_X \to 0, \quad \text{as } \ s \to -\infty.$$

(iv)$\mathscr{A}(\cdot)$ is the minimal family of closed sets with property(iii).

The following proposition states the basic result on the existence of a pullback attractor for a process, cf. [11, 40].

Proposition 3.2.5. Let $\{S(\tau, \tau - t)\}_{t \geq 0}$ be a process on a Banach space X. If the following conditions are satisfied:

(i) $\{S(\tau, \tau - t)\}_{t \geq 0}$ has a family of absorbing sets $B(\tau)$in X, and

(ii) $\{S(\tau, \tau - t)\}_{t \geq 0}$ is asymptotically compact in X,

then there exists a pullback attractor $\mathscr{A}(\tau)$ in X for this process, which is given by

$$\mathscr{A}(\tau) = \bigcap_{l \geq 0} \text{Cl}_X \bigcup_{t \geq l} S(\tau, \tau - t)B(\tau - t).$$

3.3 Pullback Absorbing Property

In this section we prove the global existence of weak solution to (3.1.7) and the pullback absorbing properties of the generated process.

Lemma 3.3.1. For every $\tau \in \mathbb{R}$ and any initial data $g_0 = (u_0(\tau - t), v_0(\tau - t)) \in H$, there exists a unique global weak solution $g(\tau, \tau - t, g_0(\tau - t)) = (u(\tau, \tau - t, u_0(\tau - t)), v(\tau, \tau - t, u_0(\tau - t)))$, $t \in [0, \infty)$, of the IVP of the non-autonomous reversible Selkov equation (3.1.7) and it becomes a strong solution on the time interval $(\tau - t, \infty)$. Moreover, there exists a constant $K_\tau > 0$, such that the set

$$B_\tau = \left\{ g \in H : \|g\|^2 \le K_\tau \right\} \tag{3.3.1}$$

is a pullback absorbing set in H for the non-autonomous reversible Selkov process $\{S(\tau, \tau - t)\}_{t \ge 0}$.

Proof. The proof of the local existence of weak solution is similar with the autonomous case by conducting *a priori* estimates on the Galerkin approximations of the initial value problem (3.1.7) and the argument of the weak/weak* convergence, see [47] for details. We conduct *a priori* estimates as follows.

Multiplying (3.1.2) with $Gu(t)$, (3.1.3) with $v(t)$ respectively, integrating on Γ, and summing them up, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(G\|u\|^2 + \|v\|^2\right) + d_1 G\|\nabla u\|^2 + d_2\|\nabla v\|^2 \\
=&\int_\Gamma (\rho G u + \beta v)\, dx - \left(aG\|u\|^2 + b\|v\|^2\right) - \int_\Gamma \left(G^2u^4 - 2Gu^3v + u^2v^2\right) dx \\
&+ \int_\Gamma (Ghu + kv)\, dx \\
\le&\rho G|\Gamma|^{\frac{1}{2}}\|u\| + \beta|\Gamma|^{\frac{1}{2}}\|v\| - \left(aG\|u\|^2 + b\|v\|^2\right) + G\|h\|\|u\| + \|k\|\|v\| \\
\le&\frac{\rho^2 G|\Gamma|}{a} + \frac{aG}{4}\|u\|^2 + \frac{\beta^2|\Gamma|}{b} + \frac{b}{4}\|v\|^2 \\
&- \left(aG\|u\|^2 + b\|v\|^2\right) + \frac{aG}{4}\|u\|^2 + \frac{G}{a}\|h\|^2 + \frac{1}{b}\|k\|^2 + \frac{b}{4}\|v\|^2.
\end{aligned} \tag{3.3.2}$$

It follows that

$$
\begin{aligned}
&\frac{d}{dt}\left(G\|u\|^2+\|v\|^2\right)+2d_1G\|\nabla u\|^2+2d_2\|\nabla v\|^2+\left(aG\|u\|^2+b\|v\|^2\right)\\
&\le \frac{2\rho^2G|\Gamma|}{a}+\frac{2\beta^2|\Gamma|}{b}+\frac{2G}{a}\|h\|^2+\frac{2}{b}\|k\|^2.
\end{aligned}
\tag{3.3.3}
$$

Let $\gamma=\min\{a,b\}$, $d_0=\min\{d_1,d_2\}$, $M_1=\frac{2\rho^2G|\Gamma|}{a}+\frac{2\beta^2|\Gamma|}{b}$, we end up with

$$
\begin{aligned}
&\frac{d}{dt}\left(G\|u\|^2+\|v\|^2\right)+2d_0(G\|\nabla u\|^2+\|\nabla v\|^2)+\gamma\left(G\|u\|^2+\|v\|^2\right)\\
&\le\ M_1+\frac{2G}{a}\|h\|^2+\frac{2}{b}\|k\|^2.
\end{aligned}
\tag{3.3.4}
$$

Multiplying by $e^{\gamma t}$, then integrating over $[\tau-t,\tau]$ where $t\ge 0$, we have

$$
\begin{aligned}
&G\|u(\tau,\tau-t,u_0(\tau-t))\|^2+\|v(\tau,\tau-t,v_0(\tau-t))\|^2\\
&+2d_0e^{-\gamma\tau}\int_{\tau-t}^{\tau}\left(G\|\nabla u(\tau,\tau-t,u_0(\tau-t))\|^2+\|\nabla v(\tau,\tau-t,u_0(\tau-t))\|^2\right)e^{\gamma s}ds\\
&\le e^{-\gamma t}(G\|u_0(\tau-t)\|^2+\|v_0(\tau-t)\|^2)+e^{-\gamma\tau}M_1\int_{\tau-t}^{\tau}e^{\gamma s}ds\\
&\quad +e^{-\gamma\tau}\int_{\tau-t}^{\tau}\left(\frac{2G}{a}\|h\|^2+\frac{2}{b}\|k\|^2\right)e^{\gamma s}ds\\
&\le e^{-\gamma t}(G\|u_0(\tau-t)\|^2+\|v_0(\tau-t)\|^2)+\frac{M_1}{\gamma}+e^{-\gamma\tau}\int_{-\infty}^{\tau}\left(\frac{2G}{a}\|h\|^2+\frac{2}{b}\|k\|^2\right)e^{\gamma s}ds.
\end{aligned}
\tag{3.3.5}
$$

It follows that

$$
\begin{aligned}
&\|u(\tau,\tau-t,u_0(\tau-t))\|^2+\|v(\tau,\tau-t,v_0(\tau-t))\|^2\\
&\le\frac{max\{1,G\}}{min\{1,G\}}e^{-\gamma t}(\|u_0(\tau-t)\|^2+\|v_0(\tau-t)\|^2)+\frac{M_1|\Gamma|}{min\{1,G\}\gamma}\\
&+\frac{max\{\frac{2G}{a},\frac{2}{b}\}}{min\{1,G\}}e^{-\gamma\tau}\int_{-\infty}^{\tau}\left(\|h\|^2+\|k\|^2\right)e^{\gamma s}ds.
\end{aligned}
\tag{3.3.6}
$$

The inequality (3.3.6) shows that the weak solution $g(\tau,\tau-t)$ will never blow up as long as we pullback the initial time. The family of all the global weak solutions $\{g(\tau,\tau-t;g_0(\tau-t)):t\ge 0, g_0\in H\}$ defines a process on H,

$$
S(\tau,\tau-t): g_0\mapsto g(\tau,\tau-t;g_0),\quad g_0\in H,\ t\ge 0,
$$

which will be called the *non-autonomous reversible Selkov process* associated with the evolutionary equation (3.1.7).

Moreover, the inequality (3.3.6) shows that the ball $B(\tau)$ shown in (3.3.1) with

$$K_\tau = 1 + \frac{M_1|\Gamma|}{min\{1,G\}\gamma} + \frac{max\{\frac{2G}{a},\frac{2}{b}\}}{min\{1,G\}} e^{-\gamma\tau}\int_{-\infty}^{\tau}\left(\|h\|^2+\|k\|^2\right)e^{\gamma s}ds$$

is a pullback absorbing set. Indeed,

$$\begin{aligned}&\limsup_{t\to+\infty}\|u(\tau,\tau-t,u_0(\tau-t))\|^2+\|v(\tau,\tau-t,v_0(\tau-t))\|^2\\ &\le 1+\frac{M_1|\Gamma|}{min\{1,G\}\gamma}+\frac{max\{\frac{2G}{a},\frac{2}{b}\}}{min\{1,G\}}e^{-\gamma\tau}\int_{-\infty}^{\tau}\left(\|h\|^2+\|k\|^2\right)e^{\gamma s}ds,\end{aligned} \tag{3.3.7}$$

and for any ball $B\subset H$ centered at the origin with radius R, there is a finite time

$$t_0(R)=\frac{1}{\gamma}log\frac{R^2max\{1,G\}}{min\{1,G\}}$$

such that $S(\tau,\tau-t)B\subset B_\tau$ for any $t\ge t_0$. The proof is completed. □

Integrating (3.3.4) over $[\tau-T,\tau-T+1]$ where $0<T\le t$, we have an instrumental inequality to prove the pullback asymptotic compactness as follows,

$$\begin{aligned}&\int_{\tau-T}^{\tau-T+1}(\|G\nabla u(s)\|^2+\|\nabla v(s)\|^2)ds\\ &\qquad\le\frac{M_1}{2d_0}+\frac{\max\{2Ga^{-1},2b^{-1}\}}{2d_0}H+G\|u_0(\tau-T)\|^2+\|v_0(\tau-T)\|^2.\end{aligned}\tag{3.3.8}$$

Note that we use the second standing assumption on perturbation terms $h(t,x),k(t,x)$ above.

Next we show the absorbing properties of the (u,v) components of the non-autonomous reversible Selkov process in the product Banach spaces $[L^{2p}(\Gamma)]^2$, for any integer $1\le p\le 3$.

Lemma 3.3.2. For any given integer $1\le p\le 3$, there exists a positive constant $K_{\tau,p}$ such that the absorbing inequality

$$\limsup_{t\to\infty}\|(u(\tau,\tau-t,u_0(\tau-t)),v(\tau,\tau-t,v_0(\tau-t)))\|_{L^{2p}}^{2p}<K_{\tau,p} \tag{3.3.9}$$

is satisfied by the (u,v) components of the non-autonomous reversible Selkov process $\{S(\tau,\tau-t)\}_{t\ge0}$ for any initial data $g_0\in H$.

Proof. The case $p = 1$ has been shown in Lemma 3.3.1. According to the smoothing property of parabolic PDEs [47], for any given initial data $g_0 \in H$ there exists a time $t_0 \in (0, 1)$ such that

$$S(\tau, \tau - t + t_0)g_0 \in E = [H_0^1(\Gamma)]^2 \hookrightarrow \mathbb{L}^6(\Gamma) \hookrightarrow \mathbb{L}^4(\Gamma). \tag{3.3.10}$$

Then the weak solution $g(\tau, \tau - t) = S(\tau, \tau - t)g_0$ becomes a strong solution on $[\tau - t + t_0, \tau]$ and satisfies

$$\begin{aligned} S(\cdot, \cdot)g_0 &\in C([\tau - t + t_0, \tau); E) \cap L^2(\tau - t + t_0, \tau; \Pi) \subset C([\tau - t + t_0, \tau); \mathbb{L}^6(\Gamma)) \\ &\subset C([\tau - t + t_0, \tau); \mathbb{L}^4(\Gamma)), \end{aligned} \tag{3.3.11}$$

for $n \leq 3$. Based on this observation, without loss of generality, we can *assume* that $g_0 \in \mathbb{L}^6(\Gamma)$ for the purpose of studying the long-time dynamics. Thus parabolic regularity (3.3.11) of strong solutions implies the $S(\tau, \tau - t)g_0 \in E \subset \mathbb{L}^6(\Gamma), t \geq 0$. By the bootstrap argument, we can *assume* that $g_0 \in \Pi \subset \mathbb{L}^8(\Gamma)$ so that $S(\tau, \tau - t)g_0 \in \Pi \subset \mathbb{L}^8(\Gamma), t \geq 0$.

Now we use another rescaling to conduct estimate. Let $\tilde{u} = u, \tilde{v} = v/G$. Then the original system (3.1.2)-(3.1.3) becomes a rescaled system(we omit the tilde for brevity):

$$\frac{\partial u}{\partial t} = d_1 \Delta u + \rho - au + Gu^2 v - Gu^3 + h(t, x), \tag{3.3.12}$$

$$\frac{\partial v}{\partial t} = d_2 \Delta v + \frac{\beta}{G} - bv + u^2 v - u^3 + \frac{k(t, x)}{G}. \tag{3.3.13}$$

Take the L^2 inner-product $\langle \cdot, \cdot \rangle$ of (3.3.12) with u^3, we obtain

$$\begin{aligned} &\frac{1}{4} \frac{d}{dt} \int_\Gamma u^4(t, x) dx + 3d_1 \|u(t) \nabla u(t)\|^2 \\ = &\int_\Gamma \left(\rho u^3(t, x) - au^4(t, x) + Gu^5(t, x)v(t, x) - Gu^6(t, x) + h(t, x)u^3(t, x)\right) dx. \end{aligned} \tag{3.3.14}$$

Then taking L^2 inner-product $\langle \cdot, \cdot \rangle$ of (3.3.13) with Gv^3, we get

$$\begin{aligned} &\frac{G}{4} \frac{d}{dt} \int_\Gamma v^4(t, x) dx + 3Gd_2 \|v(t) \nabla v(t)\|^2 \\ = &\int_\Gamma \left(\beta v^3(t, x) - bGv^4(t, x) - Gu^2(t, x)v^4(t, x) + Gu^3(t, x)v^3(t, x) + k(t, x)v^3(t, x)\right) dx. \end{aligned} \tag{3.3.15}$$

Recall $d_0 = \min\{d_1, d_2\}$, add up the above two equations (3.3.14) and (3.3.15) to obtain

$$\begin{aligned}
&\frac{1}{4}\frac{d}{dt}\left(\|u\|_{L^4}^4 + G\|v\|_{L^4}^4\right) + 3d_0\left(\|u\nabla u\|^2 + G\|v\nabla v\|^2\right) \\
&= \int_\Gamma \left(\rho u^3 - au^4 + \beta v^3 - bGv^4\right) dx + \int_\Gamma \left(hu^3 + kv\right) dx \\
&-G\int_\Gamma \left(u^6 - u^5 v - u^3 v^3 + u^2 v^4\right) dx \\
&= \int_\Gamma \left(\rho u^3 - au^4 + \beta v^3 - bGv^4\right) dx + \int_\Gamma \left(hu^3 + kv^3\right) dx \\
&+G\int_\Gamma u^2\left(-u^4 + u^3 v + uv^3 - v^4\right) dx.
\end{aligned} \tag{3.3.16}$$

Thanks to the Young inequality, we have

$$-u^4 + u^3 v + uv^3 - v^4 \le -u^4 + \left(\frac{3}{4}u^4 + \frac{1}{4}v^4\right) + \left(\frac{1}{4}u^4 + \frac{3}{4}v^4\right) - v^4 = 0. \tag{3.3.17}$$

and

$$\begin{aligned}
&\int_\Gamma \left((\rho + h)u^3 - au^4 + (\beta + k)v^3 - bGv^4\right) dx \\
&\le -\frac{1}{4}\left(a\|u\|_{L^4}^4 + bG\|v\|_{L^4}^4\right) + \frac{1}{4}\int_\Gamma \left(\frac{(\rho + h)^4}{a^3} + \frac{(\beta + k)^4}{b^3 G^3}\right) dx \\
&\le -\frac{1}{4}\left(a\|u\|_{L^4}^4 + bG\|v\|_{L^4}^4\right) + \int_\Gamma \left(\frac{2h^4}{a^3} + \frac{2k^4}{b^3 G^3}\right) dx + M_2|\Gamma|,
\end{aligned} \tag{3.3.18}$$

where

$$M_2 = \frac{2\rho^4}{a^3} + \frac{2\beta^4}{b^3 G^3}. \tag{3.3.19}$$

Substituting (3.3.17) and (3.3.18) into (3.3.16), we end up with

$$\begin{aligned}
&\frac{d}{dt}\left(\|u(t)\|_{L^4}^4 + G\|v(t)\|_{L^4}^4\right) + \gamma\left(\|u(t)\|_{L^4}^4 + G\|v(t)\|_{L^4}^4\right) \\
&\le \int_\Gamma \left(\frac{2h^4}{a^3} + \frac{2k^4}{b^3 G^3}\right) dx + M_2|\Gamma|.
\end{aligned}$$

Multiplying by $e^{\gamma t}$, then integrating over $[\tau - t, \tau]$ where $t \geq 0$, we have

$$\begin{aligned}
&\|u(\tau, \tau-t, u_0(\tau-t))\|_{L^4}^4 + G\|v(\tau, \tau-t, v_0(\tau-t))\|_{L^4}^4 \\
\leq\, & e^{-\gamma t}(\|u_0(\tau-t)\|_{L^4}^4 + G\|v_0(\tau-t)\|_{L^4}^4) + e^{-\gamma\tau} M_2|\Gamma| \int_{\tau-t}^{\tau} e^{\gamma s} ds \\
&+ e^{-\gamma\tau} \int_{\tau-t}^{\tau} \left(\frac{2}{a^3}\|h\|_{L^4}^4 + \frac{2}{b^3G^3}\|k\|_{L^4}^4 \right) e^{\gamma s} ds \\
\leq\, & e^{-\gamma t}(\|u_0(\tau-t)\|_{L^4}^4 + G\|v_0(\tau-t)\|_{L^4}^4) + \frac{M_2|\Gamma|}{\gamma} \\
&+ e^{-\gamma\tau} \int_{-\infty}^{\tau} \left(\frac{2}{a^3}\|h\|_{L^4}^4 + \frac{2}{b^3G^3}\|k\|_{L^4}^4 \right) e^{\gamma s} ds.
\end{aligned} \tag{3.3.20}$$

Recall that $\tilde{u} = u, \tilde{v} = v/G$, returning to the original u, v, we have

$$\begin{aligned}
&\|u(\tau, \tau-t, u_0(\tau-t))\|_{L^4}^4 + \frac{1}{G^3}\|v(\tau, \tau-t, v_0(\tau-t))\|_{L^4}^4 \\
\leq\, & e^{-\gamma t}(\|u_0(\tau-t)\|_{L^4}^4 + \frac{1}{G^3}\|v_0(\tau-t)\|_{L^4}^4) + \frac{M_2|\Gamma|}{\gamma} \\
&+ e^{-\gamma\tau} \int_{-\infty}^{\tau} \left(\frac{2}{a^3}\|h\|_{L^4}^4 + \frac{2}{b^3G^3}\|k\|_{L^4}^4 \right) e^{\gamma s} ds,
\end{aligned} \tag{3.3.21}$$

which implies,

$$\begin{aligned}
&\|u(\tau, \tau-t, u_0(\tau-t))\|_{L^4}^4 + \|v(\tau, \tau-t, v_0(\tau-t))\|_{L^4}^4 \\
\leq\, & \frac{max\{1, G^{-3}\}}{min\{1, G^{-3}\}} e^{-\gamma t}(\|u_0(\tau-t)\|_{L^4}^4 + \|v_0(\tau-t)\|_{L^4}^4) + \frac{M_2|\Gamma|}{min\{1, G^{-3}\}\gamma} \\
+ & \frac{max\{\frac{2}{a^3}, \frac{2}{b^3G^3}\}}{min\{1, G^{-3}\}} e^{-\gamma\tau} \int_{-\infty}^{\tau} \left(\|h\|_{L^4}^4 + \|k\|_{L^4}^4 \right) e^{\gamma s} ds.
\end{aligned} \tag{3.3.22}$$

Let $t \to +\infty$, we have (3.3.9) holds for $p = 2$ with

$$K_{2,\tau} = \frac{M_2|\Gamma|}{min\{1, G^{-3}\}\gamma} + \frac{max\{\frac{2}{a^3}, \frac{2}{b^3G^3}\}}{min\{1, G^{-3}\}} e^{-\gamma\tau} \int_{-\infty}^{\tau} \left(\|h\|_{L^4}^4 + \|k\|_{L^4}^4 \right) e^{\gamma s} ds.$$

Similarly taking L^2 inner-product $\langle \cdot, \cdot \rangle$ of (3.3.12) with u^5 and (3.3.13) with Gv^5 respectively, we get

$$\begin{aligned}
&\frac{d}{dt}\left(\|u(t)\|_{L^6}^6 + G\|v(t)\|_{L^6}^6 \right) + \gamma \left(\|u(t)\|_{L^6}^6 + G\|v(t)\|_{L^6}^6 \right) \\
\leq\, & \int_{\Gamma} \left(\frac{16h^6}{3a^5} + \frac{16k^6}{3b^5G^5} \right) dx + M_3|\Gamma|,
\end{aligned}$$

where

$$M_3 = \frac{16\rho^6}{3a^5} + \frac{16\beta^6}{3b^5G^5}.$$

Multiplying by $e^{\gamma t}$, then integrating over $[\tau - t, \tau]$ where $t \geq 0$, we have

$$\begin{aligned}
&\|u(\tau, \tau - t, u_0(\tau - t))\|_{L^6}^6 + G\|v(\tau, \tau - t, v_0(\tau - t))\|_{L^6}^6 \\
\leq & e^{-\gamma t}(\|u_0(\tau - t)\|_{L^6}^6 + G\|v_0(\tau - t)\|_{L^6}^6) + e^{-\gamma\tau} M_3|\Gamma| \int_{\tau-t}^{\tau} e^{\gamma s} ds \\
&+ e^{-\gamma\tau} \int_{\tau-t}^{\tau} \left(\frac{16}{3a^5}\|h\|_{L^6}^6 + \frac{16}{3b^5G^5}\|k\|_{L^6}^6 \right) e^{\gamma s} ds \\
\leq & e^{-\gamma t}(\|u_0(\tau - t)\|_{L^6}^6 + G\|v_0(\tau - t)\|_{L^6}^6) + \frac{M_3|\Gamma|}{\gamma} \\
&+ e^{-\gamma\tau} \int_{-\infty}^{\tau} \left(\frac{16}{3a^5}\|h\|_{L^6}^6 + \frac{16}{3b^5G^5}\|k\|_{L^6}^6 \right) e^{\gamma s} ds.
\end{aligned} \tag{3.3.23}$$

Returning to the original u, v, we have

$$\begin{aligned}
&\|u(\tau, \tau - t, u_0(\tau - t))\|_{L^6}^6 + \frac{1}{G^5}\|v(\tau, \tau - t, v_0(\tau - t))\|_{L^6}^6 \\
\leq & e^{-\gamma t}(\|u_0(\tau - t)\|_{L^6}^6 + \frac{1}{G^5}\|v_0(\tau - t)\|_{L^6}^6) + \frac{M_3|\Gamma|}{\gamma} \\
&+ e^{-\gamma\tau} \int_{-\infty}^{\tau} \left(\frac{16}{3a^5}\|h\|_{L^6}^6 + \frac{16}{3b^5G^5}\|k\|_{L^6}^6 \right) e^{\gamma s} ds
\end{aligned} \tag{3.3.24}$$

Therefore,

$$\begin{aligned}
&\|u(\tau, \tau - t, u_0(\tau - t))\|_{L^6}^6 + \|v(\tau, \tau - t, v_0(\tau - t))\|_{L^6}^6 \\
\leq & \frac{max\{1, G^{-5}\}}{min\{1, G^{-5}\}} e^{-\gamma t}(\|u_0(\tau - t)\|_{L^6}^6 + \|v_0(\tau - t)\|_{L^6}^6 + \frac{M_3|\Gamma|}{min\{1, G^{-5}\}\gamma} \\
+ & \frac{max\{\frac{16}{3a^5}, \frac{16}{3b^5G^5}\}}{min\{1, G^{-5}\}} e^{-\gamma\tau} \int_{-\infty}^{\tau} \left(\|h\|_{L^6}^6 + \|k\|_{L^6}^6 \right) e^{\gamma s} ds.
\end{aligned} \tag{3.3.25}$$

Let $t \to +\infty$, we have (3.3.9) holds for $p = 3$ with

$$K_{3,\tau} = \frac{M_3|\Gamma|}{min\{1, G^{-5}\}\gamma} + \frac{max\{\frac{16}{3a^5}, \frac{16}{3b^5G^5}\}}{min\{1, G^{-5}\}} e^{-\gamma\tau} \int_{-\infty}^{\tau} \left(\|h\|_{L^6}^6 + \|k\|_{L^6}^6 \right) e^{\gamma s} ds.$$

□

3.4 Pullback Asymptotic Compactness

In this section, we show that the non-autonomous reversible Selkov process $\{S(\tau, \tau - t)\}_{t \geq 0}$ is asymptotically compact through the following two lemmas. Note that $\|(y_1, y_2)\|^2 = \|y_1\|^2 + \|y_2\|^2$.

Lemma 3.4.1. For any given $R > 0, \tau \in \mathbb{R}$, there exists a constant $Q_\tau(R)$ such that if the initial data $g_0 \in E$ and $\|g_0\|_E \le R$, then $(u(\tau, \tau-t), v(\tau, \tau-t))$ components of the solution trajectories $g(t) = S(\tau, \tau-t)g_0$ of the IVP (3.1.7) satisfy

$$\|\nabla(u(\tau, \tau-t, u_0(\tau-t)), v(\tau, \tau-t, v_0(\tau-t)))\|^2 \le Q_\tau(R), \quad \text{for } t > T_1, \tag{3.4.1}$$

where $Q_\tau(R) > 0$ is a constant depending only on R, τ, and $T_1 > 0$ is finite and only depends on the radius R.

Proof. Taking the L^2 inner-products of (3.1.2) with $-\Delta u(t)$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 + d_1\|\Delta u\|^2 + a\|\nabla u\|^2 \\
&= \int_\Gamma \left(-\rho\Delta u - u^2 v\Delta u + Gu^3\Delta u - h\Delta u\right)\, dx \\
&\le (\frac{d_1}{4} + \frac{d_1}{4} + \frac{d_1}{2})\|\Delta u\|^2 + \frac{1}{d_1}\int_\Gamma \left(u^4v^2 + G^2u^6\right)\, dx + \frac{1}{2d_1}(\rho+h)^2|\Gamma| \\
&\le d_1\|\Delta u\|^2 + \frac{1}{3d_1}\left(\int_\Gamma (2+3G^3)u^6 dx + \int_\Gamma v^6 dx\right) + \frac{1}{d_1}(\rho^2+h^2)|\Gamma|.
\end{aligned}$$

Taking the L^2 inner-products of (3.1.3) with $-\Delta v(t)$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla v\|^2 + d_2\|\Delta v\|^2 + a\|\nabla v\|^2 \\
&= \int_\Gamma \left(-\beta\Delta v - u^2 v\Delta v - Gu^3\Delta v - k\Delta v\right)\, dx \\
&\le (\frac{d_2}{4} + \frac{d_2}{4} + \frac{d_2}{2})\|\Delta v\|^2 + \frac{1}{d_2}\int_\Gamma \left(u^4v^2 + G^2u^6\right)\, dx + \frac{1}{2d_2}(\beta+k)^2|\Gamma| \\
&\le d_2\|\Delta v\|^2 + \frac{1}{3d_2}\left(\int_\Gamma (2+3G^3)u^6 dx + \int_\Gamma v^6 dx\right) + \frac{1}{d_2}(\beta^2+k^2)|\Gamma|.
\end{aligned}$$

Adding up the two components gives

$$\begin{aligned}
&\frac{d}{dt}\|(\nabla u, \nabla v)\|^2 + 2\gamma\|(\nabla u, \nabla v)\|^2 \\
&\le \frac{2}{3}\left(\frac{1}{d_1} + \frac{1}{d_2}\right)(2+3G^2)\left(\|u\|_{L^6}^6 + \|v\|_{L^6}^6\right) + \frac{2}{d_1}(\rho^2+h^2)|\Gamma| + \frac{2}{d_2}(\beta^2+k^2)|\Gamma| \qquad (3.4.2) \\
&\le \frac{4}{3d_0}(2+3G^2)\left(\|u\|_{L^6}^6 + \|v\|_{L^6}^6\right) + \frac{2}{d_0}(\rho^2+\beta^2+h^2+k^2)|\Gamma|.
\end{aligned}$$

Note that we have taken $\|\nabla\varphi\|$ as the norm of E and there is a positive constant $\eta > 0$ associated with the Sobolev imbedding inequality

$$\|\varphi\|_{L^6} \le \eta\|\varphi\|_E = \eta\|\nabla\varphi\|, \quad \text{for any } \varphi \in E. \tag{3.4.3}$$

By (3.3.25), for any given $g_0 = (u_0(\tau - t), v_0(\tau - t)) \in E$, we have

$$\begin{aligned} &\|u(\tau, \tau - t, u_0(\tau - t))\|_{L^6}^6 + \|v(\tau, \tau - t, v_0(\tau - t))\|_{L^6}^6 \\ &\le \frac{\eta^6 max\{1, G^{-5}\}}{min\{1, G^{-5}\}} e^{-\gamma t} \|u_0(\tau - t), v_0(\tau - t)\|_E^6 + \frac{M_3|\Gamma|}{min\{1, G^{-5}\}\gamma} \\ &+ \frac{max\{\frac{16}{3a^5}, \frac{16}{3b^5 G^5}\}}{min\{1, G^{-5}\}} e^{-\gamma\tau} \int_{-\infty}^{\tau} \left(\|h\|_{L^6}^6 + \|k\|_{L^6}^6\right) e^{\gamma s} ds. \end{aligned} \tag{3.4.4}$$

Let

$$M_4(\tau) = \frac{M_3|\Gamma|}{min\{1, G^{-5}\}\gamma} + \frac{max\{\frac{16}{3a^5}, \frac{16}{3b^5 G^5}\}}{min\{1, G^{-5}\}} e^{-\gamma\tau} \int_{-\infty}^{\tau} \left(\|h\|_{L^6}^6 + \|k\|_{L^6}^6\right) e^{\gamma s} ds. \tag{3.4.5}$$

Then (3.4.2) along with these facts shows that for any initial datum $\|g_0\|_E \le R$, we arrive at

$$\begin{aligned} &\frac{d}{dt}\|(\nabla u, \nabla v)\|^2 + 2\gamma\|(\nabla u, \nabla v)\|^2 \\ &\le \frac{4}{3d_0}(2 + 3G^2)\left(\frac{\eta^6 R^6 max\{1, G^{-5}\}}{min\{1, G^{-5}\}} e^{-\gamma t} + M_4(\tau)\right) + \frac{2}{d_0}(\rho^2 + \beta^2 + h^2 + k^2)|\Gamma|. \end{aligned} \tag{3.4.6}$$

Multiplying by $e^{2\gamma t}$, then integrating over $[\tau - t, \tau]$ where $t \ge 0$, we have

$$\begin{aligned} &\|(\nabla u(\tau, \tau - t, u_0(\tau - t)), \nabla v(\tau, \tau - t, v_0(\tau - t)))\|^2 \\ &\le e^{-2\gamma t}(\|\nabla u_0(\tau - t), \nabla v_0(\tau - t))\|^2) + \frac{4(2 + 3G^2)\eta^6 R^6 max\{1, G^{-5}\}}{3d_0 min\{1, G^{-5}\}} e^{-2\gamma\tau} \int_{\tau - t}^{\tau} e^{\gamma s} ds \\ &\quad + e^{-2\gamma\tau}\left(\frac{4(2 + 3G^2)}{3d_0} M_4(\tau) + \frac{2|\Gamma|}{d_0}(\rho^2 + \beta^2)\right) \int_{\tau - t}^{\tau} e^{2\gamma s} ds \\ &+ \frac{2|\Gamma|}{d_0} e^{-2\gamma\tau} \int_{\tau - t}^{\tau} \left(\|h\|^2 + \|k\|^2\right) e^{\gamma s} ds \\ &\le e^{-2\gamma t}(\|(\nabla u_0(\tau - t), \nabla v_0(\tau - t))\|^2) + \frac{4(2 + 3G^2)\eta^6 R^6 max\{1, G^{-5}\}}{3\gamma d_0 min\{1, G^{-5}\}} e^{-\gamma\tau} \\ &\quad + \frac{1}{2\gamma}\left(\frac{4(2 + 3G^2)}{3d_0} M_4(\tau) + \frac{2|\Gamma|}{d_0}(\rho^2 + \beta^2)\right) + \frac{2|\Gamma|}{d_0} e^{-2\gamma\tau} \int_{-\infty}^{\tau} \left(\|h\|^2 + \|k\|^2\right) e^{\gamma s} ds. \end{aligned} \tag{3.4.7}$$

Put

$$\begin{aligned} Q_\tau(R) = 1 &+ \frac{4(2 + 3G^2)\eta^6 R^6 max\{1, G^{-5}\}}{3\gamma d_0 min\{1, G^{-5}\}} e^{-\gamma\tau} \\ &+ \frac{1}{2\gamma}\left(\frac{4(2 + 3G^2)}{3d_0} M_4(\tau) + \frac{2|\Gamma|}{d_0}(\rho^2 + \beta^2)\right) + \frac{2|\Gamma|}{d_0} e^{-2\gamma\tau} \int_{-\infty}^{\tau} \left(\|h\|^2 + \|k\|^2\right) e^{\gamma s} ds, \end{aligned} \tag{3.4.8}$$

and for any ball $B \subset E$ centered at the origin with radius R, there is a finite time $T_1(R) = \frac{1}{\gamma} log R$ such that (3.4.1) holds for any $t \geq T_1$. $\square$

Lemma 3.4.2. For any $\tau \in \mathbb{R}$,there exists a universal constant $P(\tau)$, for any given $R > 0$ there exists a constant $T(R) > 0$ such that if $g_0 \in H$ and $\|g_0\|^2 \leq R$, then the weak solution of the non-autonomous reversible Selkov process satisfies $S(\tau, \tau - t)g_0 \in E$ for $t \geq T(R)$ and

$$\|S(\tau, \tau - t)g_0\|_E^2 \leq P(\tau), \quad t \geq T(R).$$

Proof. By the absorbing property shown in Lemma 3.3.1, there exists a time $T_0(R) > 0$ such that for any $g_0 \in H$ with $\|g_0\| \leq R$ we have $S(\tau, \tau - t)g_0 \in B_\tau$ for $t \geq T_0(R)$, where B_τ is the bounded pullback absorbing set in H. Thus we have

$$\|S(\tau - T_1, \tau - T_1 - t)g_0\|^2 \leq K_\tau, \quad t \geq T_0(R).$$

From the inequality (3.3.8), we have

$$\begin{aligned} &\int_{\tau - T_1}^{\tau - T_1 + 1} (\|G\nabla u(s)\|^2 + \|\nabla v(s)\|^2) ds \\ &\qquad \leq \frac{M_1}{2d_0} + \frac{\max\{2Ga^{-1}, 2b^{-1}\}}{2d_0} H + \max\{1, G\} K_\tau^2, \end{aligned} \tag{3.4.9}$$

By Lemma 3.3.1, for any $g_0 \in H$, the weak solution $S(\tau, \tau - t)g_0$ is a strong solution on $(\tau - t, \tau]$. This fact and the solution regularity confirm that

$$(u, v) \in C((\tau - t, \tau); E) \cap L^2_{loc}(\tau - t, \tau; E). \tag{3.4.10}$$

Via the mean value theorem in calculus, it follows from (3.4.9) that there exists a time $T_2 \in (\tau - T_1, \tau - T_1 + 1]$ such that

$$(\|G\nabla u(T_2)\|^2 + \|\nabla v(T_2)\|^2) \leq \frac{M_1}{2d_0} + \frac{\max\{2Ga^{-1}, 2b^{-1}\}}{2d_0} H + \max\{1, G\} K_\tau^2, \tag{3.4.11}$$

that is,

$$\|u(T_2), v(T_2)\|_E^2 \leq \frac{M_1}{2d_0 \min\{1, G\}} + \frac{\max\{2Ga^{-1}, 2b^{-1}\}}{2d_0 \min\{1, G\}} H + \frac{\max\{1, G\} K_\tau^2}{\min\{1, G\}}, \tag{3.4.12}$$

Finally, set

$$T(R) = T_1 + T_2 + 1.$$

By (3.4.12), we can apply Lemma 3.4.1 to conclude that, for any $g_0 \in H$ with $\|g_0\| \leq R$,

$$\|S(\tau, \tau - t)g_0\|_E^2 \leq Q_\tau\left(\frac{M_1}{2d_0 \min\{1, G\}} + \frac{\max\{2Ga^{-1}, 2b^{-1}\}}{2d_0 \min\{1, G\}} H + \frac{\max\{1, G\} K_\tau^2}{\min\{1, G\}}\right), \quad t \geq T(R)$$

where $Q_\tau(\cdot)$ is the function given explicitly by (3.4.8), which does not depend on R.

Put

$$P(\tau) = Q_\tau\left(\frac{M_1}{2d_0 \min\{1, G\}} + \frac{\max\{2Ga^{-1}, 2b^{-1}\}}{2d_0 \min\{1, G\}} H + \frac{\max\{1, G\} K_\tau^2}{\min\{1, G\}}\right),$$

we complete the proof.

Note that we run the process from $\tau - t$, after time $\tau - T_2$ the E-norm of (u, v) can be bounded in the universal ball, then we can apply Lemma 3.4.1 to reach the conclusion.

□

Finally we reach the proof of main result on the existence of pullback attractor, which will be denoted by $\mathscr{A}(\tau)$, for the non-autonomous reversible Selkov process $\{S(\tau, \tau - t)\}_{t \geq 0}$.

Theorem 3.4.3. For any positive parameters $d_1, d_2, \rho, \beta, a, b, G$, there exist a pullback attractor $\mathscr{A}(\tau)$ in the phase space H for the non-autonomous Selkov process $\{S(\tau, \tau - t)\}_{t \geq 0}$.

Proof. In Lemma 3.3.1 we have shown that the non-autonomous reversible Selkov process $\{S(\tau, \tau - t)\}_{t \geq 0}$ has a pullback absorbing set B_τ in H. Via Lemma 3.4.2 we proved that

$$\|S(\tau, \tau - t)g_0\|_E^2 \leq P(\tau), \quad t \geq T(R) \text{ and for } \|g_0\|^2 \leq R,$$

which implies that $S(\tau, \tau - t)$ is pullback asymptotically compact in H. Thus we apply Proposition 3.2.5 to reach the conclusion that there exist a pullback attractor $\mathscr{A}(\tau)$ in H for this non-autonomous reversible Selkov process $\{S(\tau, \tau - t)\}_{t \geq 0}$. □

Chapter 4
Random attractor of stochastic Brusselator system

4.1 Introduction

In this work, we shall prove the existence of the random attractor for the following stochastic Brusselator system with multiplicative noise,

$$du = \left(d_1 \Delta u + a - (b+1)u + u^2 v\right) dt + \rho u \circ dW(t), \tag{4.1.1}$$

$$dv = \left(d_2 \Delta v + bu - u^2 v\right) dt + \rho v \circ dW(t), \tag{4.1.2}$$

for $(t, x) \in \mathbb{R} \times \Gamma$, where $\Gamma \subset \mathbb{R}^n$ $(n \leq 3)$ is a bounded domain with a locally Lipschitz continuous boundary, given the homogeneous Dirichlet boundary condition

$$u(t, x) = v(t, x) = 0, \quad t > t_0,\ x \in \partial\Gamma, \tag{4.1.3}$$

and an initial condition

$$u(t_0, x) = u_0(x),\ v(t_0, x) = v_0(x). \tag{4.1.4}$$

All the coefficients d_1, d_2, a, b and ρ are arbitrarily given positive constants. $\{W(t)\}_{t \in \mathbb{R}}$ is a one-dimensional, two-sided standard Wiener process (Brownian motion) on a probability space which will be specified later. The term $\rho u \circ dW(t), \rho v \circ dW(t)$ indicate that the stochastic PDEs (4.1.1)-(4.1.2) are interpreted as the corresponding stochastic integral equations in the Stratonovich sense.

The original Brusselator equations were proposed in [31] as a system of ODEs and the diffusive Brusselator equations have been used as a typical mathematical model for morphogenesis and trimolecular autocatalytic reactions in physical chemistry and mathematical biology, cf. [21] and the references in [47].

The concept of random attractor for random dynamical system was first introduced in [14, 19] in the study of the asymptotic dynamics of Navier-Stokes equations and other PDEs with multiplicative and additive white noise. The fundamental results on random dynamical systems and related topics have been summarized in [2].

When dealing with random dynamics and the existence of random attractor for stochastic partial differential equations with multiplicative noise, we usually transform the stochastic PDEs into deterministic ones with random parameters and random initial data through the exponential transformation of Brownian motion. In this chapter, however, we take the approach of the exponential transformation of the Ornstein-Uhlenbeck process. This transformation does change the structure of the original equations and produces the non-autonomous terms in (4.2.6) and (4.2.7). It demands more challenging and sophisticated pullback *a priori* estimates, other than the non-dynamical substitution of ω by $\theta_{-t}\omega$ as in some other publications.

Another notable aspect is that the Brusselator reaction-diffusion system does not satisfy the usual dissipative condition, cf. [47], and the bootstrap method used here is also different from the decomposition method for the deterministic global attractors in [47]. The result of this chapter shows that the approach of the Ornstein-Uhlenbeck transformation unifies the treatment of the stochastic RDEs with multiplicative noise and with additive noise in regard to pullback asymptotic dynamics, especially for the reaction-diffusion systems with some sort of weak and hidden dissipativity.

As always the initial data or solutions are not restricted to be nonnegative and there are no restrictions on any of the positive parameters in the equations (4.1.1)–(4.1.2).

The rest of the chapter is organized as follows. In Section 2 we present preliminary concepts on random dynamical system and random attractors. In Section 3 we prove the pullback absorbing property of the Brusselator random dynamical system. In Section 4 we show the pullback asymptotic compactness. In Sections 5 we reach the main results on the existence of a random attractor and its L^2 to H^1 attracting regularity.

4.2 Preliminaries and Formulation

In this section, we recall the concepts of random dynamical system and random attractor. We refer to [2, 5, 14, 16] for more details. Let $(X, \|\cdot\|_X)$ be a real separable Banach space with Borel σ-algebra $\mathcal{B}(X)$ and let $(\Omega, \mathcal{F}, P)$ be a probability space. $\mathbb{R}^+ = [0, \infty)$.

Definition 4.2.1. $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$ is called a *metric dynamical system* (MDS) if $\theta : \mathbb{R} \times \Omega \to \Omega$ is $(\mathcal{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{F})$-measurable, θ_0 is the identity on Ω, $\theta_{t+s} = \theta_t \circ \theta_s$ for all $s, t \in \mathbb{R}$ and $\theta_t P = P$ for all $t \in \mathbb{R}$ on Ω.

Definition 4.2.2. A continuous *random dynamical system* (RDS) on X over a metric dynamical system $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ is a mapping

$$\varphi : \mathbb{R}^+ \times \Omega \times X \to X, \quad (t, \omega, x) \mapsto \varphi(t, \omega, x),$$

which is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable such that for every $\omega \in \Omega$, the following conditions hold:

(i) $\varphi(0, \omega, \cdot)$ is the identity on X;

(ii) Cocycle property: $\varphi(t + s, \omega, \cdot) = \varphi(t, \theta_s\omega, \varphi(s, \omega, \cdot))$ for all $t, s \in \mathbb{R}^+$;

(iii) $\varphi(\cdot, \omega, \cdot) : \mathbb{R}^+ \times X \to X$ is strongly continuous.

Definition 4.2.3. A continuous *stochastic flow* on a Banach space X over a metric dynamical system $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ is a family of mappings $S(t, \tau, \omega) : X \to X$ for $t \geq \tau \in \mathbb{R}$ and $\omega \in \Omega$ with the following conditions:

(i) $S(t, t, \omega)$ is the identity on X;

(ii) $S(t, s, \omega)S(s, \tau, \omega) = S(t, \tau, \omega)$ for all $\tau \leq s \leq t$ and $\omega \in \Omega$;

(iii) $S(t, \tau, \omega) = S(t - \tau, 0, \theta_\tau\omega)$ for all $\tau \leq t$ and $\omega \in \Omega$;

(iv) The mapping $S(t, \tau, \omega)x$ is measurable in (t, τ, ω) and continuous in $x \in X$.

Definition 4.2.4. A random set in X is a set-valued function $B(\omega) : \Omega \to 2^X$ whose graph $\{(\omega, x) : x \in B(\omega)\} \subset \Omega \times X$ is an element of the product σ-algebra $\mathcal{F} \times \mathcal{B}(X)$. A bounded random set $B(\omega) \subset X$ means that there is a random variable $r(\omega) \in \mathbb{R}^+$ such that $||B(\omega)|| :=$

$\sup_{x\in B(\omega)} ||x|| \leq r(\omega)$ for all $\omega \in \Omega$. A random set $B(\omega)$ is called compact (respectively precompact) if for each $\omega \in \Omega$ the set $B(\omega)$ is compact (respectively precompact) in X. A bounded random set is called *tempered* with respect to $(\theta_t)_{t\in\mathbb{R}}$ on $(\Omega, \mathcal{F}, P)$, if for each ω and for any constant $c > 0$,

$$\lim_{t\to\infty} e^{-ct}||B(\theta_{-t}\omega)|| = 0.$$

A random variable $R : (\Omega, \mathcal{F}, P) \to (0, \infty)$ is called *tempered* with respect to $\{\theta_t\}_{t\in\mathbb{R}}$ if for each ω ,

$$\lim_{t\to\pm\infty} \frac{1}{t} \log R(\theta_{-t}\omega) = 0.$$

A collection $\mathcal{D}$ of random subsets of X is called *inclusion-closed*, if $D = \{D(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and $\hat{D} = \{\hat{D}(\omega) \subset D(\omega) : \omega \in \Omega\}$ imply that $\hat{D} \in \mathcal{D}$. In this case, the collection $\mathcal{D}$ is called *a universe*. In the paper, we define $\mathcal{D}$ to be the universe of all the tempered random sets in a specified phase space X. Note that all bounded non-random sets are included in $\mathcal{D}$.

Definition 4.2.5. Let $\mathcal{D}$ be a collection of random subsets of X. A random set $K \in \mathcal{D}$ is called a $\mathcal{D}$-pullback absorbing set with respect to an RDS φ over the MDS $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$, if for any $\omega \in \Omega$ and any bounded set $B(\omega) \in \mathcal{D}$ there exists a finite time $t_B(\omega) > 0$ such that

$$\varphi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)) \subset K(\omega) \quad \text{for all } t \geq t_B(\omega).$$

Definition 4.2.6. Let $\mathcal{D}$ be a collection of random subsets of X. Then an RDS φ is $\mathcal{D}$-pullback asymptotically compact in X if for each $\omega \in \Omega$, $\{\varphi(t_n, \theta_{-t_n}\omega, x_n)\}_{n=1}^{\infty}$ has a convergent subsequence in X whenever $t_n \to \infty$, and $x_n \in B(\theta_{-t_n}\omega)$ for any given $B \in \mathcal{D}$.

Definition 4.2.7. Let a universe $\mathcal{D}$ of tempered random sets in a Banach space X be given. A random set $\mathcal{A} \in \mathcal{D}$ is called *a random attractor* in $\mathcal{D}$ for a given RDS φ on X over the MDS $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$, if the following conditions are satisfied for each $\omega \in \Omega$:

(i) $\mathcal{A}$ is a compact random set.

(ii) $\mathcal{A}$ is invariant in the sense that

$$\varphi(t, \omega, \mathcal{A}(\omega)) = \mathcal{A}(\theta_t\omega), \ \ \forall\, t \geq 0;$$

(iii) $\mathcal{A}$ attracts every set $B \in \mathcal{D}$ in the sense that for every $\omega \in \Omega$ one has

$$\lim_{t\to\infty} dist_X(\varphi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)), \mathcal{A}(\omega)) = 0,$$

where the Hausdorff semi-distance is given by $dist_X(Y, Z) = \sup_{y\in Y} \inf_{z\in Z} \|y-z\|_X$ for subsets Y and Z in X.

We have the following proposition on the existence of random attractor due to Crauel and Flandoli [14, Theorem 3.11].

Proposition 4.2.8. *Given a Banach space X and a collection $\mathcal{D}$ of random sets of X, let φ be a continuous RDS on X over an MDS $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$. Suppose that there exists a closed pullback absorbing set $\{K(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and φ is pullback asymptotically compact with respect to $\mathcal{D}$, then the RDS φ has a unique random attractor $\mathcal{A} = \{\mathcal{A}(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ whose basin is $\mathcal{D}$ and given by*

$$\mathcal{A}(\omega) = \bigcap_{\tau\geq 0} \overline{\bigcup_{t\geq\tau} \varphi(t, \theta_{-t}\omega, K(\theta_{-t}\omega))}.$$

Let $\{W(t)\}_{t\in\mathbb{R}}$ be the standard one-dimensional two-sided Wiener process in the probability space $(\Omega, \mathcal{F}, P)$, where

$$\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\},$$

the σ-algebra $\mathcal{F}$ is generated by the compact-open topology on Ω, and P is the corresponding Wiener measure on $\mathcal{F}$. The shift mapping θ_t is defined by

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad t \in \mathbb{R}.$$

Then $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$ is the canonical MDS and the stochastic process $\{W(t, \omega) = \omega(t) : t \in \mathbb{R}, \omega \in \Omega\}$ is the canonical Wiener process (Brownian motion).

Consider the Ornstein-Uhlenbeck process

$$z(\theta_t\omega) = -\int_{-\infty}^{0} e^s(\theta_t\omega)(s)ds = -\int_{-\infty}^{0} e^s\omega(t+s)ds + \omega(t), \tag{4.2.1}$$

which solves the linear stochastic differential equation

$$dz + zdt = dW(t). \tag{4.2.2}$$

The following proposition is quoted from [5].

Proposition 4.2.9. *Let the metric dynamical system $(\Omega, \mathcal{F}, P, \theta_t)$ and the Ornstein-Uhlenbeck process $\{z(\theta_t\omega)\}_{t\in\mathbb{R}}$ be defined as above. Then there is a θ_t-invariant set $\tilde{\Omega} \in \Omega$ of full P-measure such that for every $\omega \in \tilde{\Omega}$, the following statements hold.*

1. *The Ornstein-Uhlenbeck process $\{z(\theta_t\omega)\}_{t\in\mathbb{R}}$ has the asymptotically sublinear growth property, i.e.*
$$\lim_{t\to\pm\infty} \frac{|z(\theta_t\omega)|}{|t|} = 0, \tag{4.2.3}$$
2. *$z(\theta_t\omega)$ is continuous in t and, for any fixed $t_0 \in \mathbb{R}$,*
$$\lim_{t\to\pm\infty} \frac{1}{t-t_0} \int_{t_0}^{t} z(\theta_s\omega)\, ds = 0, \tag{4.2.4}$$

In the sequel we consider $\omega \in \tilde{\Omega}$ only and will always write Ω for $\tilde{\Omega}$.

As the main approach to investigating the random dynamics of stochastic PDEs, we convert the stochastic Brusselator system (4.1.1)-(4.1.2) to a system of pathwise PDEs with the random parameter $\omega(t)$ and random initial data. Make the transformation

$$U = e^{-\rho z(\theta_t\omega)} u, \quad V = e^{-\rho z(\theta_t\omega)} v, \tag{4.2.5}$$

where $z(\theta_t\omega)$ is the Ornstein-Uhlenbeck process in (4.2.1). Then

$$dU = -\rho e^{-\rho z(\theta_t\omega)} u \circ dz + e^{-\rho z(\theta_t\omega)} du,$$
$$dV = -\rho e^{-\rho z(\theta_t\omega)} v \circ dz + e^{-\rho z(\theta_t\omega)} dv.$$

In view of the equation (4.2.2), $dz + zdt = dW(t)$, the system (4.1.1)-(4.1.2) is transformed by (4.2.5) to the following random PDE problem:

$$\frac{dU}{dt} = d_1 \Delta U + a e^{-\rho z(\theta_t\omega)} - (b+1)U + e^{2\rho z(\theta_t\omega)} U^2 V + \rho z(\theta_t\omega) U, \tag{4.2.6}$$
$$\frac{dV}{dt} = d_2 \Delta V + bU - e^{2\rho z(\theta_t\omega)} U^2 V + \rho z(\theta_t\omega) V, \tag{4.2.7}$$

for $\omega \in \Omega, x \in \Gamma$ and $t > t_0$, with the homogeneous Dirichlet boundary condition

$$U(t,\omega,x) = V(t,\omega,x) = 0, \quad t > t_0 \in \mathbb{R},\ x \in \partial\Gamma,\ \omega \in \Omega, \tag{4.2.8}$$

and the initial condition at $t = t_0 \in \mathbb{R}$,

$$U(t_0, \omega, x) = U_0(\omega, x) = e^{-\rho z(\theta_{t_0}\omega)} u_0(x), \quad V(t_0, \omega, x) = V_0(\omega, x) = e^{-\rho z(\theta_{t_0}\omega)} v_0(x). \tag{4.2.9}$$

For every $\omega \in \Omega$, the problem (4.2.6)-(4.2.9) of the pathwise nonautonomous partial differential equations can be written as

$$\begin{aligned} \frac{dg}{dt} &= Ag + F(g, \theta_t \omega), \\ g(t_0, \omega, x) &= g_0 = (U_0(\omega, x), V_0(\omega, x))^T, \end{aligned} \tag{4.2.10}$$

where $g(t, \omega; t_0, g_0) = (U(t, \omega; t_0, U_0), V(t, \omega; t_0, V_0))^T$ and

$$F(g, \theta_t \omega) = \begin{pmatrix} a e^{-\rho z(\theta_t \omega)} - (b+1)U + e^{2\rho z(\theta_t \omega)} U^2 V + \rho z(\theta_t \omega) U \\ \\ bU - e^{2\rho z(\theta_t \omega)} U^2 V + \rho z(\theta_t \omega) V \end{pmatrix}$$

for any $t \geq t_0$ with initial data

$$g_0(\omega) = (U_0(\omega, \cdot), V_0(\omega, \cdot))^T = (e^{-\rho z(\theta_{t_0}\omega)} u_0(\cdot), e^{-\rho z(\theta_{t_0}\omega)} v_0(\cdot))^T.$$

By conducting *a priori* estimates on the Galerkin approximations of the initial value problem (4.2.10) and the compactness argument, c.f. [12], but with the extra care on the non-autonomous terms from the random noise, we can prove the local existence and uniqueness of the weak solution $g(t, \omega; t_0, g_0),\ t \in [t_0, T(\omega, g_0)]$ for some $T(\omega, g_0) > t_0$, which depends continuously on the initial data.

By the parabolic regularity [37, Theorem 48.5], every weak solution turns out to be a strong solution for $t > t_0$ in the existence interval. Similar to Lemma 1.2 in [47], every weak solution $g(t, \omega; t_0, g_0)$ of (4.2.10) on the maximal interval of existence has the property

$$g(t, \omega; t_0, g_0) \in C([t_0, T_{max}); H) \cap C^1((t_0, T_{max}); H) \cap L^2([t_0, T_{max}); E).$$

Below we shall study the global existence and the asymptotic dynamics of the weak solutions of the problem (4.2.10).

4.3 Pullback Absorbing Property

For brevity, we write $U(t,\omega;t_0,U_0), V(t,\omega;t_0,V_0)$ as $U(t,\omega), V(t,\omega)$ or simply as U, V, similarly we write weak solution $g(t,\omega;t_0,g_0)$ as $g(t,\omega)$ or g.

Lemma 4.3.1. For any given tempered random variable $R(\omega) > 0$ and any initial data $(u_0, v_0) \in H$ with $\|(u_0,v_0)\| \leq R(\omega)$, there exists a time $-\infty < T(R,\omega) \leq -1$ such that the weak solution $g(t,\omega) = (U(t,\omega;t_0,U_0), V(t,\omega;t_0,V_0))$ of the problem of the random Brusselator system (4.2.6)-(4.2.9) exists on $[t_0, 0]$ for any initial time $t_0 \leq T(R,\omega)$.

Moreover, for terminal time $t \in [-4, 0]$ when $t_0 \leq \min\{T(R,\omega), -4\}$, there exists a random variable $M(t,\omega)$ independent of initial data such that the weak solution satisfies

$$\left\| g\left(t,\omega;t_0,e^{-\rho z(\theta_{t_0}\omega)}g_0\right)\right\|^2 \leq M(t,\omega), \quad t \geq t_0,\ \omega \in \Omega. \tag{4.3.1}$$

Proof. Taking the inner product of (4.2.7) with $V(t,\omega)$, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|V\|^2 + d_2\|\nabla V\|^2 \\ &= -e^{2\rho z(\theta_t\omega)}\int_\Gamma \left(UV - \frac{1}{2}\,be^{-2\rho z(\theta_t\omega)}\right)^2 dx + \frac{1}{4}b^2|\Gamma|e^{-2\rho z(\theta_t\omega)} + \rho z(\theta_t\omega)\|V\|^2. \end{aligned} \tag{4.3.2}$$

It follows that, in the maximal interval of existence $[t_0, T_{max})$,

$$\frac{d}{dt}\|V\|^2 + 2\lambda d_2\|V\|^2 \leq \frac{d}{dt}\|V\|^2 + 2d_2\|\nabla V\|^2 \leq 2\rho z(\theta_t\omega)\|V\|^2 + \frac{1}{2}b^2|\Gamma|e^{-2\rho z(\theta_t\omega)}. \tag{4.3.3}$$

Multiplying the above inequality by $e^{\int_{t_0}^t (2\rho z(\theta_s\omega)-2\lambda d_2)ds}$ and then integrating it over $[t_0, t]$ where $t_0 < -4 \leq t \leq 0$, we obtain

$$\begin{aligned} \|V(t,\omega;t_0,g_0)\|^2 \leq\ & \|V_0\|^2 e^{\int_{t_0}^t (2\rho z(\theta_s\omega)ds - 2\lambda d_2(t-t_0)} \\ & + \frac{1}{2}b^2|\Gamma|\int_{t_0}^t e^{\int_\tau^t (2\rho z(\theta_s\omega)-2\lambda d_2)ds - 2\rho z(\theta_\tau\omega)}d\tau. \end{aligned} \tag{4.3.4}$$

The next step is the key to the pullback estimates. We want to get rid of the initial time and data by the asymptotic decay of the Ornstein-Uhlenbeck process. The arguments go as follows.

From (4.2.3) and (4.2.4), for every random variable $R(\omega) > 0$, there exists a time $T_1(R, \omega) < -4$ such that for any $t_0 \leq T_1(R, \omega)$, and $t \in [-4, 0]$, we have

$$\begin{gathered} \frac{1}{(t-t_0)} \int_{t_0}^{t} 2\rho z(\theta_s \omega)\, ds - 2\lambda d_2 \leq -\lambda d_2, \\ e^{-\lambda d_2 (t-t_0)} e^{-\rho z(\theta_{t_0}\omega)} R^2(\omega) \leq 1. \end{gathered} \tag{4.3.5}$$

Note that the improper integral on the right-hand side of (4.3.4),

$$\int_{-\infty}^{t} \exp\left\{ \int_{\tau}^{t} (2\rho z(\theta_s \omega) - 2\lambda d_2 s) ds - 2\rho z(\theta_\tau \omega) \right\} d\tau, \tag{4.3.6}$$

is convergent for the following reason. By (4.2.3) and (4.2.4) there exists $T_2(\omega) < -4$ such that for any $\tau \leq T_2(\omega)$, we have

$$\begin{aligned} &\exp\left[\int_{\tau}^{t} (2\rho z(\theta_s \omega) - 2\lambda d_2 s) ds - 2\rho z(\theta_\tau \omega) \right] \\ = &\exp\left[(t-\tau) \left(\frac{\int_{\tau}^{t} 2\rho z(\theta_s \omega) ds}{t-\tau} - 2\lambda d_2 - \frac{2\rho z(\theta_\tau \omega)}{t-\tau} \right) \right] \leq e^{-\lambda d_2 (t-\tau)}, \end{aligned} \tag{4.3.7}$$

and

$$\int_{-\infty}^{\tau} e^{-\lambda d_2 (t-\tau)} d\tau \leq \int_{-\infty}^{T_2} e^{-\lambda d_2 (t-\tau)} d\tau = \frac{1}{\lambda d_2} e^{\lambda d_2 (T_2 - t)}. \tag{4.3.8}$$

Therefore, we have the following estimates, for $t_0 \leq T_1(R, \omega)$, and $t \in [-4, 0]$,

$$\begin{aligned} \|V(t, \omega; t_0, g_0)\|^2 \leq\ & \|V_0\|^2 e^{\int_{t_0}^{t} (2\rho z(\theta_s \omega) ds - 2\lambda d_2 (t-t_0)} \\ & + \frac{1}{2} b^2 |\Gamma| \int_{t_0}^{t} e^{\int_{\tau}^{t} (2\rho z(\theta_s \omega) - 2\lambda d_2) ds - 2\rho z(\theta_\tau \omega)} d\tau \\ \leq\ & 1 + \frac{1}{2} b^2 |\Gamma| \int_{t_0}^{t} e^{\int_{\tau}^{t} (2\rho z(\theta_s \omega) - 2\lambda d_2) ds - 2\rho z(\theta_\tau \omega)} d\tau \\ \leq\ & 1 + \frac{1}{2} b^2 |\Gamma| \int_{-\infty}^{t} e^{\int_{\tau}^{t} (2\rho z(\theta_s \omega) - 2\lambda d_2) ds - 2\rho z(\theta_\tau \omega)} d\tau, \end{aligned} \tag{4.3.9}$$

Next we deal with U component through the transformation

$$y(t, x, \omega) = U(t, x, \omega) + V(t, x, \omega). \tag{4.3.10}$$

Adding up (4.2.6) and (4.2.7), we get

$$\frac{dy}{dt} = d_1 \Delta y + (d_1 - d_2) \Delta V - y + V + a e^{-\rho z(\theta_t \omega)} + \rho z(\theta_t \omega) y, \tag{4.3.11}$$

Take the inner product of (4.3.11) with $y(t)$ to obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|y\|^2 + d_1\|\nabla y\|^2 \\
&= (d_2 - d_1)\int_\Gamma |\nabla V \cdot \nabla y|\, dx + (\rho z(\theta_t\omega) - 1)\|y\|^2 + \int_\Gamma V y\, dx + a e^{-\rho z(\theta_t\omega)}\int_\Gamma y\, dx \\
&\le \frac{|d_2 - d_1|^2}{2d_1}\|\nabla V\|^2 + \frac{d_1}{2}\|\nabla y\|^2 + (\rho z(\theta_t\omega) - 1)\|y\|^2 + \|V\|^2 + \frac{1}{2}\|y\|^2 + a^2|\Gamma| e^{-2\rho z(\theta_t\omega)}.
\end{aligned}
\tag{4.3.12}
$$

It follows that

$$
\begin{aligned}
\frac{d}{dt}\|y\|^2 + d_1\|\nabla y\|^2 \le{}& \frac{|d_2 - d_1|^2}{d_1}\|\nabla V\|^2 \\
&+ (2\rho z(\theta_t\omega) - 1)\|y\|^2 + 2\|V\|^2 + 2a^2|\Gamma| e^{-2\rho z(\theta_t\omega)}.
\end{aligned}
\tag{4.3.13}
$$

Multiplying the above inequality by $e^{\int_{t_0}^t (2\rho z(\theta_s\omega)-1)ds}$ and then integrating it over $[t_0, t]$, where $t_0 < -4 \le t \le 0$. Then there exists a time $T_3(R,\omega) < -4$ such that for any $t_0 \le T_3(R,\omega)$, $t \in [-4, 0]$, we have

$$
\begin{aligned}
&\|y(t,\omega; t_0, g_0)\|^2 \le \|y_0\|^2 e^{\int_{t_0}^t 2\rho z(\theta_s\omega)ds - (t - t_0)} \\
&\quad + \int_{t_0}^t e^{\int_\tau^t (2\rho z(\theta_s\omega)-1)ds}\left[\frac{|d_2 - d_1|^2}{d_1}\|\nabla V(\tau)\| + 2\|V(\tau)\|^2 + 2a^2|\Gamma| e^{-2\rho z(\theta_\tau\omega)}\right] d\tau \\
&\le 1 + \int_{t_0}^t e^{\int_\tau^t (2\rho z(\theta_s\omega)-1)ds}\left[\frac{|d_2 - d_1|^2}{d_1}\|\nabla V(\tau)\| + 2\lambda\|\nabla V(\tau)\|^2 + 2a^2|\Gamma| e^{-2\rho z(\theta_\tau\omega)}\right] d\tau.
\end{aligned}
\tag{4.3.14}
$$

Now we treat the integral term on the right-hand side of (4.3.14). Multiplying (4.3.3) by $e^{\int_\tau^t (2\rho z(\theta_s\omega)-1)ds}$ and then integrating it by parts on $[t_0, t]$, where $t_0 < -4 \le t \le 0$, we find that there exists $T_4(R,\omega) < -4$, such that the following inequality holds,

$$
\begin{aligned}
&2d_2\int_{t_0}^t e^{\int_\tau^t (2\rho z(\theta_s\omega)-1)ds}\|\nabla V(\tau,\omega; t_0, g_0)\|^2 d\tau \\
&\le \int_{t_0}^t e^{\int_\tau^t (2\rho z(\theta_s\omega)-1)ds}(2\rho z(\theta_\tau\omega)\|V(\tau)\|^2 + \frac{1}{2}\, b^2|\Gamma|\, e^{-2\rho z(\theta_\tau\omega)})\, d\tau \\
&\quad + e^{\int_{t_0}^t (2\rho z(\theta_s\omega)-1)ds}\|V(t_0)\|^2 + \int_{t_0}^t \|V(\tau)\|(-2\rho z(\theta_\tau\omega) + 1)\, e^{\int_\tau^t (2\rho z(\theta_s\omega)-1)ds} d\tau \\
&\le \frac{1}{2}\, b^2|\Gamma| \int_{t_0}^t e^{\int_\tau^t (2\rho z(\theta_s\omega)-1)ds}\, e^{-2\rho z(\theta_\tau\omega)} d\tau + 1 + \int_{t_0}^t \|V(\tau)\|\, e^{\int_\tau^t (2\rho z(\theta_s\omega)-1)ds} d\tau,
\end{aligned}
\tag{4.3.15}
$$

when $t_0 \le T_4(R,\omega)$. As for the term $\int_{t_0}^t \|V(\tau)\| e^{\int_\tau^t (2\rho z(\theta_s\omega)-1)ds} d\tau$ in (4.3.15), from (4.3.4), there exists $T_5(R,\omega) \le T_1(R,\omega)$ such that for $t_0 \le T_5(R,\omega)$ we have

$$
\begin{aligned}
&\int_{t_0}^t e^{\int_\tau^t (2\rho z(\theta_s\omega)-1)ds} \|V(\tau,\omega;t_0,V_0))\|^2 d\tau \\
\le & \int_{t_0}^t e^{\int_\tau^t (2\rho z(\theta_s\omega)-1)ds} \|V_0\|^2 e^{\int_{t_0}^\tau (2\rho z(\theta_s\omega) ds - 2\lambda d_2(\tau - t_0)} d\tau \\
& + \frac{b^2|\Gamma|}{2} \int_{t_0}^t e^{\int_\tau^t (2\rho z(\theta_s\omega)-1)ds} \int_{t_0}^\tau e^{\int_\xi^\tau (2\rho z(\theta_s\omega)-2\lambda d_2)ds - 2\rho z(\theta_\upsilon\omega)} d\xi d\tau \\
\le & \int_{t_0}^t e^{\int_{t_0}^t (2\rho z(\theta_s\omega)+\max\{-1,-2\lambda d_2\})ds} \|V_0\|^2 d\tau \\
& + \frac{b^2|\Gamma|}{2} \int_{t_0}^t \int_\xi^t e^{\int_\tau^t (2\rho z(\theta_s\omega)-1)ds} e^{\int_\xi^\tau (2\rho z(\theta_s\omega)-2\lambda d_2)ds - 2\rho z(\theta_\xi\omega)} d\tau d\xi \qquad (4.3.16)\\
\le & (t-t_0)\|V_0\|^2 e^{\int_{t_0}^t (2\rho z(\theta_s\omega)+\max\{-1,-2\lambda d_2\})ds} \\
& + \frac{b^2|\Gamma|}{2} \int_{t_0}^t \int_\xi^t e^{\int_\xi^t 2\rho z(\theta_s\omega)ds + \int_\xi^t \max\{-1,-2\lambda d_2\}ds - 2\rho z(\theta_\xi\omega)} d\tau d\xi \\
= & (t-t_0)\|V_0\|^2 e^{\int_{t_0}^t (2\rho z(\theta_s\omega)+\max\{-1,-2\lambda d_2\})ds} \\
& + \frac{b^2|\Gamma|}{2} \int_{t_0}^t (t-\xi) e^{\int_\xi^t 2\rho z(\theta_s\omega)ds + \int_\xi^t \max\{-1,-2\lambda d_2\}ds - 2\rho z(\theta_\xi\omega)} d\xi \\
\le & 1 + \frac{b^2|\Gamma|}{2} \int_{-\infty}^t (t-\xi) e^{\int_\xi^t 2\rho z(\theta_s\omega)ds + \max\{-1,-2\lambda d_2\}(t-\xi) - 2\rho z(\theta_\upsilon\omega)} d\xi,
\end{aligned}
$$

Note that the last improper integral above is convergent by the similar calculation as in (4.3.7) and (4.3.8). The stochastic process given by

$$
\begin{aligned}
C_1(t,\omega) = & \frac{b^2|\Gamma|}{4d_2} \int_{-\infty}^t e^{\int_\tau^t (2\rho z(\theta_s\omega)-1)ds - 2\rho z(\theta_\tau\omega)} d\tau + \frac{1}{2d_2} \\
& + 1 + \frac{1}{2} b^2|\Gamma| \int_{-\infty}^t (t-\xi) e^{\int_\xi^t 2\rho z(\theta_s\omega)ds + \max\{-1,-2\lambda d_2\}(t-\xi) - 2\rho z(\theta_\upsilon\omega)} d\xi
\end{aligned}
$$

is tempered by (4.2.3) and (4.2.4).

By (4.3.14), (4.3.15) and (4.3.16), for $t_0 \le \min\{T_3(R,\omega), T_4(R,\omega), T_5(R,\omega)\}$, we have

$$
\|y(t,\omega;t_0,y_0))\|^2 \le C_2(t,\omega), \quad \text{for } -4 \le t \le 0, \tag{4.3.17}
$$

where

$$
C_2(t,\omega) = 1 + \left(\frac{|d_2 - d_1|^2}{d_1} + 2\lambda \right) C_1(t,\omega) + 2a^2 \int_{-\infty}^t e^{\int_\tau^t (2\rho z(\theta_s\omega)-1)ds - 2\rho z(\theta_\tau\omega)} d\tau. \tag{4.3.18}
$$

Set $T(R,\omega) = \min\{T_1(R,\omega), T_2(R,\omega), T_3(R,\omega), T_4(R,\omega), T_5(R,\omega)\}$. Then we have, for $t_0 \le T(R,\omega)$ and $t \in [-4, 0]$,

$$\begin{aligned}
\|g(t,\omega;t_0,g_0)\|^2 &= \|U(t,\omega;t_0,g_0)\|^2 + \|V(t,\omega;t_0,g_0)\|^2 \\
&= \|y(t,\omega;t_0,g_0) - V(t,\omega;t_0,g_0)\|^2 + \|V(t,\omega;t_0,g_0)\|^2 \\
&\le 2\|y(t,\omega;t_0,g_0)\|^2 + 3\|V(t,\omega;t_0,g_0)\|^2 \le M(t,\omega),
\end{aligned} \tag{4.3.19}$$

where

$$M(t,\omega) = 3C_2(t,\omega) + 3\left(1 + \frac{b^2|\Gamma|}{2}\int_{-\infty}^{t} e^{\int_\tau^t (2\rho z(\theta_s\omega) - 2\lambda d_2)ds - 2\rho z(\theta_\tau\omega)} d\tau\right).$$

The proof is completed. □

If $g(t,\omega;\tau,g_0)$ is the weak solution to problem (4.2.6)-(4.2.9), then

$$h(t,\omega;\tau,h_0) = S(t,\tau,\omega)h_0 = e^{\rho z(\theta_t\omega)} g(t,\omega;\tau,g_0), \quad t \ge \tau, \tag{4.3.20}$$

where

$$h_0 = (u_0, v_0), \ \ g_0 = e^{-\rho z(\theta_\tau\omega)} h_0,$$

is the solution to the original stochastic Brusselator problem (4.1.1)-(4.1.4).

By the uniqueness of weak solution of $g(t,\omega;\tau,g_0)$ and the stationary increment of Brownian Motion, we can verify that $S(t,\tau,\omega)$ is a stochastic flow on H, namely,

$$S(t,s,\omega)S(s,\tau,\omega)) = S(t,\tau,\omega), \quad \text{for } \tau \le s \ \le t,$$

$$S(t,s,\omega) = S(t-s,0,\theta_s\omega), \qquad \text{for } s \le t.$$

The second equality means that

$$\begin{aligned}
&e^{\rho z(\theta_t\omega)} g(t,\omega;s,e^{-\rho z(\theta_s\omega)}h_0) \\
&= e^{\rho z(\theta_{t-s}\omega)} g(t-s,\theta_s\omega;0,e^{-\rho z(\omega)}h_0) \quad \text{for all } s \le t.
\end{aligned} \tag{4.3.21}$$

Define the Brusselator random dynamical system $\varphi : \mathbb{R}^+ \times \Omega \times H \to H$ over the MDS $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$ by

$$\varphi(t-\tau, \theta_\tau\omega, h_0) = S(t,\tau,\omega)h_0 = e^{\rho z(\theta_t\omega)} g(t,\omega;\tau,g_0), \tag{4.3.22}$$

where $t \geq \tau \in \mathbb{R}, \omega \in \Omega, h_0 \in H$. The cocycle property of the mapping φ can be checked by (4.3.22), (4.3.21) and the properties of stochastic flow in Definition 2.3.

From (4.3.22), we have the pullback relation

$$\varphi(t, \theta_{-t}\omega, (u_0, v_0)) = e^{\rho z(\omega)} g(0, \omega; -t, g_0), \tag{4.3.23}$$

in which $g(0, \omega; -t, g_0), t \geq 0$, can be called the *pullback quasi-trajectory* from g_0, which is not a trajectory but the terminal values at time $t = 0$ of the bunch of weak solutions $g(0, \omega; -t, g_0)$ starting from g_0 more and more backward at time $-t$. We shall deal with the pullback quasi-trajectories to investigate the pullback asymptotic behavior of the Brusselator random dynamical system φ.

Lemma 4.3.2. For the Brusselator random dynamical system φ on H over the MDS $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$, there exists a $\mathcal{D}$-pullback absorbing set $B_0(\omega)$, which is the random ball centered at the origin with the radius $M_0(\omega)$ given by

$$M_0(\omega) = e^{\rho z(\omega)} \left[3C_2(0, \omega) + 3 \left(1 + \frac{b^2|\Gamma|}{2} \int_{-\infty}^{0} e^{\int_\tau^0 (2\rho z(\theta_s\omega) - 2\lambda d_2) ds - 2\rho z(\theta_\tau \omega)} d\tau \right) \right].$$

Proof. This is a direct consequence of Lemma 4.3.1 and the characterization of the pullback quasi-trajectories of this Brusselator RDS φ. Note that $M_0(\omega)$ is a tempered random variable and $B_0 \in \mathcal{D}$. $\square$

Furthermore we show the pullback absorbing property of the V-component of the random Brusselator system (4.2.6)-(4.2.9) in the Banach space $L^6(\Gamma)$. This is a key step to pave the way toward the proof of the pullback asymptotic compactness in the next section.

Lemma 4.3.3. For any given initial data $(u_0, v_0) \in E$, and terminal time $t \in [-4, 0]$, there exists a random time $T_6(\|g_0\|_{L^6}, \omega) \leq -4$ and a positive random variables $P(t, \omega)$ such that for any initial time $t_0 \leq T_6(\|g_0\|_{L^6}, \omega)$, we have

$$\|V(t, \omega; t_0, g_0)\|_{L^6}^6 \leq P(t, \omega), \quad -4 \leq t \leq 0. \tag{4.3.24}$$

Proof. Taking the inner product of (4.2.7) with V^3, we obtain

$$\begin{aligned}
&\frac{1}{4}\frac{d}{dt}\int_\Gamma V^4(t,x)dx + 3d_2\|V(t)\nabla V(t)\|^2 \\
&= \int_\Gamma \left(bUV^3 - e^{2\rho z(\theta_t\omega)}U^2V^4 + \rho z(\theta_t\omega)V^4\right)dx \qquad (4.3.25)\\
&\le \int_\Gamma \left(\frac{1}{2}b^2e^{-2\rho z(\theta_t\omega)}V^2 + \frac{1}{2}e^{2\rho z(\theta_t\omega)}U^2V^4 - e^{2\rho z(\theta_t\omega)}U^2V^4 + \rho z(\theta_t\omega)V^4\right)dx.
\end{aligned}$$

It follows that

$$\begin{aligned}
\frac{d}{dt}\|V(t)\|_{L^4}^4 + 6\lambda d_2\|V(t)\|_{L^4}^4 \le& \frac{d}{dt}\|V(t)\|_{L^4}^4 + 6d_2\|\nabla V^2(t)\|^2 \\
\le& 2\rho z(\theta_t\omega)\|V(t)\|_{L^4}^4 + \frac{1}{2}\,b^2e^{-2\rho z(\theta_t\omega)}\|V(t)\|^2.
\end{aligned} \qquad (4.3.26)$$

Multiply the inequality (4.3.26) by $e^{\int_{t_0}^t(2\rho z(\theta_s\omega)-6\lambda d_2)ds}$ and then integrate the resulting inequality over $[t_0,t]$, where $t_0<t$. By virtue of (4.3.4), we have

$$
\begin{aligned}
&\|V(t,\omega;t_0,g_0)\|_{L^4}^4 \le \|V_0\|_{L^4}^4 e^{\int_{t_0}^t (2\rho z(\theta_s\omega)-6\lambda d_2)ds} \\
&\quad + \frac{b^2}{2}\int_{t_0}^t e^{\int_\tau^t (2\rho z(\theta_s\omega)-6\lambda d_2)ds - 2\rho z(\theta_\tau\omega)} \|V(\tau,\omega;t_0,V_0))\|^2 d\tau \\
&\le \|V_0\|_{L^4}^4 e^{\int_{t_0}^t (2\rho z(\theta_s\omega)-6\lambda d_2)ds} + \frac{b^2}{2}\int_{t_0}^t e^{\int_\tau^t (2\rho z(\theta_s\omega)-6\lambda d_2)ds - 2\rho z(\theta_\tau\omega)} \|V_0\|^2 e^{\int_{t_0}^\tau (2\rho z(\theta_s\omega)-2\lambda d_2)ds} d\tau \\
&\quad + \frac{b^4|\Gamma|}{4}\int_{t_0}^t e^{\int_\tau^t (2\rho z(\theta_s\omega)-6\lambda d_2)ds - 2\rho z(\theta_\tau\omega)} \int_{t_0}^\tau e^{\int_\upsilon^\tau (2\rho z(\theta_s\omega)-2\lambda d_2)ds - 2\rho z(\theta_\upsilon\omega)} d\upsilon d\tau \\
&= \|V_0\|_{L^4}^4 e^{\int_{t_0}^t (2\rho z(\theta_s\omega)-6\lambda d_2)ds} + \frac{b^2\|V_0\|^2}{2}\int_{t_0}^t e^{\int_\tau^t (2\rho z(\theta_s\omega)-6\lambda d_2)ds - 2\rho z(\theta_\tau\omega)} e^{\int_{t_0}^\tau (2\rho z(\theta_s\omega)-2\lambda d_2)ds} d\tau \\
&\quad + \frac{b^4|\Gamma|}{4}\int_{t_0}^t e^{-2\rho z(\theta_\tau\omega)} e^{\int_\tau^t (2\rho z(\theta_s\omega)-6\lambda d_2)ds} \int_{t_0}^\tau e^{\int_\xi^\tau (2\rho z(\theta_s\omega)-2\lambda d_2)ds - 2\rho z(\theta_\xi\omega)} d\xi d\tau \\
&= \|V_0\|_{L^4}^4 e^{\int_{t_0}^t (2\rho z(\theta_s\omega)-6\lambda d_2)ds} + \frac{b^2\|V_0\|^2}{2}\int_{t_0}^t e^{\int_\tau^t (2\rho z(\theta_s\omega)-6\lambda d_2)ds - 2\rho z(\theta_\tau\omega)} e^{\int_{t_0}^\tau (2\rho z(\theta_s\omega)-2\lambda d_2)ds} d\tau \\
&\quad + \frac{b^4|\Gamma|}{4}\int_{t_0}^t \int_\xi^t e^{-2\rho z(\theta_\tau\omega)} e^{\int_\tau^t (2\rho z(\theta_s\omega)-6\lambda d_2)ds} e^{\int_\xi^\tau (2\rho z(\theta_s\omega)-2\lambda d_2)ds - 2\rho z(\theta_\xi\omega)} d\tau d\xi \\
&\le \|V_0\|_{L^4}^4 e^{\int_{t_0}^t (2\rho z(\theta_s\omega)-6\lambda d_2)ds} + \frac{b^2\|V_0\|^2}{2}\int_{t_0}^t e^{\int_{t_0}^t (2\rho z(\theta_s\omega)-2\lambda d_2)ds} e^{-2\rho z(\theta_\tau\omega)} d\tau \\
&\quad + \frac{b^4|\Gamma|}{4}\int_{t_0}^t \int_\xi^t e^{-2\rho z(\theta_\tau\omega)} e^{\int_\xi^t 2\rho z(\theta_s\omega)ds - \int_\xi^t 2\lambda d_2 ds - 2\rho z(\theta_\xi\omega)} d\tau d\xi \\
&\le \|V_0\|_{L^4}^4 e^{\int_{t_0}^t (2\rho z(\theta_s\omega)-6\lambda d_2)ds} + \frac{b^2\|V_0\|^2}{2} e^{\int_{t_0}^t (2\rho z(\theta_s\omega)-2\lambda d_2)ds} \int_{t_0}^t e^{-2\rho z(\theta_\tau\omega)} d\tau \\
&\quad + \frac{b^4|\Gamma|}{4}\int_{t_0}^t \int_{t_0}^t e^{-2\rho z(\theta_\tau\omega)} e^{\int_\xi^t 2\rho z(\theta_s\omega)ds - \int_\xi^t 2\lambda d_2 ds - 2\rho z(\theta_\xi\omega)} d\tau d\xi.
\end{aligned}
\tag{4.3.27}
$$

Next we use the bootstrap method to take the inner product of (4.2.7) with V^5 and obtain

$$\begin{aligned}
&\frac{1}{6}\frac{d}{dt}\int_\Gamma V^6(t,x)dx + 5d_2\|V^2(t)\nabla V(t)\|^2 \\
&= \int_\Gamma \left(bUV^5 - e^{2\rho z(\theta_t\omega)}U^2V^6 + \rho z(\theta_t\omega)V^6\right) dx \\
&\le \int_\Gamma \left(\frac{1}{2}b^2 e^{-2\rho z(\theta_t\omega)}V^4 + \frac{1}{2}e^{2\rho z(\theta_t\omega)}U^2V^6 - e^{2\rho z(\theta_t\omega)}U^2V^6 + \rho z(\theta_t\omega)V^6\right) dx.
\end{aligned} \tag{4.3.28}$$

It follows that

$$\begin{aligned}
\frac{d}{dt}\|V(t)\|_{L^6}^6 + 10\lambda d_2\|V(t)\|_{L^6}^6 \le& \frac{d}{dt}\|V(t)\|_{L^6}^6 + 10d_2\|\nabla V^3(t)\|^2 \\
\le& 2\rho z(\theta_t\omega)\|V(t)\|_{L^6}^6 + \frac{1}{2}b^2 e^{-2\rho z(\theta_t\omega)}\|V(t)\|_{L^4}^4.
\end{aligned} \tag{4.3.29}$$

Then there is a random variable $T_6(\|g_0\|_{L^6},\omega) \le -4$ such that for every $\omega \in \Omega$, $t_0 \le T_6(\|g_0\|_{L^6},\omega)$ and $t \in [-4,0]$, we have

$$\begin{aligned}
&\|V(t,\omega;t_0,g_0)\|_{L^6}^6 \\
&\le \|V_0\|_{L^6}^6 e^{\int_{t_0}^t (2\rho z(\theta_s\omega)ds - 10\lambda d_2(t-t_0)} + \frac{b^2}{2}\int_{t_0}^t e^{\int_\eta^t (2\rho z(\theta_s\omega)-10\lambda d_2)ds - 2\rho z(\theta_\eta\omega)}\|V(\eta)\|_{L^4}^4 d\eta \\
&\le \|V_0\|_{L^6}^6 e^{\int_{t_0}^t (2\rho z(\theta_s\omega)ds - 10\lambda d_2(t-t_0)} + \frac{b^2}{2}\int_{t_0}^t e^{\int_\eta^t (2\rho z(\theta_s\omega)-10\lambda d_2)ds - 2\rho z(\theta_\eta\omega)}\cdot \\
&\quad \cdot \|V_0\|_{L^4}^4 e^{\int_{t_0}^\eta (2\rho z(\theta_s\omega)-6\lambda d_2)ds} d\eta \\
&\quad + \frac{b^4\|V_0\|^2}{4}\int_{t_0}^t e^{\int_\eta^t (2\rho z(\theta_s\omega)-10\lambda d_2)ds - 2\rho z(\theta_\eta\omega)} e^{\int_{t_0}^\eta (2\rho z(\theta_s\omega)-2\lambda d_2)ds}\int_{t_0}^\eta e^{-2\rho z(\theta_\tau\omega)}d\tau d\eta \\
&\quad + \frac{b^6|\Gamma|}{8}\int_{t_0}^t e^{\int_\eta^t (2\rho z(\theta_s\omega)-10\lambda d_2)ds - 2\rho z(\theta_\eta\omega)}\int_{t_0}^\eta e^{-2\rho z(\theta_\tau\omega)}d\tau \cdot \\
&\quad \cdot \int_{t_0}^\eta e^{\int_\xi^\eta 2\rho z(\theta_s\omega)ds - \int_\xi^\eta 2\lambda d_2 ds - 2\rho z(\theta_\xi\omega)}d\xi d\eta
\end{aligned} \tag{4.3.30}$$

$$
\begin{aligned}
&\leq \|V_0\|_{L^6}^6 e^{\int_{t_0}^t (2\rho z(\theta_s\omega)ds - 10\lambda d_2(t-t_0)} + \frac{b^2\|V_0\|_{L^4}^4}{2}\int_{t_0}^t e^{-2\rho z(\theta_\eta\omega)} e^{\int_{t_0}^t (2\rho z(\theta_s\omega)-6\lambda d_2)ds} d\eta \\
&\quad + \frac{b^6\|V_0\|^2}{4}\int_{t_0}^t e^{-2\rho z(\theta_\eta\omega)} e^{\int_{t_0}^t (2\rho z(\theta_s\omega)-2\lambda d_2)ds}\int_{t_0}^\eta e^{-2\rho z(\theta_\tau\omega)} d\tau d\eta \\
&\quad + \frac{b^6|\Gamma|}{8}\int_{t_0}^t e^{-2\rho z(\theta_\tau\omega)} d\tau \int_{t_0}^t \int_{t_0}^\eta e^{\int_\eta^t (2\rho z(\theta_s\omega)-10\lambda d_2)ds - 2\rho z(\theta_\eta\omega)} \cdot \\
&\quad \cdot e^{\int_\xi^\eta 2\rho z(\theta_s\omega)ds - \int_\xi^\eta 2\lambda d_2 ds - 2\rho z(\theta_\xi\omega)} d\xi d\eta \\
&\leq \|V_0\|_{L^6}^6 e^{\int_{t_0}^t (2\rho z(\theta_s\omega)ds - 10\lambda d_2(t-t_0)} + \frac{b^2\|V_0\|_{L^4}^4}{2} e^{\int_{t_0}^t (2\rho z(\theta_s\omega)-6\lambda d_2)ds}\int_{t_0}^t e^{-2\rho z(\theta_\eta\omega)} d\eta \\
&\quad + \frac{b^6\|V_0\|^2}{4} e^{\int_{t_0}^t (2\rho z(\theta_s\omega)-2\lambda d_2)ds}\int_{t_0}^t e^{-2\rho z(\theta_\eta\omega)} \int_{t_0}^\eta e^{-2\rho z(\theta_\tau\omega)} d\tau d\eta \\
&\quad + \frac{b^6|\Gamma|}{8}\int_{t_0}^t e^{-2\rho z(\theta_\tau\omega)} d\tau \int_{t_0}^t \int_\xi^t e^{-2\lambda d_2(t-\eta) - 2\rho z(\theta_\eta\omega)} e^{\int_\xi^t (2\rho z(\theta_s\omega)-2\lambda d_2)ds - 2\rho z(\theta_\xi\omega)} d\eta d\xi \\
&\leq \|V_0\|_{L^6}^6 e^{\int_{t_0}^t (2\rho z(\theta_s\omega)ds - 10\lambda d_2(t-t_0)} + \frac{b^2\|V_0\|_{L^4}^4}{2} e^{\int_{t_0}^t (2\rho z(\theta_s\omega)-6\lambda d_2)ds}\int_{t_0}^t e^{-2\rho z(\theta_\eta\omega)} d\eta \\
&\quad + \frac{b^6\|V_0\|^2}{4} e^{\int_{t_0}^t (2\rho z(\theta_s\omega)-2\lambda d_2)ds}\int_{t_0}^t e^{-2\rho z(\theta_\eta\omega)} d\eta \int_{t_0}^t e^{-2\rho z(\theta_\tau\omega)} d\tau \\
&\quad + \frac{b^6|\Gamma|}{8}\int_{t_0}^t e^{-2\rho z(\theta_\tau\omega)} d\tau \int_{t_0}^t e^{-2\lambda d_2(t-\eta) - 2\rho z(\theta_\eta\omega)} d\eta \cdot \int_{t_0}^t e^{\int_\xi^t (2\rho z(\theta_s\omega)-2\lambda d_2)ds - 2\rho z(\theta_\xi\omega)} d\xi \\
&\leq P(t,\omega),
\end{aligned}
$$

where

$$
\begin{aligned}
P(t,\omega) = 3 + \frac{b^6|\Gamma|}{8}\int_{-\infty}^t e^{-2\rho z(\theta_\tau\omega)} d\tau \int_{-\infty}^t e^{-2\lambda d_2(t-\eta) - 2\rho z(\theta_\eta\omega)} d\eta \cdot \\
\cdot \int_{-\infty}^t e^{\int_\xi^t (2\rho z(\theta_s\omega)-2\lambda d_2)ds - 2\rho z(\theta_\xi\omega)} d\xi.
\end{aligned}
$$

Note that the three improper integrals above are convergent by the similar calculations as shown in (4.3.7) and (4.3.8). The proof is completed. □

The next lemma is instrumental to the proof of pullback asymptotic compactness in the next section.

Lemma 4.3.4. Let (t_0, t_1) satisfy $t_0 < -4 \le t_1 < 0$ and $(u_0, v_0) \in H$ with $\|(u_0, v_0)\| \le R(\omega)$, where $R(\omega) > 0$ is any given random variable as in Lemma 4.3.1. If the weak solution $g(t, \omega; t_0, g_0)$ satisfies $\|g(t_1, \omega; t_0, g_0)\| \in E$ with

$$\|g(t_1, \omega; t_0, g_0)\|_E \le G(\omega),$$

where $G(\omega) > 0$ is any given random variable, then there exists a random variable $D(t, G, \omega) > 0$ such that

$$\|V(t, \omega; t_0, g_0)\|_{L^6}^6 \le D(t, G, \omega), \quad \text{for any } t \in [t_1, 0],\ t_0 \le \min\{T(R, \omega), -4\}, \tag{4.3.31}$$

where $T(R, \omega)$ is the same as in Lemma 4.3.1.

Proof. Fix the initial time $t_0 \le \min\{T(R, \omega), -4\}$. Integrate (4.3.26) over $[t_1, t]$ to get

$$\begin{aligned}
&\|V(t, \omega; t_0, g_0)\|_{L^4}^4 \le \|V(t_1, \omega; t_0, g_0)\|_{L^4}^4 e^{\int_{t_1}^t (2\rho z(\theta_s\omega) - 6\lambda d_2) ds} \\
&\quad + \frac{1}{2} b^2 \int_{t_1}^t e^{\int_\tau^t (2\rho z(\theta_s\omega) - 6\lambda d_2) ds - 2\rho z(\theta_\tau\omega)} \|V(\tau, \omega; t_0, g_0)\|^2 d\tau \\
&\le \delta^4 G^4(\omega) e^{\int_{t_1}^t (2\rho z(\theta_s\omega) - 6\lambda d_2) ds} \\
&\quad + \frac{1}{2} b^2 \int_{t_1}^t e^{\int_\tau^t (2\rho z(\theta_s\omega) - 6\lambda d_2) ds - 2\rho z(\theta_\tau\omega)} \|V(\tau, \omega; t_0, g_0)\|^2 d\tau \\
&\le \delta^4 G^4(\omega) e^{\int_{t_1}^t (2\rho z(\theta_s\omega) - 6\lambda d_2) ds} + \frac{b^2}{2} \int_{t_1}^t e^{\int_\tau^t (2\rho z(\theta_s\omega) - 6\lambda d_2) ds - 2\rho z(\theta_\tau\omega)} d\tau \\
&\quad + \frac{1}{4} b^4 |\Gamma| \int_{t_1}^t e^{-2\rho z(\theta_\tau\omega)} e^{\int_\tau^t (2\rho z(\theta_s\omega) - 6\lambda d_2) ds} \int_{-\infty}^\tau e^{\int_\xi^\tau (2\rho z(\theta_s\omega) - 2\lambda d_2) ds - 2\rho z(\theta_\xi\omega)} d\xi d\tau,
\end{aligned} \tag{4.3.32}$$

in which the last inequality follows from the use of (4.3.4) and δ is the constant of the Sobolev embedding $H_0^1(\Gamma) \hookrightarrow L^4(\Gamma)$,

$$\|\varphi\|_{L^4(\Gamma)} \le \delta \|\varphi\|_E, \quad \text{for any } \varphi \in E.$$

Put

$$\begin{aligned}
\Pi(t, \omega) &= \delta^4 \tilde{P}^4(\omega) e^{\int_{t_1}^t (2\rho z(\theta_s\omega) - 6\lambda d_2) ds} + \frac{1}{2} b^2 \int_{t_1}^t e^{\int_\tau^t (2\rho z(\theta_s\omega) - 6\lambda d_2) ds - 2\rho z(\theta_\tau\omega)} d\tau \\
&\quad + \frac{1}{4} b^4 |\Gamma| \int_{t_1}^t e^{-2\rho z(\theta_\tau\omega)} e^{\int_\tau^t (2\rho z(\theta_s\omega) - 6\lambda d_2) ds} \int_{-\infty}^\tau e^{\int_\xi^\tau (2\rho z(\theta_s\omega) - 2\lambda d_2) ds - 2\rho z(\theta_\xi\omega)} d\xi d\tau.
\end{aligned} \tag{4.3.33}$$

Fix any initial time $t_0 \leq \min\{T(R,\omega), -4\}$. By integrating (4.3.29) over $[t_1, t]$ and using (2.3.3), we get

$$\begin{aligned}
&\|V(t,\omega;t_0,g_0)\|_{L^6}^6 \\
&\leq \|V(t_1,\omega;t_0,g_0)\|_{L^6}^6\, e^{\int_{t_1}^t (2\rho z(\theta_s\omega)ds - 10\lambda d_2(t-t_0)} \\
&\quad + \frac{1}{2}b^2 \int_{t_1}^t e^{\int_\tau^t (2\rho z(\theta_s\omega)-10\lambda d_2)ds - 2\rho z(\theta_\tau\omega)} \|V(\tau)\|_{L^4}^4 d\tau \\
&\leq \zeta^6 G^6(\omega) e^{\int_{t_1}^t (2\rho z(\theta_s\omega)-6\lambda d_2)ds} + \frac{1}{2}b^2 \int_{t_1}^t e^{\int_\tau^t (2\rho z(\theta_s\omega)-6\lambda d_2)ds - 2\rho z(\theta_\tau\omega)} \Pi(\tau,\omega) d\tau.
\end{aligned} \tag{4.3.34}$$

Then (4.3.31) is valid with

$$D(t,G,\omega) = \zeta^6 G^6(\omega) e^{\int_{t_1}^t (2\rho z(\theta_s\omega)-6\lambda d_2)ds} + \frac{b^2}{2} \int_{t_1}^t e^{\int_\tau^t (2\rho z(\theta_s\omega)-6\lambda d_2)ds - 2\rho z(\theta_\tau\omega)} \Pi(\tau,\omega) d\tau.$$

The proof is completed. □

4.4 Pullback Asymptotic Compactness

In this section, we show that the Brusselator random dynamical system φ is pullback asymptotically compact in H via the following modified uniform Gronwall inequality, cf. [37].

Proposition 4.4.1. Given a natural number $n > 1$, let β, ζ, and h be nonnegative functions in $L^1([-n,0];\mathbb{R}^+)$. Assume that β is absolutely continuous on $[-n,0]$ and the following differential inequality is satisfied,

$$\frac{d\beta}{dt} \leq \zeta\beta + h, \quad \text{for } t \in [-n,0].$$

If

$$\int_t^{t+1} \zeta(\tau)\, d\tau \leq A, \quad \int_t^{t+1} \beta(\tau)\, d\tau \leq B, \quad \int_t^{t+1} h(\tau)\, d\tau \leq C,$$

for any $t \in [-n,-1]$, where A, B, and C are some positive constants, then

$$\beta(t) \leq \big(B + C\big) e^A \quad \text{for} \quad t \in [-n+1, 0].$$

Lemma 4.4.2. For any given random variable $R(\omega) > 0$ and any initial data $(u_0, v_0) \in H$ with $\|(u_0,v_0)\| \leq R(\omega)$, there exists a tempered random variable $K(\omega) > 0$, and a finite time

$T(R,\omega) < 0$ such that if the initial time $t_0 \le T(R,\omega)$, then the weak solution $g(t,\omega;\tau,g_0)$, where $g_0 = e^{-\rho z(\theta_\tau \omega)}(u_0, v_0)$, of the problem of the random Brusselator reaction-diffusion system (4.2.6)-(4.2.9) satisfies $g(0,\omega;t_0,g_0) \in E$ and

$$\|g(0,\omega;t_0,g_0)\|_E^2 \le K(\omega), \quad t_0 \le T(R,\omega). \tag{4.4.1}$$

Proof. The proof is divided into three bootstrap steps. First we conduct estimates of the time average of the $H_0^1(\Gamma)$-norm for both U-component and V-component solutions on the time interval $[-4,-1]$. Second we apply the uniform Gronwall inequality (Proposition 4.4.1) to get the pointwise estimate of U-component in the time interval $[-2,0]$. Third we use the results of previous two steps to get the pointwise estimate of V-component in the time interval $[-1,0]$.

STEP 1. In this step, we establish the time-average estimates of the of E-norm for the weak solutions (U,V). Note that the estimate of $L^6(\Gamma)$-norm of the V-component of the weak solution has been obtained in Lemma 4.3.4. Since $z(\theta_t\omega)$ is continuous in t, we see that $Z(\omega) = \max_{-4\le\tau\le-1}|z(\theta_\tau\omega)|$ is a positive constant for every given $\omega \in \Omega$. Fix the initial time $t_0 \le \min\{T(R,\omega),-4\}$, here $T(R,\omega)$ comes from Lemma 4.3.1, integrate the second inequality of (4.3.3) over $[t,t+1]$, where $-4 \le t \le -1$, and by (4.3.9) we have

$$\begin{aligned}
&\int_t^{t+1} 2d_2\|\nabla V(\tau,\omega;t_0,g_0)\|^2 d\tau \\
\le& \int_t^{t+1} 2\rho z(\theta_\tau\omega)\left(1+\frac{b^2|\Gamma|}{2}\int_{-\infty}^{\tau} e^{\int_\xi^\tau (2\rho z(\theta_s\omega)-2\lambda d_2 s)ds-2\rho z(\theta_\xi\omega)}d\xi\right)d\tau \\
&+\frac{b^2|\Gamma|}{2}\int_t^{t+1} e^{-2\rho z(\theta_\tau\omega)}d\tau + \|V(t)\|^2 \\
\le& \int_{-4}^{0} 2c|z(\theta_\tau\omega)|\left(1+\frac{b^2|\Gamma|}{2}\int_{-\infty}^{\tau} e^{\int_\xi^\tau (2\rho z(\theta_s\omega)-2\lambda d_2 s)ds-2\rho z(\theta_\xi\omega)}d\xi\right)d\tau \\
&+\frac{b^2|\Gamma|}{2}\int_{-4}^{0} e^{-2\rho z(\theta_\tau\omega)}d\tau + 1 \\
&+\frac{b^2|\Gamma|}{2}\max_{-4\le t\le -1}\int_{-\infty}^{t} e^{\int_\tau^t (2\rho z(\theta_s\omega)-2\lambda d_2 s)ds-2\rho z(\theta_\tau\omega)}d\tau.
\end{aligned} \tag{4.4.2}$$

Then for $t_0 \le \min\{T(R,\omega),-4\}$ and $-4 \le t \le -1$, we have

$$\int_t^{t+1}\|\nabla V(\tau,\omega;t_0,g_0)\|^2 d\tau \le \frac{K_1(\omega)}{2d_2}, \tag{4.4.3}$$

where

$$K_1(\omega) = \int_{-4}^{0} 2\rho|z(\theta_\tau\omega)| \left(1 + \frac{b^2|\Gamma|}{2} \int_{-\infty}^{\tau} e^{\int_\xi^\tau (2\rho z(\theta_s\omega) - 2\lambda d_2 s) ds - 2\rho z(\theta_\xi\omega)} d\xi\right) d\tau$$
$$+ \frac{b^2|\Gamma|}{2} \int_{-4}^{0} e^{-2\rho z(\theta_\tau\omega)} d\tau + 1 + \frac{b^2|\Gamma|}{2} \max_{-4\le t\le -1} \int_{-\infty}^{t} e^{\int_\tau^t (2\rho z(\theta_s\omega) - 2\lambda d_2 s) ds - 2\rho z(\theta_\tau\omega)} d\tau.$$

In particular, let $t = -4$ and we have

$$\int_{-4}^{-3} \|\nabla V(\tau,\omega;t_0,g_0)\|^2 d\tau \le \frac{K_1(\omega)}{2d_2}. \tag{4.4.4}$$

By the Mean Value Theorem, there is a time $t_1 \in [-4,-3]$ such that

$$\|V(t_1,\omega;t_0,g_0)\|_E \le \frac{K_1(\omega)}{2d_2}. \tag{4.4.5}$$

Then by Lemma 4.3.4, there is a random variable $D(t, K_1/(2d_2), \omega) > 0$ such that

$$\|V(t,\omega;t_0,g_0)\|_{L^6}^6 \le D(t, K_1/(2d_2), \omega), \quad \text{for any } t \in [t_1, 0],\ t_0 \le T(R,\omega). \tag{4.4.6}$$

Fix any initial time $t_0 \le \min\{T(R,\omega), -4\}$. Integrating the inequality of (4.3.13) over $[t, t+1]$, where $-4 \le t \le -1$, in view of (4.3.17) and (4.4.5) we have

$$\begin{aligned}
&\int_t^{t+1} d_1 \|\nabla y(\tau,\omega;t_0,g_0)\|^2 d\tau \\
&\le \left(\frac{|d_2-d_1|^2}{d_1} + 2\lambda\right) \int_t^{t+1} \|\nabla V\| d\tau + \int_t^{t+1} (2\rho z(\theta_\tau\omega) - 1)\|y(\tau)\|^2 d\tau + \|y(t)\|^2 \\
&\quad + 2a^2|\Gamma| \int_t^{t+1} e^{-2\rho z(\theta_\tau\omega)} d\tau \\
&\le \left(\frac{|d_2-d_1|^2}{d_1} + 2\lambda\right) \frac{K_1(\omega)}{2d_2} + \max_{-4\le t\le 0} C_2(t,\omega) \int_{-4}^{0} |2\rho z(\theta_\tau\omega) - 1| d\tau \\
&\quad + \max_{-4\le t\le -1} C_2(t,\omega) + 2a^2|\Gamma| \int_{-4}^{0} e^{-2\rho z(\theta_\tau\omega)} d\tau.
\end{aligned} \tag{4.4.7}$$

Consequently, for $t_0 \le \min\{T(R,\omega), -4\}$ and $-4 \le t \le -1$, it holds that

$$\begin{aligned}
\int_t^{t+1} \|\nabla U(\tau,\omega;t_0,g_0)\|^2 d\tau &= \int_t^{t+1} \|\nabla y(\tau,\omega;t_0,g_0) - \nabla V(\tau,\omega;t_0,g_0)\|^2 d\tau \\
&\le \int_t^{t+1} 2\left(\|\nabla y(\tau,t_0,\omega,y_0)\|^2 + \|\nabla V(\tau,t_0,\omega,V_0)\|^2\right) d\tau \\
&\le \frac{K_1(\omega)}{d_2} + \frac{2K_2(\omega)}{d_1},
\end{aligned} \tag{4.4.8}$$

where

$$K_2(\omega) = \left(\frac{|d_2 - d_1|^2}{d_1} + 2\lambda \right) \frac{K_1(\omega)}{2d_2} + \max_{-4 \le t \le 0} C_2(t,\omega) \int_{-4}^{0} |2\rho z(\theta_\tau \omega) - 1| d\tau$$
$$+ \max_{-4 \le t \le -1} C_2(t,\omega) + 2a^2|\Gamma| \int_{-4}^{0} e^{-2\rho z(\theta_\tau \omega)} d\tau.$$

STEP 2. Now we conduct the estimates of $H_0^1(\Gamma)$-norm for the U-component of the weak solutions. Taking the inner product of (4.2.6) with $-\Delta U(t)$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla U\|^2 + d_1\|\Delta U\|^2 + (b+1)\|\nabla U\|^2 \\
=& \int_\Gamma \left(-ae^{-\rho z(\theta_t\omega)} \Delta U - e^{2\rho z(\theta_t \omega)} U^2 V \Delta U \right) \, dx + \rho z(\theta_t \omega)\|\nabla U\|^2 \\
\le& \left(\frac{d_1}{4} + \frac{d_1}{4} \right) \|\Delta U\|^2 + \frac{a^2|\Gamma|}{d_1} e^{-2\rho z(\theta_t \omega)} + \frac{1}{d_1} e^{4\rho z(\theta_t\omega)} \int_\Gamma U^4 V^2 \, dx + \rho z(\theta_t\omega)\|\nabla U\|^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\frac{d}{dt}\|\nabla U\|^2 + d_1\|\Delta U\|^2 + 2(b+1)\|\nabla U\|^2 \\
\le& \frac{2a^2|\Gamma|}{d_1} e^{-2\rho z(\theta_t\omega)} + \frac{2}{d_1} e^{4\rho z(\theta_t \omega)} \|U\|_{L^6}^4 \|V\|_{L^6}^2 + 2\rho z(\theta_t\omega)\|\nabla U\|^2 \\
\le& \frac{2a^2|\Gamma|}{d_1} e^{-2\rho z(\theta_t\omega)} + \frac{2}{d_1} e^{4\rho z(\theta_t \omega)} \zeta^4 \|\nabla U\|^4 \|V\|_{L^6}^2 + 2\rho z(\theta_t\omega)\|\nabla U\|^2 \\
\le& \frac{2a^2|\Gamma|}{d_1} e^{-2\rho z(\theta_t\omega)} + \left(\frac{2}{d_1} e^{4\rho z(\theta_t \omega)} \zeta^4 \|\nabla U\|^2 \|V\|_{L^6}^2 + 2\rho z(\theta_t\omega) \right) \|\nabla U\|^2.
\end{aligned} \tag{4.4.9}$$

After dropping the terms $d_1\|\Delta U\|^2$ and $2(b+1)\|\nabla U\|^2$ from the left-hand side of the inequality (4.4.9), it can be written as

$$\frac{d\beta}{dt} \le \alpha(t)\beta(t) + \gamma(t), \quad t \in [-3, 0], \tag{4.4.10}$$

where

$$\begin{aligned}
\beta(t) &= \|\nabla U\|^2, \\
\alpha(t) &= \frac{2\zeta^4}{d_1} e^{4\rho z(\theta_t\omega)} \|\nabla U\|^2 \|V\|_{L^6}^2 + 2\rho z(\theta_t \omega), \quad \text{and} \\
\gamma(t) &= \frac{2a^2|\Gamma|}{d_1} e^{-2\rho z(\theta_t\omega)}.
\end{aligned}$$

For $t_0 \le T(R,\omega)$ and $-3 \le t \le -1$ and, we obtain the following estimates: By (4.4.8),

$$\int_t^{t+1} \beta(\tau) d\tau = \int_t^{t+1} \|\nabla U\|^2 d\tau \le \frac{K_1(\omega)}{d_2} + \frac{2K_2(\omega)}{d_1}.$$

By (4.4.8) and (4.4.6)

$$\begin{aligned}\int_t^{t+1} \alpha(\tau)\, d\tau &= \int_t^{t+1} \frac{2\zeta^4}{d_1} e^{4\rho z(\theta_\tau\omega)} \|\nabla U\|^2 \|V\|_{L^6}^2 d\tau + \int_t^{t+1} 2\rho z(\theta_\tau \omega) d\tau \\ &\le \frac{2\zeta^4}{d_1} \max_{-3\le\tau\le 0} \left[D^{1/3}\left(\tau, \frac{K_1}{2d_2}, \omega\right) e^{4\rho z(\theta_\tau\omega)}\right] \left[\frac{K_1(\omega)}{d_2} + \frac{2K_2(\omega)}{d_1}\right] + 2\int_{-3}^0 c|z(\theta_\tau\omega)| d\tau,\end{aligned}$$

and

$$\int_t^{t+1} \gamma(\tau) d\tau = \frac{2a^2|\Gamma|}{d_1} \int_t^{t+1} e^{-2\rho z(\theta_\tau\omega)} d\tau \le \frac{2a^2|\Gamma|}{d_1} \int_{-3}^0 e^{-2\rho z(\theta_\tau\omega)} d\tau.$$

Apply the Uniform Gronwall Inequality (Proposition 4.4.1) with the above three estimates to get

$$\|U(t,\omega; t_0, g_0)\|_E^2 \le K_3(\omega), \quad t \in [-2, 0],\ t_0 \le T(R,\omega), \tag{4.4.11}$$

where

$$\begin{aligned} K_3(\omega) &= \left(\frac{K_1(\omega)}{d_2} + \frac{2K_2(\omega)}{d_1} + \int_{-3}^0 \frac{2a^2|\Gamma|}{d_1} e^{-2\rho z(\theta_\tau\omega)} d\tau \right) \cdot \\ &\cdot \exp\left\{ \frac{2\zeta^4}{d_1} \max_{-3\le\tau\le 0} \left[D^{\frac{1}{3}}\left(\tau, \frac{K_1}{2d_2}, \omega\right) e^{4\rho z(\theta_\tau\omega)}\right] \left[\frac{K_1(\omega)}{d_2} + \frac{2K_2(\omega)}{d_1}\right] + 2\int_{-3}^0 c|z(\theta_\tau\omega)| d\tau \right\}. \end{aligned}$$

STEP 3. In this step, we wrap up the proof by conducting the estimates of $H_0^1(\Gamma)$-norm for the V-component of the weak solutions based on the results of Step 1 and Step 2. Taking the inner product of (4.2.7) with $-\Delta V(t)$, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\nabla V\|^2 + d_2\|\Delta V\|^2 \\ &= \int_\Gamma \left(-bU\Delta V + e^{2\rho z(\theta_t\omega)} U^2 V \Delta V\right)\, dx + \rho z(\theta_t\omega)\|\nabla V\|^2 \\ &\le \left(\frac{d_2}{4} + \frac{d_2}{4}\right) \|\Delta V\|^2 + \frac{b^2}{d_2}\|U\|^2 + \frac{1}{d_2} e^{4\rho z(\theta_t\omega)} \int_\Gamma U^4 V^2\, dx + \rho z(\theta_t\omega)\|\nabla V\|^2 \end{aligned}$$

It follows that

$$\begin{aligned} &\frac{d}{dt}\|\nabla V\|^2 + d_2\|\Delta V\|^2 \\ \le &\frac{2b^2}{d_2}\|U\|^2 + \frac{2}{d_2} e^{4\rho z(\theta_t\omega)} \|U\|_{L^6}^4 \|V\|_{L^6}^2 + 2\rho z(\theta_t\omega)\|\nabla V\|^2 \\ \le &\frac{2b^2}{d_2}\|U\|^2 + \frac{2\zeta^6}{d_2} e^{4\rho z(\theta_t\omega)} \|\nabla U\|^4 \|\nabla V\|^2 + 2\rho z(\theta_t\omega)\|\nabla V\|^2. \end{aligned} \tag{4.4.12}$$

After dropping the terms $d_2\|\Delta V\|^2$ from the left-hand side, the inequality (4.4.12) can be written as

$$\frac{d\tilde{\beta}}{dt} \leq \tilde{\alpha}(t)\tilde{\beta}(t) + \tilde{\gamma}(t), t \in [-2, 0], \tag{4.4.13}$$

where

$$\begin{aligned}
\tilde{\beta}(t) &= \|\nabla V\|^2, \\
\tilde{\alpha}(t) &= \frac{2\zeta^6}{d_2} e^{4\rho z(\theta_t \omega)} \|\nabla U\|^4 + 2\rho z(\theta_t \omega), \quad \text{and} \\
\tilde{\gamma}(t) &= \frac{2b^2}{d_2} \|U\|^2.
\end{aligned}$$

For $-2 \leq t \leq -1$ and $t_0 \leq \min\{T(R, \omega), -4\}$, we have the following estimates: It follows from (4.4.3) that

$$\int_t^{t+1} \tilde{\beta}(\tau) d\tau = \int_t^{t+1} \|\nabla V\|^2 d\tau \leq \frac{K_1(\omega)}{2d_2}.$$

By (4.4.11),

$$\begin{aligned}
\int_t^{t+1} \tilde{\alpha}(\tau) d\tau &= \frac{2\zeta^6}{d_2} \int_t^{t+1} e^{4\rho z(\theta_\tau \omega)} \|\nabla U\|^4 d\tau + 2 \int_t^{t+1} \rho z(\theta_\tau \omega) d\tau \\
&\leq \frac{2\zeta^6}{d_2} \max_{-2 \leq \tau \leq 0} (e^{4\rho z(\theta_\tau \omega)}) K_3^2(\omega) + 2 \int_{-2}^0 c|z(\theta_\tau \omega)| d\tau,
\end{aligned}$$

and by (4.4.8)

$$\int_t^{t+1} \tilde{\gamma}(\tau) d\tau = \frac{2b^2}{d_2} \int_t^{t+1} \|U\|^2 \leq \frac{2b^2\lambda^2}{d_2} \int_t^{t+1} \|\nabla U\|^2 \leq \frac{2b^2\lambda^2}{d_2} \left(\frac{K_1(\omega)}{d_2} + \frac{2K_2(\omega)}{d_1} \right).$$

Apply the Uniform Gronwall Inequality again with the three estimates above to obtain

$$\|V(t, \omega; t_0, g_0)\|_E^2 \leq K_4(\omega), \quad t \in [-1, 0], \ t_0 \leq \min\{T(R, \omega), -4\}, \tag{4.4.14}$$

where

$$\begin{aligned}
K_4(\omega) \ &= \left(\frac{K_1(\omega)}{2d_2} + \frac{2b^2\lambda^2}{d_2} \left[\frac{K_1(\omega)}{d_2} + \frac{2K_2(\omega)}{d_1} \right] \right) \cdot \\
&\cdot \exp \left[\frac{2\zeta^6}{d_2} \max_{-2 \leq \tau \leq -1} \left(e^{4\rho z(\theta_\tau \omega)} \right) K_3^2(\omega) + 2 \int_{-2}^0 c|z(\theta_\tau \omega)| d\tau \right].
\end{aligned}$$

Finally, put $t = 0$ in (4.4.11) and (4.4.14). Thus (4.4.1) holds with $K(\omega) = K_3(\omega) + K_4(\omega)$. The proof is completed. $\square$

4.5 Main Results on Random Attractor

In this section, we finally prove the existence of a random attractor for the Brusselator random dynamical system φ in the phase space H. Moreover, we show that it has the H to E attracting regularity.

Theorem 4.5.1. For any positive parameters d_1, d_2, a, b and ρ, there exists a unique random attractor $\mathscr{A} = \{\mathscr{A}(\omega)\}_{\omega\in\Omega}$ in the phase space H for the Brusselator random dynamical system φ over the MDS $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$.

Proof. By Lemma 4.3.2, the RDS φ has a pullback absorbing set with respect to the universe $\mathcal{D}$, which is the closed random ball $B_0(\omega)$ centered at the origin with radius $M_0(\omega)$ in H.

Lemma 4.4.2 and the compact imbedding $E \hookrightarrow H$ imply that the RDS φ is pullback asymptotically compact in H with respect to $\mathcal{D}$.

By Proposition 4.2.8, there exists a unique random attractor $\mathscr{A} = \{\mathscr{A}(\omega)\}_{\omega\in\Omega}$ in H for this RDS φ, which is given by

$$\mathscr{A}(\omega) = \bigcap_{\tau\geq 0} \overline{\bigcup_{t\geq\tau} \varphi(t, \theta_{-t}\omega, B_0(\theta_{-t}\omega))}, \quad \omega\in\Omega$$

Therefore, we reach the conclusion. □

Now we show that the random attractor $\mathscr{A}(\omega)$ is an (H, E) random attractor. This concept is a generalization of (H, E) global attractor introduced in Chapter 2.

Definition 4.5.2. Let $\{\Sigma(t,\omega)\}_{t\geq 0}$ be a random dynamical system on a Banach space $\mathbb{X}$ over a given metric dynamical system and let $\mathbb{Y}$ be a compactly imbedded subspace of $\mathbb{X}$. Let a universe $\mathcal{D}$ of tempered random sets in a Banach space $\mathbb{X}$ be given. A subset $\mathcal{A} \in \mathcal{D}$ is called an $(\mathbb{X}, \mathbb{Y})$ random attractor for this RDS, if $\mathcal{A}(\omega)$ has the following properties:

(i) $\mathcal{A}$ is a nonempty, compact, and invariant random set in $\mathbb{Y}$.

(ii) $\mathcal{A}$ attracts any set $B \in \mathcal{D}$ with respect to the $\mathbb{Y}$-norm. Namely, there exists $\tau = \tau_B > 0$ such that $\Sigma(t, \theta_{-t}\omega, B(\theta_{-t}\omega)) \subset \mathbb{Y}$ for $t > \tau$ and

$$\mathrm{dist}_{\mathbb{Y}}(\Sigma(t, \theta_{-t}\omega, B(\theta_{-t}\omega)), \mathcal{A}(\omega)) \to 0, \quad \text{as } t\to\infty.$$

Definition 4.5.3. Let $\mathcal{D}$ be a universe of sets in a Banach space X. A random dynamical system (φ, θ) on X is said to be $\mathcal{D}$-flattening if for every $B \in \mathcal{D}$, $\epsilon > 0$ and $\omega \in \Omega$, there exists a $T_0(B, \omega, \epsilon) > 0$ and a finite dimensional subspace of $X_1(\epsilon)$ (which may depend on ϵ) of X, such that the following two conditions are satisfied.

(i) $\bigcup_{t \geq T_0} Q\, \varphi(t, \theta_{-t}\omega, B(\theta_{-t}\omega))$ is bounded in X,

(ii) $||(I - Q)(\varphi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)))||_X < \epsilon$.

where $Q : X \to X_1(\epsilon)$ is a bounded projection.

Let $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots \to \infty$ as $j \to \infty$ be the complete set of eigenvalues each repeated to its multiplicity of the differential operator $-A : [H^2(\Gamma) \cap H_0^1(\Gamma)]^2 \to [L^2(\Gamma)]^2$ defined by (1.0.2) and $\{e_j\}_{j=1}^{\infty}$ be the corresponding eigenvectors. Let $Q_n : H \to \text{Span}\{e_1, \cdots, e_n\}$ be the orthogonal projection from H onto the subspace spanned by the first n eigenvectors. Then every $u \in H$ has a unique orthogonal decomposition $u = u_1 + u_2$, where $u_1 = Q_n u$ and $u_2 = (I - Q_n)u$ are called low modes and high modes, respectively.

The following proposition is seen in [50].

Proposition 4.5.4. Given a uniformly convex Banach space $\mathbb{X}$ and a universe of random sets $\mathcal{D}$ in $\mathbb{X}$, let φ be a continuous RDS on $\mathbb{X}$ over an MDS $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t \in \mathbb{R}})$. Suppose that there exists a closed pullback absorbing set $\{K(\omega)\}_{\omega \in \Omega} \in \mathcal{D}$ and the RDS φ is $\mathcal{D}$-flattening, then the RDS φ has a unique random attractor $\mathcal{A} = \{\mathcal{A}(\omega)\}_{\omega \in \Omega} \in \mathcal{D}$ in $\mathbb{X}$, which is given by

$$\mathcal{A}(\omega) = \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} \varphi(t, \theta_{-t}\omega, K(\theta_{-t}\omega))}.$$

Next we show that the Brusselator random dynamical system φ possesses the flattening property in the more regular space E. The first condition of the flattening property is clearly satisfied due to Lemma 4.4.2. To prove the second condition, we decompose the weak solution $g(t, \omega; t_0, g_0)$ of the random Brusselator reaction-diffusion system (4.2.6)-(4.2.9) as the high modes and the low modes,

$$g(t, \omega) = g_1(t, \omega) + g_2(t, \omega), \quad \text{where } g_1 = Q_n g,\ g_2 = (I - Q_n)g.$$

Then we show the following lemma.

Lemma 4.5.5. For any given $\epsilon > 0$ and any initial data $(u_0, v_0) \in H$ with $\|(u_0, v_0)\| \le R(\omega)$, where $R(\omega)$ is an arbitrarily given tempered positive random variable, there exists a finite $T_E(R,\omega) = \min\{T(R,\omega), -4\} < 0$, where $T(R,\omega)$ is given in Lemma 4.3.1, and a positive integer $N(\epsilon,\omega)$ such that the high modes $g_2(t,\omega; t_0, g_0)$ satisfy

$$\|e^{\rho z(\omega)} g_2(0,\omega; t_0, g_0)\|_E < \epsilon, \quad \text{for all } \ t_0 \le T_E(R,\omega), \ n > N(\epsilon,\omega). \tag{4.5.1}$$

Proof. Taking the inner product of (4.2.6) with $-\Delta U_2(t)$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla U_2\|^2 + d_1\|\Delta U_2\|^2 + (b+1)\|\nabla U_2\|^2 \\
&= \int_\Gamma \left(-a e^{-\rho z(\theta_t\omega)}\Delta U_2 - e^{2\rho z(\theta_t\omega)} U^2 V \Delta U_2\right) dx + \rho z(\theta_t\omega)\|\nabla U_2\|^2 \\
&\le \left(\frac{d_1}{4} + \frac{d_1}{4}\right)\|\Delta U_2\|^2 + \frac{a^2|\Gamma|}{d_1} e^{-2\rho z(\theta_t\omega)} + \frac{1}{d_1} e^{4\rho z(\theta_t\omega)} \int_\Gamma U^4 V^2\, dx + \rho z(\theta_t\omega)\|\nabla U_2\|^2.
\end{aligned}$$

It follows that for any $t_0 \le T_E(R,\omega)$ (where $T(R,\omega)$ is given in Lemma 4.3.1) and $t \in [-1, 0]$,

$$\begin{aligned}
&\frac{d}{dt}\|\nabla U_2\|^2 + d_1\lambda_{n+1}\|\nabla U_2\|^2 \\
\le& \frac{2a^2|\Gamma|}{d_1} e^{-2\rho z(\theta_t\omega)} + \frac{2}{d_1} e^{4\rho z(\theta_t\omega)} \|U\|_{L^6}^4 \|V\|_{L^6}^2 + 2\rho z(\theta_t\omega)\|\nabla U_2\|^2 \\
\le& \frac{2a^2|\Gamma|}{d_1} e^{-2\rho z(\theta_t\omega)} + \frac{2}{d_1} e^{4\rho z(\theta_t\omega)} \zeta^6 \|\nabla U\|^4 \|\nabla V\|^2 + 2\rho z(\theta_t\omega)\|\nabla U_2\|^2 \\
\le& \frac{2a^2|\Gamma|}{d_1} e^{-2\rho z(\theta_t\omega)} + \frac{2}{d_1} e^{4\rho z(\theta_t\omega)} \zeta^6 K_3^2(\omega) K_4(\omega) + 2\rho z(\theta_t\omega)\|\nabla U_2\|^2.
\end{aligned} \tag{4.5.2}$$

Multiply (4.5.2) by $e^{\int_\sigma^t (2\rho z(\theta_s\omega) - d_1\lambda_{n+1}) ds}$ and integrate the resulting inequality over $[\sigma, t]$, for $-1 \le \sigma < t \le 0$,

$$\begin{aligned}
\|\nabla U_2(t,\omega; t_0, g_0)\|^2 \le& \|\nabla U_2(\sigma,\omega; t_0, g_0)\|^2 e^{\int_\sigma^t (2\rho z(\theta_s\omega) - d_1\lambda_{n+1}) ds} \\
&+ \frac{2a^2|\Gamma|}{d_1} \int_\sigma^t e^{\int_\tau^t (2\rho z(\theta_s\omega) - d_1\lambda_{n+1}) ds - 2\rho z(\theta_\tau\omega)} d\tau \\
&+ \frac{2}{d_1}\zeta^4 K_3^2(\omega) K_4(\omega) \int_\sigma^t e^{\int_\tau^t (2\rho z(\theta_s\omega) - d_1\lambda_{n+1}) ds + 4\rho z(\theta_\tau\omega)} d\tau \\
\le& \|\nabla U_2(\sigma,\omega; t_0, g_0)\|^2 e^{\int_\sigma^t (2\rho z(\theta_s\omega)) ds - d_1\lambda_{n+1}(t-\sigma)} \\
&+ \frac{2a^2|\Gamma|}{d_1} \int_\sigma^t e^{\int_\tau^t (2\rho z(\theta_s\omega)) ds - d_1\lambda_{n+1}(t-\tau) - 2\rho z(\theta_\tau\omega)} d\tau \\
&+ \frac{2}{d_1}\zeta^4 K_3^2(\omega) K_4(\omega) \int_\sigma^t e^{\int_\tau^t (2\rho z(\theta_s\omega)) ds - d_1\lambda_{n+1}(t-\tau) + 4\rho z(\theta_\tau\omega)} d\tau,
\end{aligned} \tag{4.5.3}$$

where

$$\|\nabla U_2(\sigma,\omega;t_0,g_0)\|^2 e^{\int_\sigma^t 2\rho z(\theta_s\omega)ds} \le K_3(\omega)e^{\int_{-1}^0 2\rho|z(\theta_s\omega)|ds},$$

$$e^{-d_1\lambda_{n+1}(t-\sigma)} \to 0 \ \text{ as } n\to 0,$$

$$\frac{2a^2|\Gamma|}{d_1}\int_\sigma^t e^{\int_\tau^t(2\rho z(\theta_s\omega))ds-2\rho z(\theta_\tau\omega)}d\tau \le \frac{2a^2|\Gamma|}{d_1}\int_{-1}^0 e^{\int_{-1}^0 2\rho|z(\theta_s\omega)|ds-2\rho z(\theta_\tau\omega)}d\tau,$$

$$\int_\sigma^t e^{-d_1\lambda_{n+1}(t-\tau)}d\tau = \frac{1}{d_1\lambda_{n+1}}(1-e^{-d_1\lambda_{n+1}(t-\sigma)}) \to 0 \ \text{ as } n\to 0.$$

Consequently, there exists an integer $N_1(\epsilon,\omega)\ge 1$ such that for every $\omega\in\Omega$ and $t\in[-1,0]$,

$$\|e^{\rho z(\theta_t\omega)}\nabla U_2(t,\omega;t_0,g_0)\|^2 \le \max_{t\in[-1.0]} e^{2\rho z(\theta_t\omega)}\|\nabla U_2(t,\omega;t_0,g_0)\|^2 < \frac{\epsilon^2}{2}, \tag{4.5.4}$$

whenever $t_0\le T_E(R,\omega)$, $n>N_1(\epsilon,\omega)$.

On the other hand, taking the inner product of (4.2.7) with $-\Delta V_2(t)$, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla V_2\|^2 + d_2\|\Delta V_2\|^2\\ =&\int_\Gamma \left(-bU\Delta V_2 + e^{2\rho z(\theta_t\omega)}U^2V\Delta V_2\right)\,dx + \rho z(\theta_t\omega)\|\nabla V_2\|^2\\ \le&\left(\frac{d_2}{4}+\frac{d_2}{4}\right)\|\Delta V_2\|^2 + \frac{b^2}{d_2}\|U\|^2 + \frac{1}{d_2}e^{4\rho z(\theta_t\omega)}\int_\Gamma U^4V^2\,dx + \rho z(\theta_t\omega)\|\nabla V_2\|^2.\end{aligned}$$

It follows that

$$\begin{aligned}&\frac{d}{dt}\|\nabla V_2\|^2 + d_2\lambda_{n+1}\|\nabla V_2\|^2\\ \le&\frac{2b^2}{d_2}\|U\|^2 + \frac{2}{d_2}e^{4\rho z(\theta_t\omega)}\|U\|_{L^6}^4\|V\|_{L^6}^2 + 2\rho z(\theta_t\omega)\|\nabla V_2\|^2\\ \le&\frac{2b^2}{d_2}C_2(t,\omega) + \frac{2\zeta^6}{d_2}e^{4\rho z(\theta_t\omega)}K_3^2(\omega)K_4(\omega) + 2\rho z(\theta_t\omega)\|\nabla V_2\|^2,\end{aligned} \tag{4.5.5}$$

for $t\in[-1,0]$ and $t_0\le T_E(R,\omega)$. Multiply (4.5.5) by $e^{\int_\sigma^t(2\rho z(\theta_s\omega)-d_2\lambda_{n+1})ds}$ and integrate the resulting inequality over $[\sigma, t]$ where $-1 \le \sigma < t \le 0$,

$$
\begin{aligned}
\|\nabla V_2(t,\omega;t_0,g_0)\|^2 \le\, & \|\nabla V_2(\sigma,\omega;t_0,g_0)\|^2 e^{\int_\sigma^t (2\rho z(\theta_s\omega - d_2\lambda_{n+1})ds} \\
& + \frac{2b^2}{d_2}\int_\sigma^t e^{\int_\tau^t (2\rho z(\theta_s\omega)-d_2\lambda_{n+1})ds} C_2(\tau,\omega)d\tau \\
& + \frac{2\zeta^6}{d_2} K_3^2(\omega)K_4(\omega)\int_\sigma^t e^{\int_\tau^t (2\rho z(\theta_s\omega)-d_2\lambda_{n+1})ds + 4\rho z(\theta_\tau\omega)} d\tau \\
\le\, & \|\nabla V_2(\sigma,\omega;t_0,g_0)\|^2 e^{\int_\sigma^t (2\rho z(\theta_s\omega))ds - d_2\lambda_{n+1}(t-\sigma)} \\
& + \frac{2b^2}{d_2}\int_\sigma^t e^{\int_\tau^t (2\rho z(\theta_s\omega))ds - d_2\lambda_{n+1}(t-\tau)} C_2(\tau,\omega)d\tau \\
& + \frac{2\zeta^6}{d_2} K_3^2(\omega)K_4(\omega)\int_\sigma^t e^{\int_\tau^t (2\rho z(\theta_s\omega))ds - d_2\lambda_{n+1}(t-\tau) + 4\rho z(\theta_\tau\omega)} d\tau.
\end{aligned}
\tag{4.5.6}
$$

Similar to (4.5.3) and (4.5.4), there exists an integer $N_2(\epsilon,\omega) \ge 1$ such that for every $\omega \in \Omega$ and $t \in [-1, 0]$,

$$
\|e^{\rho z(\theta_t\omega)}\nabla V_2(t,\omega;t_0,g_0)\|^2 \le \max_{t\in[-1,0]} e^{2\rho z(\theta_t\omega)}\|\nabla V_2(t,\omega;t_0,g_0)\|^2 < \frac{\epsilon^2}{2}, \tag{4.5.7}
$$

whenever $t_0 \le T_E(R,\omega)$, $n > N_2(\epsilon,\omega)$. Adding up (4.5.3) and (4.5.6), we reach (4.5.1). The proof is completed. □

Theorem 4.5.6. The random attractor $\mathscr{A} = \{\mathscr{A}(\omega)\}_{\omega\in\Omega}$ for the Brusselator random dynamical system φ shown in Theorem 4.5.1 is indeed an (H, E) random attractor.

Proof. Since Lemma 4.5.5 along with (4.3.23) shows that the flattening property with respect to E is satisfied by the RDS φ and Lemma 4.4.2 together with (4.3.23) confirms that there exists a pullback absorbing ball centered at the origin with the radius $e^{\rho z(\omega)}K(\omega)$ in the space E. Therefore, by Proposition 4.5.4 with $X = E$, there exists a unique random attractor $\mathscr{A}_E$ for the Brusselator random dynamical system φ in E.

By the mutual attraction and the invariance of both random attractors $\mathscr{A}$ in H from Theorem 4.5.1 and $\mathscr{A}_E$ in E, we see that $\mathscr{A}_E = \mathscr{A}$ and $\mathscr{A}$ is an (H, E) random attractor. The proof is completed. □

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