6-20-2016

Hamiltonian Formulations and Symmetry Constraints of Soliton Hierarchies of (1+1)-Dimensional Nonlinear Evolution Equations

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by

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A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
Department of Mathematics and Statistics
College of Arts and Sciences
University of South Florida

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Date of Approval:
June 09, 2016

Keywords: Spectral problem, Soliton hierarchy, Hamiltonian formulation, Liouville integrability, Symmetry constraint

Dedication

This doctoral dissertation is dedicated to my mother and the memory of my father.

Acknowledgments

First and foremost, I would like to thank God Almighty for guiding me successfully through this journey. Without His grace and manifold blessings, I would not have been able to accomplish this feat.

I am deeply grateful to my dissertation advisor, Dr. Wen-Xiu for his immeasurable advice and guidance without which this dissertation would not have been possible. I am also grateful to the committee members, Dr. Razvan Teodorescu, Dr. Gangaram Ladde and Dr. Mohammed Elhamdadi for their support and advice throughout this process. My sincere appreciation goes to Dr. Kwabena Gyimah-Brempong of the Department of Economics for taking time out of his busy schedule to chair the defense committee.

I acknowledge Dr. Marcus McWaters and Dr. Stephen Suen for their advice and encouragement during this journey. Words are not enough to thank them. I am also thankful to Dr. Ustinov Nikolai of the theoretical physics department at the Kaliningrad branch of MIIT in Russua for the many email discussions which helped to make this work a success. My warm appreciation also goes to Dr. Ana Torres of the Smart lab and all the faculty and staff of the mathematics department who contributed in diverse ways to make my experience in this department a memorable one.

I am indebted to all my friends, especially Emmanuel Appiah, Neranga Fernando, David Kabore, Richard Boampong, Junyi Tu, Stephen Lapanno, Seyed Zoarolshd, Tharu Bhikari, NawaRaj Pokhrel and Kumar Vijay Garapati for the support and encouragement they offered me on this journey. I am also indebted to the Foster family (Paul, Sue, Kathrine, Sam and Ben) for their love and friendship.

Most importantly, I would like to express my heartfelt gratitude to my mother Ernestina Bonsrah, my late father Emmanuel Okwanning Manukure, my sisters Diana

Osafo and Abigail Manukure and my brother David Ansah Manukure. I owe them a debt of gratitude for the unflinching love and prayers they offered me during this journey. I also thank the rest of my family especially my aunt, Lydia Bonsrah and her husband Michael Antonio, my Uncle Victor Bonsrah, my Aunt Victoria Bonsrah and my grandmother Paulina Ansah for their kindness and support.

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Abstract

We derive two hierarchies of 1+1 dimensional soliton-type integrable systems from two spectral problems associated with the Lie algebra of the special orthogonal Lie group $\mathrm{SO}(3, \mathbb{R})$. By using the trace identity, we formulate Hamiltonian structures for the resulting equations. Further, we show that each of these equations can be written in Hamiltonian form in two distinct ways, leading to the integrability of the equations in the sense of Liouville. We also present finite-dimensional Hamiltonian systems by means of symmetry constraints and discuss their integrability based on the existence of sufficiently many integrals of motion.

1 Introduction

1.1 Background

One aspect of modern mathematical physics that has been widely studied in the last few decades is the theory of integrable systems. The origin of the theory dates back to the 19th century when the Korteweg-de Vries (KdV) equation was derived for the description of solitary waves in shallow water. The initial observation of a solitary wave in shallow water was made by John Scott Russell [33, 34], a young Scottish engineer and naval architect, in experiments to design a more efficient canal boat for the Union Canal Company in 1834. Russell's work was not given much attention until in the mid 1960's when scientists began to study nonlinear wave propagation using computers. A major breakthrouh that motivated the revolution in soliton theory was the work of Zabusky and Kruskal [41] in the discovery of a numerical computation of solutions for the KdV equation in 1965. In an attempt to resolve the Fermi-Pasta-Ulam paradox [7], Zabusky and Kruskal obseverd that solitary waves retain their shapes and speed after collision in a way analogous to colliding particles. Because of this particle-like behavior, Zabusky and Kruskal called solitary waves "solitons". Following this landmark, Gardner, Greene, Kruskal and Miura discovered a method for finding soliton solutions for the KdV equation. This method, now known as the Inverse Scattering Transform (IST) [10], is considered one of the most important discoveries of the 20th century. Further work in relation to the IST by Lax [13] in 1968 revealed more remarkable properties of the KdV equation. It turned out that the KdV equation is the compatibility condition between a pair of two linear operators which are now called the Lax Pair. In 1974, the work of Gardner et al. and Lax was extended to an infinite number of integrable equations by Ablowitz, Kaup, Newell and Surgur [2]. In particular, they showed that one can derive, from a matrix spectral problem, nonlinear evolution equations that are solvable by the IST. This technique underscores the importance of spectral problems in soliton theory.

Solitons are known to arise as solutions to integrable systems. There are many notions of integrability (complete integrability, Liouville integrability, algebraic integrability, analytic integrability, etc.), but a universally accepted definition does not exist. Many of these notions of integrability do not involve explicit solutions since in general, obtaining explicit solutions of integrable systems is a very difficult task, although many new solution techniques such as the Hirota bilinear method, the algebro-geometric method and the Lie-algebraic method have recently been introduced. Motivated by the Liouville-Arnold theorem [4], the notion of integrability in the sense of Liouville (Liouville integrability) is the existence of infinitely many conservation laws and commuting symmetries. There are many effective methods for finding symmetries and conservation laws, although constructing nonlinear partial differential systems, especially multi-component ones, possessing this property is not easy. One such method lies in a result on bi-Hamiltonian systems which is due to Magri [25]. In his seminal paper, Magri demonstrates that if a partial differential equation can be written as a Hamiltonian system in two different but compatible ways, then the system possesses infinitely many conservation laws and symmetries, and thus, integrable in the sense of Liouville. In this dissertation, we focus on hierarchies of Liouville integrable Hamiltonian systems.

The dissertation is organized as follows: In Chapter 2, we study a few methods for constructing integrable systems and illustrate how some well-known equations such as the KdV equation, the sine-Gordon equation and the nonlinear Schrödinger equation can be derived from these methods. Further, we introduce a spectral problem and derive its associated hierarchy of integrable systems using the so-called Tu scheme. We then provide an extension of this spectral problem and subsequently construct another hierarchy of integrable systems which contains the earlier hierarchy as a subsystem. In chapter 3, we focus on Hamiltonian structures and integrability of nonlinear evolution equations. In particular, we show that the newly constructed hierarchies have bi-Hamiltonian and tri-Hamiltonian structures and are thus integrable in the sense that they possess infinitely many conserved functionals in involution. The fourth chapter deals with ordinary differential equations in Hamiltonian form associated with soliton hierarchies. In this chapter, we derive finite-dimensional Hamiltonian systems from a soliton hierarchy by means of symmetry constraint and discuss their integrability based on the existence of sufficiently many integrals of motion. Finally, we give some concluding remarks on our results in chapter 5.

1.2 Preliminaries

We now introduce some basic notations and definitions. Let M be an open subset of $X \times U$, where X is the space of independent variables $x = (x^1, \cdots, x^p)$ and U is the space of dependent variables $u = (u^1, \cdots, u^q)^T$. We denote by $\mathcal{A}$, the algebra of smooth functions $P(x, u^{(n)})$ depending on x, u and derivatives of u up to a finite order n. The functions in $\mathcal{A}$ are called differential functions. For convenience, we will use the notation $P[u]$ or simply P for the differential function $P(x, u^{(n)})$. We also denote the quotient space of $\mathcal{A}$ under the image of the total divergence by $\mathcal{F}$. This is the space of functionals $\mathcal{P} = \int P dx$. In most of our examples, we will assume that $p = q = 1$, i.e., $X = U = \mathbb{R}$.

Definition 1.2.1 Given

$$P[u] = P(x, u^{(n)}), \tag{1.2.1}$$

the *i-th total derivative* of P is defined as

$$D_i P = \frac{\partial P}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{J} u^\alpha_{J,i} \frac{\partial P}{\partial u^\alpha_J} \tag{1.2.2}$$

where $J = (j_1, \cdots, j_k)$ with $1 \leq j_k \leq p, k \geq 0$ and

$$u^\alpha_{J,i} = \frac{\partial u^\alpha_J}{\partial x^i} = \frac{\partial^{k+1} u^\alpha}{\partial x^i \partial x^{j_1} \cdots \partial x^{j_k}}. \tag{1.2.3}$$

If $X = \mathbb{R}$ and $U = \mathbb{R}$, then we have the functional $P[u] = P(x, u, u_x, \cdots)$. The total derivative of P is thus

$$D_x P = \frac{\partial P}{\partial x} + u_x \frac{\partial P}{\partial u} + u_{xx} \frac{\partial P}{\partial u_x} + \cdots. \tag{1.2.4}$$

Example 1.2.2 *If*

$$P = xuu_x, \tag{1.2.5}$$

then

$$D_x P = uu_x + xu_x^2 + xuu_{xx}. \tag{1.2.6}$$

Definition 1.2.3 Let

$$P[u] = P(x, u^{(n)}) \in \mathcal{A}^r \tag{1.2.7}$$

be an $r-$tuple of differential functions. The *Fréchet derivative* of P is the differential operator $d_P : \mathcal{A}^q \to \mathcal{A}^r$ defined as

$$d_P(Q) \equiv P'[Q] = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} P[u + \varepsilon Q[u]] \tag{1.2.8}$$

for any $Q \in \mathcal{A}^q$.

Example 1.2.4 *If*

$$P = uu_x, \tag{1.2.9}$$

then

$$d_P(Q) = P'[Q] = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}(u+\varepsilon Q)(u_x+\varepsilon D_x Q) = u_x Q + u D_x Q. \tag{1.2.10}$$

Definition 1.2.5 For $1 \le \alpha \le q$, the α-th *Euler operator* is given by

$$E_\alpha = \sum_J (-D)_J \frac{\partial}{\partial u_J^\alpha}, \tag{1.2.11}$$

the sum extending over all multi-indices $J = (j_1, ..., j_k)$ with $1 \le j_k \le p,\ k \ge 0$.

If $p = q = 1$, we have u as a function of a single variable x and thus

$$E = \sum_{j=0}^{\infty} (-D_x)^j \frac{\partial}{\partial u_j} = \frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u_x} + D_x^2 \frac{\partial}{\partial u_{xx}} - \cdots . \tag{1.2.12}$$

Definition 1.2.6 The variational derivative of a functional

$$\mathcal{L} = \int_\Omega L(x, u^{(n)}),\ \Omega \subset X \tag{1.2.13}$$

is defined as

$$\frac{\delta \mathcal{L}[u]}{\delta u} = E(L) = (E_1(L), ..., E_q(L))^T. \tag{1.2.14}$$

If $p = q = 1$, the n-th order variational derivative of the functional

$$\mathcal{L} = \int_a^b L(x, u^{(n)}) dx$$

is given by

$$\frac{\delta \mathcal{L}}{\delta u} = E(L) = \sum_{j=0}^{\infty} (-D_x)^j \frac{\partial L}{\partial u_j} = \frac{\partial L}{\partial u} - D_x \frac{\partial L}{\partial u_x} + D_x^2 \frac{\partial L}{\partial u_{xx}} - \cdots + (-1)^n D_x^n \frac{\partial L}{\partial u_n}$$

Example 1.2.7 *Let*

$$\mathcal{L} = \int (u_{xx}^2 - uu_x)dx. \tag{1.2.15}$$

Then

$$\frac{\delta \mathcal{L}}{\delta u} = E(L) = -u_x - D_x(-u) + D_x^2(2u_{xx}) = 2u_{xxxx}. \tag{1.2.16}$$

Definition 1.2.8 An *evolution equation* is a partial differential equation of the form

$$u_t = K[u] \tag{1.2.17}$$

where $u(x,t)$ is a dependent variable, and $K[u]$ is a differential function. If K is nonlinear, the above equation is called a nonlinear evolution equation.

Definition 1.2.9 A *differential operator* is a finite sum

$$\mathcal{D} = \sum_{i=0}^{n} P_i[u]D_x^i, \tag{1.2.18}$$

where the coefficients $P_i[u]$ are differential functions. We say that $\mathcal{D}$ has order n provided its leading coefficient is not identically zero: $P_n \not\equiv 0$.

Definition 1.2.10 A (formal) *pseudo-differential operator* is a formal infinite series

$$\mathcal{D} = \sum_{i=-\infty}^{n} P_i[u]D_x^i, \tag{1.2.19}$$

whose coefficients P_i are differential functions. We say that $\mathcal{D}$ has order n if its leading coefficient is not identically zero: $P_n \not\equiv 0$.

Example 1.2.11 *The operator*

$$\mathcal{D} = D_x^2 + \frac{2}{3}u + \frac{1}{3}u_x D_x^{-1} \tag{1.2.20}$$

is a pseudo-differential of order 2.

Definition 1.2.12 If

$$\mathcal{D} = \sum_J P_J[u] D_J, \quad P_J \in \mathcal{A} \tag{1.2.21}$$

is a differential operator, its (formal) adjoint is the differential operator $\mathcal{D}^*$ which satisfies

$$\int_\Omega P \cdot \mathcal{D} Q dx = \int_\Omega Q \cdot \mathcal{D}^* P dx \tag{1.2.22}$$

for every pair of differential functions $P, Q \in \mathcal{A}$ which vanish when $u \equiv 0$, every domain $\Omega \in \mathbb{R}^p$ and every function $u = f(x)$ of compact support in Ω. It follows from integration by parts that

$$\mathcal{D}^* = \sum_J (-D)_J P_J. \tag{1.2.23}$$

That is, for any $Q \in \mathcal{A}$,

$$\mathcal{D}^* Q = \sum_J (-D)_J (P_J Q). \tag{1.2.24}$$

Similarly, a matrix differential operator $\mathcal{D} : \mathcal{A}^k \to \mathcal{A}^l$ with entries $D_{\mu\nu}$ has adjoint $\mathcal{D}^* : \mathcal{A}^l \to \mathcal{A}^k$ with entries $D^*_{\mu\nu} = (D_{\nu\mu})^*$, the adjoint of the transpose entries of $\mathcal{D}$. Note that $(\mathcal{D}\mathcal{E})^* = \mathcal{E}^* \mathcal{D}^*$.

Example 1.2.13 *If*

$$\mathcal{D} = D_x^2 + u D_x, \tag{1.2.25}$$

then its adjoint is

$$\mathcal{D}^* = (-D_x)^2 + (-D)_x u = D_x^2 - u D_x - u_x. \tag{1.2.26}$$

Definition 1.2.14 Let $Q[u] = (Q_1[u], Q_2[u], ..., Q_q[u])^T \in \mathcal{A}^q$ be a $q-$tuple of differential functions. The generalized vector field

$$\mathrm{v}_Q = \sum_{\alpha=1}^{q} Q_\alpha[u] \frac{\partial}{\partial u^\alpha} \tag{1.2.27}$$

is called an evolutionary vector field, and Q is called its characteristic.

The space of evolutionary vector fields is a Lie algebra with Lie bracket

$$[\mathrm{v}_P, \mathrm{v}_Q] = \mathrm{v}_{[P,Q]}. \tag{1.2.28}$$

Thus, the generalized vector field $\mathrm{v}_{[P,Q]}$ is also an evolutionary vector field with characteristic

$$[P, Q] = \mathrm{v}_P(Q) - \mathrm{v}_Q(P) = d_Q[P] - d_P[Q]. \tag{1.2.29}$$

where $d_P[Q]$ is the Fréchet derivative of P in the direction of Q.

Definition 1.2.15 An evolutionary vector field $\mathrm{v_Q}$ with characteristic Q is a symmetry of the system of evolution equation $u_t = P[u]$ if and only if

$$\frac{\partial Q}{\partial t} + [P, Q] = 0 \tag{1.2.30}$$

holds identically in $(x, t, u^{(m)})$. By abuse of language, we will usually say that Q is a symmetry of $u_t = K[u]$.

In particular, if $Q[u] = Q(x, u^{(m)})$ does not depend explicitly on t, then the above equation (1.2.30) reduces to the condition that

$$[P, Q] = 0. \tag{1.2.31}$$

Example 1.2.16 *Consider the well-known KdV equation*

$$u_t = u_{xxx} + 6uu_x. \tag{1.2.32}$$

Denote $K[u] = u_{xxx} + 6uu_x$ *and let* $P[u] = u_x$*. Then,*

$$\begin{aligned} K'[P] = \frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0} K[u+\varepsilon P] &= \frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0} \left((u_{xxx} + \varepsilon P_{xxx}) + 6(u+\varepsilon P)(u_x + \varepsilon P_x)\right) \\ &= P_{xxx} + 6Pu_x + 6uP_x \\ &= u_{xxxx} + 6u_x^2 + 6uu_{xx} \end{aligned} \tag{1.2.33}$$

Similarly,

$$\begin{aligned} P'[K] = \frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0} P[u+\varepsilon K] &= \frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0} (u_x + \varepsilon K_x) \\ &= K_x = u_{xxxx} + 6u_x^2 + 6uu_{xx} \end{aligned} \tag{1.2.34}$$

It follows that $[P, K] = 0$*. Thus,* $P[u] = u_x$ *is a symmetry of the KdV equation. It is known that the KdV equation possesses infinitely many symmetries and* $P[u] = u_x$ *is only one of those symmetries.*

Definition 1.2.17 A recursion operator for a system of evolution equations $u_t = K[u]$ is a linear operator $\Phi : \mathcal{A}^q \to \mathcal{A}^q$ in the space of $q-$tuples of differential functions with the property that whenever v_Q is an evolutionary symmetry of $u_t = K[u]$, so is v_P with $P = \Phi Q$.

Thus, given a characteristic Q_0 and a recursion operator Φ for a system of differential equations, one can generate infinitely many symmetries $Q_n = \Phi^n Q_0,\ n = 0, 1, 2, ...$, by recursively applying Φ to the characteristic Q_0.

Example 1.2.18 *The recursion operator for the KdV equation is*

$$\Phi = D_x^2 + 4u + 2u_x D_x^{-1}, \tag{1.2.35}$$

where D_x^{-1} *is the inverse of the differential operator* D_x*. Applying* Φ *recursively to the symmetry* $Q_0 = u_x$ *yields infinitely many symmetries* $Q_n = \Phi^n Q_0,\ n \geq 0,$ *of which* *the first two are*

$$Q_1 = \Phi Q_0 = (D_x^2 + 4u + 2u_x D_x^{-1})u_x = u_{xxx} + 6uu_x, \tag{1.2.36}$$

and

$$\begin{aligned} Q_2 = \Phi^2 Q_0 = \Phi Q_1 &= (D_x^2 + 4u + 2u_x D_x^{-1})(u_{xxx} + 6uu_x) \\ &= u_{xxxxx} + 10uu_{xxx} + 20u_x u_{xx} + 30u^2 u_x. \end{aligned} \tag{1.2.37}$$

In this work, we will often encounter recursion operators that are $q \times q$ matrices of integro-differential operators.

2 Spectral Problems and Soliton Hierarchies

2.1 Introduction

Searching for new hierarchies of soliton equations remains a very important aspect of soliton theory. It is well known that systems of soliton equations usually come in hierarchies which are constructed from spectral problems associated with matrix Lie algebras (see, e.g. [1, 5]). In their work on the inverse scattering transform, Ablowitz, Kaup, Newell and Segur introduced the so-called AKNS spectral problem [2] as a starting point for deriving nonlinear evolution equations solvable by the IST. This has given rise to several other spectral problems, some of which arise from modifications of existing ones. Notable examples include the Kaup-Newell spectral problem [12], the Wadati-Konno-Ichikawa spectral problem [40] and the Dirac spectral problem [11, 21]. In this section, we introduce a spectral problem as a modification of the well-known Dirac spectral problem. Our interest lies in real soliton equations and as a consequence we consider spectral problems associated with the semisimple matrix Lie algebra $\mathrm{so}(3,\mathbb{R})$.

2.2 Methods for Constructing Integrable Systems

In the past few decades, a large number of methods for constructing integrable systems have been proposed, some of which include Lax pairs, the zero-curvature formulation scheme, Nijenhuis operators (hereditary symmetries), the Hirota bilinear method and the Sato formalism and the Gelfand-Dickey approach. We consider a few of them below.

2.2.1 Lax Pairs

The concept of Lax pair nonlinear evolution systems is due to P. D. Lax [13]. According to Lax, completely integrable nonlinear partial differential equations

$$u_t = K[u] \tag{2.2.1}$$

have an associated system of linear partial differential equations:

$$L\Psi = \lambda\Psi, \quad \Psi_t = M\Psi, \tag{2.2.2}$$

where L and M are linear differential operators and Ψ is an eigenfunction of L corresponding to the eigenvalue λ. The pair of operators L and M are known as a Lax pair for (2.2.1). The property that (2.2.1) is completely integrable lies in the fact that the eigenvalues are independent of time (i.e. $\lambda_t = 0$). In this case, we say that the eigenvalue problem (2.2.2) is isospectral. The compatibility of the equations (2.2.2) leads to the operator equation

$$\frac{dL}{dt} = [M, L], \tag{2.2.3}$$

where $[L, M] \equiv LM - ML$ is the operator commutator. This equation (2.2.3) is known as the *Lax equation*. It can easily be derived from the fact that

$$\frac{d}{dt}(L\Psi) = \frac{dL}{dt}\Psi + L\Psi_t = \frac{dL}{dt}\Psi + LM\Psi \tag{2.2.4}$$

and

$$\frac{d}{dt}(L\Psi) = \frac{dL}{dt}(\lambda\Psi) = \lambda\Psi_t = M(\lambda\Psi) = ML\Psi. \tag{2.2.5}$$

Example 2.2.1 *The KdV equation has the Lax pair*

$$L = -D_x^2 + u, \quad M = -4D_x^3 u + 6uD_x + 3u_x. \tag{2.2.6}$$

One can easily show that the Lax condition (2.2.3) *is equivalent to the equation*

$$(L_t + [L, M])\Psi = (u_t - 6uu_x + u_{xxx})\Psi = 0 \tag{2.2.7}$$

which gives rise to the KdV equation,

$$u_t - 6uu_x + u_{xxx} = 0. \tag{2.2.8}$$

2.2.2 Zero-Curvature Representation

Let $U(\lambda)$ and $V(\lambda)$ be matrix valued functions of x and t depending on the auxiliary variable λ called the spectral parameter. Consider a system of linear partial differential equations

$$\begin{cases} \Phi_x = U(\lambda)\Phi, \\ \Phi_t = V(\lambda)\Phi, \end{cases} \tag{2.2.9}$$

where Φ is a column vector whose components depend on (x, t, λ). The consistency condition $\Phi_{xt} = \Phi_{tx}$ gives rise to the compatibility conditions

$$\frac{\partial}{\partial t}(U(\lambda)\Phi) - \frac{\partial}{\partial x}(V(\lambda)\Phi) = \left(\frac{\partial}{\partial t}U(\lambda) - \frac{\partial}{\partial x}V(\lambda) + [U(\lambda), V(\lambda)]\right)\Phi = 0, \tag{2.2.10}$$

which gives rise to the equation

$$U_t - V_x + [U, V] = 0. \tag{2.2.11}$$

This equation is referred to as the *zero curvature equation* and the scheme is known as the zero curvature representation [37].

Example 2.2.2 *If*

$$U = \frac{i}{2}\begin{bmatrix} 2\lambda & u_x \\ u_x & -2\lambda \end{bmatrix}, \qquad V = \frac{1}{4i\lambda}\begin{bmatrix} \cos(u) & -i\sin(u) \\ i\sin(u) & -\cos(u) \end{bmatrix}, \tag{2.2.12}$$

where $u = u(x,t)$*, then the zero curvature equation* (2.2.11) *is equivalent to the sine-Gordon equation*

$$u_{xt} = \sin(u). \tag{2.2.13}$$

2.2.3 The ZS-AKNS Scheme

Another method for deriving integrable systems is the AKNS scheme [3, 2, 36] which was introduced by Ablowitz, Kaup, Newell and Segur in 1973 and Zakharov and Shabat in 1972. Integrable systems such as the KdV equation, the sine-Gordon equation and the mKdV equation can be derived from this method. The method is formulated by considering the linear systems

$$\begin{cases} \phi_x = U\phi, \\ \phi_t = V\phi, \end{cases} \tag{2.2.14}$$

where ϕ is a column vector and U and V are matrices. The consistency condition $\phi_{xt} = \phi_{tx}$ (under the isospectral condition, $\lambda_t = 0$) gives rise to the zero-curvature equation (2.2.11). The matrices U and V which are said to form an AKNS pair depend on the spectral parameter λ. This is in contrast with the Lax method in the sense that the Lax pair may not depend on λ.

Example 2.2.3 *Using the ZS-AKNS method, we derive the KdV equation from the* *one-dimensional Schrödinger equation,*

$$L\psi = \lambda\psi, \tag{2.2.15}$$

where $L = -\frac{d^2}{dx^2} + u(x,t)$ *and* $\lambda = k^2$*. Letting*

$$\phi = \begin{bmatrix} \psi_x \\ \psi \end{bmatrix}, \quad U = \begin{bmatrix} 0 & u-\lambda \\ 1 & 0 \end{bmatrix}, \tag{2.2.16}$$

we formulate the linear systems

$$\begin{cases} \phi_x = U\phi, \\ \phi_t = V\phi, \end{cases} \tag{2.2.17}$$

where V *is assumed to be of the form*

$$V = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{2.2.18}$$

The entries a, b, c *and* d *may depend on* x, t *and* λ*. The zero curvature equation* (2.2.11) *yields*

$$\begin{cases} b = -a_x + (u-\lambda)c \\ d = a - c_x \\ d_x = -a_x \\ u_t - b_x + (d-a)(u-\lambda) = 0 \end{cases} \tag{2.2.19}$$

From the first three equations in (2.2.19)*, we can rewrite the last equation in* (2.2.19) *as*

$$u_t + \frac{1}{2}c_{xxx} - u_x c - 2c_x(u-\lambda) = 0 \tag{2.2.20}$$

Assuming that c depends linearly on the spectral parameter λ, we can let $c = \lambda\alpha + \beta$ and obtain

$$2\alpha_x\lambda^2 + \left(\frac{1}{2}\alpha_{xxx} - 2\alpha_x u + 2\beta_x - u_x\alpha\right)\lambda + \left(u_t + \frac{1}{2}\beta_{xxx} - 2\beta_x u - u_x\beta\right) = 0. \tag{2.2.21}$$

Equating the coefficients of the powers of λ to zero, we have

$$\begin{cases} \alpha = k_1, \\ \beta = \frac{1}{2}k_1 u + k_2, \\ u_t + \frac{1}{4}k_1 u_{xxx} - k_2 u_x - \frac{3}{2}k_1 u u_x = 0, \end{cases} \tag{2.2.22}$$

where k_1 and k_2 are arbitrary constants. Choosing $k_1 = 4$ and $k_2 = 0$, we obtain the KdV equation (1.2.32)*. From the first three equations in* (2.2.19) *and the fact that $c = 4\lambda + 2u$, we have*

$$\begin{cases} a = u_x + k_3, \\ b = -4\lambda^2 + 2\lambda u + 2u^2 - u_{xx}, \\ d = k_3 - u_x, \end{cases} \tag{2.2.23}$$

where k_3 is an arbitrary constant. Choosing $k_3 = 0$, we find an explicit form for the matrix V as

$$V = \begin{bmatrix} u_x & -4\lambda^2 + 2\lambda u + 2u^2 - u_{xx} \\ 4\lambda + 2u & -u_x \end{bmatrix}. \tag{2.2.24}$$

2.2.4 The Tu-Ma Scheme

One of the most widely used methods for generating integrable hierarchies is the so-called Tu-Ma scheme [38, 39]. We give a brief outline of this scheme below.

Introduce an isospectral problem

$$\phi_x = U\phi = U(u,\lambda)\phi \in \tilde{\mathfrak{g}}, \tag{2.2.25}$$

where $\tilde{\mathfrak{g}}$ is a simple matrix loop algebra based on a given matrix Lie algebra $\mathfrak{g}$, u, a dependent variable, and λ, the spectral parameter.

Then, search for a solution of the form

$$W = W(u,\lambda) = \sum_{i\geq 0} W_i \lambda^{-i}, W_i \in \tilde{\mathfrak{g}},\ i \geq 0, \tag{2.2.26}$$

to the stationary zero curvature equation

$$W_x = [U, W], \tag{2.2.27}$$

and introduce the Lax matrices

$$V^{[m]} = V^{[m]}(u,\lambda) = (\lambda^m W)_+ + \Delta_m \in \tilde{\mathfrak{g}},\ m \geq 0, \tag{2.2.28}$$

to formulate the temporal spectral problems

$$\phi_{t_m} = V^{[m]}\phi = V^{[m]}(u,\lambda)\phi,\ m \geq 0. \tag{2.2.29}$$

The term $(\lambda^m W)_+$ denotes the polynomial part of $\lambda^m W$ in λ, and the modification terms $\Delta_m \in \tilde{\mathfrak{g}}$ guarantee that the zero curvature equations

$$U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0,\ m \geq 0, \tag{2.2.30}$$

produce a hierarchy of soliton equations

$$u_{t_m} = K_m(u),\ m \geq 0. \tag{2.2.31}$$

In this dissertation, we will usually take $\tilde{\mathfrak{g}}$ to be

$$\widetilde{\mathrm{so}}(3,\mathbb{R}) = \left\{ \sum_{i\geq 0}^{\infty} A_i \lambda^{n-i} | A_i \in \mathrm{so}(3,\mathbb{R}),\ i \geq 0\ , n \in \mathbb{Z} \right\}. \tag{2.2.32}$$

This loop algebra $\widetilde{\mathrm{so}}(3,\mathbb{R})$ [16] is based on the three-dimensional special orthogonal Lie algebra $\mathrm{so}(3,\mathbb{R})$, which consists of 3×3 skew-symmetric real matrices. We choose the matrices

$$e_1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},\ e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix},\ e_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.2.33}$$

having the commutator relations

$$[e_1, e_2] = e_3,\ [e_2, e_3] = e_1,\ [e_3, e_1] = e_2 \tag{2.2.34}$$

as a basis for $\mathrm{so}(3,\mathbb{R})$.

Example 2.2.4 *Consider the spectral problem*

$$\phi_x = \begin{bmatrix} \lambda^2 & \lambda p \\ \lambda q & -\lambda^2 \end{bmatrix} \phi \in \widetilde{sl}(2,\mathbb{R}), \quad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix},\ u = \begin{bmatrix} p \\ q \end{bmatrix} \tag{2.2.35}$$

and suppose W in (2.2.27) is of the form

$$W = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} = \begin{bmatrix} \sum_{i\geq 0} a_i \lambda^{-2i} & \sum_{i\geq 0} b_i \lambda^{-2i-1} \\ \sum_{i\geq 0} c_i \lambda^{-2i-1} & -\sum_{i\geq 0} a_i \lambda^{-2i} \end{bmatrix} \in \widetilde{sl}(2,\mathbb{R}), \tag{2.2.36}$$

where

$$\widetilde{sl}(2,\mathbb{R}) = \left\{ \sum_{i\geq 0}^{\infty} A_i \lambda^{n-i} | A_i \in sl(2,\mathbb{R}),\ i \geq 0\ , n \in \mathbb{Z} \right\}. \tag{2.2.37}$$

Then, (2.2.27) *gives rise to the systems*

$$\begin{cases} a_0 = 1, \\ b_0 = p, \\ c_0 = q, \end{cases} \tag{2.2.38}$$

and

$$\begin{cases} a_{i+1,x} = -\frac{1}{2}(qb_{i,x} + pc_{i,x}), \\ b_{i+1} = \frac{1}{2}b_{i,x} + pa_{i+1}, & , i \geq 1. \\ c_{i+1} = -\frac{1}{2}c_{i,x} + qa_{i+1} \end{cases} \tag{2.2.39}$$

The corresponding zero curvature equations (2.2.30) *with a modification term* $\Delta_m = -a_{m+1}e_1$ *generate the following hierarchy of soliton equations:*

$$u_{t_m} = K_m = \begin{bmatrix} b_{m,x} \\ c_{m,x} \end{bmatrix}, \ m \geq 0, \tag{2.2.40}$$

This hierarchy is usually referred to as the real form of the Kaup-Newell hierarchy since its underlying Lie algebra is real. The original Kaup-Newell hierarchy is associated with the complex Lie algebra $sl(2, \mathbb{C})$*. Under the transformations* $\frac{\partial}{\partial x} \to -i\frac{\partial}{\partial x}$ *and* $\frac{\partial}{\partial t_m} \to -i\frac{\partial}{\partial t_m}$, (2.2.40) *is transformed into the original Kaup-Newell soliton hierarchy [12]. The first nonlinear system*

$$u_{t_1} = \begin{bmatrix} p \\ q \end{bmatrix}_{t_1} = K_1 = \begin{bmatrix} \frac{1}{2}[p_{xx} - (p^2q)_x] \\ -\frac{1}{2}[q_{xx} + (pq^2)_x] \end{bmatrix}, \tag{2.2.41}$$

transforms into the derivative nonlinear Schrödinger equation

$$iq_{t_1} = -\frac{1}{2}q_{xx} + \frac{1}{2}i(q^*q^2)_x \tag{2.2.42}$$

under the transformations $\frac{\partial}{\partial x} \to -i\frac{\partial}{\partial x}$, $\frac{\partial}{\partial t_1} \to -i\frac{\partial}{\partial t_1}$ *and* $p = -q^*$.

2.3 A Spectral Problem and Soliton Hierarchy

Consider the following spectral problem:

$$\phi_x = U\phi = U(u,\lambda)\phi \in \tilde{\mathfrak{g}}. \tag{2.3.1}$$

If we define

$$U = \lambda e_1 + pe_2 + qe_3, \tag{2.3.2}$$

where

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \; e_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; e_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \tag{2.3.3}$$

are a basis of the special linear Lie algebra, sl(2,$\mathbb{R}$), then the above spectral problem (2.3.1), usually called the ZS-AKNS spectral problem [36], generates the real form of the AKNS soliton hierarchy. If

$$U = \lambda^2 e_1 + \lambda pe_2 + \lambda qe_3, \tag{2.3.4}$$

then (2.3.1) generates the real form of the Kaup-Newell hierarchy. Although, this is not the original spectral problem introduced by Kaup and Newll, we still refer to it as the Kaup-Newell spectral problem. Evidently, the matrix (2.3.4) is a modification of (2.3.2). More precisely, (2.3.4) is the product of (2.3.2) and the spectral parameter λ. This gives rise to a very interesting question: *can one generate a new spectral problem from any of the existing ones by a similar modification?* As a consequence, we present the following main theorem:

Theorem 2.3.1 *[27] The spectral problem*

$$\phi_x = U\phi = U(u,\lambda)\phi \in \tilde{\mathfrak{g}}, \tag{2.3.5}$$

with spectral matrix

$$U = \lambda q e_1 + (\lambda^2 + \lambda p)e_2 + (-\lambda^2 + \lambda p)e_3 \tag{2.3.6}$$

where e_1, e_2 and e_3 are a basis of the Lie algebra $so(3,\mathbb{R})$ given by (2.2.33), and $\widetilde{\mathfrak{g}} = \widetilde{so}(3,\mathbb{R})$, produces a hierarchy soliton equations

$$u_{t_m} = K_m = \Phi^m \begin{bmatrix} p_x \\ q_x \end{bmatrix}, \ m \geq 0, \tag{2.3.7}$$

where the operator Φ given by

$$\Phi = \begin{bmatrix} -\partial p \partial^{-1} p & \frac{1}{2}\partial - \frac{1}{2}\partial p \partial^{-1} q \\ -\partial - \partial q \partial^{-1} p & -\frac{1}{2}\partial q \partial^{-1} q \end{bmatrix}, \ \partial = \frac{\partial}{\partial x}. \tag{2.3.8}$$

This spectral problem introduced in this theorem is a modification of the well-known Dirac spectral problem [11, 21]:

$$\phi_x = U\phi = U(u,\lambda)\phi \in \tilde{\mathfrak{g}} \tag{2.3.9}$$

with spectral matrix

$$U = q e_1 + (\lambda + p)e_2 + (-\lambda + p)e_3 \tag{2.3.10}$$

where e_1, e_2 and e_3 are a basis of the Lie algebra $\mathrm{sl}(2,\mathbb{R})$ given by (2.3.3).

In what follows we employ the so-called Tu-Ma Scheme [15, 38, 39] to prove the above theorem.

Proof. The spectral problem (2.3.5) associated with the loop algebra (2.2.32) reads

$$\phi_x = U(u,\lambda)\phi, \quad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}, \; u = \begin{bmatrix} p \\ q \end{bmatrix}, \tag{2.3.11}$$

where the spectral matrix U is

$$U = \lambda q e_1 + (\lambda^2 + \lambda p)e_2 + (-\lambda^2 + \lambda p)e_3 = \begin{bmatrix} 0 & \lambda^2 - \lambda p & -\lambda q \\ -\lambda^2 + \lambda p & 0 & -\lambda^2 - \lambda p \\ \lambda q & \lambda^2 + \lambda p & 0 \end{bmatrix}. \tag{2.3.12}$$

If we suppose that W, a solution of the stationary zero curvature equation (2.2.27) is of the form

$$W = ce_1 + (a+b)e_2 + (a-b)e_3 = \begin{bmatrix} 0 & -a+b & -c \\ a-b & 0 & -a-b \\ c & a+b & 0 \end{bmatrix} \in \widetilde{\mathrm{so}}(3,\mathbb{R}), \tag{2.3.13}$$

then, it follows from (2.2.27) that

$$\begin{cases} a_x = \lambda q b - \lambda^2 c, \\ b_x = -\lambda q a + \lambda p c, \\ c_x = 2\lambda^2 a - 2\lambda p b. \end{cases} \tag{2.3.14}$$

Letting

$$\begin{cases} a = \sum_{i\geq 0} a_i\lambda^{-2i-1}, \\ b = \sum_{i\geq 0} b_i\lambda^{-2i}, \\ c = \sum_{i\geq 0} c_i\lambda^{-2i-1}, \end{cases} \tag{2.3.15}$$

and taking the initial values

$$\begin{cases} a_0 = p, \\ b_0 = 1, \\ c_0 = q, \end{cases} \tag{2.3.16}$$

obtained by solving the equations

$$\begin{cases} a_0 - pb_0 = 0, \\ c_0 - qb_0 = 0, \\ b_{0,x} = -qa_0 + pc_0, \end{cases} \tag{2.3.17}$$

we obtain from (2.3.14), the relations

$$\begin{cases} a_{i,x} = qb_{i+1} - c_{i+1}, \\ b_{i+1,x} = -qa_{i+1} + pc_{i+1}, \\ c_{i,x} = 2a_{i+1} - 2pb_{i+1}, \end{cases} \quad i \geq 0. \tag{2.3.18}$$

The first and last equations give rise to

$$b_{i+1,x} = -\frac{q}{2}c_{i,x} - pa_{i,x},\ i \geq 0. \tag{2.3.19}$$

By imposing the following conditions (i.e., choosing constants of integration to be zero):

$$a_i|_{u=0} = b_i|_{u=0} = c_i|_{u=0} = 0,\ i \geq 1, \tag{2.3.20}$$

(2.3.18) uniquely determines the sequence $\{a_i, b_i, c_i | i \geq 1\}$, of which the first two sets are presented as follows:

$$\begin{cases} a_1 = \frac{1}{2}q_x - \frac{1}{2}p^3 - \frac{1}{4}pq, \\ b_1 = -\frac{1}{2}p^2 - \frac{1}{4}q^2, \\ c_1 = -p_x - \frac{1}{2}qp^2 - \frac{1}{4}q^3, \end{cases} \tag{2.3.21}$$

and

$$\begin{cases} a_2 = -\frac{1}{2}p_{xx} - \frac{3}{4}q_x p^2 - \frac{3}{8}q^2 q_x + \frac{3}{8}q^2 p^3 + \frac{3}{8}p^5 + \frac{3}{32}pq^4, \\ b_2 = -\frac{3}{8}q^2 p^2 + \frac{3}{8}p^4 - \frac{1}{2}q_x p + \frac{1}{2}qp_x + \frac{3}{32}q^4, \\ c_2 = -\frac{1}{2}q_{xx} + \frac{3}{2}p_x p^2 + \frac{3}{4}p_x q^2 + \frac{3}{8}q^3 p^2 + \frac{3}{8}qp^4 + \frac{3}{32}q^5. \end{cases} \tag{2.3.22}$$

Now, by virtue of (2.3.18) and the structure of the spectral matrix U in (2.3.12), we introduce a sequence of Lax operators

$$V^{[m]} = \lambda(\lambda^{(2m+1)}W)_+,\ m \geq 0, \tag{2.3.23}$$

and as a result, the corresponding zero curvature equations

$$U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0,\ m \geq 0, \tag{2.3.24}$$

generate the following hierarchy of soliton equations:

$$u_{t_m} = K_m = \begin{bmatrix} a_{m,x} \\ c_{m,x} \end{bmatrix} = \Phi^m \begin{bmatrix} a_{0,x} \\ c_{0,x} \end{bmatrix} = \Phi^m \begin{bmatrix} p_x \\ q_x \end{bmatrix},\ m \geq 0, \tag{2.3.25}$$

with Φ determined by (2.3.18), and given as

$$\Phi = \begin{bmatrix} -\partial p \partial^{-1} p & \frac{1}{2}\partial - \frac{1}{2}\partial p \partial^{-1} q \\ -\partial - \partial q \partial^{-1} p & -\frac{1}{2}\partial q \partial^{-1} q \end{bmatrix}, \ \partial = \frac{\partial}{\partial x}, \tag{2.3.26}$$

where ∂^{-1} is the inverse of the differential operator ∂.

■

Proposition 2.3.2 *The functions $a_i, b_i, c_i,\ i \geq 1$ are local.*

Proof. The identities

$$\text{tr}(W^2) = -4a^2 - 4b^2 - 2c^2 \tag{2.3.27}$$

and

$$\frac{d}{dx}\text{tr}(W^2) = 2\text{tr}(WW_x) = 2\text{tr}(W[U, W]) = 0, \tag{2.3.28}$$

as well as the initial data (2.3.16) give rise to the equation

$$2a^2 + 2b^2 + c^2 = (2a^2 + 2b^2 + c^2)|_{u=0} = 2. \tag{2.3.29}$$

Rewriting this equation (2.3.29) in terms of the Laurent series of the functions a, b and c in (2.3.15) and balancing coefficients of λ^i for each $i \geq 1$ gives

$$b_i = -\frac{1}{2}\sum_{k+l=i,k,l\geq 1} b_k b_l - \frac{1}{2}\sum_{k+l=i,k,l\geq 0}\left(a_k a_l + \frac{1}{2}c_k c_l\right),\ i \geq 1. \tag{2.3.30}$$

Based on this recursion relation and the first and last relations of (2.3.18), an application of mathematical induction shows that all the functions $a_i, b_i, c_i,\ i \geq 1$ are differential polynomials in u, and thus are local functions. This completes the proof.

■

The first system in (2.3.25) reads

$$u_{t_1} = \begin{bmatrix} p \\ q \end{bmatrix}_{t_1} = K_1 = \begin{bmatrix} q_{xx} - 3p^2p_x - \frac{1}{2}p_xq^2 - pqq_x \\ -p_{xx} - \frac{1}{2}q_xp^2 - qpp_x - \frac{3}{4}q^2q_x \end{bmatrix}. \tag{2.3.31}$$

It is worth mentioning that these equations (2.3.25) have very recently been found to be connected to the original Kaup-Newell system [12] by some gauge transformation, although, interestingly enough, the newly introduced spectral matrix (2.3.12) cannot be directly transformed to the Kaup-Newell spectral matrix [12]. In spite of such a connection, there may still be some remarkable differences between these two spectral problems. We elaborate this point in the next chapter.

2.4 An Extended Spectral Problem and Soliton Hierarchy

In this section, we present a generalization of the soliton hierarchy (2.3.25).

Theorem 2.4.1 *[26] The spectral problem*

$$\phi_x = U\phi = U(u,\lambda)\phi \in \tilde{\mathfrak{g}}, \tag{2.4.1}$$

with spectral matrix

$$U = \lambda q e_1 + [\lambda^2 + \lambda p + \alpha(p^2 + \frac{q^2}{2})]e_2 + [-\lambda^2 + \lambda p - \alpha(p^2 + \frac{q^2}{2})]e_3, \tag{2.4.2}$$

where $\alpha \in \mathbb{R}$, e_1, e_2 and e_3 are a basis of the Lie algebra, $so(3,\mathbb{R})$ and $\widetilde{\mathfrak{g}} = \widetilde{so}(3,\mathbb{R})$ produces a hierarchy of soliton equations

$$u_{t_m} = K_m = \begin{bmatrix} a_{m,x} + \alpha(p^2 + \frac{q^2}{2})c_m + 2\alpha q b_{m+1} \\ c_{m,x} - 2\alpha(p^2 + \frac{q^2}{2})a_m - 4\alpha p b_{m+1} \end{bmatrix}, \ m \geq 0, \tag{2.4.3}$$

where the vector components of K_m are determined by the recursion relations

$$\begin{cases} b_{m+1,x} = -\frac{q}{2}c_{m,x} - pc_{m,x} + \alpha(p^2 + \frac{q^2}{2})(qa_m - pc_m), \\ c_{m+1} = -a_{m,x} + qb_{m+1} - \alpha(p^2 + \frac{q^2}{2})c_m, \\ a_{m+1} = \frac{1}{2}c_{m,x} + pb_{m+1} - \alpha(p^2 + \frac{q^2}{2})a_m, \end{cases} \tag{2.4.4}$$

with initial values

$$a_0 = p, \ b_0 = 1, \ c_0 = q.$$

Proof. The spectral problem associated with the loop algebra (2.2.32) reads

$$\phi_x = U\phi = U(u,\lambda)\phi, \ u = \begin{bmatrix} p \\ q \end{bmatrix}, \ \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}, \tag{2.4.5}$$

with the spectral matrix

$$U = \begin{bmatrix} 0 & \lambda^2 - \lambda p + \alpha(p^2 + \frac{q^2}{2}) & -\lambda q \\ -\lambda^2 + \lambda p - \alpha(p^2 + \frac{q^2}{2}) & 0 & -\lambda^2 - \lambda p - \alpha(p^2 + \frac{q^2}{2}) \\ \lambda q & \lambda^2 + \lambda p + \alpha(p^2 + \frac{q^2}{2}) & 0 \end{bmatrix}. \tag{2.4.6}$$

If we take W in (2.2.27) to be of the form

$$W = ce_1 + (a+b)e_2 + (a-b)e_3 = \begin{bmatrix} 0 & -a+b & -c \\ a-b & 0 & -a-b \\ c & a+b & 0 \end{bmatrix} \in \widetilde{\text{so}}(3,\mathbb{R}), \tag{2.4.7}$$

then (2.2.27) gives rise to

$$\begin{cases} a_x = \lambda q b - \lambda^2 c - \alpha(p^2 + \frac{q^2}{2})c, \\ b_x = -\lambda q a + \lambda p c, \\ c_x = 2\lambda^2 a - 2\lambda p b + 2\alpha(p^2 + \frac{q^2}{2})a. \end{cases} \tag{2.4.8}$$

Letting

$$\begin{cases} a = \sum_{i\geq 0} a_i \lambda^{-2i-1}, \\ b = \sum_{i\geq 0} b_i \lambda^{-2i}, \\ c = \sum_{i\geq 0} c_i \lambda^{-2i-1}, \end{cases} \tag{2.4.9}$$

and taking the initial values

$$\begin{cases} a_0 = p, \\ b_0 = 1, \\ c_0 = q, \end{cases} \tag{2.4.10}$$

which are obtained by solving the equations,

$$\begin{cases} a_0 - p b_0 = 0, \\ c_0 - q b_0 = 0, \\ b_{0,x} = -q a_0 + p c_0, \end{cases}$$

the system (2.4.8) reduces to

$$\begin{cases} b_{i+1,x} = -q a_{i+1} + p c_{i+1}, \\ a_{i,x} = q b_{i+1} - c_{i+1} - \alpha(p^2 + \frac{q^2}{2})c_i, \\ c_{i,x} = 2a_{i+1} - 2p b_{i+1} + 2\alpha(p^2 + \frac{q^2}{2})a_i. \end{cases} \tag{2.4.11}$$

The last two equations lead to

$$b_{i+1,x} = -\frac{q}{2}c_{i,x} - pc_{i,x} + \alpha(p^2 + \frac{q^2}{2})(qa_i - pc_i),\ i \geq 0. \tag{2.4.12}$$

By choosing constants of integration to be zero, (2.4.11) consequently determines uniquely the sequence $\{a_i, b_i, c_i | i \geq 1\}$, of which the first two sets are presented as follows:

$$\begin{cases} a_1 = \frac{1}{2}q_x - \frac{1}{2}p^3 - \frac{1}{4}pq - \alpha p^3 - \frac{\alpha}{2}pq^2, \\ b_1 = -\frac{1}{2}p^2 - \frac{1}{4}q^2, \\ c_1 = -p_x - \frac{1}{2}qp^2 - \frac{1}{4}q^3 - \alpha qp^2 - \frac{\alpha}{2}q^3, \end{cases} \tag{2.4.13}$$

$$\begin{cases} a_2 = -\frac{1}{2}p_{xx} - \frac{3}{4}q_xp^2 - \frac{3}{8}q^2q_x + \frac{3}{8}q^2p^3 + \frac{3}{8}p^5 + \frac{3}{32}pq^4 - \alpha q_xp^2 - \alpha qpp_x \\ \qquad - \alpha q^2q_x + \frac{3}{2}\alpha p^5 + \frac{3}{2}\alpha p^3q^2 + \frac{3}{8}\alpha pq^4 + \alpha^2p^5 + \alpha^2p^3q^2 + \frac{1}{4}\alpha^2pq^4, \\ b_2 = -\frac{3}{8}q^2p^2 + \frac{3}{8}p^4 - \frac{1}{2}q_xp + \frac{1}{2}qp_x + \frac{3}{32}q^4 + \alpha p^4 + \alpha p^2q^2 - \frac{\alpha}{4}q^4, \\ c_2 = -\frac{1}{2}q_{xx} + \frac{3}{2}p_xp^2 + \frac{3}{4}p_xq^2 + \frac{3}{8}q^3p^2 + \frac{3}{8}qp^4 + \frac{3}{32}q^5 + 4\alpha p^2p_x + \alpha p_xq^2 \\ \qquad + \alpha pqq_x + \frac{3}{2}\alpha qp^4 + \frac{3}{2}\alpha p^2q^3 + \frac{3}{8}\alpha q^5 + \alpha^2qp^4 + \alpha^2p^2q^3 + \frac{\alpha^2}{4}q^5. \end{cases} \tag{2.4.14}$$

Now, the system (2.4.11) leads to

$$\begin{bmatrix} a_{i+1} \\ b_{i+1} \end{bmatrix} = Q \begin{bmatrix} a_i \\ b_i \end{bmatrix},\ Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}, \tag{2.4.15}$$

where

$$\begin{cases} Q_{11} = -p\partial^{-1}p\partial + \alpha p\partial^{-1}(p^2q + \frac{q^3}{2}) - \alpha(p^2 + \frac{q^2}{2}), \\ Q_{12} = -\frac{1}{2}\partial - \frac{p}{2}\partial^{-1}q\partial - \alpha p\partial^{-1}(p^3 + p\frac{q^2}{2}), \\ Q_{21} = -\partial - q\partial^{-1}p\partial + \alpha q\partial^{-1}(p^2q + \frac{q^3}{2}), \\ Q_{22} = -\frac{q}{2}\partial^{-1}q\partial - \alpha q\partial^{-1}(p^3q + p\frac{q^2}{2}) - \alpha(p^2 + \frac{q^2}{2}). \end{cases} \tag{2.4.16}$$

We now introduce a sequence of Lax operators with modification terms:

$$V^{[m]} = \lambda(\lambda^{(2m+1)}W)_+ + \Delta_m, \; m \geq 0, \tag{2.4.17}$$

where Δ_m is chosen as

$$\Delta_m = \delta_m(e_2 - e_3), \; m \geq 0. \tag{2.4.18}$$

Consequently, we obtain

$$V_x^{[m]} - [U, V^{[m]}] = \lambda(\lambda^{2m+1}W_x)_+ + \delta_{m,x}(e_2 - e_3) - \lambda[U, (\lambda^{2m+1}W)_+] - [U, \delta_m(e_2 - e_3)]. \tag{2.4.19}$$

Now, we see that

$$\begin{aligned} &(\lambda^{2m+1}W_x)_+ - [U, (\lambda^{2m+1}W)_+] \\ &= \begin{bmatrix} 0 & -a_{m,x} - \alpha(p^2 + \frac{q^2}{2})c_m & -c_{m,x} + 2\alpha(p^2 + \frac{q^2}{2})a_m \\ a_{m,x} + \alpha(p^2 + \frac{q^2}{2})c_m & 0 & -a_{m,x} - \alpha(p^2 + \frac{q^2}{2})c_m \\ c_{m,x} - 2\alpha(p^2 + \frac{q^2}{2})a_m & a_{m,x} + \alpha(p^2 + \frac{q^2}{2})c_m & 0 \end{bmatrix} \end{aligned} \tag{2.4.20}$$

and also

$$[U, \delta_m(e_2 - e_3)] = \lambda\delta_m(-2pe_1 + qe_2 + qe_3). \tag{2.4.21}$$

Thus, (2.4.19) becomes

$$
\begin{aligned}
&V_x^{[m]} - [U, V^{[m]}] \\
&= \lambda \begin{bmatrix} 0 & -a_{m,x} - \alpha(p^2 + \frac{q^2}{2})c_m & -c_{m,x} + 2\alpha(p^2 + \frac{q^2}{2})a_m \\ a_{m,x} + \alpha(p^2 + \frac{q^2}{2})c_m & 0 & -a_{m,x} - \alpha(p^2 + \frac{q^2}{2})c_m \\ c_{m,x} - 2\alpha(p^2 + \frac{q^2}{2})a_m & a_{m,x} + \alpha(p^2 + \frac{q^2}{2})c_m & 0 \end{bmatrix} \\
&\quad + \lambda \begin{bmatrix} 0 & q\delta_m & -2p\delta_m \\ -q\delta_m & 0 & q\delta_m \\ 2p\delta_m & -q\delta_m & 0 \end{bmatrix} + \begin{bmatrix} 0 & \delta_{m,x} & 0 \\ \delta_{m,x} & 0 & -\delta_{m,x} \\ 0 & \delta_{m,x} & 0 \end{bmatrix},
\end{aligned}
\tag{2.4.22}
$$

and as a result the corresponding zero curvature equations

$$
U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0, \ m \geq 0, \tag{2.4.23}
$$

generate the equations:

$$
\begin{cases} p_{t_m} = a_{m,x} + \alpha(p^2 + \frac{q^2}{2})c_m - q\delta_m \\ q_{t_m} = c_{m,x} - 2\alpha(p^2 + \frac{q^2}{2})a_m + 2p\delta_m \\ \delta_{m,x} = \alpha(p^2 + \frac{q^2}{2})_{t_m}. \end{cases} \tag{2.4.24}
$$

To find δ_m, we observe that

$$
\begin{aligned}
\delta_{m,x} &= \alpha(2pp_{t_m} + qq_{t_m}) \\
&= \alpha[2p(a_{m,x} + \alpha(p^2 + \frac{q^2}{2})c_m - q\delta_m) + q(c_{m,x} - 2\alpha(p^2 + \frac{q^2}{2})a_m + 2p\delta_m)] \\
&= \alpha[2p(a_{m,x} + \alpha(p^2 + \frac{q^2}{2})c_m) + q(c_{m,x} - 2\alpha(p^2 + \frac{q^2}{2})a_m)] \\
&= -2\alpha b_{m+1,x}.
\end{aligned}
$$

Hence, upon taking $\delta_m = -2\alpha b_{m+1,x}$, (2.4.24) produces

$$\begin{cases} p_{t_m} = a_{m,x} + \alpha(p^2 + \frac{q^2}{2})c_m + 2\alpha q b_{m+1}, \\ q_{t_m} = c_{m,x} - 2\alpha(p^2 + \frac{q^2}{2})a_m - 4\alpha p b_{m+1}, \end{cases} \tag{2.4.25}$$

and thus, we obtain the following hierarchy of soliton equations:

$$u_{t_m} = K_m = \begin{bmatrix} a_{m,x} + \alpha(p^2 + \frac{q^2}{2})c_m + 2\alpha q b_{m+1} \\ c_{m,x} - 2\alpha(p^2 + \frac{q^2}{2})a_m - 4\alpha p b_{m+1} \end{bmatrix}, \ m \geq 0. \tag{2.4.26}$$

■

It should be noted that when $\alpha = 0$, we obtain the soliton hierarchy (2.3.25).

Proposition 2.4.2 *The functions $a_i, b_i, c_i, \ i \geq 1$ are local.*

The proof is similar to that of proposition 2.3.2. We leave out the details.

3 Hamiltonian Formulations of Soliton Hierarchies

3.1 Introduction

In the theory of integrable systems, the Hamiltonian formalism, originally introduced by Gardner [9] and Faddeev and Zakharov [42] in the early 1970's, is closely related to the concept of integrability. For a system of first-order ordinary differential equations, integrability in the sense of Liouville is based on the existence of a Hamiltonian structure and sufficiently many functionally independent conserved quantities (first integrals) which are in involution. This is the content of the Liouville-Arnold thorem. In the case of partial differential equations (evolution equations) in Hamiltonian form, the phase space is infinite-dimensional and so by extension we would require the existence of an infinite number of conserved quantities to guarantee integrability in the sense of Liouville [15, 17, 37, 39] . For systems that can be written in two different Hamiltonian forms (bi-Hamiltonian systems), integrability is guaranteed by a result due to Magri [25]. Our goal in this section is to discuss integrability within the framework of bi-Hamiltonian structures.

Throughout this section, we assume that each function in the potential vector u is in the Schwartz space (the space of rapidly decreasing functions on $\mathbb{R}^n$).

Theorem 3.1.1 *[29, 30] Let $P[u] \in \mathcal{A}^p$ be defined over a vertically star-shaped region $M \subset X \times U$. Then P is the Euler-Lagrange expression for some variational problem $\mathcal{L} = \int L dx$, i.e. $P = E(L)$, if and only if the Fréchet derivative d_P is self adjoint: $d_P^* = d_P$. In this case, a Lagrangian for P can be explicitly constructed using the* *homotopy formula*

$$L[u] = \int_0^1 u \cdot P[\lambda u] d\lambda. \tag{3.1.1}$$

When $P[u]$ *contains irrational terms, the above integral may diverge due to a possible singularity at* $\lambda = 0$*. In this case we evaluate the integral*

$$L[u] = \int_{\lambda_0}^1 u \cdot P[\lambda u] d\lambda \tag{3.1.2}$$

and take the limit as $\lambda_0 \to \infty$*.*

Example 3.1.2 *Consider the functional*

$$\mathcal{L} = \int (u_{xx}^2 - uu_x) dx. \tag{3.1.3}$$

Its Euler-Lagrange expression is

$$E(L)[u] = P[u] = -u_x - D_x(-u) + D_x^2(2u_{xx}) = 2u_{xxxx}. \tag{3.1.4}$$

and it's Fréchet derivative is

$$d_P(Q) = P'[Q] = 2\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}[(u_{xxxx} + \varepsilon D_x^4 Q)] = 2D_x^4 Q. \tag{3.1.5}$$

Thus,

$$d_P = 2D_x^4, \tag{3.1.6}$$

which is obviously self-adjoint.

On the other hand, given $P[u] = 2u_{xxxx}$*, we can obtain the functional* (3.1.3) *using the homotopy formula* (3.1.1) *as follows*

$$\begin{aligned}\mathcal{L}[u] &= \int \Big\{ \int_0^1 u \cdot P[\lambda u] d\lambda \Big\} dx \\ &= 2 \int \Big\{ \int_0^1 u \cdot (\lambda u_{xxxx}) d\lambda \Big\} dx \\ &= \int u u_{xxxx} dx. \end{aligned} \tag{3.1.7}$$

The Lagrangian $L = uu_{xxxx}$ *is not the same as the original one in* (3.1.3), *but the two are still equivalent since*

$$uu_{xxxx} = u_{xx}^2 - uu_x + D_x(uu_{xxx} - u_x u_{xx} + \frac{1}{2}u^2). \tag{3.1.8}$$

Definition 3.1.3 A conservation law of a system of evolution equations takes the form

$$D_t H + \text{Div} P = 0, \tag{3.1.9}$$

which vanishes for all solutions $u = f(x,t)$ of the given system. Here Div denotes the spatial divergence and the conserved density $H(x,t,u^{(n)})$ is assumed without loss of generality to depend only on $x-$derivatives of u. Equivalently, for $\Omega \subset X$, the functional

$$\mathcal{H} = \int_\Omega H(x,t,u^{(n)}) \tag{3.1.10}$$

is a constant, independent of t, for all solutions u such that $H(x,t,u^{(n)}) \to 0$ as $x \to \partial\Omega$.

Example 3.1.4 *As an example, let's consider the KdV equation* (1.2.32). *This equation is known to have infinitely many conservation laws. We show how to derive the first two. In the one-dimensional case, equation* (3.1.9) *becomes*

$$D_t H + D_x P = 0. \tag{3.1.11}$$

Thus, we have

$$\int D_t H dx = -\int D_x P dx, \tag{3.1.12}$$

which reduces to

$$D_t \int H dx = 0, \tag{3.1.13}$$

due to the boundary conditions. Thus the quantity $\mathcal{H}$ given by

$$\mathcal{H} = \int H dx \tag{3.1.14}$$

is conserved. Now the KdV equation can be rewritten in the form

$$D_t u + D_x \left(3u^2 + u_{xx}\right) = 0, \tag{3.1.15}$$

and so we have that the quantity

$$\mathcal{H}_1 = \int u dx \tag{3.1.16}$$

is a conserved functional for the KdV equation.

A second conservation law is derived as

$$D_t \left(u^2\right) + D_x \left(4u^3 + 2uu_{xx} - u_x^2\right) = 0 \tag{3.1.17}$$

and so, we have that the quantity

$$\mathcal{H}_2 = \int u^2 dx \tag{3.1.18}$$

is also a conserved functional for the KdV equation.

Definition 3.1.5 Let $\mathcal{D} : \mathcal{A}^q \to \mathcal{A}^q$ be a linear operator. We define a bracket on $\mathcal{F}$

as follows

$$\{\mathcal{P},\mathcal{Q}\}_J = \int \left(\frac{\delta\mathcal{P}}{\delta u}\right)^T \mathcal{D}\frac{\delta\mathcal{Q}}{\delta u}dx \tag{3.1.19}$$

where $\mathcal{P},\mathcal{Q}\in\mathcal{F}$.

Definition 3.1.6 A linear operator $\mathcal{D}:\mathcal{A}^q\to\mathcal{A}^q$ is called Hamiltonian if its bracket satisfies the condition of *skew-symmetry*

$$\{\mathcal{P},\mathcal{Q}\} = -\{\mathcal{Q},\mathcal{P}\} \tag{3.1.20}$$

and the *Jacobi identity*

$$\{\{\mathcal{P},\mathcal{Q}\},\mathcal{R}\}+\{\{\mathcal{R},\mathcal{P}\},\mathcal{Q}\}+\{\{\mathcal{Q},\mathcal{R}\},\mathcal{P}\} = 0 \tag{3.1.21}$$

for all functionals $\mathcal{P},\mathcal{Q},\mathcal{R}\in\mathcal{F}$. In this case the bracket (3.1.19) is called a *Poisson bracket*.

Definition 3.1.7 A Hamiltonian system of evolution equation is a system of the form

$$\frac{\partial u}{\partial t} = \mathcal{D}\frac{\delta\mathcal{H}}{\delta u}, \tag{3.1.22}$$

where $\mathcal{H}\in\mathcal{F}$ and $\mathcal{D}$ is a Hamiltonian operator. The functional $\mathcal{H}$ is called a Hamiltonian functional.

Definition 3.1.8 A nonlinear evolution equation is called *Liouville integrable* if it can be written as a Hamiltonian system with a well defined Poisson bracket $\{\cdot,\cdot\}$, such that it possesses an infinite number of conserved functionals, $\{\mathcal{H}_n\}$ which are in involution in pairs $\{\mathcal{H}_m,\mathcal{H}_m\}=0$.

Proposition 3.1.9 *[29] Let $\mathcal{D}$ be a $q\times q$ matrix differential operator with bracket*

(3.1.19) *on the space of functionals. Then the bracket is skew-symmetric, i.e.,* (3.1.20) *holds, if and only if* $\mathcal{D}$ *is skew-adjoint:* $\mathcal{D}^* = -\mathcal{D}$.

Corollary 3.1.10 *[29] If* $\mathcal{D}$ *is a skew-adjoint* $q \times q$ *matrix differential operator whose coefficients do not depend on* u*, then* $\mathcal{D}$ *is automatically a Hamiltonian operator.*

Definition 3.1.11 A pair of skew-adjoint $q \times q$ matrix differential operators $\mathcal{D}$ and $\mathcal{E}$ is said to form a Hamiltonian pair if every linear combination $a\mathcal{D} + b\mathcal{E},\ a, b \in \mathbb{R}$, is a Hamiltonian operator.

Definition 3.1.12 A system of evolution equations is a bi-Hamiltonian system if it can be written in the form

$$u_t = K_1[u] = \mathcal{D}\frac{\delta \mathcal{H}_1}{\delta u} = \mathcal{E}\frac{\delta \mathcal{H}_0}{\delta u}, \tag{3.1.23}$$

where $\mathcal{D}$ and $\mathcal{E}$ form a Hamiltonian pair.

Lemma 3.1.13 *[29] If* $\mathcal{D}, \mathcal{E}$ *are skew-adjoint operators, then they form a Hamiltonian pair if and only if* $\mathcal{D}, \mathcal{E}$ *and* $\mathcal{D} + \mathcal{E}$ *are all Hamiltonian operators.*

Theorem 3.1.14 *[29] Let* $u_t = K_1[u] = \mathcal{D}\frac{\delta \mathcal{H}_1}{\delta u} = \mathcal{E}\frac{\delta \mathcal{H}_0}{\delta u}$ *be a bi-Hamiltonian system of evolution equations. Then the operator* $\mathcal{R} = \mathcal{E} \cdot \mathcal{D}^{-1}$ *is a recursion operator for the system.*

Theorem 3.1.15 *[29] Let*

$$u_t = K_1[u] = \mathcal{D}\frac{\delta \mathcal{H}_1}{\delta u} = \mathcal{E}\frac{\delta \mathcal{H}_0}{\delta u} \tag{3.1.24}$$

be a bi-Hamiltonian system of evolution equations. Assume that the operator $\mathcal{D}$ *of the Hamiltonian pair is nondegenerate. Let* $\mathcal{R} = \mathcal{E} \cdot \mathcal{D}^{-1}$ *be the corresponding recursion operator, and let* $K_0 = \mathcal{D}\frac{\delta \mathcal{H}_0}{\delta u}$*. Assume that for each* $n = 1, 2, \ldots,$ *we can recursively define*

$$K_n = \mathcal{R}K_{n-1}, \quad n \geq 1, \tag{3.1.25}$$

meaning that for each n, K_{n-1} *lies in the image of* $\mathcal{D}$. *Then there exists a sequence of functionals* $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2, \ldots,$ *such that*

1. *for each* $n \geq 1$, *the evolution equation*

$$u_t = K_1[u] = \mathcal{D}\frac{\delta \mathcal{H}_n}{\delta u} = \mathcal{E}\frac{\delta \mathcal{H}_{n-1}}{\delta u} \qquad (3.1.26)$$

   is a bi-Hamiltonian system;

2. *the corresponding evolutionary vector fields* $\mathrm{v}_n = \mathrm{v}_{K_n}$ *all mutually commute:*

$$[\mathrm{v}_n, \mathrm{v}_m] = 0, \quad n, m \geq 0; \qquad (3.1.27)$$

3. *the Hamiltonian functionals* $\mathcal{H}_n$ *are a all in involution with respect to either Poisson bracket:*

$$\{\mathcal{H}_n, \mathcal{H}_m\}_{\mathcal{D}} = \{\mathcal{H}_n, \mathcal{H}_m\}_{\mathcal{E}} = 0, \quad n, m \geq 0, \qquad (3.1.28)$$

and hence provide an infinite collection of conservation laws for each of the bi-Hamiltonian systems (3.1.26).

Definition 3.1.16 Let $\mathcal{D}$ be a skew-adjoint differential operator. We define a functional bi-vector, Θ associated with $\mathcal{D}$ as

$$\Theta_{\mathcal{D}} = \frac{1}{2}\int \{\theta \wedge \mathcal{D}\theta\} dx, \qquad (3.1.29)$$

where θ is a a "uni-vector" corresponding to the one-form du.

Definition 3.1.17 Let v_Q be an evolutionary vector field with characteristics Q. The *prolongation* of v_Q is the vector field, $\mathrm{pr}\ \mathrm{v}_Q$, defined by

$$\mathrm{pr}\ \mathrm{v}_Q = \sum_J D_J(Q)\frac{\partial}{\partial u_J}. \qquad (3.1.30)$$

If $\mathcal{D}$ is a differential operator, then the vector field $\mathrm{v}_{\mathcal{D}\theta}$ is a formal evolutionary vector field with characteristic $\mathcal{D}\theta$, and thus

$$\text{pr } \mathrm{v}_{\mathcal{D}\theta} = \sum_J D_J(\mathcal{D}\theta)\frac{\partial}{\partial u_J}. \tag{3.1.31}$$

So, if $P \in \mathcal{A}$ is a differential function, then

$$\text{pr } \mathrm{v}_{\mathcal{D}\theta}(P) = \sum_J \frac{\partial P}{\partial u_J} D_J(\mathcal{D}\theta). \tag{3.1.32}$$

The vector field pr $\mathrm{v}_{\mathcal{D}\theta}$, can also act on differential operators (see, e.g., [29]) in such a way that it acts only on coefficients that are functionally dependent on u.

Example 3.1.18 *If $\mathcal{D} = \mathcal{D}_x^3 + u\mathcal{D}_x$ and $P = u + u_x$, then*

$$\mathcal{D}\theta = \theta_{xxx} + u\theta_x \tag{3.1.33}$$

and so

$$\begin{aligned} \text{pr } \mathrm{v}_{\mathcal{D}\theta}(P) &= \text{pr } \mathrm{v}_{\mathcal{D}\theta}(u) + \text{pr } \mathrm{v}_{\mathcal{D}\theta}(u_x) \\ &= (\mathcal{D}\theta) + (\mathcal{D}\theta)_x \\ &= \theta_{xxxx} + \theta_{xxx} + u\theta_x + u_x\theta_x + u\theta_{xx}. \end{aligned} \tag{3.1.34}$$

Also,

$$\begin{aligned} \text{pr } \mathrm{v}_{\mathcal{D}\theta}(\mathcal{D}) &= \text{pr } \mathrm{v}_{\mathcal{D}\theta}(\mathcal{D}_x^3 + u\mathcal{D}_x) \\ &= \text{pr } \mathrm{v}_{\mathcal{D}\theta}(u)D_x \\ &= (\mathcal{D}\theta)D_x \\ &= (\theta_{xxx} + u\theta_x)D_x. \end{aligned} \tag{3.1.35}$$

Proposition 3.1.19 *[29] Let $\mathcal{D}$ be a skew-adjoint differential operator. Then $\mathcal{D}$ is*

Hamiltonian if and only if the functional tri-vector

$$\Psi = \int \{\theta \wedge \text{pr } \mathrm{v}_{\mathcal{D}\theta}(\mathcal{D}) \wedge \theta\} dx \tag{3.1.36}$$

vanishes.

Corollary 3.1.20 *[29] Let $\mathcal{D}$ be a skew-adjoint differential operator and $\Theta_{\mathcal{D}} = \frac{1}{2}\int\{\theta \wedge \mathcal{D}\theta\}dx$ be the corresponding functional bi-vector. Then*

$$\text{pr } \mathrm{v}_{\mathcal{D}\theta}(\Theta_{\mathcal{D}}) = \frac{1}{2}\int \{\text{pr } \mathrm{v}_{\mathcal{D}\theta}(\theta \wedge \mathcal{D}\theta)\} dx, \tag{3.1.37}$$

where

$$-\text{pr } \mathrm{v}_{\mathcal{D}\theta}(\theta \wedge \mathcal{D}\theta) = \theta \wedge \text{pr } \mathrm{v}_{\mathcal{D}\theta}(\mathcal{D}) \wedge \theta. \tag{3.1.38}$$

Example 3.1.21 *From example 3.1.18, we have*

$$\begin{aligned} \theta \wedge \text{pr } \mathrm{v}_{\mathcal{D}\theta}(\mathcal{D}) \wedge \theta &= \theta \wedge (\theta_{xxx} D_x + u\theta_x D_x) \wedge \theta \\ &= \theta \wedge \theta_{xxx} \wedge \theta_x + u\theta \wedge \theta_x \wedge \theta_x \\ &= -(\theta \wedge \theta_x \wedge \theta_{xxx}). \end{aligned} \tag{3.1.39}$$

Here we used the fact that $\theta \wedge \theta_x \wedge \theta_x = 0$, by the properties of the wedge product. Thus,

$$\text{pr } \mathrm{v}_{\mathcal{D}\theta}(\Theta_{\mathcal{D}}) = \frac{1}{2}\int \{\theta \wedge \theta_x \wedge \theta_{xxx}\} dx. \tag{3.1.40}$$

By the following property of the wedge products

$$\begin{aligned} (\theta \wedge \theta_x) \wedge \theta_{xxx} &= (\theta \wedge \theta_x) \wedge D_x\theta_{xx} \\ &= D_x(\theta \wedge \theta_x \wedge \theta_{xx}) - D_x(\theta \wedge \theta_x) \wedge \theta_{xx}, \end{aligned} \tag{3.1.41}$$

we obtain

$$\begin{aligned} \text{pr } \mathrm{v}_{\mathcal{D}\theta}(\Theta_{\mathcal{D}}) &= \frac{1}{2}\int \{\theta_{xxx} \wedge \theta \wedge \theta_x\} dx \\ &= -\frac{1}{2}\int D_x(\theta \wedge \theta_x) \wedge \theta_{xx} dx \\ &= -\frac{1}{2}\int (\theta_x \wedge \theta_x + \theta \wedge \theta_{xx}) \wedge \theta_{xx} dx \\ &= 0 \end{aligned} \tag{3.1.42}$$

again, by the properties of the wedge products.

Theorem 3.1.22 *[29] Let $\mathcal{D}$ be a skew-adjoint differential operator and $\Theta_{\mathcal{D}} = \frac{1}{2}\int\{\theta\wedge \mathcal{D}\theta\}dx$ be the corresponding functional bi-vector. Then $\mathcal{D}$ is Hamiltonian if and only if*

$$\text{pr } \mathrm{v}_{\mathcal{D}\theta}(\Theta_{\mathcal{D}}) = 0. \tag{3.1.43}$$

Example 3.1.23 *Let $\mathcal{D}$ be given as in example 3.1.18. By proposition 3.1.19, we only need to show that* pr $\mathrm{v}_{\mathcal{D}\theta}(\Theta_{\mathcal{D}}) = 0$:

$$\begin{aligned} \text{pr } \mathrm{v}_{\mathcal{D}\theta}(\Theta_{\mathcal{D}}) &= \frac{1}{2}\int \{\text{pr } \mathrm{v}_{\mathcal{D}\theta}(\theta \wedge \mathcal{D}\theta)\} dx \\ &= \frac{1}{2}\int \{\text{pr } \mathrm{v}_{\mathcal{D}\theta}(\theta \wedge (\theta_{xxx} + u\theta_x))\} dx \\ &= \frac{1}{2}\int \{\text{pr } \mathrm{v}_{\mathcal{D}\theta}(\theta \wedge \theta_{xxx} + u\theta \wedge \theta_x)\} dx \\ &= \frac{1}{2}\int ((\mathcal{D}\theta) \wedge \theta \wedge \theta_x) dx \\ &= \frac{1}{2}\int (\theta_{xxx} \wedge \theta \wedge \theta_x + u\theta_x \wedge \theta \wedge \theta_x) dx \\ &= 0 \end{aligned} \tag{3.1.44}$$

by previous calculations. Thus we have shown that $\mathcal{D} = \mathcal{D}_x^3 + u\mathcal{D}_x$ is a Hamiltonian operator.

Corollary 3.1.24 *[29] Let $\mathcal{D}$ and $\mathcal{E}$ be Hamiltonian differential operators. Then $\mathcal{D}, \mathcal{E}$ form a Hamiltonian pair if and only if*

$$\text{pr } \mathrm{v}_{\mathcal{D}\theta}(\Theta_{\mathcal{E}}) + \text{pr } \mathrm{v}_{\mathcal{E}\theta}(\Theta_{\mathcal{D}}) = 0, \tag{3.1.45}$$

where

$$\Theta_{\mathcal{D}} = \frac{1}{2}\int \{\theta \wedge \mathcal{D}\theta\}dx, \quad \Theta_{\mathcal{E}} = \frac{1}{2}\int \{\theta \wedge \mathcal{E}\theta\}dx. \tag{3.1.46}$$

Corollary 3.1.25 *[29] If $P \in \mathcal{A}^r$ and $Q \in \mathcal{A}^q$, then*

$$\text{pr } \mathrm{v}_Q(P) = d_P(Q), \tag{3.1.47}$$

where $d_P(Q)$ is the Fréchet derivative.

Lemma 3.1.26 *[29] Let $\mathcal{P}, \mathcal{L}, \mathcal{R}$ be functionals with variational derivatives $\dfrac{\delta \mathcal{P}}{\delta u} = P, \dfrac{\delta \mathcal{Q}}{\delta u} = Q, \dfrac{\delta \mathcal{R}}{\delta u} = R \in \mathcal{A}^q$. Then the Jacobi identity* (3.1.21) *is equivalent to the expression*

$$\int \Big[P \cdot \text{pr } \mathrm{v}_{\mathcal{D}R}(\mathcal{D})Q + R \cdot \text{pr } \mathrm{v}_{\mathcal{D}Q}(\mathcal{D})P + Q \cdot \text{pr } \mathrm{v}_{\mathcal{D}P}(\mathcal{D})R\Big]dx = 0. \tag{3.1.48}$$

By corollary 3.1.25, this is equivalent to

$$\int \Big[P \cdot d_{\mathcal{D}}(\mathcal{D}R)Q + R \cdot d_{\mathcal{D}}(\mathcal{D}Q)P + Q \cdot d_{\mathcal{D}}(\mathcal{D}P)R\Big]dx = 0, \tag{3.1.49}$$

which can also be written as

$$\int \Big[P \cdot \mathcal{D}'[\mathcal{D}R]Q + R \cdot \mathcal{D}'[\mathcal{D}Q]P + Q \cdot \mathcal{D}'[\mathcal{D}P]R\Big] dx = 0 \tag{3.1.50}$$

or more commonly

$$\langle P, D'[DR]Q\rangle + \langle R, D'[DQ]P\rangle + \langle Q, D'[DP]R\rangle = \langle P, D'[DR]Q\rangle + \mathrm{cycle}(P, Q, R) = 0. \tag{3.1.51}$$

Proposition 3.1.27 *[29] Let $\mathcal{D}$ be a skew-adjoint $q \times q$ matrix differential operator. Then the bracket* (3.1.19) *satisfies the Jacobi identity if and only if* (3.1.50) *vanishes for all $q-$tuples $P, Q, R \in \mathcal{A}^q$.*

3.1.1 A Classic Example-The Harry Dym Equation

As an example, we consider the well-known Harry Dym equation

$$u_t = -\frac{1}{2}u^3 u_{xxx}. \tag{3.1.52}$$

Under the transformation $v = u^{-2}$, this equation can written in the equivalent form

$$v_t = (v^{-\frac{1}{2}})_{xxx}. \tag{3.1.53}$$

We show that this equation has a bi-Hamiltonian structure

$$v_t = D_1 \frac{\delta \mathcal{H}_1}{\delta v} = D_2 \frac{\delta \mathcal{H}_2}{\delta v}, \tag{3.1.54}$$

where D_1 and D_2 are the Hamiltonian operators and $\mathcal{H}_1$ and $\mathcal{H}_2$ are the Hamiltonian functionals.

Observe that

$$v_t = D_x^3(v^{-\frac{1}{2}}) \equiv D_1 \frac{\delta \mathcal{H}_1}{\delta v}. \tag{3.1.55}$$

Let $K_1[v] = v^{-\frac{1}{2}}$. Then by the formula (3.1.1) in theorem 3.1.1, we have

$$\begin{aligned} \mathcal{H}_1 &= \int \Big(\int_0^1 v \cdot K_1[\lambda v] d\lambda \Big) dx \\ &= \int \Big(\int_0^1 v^{\frac{1}{2}} \lambda^{\frac{1}{2}} d\lambda \Big) dx \\ &= 2 \int v^{\frac{1}{2}} dx. \end{aligned} \tag{3.1.56}$$

Also,

$$\begin{aligned} D_x^3(v^{-\frac{1}{2}}) &= -\frac{1}{8} \cdot \frac{4v_{xxx}v^2 - 18v_x v_{xx} + 15v_x^3}{v^{\frac{7}{2}}} \\ &= (2vD_x + v_x)\Big(\frac{5v_x^2}{16v^{\frac{7}{2}}} - \frac{v_{xx}}{4v^{\frac{5}{2}}} \Big) \\ &\equiv D_2 \frac{\delta \mathcal{H}_2}{\delta v}. \end{aligned} \tag{3.1.57}$$

Let $K_2[v] = \dfrac{5v_x^2}{16v^{\frac{7}{2}}} - \dfrac{v_{xx}}{4v^{\frac{5}{2}}}$. Then by the formula (3.1.2) in theorem 3.1.1, we have

$$\begin{aligned} \mathcal{H}_2 &= \lim_{\lambda_0 \to \infty} \int \Big(\int_{\lambda_0}^1 v \cdot K_2[\lambda v] d\lambda \Big) dx \\ &= \int \Big(\lim_{\lambda_0 \to \infty} \int_{\lambda_0}^1 \lambda^{-\frac{3}{2}} \Big(\frac{5}{16} v_x^2 v^{-\frac{5}{2}} - \frac{1}{4} v_{xx} v^{-\frac{3}{2}} \Big) d\lambda \Big) dx \\ &= \int \Big(-\frac{5}{8} v_x^2 v^{-\frac{5}{2}} + \frac{1}{2} v_{xx} v^{-\frac{3}{2}} \Big) dx. \end{aligned} \tag{3.1.58}$$

The differential function $-\frac{5}{8}v_x^2v^{-\frac{5}{2}} + \frac{1}{2}v_{xx}v^{-\frac{3}{2}}$ can be rewritten as

$$-\frac{5}{8}v_x^2v^{-\frac{5}{2}} + \frac{1}{2}v_{xx}v^{-\frac{3}{2}} = -\frac{1}{8}v_x^2v^{-\frac{5}{2}} + D_x\Big(-\frac{1}{2}v_xv^{-\frac{3}{2}}\Big) \tag{3.1.59}$$

Thus, we have

$$\mathcal{H}_2 = \int\Big(-\frac{5}{8}v_x^2v^{-\frac{5}{2}} + \frac{1}{2}v_{xx}v^{-\frac{3}{2}}\Big)dx = -\frac{1}{8}\int v^{-\frac{5}{2}}v_x^2dx. \tag{3.1.60}$$

Now we show that the operators D_1 and D_2 are Hamiltonian. By formula 3.1.29,

$$\begin{aligned}\Theta_{D_1} &= \frac{1}{2}\int\{\theta \wedge D_1\theta\}dx\\ &= \frac{1}{2}\int\{\theta \wedge D_x^3\theta\}dx\\ &= \frac{1}{2}\int\{\theta \wedge \theta_{xxx}\}dx\end{aligned} \tag{3.1.61}$$

and

$$\begin{aligned}\Theta_{D_2} &= \frac{1}{2}\int\{\theta \wedge D_2\theta\}dx\\ &= \frac{1}{2}\int\{\theta \wedge (2vD_x + v_x)\theta\}dx\\ &= \frac{1}{2}\int\{\theta \wedge (2v\theta_x + v_x\theta)\}dx\\ &= \frac{1}{2}\int\{2v\theta \wedge \theta_x\}dx.\end{aligned} \tag{3.1.62}$$

So by corollary 3.1.20, we have

$$\begin{aligned}\text{pr }\mathbf{v}_{D_1}(\Theta_{D_1}) &= \frac{1}{2}\text{pr }\mathbf{v}_{D_1}\int\{\theta \wedge \theta_{xxx}\}dx\\ &= 0\end{aligned} \tag{3.1.63}$$

trivially, since there are no terms in v. Thus $D_1 = D_x^3$ is a Hamiltonian operator.

Similarly,

$$\begin{aligned}
\text{pr } \mathbf{v}_{D_2}(\Theta_{D_2}) &= \text{pr } \mathbf{v}_{D_2} \int \{v\theta \wedge \theta_x\} dx \\
&= \int \{D_2(\theta) \wedge \theta \wedge \theta_x\} dx \\
&= \int \{(2v\theta_x + v_x\theta) \wedge \theta \wedge \theta_x\} dx \\
&= \int \{2v\theta_x \wedge \theta \wedge \theta_x + v_x\theta \wedge \theta \wedge \theta_x\} dx \\
&= 0
\end{aligned} \tag{3.1.64}$$

by the properties of the wedge product. So $D_2 = 2vD_x + u_x$ is also a Hamiltonian operator.

Now we show that D_1 and D_2 form a Hamiltonian pair. Due to lemma 3.1.13, we only need to show that $D_1 + D_2$ is also a Hamiltonian operator. By corollary 3.1.20, we have

$$\begin{aligned}
\text{pr } \mathbf{v}_{D_2}(\Theta_{D_1}) &= \frac{1}{2}\text{pr } \mathbf{v}_{D_2} \int \{\theta \wedge \theta_{xxx}\} dx \\
&= 0
\end{aligned} \tag{3.1.65}$$

and

$$\begin{aligned}
\text{pr } \mathbf{v}_{D_1}(\Theta_{D_2}) &= \text{pr } \mathbf{v}_{D_1} \int \{v\theta \wedge \theta_x\} dx \\
&= \int \{D_1(\theta) \wedge \theta \wedge \theta_x\} dx \\
&= \int \{\theta_{xxx} \wedge \theta \wedge \theta_x\} dx \\
&= \int \{\theta \wedge \theta_x \wedge \theta_{xxx}\} dx \\
&= 0
\end{aligned} \tag{3.1.66}$$

by previous calculations. We have thus shown that the Harry Dym equation is bi-

Hamiltonian.

3.2 Hamiltonian Formulation and Liouville Integrability of the First Hierarchy

We now present the first hierarchy (2.3.25) in a Hamiltonian form using the identity:

$$\frac{\delta}{\delta u}\int \mathrm{tr}\left(\frac{\partial U}{\partial \lambda}W\right)dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\mathrm{tr}\left(\frac{\partial U}{\partial u}W\right),\ \gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\mathrm{tr}(W^2)|, \tag{3.2.67}$$

This identity is called the trace identity [17, 38].

3.2.1 Hamiltonian Structure

Observe that

$$\frac{\partial U}{\partial p} = \begin{bmatrix} 0 & -\lambda & 0 \\ \lambda & 0 & -\lambda \\ 0 & \lambda & 0 \end{bmatrix},\ \frac{\partial U}{\partial q} = \begin{bmatrix} 0 & 0 & -\lambda \\ 0 & 0 & 0 \\ \lambda & 0 & 0 \end{bmatrix} \text{ and } \frac{\partial U}{\partial \lambda} = \begin{bmatrix} 0 & 2\lambda - p & -q \\ -2\lambda + p & 0 & -2\lambda - p \\ q & 2\lambda + p & 0 \end{bmatrix},$$

and as a result, we have

$$\mathrm{tr}\left(W\frac{\partial U}{\partial \lambda}\right) = -4ap - 8b\lambda - 2qc,\ \mathrm{tr}\left(W\frac{\partial U}{\partial p}\right) = -4a\lambda \text{ and } \mathrm{tr}\left(W\frac{\partial U}{\partial q}\right) = -2c\lambda.$$

Consequently, we have

$$\frac{\delta}{\delta u}\int (-4ap - 8\lambda b - 2qc)\,dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\begin{bmatrix} -4\lambda a \\ -2\lambda c \end{bmatrix}. \tag{3.2.68}$$

Equating coefficients of λ^m in the above equation gives

$$\frac{\delta}{\delta u}\int (2pa_m + 4b_{m+1} + qc_m)dx = (\gamma - 2m)\begin{bmatrix} 2a_m \\ c_m \end{bmatrix},\ m \geq 0,$$

and considering a particular case with $m = 1$ yields $\gamma = 0$, and thus we obtain

$$\frac{\delta}{\delta u}\int\left(-\frac{2pa_m + 4b_{m+1} + qc_m}{2m}\right)dx = \begin{bmatrix} 2a_m \\ c_m \end{bmatrix},\ m \geq 1. \tag{3.2.69}$$

We have thus proved the following proposition

Proposition 3.2.1 *The soliton hierarchy* (2.3.25) *has the Hamiltonian structure:*

$$u_{t_m} = K_m = \begin{bmatrix} a_{m,x} \\ c_{m,x} \end{bmatrix} = J\begin{bmatrix} 2a_m \\ c_m \end{bmatrix} = J\frac{\delta \mathcal{H}_m}{\delta u},\ m \geq 0, \tag{3.2.70}$$

with Hamiltonian operator

$$J = \begin{bmatrix} \frac{1}{2}\partial & 0 \\ 0 & \partial \end{bmatrix}, \tag{3.2.71}$$

and Hamiltonian functionals

$$\mathcal{H}_0 = \int\left(p^2 + \frac{q^2}{2}\right)dx,\ \mathcal{H}_m = \int\left(-\frac{2pa_m + 4b_{m+1} + qc_m}{2m}\right)dx,\ m \geq 1. \tag{3.2.72}$$

Using formula 3.1.1 in theorem 3.1.1, $\mathcal{H}_0$ is obtained by direct computation as follows:
Observing that

$$K_0 = \frac{\delta \mathcal{H}_0}{\delta u} = \begin{bmatrix} 2p \\ q \end{bmatrix}, \tag{3.2.73}$$

we have

$$K_0(\lambda u) = \lambda\begin{bmatrix} 2p \\ q \end{bmatrix}. \tag{3.2.74}$$

Thus,

$$\begin{aligned}\mathcal{H}_0 &= \int \Big(\int_0^1 u^T \cdot K_0(\lambda u) d\lambda\Big) dx \\ &= \int \Big(\int_0^1 \lambda \cdot \begin{bmatrix} p \\ q \end{bmatrix}^T \cdot \begin{bmatrix} 2p \\ q \end{bmatrix} d\lambda\Big) dx \\ &= \int \Big(p^2 + \frac{q^2}{2}\Big) dx.\end{aligned} \tag{3.2.75}$$

3.2.2 Bi-Hamiltonian Structure and Liouville Integrability

Proposition 3.2.2 *The operator D defined by*

$$D = \alpha \begin{bmatrix} 0 & \frac{1}{2}\partial^2 \\ -\frac{1}{2}\partial^2 & 0 \end{bmatrix} + \beta \begin{bmatrix} \frac{1}{2}\partial & 0 \\ 0 & \partial \end{bmatrix} + \begin{bmatrix} -\frac{1}{2}\partial p \partial^{-1} p \partial & -\frac{1}{2}\partial p \partial^{-1} q \partial \\ -\frac{1}{2}\partial q \partial^{-1} p \partial & -\frac{1}{2}\partial q \partial^{-1} q \partial \end{bmatrix}, \tag{3.2.76}$$

for arbitrary constants $\alpha, \beta \in \mathbb{R}$ is a Hamiltonian operator.

Proof. It is easy to see that $D^* = -D$. Thus we only need to verify lemma 3.1.26:

$$\langle P, D'[DQ]R\rangle + \text{cycle}(P,Q,R) = \langle P, D'[DQ]R\rangle + \langle Q, D'[DR]P\rangle + \langle R, D'[DP]Q\rangle = 0 \tag{3.2.77}$$

for arbitrary symmetries

$$P = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}, \ Q = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}, \ R = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}. \tag{3.2.78}$$

Here, cycle(P,Q,R) denotes the cyclic permutation of P, Q, R.

Let

$$X = \partial^{-1}(pP_{1,x} + qP_{2,x}), \ Y = \partial^{-1}(pQ_{1,x} + qQ_{2,x}), \ Z = \partial^{-1}(pR_{1,x} + qR_{2,x}) \tag{3.2.79}$$

and

$$\begin{pmatrix} U \\ V \end{pmatrix} = DQ = \frac{1}{2}\begin{pmatrix} \alpha Q_{2,xx} + \beta Q_{1,xx} - \partial p \partial^{-1}(pQ_{1,x} + qQ_{2,x}) \\ -\alpha Q_{1,xx} + 2\beta Q_{2,xx} - \partial q \partial^{-1}(pQ_{1,x} + qQ_{2,x}) \end{pmatrix} \\ = \frac{1}{2}\begin{pmatrix} \alpha Q_{2,xx} + \beta Q_{1,xx} - \partial p Y \\ -\alpha Q_{1,xx} + 2\beta Q_{2,xx} - \partial q Y \end{pmatrix}. \tag{3.2.80}$$

Then

$$D'[DQ] = -\frac{1}{2}\begin{bmatrix} \partial U \partial^{-1} p \partial + \partial p \partial^{-1} U \partial & \partial U \partial^{-1} q \partial + \partial p \partial^{-1} V \partial \\ \partial V \partial^{-1} p \partial + \partial q \partial^{-1} U \partial & \partial V \partial^{-1} q \partial + \partial q \partial^{-1} V \partial \end{bmatrix}, \tag{3.2.81}$$

and

$$D'[DQ]R = -\frac{1}{2}\begin{pmatrix} \partial U \partial^{-1}(pR_{1,x} + qR_{2,x}) + \partial p \partial^{-1}(UR_{1,x} + VR_{2,x}) \\ \partial V \partial^{-1}(pR_{1,x} + qR_{2,x}) + \partial q \partial^{-1}(UR_{1,x} + VR_{2,x}) \end{pmatrix}. \tag{3.2.82}$$

So, we have

$$\begin{aligned} &\langle P, D'[DQ]R \rangle \\ =& -\frac{1}{2}\int \{P_1[\partial U \partial^{-1}(pR_{1,x} + qR_{2,x})] + P_1[\partial p \partial^{-1}(UR_{1,x} + VR_{2,x})]\}dx \\ &-\frac{1}{2}\int \{P_2[\partial V \partial^{-1}(pR_{1,x} + qR_{2,x})] + P_2[\partial q \partial^{-1}(UR_{1,x} + VR_{2,x})]\}dx. \end{aligned} \tag{3.2.83}$$

Applying integration by parts to (3.2.83), we obtain

$$
\begin{aligned}
&\langle P, D'[DQ]R\rangle \\
=&-\frac{1}{2}\int\{-P_{1,x}U\partial^{-1}(pR_{1,x}+qR_{2,x})-P_{1,x}p\partial^{-1}(UR_{1,x}+VR_{2,x})\}dx \\
&-\frac{1}{2}\int\{-P_{2,x}V\partial^{-1}(pR_{1,x}+qR_{2,x})-P_{2,x}q\partial^{-1}(UR_{1,x}+VR_{2,x})\}dx \\
=&-\frac{1}{2}\int\{-(UP_{1,x}+VP_{2,x})\partial^{-1}(pR_{1,x}+qR_{2,x})-(pP_{1,x}+qP_{2,x})\partial^{-1}(UR_{1,x}+VR_{2,x})\}dx \\
=&-\frac{1}{2}\int\{-(UP_{1,x}+VP_{2,x})Z-(pP_{1,x}+qP_{2,x})\partial^{-1}(UR_{1,x}+VR_{2,x})\}dx.
\end{aligned}
\tag{3.2.84}
$$

Again, applying integration by parts to the second term of the last expression in (3.2.84), we have

$$
\begin{aligned}
&\langle P, D'[DQ]R\rangle \\
=&-\frac{1}{2}\int\{(UR_{1,x}+VR_{2,x})\partial^{-1}(pP_{1,x}+qP_{2,x})-(UP_{1,x}+VP_{2,x})Z\}dx \\
=&-\frac{1}{2}\int\{(UR_{1,x}+VR_{2,x})X-(UP_{1,x}+VP_{2,x})Z\}dx \\
=&-\frac{\alpha}{2}\int\{(Q_{2,xx}R_{1,x}-Q_{1,xx}R_{2,x})X-(Q_{2,xx}P_{1,x}-Q_{1,xx}P_{2,x})Z\}dx \\
&-\frac{\beta}{2}\int\{(Q_{1,x}R_{1,x}+2Q_{2,x}R_{2,x})X-(Q_{1,x}P_{1,x}+2Q_{2,x}P_{2,x})Z\}dx \\
&+\frac{1}{2}\int\{XY_xZ_x-X_xY_xZ+(p_xR_{1,x}+q_xR_{2,x})XY-(p_xP_{1,x}+q_xP_{2,x})YZ\}dx.
\end{aligned}
\tag{3.2.85}
$$

It follows that

$$
\begin{aligned}
&\int \{(Q_{2,xx}R_{1,x} - Q_{1,xx}R_{2,x})X - (Q_{2,xx}P_{1,x} - Q_{1,xx}P_{2,x})Z\}dx + \text{cycle}(P,Q,R)\\
&= \int \{(Q_{2,xx}R_{1,x} - Q_{1,xx}R_{2,x})X - (R_{2,xx}Q_{1,x} - R_{1,xx}Q_{2,x})X\}dx + \text{cycle}(P,Q,R)\\
&= \int \{(Q_{2,x}R_{1,x} - Q_{1,x}R_{2,x})_x X\}dx + \text{cycle}(P,Q,R)\\
&= -\int \{(Q_{2,x}R_{1,x} - Q_{1,x}R_{2,x})X_x\}dx + \text{cycle}(P,Q,R)\\
&= -\int \{p(P_{1,x}Q_{2,x}R_{1,x} - P_{1,x}Q_{1,x}R_{2,x}) + q(P_{2,x}Q_{2,x}R_{1,x} - P_{2,x}Q_{1,x}R_{2,x})\}dx + \text{cycle}(P,Q,R)\\
&= -\int \{p(P_{1,x}Q_{2,x}R_{1,x} - R_{1,x}P_{1,x}Q_{2,x}) + q(P_{2,x}Q_{2,x}R_{1,x} - Q_{2,x}R_{1,x}P_{2,x})\}dx + \text{cycle}(P,Q,R)\\
&= 0.
\end{aligned}
\tag{3.2.86}
$$

Similarly, by cyclic permutation, we have

$$
\begin{aligned}
&\int \{(Q_{1,x}R_{1,x} + 2Q_{2,x}R_{2,x})X - (Q_{1,x}P_{1,x} + 2Q_{2,x}P_{2,x})Z\}dx + \text{cycle}(P,Q,R)\\
&= \int \{(Q_{1,x}R_{1,x} + 2Q_{2,x}R_{2,x})X - (R_{1,x}X_{1,x} + 2R_{2,x}Q_{2,x})X\}dx + \text{cycle}(P,Q,R)\\
&= 0
\end{aligned}
\tag{3.2.87}
$$

and

$$
\begin{aligned}
&\frac{1}{2}\int \{XY_xZ_x - X_xY_xZ + (p_xR_{1,x} + q_xR_{2,x})XY - (p_xP_{1,x} + q_xP_{2,x})YZ\}dx\\
&= \frac{1}{2}\int \{XY_xZ_x - Y_xZ_xX + (p_xR_{1,x} + q_xR_{2,x})XY - (p_xR_{1,x} + q_xR_{2,x})XY\}dx\\
&= 0.
\end{aligned}
\tag{3.2.88}
$$

■

This implies that the operator $M = D(\alpha = 1, \beta = 0)$ given by

$$M = \Phi J = \begin{bmatrix} -\frac{1}{2}\partial p \partial^{-1} p \partial & \frac{1}{2}\partial^2 - \frac{1}{2}\partial p \partial^{-1} q \partial \\ -\frac{1}{2}\partial^2 - \frac{1}{2}\partial q \partial^{-1} p \partial & -\frac{1}{2}\partial q \partial^{-1} q \partial \end{bmatrix} \tag{3.2.89}$$

is Hamiltonian, and $J + M = D(\alpha = \beta = 1)$ given by

$$J + M = \begin{bmatrix} \frac{1}{2}\partial & 0 \\ 0 & \partial \end{bmatrix} + \begin{bmatrix} -\frac{1}{2}\partial p \partial^{-1} p \partial & \frac{1}{2}\partial^2 - \frac{1}{2}\partial p \partial^{-1} q \partial \\ -\frac{1}{2}\partial^2 - \frac{1}{2}\partial q \partial^{-1} p \partial & -\frac{1}{2}\partial q \partial^{-1} q \partial \end{bmatrix}, \tag{3.2.90}$$

is also Hamiltonian. It then follows from lemma 3.1.13 that J and M form a Hamiltonian pair. Thus we have proved the following proposition

Proposition 3.2.3 *The soliton hierarchy* (2.3.25) *is bi-Hamiltonian:*

$$u_{t_m} = K_m = J \frac{\delta \mathcal{H}_m}{\delta u} = M \frac{\delta \mathcal{H}_{m-1}}{\delta u},\ m \geq 1, \tag{3.2.91}$$

where J*,* $\mathcal{H}_m$ *and* M *and are defined by* (3.2.71)*,* (3.2.72) *and* (3.2.89)*, respectively.*

It follows from Magri's result [25] that the bi-Hamiltonian hierarchy (3.2.91) is Liouville integrable. i.e., it possesses infinitely many commuting conserved functionals:

$$\{\mathcal{H}_k, \mathcal{H}_l\}_J = \int \left(\frac{\delta \mathcal{H}_k}{\delta u} \right)^T J \frac{\delta \mathcal{H}_l}{\delta u} dx = 0,\ k, l \geq 0, \tag{3.2.92}$$

$$\{\mathcal{H}_k, \mathcal{H}_l\}_M = \int \left(\frac{\delta \mathcal{H}_k}{\delta u} \right)^T M \frac{\delta \mathcal{H}_l}{\delta u} dx = 0,\ k, l \geq 0, \tag{3.2.93}$$

and commuting symmetries:

$$[K_k, K_l] = K_k'(u)[K_l] - K_l'(u)[K_k] = 0,\ k, l \geq 0. \tag{3.2.94}$$

We present the first nonlinear integrable system in this hierarchy (3.2.91) as follows:

$$u_{t_1} = \begin{bmatrix} p \\ q \end{bmatrix}_{t_1} = K_1 = \begin{bmatrix} q_{xx} - 3p^2 p_x - \frac{1}{2} p_x q^2 - pq q_x \\ -p_{xx} - \frac{1}{2} q_x p^2 - qp p_x - \frac{3}{4} q^2 q_x \end{bmatrix} = J \frac{\delta \mathcal{H}_1}{\delta u} = M \frac{\delta \mathcal{H}_0}{\delta u},$$

where the Hamiltonian functional $\mathcal{H}_0$ is defined by (3.2.72) and $\mathcal{H}_1$ is given by

$$\mathcal{H}_1 = \int \left(\frac{1}{2} q_x p - \frac{1}{2} q p_x - \frac{1}{4} p^4 - \frac{1}{4} q^2 p^2 - \frac{1}{16} q^4 \right) dx. \tag{3.2.95}$$

3.2.3 Tri-Hamiltonian Formulation

In this subsection, we formulate a tri-Hamiltonian structure [28] for the soliton hierarchy (2.3.25). We begin with the following proposition.

Proposition 3.2.4 *The operator*

$$I_{\alpha,\beta} = \begin{bmatrix} \alpha\partial - \frac{1}{2} q \partial^{-1} q & -1 + q\partial^{-1} p \\ 1 + p\partial^{-1} q & \beta\partial - 2p\partial^{-1} p \end{bmatrix}, \tag{3.2.96}$$

is a Hamiltonian operator for $\alpha = \dfrac{\beta}{2}$ *where* $\alpha, \beta \in \mathbb{R}$.

Proof. For $\alpha = \dfrac{\beta}{2}$, denote this operator (3.2.96) by

$$D = \beta \begin{bmatrix} \frac{1}{2}\partial & 0 \\ 0 & \partial \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} q\partial^{-1} q & -1 + q\partial^{-1} p \\ 1 + p\partial^{-1} q & -2p\partial^{-1} p \end{bmatrix}, \tag{3.2.97}$$

We similarly verify that (3.2.77) holds for arbitrary symmetries

$$P = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}, \ Q = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}, \ R = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}. \tag{3.2.98}$$

Let

$$A=\partial^{-1}(qP_1-2pP_2),\ Y=\partial^{-1}(qQ_1-2pQ_2),\ Z=\partial^{-1}(qR_1-2pR_2) \tag{3.2.99}$$

and

$$\begin{aligned}\begin{pmatrix}U\\V\end{pmatrix}=DQ&=\begin{pmatrix}-Q_2+\frac{\beta}{2}Q_{1,x}-\frac{1}{2}q\partial^{-1}(qQ_1-2pQ_2)\\Q_1+\beta Q_{2,x}+p\partial^{-1}(qQ_1-2pQ_2)\end{pmatrix}\\&=\begin{pmatrix}-Q_2+\frac{\beta}{2}Q_{1,x}-\frac{1}{2}qY\\Q_1+\beta Q_{2,x}+pY\end{pmatrix}.\end{aligned} \tag{3.2.100}$$

Then

$$D'[DQ]=\begin{bmatrix}-\frac{1}{2}V\partial^{-1}q-\frac{1}{2}q\partial^{-1}V & V\partial^{-1}p+q\partial^{-1}U\\U\partial^{-1}q+p\partial^{-1}V & -2U\partial^{-1}p-2p\partial^{-1}U\end{bmatrix}, \tag{3.2.101}$$

and

$$D'[DQ]R=\begin{pmatrix}-\frac{1}{2}V\partial^{-1}(qR_1-2pR_2)-\frac{1}{2}q\partial^{-1}(VR_1-2UR_2)\\U\partial^{-1}(qR_1-2pR_2)+p\partial^{-1}(VR_1-2UR_2)\end{pmatrix}. \tag{3.2.102}$$

So, we have

$$\begin{aligned}&\langle P,D'[DQ]R\rangle\\&=-\frac{1}{2}\int\{P_1[V\partial^{-1}(qR_1-2pR_2)]+P_1[q\partial^{-1}(VR_1-2UR_2)]\}dx\\&\quad+\int\{P_2[U\partial^{-1}(qR_1-2pR_2)]+P_2[p\partial^{-1}(VR_1-2UR_2)]\}dx.\end{aligned} \tag{3.2.103}$$

Applying integration by parts to (3.2.103), we obtain

$$\begin{aligned}
&\langle P, D'[DQ]R\rangle \\
=&-\frac{1}{2}\int\{-(\partial^{-1}P_1V)(qR_1-2pR_2)-(\partial^{-1}P_1q)(VR_1-2UR_2)\}dx \\
&+\int\{-(\partial^{-1}P_2U)(qR_1-2pR_2)-(\partial^{-1}P_2p)(VR_1-2UR_2)\}dx \\
=&\frac{1}{2}\int\{(qR_1-2pR_2)\partial^{-1}(P_1V-2P_2U)+(VR_1-2UR_2)\partial^{-1}(P_1q-2P_2p)\}dx \\
=&\frac{1}{2}\int\{(qR_1-2pR_2)\partial^{-1}(P_1V-2P_2U)+(VR_1-2UR_2)X\}dx.
\end{aligned} \tag{3.2.104}$$

Applying integration by parts to the first term of the last expression in (3.2.104), we have

$$\begin{aligned}
&\langle P, D'[DQ]R\rangle \\
=&\frac{1}{2}\int\{-(P_1V-2P_2U)\partial^{-1}(qR_1-2pR_2)+(VR_1-2UR_2)X\}dx \\
=&\frac{1}{2}\int\{(VR_1-2UR_2)X-(P_1V-2P_2U)Z\}dx \\
=&\frac{\beta}{2}\int\{(R_1Q_{2,x}-R_2Q_{1,x})X-(P_1Q_{2,x}-P_2Q_{1,x})Z\}dx \\
&+\frac{1}{2}\int\{(R_1Q_1+2R_2Q_2)X-(P_1Q_1+2P_2Q_2)Z\}dx \\
&+\frac{1}{2}\int\{(R_1p+R_2q)YX-(P_1p+P_2q)YZ\}dx.
\end{aligned} \tag{3.2.105}$$

It follows that

$$
\begin{aligned}
&\int \{(R_1 Q_{2,x} - R_2 Q_{1,x})X - (P_1 Q_{2,x} - P_2 Q_{1,x})Z\} dx \\
&= \int \{(R_1 Q_{2,x} - R_2 Q_{1,x})X - (Q_1 R_{2,x} - Q_2 R_{1,x})X\} dx \\
&= \int \{(R_1 Q_2 - R_2 Q_1)_x X\} dx + \text{cycle}(P, Q, R) \\
&= -\int \{(R_1 Q_2 - R_2 Q_1) X_x\} dx + \text{cycle}(P, Q, R) \\
&= -\int \{(R_1 Q_2 - R_2 Q_1)(P_1 q - 2P_2 p)\} dx + \text{cycle}(P, Q, R) \\
&= \int \{2p(P_2 R_1 Q_2 - P_2 R_2 Q_1) - q(P_1 R_1 Q_2 - P_1 R_2 Q_1)\} dx + \text{cycle}(P, Q, R) \\
&= \int \{2p(P_2 R_1 Q_2 - Q_2 P_2 R_1) - q(P_1 R_1 Q_2 - R_1 Q_2 P_1)\} dx + \text{cycle}(P, Q, R) \\
&= 0.
\end{aligned}
\tag{3.2.106}
$$

Similarly, by cyclic permutation, we have

$$
\begin{aligned}
&\int \{(R_1 Q_1 + 2R_2 Q_2)X - (P_1 Q_1 + 2P_2 Q_2)Z\} dx + \text{cycle}(P, Q, R) \\
&= \int \{(R_1 Q_1 + 2R_2 Q_2)X - (Q_1 R_1 + 2Q_2 R_2)X\} dx + \text{cycle}(P, Q, R) \\
&= 0
\end{aligned}
\tag{3.2.107}
$$

and

$$
\begin{aligned}
&\int \{(R_1 p + R_2 q)YX - (P_1 p + P_2 q)YZ\} dx \\
&= \int \{(P_1 p + P_2 q)ZY - (P_1 p + P_2 q)YZ\} dx \\
&= 0.
\end{aligned}
\tag{3.2.108}
$$

■

Thus, the operator $I = D(\beta = 0)$ given by

$$I = \begin{bmatrix} -\frac{1}{2}q\partial^{-1}q & -1 + q\partial^{-1}p \\ 1 + p\partial^{-1}q & -2p\partial^{-1}p \end{bmatrix} \tag{3.2.109}$$

is Hamiltonian and the sum $J + I = D(\beta = 1)$ given by

$$D = \begin{bmatrix} \frac{1}{2}\partial & 0 \\ 0 & \partial \end{bmatrix} + \begin{bmatrix} -\frac{1}{2}q\partial^{-1}q & -1 + q\partial^{-1}p \\ 1 + p\partial^{-1}q & -2p\partial^{-1}p \end{bmatrix}, \tag{3.2.110}$$

is also Hamiltonian. So by lemma 3.1.13, J and I form a Hamiltonian pair. Consequently, we have the following proposition.

Proposition 3.2.5 *The Hamiltonian operators I, J and M given by* (3.2.109), (3.2.71), *and* (3.2.89)*, respectively form a Hamiltonian triple: Any linear combination of I, J and M is also Hamiltonian.*

The proof of this proposition is a little tedious but very similar to the proofs of propositions 3.2.1 and 3.2.3. We leave out the details.

From the condition that, $II^{-1} = I^{-1}I = I_2$, where I_2 is the 2×2 identity matrix, we obtain, through a simple computation, the inverse operator of I as

$$I^{-1} = \begin{bmatrix} -2p\partial^{-1}p & 1 + p\partial^{-1}q \\ -1 - q\partial^{-1}p & -\frac{1}{2}q\partial^{-1}q \end{bmatrix}. \tag{3.2.111}$$

The fact that I and J form a Hamiltonian pair gives rise to a hereditary symmetry operator $\Phi = JI^{-1}$, which is exactly the hereditary recursion operator defined by (2.3.26). Note also that it is direct to see that $J = \Phi I$ and $M = \Phi J = \Phi^2 I$. It thus, follows that the soliton hierarchy (2.3.25) is tri-Hamiltonian:

$$u_{t_m} = K_m = I\frac{\delta \mathcal{H}_{m+1}}{\delta u} = J\frac{\delta \mathcal{H}_m}{\delta u} = M\frac{\delta \mathcal{H}_{m-1}}{\delta u}, \; m \geq 1, \tag{3.2.112}$$

where the Hamiltonian operators I, J and M are defined by (3.2.109), (3.2.71) and (3.2.89) respectively, and the Hamiltonian functionals by (3.2.72).

We remark that one can also verify this tri-Hamiltonian structure by direct computation (e.g. using a computer algebra system). Note that the first Hamiltonian structure arises as a consequence of the decomposition $J = \Phi I$ of the Hamiltonian operator J (see, e.g., [24]).

3.2.3.1 Inverse Soliton Hierarchy

We now present an inverse hierarchy of commuting symmetries by computing the inverse of the recursion operator Φ. Based on the relation $\Phi = JI^{-1}$, we present Φ^{-1} as

$$\Phi^{-1} = \begin{bmatrix} -q\partial^{-1}q\partial^{-1} & \partial^{-1} + q\partial^{-1}p\partial^{-1} \\ 2\partial^{-1} + 2p\partial^{-1}q\partial^{-1} & -2p\partial^{-1}p\partial^{-1} \end{bmatrix}, \tag{3.2.113}$$

which is also a hereditary recursion operator (see, e.g., [8]). As a result, we have the hierarchy

$$u_{t_m} = K_{-m} = \Phi^{-m} K_0 = \Phi^{-m} \begin{bmatrix} p_x \\ q_x \end{bmatrix}, \ m \geq 1, \tag{3.2.114}$$

with infinitely many commuting symmetries:

$$[K_{-m}, K_{-n}] = K'_{-m}(u)[K_{-n}] - K'_{-n}(u)[K_{-m}] = 0, \ m, n \geq 1.$$

The first and second symmetry systems are

$$\begin{bmatrix} p_t \\ q_t \end{bmatrix} = \begin{bmatrix} q \\ 2p \end{bmatrix} \text{ and } \begin{bmatrix} p_t \\ q_t \end{bmatrix} = \begin{bmatrix} -q\partial^{-1}q\partial^{-1}q + 2\partial^{-1}p + 2q\partial^{-1}p\partial^{-1}p \\ 2\partial^{-1}q + 2p\partial^{-1}q\partial^{-1}q - 4p\partial^{-1}p\partial^{-1}p \end{bmatrix}. \tag{3.2.115}$$

The equations in this hierarchy are nonlocal and for this reason, it may be difficult to guarantee Liouville's integrability based on Magri's scheme [25].

3.3 Hamiltonian Formulation and Liouville Integrability of the Second Hierarchy

3.3.1 Hamiltonian Formulation

We again apply the trace identity (3.2.67) to formulate a Hamiltonian structure for the extended soliton hierarchy (2.4.26). From the partial derivatives

$$\frac{\partial U}{\partial p} = \begin{bmatrix} 0 & -\lambda + 2\alpha p & 0 \\ \lambda - 2\alpha p & 0 & -\lambda - 2\alpha p \\ 0 & \lambda + 2\alpha p & 0 \end{bmatrix}, \quad \frac{\partial U}{\partial q} = \begin{bmatrix} 0 & \alpha q & -\lambda \\ -\alpha q & 0 & -\alpha q \\ \lambda & \alpha q & 0 \end{bmatrix} \text{ and}$$

$$\frac{\partial U}{\partial \lambda} = \begin{bmatrix} 0 & 2\lambda - p & -q \\ -2\lambda + p & 0 & -2\lambda - p \\ q & 2\lambda + p & 0 \end{bmatrix},$$

we easily obtain

$$\mathrm{tr}(W\frac{\partial U}{\partial \lambda}) = -4ap - 8b\lambda - 2qc, \ \mathrm{tr}(W\frac{\partial U}{\partial p}) = -4a\lambda - 8\alpha pb, \text{ and } \mathrm{tr}(W\frac{\partial U}{\partial q}) = -2c\lambda - 4\alpha qb,$$

and so the trace identity gives

$$\frac{\delta}{\delta u}\int(-4ap - 8\lambda b - 2qc)dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\begin{bmatrix} -4\lambda a - 8\alpha pb \\ -2\lambda c - 4\alpha qb \end{bmatrix}. \tag{3.3.1}$$

Equating coefficients of λ^m in the above equation, we get

$$\frac{\delta}{\delta u}\int(2pa_m + 4b_{m+1} + qc_m)dx = (\gamma - 2m)\begin{bmatrix} 2a_m + 4\alpha pb_m \\ c_m + 2\alpha qb_m \end{bmatrix}, \ m \geq 0$$

and considering a particular case with $m = 1$ yields $\gamma = 0$. Thus we obtain

$$\frac{\delta}{\delta u}\mathcal{H}_m = \begin{bmatrix} 2a_m + 4\alpha p b_m \\ c_m + 2\alpha q b_m \end{bmatrix}, \ m \geq 0, \tag{3.3.2}$$

where the Hamiltonian functionals are

$$\mathcal{H}_0 = \int (p^2(1+2\alpha) + q^2(\frac{1}{2} + \alpha q))dx, \quad \mathcal{H}_m = \int (-\frac{2pa_m + 4b_{m+1} + qc_m}{2m})dx, \ m \geq 1. \tag{3.3.3}$$

Here, $\mathcal{H}_0$ is computed similarly as in (3.2.75).

Now using the relation $b_{m+1} = \partial^{-1}(pc_{m+1} - qa_{m+1})$ from (2.4.11), the system (2.4.25) becomes

$$\begin{cases} p_{t_m} = -c_{m+1} + qb_{m+1} + 2\alpha q b_{m+1}, \\ q_{t_m} = 2a_{m+1} - 2pb_{m+1} - 4\alpha p b_{m+1}. \end{cases} \tag{3.3.4}$$

Observing that

$$\begin{bmatrix} a_m \\ b_m \end{bmatrix} = N \begin{bmatrix} 2a_m + 4\alpha p b_m \\ c_m + 2\alpha q b_m \end{bmatrix}, \tag{3.3.5}$$

where

$$N = \begin{bmatrix} \frac{1}{2} + \alpha p \partial^{-1} q & -2\alpha p \partial^{-1} p \\ \alpha q \partial^{-1} q & 1 - 2\alpha q \partial^{-1} p \end{bmatrix}, \tag{3.3.6}$$

we can easily see that

$$K_m = \begin{bmatrix} -(1+2\alpha)q\partial^{-1}q & -1 + (1+2\alpha)q\partial^{-1}p \\ 2 + 2(1+2\alpha)p\partial^{-1}q & -2(1+2\alpha)p\partial^{-1}p \end{bmatrix} \begin{bmatrix} a_{m+1} \\ c_{m+1} \end{bmatrix}$$

$$= J \begin{bmatrix} 2a_{m+1} + 4\alpha p b_{m+1} \\ c_{m+1} + 2\alpha q b_{m+1} \end{bmatrix}, \ m \geq 0, \qquad . \tag{3.3.7}$$

where

$$J = \begin{bmatrix} -\frac{1}{2}(1+4\alpha)q\partial^{-1}q & -1 + (1+4\alpha)q\partial^{-1}p \\ 1 + (1+4\alpha)p\partial^{-1}q & -2(1+4\alpha)p\partial^{-1}p \end{bmatrix} \tag{3.3.8}$$

is a Hamiltonian operator. We have thus, proved the following proposition:

Proposition 3.3.1 *The soliton hierarchy* (2.4.26) *has a Hamiltonian structure:*

$$u_{t_m} = K_m = J \frac{\delta \mathcal{H}_{m+1}}{\delta u}, \ m \geq 0, \tag{3.3.9}$$

where the Hamiltonian functionals $\mathcal{H}_m$ *are defined by* (3.3.3).

A direct computation shows

$$\frac{\delta \mathcal{H}_{m+1}}{\delta u} = \Psi \frac{\delta \mathcal{H}_m}{\delta u}, \ \Psi = N^{-1} Q N, \tag{3.3.10}$$

where the inverse of the operator N is given by

$$N^{-1} = \begin{bmatrix} 2 - 4\alpha p\partial^{-1}q & 4\alpha p\partial^{-1}p \\ -2\alpha q\partial^{-1}q & 1 + 2\alpha q\partial^{-1}p \end{bmatrix}. \tag{3.3.11}$$

The expression for Ψ is explicitly given by

$$\Psi = \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{bmatrix} \tag{3.3.12}$$

where

$$\begin{cases}\Psi_{11} = -\alpha(p^2+\frac{q^2}{2}) + \alpha\partial q\partial^{-1}q - (1+2\alpha)p\partial^{-1}p\partial + \alpha(1+2\alpha)p\partial^{-1}(p^2q+\frac{q^3}{2}) \\ \qquad -2\alpha(1+2\alpha)p\partial^{-1}p\partial p\partial^{-1}q - \alpha(1+2\alpha)p\partial^{-1}q\partial q\partial^{-1}q - 2\alpha^2(p^3+p\frac{q^2}{2})\partial^{-1}q, \\ \Psi_{12} = \partial - 2\alpha\partial q\partial^{-1}p - (1+2\alpha)p\partial^{-1}q\partial - 2\alpha(1+2\alpha)p\partial^{-1}(p^3+p\frac{q^2}{2}) \\ \qquad +2\alpha(1+2\alpha)p\partial^{-1}q\partial q\partial^{-1}p + 4\alpha(1+2\alpha)p\partial^{-1}p\partial p\partial^{-1}p + 4\alpha^2(p^3+p\frac{q^2}{2})\partial^{-1}p, \\ \Psi_{21} = -\frac{1}{2}\partial - \alpha\partial p\partial^{-1}q - \frac{1}{2}(1+2\alpha)q\partial^{-1}p\partial + \frac{\alpha}{2}(1+2\alpha)q\partial^{-1}(p^2q+\frac{q^3}{2}) \\ \qquad -\alpha(1+2\alpha)q\partial^{-1}p\partial p\partial^{-1}q - \frac{\alpha}{2}(1+2\alpha)q\partial^{-1}q\partial q\partial^{-1}q - \alpha^2(p^2q+\frac{q^3}{2})\partial^{-1}q, \\ \Psi_{22} = -\alpha(p^2+\frac{q^2}{2}) + 2\alpha\partial p\partial^{-1}p - \frac{1}{2}(1+2\alpha)q\partial^{-1}q\partial - \alpha(1+2\alpha)q\partial^{-1}(p^3+p\frac{q^2}{2}) \\ \qquad +2\alpha(1+2\alpha)q\partial^{-1}p\partial p\partial^{-1}p + \alpha(1+2\alpha)q\partial^{-1}q\partial q\partial^{-1}p + 2\alpha^2(p^2q+\frac{q^3}{2})\partial^{-1}p. \end{cases} \tag{3.3.13}$$

From the relation $K_{m+1} = \Phi K_m,\ m \geq 0$ in (3.1.25), and $J\Psi = \Phi J$ (see e.g. [22]), we obtain a common recursion operator for the soliton hierarchy (2.4.26):

$$\Phi = \Psi^\dagger = N^\dagger Q^\dagger (N^{-1})^\dagger,$$

where $N^\dagger,\ Q^\dagger$ and $(N^{-1})^\dagger$ are the adjoint operators of N, Q and N^{-1}, respectively. The expression for Φ is explicitly presented as

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix}, \tag{3.3.14}$$

where

$$\begin{cases} \Phi_{11} = -\alpha(p^2+\frac{q^2}{2}) + \alpha q\partial^{-1}q\partial - (1+2\alpha)\partial p\partial^{-1}p - \alpha(1+2\alpha)(p^2q+\frac{q^3}{2})\partial^{-1}p \\ \qquad +2\alpha(1+2\alpha)q\partial^{-1}p\partial p\partial^{-1}p + \alpha(1+2\alpha)q\partial^{-1}q\partial q\partial^{-1}p + 2\alpha^2 q\partial^{-1}(p^3+p\frac{q^2}{2}), \\ \Phi_{12} = \frac{1}{2}\partial - \alpha q\partial^{-1}p\partial - \frac{1}{2}(1+2\alpha)\partial p\partial^{-1}q - \frac{\alpha}{2}(1+2\alpha)(p^2q+\frac{q^3}{2})\partial^{-1}q \\ \qquad +\alpha(1+2\alpha)q\partial^{-1}p\partial p\partial^{-1}q + \frac{\alpha}{2}(1+2\alpha)q\partial^{-1}q\partial q\partial^{-1}q + \alpha^2 q\partial^{-1}(p^2q+\frac{q^3}{2}), \\ \Phi_{21} = -\partial - 2\alpha p\partial^{-1}q\partial - (1+2\alpha)\partial q\partial^{-1}p + 2\alpha(1+2\alpha)(p^3+p\frac{q^2}{2})\partial^{-1}p \\ \qquad -2\alpha(1+2\alpha)p\partial^{-1}q\partial q\partial^{-1}p - 4\alpha(1+2\alpha)p\partial^{-1}p\partial p\partial^{-1}p - 4\alpha^2 p\partial^{-1}(p^3+p\frac{q^2}{2}), \\ \Phi_{22} = -\alpha(p^2+\frac{q^2}{2}) + 2\alpha p\partial^{-1}p\partial - \frac{1}{2}(1+2\alpha)\partial q\partial^{-1}q + \alpha(1+2\alpha)(p^3+p\frac{q^2}{2})\partial^{-1}q \\ \qquad -2\alpha(1+2\alpha)p\partial^{-1}p\partial p\partial^{-1}q - \alpha(1+2\alpha)p\partial^{-1}q\partial q\partial^{-1}q - 2\alpha^2 p\partial^{-1}(p^2q+\frac{q^3}{2}). \end{cases} \tag{3.3.15}$$

3.3.2 Bi-Hamiltonian Structure and Liouville Integrability

Define

$$M = \Phi J = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}. \tag{3.3.16}$$

where

$$\begin{cases} M_{11} = \frac{1}{2}\partial - \alpha q\partial^{-1}p\partial + \alpha\partial p\partial^{-1}q + \alpha^2(p^2q+\frac{q^3}{2})\partial^{-1}q \\ \qquad -2\alpha^2 q\partial^{-1}p\partial p\partial^{-1}q - \alpha^2 q\partial^{-1}q\partial q\partial^{-1}q + \alpha^2 q\partial^{-1}(p^2q+\frac{q^3}{2}), \\ M_{12} = \alpha(p^2+\frac{q^2}{2}) - \alpha q\partial^{-1}q\partial - 2\alpha\partial p\partial^{-1}p - 2\alpha^2(p^2q+\frac{q^3}{2})\partial^{-1}p \\ \qquad +4\alpha^2 q\partial^{-1}p\partial p\partial^{-1}p + 2\alpha^2 q\partial^{-1}q\partial q\partial^{-1}p - 2\alpha^2 q\partial^{-1}(p^3+p\frac{q^2}{2}), \\ M_{21} = -\alpha(p^2+\frac{q^2}{2}) + 2\alpha p\partial^{-1}p\partial + \alpha\partial q\partial^{-1}q - 2\alpha^2(p^3+p\frac{q^2}{2})\partial^{-1}q \\ \qquad +2\alpha^2 p\partial^{-1}q\partial q\partial^{-1}q + 4\alpha^2 p\partial^{-1}p\partial p\partial^{-1}q - 2\alpha^2 p\partial^{-1}(p^2q+\frac{q^3}{2}), \\ M_{22} = \partial + 2\alpha p\partial^{-1}q\partial - 2\alpha\partial q\partial^{-1}p + 4\alpha^2(p^3+p\frac{q^2}{2})\partial^{-1}p \\ \qquad -4\alpha^2 p\partial^{-1}q\partial q\partial^{-1}p - 8\alpha^2 p\partial^{-1}p\partial p\partial^{-1}p + 4\alpha^2 p\partial^{-1}(p^3+p\frac{q^2}{2}). \end{cases} \tag{3.3.17}$$

Proposition 3.3.2 *The operator*

$$D = \gamma J + M, \quad \gamma \in \mathbb{R}, \tag{3.3.18}$$

where J and M are given by (3.3.8) *and* (3.3.16) *respectively, is Hamiltonian.*

The proof of this proposition is rather complicated but very similar to the proofs of propositions 3.2.1 and 3.2.3. For brevity, we leave out the proof.

Choosing $\gamma = 0$ shows that M is Hamiltonian and $\gamma = 1$ shows that $J + M$ is Hamiltonian and so by lemma 3.1.13, J and M form a Hamiltonian pair. Thus we have proved that

Proposition 3.3.3 *The soliton hierarchy* (2.4.26) *has a bi-Hamiltonian structure*

$$u_{t_m} = K_m = J\frac{\delta \mathcal{H}_{m+1}}{\delta u} = M\frac{\delta \mathcal{H}_m}{\delta u}, \ m \geq 0, \tag{3.3.19}$$

where J, $\mathcal{H}_m$ and M and are defined by (3.3.8)*,* (3.3.3) *and* (3.3.16)*, respectively.*

It follows from Magri's scheme [25] that the hierarchy (2.4.26) is Liouville integrable, i.e., it possesses infinitely many commuting conserved functionals:

$$\{\mathcal{H}_m, \mathcal{H}_n\}_J = \int (\frac{\delta \mathcal{H}_m}{\delta u})^T J \frac{\delta \mathcal{H}_n}{\delta u} dx = 0, \ m, n \geq 0, \tag{3.3.20}$$

$$\{\mathcal{H}_m, \mathcal{H}_n\}_M = \int (\frac{\delta \mathcal{H}_m}{\delta u})^T M \frac{\delta \mathcal{H}_n}{\delta u} dx = 0, \ m, n \geq 0. \tag{3.3.21}$$

and commuting symmetries:

$$[K_m, K_n] = K'_m(u)[K_n] - K'_n(u)[K_m] = J\frac{\delta}{\delta u}\{\mathcal{H}_m, \mathcal{H}_n\} = 0, \ m, n \geq 0. \tag{3.3.22}$$

We point out that, similar to [22], these commuting relations can also be generated from the Virasoro algebra of Lax operators [14, 19], which is easier to prove than the existence of a bi-Hamiltonian structure.

4 Symmetry Constraints and Finite-dimensional Hamiltonian Systems

4.1 Introduction

As mentioned earlier, integrability of finite-dimensional Hamiltonian systems in the sense of Liouville is based on the existence of sufficiently many functionally independent conserved quantities or first integrals which are in involution. This is a well-developed concept in the theory of ordinary differential equations. In this chapter, we construct finite-dimensional Hamiltonian systems by means of the so-called *Bargmann symmetry constraint* [18, 23, 20] from a hierarchy of (1+1)-dimensional evolution equations associated with the spectral problem (2.3.5), and discuss their integrability in the sense of Liouville. The Bargmann symmetry constraint method corresponds to the binary nonlinearization technique [18, 23, 20] in which the Lax pairs and adjoint Lax pairs of a given soliton hierarchy are "constrained" to a nonlinearized system, yielding a finite-dimensional Liouville integrable Hamiltonian system whose solutions can be presented explicitly. For the sake of simplicity, we will use the loop algebra in equation (2.2.37) as the underlying loop algebra.

Definition 4.1.1 A *Poisson bracket* on a smooth manifold M is a bilinear skew-symmetric operation that assigns a smooth real-valued function $\{F, H\}$ on M to each pair F, H of smooth, real-valued functions, with the basic properties:

(a) Jacobi Identity: $\{\{F, H\}, P\} + \{\{P, F\}, H\} + \{\{H, P\}, F\} = 0$,

(b) Leibniz's rule : $\{F, H \cdot P\} = \{F, H\} \cdot P + H \cdot \{F, P\}$.

Definition 4.1.2 A manifold M with a Poisson bracket is called a *Poisson manifold*. The bracket is said to define a *Poisson structure* on M.

Example 4.1.3 *Let M be an even dimensional Euclidean space $\mathbb{R}^{2n}$ with coordinates $(p, q) = (p^1, \cdots, p^n, q^1, \cdots, q^n)$. If $H(p, q)$ and $F(p, q)$ are smooth functions, then the Poisson bracket of F and H, denoted by $\{F, H\}$, is given by*

$$\{F, H\} = \sum_{i=1}^{n} \Big(\frac{\partial F}{\partial q^i} \frac{\partial H}{\partial p^i} - \frac{\partial F}{\partial p^i} \frac{\partial H}{\partial q^i} \Big). \tag{4.1.1}$$

Definition 4.1.4 The system of ordinary differential equations

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p^i}, \quad \frac{dp^i}{dt} = -\frac{\partial H}{\partial q^i}, \quad i = 1, \ldots, n, \tag{4.1.2}$$

is called a finite-dimensional Hamiltonian system, where the function $H(p, q)$ is a smooth function. We call H the Hamiltonian function of the system. In Classical Mechanics, the above equations (4.1.2) are referred to as Hamilton's equations.

Definition 4.1.5 Let M be a Poisson manifold and H a smooth function. The *Hamiltonian vector* field associated with H is the unique smooth vector field X_H on M satisfying

$$X_H(F) = \{F, H\} = -\{H, F\} \tag{4.1.3}$$

for every smooth function F.

Example 4.1.6 *In the case of the Poisson bracket on $\mathbb{R}^{2n}$ in equation* (4.1.1)*, the Hamiltonian vector field corresponding to $H(p, q)$ is the function*

$$X_H = \sum_{i=1}^{n} \Big(\frac{\partial H}{\partial p^i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p^i} \Big). \tag{4.1.4}$$

Definition 4.1.7 A *symplectic manifold* is a smooth (even-dimensional) manifold, M, equipped with a simplectic form ω. A *simplectic form* is a closed nondegenerate differential 2-form. We say that ω is closed if the exterior derivative $d\omega = 0$ and non-degenerate if for all $p \in M$, there exists an $X \in T_pM$ such that whenever $\omega(X, Y) = 0$ for all $Y \in T_pM$, it follows that $X = 0$.

Definition 4.1.8 Every simplectic manifold (M, ω) is a Poisson manifold with the Poisson bracket defined by the simplectic form, ω as

$$\{F, H\} = \omega(X_F, X_H), \tag{4.1.5}$$

where X_F is a vector field associated with F.

Example 4.1.9 *The most basic example of a symplectic manifold is* $\mathbb{R}^{2n}$ *equipped with the form* $\omega = \sum_{i=1}^{n} dp^i \wedge dq^i$. *Note that according to equations* (4.1.3), (4.1.4) *and* (4.1.5)*, we have*

$$\{F, H\} = \omega(X_F, X_H) = X_H(F) = \sum_{i=1}^{n} \Big(\frac{\partial H}{\partial p^i} \frac{\partial F}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial F}{\partial p^i} \Big). \tag{4.1.6}$$

Definition 4.1.10 The number of degrees of freedom of a Hamiltonian system is the number of (p^i, q^i) pairs in Hamiltons equations (i.e., the value of n).

Proposition 4.1.11 *[29] A function $F(p, q)$ is a first integral (or an integral of motion) for the Hamiltonian system* (4.1.2) *with Hamiltonian function H if and only if F and H are in involution, i.e., $\{F, H\} = 0$.*

Definition 4.1.12 A Hamiltonian system on the $2n$-dimensional symplectic manifold, M is said to be *completely integrable* or *Liouville-integrable* if there are n smooth first integrals in involution which are functionally independent on an open and dense subset of M.

4.2 Soliton Hierarchy and Its Hamiltonian Formulation

Theorem 4.2.1 *The spectral problem*

$$\phi_x = U\phi = U(u,\lambda)\phi \in \tilde{\mathfrak{g}}, \tag{4.2.7}$$

with spectral matrix

$$U = \lambda q e_1 + (\lambda^2 + \lambda p)e_2 + (-\lambda^2 + \lambda p)e_3 = \begin{bmatrix} \lambda q & \lambda^2 + \lambda p \\ -\lambda^2 + \lambda p & -\lambda q \end{bmatrix}, \tag{4.2.8}$$

where e_1, e_2 *and* e_3 *are a basis of the Lie algebra* $sl(2,\mathbb{R})$ *given by* (2.3.3) *and* $\widetilde{\mathfrak{g}} = \widetilde{sl}(2,\mathbb{R})$ *produces a hierarchy of soliton equations*

$$u_{t_m} = K_m = \Phi^m \begin{bmatrix} p_x \\ q_x \end{bmatrix}, \ m \geq 0, \tag{4.2.9}$$

where the operator Φ *given by*

$$\Phi = \begin{bmatrix} \partial p \partial^{-1} p & \frac{1}{2}\partial + \partial p \partial^{-1} q \\ -\frac{1}{2}\partial + \partial q \partial^{-1} p & \partial q \partial^{-1} q \end{bmatrix}, \ \partial = \frac{\partial}{\partial x}. \tag{4.2.10}$$

The proof begins with W in the stationary zero-curvature equation

$$W_x = [U, W] \tag{4.2.11}$$

defined as

$$W = \sum_{i\geq 0} \begin{bmatrix} c_i \lambda^{-2i-1} & a_i \lambda^{-2i-1} + b_i \lambda^{-2i} \\ a_i \lambda^{-2i-1} - b_i \lambda^{-2i} & -c_i \lambda^{-2i-1} \end{bmatrix} \in \widetilde{\mathrm{sl}}(2,\mathbb{R}). \tag{4.2.12}$$

Equation (4.2.11) then gives the relations

$$a_m = \frac{1}{2}c_{m-1,x} + pb_m,\ c_m = qb_m - \frac{1}{2}a_{m-1,x} \ \text{ and } \ b_{m,x} = pa_{m-1,x} + qc_{m-1,x},\ m \geq 1, \tag{4.2.13}$$

with initial values

$$a_0 = p,\ b_0 = 1,\ c_0 = q. \tag{4.2.14}$$

The remainder of the proof follows similarly as in the proof of theorem 2.3.1.

We point out a well-known fact that there is a gauge transformation between the Dirac and ZS-AKNS spectral problems (see [31] for details). This is the same gauge transformation [31] that transforms the spectral problem (4.2.7) into the original Kaup-Newell spectral problem [12]. However, as pointed out in [31], the extensions of the Dirac and the ZS-AKNS spectral problems, which are also gauge equivalent under the same transformation, have different non-dynamical r-matrices [35] determined by their respective finite-dimensional Hamiltonian systems via some constraints. Thus, although two spectral problems may be gauge equivalent, they may possess certain properties that are completely different. We thus conjecture that our spectral problem and the Kaup-Newell spectral problem may be another gauge equivalent pair of spectral problems with extensions that have two different pair of r-matrices determined by their respective finite-dimensional Hamiltonian systems.

Proposition 4.2.2 *The hierarchy* (4.2.9) *has a Hamiltonian structure*

$$u_{t_m} = K_m = J\frac{\delta \mathcal{H}_m}{\delta u},\ m \geq 0, \tag{4.2.15}$$

with Hamiltonian operator

$$J = \begin{bmatrix} \partial & 0 \\ 0 & \partial \end{bmatrix}, \tag{4.2.16}$$

and Hamiltonian functionals

$$\mathcal{H}_0 = \int \left(\frac{p^2}{2} + \frac{q^2}{2}\right) dx,\ \mathcal{H}_m = \int \left(-\frac{pa_m - 2b_{m+1} + qc_m}{2m}\right) dx,\ m \geq 1. \quad (4.2.17)$$

Similar to proposition 3.2.2, one can show that this hierarchy (4.2.9) is bi-Hamiltonian and thus according to Magri's scheme [25], each system is integrable in the sense of Liouville.

4.3 Binary Nonlinearization of Lax Pairs and Adjoint Lax Pairs

4.3.1 Bargmann Symmetry Constraint

We now consider the binary nonlinearization procedure [18, 23, 20].

Proposition 4.3.1 *The Lax pairs*

$$\begin{cases} \phi_x = U\phi = U(u,\lambda)\phi, \\ \phi_{t_m} = (V^{[m]})\phi = (V^{[m]})(u,\lambda)\phi, \end{cases} \quad (4.3.1)$$

of the the system (4.2.9) *have associated adjoint Lax pairs*

$$\begin{cases} \psi_x = -U^T\psi = -U^T(u,\lambda)\psi, \\ \psi_{t_m} = -(V^{[m]})^T\psi = -(V^{[m]})^T(u,\lambda)\psi, \end{cases} \quad (4.3.2)$$

where $\psi = (\psi_1, \psi_2)^T$ *and* $V^{[m]} = \lambda(\lambda^{(2m+1)}W)_+,\ m \geq 0,$ *with* W *given by* (4.2.12).

Proof. The proof easily follows from the fact that $U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0$ if and only if $(-U^T)_{t_m} - ((-V^{[m]})^T)_x + [-U^T, -(V^{[m]})^T] = 0.$

∎

Now recall from [18] that the variational derivatives of the spectral parameter $\lambda = \lambda(u)$ with respect to the potentials p and q can be expressed as

$$\frac{\delta\lambda}{\delta p} = \frac{\langle \phi\psi^T, \frac{\partial U}{\partial p}\rangle}{-\int_{-\infty}^{\infty}\langle \phi\psi^T, \frac{\partial U}{\partial \lambda}\rangle}, \frac{\delta\lambda}{\delta q} = \frac{\langle \phi\psi^T, \frac{\partial U}{\partial q}\rangle}{-\int_{-\infty}^{\infty}\langle \phi\psi^T, \frac{\partial U}{\partial \lambda}\rangle}. \tag{4.3.3}$$

It follows that

$$\frac{\delta\lambda}{\delta u} = \frac{1}{E}\begin{pmatrix} \lambda\phi_1\psi_2 + \lambda\phi_2\psi_1 \\ \lambda\phi_1\psi_1 - \lambda\phi_2\psi_2 \end{pmatrix}, \tag{4.3.4}$$

where

$$E = -\int_{-\infty}^{\infty}(q\phi_1\psi_1 - 2\lambda\phi_1\psi_2 + p\phi_1\psi_2 + 2\lambda\phi_2\psi_1 - q\phi_2\psi_2)dx. \tag{4.3.5}$$

Introducing N distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_N$, equations (4.3.1) and (4.3.2) yield the spatial and temporal systems

$$\begin{cases} \begin{pmatrix} \phi_{1j} \\ \phi_{2j}, \end{pmatrix}_x = U(u, \lambda_j)\begin{pmatrix} \phi_{1j} \\ \phi_{2j}, \end{pmatrix}, \ j = 1, 2, \cdots, N, \\ \begin{pmatrix} \psi_{1j} \\ \psi_{2j}, \end{pmatrix}_x = -U^T(u, \lambda_j)\begin{pmatrix} \psi_{1j} \\ \psi_{2j}, \end{pmatrix}, \ j = 1, 2, \cdots, N, \end{cases} \tag{4.3.6}$$

$$\begin{cases} \begin{pmatrix} \phi_{1j} \\ \phi_{2j}, \end{pmatrix}_{t_m} = V^{(m)}(u, \lambda_j)\begin{pmatrix} \phi_{1j} \\ \phi_{2j}, \end{pmatrix}, \ j = 1, 2, \cdots, N, \\ \begin{pmatrix} \psi_{1j} \\ \psi_{2j}, \end{pmatrix}_{t_m} = -(V^{(m)})^T(u, \lambda_j)\begin{pmatrix} \psi_{1j} \\ \psi_{2j}, \end{pmatrix}, \ j = 1, 2, \cdots, N. \end{cases} \tag{4.3.7}$$

Observing that the hierarchy (4.2.9) can be rewritten as

$$u_t = J \begin{bmatrix} a_m \\ c_m \end{bmatrix} = JG_m = J\Psi^m \begin{bmatrix} a_0 \\ c_0 \end{bmatrix}, \quad m \geq 0, \tag{4.3.8}$$

where

$$\Psi = \Phi^* = \begin{bmatrix} p\partial^{-1}p\partial & \frac{1}{2}\partial + p\partial^{-1}q\partial \\ -\frac{1}{2}\partial + q\partial^{-1}p\partial & q\partial^{-1}q\partial \end{bmatrix}. \tag{4.3.9}$$

the Bargmann symmetry constraint

$$JG_0 = J\sum_{i=1}^{N} E_j \frac{\delta\lambda_j}{\delta u} \text{ or } G_0 = \sum_{i=1}^{N} E_j \frac{\delta\lambda_j}{\delta u}, \tag{4.3.10}$$

where

$$E_j = -\int_{-\infty}^{\infty} (q\phi_{1j}\psi_{1j} - 2\lambda_j\phi_{1j}\psi_{2j} + p\phi_{1j}\psi_{2j} + 2\lambda_j\phi_{2j}\psi_{1j} - q\phi_{2j}\psi_{2j})dx, \ 1 \leq j \leq N, \tag{4.3.11}$$

gives rise to the constraint

$$\begin{pmatrix} a_0 \\ c_0 \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \langle AP_1, Q_2\rangle + \langle AP_2, Q_1\rangle \\ \langle AP_1, Q_1\rangle - \langle AP_2, Q_2\rangle \end{pmatrix}, \tag{4.3.12}$$

where $P_i = (\phi_{i1}, ..., \phi_{iN})^T, Q_i = (\psi_{i1}, ..., \psi_{iN})^T, i = 1, 2, \ A = \text{diag}(\lambda_1, \lambda_2, ..., \lambda_N)$ and $\langle \cdot, \cdot \rangle$ denotes the usual inner product on $\mathbb{R}^N$.

4.3.2 Finite-dimensional Hamiltonian Systems

Hereafter, we will denote u by $\tilde{u}$ under the constraint (4.3.12), to distinguish between equations.

Substituting equation (4.3.12) into equations (4.3.6) and (4.3.7), we obtain the nonlinear Lax pairs and adjoint Lax pairs

$$\begin{cases} \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \end{pmatrix}_x = \begin{pmatrix} -\lambda_j \tilde{q} & \lambda_j^2 - \lambda_j \tilde{p} \\ -\lambda_j^2 - \lambda_j \tilde{p} & \lambda_j \tilde{q} \end{pmatrix} \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \end{pmatrix}, \ j = 1, 2, \cdots, N, \\ \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \end{pmatrix}_x = \begin{pmatrix} \lambda_j \tilde{q} & \lambda_j^2 + \lambda_j \tilde{p} \\ -\lambda_j^2 + \lambda_j \tilde{p} & -\lambda_j \tilde{q} \end{pmatrix} \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \end{pmatrix}, \ j = 1, 2, \cdots, N, \end{cases} \tag{4.3.13}$$

$$\begin{cases} \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \end{pmatrix}_{t_m} = \begin{pmatrix} \sum_{i=0}^m \tilde{c}_j \lambda^{2m+1-2j} & \sum_{i=0}^m (\tilde{a}_j \lambda^{2m+1-2j} + \tilde{b}_j \lambda^{2m+2-2j}) \\ \sum_{i=0}^m (\tilde{a}_j \lambda^{2m+1-2j} - \tilde{b}_j \lambda^{2m+2-2j}) & -\sum_{i=0}^m \tilde{c}_j \lambda^{2m+1-2j} \end{pmatrix} \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \end{pmatrix}, \\ \qquad j = 1, 2, \cdots, N, \\ \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \end{pmatrix}_{t_m} = \begin{pmatrix} -\sum_{i=0}^m \tilde{c}_j \lambda^{2m+1-2j} & -\sum_{i=0}^m (\tilde{a}_j \lambda^{2m+1-2j} - \tilde{b}_j \lambda^{2m+2-2j}) \\ -\sum_{i=0}^m (\tilde{a}_j \lambda^{2m+1-2j} + \tilde{b}_j \lambda^{2m+2-2j}) & \sum_{i=0}^m \tilde{c}_j \lambda^{2m+1-2j} \end{pmatrix} \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \end{pmatrix}, \\ \qquad j = 1, 2, \cdots, N. \end{cases} \tag{4.3.14}$$

We can easily see that the system (4.3.13) is a finite-dimensional system with respect to x, where as (4.3.14) is a system of evolution equations with respect to x and t_m. However, this system (4.3.14) can be transformed into a finite-dimensional Hamiltonian system under the control of (4.3.13). The system (4.3.13) can be put in the form

$$\begin{cases} P_{1x} = \langle AP_1, Q_1 \rangle AP_1 - \langle AP_2, Q_2 \rangle AP_1 + A^2 P_2 + \langle AP_1, Q_2 \rangle AP_2 + \langle AP_2, Q_1 \rangle AP_2, \\ P_{2x} = -A^2 P_1 \langle AP_1, Q_2 \rangle AP_1 + \langle AP_2, Q_1 \rangle AP_1 - \langle AP_1, Q_1 \rangle AP_2 + \langle AP_2, Q_2 \rangle AP_2, \\ Q_{1x} = -\langle AP_1, Q_1 \rangle AQ_1 + \langle AP_2, Q_2 \rangle AQ_1 + A^2 Q_2 - \langle AP_1, Q_2 \rangle AQ_2 - \langle AP_2, Q_1 \rangle AQ_2, \\ Q_{2x} = -A^2 Q_1 - \langle AP_1, Q_2 \rangle AQ_1 - \langle AP_2, Q_1 \rangle AQ_1 + \langle AP_1, Q_1 \rangle AQ_2 - \langle AP_2, Q_2 \rangle AQ_2. \end{cases} \tag{4.3.15}$$

It follows that

Proposition 4.3.2 *The system* (4.3.13) *has the Hamiltonian form*

$$P_{ix} = \frac{\partial H}{\partial Q_i}, Q_{ix} = -\frac{\partial H}{\partial P_i},\ \ i = 1, 2, \tag{4.3.16}$$

where the Hamiltonian function is

$$H = \frac{1}{2}(\langle AP_1, Q_1\rangle - \langle AP_2, Q_2\rangle)^2 + \frac{1}{2}(\langle AP_1, Q_2\rangle + \langle AP_2, Q_1\rangle)^2 + \langle A^2P_1, Q_2\rangle - \langle A^2P_2, Q_1\rangle. \tag{4.3.17}$$

In the next subsections, we want to discuss integrability. We will consider two main cases:

1. When zero boundary conditons are imposed on the eigenfunctions and the adjoint eigenfunctions and
2. When no boundary conditions are imposed on the eigenfunctions and the adjoint eigenfunctions.

4.3.3 Involutive and Independent Systems

We begin by deriving integrals of motion for the systems (4.3.13) and (4.3.14). First we consider the case where the zero boundary conditions, $\lim_{|x|\to+\infty}\phi = \lim_{|x|\to+\infty}\psi = 0$ are imposed. Under these conditions, we obtain the identity

$$\Psi\frac{\delta\lambda}{\delta u} = \lambda^2\frac{\delta\lambda}{\delta u}, \tag{4.3.18}$$

where Ψ is defined by equation (4.3.9). In this case, the above identity (4.3.18) gives rise to

$$\begin{pmatrix} \tilde{a}_m \\ \tilde{c}_m \end{pmatrix} = \begin{pmatrix} \langle A^{2m+1}P_1, Q_2\rangle + \langle A^{2m+1}P_2, Q_1\rangle \\ \langle A^{2m+1}P_1, Q_1\rangle - \langle A^{2m+1}P_2, Q_2\rangle \end{pmatrix},\ m \geq 0. \tag{4.3.19}$$

Observing from equation (4.2.13) that

$$b_{m,x} = 2qa_m - pc_m,\ m \geq 0, \tag{4.3.20}$$

we obtain

$$\tilde{b}_0 = 1,\ \tilde{b}_m = \langle A^{2m}P_1, Q_2\rangle - \langle A^{2m}P_2, Q_1\rangle,\ m \geq 1. \tag{4.3.21}$$

Corollary 4.3.3 *[23] If $\tilde{U}$ and $\tilde{W}$ satisfy the adjoint representation equation $\tilde{W}_x = [\tilde{U}, \tilde{W}]$, then identity*

$$(\tilde{W}^2)_x = [\tilde{U}, \tilde{W}^2] \tag{4.3.22}$$

holds. It follows from the above corollary that

$$F_x = \frac{1}{2}(\mathrm{tr}\tilde{W}^2)_x = \frac{d}{dx}(\tilde{a}^2 - \tilde{b}^2 + \tilde{c}^2) = \mathrm{tr}[\tilde{U}, \tilde{W}^2] = 0. \tag{4.3.23}$$

Thus, F is a generating function of integrals of motion for the system (4.3.16).

Now, considering $F = \sum_{n\geq 0} F_n\lambda^{2n}$, we obtain

$$F_n = \sum_{i=0}^{n-1} \tilde{a}_i\tilde{a}_{n-1-i} - \sum_{i=0}^{n} \tilde{b}_i\tilde{b}_{n-i} + \sum_{i=0}^{n-1} \tilde{c}_i\tilde{c}_{n-1-i},\ n \geq 1, \tag{4.3.24}$$

and substituting (4.3.19) and (4.3.21) into (4.3.24), we have the following expressions for $F_n, n \geq 0$:

$$\begin{cases} F_0 = -1, \\ F_1 = (\langle AP_1, Q_1\rangle - \langle AP_2, Q_2\rangle)^2 + (\langle AP_1, Q_2\rangle + \langle AP_2, Q_1\rangle)^2 + 2(\langle A^2P_1, Q_2\rangle - \langle A^2P_2, Q_1\rangle) \\ \quad = 2H, \\ F_n = \sum_{i=0}^{n-1}[(\langle A^{2i+1}P_1, Q_2\rangle + \langle A^{2i+1}P_2, Q_1\rangle)(\langle A^{2n-1-2i}P_1, Q_2\rangle + \langle A^{2n-1-2i}P_2, Q_1\rangle) \\ \quad +(\langle A^{2i+1}P_1, Q_2\rangle - \langle A^{2i+1}P_2, Q_2\rangle)(\langle A^{2n-1-2i}P_1, Q_1\rangle - \langle A^{2n-1-2i}P_2, Q_2\rangle)] \\ \quad -\sum_{i=1}^{n-1}[(\langle A^{2i}P_1, Q_2\rangle - \langle A^{2i}P_2, Q_1\rangle)(\langle A^{2n-2i}P_1, Q_2\rangle - \langle A^{2n-2i}P_2, Q_1\rangle) \\ \quad -2(\langle A^{2n}P_1, Q_2\rangle - \langle A^{2n}P_2, Q_1\rangle)\ , n \geq 2. \end{cases} \tag{4.3.25}$$

It should be noted that the above functions are all polynomials of $4N$ dependent variables ϕ_{ij}, ψ_{ij}.

At this point, we also want to present the temporal systems (4.3.14) in a Hamiltonian form. Observing the equality

$$\Big(\frac{\partial F}{\partial P_i}, \frac{\partial F}{\partial Q_i}\Big)^T = \Big(\mathrm{tr}\Big(\tilde{W}\frac{\partial \tilde{W}}{\partial P_i}\Big), \mathrm{tr}\Big(\tilde{W}\frac{\partial \tilde{W}}{\partial Q_i}\Big)\Big)^T, \tag{4.3.26}$$

we find that

$$\begin{aligned} \mathrm{tr}\Big(\tilde{W}\frac{\partial \tilde{W}}{\partial P_1}\Big) &= \mathrm{tr}\sum_{i=0}^{\infty}\begin{pmatrix} \tilde{c}_i\lambda^{-2i-1} & \tilde{a}_i\lambda^{-2i-1} + \tilde{b}_i\lambda^{-2i} \\ \tilde{a}_i\lambda^{-2i-1} - \tilde{b}_i\lambda^{-2i} & \tilde{c}_i\lambda^{-2i-1} \end{pmatrix} \sum_{j=0}^{\infty}\frac{\partial}{\partial P_1}\begin{pmatrix} \tilde{c}_j\lambda^{-2j-1} & \tilde{a}_j\lambda^{-2j-1} + \tilde{b}_j\lambda^{-2j} \\ \tilde{a}_j\lambda^{-2j-1} - \tilde{b}_j\lambda^{-2j} & \tilde{c}_j\lambda^{-2j-1} \end{pmatrix} \\ &= \mathrm{tr}\sum_{\substack{i\geq 0\\ j\geq 0}}\begin{pmatrix} \tilde{c}_i\lambda^{-2i-1} & \tilde{a}_i\lambda^{-2i-1} + \tilde{b}_i\lambda^{-2i} \\ \tilde{a}_i\lambda^{-2i-1} - \tilde{b}_i\lambda^{-2i} & \tilde{c}_i\lambda^{-2i-1} \end{pmatrix}\begin{pmatrix} A^{2j+1}Q_1\lambda^{-2j-1} & A^{2j+1}Q_2\lambda^{-2j-1} + A^{2j}Q_2\lambda^{-2j} \\ A^{2j+1}Q_2\lambda^{-2j-1} - A^{2j}Q_2\lambda^{-2j} & -A^{2j+1}Q_1\lambda^{-2j-1} \end{pmatrix} \\ &= \sum_{\substack{i\geq 0\\ j\geq 0}}[2\tilde{c}_iA^{2j+1}Q_1\lambda^{-2i-2j-2} + 2\tilde{a}_iA^{2j+1}Q_2\lambda^{-2i-2j-2} - 2\tilde{b}_iA^{2j}Q_2\lambda^{-2i-2j}] \\ &= \sum_{\substack{i\geq 0\\ j\geq 1}}[2\tilde{c}_iA^{2j-1}Q_1 + 2\tilde{a}_iA^{2j-1}Q_2 - 2\tilde{b}_iA^{2j}Q_2]\lambda^{-2i-2j}. \end{aligned} \tag{4.3.27}$$

Similarly, we have

$$\mathrm{tr}\Big(\tilde{W}\frac{\partial\tilde{W}}{\partial P_2}\Big) = \sum_{\substack{i\geq 0\\ j\geq 1}}[-2\tilde{c}_i A^{2j-1}Q_2 + 2\tilde{a}_i A^{2j-1}Q_1 + 2\tilde{b}_i A^{2j}Q_1]\lambda^{-2i-2j}. \tag{4.3.28}$$

It then follows that

$$\begin{pmatrix} \frac{1}{2}\frac{\partial F_{m+1}}{\partial P_1} \\ \frac{1}{2}\frac{\partial F_{m+1}}{\partial P_2} \end{pmatrix} = -(-\tilde{V}^{[m]})^T \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}. \tag{4.3.29}$$

A similar argument leads to

$$\begin{pmatrix} \frac{1}{2}\frac{\partial F_{m+1}}{\partial Q_1} \\ \frac{1}{2}\frac{\partial F_{m+1}}{\partial Q_2} \end{pmatrix} = \tilde{V}^{[m]} \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}. \tag{4.3.30}$$

Thus we have the following

Proposition 4.3.4 *The system* (4.3.14) *has a Hamiltonian form*

$$P_{it_m} = \frac{\partial(\frac{1}{2}F_{m+1})}{\partial Q_i}, Q_{it_m} = -\frac{\partial(\frac{1}{2}F_{m+1})}{\partial P_i}, \tag{4.3.31}$$

where $F_{m+1},\ m \geq 0$ *is given by* (4.3.25).

Definition 4.3.5 As in example 4.1.9, we define the Poisson bracket of two functions F and H on the symplectic manifold $(\mathbb{R}^{4N}, \omega)$ in terms of local coordinates (i.e., $\omega = \sum_{i=1}^{2}\sum_{j=1}^{N} d\psi_{ij}\wedge d\phi_{ij} = \sum_{i=1}^{2} dQ_i \wedge dP_i$) as

$$\begin{aligned}\{F, H\} &= \sum_{i=1}^{2}\sum_{j=1}^{N}\Big(\frac{\partial F}{\partial\psi_{ij}}\frac{\partial H}{\partial\phi_{ij}} - \frac{\partial F}{\partial\phi_{ij}}\frac{\partial H}{\partial\psi_{ij}}\Big) \\ &= \sum_{i=1}^{2}\Big(\Big(\frac{\partial F}{\partial Q_i}\Big)^T\frac{\partial H}{\partial P_i} - \Big(\frac{\partial F}{\partial P_i}\Big)^T\frac{\partial H}{\partial Q_i}\Big) \\ &= \sum_{i=1}^{2}\Big(\Big\langle\frac{\partial F}{\partial Q_i}, \frac{\partial H}{\partial P_i}\Big\rangle - \Big\langle\frac{\partial F}{\partial P_i}, \frac{\partial H}{\partial Q_i}\Big\rangle\Big),\end{aligned} \tag{4.3.32}$$

where $\langle\cdot,\cdot\rangle$ still denotes the usual inner product on $\mathbb{R}^N$.

Corollary 4.3.6 *[23] If $\tilde{U}$ and $\tilde{V}^{[m]}$ satisfy the zero curvature equation $\tilde{U}_{t_m} - \tilde{V}_x^{[m]} + [\tilde{U}, \tilde{V}^{[m]}] = 0$, then the identity*

$$\tilde{W}_{t_m} = [\tilde{V}^{[m]}, \tilde{W}], \;\; m \geq 0, \tag{4.3.33}$$

holds.

It follows from the above corollary that $F_{t_m} = \frac{1}{2}(\mathrm{tr}\tilde{W}^2)_{t_m} = 0, m \geq 0$. Thus $F = \frac{1}{2}(\mathrm{tr}\tilde{W}^2)$ is also a generating function of integrals of motion for (4.3.31). Hence we have the following:

Proposition 4.3.7 *The sequence of functions $\{F_n\}_{n=0}^{\infty}$ where F_n is given by* (4.3.25) *constitutes an involutive system with respect to* (4.3.32):

$$\{F_{n+1}, F_{m+1}\} = \frac{\partial}{\partial t_m} F_{n+1} = 0, \;\; m, n \geq 0. \tag{4.3.34}$$

The following theorem concludes that the nonlinearized spatial system (4.3.13) is Liouville integrable [4] in some region of the phase space.

Theorem 4.3.8 *Let $\bar{F}_k, 1 \leq k \leq N$, be defined by*

$$\bar{F}_k = \sum_{i=1}^{2} \phi_{ik}\psi_{ik}, \;\; 1 \leq k \leq N. \tag{4.3.35}$$

Then $\bar{F}_k, 1 \leq k \leq N,\; F_n, n \geq 1$, constitute an involutive system of which $\bar{F}_k, F_n, 1 \leq k, n \leq N$ are functionally independent in some open subset of $\mathbb{R}^{4N}$.

Proof. Let $\bar{V}(\lambda_k) = (\phi_{ik}\psi_{jk})_{i,j=1,2},\; 1 \leq k \leq N$. We see that $\bar{F}_k = \mathrm{tr}(\bar{V}(\lambda_k))$ and since $\bar{V}(\lambda_k)$ satisfies the equation $\bar{V}(\lambda_k)_x = [U(\tilde{u}, \lambda_k), \bar{V}(\lambda_k)]$ (see e.g., [20] for details) we have that $\bar{F}_{kx} = \mathrm{tr}(\bar{V}(\lambda_k)_x) = \mathrm{tr}[U(\tilde{u}, \lambda_k), \bar{V}(\lambda_k)] = 0$ which shows that $\bar{F}_k, 1 \leq k \leq N$ are all integrals of motion of the nonlinearized spatial system (4.3.13).

Upon observing that

$$\frac{\partial \bar{F}_k}{\partial P_i} = (\underbrace{0,\cdots,0}_{k-1}, \psi_{ik}, \underbrace{0,\cdots,0}_{N-k})^T,\ \frac{\partial \bar{F}_k}{\partial Q_i} = (\underbrace{0,\cdots,0}_{k-1}, \phi_{ik}, \underbrace{0,\cdots,0}_{N-k})^T, i=1,2,\ 1 \le k \le N, \tag{4.3.36}$$

it can be easily shown that

$$\{\bar{F}_k, \bar{F}_l\} = \sum_{i=1}^{2} \Big(\Big(\frac{\partial \bar{F}_k}{\partial Q_i}\Big)^T \frac{\partial \bar{F}_l}{\partial P_i} - \Big(\frac{\partial \bar{F}_k}{\partial P_i}\Big)^T \frac{\partial \bar{F}_l}{\partial Q_i} \Big) = 0, \tag{4.3.37}$$

which implies that the functions $\bar{F}_k, 1 \le k \le N$ are in involution. It can also be shown that

$$\{\bar{F}_k, F_n\} = \sum_{i=1}^{2} \Big(\Big(\frac{\partial \bar{F}_k}{\partial Q_i}\Big)^T \frac{\partial F_n}{\partial P_i} - \Big(\frac{\partial \bar{F}_k}{\partial P_i}\Big)^T \frac{\partial F_n}{\partial Q_i} \Big) = 0. \tag{4.3.38}$$

This establishes the involution of $\bar{F}_k, 1 \le k \le N,\ F_n, n \ge 1$.

Next, we demonstrate that these functions $\bar{F}_k, 1 \le k \le N,\ F_n, n \ge 1$ are functionally independent in some open subset $U \subset \mathbb{R}^{4N}$. This is equivalent to showing that the $2N \times 4N$ Jacobian matrix

$$J = \frac{\partial(\bar{F}_1, \cdots, \bar{F}_k, F_1, \cdots, F_n)}{\partial(P_1, P_2, Q_1, Q_2)} \tag{4.3.39}$$

has rank $2N$, i.e., the column vectors $\{\nabla \bar{F}_k, \nabla F_n\}$ of J are linearly independent at each point of U.

Consider only the first column of J: $\Big\{\frac{\partial \bar{F}_1}{\partial P_1}, \cdots, \frac{\partial \bar{F}_n}{\partial P_1}, \frac{\partial F_1}{\partial P_1}, \cdots, \frac{\partial F_n}{\partial P_1}\Big\}$, and let

$$\sum_{k=1}^{N} \alpha_k \Big(\frac{\partial \bar{F}_k}{\partial P_1}\Big)^T + \sum_{n=1}^{N} \beta_n \Big(\frac{\partial F_n}{\partial P_1}\Big)^T = 0, \tag{4.3.40}$$

where $\alpha_k, \beta_n, 1 \leq k, n \leq N$ are constants. Now suppose U contains points such that $P_1 = P_2 = 0$ and $Q_1, Q_2 \neq 0$. To this end, we find

$$\left.\frac{\partial \bar{F}_k}{\partial P_i}\right|_{P_1=P_2=0} = (\underbrace{0, \cdots, 0}_{k-1}, \psi_{ik}, \underbrace{0, \cdots, 0}_{N-k})^T, i = 1, 2, \ 1 \leq k \leq N, \tag{4.3.41}$$

$$\left.\frac{\partial F_1}{\partial P_1}\right|_{P_1=P_2=0} = -2A^2 Q_2, \quad \left.\frac{\partial F_n}{\partial P_1}\right|_{P_1=P_2=0} = -2A^{2n} Q_2, n \geq 2. \tag{4.3.42}$$

Under the condition $P_1 = P_2 = 0$, and for $1 \leq k \leq N$, equation (4.3.40) reduces to

$$\alpha_k \psi_{1k} + \sum_{n=1}^{N} (-2\beta_n \lambda_k^{2n}) \psi_{2k} = 0. \tag{4.3.43}$$

It follows that $\alpha_k = \beta_n = 0, \ 1 \leq k, n \leq N$. The same conclusion can be be arrived at from the remaining columns in the Jacobian matrix (4.3.39). Thus, all the column vectors are linearly independent, which completes the proof.

■

4.3.4 Integrability with No Boundary Conditions

We now discuss the Liouville integrability of the systems (4.3.13) and (4.3.14) in the case where no boundary conditions are imposed on the eigenfunctions and the adjoint eigenfunctions.

Now, consider again the spectral and adjoint spectral problems

$$\begin{cases} \phi_x = U(u, \lambda)\phi, \quad \phi = (\phi_1, \phi_2)^T, \\ \psi_x = -U(u, \lambda)^T \psi, \quad \psi = (\psi_1, \psi_2)^T, \end{cases} \tag{4.3.44}$$

where U is given by equation (4.2.8). Then from equation (4.3.4), we have

$$\Psi \begin{pmatrix} \lambda\phi_1\psi_2 + \lambda\phi_2\psi_1 \\ \lambda\phi_1\psi_1 - \lambda\phi_2\psi_2 \end{pmatrix} = \lambda^2 \begin{pmatrix} \lambda\phi_1\psi_2 + \lambda\phi_2\psi_1 \\ \lambda\phi_1\psi_1 - \lambda\phi_2\psi_2 \end{pmatrix} + I \begin{pmatrix} p \\ q \end{pmatrix}, \tag{4.3.45}$$

where I is an integral of motion of (4.3.44). Applying (4.3.45) m times under the constraint (4.3.12) yields

$$\begin{pmatrix} \tilde{a}_m \\ \tilde{c}_m \end{pmatrix} = \bar{\Psi}^m \begin{pmatrix} \tilde{a}_0 \\ \tilde{c}_0 \end{pmatrix} = \sum_{i=0}^{m} I_i \begin{pmatrix} \langle A^{2m+1-2i}P_1, Q_2\rangle + \langle A^{2m+1-2i}P_2, Q_1\rangle \\ \langle A^{2m+1-2i}P_1, Q_1\rangle - \langle A^{2m+1-2i}P_2, Q_2\rangle \end{pmatrix}, m \geq 0, \tag{4.3.46}$$

where $I_0 = 1$ and $I_i, 1 \leq i \leq m$, are all integrals of motion of (4.3.13). From (4.3.20) we obtain

$$\begin{aligned} \bar{b}_m &= 2\partial^{-1}(q\bar{a}_m - p\bar{c}_m) \\ &= \sum_{i=0}^{m-1} I_i(\langle A^{2m-2i}P_1, Q_2\rangle - \langle A^{2m-2i}P_2, Q_1\rangle) + T_m,\ m \geq 1, \end{aligned} \tag{4.3.47}$$

where T_m is also an integral of motion of (4.3.13). The first two equations in (4.2.13) lead to $T_m = I_m, m \geq 1$.

Now, from $\frac{1}{2}\mathrm{tr}(\tilde{W}^2)_x = (\tilde{a}^2 - \tilde{b}^2 + \tilde{c}^2)_x = 0$ (corollary 4.3.3) and the fact that $(\frac{1}{2}\mathrm{tr}\tilde{W}^2)|_{u=0} = 1$, which imply $\tilde{a}^2 - \tilde{b}^2 + \tilde{c}^2 = 1$, we obtain the following

$$2\tilde{b}_m = \sum_{i=0}^{m-1} \tilde{a}_i\tilde{a}_{m-1-i} - \sum_{i=1}^{m-1} \tilde{b}_i\tilde{b}_{m-i} + \sum_{i=0}^{m-1} \tilde{c}_i\tilde{c}_{m-1-i},\ m \geq 2, \tag{4.3.48}$$

in a way similar to (4.3.24). Substituting (4.3.46) and (4.3.47) into (4.3.48), we get

$$\begin{aligned}
&2\sum_{i=0}^{m-1} I_i(\langle A^{2m-2i}P_1, Q_2\rangle - \langle A^{2m-2i}P_2, Q_1\rangle) + 2I_m \\
&= \sum_{i=0}^{m-1}\Big[\sum_{k=0}^{i} I_k\Big(\langle A^{2i-2k+1}P_1, Q_2\rangle + \langle A^{2i-2k+1}P_2, Q_1\rangle\Big)\Big] \times \\
&\qquad \Big[\sum_{l=0}^{m-1-i} I_l\Big(\langle A^{2m-2i-1-2l}P_1, Q_2\rangle + \langle A^{2m-2i-1-2l}P_2, Q_1\rangle\Big)\Big] \\
&- \sum_{i=1}^{m-1}\Big[\sum_{k=0}^{i-1} I_k\Big(\langle A^{2i-2k}P_1, Q_2\rangle - \langle A^{2i-2k}P_2, Q_1\rangle\Big) + I_i\Big] \times \qquad (4.3.49)\\
&\qquad \Big[\sum_{l=0}^{m-1-i} I_l\Big(\langle A^{2m-2i-2l}P_1, Q_2\rangle + \langle A^{2m-2i-2l}P_2, Q_1\rangle\Big) + I_{m-i}\Big] \\
&+ \sum_{i=0}^{m-1}\Big[\sum_{k=0}^{i} I_k\Big(\langle A^{2i-2k+1}P_1, Q_1\rangle - \langle A^{2i-2k+1}P_2, Q_2\rangle\Big)\Big] \times \\
&\qquad \Big[\sum_{l=0}^{m-1-i} I_l\Big(\langle A^{2m-2i-1-2l}P_1, Q_1\rangle - \langle A^{2m-2i-1-2l}P_2, Q_2\rangle\Big)\Big].
\end{aligned}$$

Interchanging the summation in the above equality in the following way (see, e.g., in [23, 20]):

$$\sum_{i=1}^{m-1}\sum_{k=0}^{i-1} = \sum_{k=0}^{m-2}\sum_{i=k+1}^{m-1}, \sum_{i=1}^{m-1}\sum_{l=0}^{m-i-1} = \sum_{l=0}^{m-2}\sum_{i=1}^{m-1-l}, \sum_{i=1}^{m-1}\sum_{k=0}^{i-1}\sum_{l=0}^{m-i-1} = \sum_{k=0}^{m-2}\sum_{l=0}^{(m-2)-k}\sum_{i=k+1}^{m-(l+1)}, \qquad (4.3.50)$$

we deduce that (see, e.g., in [20]):

$$I_m = -\frac{1}{2}\sum_{i=1}^{m-1} I_i I_{m-i} + \frac{1}{2}\sum_{\substack{k+l\leq m-1\\ k,l\geq 0}} I_k I_l F_{m-(k+l)}. \qquad (4.3.51)$$

From the expression $\tilde{b}_1 = \frac{1}{2}(\tilde{q}^2 + \tilde{p}^2)$, we obtain $I_1 = \frac{1}{2}F_1$, which, together with the above formula (4.3.51) gives

$$I_m = \sum_{n=1}^{m} d_n \sum_{\substack{i_1+i_2+...+i_n=m \\ i_1,...,i_n \geq 1}} F_{i_1}...F_{i_n}, m \geq 1, \tag{4.3.52}$$

where the constants d_n are given by

$$d_1 = \frac{1}{2}, d_2 = \frac{3}{8}, d_n = d_{n-1} + \frac{1}{2}\sum_{s=1}^{n-2} d_s d_{n-s-1} - \frac{1}{2}\sum_{s=1}^{n-1} d_s d_{n-s}. \tag{4.3.53}$$

Thus, by direct computation we can verify that

Proposition 4.3.9 *The temporal systems* (4.3.14) *have the Hamiltonian form*

$$P_{it_m} = \frac{\partial H_m}{\partial Q_i}, Q_{it_m} = -\frac{\partial H_m}{\partial P_i}, \tag{4.3.54}$$

where

$$H_m = \frac{1}{2}\sum_{n=0}^{m} \frac{d_n}{n+1} \sum_{\substack{i_1+i_2+...+i_n=m \\ i_1,...,i_n \geq 1}} F_{i_1}...F_{i_n}, m \geq 1. \tag{4.3.55}$$

The functions $H_m, m \geq 1$ are all functions of $F_m, m \geq 1$, and since $F_m, m \geq 1$ commute, $H_m, m \geq 1$ commute as well, which implies the commutability of the Hamiltonian phase flows $g_{H_m}^{tm}$. We thus, have the following theorem on Liouville integrability.

Theorem 4.3.10 *The spatial systems* (4.3.13) *and the temporal systems* (4.3.14) *under the control of* (4.3.14) *are all finite-dimensional Liouville integrable Hamiltonian systems with Hamiltonian functions H and H_m respectively, which are given by equation* (4.3.17) *and* (4.3.55)*, respectively. Additionally, these systems possess $2N$ involutive and independent integrals of motion: $\bar{F}_k, F_n, 1 \leq k, n \leq N$.*

4.3.5 Involutive Solutions

The above theorem leads to the following result:

Theorem 4.3.11 *The m-th soliton equation $u_{t_m} = K_m$, has the following involutive solution with separated variables x, t_m:*

$$\begin{cases} p(x,t_m) = \langle A g_H^x g_{H_m}^{tm} P_1(0,0), g_H^x g_{H_m}^{tm} Q_2(0,0)\rangle + \langle A g_H^x g_{H_m}^{tm} P_2(0,0), g_H^x g_{H_m}^{tm} Q_1(0,0)\rangle, \\ q(x,t_m) = \langle A g_H^x g_{H_m}^{tm} P_1(0,0), g_H^x g_{H_m}^{tm} Q_1(0,0)\rangle - \langle A g_H^x g_{H_m}^{tm} P_2(0,0), g_H^x g_{H_m}^{tm} Q_2(0,0)\rangle, \end{cases} \tag{4.3.56}$$

where $g_H^x, g_{H_m}^{tm}$ are the Hamiltonian phase flows [4] associated with the Hamiltonian systems (4.3.16) *and equation* (4.3.54) *respectively, and $P_i(0,0), Q_i(0,0), i = 1, 2,$ are arbitrary initial value vectors.*

Proof. Define $P_i(x,t_m) = g_H^x g_{H_m}^{t_m} P_i(0,0)$ and $Q_i(x,t_m) = g_x^H g_{H_m}^{t_m} Q_i(0,0),\ 1 \leq i \leq 2$. It follows that $P_i(x,t_m)$ and $Q_i(x,t_m), 1 \leq i \leq 2$, solve the systems (4.3.16) and (4.3.54). We know that the system (4.3.54) is equivalent to the nonlinearized temporal system (4.3.14) under the control of (4.3.13) , This implies that $P_i(x,t_m)$ and $Q_i(x,t_m)$ should also solve (4.3.13) and (4.3.14). Thus (3.47) determines a solution to $u_{t_m} = K_m$. In addition, since $\{H, H_m\} = 0$, the systems (4.3.13) and (4.3.14) are compatible and hence the Hamiltonian phase flows $g_H^x, g_{H_m}^{t_m}$ commute (i.e., $g_H^x g_{H_m}^{t_m} = g_{H_m}^{t_m} g_H^x$). This shows that the solution (4.3.56) is involutive.

■

5 Concluding Remarks

In this dissertation we have presented a spectral problem and its extension based on a modification of the well-known Dirac spectral problem. Through the so-called Tu-Ma scheme, we have obtained two soliton hierarchies with one as a subsystem of the other. We have further expressed all the equations in Hamiltonian and bi-Hamiltonian forms and have consequently proved the Liouville integrability of these equations over the real field based on Magri's scheme [25].

We have also derived a third Hamiltonian structure for the first hierarchy and shown the compatibility of the three Hamiltonian structures making this hierarchy tri-Hamiltonian. Although the existence of such a structure is not needed to prove integrability, it is still interesting due to the fact that not every integrable system is tri-Hamiltonian. Additionally, we have also derived an inverse hierarchy of commuting symmetries whose integrability, as remarked earlier, may be difficult to establish due to the nonlocalness of the equations. The success in obtaining this hierarchy is a direct consequence of the existence of the inverse of the recursion operator. For many soliton hierarchies, the inverse of the recursion operator may not exist and as a result these hierarchies may not have an inverse hierarchy of commuting symmetries. Examples of such hierarchies include the Wadati-Konno-Ichikawa hierarchy and the Jaulent-Miodek hierarchy [32].

Furthermore, we have presented finite-dimensional Hamiltonian systems by means of the Bargmann symmetry constraint and discussed their integrability in terms of the existence of integrals of motion. Most importantly, we have presented involutive solutions to the nonlinear evolution equations in the soliton hierarchy based on the integrability of the finite-dimensional integrable systems.

Finally, we remark that except for the inverse hierarchy, all equations generated are local although the recursion operators for the hierarchies involve nonlocal terms. This is due to the locality property of the functions $\{a_i, b_i, c_i | i \geq 1\}$. Nonlocality is a troubling element in the theory of integrable systems as many recursion operators usually involve nonlocal terms. One may see the recent paper by De Sole, Kac and Valeri [6] to realize the disturbing nature of nonlocality.

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