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Putnam's Inequality and Analytic Content in the Bergman Space

Matthew Fleeman
University of South Florida, mcfleema@mail.usf.edu

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by

Matthew C. Fleeman

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University of South Florida

Major Professor: Dmitry Khavinson, Ph.D.
Catherine Bénéteau, Ph.D.
Sherwin Kouchekian, Ph.D.
Razvan Teodorescu, Ph.D.

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Dedication

I would like to dedicate this dissertation to my parents. This has been a long time in the making, and they have never stopped encouraging and supporting me.

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Finally, I would like to thank Jan-Fredrick Olsen. He first suggested the main equality in chapter four during our conversations at the Mittag-Leffler institute in 2015.

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Abstract

In this dissertation we are interested in studying two extremal problems in the Bergman space. The topics are divided into three chapters.

In Chapter 2, we study Putnam's inequality in the Bergman space setting. In [32], the authors showed that Putnam's inequality for the norm of self-commutators can be improved by a factor of $\frac{1}{2}$ for Toeplitz operators with analytic symbol φ acting on the Bergman space $A^2(\Omega)$. This improved upper bound is sharp when $\varphi(\Omega)$ is a disk. We show that disks are the only domains for which the upper bound is attained

In Chapter 3, we consider the problem of finding the best approximation to $\bar{z}$ in the Bergman space $A^2(\Omega)$. We show that this best approximation is the derivative of the solution to the Dirichlet problem on $\partial\Omega$ with data $|z|^2$ and give examples of domains where the best approximation is a polynomial, or a rational function.

Finally, in Chapter 4 we study Bergman analytic content, which measures the $L^2(\Omega)$-distance between $\bar{z}$ and the Bergman space $A^2(\Omega)$. We compute the Bergman analytic content of simply connected quadrature domains with quadrature formula supported at one point, and we also determine the function $f \in A^2(\Omega)$ that best approximates $\bar{z}$. We show that, for simply connected domains, the square of Bergman analytic content is equal to torsional rigidity from classical elasticity theory, while for multiply connected domains these two domain constants are not equivalent in general.

Chapter 1
Introduction

In this dissertation we study two extremal problems in the Bergman space. In both cases we consider problems which have been studied in the context of other analytic function spaces, and examine them in the Bergman space setting. We let $\mathbb{C}$ denote the complex plane. For a bounded domain $\Omega \subset \mathbb{C}$, the Bergman space $A^2(\Omega)$ is defined by:

$$A^2(\Omega) := \{f \in \mathrm{Hol}(\Omega) : \|f\|^2_{A^2(\Omega)} = \int_\Omega |f(z)|^2 dA(z) < \infty\},$$

where dA denotes the area measure on Ω.

Chapter 2 concerns the study of self-commutators acting on the Bergman space and is based on the paper [17], which has been published in Complex Analysis and Operator Theory.

Chapter 3 studies the best approximation to $\bar{z}$ in $A^2(\Omega)$, and is based on [18], which has been accepted for publication.

Chapter 4 studies the Bergman analytic content of Ω, and is based on [19], which has been submitted for publication.

1.1 Self-Commutators

Let $\mathcal{H}$ be a complex, separable Hilbert space and $T : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator on $\mathcal{H}$. The self-commutator of T is defined by

$$[T^*, T] := T^*T - TT^*,$$

where T^* is the adjoint of T. We say that the operator T is *positive* if $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{H}$ (cf. [36, p. 330]), and that T is *hyponormal* if $[T^*, T]$ is positive (cf. [10, p. 46]). Recall that $\lambda \in \mathbb{C}$ is in $sp(T)$, the *spectrum of* T, if $T - \lambda I$ is not invertible (cf. [36, p. 104]). The celebrated Putnam inequality (cf. [5] and [34]) states that if T is hyponormal, then

$$\|[T^*, T]\| \leq \frac{\mathrm{Area}(sp(T))}{\pi}.$$

Let φ be in $H^{\infty}(\Omega)$, the space of bounded analytic functions. The *Toeplitz operator with symbol* φ, denoted by T_{φ}, is given by

$$T_{\varphi}f = \varphi f,$$

and is a bounded hyponormal operator on $A^2(\Omega)$. By the Spectral Mapping Theorem (cf. [36, p. 263]), if φ is analytic in Ω, then $sp(T_{\varphi}) = \overline{\varphi(\Omega)}$.

A function f analytic in Ω is said to belong to the Smirnov space $E^2(\Omega)$ if there is a sequence of rectifiable Jordan curves $\{\Gamma_n\}_{n=0}^{\infty} \subset \Omega$ tending to $\Gamma = \partial\Omega$ such that

$$\int_{\Gamma_n} |f(z)|^2 \, |dz| \leq M < \infty,$$

with $\|f\|^2_{E^2(\Omega)} = \int_{\Gamma} |f(z)|^2 \, |dz|$ (cf. [14, p. 168]). In [25], D. Khavinson studied the norms of self-commutators of Toeplitz operators acting on the Smirnov space. There it was shown that the following lower bound holds:

$$\|[T_{\varphi}^*, T_{\varphi}]\| \geq \frac{4\text{Area}(\varphi(\Omega))^2}{\|\varphi'\|^2_{E^2(\Omega)} \cdot \text{Per}(\Omega)}, \tag{1.1.1}$$

where $\text{Per}(\Omega)$ denotes the perimeter of the boundary of Ω.

Since $[T_{\varphi}^*, T_{\varphi}]$ is a positive operator, an interesting consequence of (1.1.1) follows by setting $\varphi(z) = z$, so that $\|\varphi'\|^2_{E^2(\Omega)} = \|1\|^2_{E^2(\Omega)} = \text{Per}(\Omega)$, and combining (1.1.1) with Putnam's inequality, we obtain

$$\text{Per}(\Omega)^2 \geq 4\pi Area(\Omega), \tag{1.1.2}$$

which is the classical isoperimetric inequality. The equality in (1.1.2) holds if and only if Ω is a disk, and consequently shows that Putnam's inequality is sharp in the Smirnov space setting.

In Chapter 2 we examine self-commutators acting on the Bergman space. In particular, we examine $\left\|[T_{\varphi}^*, T_{\varphi}]\right\|$ when T_{φ} acts on $A^2(\Omega)$ and φ is univalent in Ω. We show that in such cases, Putnam's inequality can be improved by a factor of $\frac{1}{2}$. This upper bound is sharp, and is achieved if and only if $\varphi(\Omega)$ is a disk.

1.2 Torsional Rigidity

Throughout this dissertation, several of our results will be connected to the notion of torsional rigidity (cf. [33]). There are several equivalent definitions (cf. [6, pp.63-66] and [33, pp. 87-89]). If Ω is a simply connected domain, the *torsional rigidity of* Ω, $\rho = \rho(\Omega)$, is

$$\rho = 2\int_{\Omega} \nu dA,$$

where ν is the unique solution to the Dirichlet problem

$$\begin{cases} \Delta\nu &= -2 \\ \nu|_{\partial\Omega} &= 0. \end{cases}$$

For multiply connected domains, the following definition comes from [6, pp. 63-66]. If Ω is multiply connected, bounded by finitely many Jordan curves $\Gamma_0, \ldots, \Gamma_n$, with Γ_0 being the outer boundary curve, then

$$\rho(\Omega) := \int_\Omega |\nabla\nu|^2 \, dA, \tag{1.2.1}$$

where ν solves the Dirichlet problem

$$\begin{cases} \Delta\nu = -2 \quad \text{in } \Omega \\ \nu|_{\Gamma_0} = 0 \\ \nu|_{\Gamma_i} = c_i \quad i = 1, \ldots, n, \end{cases}$$

where the constants c_i are not known *a priori* but are determined by the conditions

$$\int_{\Gamma_i} \partial_n \nu ds = 2a_i, \qquad i = 1, \ldots, n,$$

where ∂_n denotes differentiation in the direction of the outward normal, ds is the arclength element, and a_i is the area enclosed by Γ_i. Note that these definitions coincide for simply connected domains. The function ν is referred to as the “Prandtl stress function” in elasticity theory, and this is known as the "St. Venant torsion theory". Intuitively, if we imagine a cylindrical object with cross-section Ω, then the torsional rigidity measures the resistance to twisting.

1.3 Analytic Content

Let $K \subset \mathbb{C}$ be compact. The space $R(K)$ is the uniform closure of the space of rational functions whose poles lie off K. In [24], D. Khavinson studied the question of “how far” $\overline{z}$ is from $R(K)$. The analytic content of K, denoted by $\lambda(K)$, is defined by:

$$\lambda(K) := \inf_{f \in R(K)} \|\bar{z} - f\|_\infty .$$

The extremal function $f \in R(K)$ for which $\lambda(K)$ is attained is called the best approximation to $\bar{z}$ in $R(K)$. The author proved in [24] the following “isoperimetric sandwich”:

$$\frac{2\text{Area}(\Omega)}{\text{Per}(\Omega)} \leq \lambda(\Omega) \leq \sqrt{\frac{\text{Area}(\Omega)}{\pi}}.$$

Here the upper bound is due to Alexander (cf. [2]), and the lower bound is due to Khavinson (cf. [9],[20], and [24]).

In Chapter 3, we will study the best approximation to $\bar{z}$ in $A^2(\Omega)$. In Section 3.1 we characterize the best approximation to $\bar{z}$ as the derivative of the solution to the Dirichlet problem on Γ with data $|z|^2$. This shows an interesting connection between the Dirichlet problem and the Bergman projection. Recently in [29], A. Legg noted independently another such connection via the Khavinson-Shapiro conjecture. (Recall that the latter conjecture states that ellipsoids are the only domains where the solution to the Dirichlet problem with polynomial data is always a polynomial, cf. [30] and [35]. In [29, Proposition 2.1], the author showed that in the plane this happens if and only if the Bergman projection maps polynomials to polynomials.) In Section 3.2 we look at specific examples. In particular, we look at domains for which the best approximation is a monomial Cz^k, some examples where the best approximation is a rational function with simple poles, as well as examples where the best approximation is a rational function with non-simple poles. In Chapter 4, we study Bergman analytic content, denoted by $\lambda_{A^2}(\Omega)$. We show that when Ω is simply connected, then $\lambda_{A^2}(\Omega) = \sqrt{\rho(\Omega)}$. In Section 4.3, we show that for multiply connected domains this equality fails in general.

1.4 Quadrature Domains

A bounded domain $\Omega \subset \mathbb{C}$ is called a *quadrature domain* if it admits a formula expressing the area integral of any test function $g \in A^2(\Omega)$ as a finite sum of weighted point evaluations of g and its derivatives:

$$\int_D g(z)dA(z) = \sum_{m=1}^{N}\sum_{k=0}^{n_m} a_{m,k}g^{(k)}(z_m), \tag{1.4.1}$$

where the points $z_m \in \Omega$ and constants $a_{m,k}$ are fixed and independent of g. A simply connected domain Ω is a quadrature domain if and only if the conformal map $\phi : \mathbb{D} \to \Omega$ is a rational function. (Cf. [37, pp.17-19] for a quick background on quadrature domains.)

In Chapter 4, we give an explicit calculation of $\lambda_{A^2}(\Omega)$ when Ω is a quadrature domain whose conformal map from the disk is a polynomial. In Section 4.3 we look at specific examples.

Chapter 2

Self-Commutators Acting on the Bergman Space

Recall that if T is a hyponormal operator, then Putnam's inequality states

$$\|[T^*, T]\| \leq \frac{Area(sp(T))}{\pi},$$

where $sp(T)$ denotes the spectrum of T (cf. [5]).

This inequality is sharp, as was shown by Khavinson in [25]. We are interested in whether this inequality is sharp in the context of the Bergman space. In Section 2.1, we show that Putnam's inequality can only be sharp when $sp(T)$ is a disk.

Recall that the Toeplitz operator with symbol φ, denoted by T_φ, is given by

$$T_\varphi f = \varphi f.$$

In [8], Bell, Ferguson, and Lundberg showed that if T_z acts on the Bergman space, $A^2(\Omega)$, then

$$\|[T_z^*, T_z]\| \geq \frac{\rho(\Omega)}{\text{Area}(\Omega)}.$$

The authors also conjectured that in the Bergman space setting, Putnam's inequality could be improved by a factor of $\frac{1}{2}$ for Toeplitz operators with analytic symbol φ. This conjecture was recently proven by Olsen and Reguera in [32] for univalent φ. Combined with the lower bound given by Bell, Ferguson, and Lundberg, this yields a new proof of the St. Venant inequality

$$\rho(\Omega) \leq \frac{\text{Area}(\Omega)^2}{2\pi}.$$

In Section 2.2, we give a sketch of Olsen and Reguera's proof of the improved upper bound in the Bergman setting for self-commutators of Toeplitz operators with symbol φ univalent in Ω. This is needed for our argument in Section 2.3, where we show that the upper bound for $\|[T_\varphi^*, T_\varphi]\|$ is achieved if and only if $\varphi(\Omega)$ is a disk. This gives another proof, similar in spirit to that of Davenport in [33, pp. 121-136], of the well known fact that St. Venant's inequality becomes equality only for disks.

2.1 Non-Sharpness of Putnam's Inequality

In this Section, we illustrate why Putnam's inequality is not sharp in a Bergman space setting. We start with the following elementary Lemma found in [15, p.13].

Lemma 2.1.1. *Suppose $\omega = \varphi(z)$ maps a domain D conformally onto a domain Ω. Then the linear map $T(f) = g$ defined by*

$$g(z) = f(\varphi(z))\varphi'(z)$$

defines an isometry of $A^2(\Omega)$ onto $A^2(D)$.

Proof. That T is an isometry is clear from the fact that

$$\int_{\Omega} |f(\omega)|^2 dA(\omega) = \int_{D} |f(\varphi(z))|^2 |\varphi'(z)|^2 dA(z),$$

where $|\varphi'(z)|^2$ is the Jacobian of the conformal map φ.

To see that T is onto, let $g \in A^2(D)$ and let $z = \psi(\omega)$ be the inverse mapping. Then $f(\omega) = g(\psi(\omega))\psi'(\omega)$ is in $A^2(\Omega)$ and $T(f) = g$ since

$$T(f) = f(\varphi(z))\varphi'(z) = g(\psi(\varphi(z))\varphi'(\psi(\omega))\psi'(\omega),$$

and we can write $\psi'(\omega) = \frac{1}{\varphi'(\psi(\omega))}$, which is well defined on D because $\varphi'|_D \neq 0$. So $T(f) = g$ and T is onto as claimed. ☐

The following statement is now straightforward.

Theorem 2.1.2. *Suppose Ω is a bounded Jordan domain and $\varphi : \mathbb{D} \to \Omega$ is a conformal mapping. Then*

$$\|[T_\varphi^*, T_\varphi]\|_{A^2(\mathbb{D}) \to A^2(\mathbb{D})} = \|[T_z^*, T_z]\|_{A^2(\Omega) \to A^2(\Omega)}.$$

Proof. We start with the following straightforward calculation (cf. [5]). If we take $A_1^2(\mathbb{D})$ to be the unit ball of $A^2(\mathbb{D})$, we have that

$$\begin{aligned}\|[T_\varphi^*, T_\varphi]\| &= \sup_{f \in A_1^2(\mathbb{D})} \langle [T_\varphi^*, T_\varphi] f, f \rangle \\ &= \sup_{f \in A_1^2(\mathbb{D})} \left(\|T_\varphi f\|_{A^2(\mathbb{D})}^2 - \|T_\varphi^* f\|_{A^2(\mathbb{D})}^2 \right) \\ &= \sup_{f \in A_1^2(\mathbb{D})} \left(\|\varphi f\|_{A^2(\mathbb{D})}^2 - \|P(\bar{\varphi} f)\|_{A^2(\mathbb{D})}^2 \right)\end{aligned}$$

$$= \sup_{f \in A_1^2(\mathbb{D})} \left(\|\varphi f\|_{L^2(\mathbb{D})}^2 - \|P(\bar{\varphi} f)\|_{A^2(\mathbb{D})}^2 \right).$$

Thus we have that

$$\|[T_\varphi^*, T_\varphi]\| = \sup_{g \in A_1^2(\mathbb{D})} \left(\|\varphi f\|_{L^2(\mathbb{D})}^2 - \|P(\bar{\varphi} g)\|_{A^2(\mathbb{D})}^2 \right)$$

$$= \sup_{g \in A_1^2(\mathbb{D})} \left\{ \left\{ \inf_{f \in A^2(\mathbb{D})} \|\bar{\varphi} g - f\|_{L^2(\mathbb{D})}^2 \right\} \right\}.$$

Fixing $f, g \in A^2(\mathbb{D})$, with $g \in A_1^2(\mathbb{D})$, and letting $\psi = \varphi^{-1}$, we see that

$$\|\bar{\varphi} g - f\|_{L^2(\mathbb{D})}^2 = \int_{\mathbb{D}} |\bar{\varphi} g - f|^2 dA$$

$$= \int_\Omega |\bar{z} g(\psi(\omega)) - f(\psi(\omega))|^2 |\psi'(\omega)|^2 dA(\omega)$$

$$= \int_\Omega |\bar{z} g(\psi(\omega))\psi'(\omega) - f(\psi(\omega))\psi'(\omega)|^2 dA(\omega).$$

By Lemma 2.1.1, $T(f) = f(\psi(\omega))\psi'(\omega)$ is a surjective isometry from $A^2(\mathbb{D})$ onto $A^2(\Omega)$. So, we have that

$$\sup_{g \in A_1^2(\mathbb{D})} \left\{ \inf_{f \in A^2(\mathbb{D})} \left\{ \|\bar{\varphi} g - f\|_{L^2(\mathbb{D})}^2 \right\} \right\} = \sup_{g \in A_1^2(\Omega)} \left\{ \left\{ \inf_{f \in A^2(\Omega)} \|\bar{z} g - f\|_{L^2(\Omega)}^2 \right\} \right\},$$

and the proof is complete. □

This leads to the following interesting observation.

Theorem 2.1.3. *Let φ and Ω be as in Theorem 2.1.2. Then $\|[T_\varphi^*, T_\varphi]\|$ can only achieve the upper bound stated in Putnam's inequality (cf. [34]) if $\varphi(\mathbb{D})$ is a disk.*

Proof. Let $A_1^2(\Omega)$ be the unit ball in $A^2(\Omega)$. By Theorem 2.1.2, we have that

$$\|[T_\varphi^*, T_\varphi]\|_{A^2(\mathbb{D}) \to A^2(\mathbb{D})} = \|[T_z^*, T_z]\|_{A^2(\Omega) \to A^2(\Omega)} = \sup_{g \in A_1^2(\Omega)} \left\{ \left\{ \inf_{f \in A^2(\Omega)} \|\bar{z} g - f\|_{L^2(\Omega)}^2 \right\} \right\}.$$

Fix $g \in A_1^2(\Omega)$, we have

$$\inf_{f \in A^2(\Omega)} \|\bar{z} g - f\|_{L^2(\Omega)}^2 = \inf_{f \in A^2(\Omega)} \int_\Omega |\bar{z} g - f|^2 dA$$

$$\leq \inf_{h:\, gh \in A^2(\Omega)} \int_\Omega |\bar{z} - f|^2 |g|^2 dA \leq \inf_{h:\, gh \in A^2(\Omega)} \|\bar{z} - h\|_\infty^2$$

since $g \in A^2_1(\Omega)$. Further, since the polynomials $\mathcal{P}$ are dense in $H^\infty(\Omega)$ for any bounded Jordan domain Ω, and since for all $g \in A^2_1(\Omega)$, and all $p \in \mathcal{P}$, we have that $gp \in A^2(\Omega)$, we obtain from the last inequality that

$$\inf_{f \in A^2(\Omega)} \|\bar{z}g - f\|^2_{L^2(\Omega)} \leq \inf_{h \in R(\overline{\Omega})} \|\bar{z} - h\|^2_{L^\infty(\Omega)}$$

where $R(\overline{\Omega})$ is the uniform closure of the algebra of rational functions in Ω with poles outside $\overline{\Omega}$. In [2], Alexander proved that

$$\inf_{f \in R(\overline{\Omega})} \|\bar{z} - f\|_{L^\infty(\Omega)} \leq \sqrt{\frac{Area(\Omega)}{\pi}},$$

and further that equality is achieved if and only if Ω is a disk (cf. [5, 20]). The theorem now immediately follows. □

Remark. If we take $\mathcal{H}$ to be any Hilbert space and T to be any subnormal operator with a rationally cyclic vector, then there is a positive finite Borel measure μ on $sp(T)$ such that T is unitarily equivalent to multiplication by z on $R^2(sp(T),\, \mu)$ which is the closure of $R(sp(T))$ in $L^2(sp(T),\, \mu)$ (cf. [5]). From this, repeating the above argument word for word, we obtain that if

$$\|[T^*, T]\| = \frac{Area(sp(T))}{\pi},$$

then $sp(T)$ must be a disk. The case when T does not have a rationally cyclic vector follows from the above case as in [5], so that the above theorem extends to all Hilbert spaces and any subnormal operator T.

The following example shows that the converse fails, and in particular fails for Bergman spaces.

Example 2.1.4. Let $\varphi(z) = z^k$ for some $k \in \mathbb{N}$, and let $T_\varphi : A^2(\mathbb{D}) \to A^2(\mathbb{D})$, and recall that $P : L^2(\mathbb{D}) \to A^2(\mathbb{D})$ is the orthogonal projection of $L^2(\mathbb{D})$ onto $A^2(\mathbb{D})$. As we showed in Theorem 2.1.2,

$$\|[T^*_\varphi, T_\varphi]\| = \sup_{g \in A^2_1(\mathbb{D})} \left(\|\varphi g\|^2_{L^2(\mathbb{D})} - \|P(\bar{\varphi} g)\|^2_{A^2(\mathbb{D})} \right).$$

Let $\psi_n(z) = (\frac{n+1}{\pi})^{\frac{1}{2}} z^n$, where $n = 0, 1, 2, \ldots$. The collection $\{\psi_n(z)\}_{n=0}^\infty$ forms an orthonormal basis for $A^2(\mathbb{D})$ (cf. [15, p. 11]). For $g \in A^2_1(\mathbb{D})$, we can write

$$g(z) = \sum_{n=0}^\infty \hat{g}(n) \psi_n(z),$$

where $\hat{g}(n) := \langle g, \psi_n \rangle$ and $\sum_{n=0}^\infty |\hat{g}(n)|^2 = 1$. Since we have an orthonormal basis at hand, we can calculate $P(\bar{\varphi} g)$ explicitly:

$$P(\bar{z}^k g) = \sum_{n=0}^\infty \langle \bar{z}^k g, \psi_n \rangle \psi_n.$$

Calculating $\langle \bar{z}^k g, \psi_n \rangle$, we find that

$$\langle \bar{z}^k g, \psi_n \rangle = \langle \bar{z}^k \sum_{m=0}^{\infty} \hat{g}(m) \psi_m, \psi_n \rangle,$$

where

$$\begin{aligned} \langle \bar{z}^k \hat{g}(m) \psi_m, \psi_n \rangle &= \int_{\mathbb{D}} \frac{\sqrt{(m+1)(n+1)}}{\pi} \hat{g}(m) z^m \bar{z}^{n+k} dA \\ &= \frac{\sqrt{(m+1)(n+1)}}{\pi} \hat{g}(m) \frac{2\pi}{m+n+k+2} \delta_{m,n+k}, \end{aligned}$$

where $\delta_{i,j}$ is the Kronecker symbol. Thus,

$$\langle \bar{z}^k g, \psi_n \rangle = \left(\frac{n+1}{n+k+1} \right)^{\frac{1}{2}} \hat{g}(n+k),$$

and so we obtain that

$$\|P(\bar{z}^k g)\|^2_{A^2(\mathbb{D})} = \sum_{n=0}^{\infty} \frac{n+1}{n+k+1} |\hat{g}(n+k)|^2. \tag{2.1.1}$$

Similarly, when we calculate $\|z^k g\|^2_{A^2(\mathbb{D})}$, we find that

$$\begin{aligned} \langle z^k \hat{g}(m) \psi_m, \psi_n \rangle &= \int_{\mathbb{D}} \frac{\sqrt{(m+1)(n+1)}}{\pi} \hat{g}(m) z^{m+k} \bar{z}^{n} dA \\ &= \frac{\sqrt{(m+1)(n+1)}}{\pi} \hat{g}(m) \frac{2\pi}{m+n+k+2} \delta_{m+k,n}. \end{aligned}$$

Thus

$$\langle z^k g, \psi_n \rangle = \begin{cases} \sqrt{\frac{n-k+1}{n+1}} \hat{g}(n-k) & n \geq k \\ 0 & n < k. \end{cases}$$

Hence,

$$\|z^k g\|^2_{L^2(\mathbb{D})} = \sum_{n=k}^{\infty} \frac{n-k+1}{n+1} |\hat{g}(n-k)|^2. \tag{2.1.2}$$

Combining (2.1.1) and (2.1.2), we obtain that

$$\begin{aligned} \|[T^*_\varphi, T_\varphi]\| &= \sup_{g \in A^2_1(\mathbb{D})} \left\{ \sum_{n=k}^{\infty} \frac{n-k+1}{n+1} |\hat{g}(n-k)|^2 - \sum_{n=0}^{\infty} \frac{n+1}{n+k+1} |\hat{g}(n+k)|^2 \right\} \\ &= \sup_{g \in A^2_1(\mathbb{D})} \left\{ \sum_{n=0}^{k-1} \frac{n+1}{n+k+1} |\hat{g}(n)|^2 + \sum_{n=k}^{\infty} \left(\frac{n+1}{n+k+1} - \frac{n-k+1}{n+1} \right) |\hat{g}(n)|^2 \right\} \\ &\leq \sup_{g \in A^2_1(\mathbb{D})} \left\{ \sum_{n=0}^{k-1} \frac{n+1}{n+k+1} |\hat{g}(n)|^2 + \sum_{n=k}^{\infty} \frac{k}{n+k+1} |\hat{g}(n)|^2 \right\} \end{aligned}$$

since

$$\frac{n+1}{n+k+1} - \frac{n-k+1}{n+1} \leq \frac{n+1}{n+k+1} - \frac{n-k+1}{n+k+1} = \frac{k}{n+k+1}, \qquad k \geq 0.$$

Further, since $\frac{n+1}{n+k+1} \leq \frac{k}{2k}$ for $0 \leq n \leq k-1$, we obtain that

$$\|[T_\varphi^*, T_\varphi]\| \leq \sup_{g \in A_1^2(\mathbb{D})} \frac{k}{2k} \sum_{n=0}^{\infty} |\hat{g}(n)|^2 = \frac{1}{2}.$$

This upper bound is achieved if we take $g = \psi_{k-1}$, so that $\|[T_\varphi^*, T_\varphi]\| = \frac{1}{2}$, whenever $\varphi(z) = z^k$ for any $k \in \mathbb{N}$. Thus, we see that the converse to Theorem 2.1.3 fails.

This calculation, independently done by T. Ferguson, leads to the conjecture, following Bell et. al. that in the Bergman space setting, Putnam's inequality can be improved by a factor of $\frac{1}{2}$. This conjecture was proven in 2013 by Olsen and Reguera in [32]. In the following section we give a sketch of their proof which will be needed in Section 2.3.

2.2 Olsen-Reguera Theorem

In their paper, Olsen and Reguera worked with the Hankel operator on $A^2(\mathbb{D})$ with symbol $\varphi \in L^2(\mathbb{D})$ defined by

$$H_\varphi(f) := (I - P)(\varphi f), \qquad f \in A^2(\mathbb{D}).$$

They proved the following theorem.

Theorem 2.2.1. *Let $\varphi \in A^2(\mathbb{D})$ be in the Dirichlet space $\mathscr{D}$, i.e., $\varphi' \in A^2(\mathbb{D})$. Then*

$$\|H_{\bar{\varphi}}\| \leq \frac{1}{\sqrt{2}} \|\varphi'\|_{A^2(\mathbb{D})}.$$

Proof. For the reader's convenience, we give here a sketch of their proof. For full details, cf. [32, Section 2]. For $f \in A^2(\mathbb{D})$,we write $f(z) = \sum_{n \geq 0} a_n z^n$, and without loss of generality we assume that $\|f\|_{A^2(\mathbb{D})} = 1$, and set $\varphi(z) = \sum_{k \geq 1} c_k z^k$. (We can also assume without loss of generality that $\varphi(0) = 0$.) The basic strategy is to calculate $H_{\bar{\varphi}} f$ in terms of these Taylor coefficients and obtain the desired norm estimate by working directly with the coefficients. Crucial to our purposes is the fact that the only inequality used in [32] is the arithmetic-geometric inequality $ab \leq \frac{a^2+b^2}{2}$.

First, by computing $P(\bar{\varphi} z^n)$ for each n, we find that

$$H_{\bar{\varphi}} f = \bar{\varphi}(z) f(z) - P(\overline{\varphi} f)(z)$$

$$= \sum_{l \geq 0} \sum_{n \geq 0} \overline{c_l} a_n \overline{z}^l z^n - \sum_{n \geq 1} \sum_{k=0}^{n-1} \frac{k+1}{n+1} a_n \overline{c_{n-k}} z^k. \tag{2.2.1}$$

Then, after rewriting the above expression to take advantage of the orthogonality of monomials, we let $z = re^{i\theta}$ and integrate the modulus squared with respect to $\frac{d\theta}{\pi}$. This yields that $\|H_{\bar{\varphi}} f\|^2_{A^2(\mathbb{D})}$ is equal to

$$2 \sum_{k \geq 1} r^{2k} \left| \sum_{n \geq 0} a_n \overline{c_{n+k}} r^{2n} \right|^2 + 2 \sum_{k \geq 0} r^{2k} \left| \sum_{n \geq k+1} a_n \overline{c_{n-k}} (r^{2(n-k)} - \frac{k+1}{n+1}) \right|^2. \tag{2.2.2}$$

This expression is once again rewritten and then integrated with respect to rdr. If we set

$$(I) := \sum_{n,m \geq 1, k \geq 0} \frac{a_n \overline{a_m} c_{k+m} \overline{c_{k+n}}}{n+m+k+1},$$

$$(II) := \sum_{k \geq 0} \sum_{n,m \geq k+1} \frac{a_n \overline{a_m} c_{m-k} \overline{c_{n-k}} (m-k)(n-k)}{(n+1)(m+1)(n+m-k+1)},$$

then we obtain that

$$\|H_{\bar{\varphi}} f\|^2_{A^2(\mathbb{D})} = (I) + (II).$$

Relabeling the indices, and setting $a_n = b_{n+1}(n+1)$, we find that

$$(I) = \sum_{n,m \geq 1, k \geq 0} b_n \overline{b_m} c_{k+m} \overline{c_{k+n}} \frac{nm}{n+m+k},$$

$$(II) = \sum_{n,m,k \geq 1} b_{n+k} \overline{b_{m+k}} c_m \overline{c_n} \frac{mn}{n+m+k}.$$

Using the symmetry in m and n we may interpret each term as being half that of its real part so that the inequality $2Re(ab) \leq |a|^2 + |b|^2$ may be applied to each term of the above expressions, and this is the only place where inequalities occur, which yields

$$(I) \leq \sum_{n,m \geq 1, k \geq 0} \left(|b_n c_{k+m}|^2 + |b_m c_{k+n}|^2 \right) \frac{nm}{2(n+m+k)} = \sum_{n,m \geq 1, k \geq 0} |b_n c_{k+m}|^2 \frac{nm}{n+m+k} =: (I_*),$$

$$(II) \leq \sum_{n,m,k \geq 1} \left(|b_{n+k} c_m|^2 + |b_{m+k} c_n|^2 \right) \frac{nm}{2(n+m+k)} = \sum_{n,m,k \geq 1} |b_{n+k} c_m|^2 \frac{mn}{m+n+k} =: (II_*).$$

By changing the order of summation we arrive, at the expression

$$(I_*) + (II_*) = \sum_{n,m \geq 1} |b_n|^2 |c_m|^2 \frac{nm}{2}.$$

Finally, replacing a_n by $b_{n+1}(n+1)$, we now see that the right hand side exactly equals

$$\frac{1}{2} \sum_{n,m \geq 0} |b_n|^2 |c_m|^2 mn = \frac{1}{2} \left(\sum_{n \geq 0} \frac{|a_n|^2}{n+1} \right) \left(\sum_{m \geq 1} |c_m|^2 m \right) = \frac{1}{2} \|f\|^2_{A^2(\mathbb{D})} \|\varphi'\|^2_{A^2(\mathbb{D})}.$$

which is what was to be shown. $\square$

Remark. From here, the inequality

$$\|[T_\varphi^*, T_\varphi]\| \leq \frac{\|\varphi'\|^2_{A^2(\Omega)}}{2} \tag{2.2.3}$$

is seen as a corollary by showing that if ψ is the conformal map from Ω to $\mathbb{D}$, then

$$\|[T_\varphi^*, T_\varphi]\|_{A^2(\Omega)\to A^2(\Omega)} = \|H_{\bar\varphi}\|^2_{A^2(\Omega)\to L^2(\Omega)} = \|H_{\overline{\varphi\circ\psi}}\|^2_{A^2(\mathbb{D})\to L^2(\mathbb{D})},$$

and thus we can apply Theorem 2.2.1, and the result follows. (Refer to [32] for more details.) Taking $\varphi(z) = z$, and combining (2.2.3) with the result of Bell, Ferguson, and Lundberg, one arrives at a proof of the sharp St. Venant inequality

$$\rho(\Omega) \leq \frac{Area^2(\Omega)}{2\pi}. \tag{2.2.4}$$

It should be noted that when $\varphi = z^k$, many of the terms in (2.2.2) become zero resulting in the value we found in Example 2.1.4 of $\frac{1}{2}$ rather than the Olsen-Reguera upper bound of $\frac{k}{2}$.

2.3 Unique Extremality of the Disk

We now show that from the proof of Theorem 2.2.1, we may deduce that equality is obtained in (2.2.3) only if $\varphi(\Omega)$ is a disk. This will come as a corollary to the following Theorem.

Theorem 2.3.1. *Suppose $\varphi(z)$ is analytic in $\mathbb{D}$ such that $\varphi(z) \in \mathscr{D}$, the Dirichlet space. Further suppose that*

$$\|[T_\varphi^*, T_\varphi]\|_{A^2(\mathbb{D})\to A^2(\mathbb{D})} = \frac{\|\varphi'\|^2_{A^2(\Omega)}}{2}.$$

Then $\varphi(\mathbb{D})$ is a disk.

Proof. Since $\varphi \in \mathscr{D}$, $H_{\overline{\varphi}}$ is compact (cf.[39, p.145]), and so attains its norm on $A^2_1(\mathbb{D})$. Recall from the proof of Theorem 2.1.2 that

$$\|[T_\varphi^*, T_\varphi]\|_{A^2(\mathbb{D})\to A^2(\mathbb{D})} = \left(\sup_{f\in A^2_1(\mathbb{D})} \|\varphi f\|^2_{L^2(\mathbb{D})} - \|P(\bar\varphi f)\|^2_{A^2(\mathbb{D})}\right)$$

$$= \|H_{\bar\varphi}\|^2_{A^2(\mathbb{D})\to L^2(\mathbb{D})}.$$

We now examine the proof of Theorem 2.2.1 to find out exactly when equality may happen. Recall that if $f \in A^2_1(\mathbb{D})$, then

$$\|H_{\bar\varphi} f\|^2_{A^2(\mathbb{D})} = (I) + (II) \leq (I_*) + (II_*) = \frac{1}{2}\|f\|^2_{A^2(\mathbb{D})}\|\varphi'\|^2_{A^2(\mathbb{D})},$$

where (I), (II), (I_*), and (II_*) are as in Theorem 2.2.1. The only inequality at work here is $2Re(ab) \leq |a|^2 + |b|^2$, where equality is achieved if, and only if, $a = \bar{b}$. Thus we find that equality is achieved if $(I) = (I_*)$ and $(II) = (II_*)$, which will only happen if the following infinite system of equations is satisfied:

$$b_i c_{j+k} = b_j c_{i+k} \qquad i, j \geq 1, k \geq 0, \tag{2.3.1}$$

$$b_{i+k} c_j = b_{j+k} c_k \qquad i, j, k \geq 1, \tag{2.3.2}$$

where $\varphi(z) = \sum_{k\geq 1} c_k z^k$ is given and $f(z) = \sum_{n\geq 1} n b_n z^{n-1}$ is an extremal function in $A^2_1(\mathbb{D})$ such that the above equations are satisfied.

It is clear that if $c_k = 0$ for all but a single k, that is if $\varphi(z) = cz^k$, then the above equations can be satisfied by a non-zero $f \in A^2_1(\mathbb{D})$. In fact, we know from Example 2.1.4 that if we take $f = \psi_{k-1}$, then (2.3.1) and (2.3.2) will be trivially satisfied. As we remarked above, in this case the formula (2.2.2) is simplified, so that the resulting norm is $\frac{c^2}{2}$ instead of our expected upper bound of $\frac{c^2 k}{2}$ It is also clear that the above equations are satisfied when $\varphi(z) = \sum_{k\geq 1} r^k z^k$ for some $r < 1$. Here, the extremal $f = \frac{1}{\|\varphi\|_{A^2(\mathbb{D})}} \sum_{k\geq 0} r^k z^k$. In both cases $\varphi(\mathbb{D})$ is a disk.

We will now show that for all other φ, (2.3.1) and (2.3.2) only hold for $f \equiv 0$. We will do this by looking at two cases.

First suppose that $\varphi(z)$ has at least two non-zero Taylor coefficients, c_m, c_n, with $m < n$, and at least one zero coefficient c_{k_0} such that $k_0 > n$. This encompasses all Taylor series which do not have an infinite non-zero tail. Without loss of generality we can assume that $k_0 = n + 1$ by taking c_{k_0} to be the first zero coefficient after at least two non-zero coefficients. We now assume that we have found an $f \in A^2_1(\mathbb{D})$ whose Taylor coefficients satisfy (2.3.1) and (2.3.2). By (2.3.2), we have that

$$b_{n+k} c_m = b_{m+k} c_n \qquad k \geq 1, \tag{2.3.3}$$

$$b_{n+k+1} c_m = b_{m+k} c_{n+k+1} \qquad k \geq 1. \tag{2.3.4}$$

Hence, we can conclude that $b_j = 0$ for all $j \geq n+2$ by (2.3.4), which implies that $b_{m+k} = 0$ for all $k \geq 2$ by (2.3.3). We now let $i = m+1$, $j = m$ and choose k such that $m + k = n$. Then by (2.3.1) we have that

$$b_{m+1} c_{m+k} = b_{m+1} c_n = b_m c_{m+1+k} = b_m c_{n+1} = 0,$$

which shows that $b_{m+1} = 0$.

Now choosing $i < m+1$, $j = m+1$ and choosing k such that $m+1+k = n$, then by (2.3.1) we have that

$$b_i c_{m+1+k} = b_i c_n = b_{m+1} c_{n+k} = 0.$$

Hence, we have that in fact $b_i = 0$ for all $i \geq 1$, which means that $f \equiv 0$.

Suppose now instead that $\varphi(z)$ is such that its Taylor series does have an infinite non-zero tail, but the coefficients do not exhibit a geometric progression. This means that we can find three non-zero coefficients, c_m, c_{m+1}, and c_{m+2} such that

$$\frac{c_m}{c_{m+1}} \neq \frac{c_{m+1}}{c_{m+2}}. \tag{2.3.5}$$

By (2.3.2), we have that

$$b_{m+k} c_{m+1} = b_{m+1+k} c_m \qquad k \geq 1, \tag{2.3.6}$$

$$b_{m+1+k} c_{m+2} = b_{m+2+k} c_{m+1} \qquad k \geq 1. \tag{2.3.7}$$

In particular, choosing $k = 2$ in (2.3.6) and $k = 1$ in (2.3.7) we have that

$$b_{m+2} c_{m+1} = b_{m+3} c_m,$$

and

$$b_{m+2} c_{m+2} = b_{m+3} c_{m+1},$$

which by (2.3.5) means that $b_{m+2} = b_{m+3} = 0$. In fact, the same argument shows that $b_j = 0$ for all $j \geq m+2$. But then of course, by (2.3.6) we immediately get that $b_j = 0$ for all $j \geq m+1$. Now once again simply let $i < m+1$, $j = m+1$, and $k = 1$, and then by (2.3.1) we once again have that $b_i = 0$ for all $i \geq 1$, and so $f \equiv 0$. □

Our result now follows as a corollary.

Corollary 2.3.2. $\|[T_z^*, T_z]\|_{A^2(\Omega)\to A^2(\Omega)} = \frac{Area(\Omega)}{2\pi}$ *if and only if* Ω *is a disk.*

Proof. By Theorem 2.1.2,

$$\|[T_z^*, T_z]\|_{A^2(\Omega)\to A^2(\Omega)} = \|[T_\varphi^*, T_\varphi]\|_{A^2(\mathbb{D})\to A^2(\mathbb{D})}$$

where φ is the conformal map from $\mathbb{D}$ onto the simply connected domain Ω. The corollary now immediately follows from Theorem 2.3.1. □

By combining Theorems 2.2.1 and 2.3.1, along with [8, Theorem 1.2], the celebrated St. Venant inequality (cf. [33, p. 121]) follows immediately.

Corollary 2.3.3. *Let Ω be a simply connected domain. Then*

$$\rho(\Omega) \leq \frac{\mathrm{Area}^2(\Omega)}{2\pi},$$

with equality if and only if Ω is a disk.

2.4 Limitations of the Olsen-Reguera Theorem

In Section 2.1, it was shown that in the Bergman space setting, Putnam's inequality is strict, i.e., that

$$\|[T_\varphi^*, T_\varphi]\| < \frac{\mathrm{Area}(\varphi(\Omega))}{\pi}.$$

Theorem 2.2.1 states that for φ in the Dirichlet space,

$$\|[T_\varphi^*, T_\varphi]\| \leq \frac{\|\varphi'\|^2_{A^2(\Omega)}}{2}.$$

We would like to find a uniform bound on $\|[T_\varphi^*, T_\varphi]\|$ for all bounded φ. One possible approach is examining the dual problem

$$\max_{g \in (A^2(\Omega))^\perp} \left| \int_\Omega \overline{\varphi g} dA \right|,$$

and applying the Poincaré inequality (cf. [4]) since $(A^2(\Omega))^\perp = \{\frac{\partial u}{\partial z} : u \in W_0^{1,2}(\Omega)\}$, where $W_0^{1,2}(\Omega)$ is the Sobolev space of functions vanishing on $\partial\Omega$. In light of the Olsen-Reguera result, we cautiously conjecture that

$$\|[T_\varphi^*, T_\varphi]\| \leq \frac{\mathrm{Area}(\varphi(\Omega))}{2\pi},$$

where $\mathrm{Area}(\varphi(\Omega))$ is counted without multiplicity.

Note that the difference between this conjecture and the result of Olsen and Reugera is that their estimate is in terms of the Dirichlet norm of φ, which is area of $\varphi(\Omega)$ counting multiplicity. This estimate increases as the multiplicity of φ increases, quickly becoming worse than Putnam's inequality. I want to strengthen their result in terms of $\mathrm{Area}(\varphi(\Omega))$ not counting multiplicity. This would allow us to extend the Olsen-Reguera result to functions that aren't in the Dirichlet space. Indeed, simply by extending their result to finite Blaschke products, we would be able to state the theorem for all bounded φ. This would not only prove our conjecture, but improve Putnam's inequality in the Bergman space setting by a factor of $\frac{1}{2}$.

Chapter 3

Approximating $\overline{z}$ in the Bergman Space

Recall that the analytic content of Ω, $\lambda(\Omega) := \inf_{f \in H^\infty(\Omega)} \|\overline{z} - f\|_\infty$, measures "how far" $\bar{z}$ is from being a bounded analytic function, and that the extremal function g such that $\lambda(\Omega) = \|\overline{z} - g\|_\infty$ is called the best approximation to $\bar{z}$. In [22], Guadarrama and Khavinson extended this concept to Smirnov and Bergman spaces. They showed that the best approximation to $\bar{z}$ is 0 if and only if Ω is a disk, and that the best approximation is $\frac{c}{z}$ if and only if Ω is an annulus centered at the origin. In this chapter, we characterize the best approximation to $\bar{z}$ as the derivative of the solution to the Dirichlet problem on Γ with data $|z|^2$. In Section 3.2, we look at examples where the best approximation to $\bar{z}$ is a monomial Cz^k or a rational function.

3.1 Classifying the Best Approximation

Unless specified otherwise, we consider domains bounded by finitely many smooth Jordan curves. The following theorem is the foundation for the rest of the dissertation.

Theorem 3.1.1. *Let Ω be a bounded finitely connected domain. Then $f(z)$ is the projection of $\overline{z}$ onto $A^2(\Omega)$ if and only if $|\zeta|^2 = F(\zeta) + \overline{F(\zeta)}$ on $\Gamma = \partial\Omega$, where $F'(z) = f(z)$.*

(Although F can, in a multiply connected domain, be multivalued, $\mathrm{Re}(F)$ can be assumed to be single valued as a solution to the Dirichlet problem with data $|\zeta|^2$ on Γ.)

Proof. First suppose that $\overline{z} - f(z)$ is orthogonal to $A^2(\Omega)$ in $L^2(\Omega)$. Then for every $z \in \hat{\mathbb{C}} \backslash \overline{\Omega}$ we have that

$$\int_\Omega (\overline{\zeta} - f(\zeta)) \overline{\frac{1}{\zeta - z}} dA(\zeta) = 0 = \int_\Omega (\zeta - \overline{f(\zeta)}) \frac{1}{\zeta - z} dA(\zeta).$$

Then, by Green's Theorem, for any single valued branch of F, where $F' = f$, we have that

$$\int_\Gamma (|\zeta|^2 - \overline{F(\zeta)}) \frac{1}{\zeta - z} d\zeta = 0.$$

Since F belongs to the Dirichlet space ($F' = f \in A^2$), F also belongs to the Hardy space H^2, and therefore has well defined boundary values almost everywhere on Γ (cf. [14, p. 17] and [16, p. 88]). By the F. and M. Riesz Theorem (cf. [14, p. 41] and [16, pp. 62, 107]), vanishing of the Cauchy transform outside of Ω in the above formula occurs if and only if we have

$$|\zeta|^2 - \overline{F(\zeta)} = h(\zeta)$$

almost everywhere on Γ, where $h(\zeta)$ is analytic in Ω and belongs to the Hardy space H^2.

Now, since $|\zeta|^2$ is real and we have that $|\zeta|^2 = \overline{F(\zeta)} + h(\zeta)$ on Γ, then it must be that

$$\overline{F(\zeta)} + h(\zeta) = F(\zeta) + \overline{h(\zeta)},$$

which implies that $h = F$, and $|\zeta|^2 = F(\zeta) + \overline{F(\zeta)}$ on Γ as desired.

Conversely, if $|\zeta|^2 - \overline{F(\zeta)} = h(\zeta)$ on Γ for some h analytic in Ω, in particular for $h = F$, then we have that for all $z \in \hat{\mathbb{C}} \backslash \overline{\Omega}$,

$$\begin{aligned} 0 &= \int_\Gamma (|\zeta|^2 - \overline{F(\zeta)}) \frac{1}{\zeta - z} d\zeta \\ &= \int_\Omega (\zeta - \overline{F'(\zeta)}) \frac{1}{\zeta - z} dA(\zeta), \end{aligned}$$

and so we have that $\overline{\zeta} - F'(\zeta)$ is orthogonal to $A^2(\Omega)$. □

This argument is similar to that of Khavinson and Stylianopoulos in [28]. The following is an immediate corollary.

Corollary 3.1.2. *The best approximation to $\overline{z}$ in $A^2(\Omega)$ is a polynomial if and only if the Dirichlet problem with data $|z|^2$ has a real-valued polynomial solution. Similarly, the best approximation to $\overline{z}$ in $A^2(\Omega)$ is a rational function if and only if the Dirichlet problem with data $|z|^2$ has a solution which is the sum of a rational function and a finite linear combination of logarithmic potentials of real point charges located in the complement of Ω.*

The following theorem investigates what increasing the connectivity of the domain tells us about the best approximation.

Theorem 3.1.3. *Let Ω be a finitely connected domain and let $f(z)$ be the best approximation to $\overline{z}$ in $A^2(\Omega)$. Then f must have at least one singularity in every bounded component of the complement.*

Proof. Suppose $\partial\Omega = \Gamma = \cup_{i=1}^{n}\Gamma_i$ where Γ_i is a Jordan curve for each i. By Theorem 3.1.1, we must have that $|z|^2 - 2\mathrm{Re}F = 0$ on Γ where $F' = f$. Suppose that there is a bounded component K of the complement of Ω such that f is analytic in $G := \Omega \cup K$. Without loss of generality we will assume $\partial G = \cup_{i=1}^{n-1}\Gamma_i$. Then $|z|^2 - 2\mathrm{Re}F$ is subharmonic in G and vanishes on ∂G. However since $|z|^2 - 2\mathrm{Re}F$ cannot be constant in G, it must be that $|z|^2 - 2\mathrm{Re}F < 0$ in G. In particular it cannot vanish on Γ_n, which is a contradiction, and therefore f cannot be analytic in G. □

The following noteworthy corollary is now immediate.

Corollary 3.1.4. *If Ω is a finitely connected domain, and the best approximation to $\overline{z}$ is a polynomial, then Ω must be simply connected and $\partial\Omega$ is algebraic.*

The converse to Corollary 3.1.4 is false. In Section 3.2, we will give an example of a simply connected domain where the best approximation to $\overline{z}$ is a rational function. Corollary 3.1.4 implies that if the best approximation to $\overline{z}$ is a polynomial then the boundary of Ω, $\Gamma = \partial\Omega$, can be parametrized by a Schwarz function (cf. [37, p. 3]). Recall that the Schwarz function $S(z)$ is the function, analytic in a tubular neighborhood of Γ, which satisfies the condition that $S(z) = \overline{z}$ for all $z \in \Gamma$. There is a connection between the best approximation to $\overline{z}$ in $A^2(\Omega)$ and the Schwarz function of Γ. We record this connection in the following proposition.

Proposition 3.1.5. *If Ω is a simply connected domain, and if the best approximation to $\overline{z}$ is a polynomial of degree at least 1, then the Schwarz function of $\Gamma = \partial\Omega$ cannot be meromorphic in Ω. Further, when the best approximation is a polynomial the Schwarz function of the corresponding domain must have algebraic singularities and no finite poles unless Ω is a disk.*

Proof. Suppose that $S(z)$ is the Schwarz function of $\Gamma = \partial\Omega$ and let $p(z)$, a polynomial of degree $n-1$, be the best approximation to $\overline{z}$ in $A^2(\Omega)$ with anti-derivative $P(z)$. By Theorem 3.1.1, $zS(z) = P(z) + \overline{P(z)} = P(z) + P^{\#}(S(z))$ on Γ, where $P^{\#}(z) = \overline{P(\overline{z})}$. If S has a pole of order k at some $z_0 \neq 0$, then $zS(z)$ has a pole of order k at z_0 while $P^{\#}(S(z))$ has a pole of order nk at z_0. Thus $n \leq 1$. If $z_0 = 0$, and $k \geq 2$, then the same argument applies. If $z_0 = 0$ and $k = 1$, then p is constant and Γ is a circle. Since S is meromorphic in Ω if and only if the conformal map $\varphi : \mathbb{D} \to \Omega$ is a rational function (cf. [11, p. 158] and [37, pp. 17-19]), this shows that if Ω is a quadrature domain which is not a disk, then the best approximation to $\overline{z}$ cannot be a polynomial. □

We now look at some examples illustrating the above results.

3.2 Examples

The following examples were generated using Maple by plotting the boundary curve $|z|^2 - 1 = \mathrm{Const}\mathrm{Re}(F(z))$ where $f(z) = \frac{F'(z)}{2}$ is the best approximation to $\overline{z}$ in $A^2(\Omega)$, and $\mathrm{Re}(F(z))$ is the real part of $F(z)$ (cf. Theorem 3.1.1).

Note that in the next few examples with best approximation Cz^k, the associated domains have the $k+1$ fold symmetry inherited from the k fold symmetry of the best approximation. Note also that by the domain we mean everywhere the bounded domain.

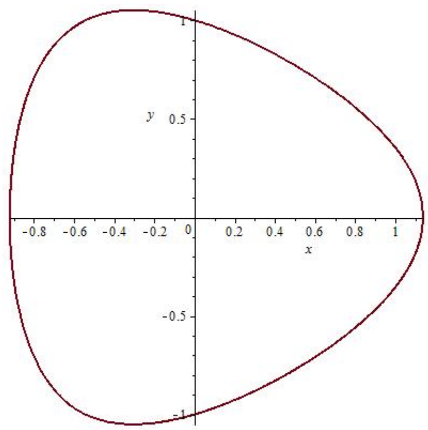

Figure 3.1.: The best approximation to $\overline{z}$ in this domain is $f(z) = \frac{3z^2}{10}$.

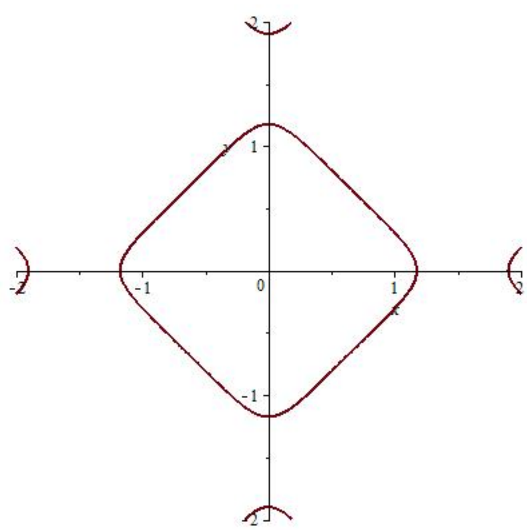

Figure 3.2.: The best approximation to $\overline{z}$ in this domain is $f(z) = \frac{2z^3}{5}$.

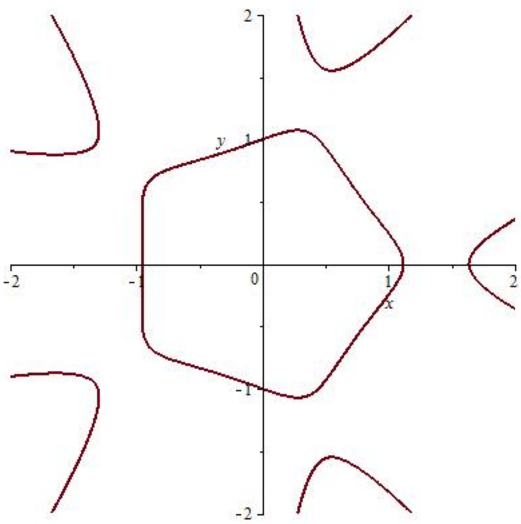

Figure 3.3.: The best approximation to $\overline{z}$ in this domain is $f(z) = \frac{5z^4}{14}$.

The following example shows that the best approximation may be a rational function even when the domain is simply connected. Thus while Corollary 3.1.4 guarantees that Ω is simply connected whenever the best approximation to $\overline{z}$ is an entire function, the converse is not true.

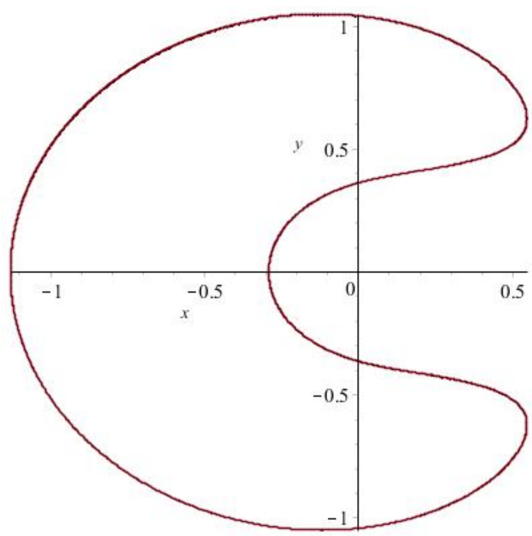

Figure 3.4.: The best approximation to $\overline{z}$ in this domain is $f(z) = \frac{1}{3z} + \frac{1}{5(z-\frac{1}{2})}$.

The constant(s) involved also play a strong role in the shape, and even connectivity of the domain, as the following pictures show.

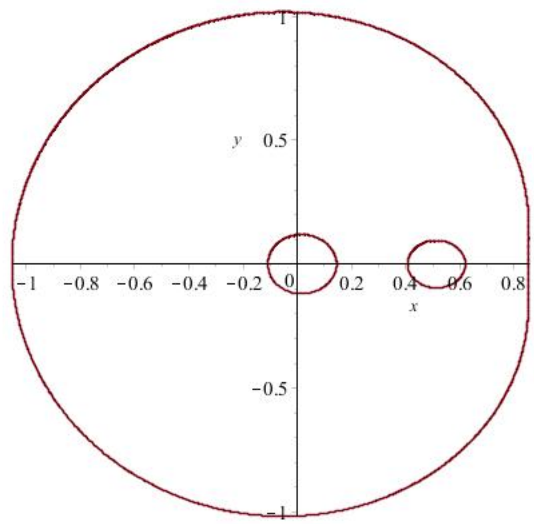

Figure 3.5.: The best approximation to $\overline{z}$ in this domain is $f(z) = \frac{1}{7z} + \frac{1}{10(z-\frac{1}{2})}$.

Note that in Figure 3.5, the best approximation has the same poles as the best approximation for the domain in Figure 3.4. Yet the domain in Figure 3.4 is simply connected, while the domain in Figure 3.5 is not.

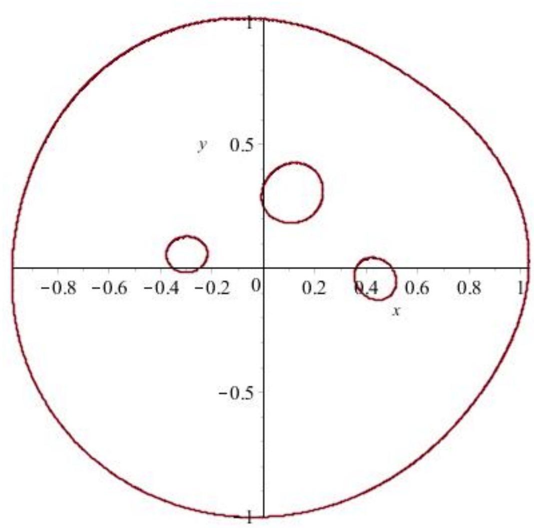

Figure 3.6.: The best approximation to $\overline{z}$ in this domain is $f(z) = -\frac{3z^2-2(\frac{1}{4}-\frac{1}{3}i)z-\frac{1}{8}+\frac{1}{12}i}{40(z-\frac{1}{2})^2(z-\frac{i}{3})^2(z+\frac{1}{4})^2}$.

In Figure 3.6, the domain is multiply connected with three holes.

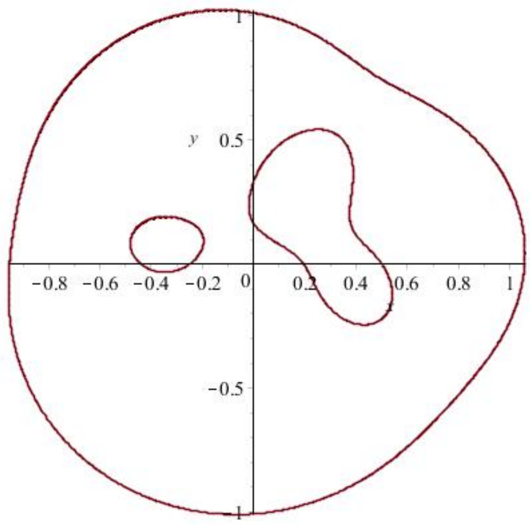

Figure 3.7.: The best approximation to $\overline{z}$ in this domain is $f(z) = -\frac{3z^2-2(\frac{1}{4}-\frac{1}{3}i)z-\frac{1}{8}+\frac{1}{12}i}{10(z-\frac{1}{2})^2(z-\frac{i}{3})^2(z+\frac{1}{4})^2}$.

In Figure 3.7, the best approximation has the same poles as the best approximation in Figure 3.6, but the resulting domain has only two holes.

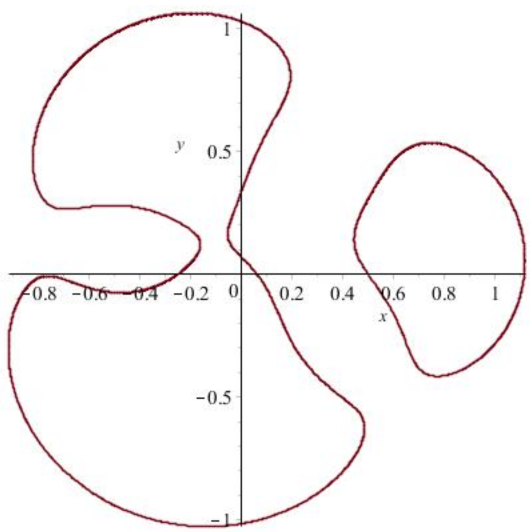

Figure 3.8.: The best approximation to $\overline{z}$ in this domain is $f(z) = -\frac{3z^2-2(\frac{1}{4}-\frac{1}{3}i)z-\frac{1}{8}+\frac{1}{12}i}{8(z-\frac{1}{2})^2(z-\frac{i}{3})^2(z+\frac{1}{4})^2}$.

Note that we actually have here two simply connected domains where the best approximation to $\overline{z}$ in both domains is $f(z) = -\frac{3z^2-2(\frac{1}{4}-\frac{1}{3}i)z-\frac{1}{8}+\frac{1}{12}i}{8(z-\frac{1}{2})^2(z-\frac{i}{3})^2(z+\frac{1}{4})^2}$. (It should be noted that in all of the above examples, the poles lie outside of $\overline{\Omega}$.)

As the order of the pole of the best approximation increases we see $k-1$ symmetric loops separating the pole from the domain. (Here k is the order of the pole of the best approximation.)

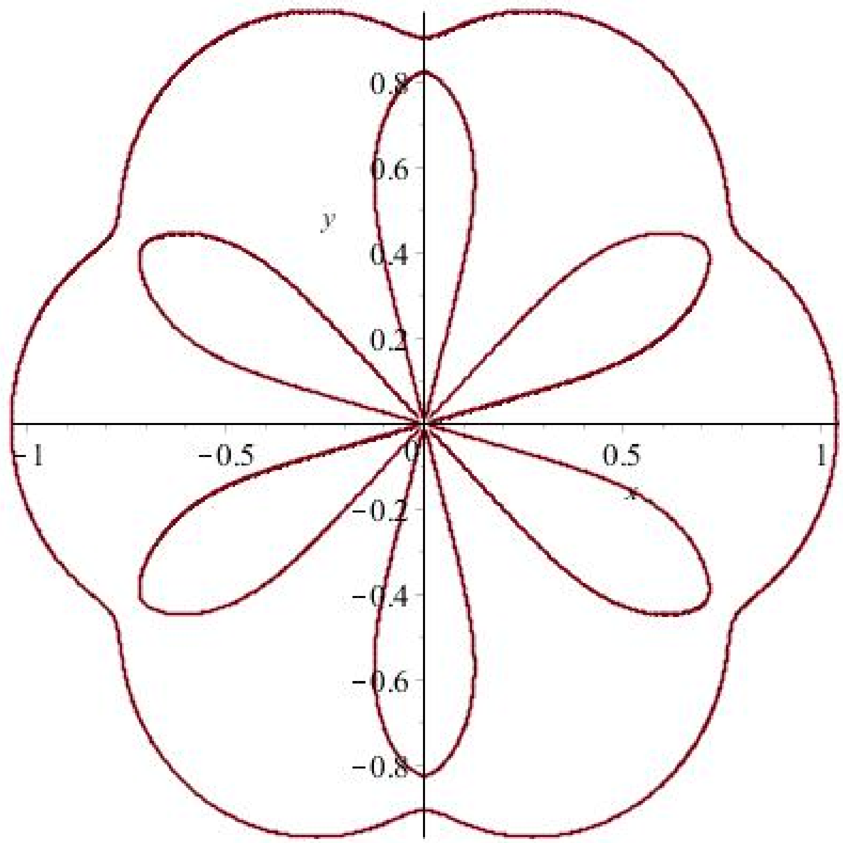

Figure 3.9.: The best approximation to $\overline{z}$ in this domain is $f(z) = \frac{-3}{10z^7}$.

(It should be noted that the loops do not pass through 0. So 0 does not belong to $\overline{\Omega}$!)

3.3 Conditions for a Bounded Component

The domains defined by $C\mathrm{Re}(z^n) - |z|^2 + 1 > 0$ represent an interesting class of examples. These are the domains for which the best approximation to $\bar{z}$ is a monomial, namely, $\frac{Cn}{2}z^{n-1}$. However, as indicated in Figure 3.10, there are values of C for which the set $\{z : C\mathrm{Re}(z^n) - |z|^2 + 1 > 0\}$ does not have a bounded component, and $\bar{z}$ is no longer in $L^2(\Omega)$. Here we address the question of what range of values of the constant C gives rise to a bounded component. Below, for example, when $C = \frac{1}{2}$, we do not get a bounded component.

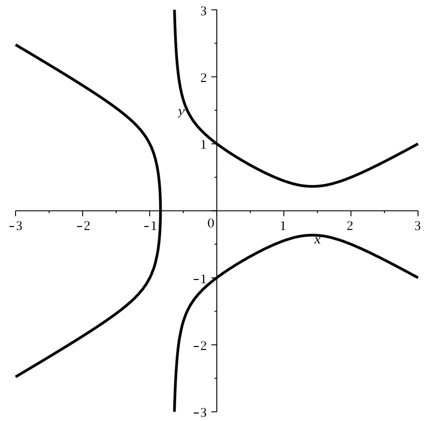

Figure 3.10.: The region $\{z : \frac{1}{2}\mathrm{Re}(z^3) - |z|^2 + 1 > 0\}$ does not have a bounded component.

Theorem 3.3.1. *The set* $\{z : C\mathrm{Re}(z^n) - |z|^2 + 1 > 0\}$ *has a bounded component whenever*

$$C \leq \frac{2(n-2)^{\frac{n-2}{2}}}{n^{\frac{n}{2}}}.$$

Proof. Take $z = re^{i\theta}$ and let $f(r,\theta) := C\cos(n\theta)r^n - r^2 + 1$ be the defining function of the domain in polar coordinates. We will show that when

$$C \leq \frac{2(n-2)^{\frac{n-2}{2}}}{n^{\frac{n}{2}}}$$

we have $f(R,\theta) \leq 0$ for all θ, where $R = (\frac{2}{nC})^{1/(n-2)}$. Since the region $\{z : C\mathrm{Re}(z^n) - |z|^2 + 1 > 0\}$ clearly contains the origin, this ensures that it has a component entirely contained in the disk $|z| < R$.

It is enough to show that $f(R,0) \leq 0$ since we have $f(R,\theta) \leq f(R,0)$.

The function $F(r) := f(r,0) = Cr^n - r^2 + 1$, has derivative $F'(r) = Cnr^{n-1} - 2r$, with a critical point at $R = (\frac{2}{nC})^{1/(n-2)}$, which by the first derivative test is a local minimum. Plugging this critical point into F, we find that

$$C\left(\frac{2}{nC}\right)^{n/(n-2)} - \left(\frac{2}{nC}\right)^{2/(n-2)} + 1 \leq 0$$

precisely when

$$C \leq \frac{2(n-2)^{\frac{n-2}{2}}}{n^{\frac{n}{2}}}.$$

$\square$

Chapter 4

Bergman Analytic Content

In [22] the authors expanded the notion of analytic content, $\lambda(\Omega) := \inf_{f\in H^\infty(\Omega)} \|\overline{z} - f\|_\infty$, defined in [9] and [24], to Bergman and Smirnov spaces contexts. Recall the following "isoperimetric sandwich", which goes back to [24]:

$$\frac{2\text{Area}(\Omega)}{\text{Per}(\Omega)} \le \lambda(\Omega) \le \sqrt{\frac{\text{Area}(\Omega)}{\pi}}.$$

Following [22], we define

$$\lambda_{A^2}(\Omega) := \inf_{f\in A^2(\Omega)} \|\overline{z} - f\|_2 .$$

In [18] the inequality

$$\sqrt{\rho(\Omega)} \le \lambda_{A^2}(\Omega). \tag{4.0.1}$$

was shown to hold for simply connected domains. In Section 4.1, we show that in fact for simply connected domains 4.0.1 is equality. In general, 4.0.1 is not equality. This follows from explicit computations for doubly-connected domains such as the annulus, which we discuss in Section 4.3.

4.1 Main Equality

Theorem 4.1.1. *If Ω is a simply connected domain with a piecewise smooth boundary, then*

$$\sqrt{\rho(\Omega)} = \lambda_{A^2}(\Omega).$$

Proof. Recall that if Ω is a simply connected domain, the torsional rigidity $\rho = \rho(\Omega)$ is given by

$$\rho = \int_\Omega |\nabla \nu|^2 dA,$$

where ν is the unique solution to the Dirichlet problem

$$\begin{cases} \Delta\nu &= -2 \\ \nu|_{\partial\Omega} &= 0 \end{cases}$$

(cf. [33, pp. 24 and 88] and [6, pp. 63-66]).

Consider the function $u(z) := \nu(z) + \frac{|z|^2}{2}$. Then u solves the Dirichlet problem:

$$\begin{cases} \Delta u & = 0 \\ u|_{\partial\Omega} & = \frac{|z|^2}{2}. \end{cases}$$

Thus, by Theorem 3.1.1, $u = \mathrm{Re}(F)$, where $f = F'$ is the best approximation to $\bar{z}$ in $A^2(\Omega)$.

Letting ν denote the torsion function, we have:

$$\begin{aligned} \rho(\Omega) &= \int_\Omega |\nabla \nu|^2 dA \\ &= \int_\Omega |2\partial_z \nu|^2 dA \\ &= \int_\Omega \left| 2\partial_z u - 2\partial_z \frac{|z|^2}{2} \right|^2 dA \\ &= \int_\Omega \left| F' - \bar{z} \right|^2 dA \\ &= \int_\Omega |\bar{z} - f|^2 \, dA \\ &= \int_\Omega |z|^2 - |f|^2 dA \\ &= \lambda_{A^2}(\Omega)^2, \end{aligned}$$

and the claim follows. □

Now by the Saint-Venant inequality (cf. Corollary 2.3.3), we immediately have the following corollary.

Corollary 4.1.2. *Let Ω be a simply connected domain. Then*

$$\lambda_{A^2}(\Omega) \leq \frac{\mathrm{Area}(\Omega)}{\sqrt{2\pi}}.$$

4.2 Bergman Analytic Content in Quadrature Domains

We now use Theorem 3.1.1 to give an explicit formula for Bergman analytic content for certain quadrature domains. Recall that a bounded domain $\Omega \subset \mathbb{C}$ is called a quadrature domain if it admits a formula expressing the area integral of any test function $g \in A^2(\Omega)$ as a finite sum of weighted point evaluations of g and its derivatives:

$$\int_D g(z) dA(z) = \sum_{m=1}^{N} \sum_{k=0}^{n_m} a_{m,k} g^{(k)}(z_m), \tag{4.2.1}$$

where the points $z_m \in \Omega$ and constants $a_{m,k}$ are fixed and independent of g. This class of domains is C^∞-dense in the space of domains having a C^∞-smooth boundary [7, Thm. 1.7], and the restricted class of quadrature domains for which $N = 1$ in (4.2.1) has the same density property. When Ω is a simply connected quadrature domain with $N = 1$, the conformal mapping $\phi : \mathbb{D} \to \Omega$ is a polynomial, and by making a translation we may assume that the quadrature distribution is supported at $\phi(0) = 0$.

Theorem 4.2.1. *Let $\Omega \subset \mathbb{C}$ be a simply connected quadrature domain with quadrature formula supported at a single point (say, the origin), and let $\phi : \mathbb{D} \to \Omega$ be the (polynomial) conformal map from the unit disk*

$$\phi(z) = \sum_{k=1}^{n} a_k z^k.$$

Then the Bergman analytic content of Ω is:

$$\pi^{1/2} \left[\sum_{m=1}^{2n-1} \frac{|c_m|^2}{m+1} - \sum_{k=1}^{n-1} k \left| \sum_{j=1}^{n-k} a_{k+j} \overline{a_j} \right|^2 \right]^{1/2},$$

where

$$c_m := \sum_{k+j=m+1} k a_k a_j \qquad 1 \le k, j \le n.$$

Moreover, the best approximation f to $\bar{z}$ is the derivative $f = F'$ of $F = P \circ \phi^{-1}$, where

$$P(\zeta) = \frac{1}{2} \sum_{k=1}^{n} |a_k|^2 + \sum_{k=1}^{n-1} \sum_{j=1}^{n-k} a_{k+j} \overline{a_j} \zeta^k.$$

Proof. By the definition of Bergman analytic content, we have $\lambda_{A^2}(\Omega) = \|\bar{z} - f\|_2$, where f is the projection of $\bar{z}$ onto $A^2(\Omega)$. By the Pythagorean theorem we then have that

$$\begin{aligned} \lambda_{A^2}(\Omega) &= \left(\int_\Omega |\bar{z}|^2 \, dA(z) - \int_\Omega |f(z)|^2 \, dA(z) \right)^{1/2} \\ &= \left(\int_{\mathbb{D}} \left|\overline{\phi}\phi'\right|^2 dA - \int_{\mathbb{D}} |f \circ \phi|^2 \left|\phi'\right|^2 dA \right)^{1/2}, \end{aligned}$$

where we have changed variables $z = \phi(\zeta)$, $dA(z) = |\phi'(\zeta)|^2 dA(\zeta)$. The first term $\int_{\mathbb{D}} \left|\overline{\phi}\phi'\right|^2 dA = \int_{\mathbb{D}} |\phi\phi'|^2 \, dA$ is simply the square of the Bergman norm of a polynomial $\phi\phi'$:

$$\int_{\mathbb{D}} \left|\overline{\phi}\phi'\right|^2 dA = \pi \sum_{m=1}^{2n-1} \frac{|c_m|^2}{m+1}, \tag{4.2.2}$$

where

$$c_m := \sum_{k+j=m+1} k a_k a_j \qquad 1 \le k, j \le n,$$

are the coefficients in the expansion of the product $\phi \cdot \phi'$.

In order to compute $\int_\Omega |f(z)|^2 \, dA(z)$, we first find f explicitly. By Theorem 3.1.1, $f = F'$, where $u = \mathrm{Re}(F)$ solves the Dirichlet problem

$$\begin{cases} \Delta u & = 0 \\ u|_{\partial\Omega} & = \frac{|z|^2}{2}. \end{cases}$$

Changing coordinates using the conformal map ϕ, we obtain a harmonic function $\tilde{u} = u \circ \phi$ that solves the following Dirichlet problem in the unit disk:

$$\begin{cases} \Delta \tilde{u} & = 0 \\ \tilde{u}|_{\mathbb{T}} & = \frac{\phi\overline{\phi}}{2}. \end{cases}$$

Now, on $\mathbb{T}$ we have that $\phi\overline{\phi} = P(\zeta) + \overline{P(\zeta)}$, where

$$P(\zeta) = \frac{1}{2}\sum_{k=1}^{n} |a_k|^2 + \sum_{k=1}^{n-1}\sum_{j=1}^{n-k} a_{k+j}\overline{a_j}\zeta^k.$$

Since $P(\zeta) + \overline{P(\zeta)}$ is a harmonic polynomial, we have that $\tilde{u}(\zeta) = \mathrm{Re}(P(\zeta))$. Thus, $F \circ \phi = P$, and so by the chain rule $(f \circ \phi)(\phi') = p$, where

$$p(\zeta) = P'(\zeta) = \sum_{k=1}^{n-1} k \sum_{j=1}^{n-k} a_{k+j}\overline{a_j}\zeta^{k-1}.$$

Calculating the Bergman norm of this polynomial, we find that

$$\int_\Omega |f(z)|^2 \, dA(z) = \int_{\mathbb{D}} |f \circ \phi|^2 \left|\phi'\right|^2 dA = \sum_{k=1}^{n-1} k \left| \sum_{j=1}^{n-k} a_{k+j}\overline{a_j} \right|^2. \tag{4.2.3}$$

Combining (4.2.2) and (4.2.3), the result follows. □

While the explicit formulas in Theorem 4.2.1 appear to be new, our proof based on conformal mapping is very similar to the procedure described in [31, Sec. 134].

4.3 Examples

In this section, we calculate some values of $\lambda_{A^2}(\Omega)$. In particular, we calculate a family of examples by applying Theorem 4.2.1, and examine two doubly connected cases.

4.3.1 Epicycloids

Let us consider the one-parameter family of domains Ω with conformal map $\phi : \mathbb{D} \to \Omega$, given by $\phi(z) = z + az^n$, with $0 \leq a \leq \frac{1}{n}$.

Applying Theorem 4.2.1, we immediately obtain:

$$\lambda_{A^2}(\Omega) = \sqrt{\frac{\pi\,(1 + 4a^2 + na^4)}{2}}.$$

When $a = \frac{1}{n}$ the domain develops cusps (the case $n = 4$ is plotted in Figure 4.1). The case $n = 2$ and $a = \frac{1}{2}$ is a cardioid (cf. [38, Sec. 58]).

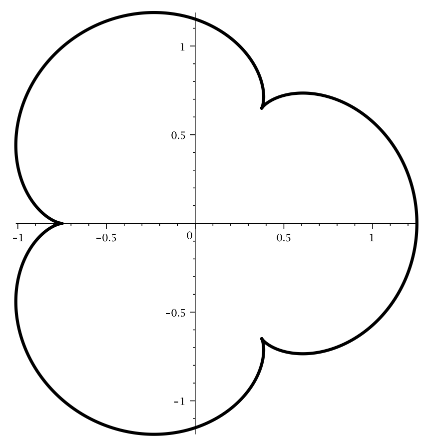

Figure 4.1.: The epicycloid domain when $n = 4$, $a = 1/4$.

4.3.2 The Annulus

The following example shows that Theorem 4.1.1 does not hold in general for multiply connected domains. Let $\Omega = \{z : \; r < |z| < R\}$ be the annulus. The best approximation to $\bar{z}$ in $A^2(\Omega)$ is $f(z) = \frac{C}{z}$, where

$$C = \frac{R^2 - r^2}{2(\log R - \log r)}$$

(cf. [22]). Following the proof of Theorem 4.2.1, we have that

$$\lambda^2_{A^2}(\Omega) = \int_\Omega |z|^2 - |f|^2 dA. \tag{4.3.1}$$

Integrating in polar coordinates we get that

$$\int_\Omega |z|^2 dA = \frac{\pi}{2}(R^4 - R^2),$$

and

$$\begin{aligned}\int_\Omega \left|\frac{C}{z}\right|^2 dA &= 2\pi C^2 \int_r^R \frac{1}{\rho} d\rho \\ &= \frac{\pi}{2}\frac{(R^2-r^2)^2}{\log R - \log r}.\end{aligned}$$

Thus, we have that

$$\lambda_{A^2}^2(\Omega) = \frac{\pi}{2}\left(\left(R^4 - r^4\right) - \frac{(R^2-r^2)^2}{\log R - \log r}\right),$$

which is smaller than the torsional rigidity [6, p. 64] of Ω:

$$\rho(\Omega) = \frac{\pi}{2}\left(R^4 - r^4\right).$$

So we find that Theorem 4.1.1 doesn't hold for multiply connected domains.

4.3.3 The Annular Region Bounded by a Pair of Confocal Ellipses

We consider the region G between two confocal ellipses that is the image of an annulus $\Omega := \{z \in \mathbb{C} : r < |z| < R\}$ under the Joukowski map $\phi(z) = z + \frac{1}{z}$.

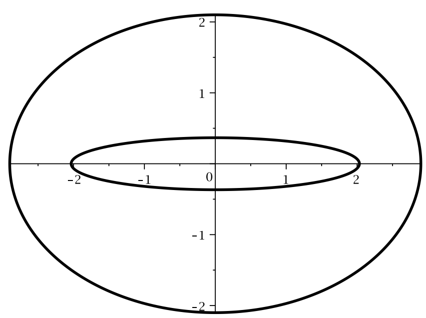

Figure 4.2.: The annular region G when $r = 1.2$, $R = 2.5$.

Following the proof of Theorem 4.2.1, the projection of $\bar{z}$ to the Bergman space is given by $f = F'$, where $u = \mathrm{Re}(F)$ solves the Dirichlet problem

$$\begin{cases}\Delta u &= 0\\ u|_{\partial G} &= \frac{|z|^2}{2}.\end{cases}$$

The function $\tilde{u} = u \circ \phi$ is harmonic and solves the following Dirichlet problem in the annulus $\Omega := \{\zeta \in \mathbb{C} : r < |\zeta| < R\}$:

$$\begin{cases} \Delta \tilde{u} & = 0 \\ \tilde{u}|_{\partial\Omega} & = \frac{\phi\bar{\phi}}{2}. \end{cases}$$

We make the ansatz

$$2\tilde{u}(\zeta) = A + B \log|\zeta| + C(\zeta^2 + \bar{\zeta}^2) + D\left(\frac{1}{\zeta^2} + \frac{1}{\bar{\zeta}^2}\right).$$

The boundary condition gives:

$$2\tilde{u}(\zeta) = |\zeta|^2 + \frac{1}{|\zeta|^2} + \frac{\zeta}{\bar{\zeta}} + \frac{\bar{\zeta}}{\zeta} \quad \text{on } \partial\Omega.$$

Using polar coordinates to parameterize the two circular boundary components $z = re^{i\theta}$ and $z = Re^{i\theta}$, we obtain two equations:

$$\begin{aligned} A + B\log r + 2\left(Cr^2 + \frac{D}{r^2}\right)\cos(2\theta) &= r^2 + \frac{1}{r^2} + 2\cos(2\theta), \\ A + B\log R + 2\left(CR^2 + \frac{D}{R^2}\right)\cos(2\theta) &= R^2 + \frac{1}{R^2} + 2\cos(2\theta), \end{aligned}$$

which implies the system of equations for A, B, C, D

$$\begin{aligned} Cr^2 + \frac{D}{r^2} &= 1, \\ CR^2 + \frac{D}{R^2} &= 1, \\ A + B\log R &= R^2 + \frac{1}{R^2}, \\ A + B\log r &= r^2 + \frac{1}{r^2}. \end{aligned}$$

Solving this (linear in A, B, C, D) system, we obtain:

$$\begin{aligned} A &= \frac{-\log r}{\log R - \log r}\left(R^2 + \frac{1}{R^2}\right) + \frac{\log R}{\log R - \log r}\left(r^2 + \frac{1}{r^2}\right), \\ B &= \frac{1}{\log R - \log r}\left(R^2 + \frac{1}{R^2} - r^2 - \frac{1}{r^2}\right), \\ C &= \frac{1}{R^2 + r^2}, \\ D &= \frac{r^2 R^2}{R^2 + r^2}. \end{aligned}$$

We have

$$(f \circ \phi)\phi' = \frac{B}{2\zeta} + C\zeta - \frac{D}{\zeta^3},$$

and thus the square of the Bergman norm of f is

$$\begin{aligned}\int_G |f(z)|^2\,dA(z) &= \int_\Omega |f\circ\phi|^2\,|\phi'|^2\,dA \\ &= \int_\Omega \left|\frac{B}{2\zeta} + C\zeta - \frac{D}{\zeta^3}\right|^2 dA \\ &= \frac{\pi}{2}\left(B^2(\log R - \log r) + C^2(R^4 - r^4) + D^2\left(\frac{1}{r^4} - \frac{1}{R^4}\right)\right).\end{aligned}$$

The square of the Bergman norm of $\bar{z}$ is

$$\begin{aligned}\int_G |z|^2\,dA(z) &= \int_\Omega |\phi\phi'|^2\,dA \\ &= \int_\Omega \left|\left(\zeta + \frac{1}{\zeta}\right)\left(1 - \frac{1}{\zeta^2}\right)\right|^2 dA(\zeta) \\ &= \int_\Omega \left|\zeta - \frac{1}{\zeta^3}\right|^2 dA(\zeta) \\ &= \frac{\pi}{2}\left(R^4 - r^4 + \frac{1}{r^4} - \frac{1}{R^4}\right).\end{aligned}$$

Thus, $\lambda_{A^2}(G)^2 = \int_G |z|^2\,dA(z) - \int_G |f(z)|^2\,dA(z)$ is given by:

$$\frac{\pi}{2}\left(R^4 - r^4 + \frac{1}{r^4} - \frac{1}{R^4} - \frac{1}{\log R - \log r}\left(R^2 + \frac{1}{R^2} - r^2 - \frac{1}{r^2}\right)^2 - 2\frac{R^2 - r^2}{R^2 + r^2}\right).$$

4.4 An Ahlfors-Beurling Type Conjecture

We conclude with a conjecture in the spirit of Ahlfors and Beurling. Recall that for all $u \in W_0^{1,2}(\Omega)$, we can write

$$\left|\int_\Omega u(z)dA(z)\right| = \left|\int_\Omega \frac{-1}{\pi}\int_\Omega \frac{\partial u}{\partial\bar{\zeta}}\frac{1}{\zeta - z}dA(\zeta)dA(z)\right|. \tag{4.4.1}$$

Applying Fubini's Theorem and the Cauchy-Schwarz inequality, we find that

$$\left|\int_\Omega u(z)dA(z)\right| \le \left\|\frac{\partial u}{\partial\bar{z}}\right\|_2 \left\|\frac{1}{\pi}\int_\Omega \frac{dA(z)}{z - \zeta}\right\|_2. \tag{4.4.2}$$

In [12] and [13], (also cf. [3]) it was proved that the Cauchy integral operator $C : L^2(\Omega) \to L^2(\Omega)$, defined by

$$Cf(z) = \frac{-1}{\pi}\int_\Omega \frac{f(\zeta)}{\zeta - z}dA(\zeta),$$

has norm $\frac{2}{\sqrt{\Lambda_1}}$ whenever Ω is a simply connected domain with a piecewise smooth boundary, and Λ_1 is the smallest positive eigenvalue of the Dirichlet Laplacian,

$$\begin{cases} -\Delta u = \Lambda u \\ u|_{\partial\Omega} = 0. \end{cases}$$

Further, by the Faber-Krahn inequality, cf. [33, pp. 18, 98] and [6, p. 104], we have that

$$\frac{2}{\sqrt{\Lambda_1}} \leq \frac{2}{j_0}\sqrt{\frac{\text{Area}(\Omega)}{\pi}},$$

where j_0 is the smallest positive zero of the Bessel function $J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2}(\frac{x}{k})^{2k}$. Combining the above inequality with (4.4.2) we obtain

$$\frac{1}{\left\|\frac{\partial u}{\partial \bar{z}}\right\|_2} \left| \int_\Omega u dA(z) \right| \leq \frac{2}{j_0} \frac{\text{Area}(\Omega)}{\sqrt{\pi}}. \tag{4.4.3}$$

This together with Theorem 4.1.1 and (4.4.2), yields an isoperimetric inequality:

$$\rho(\Omega) \leq \frac{4\text{Area}^2(\Omega)}{j_0^2 \pi}.$$

However, this is a coarser upper bound than that found above since $\frac{2}{j_0} \geq \frac{1}{\sqrt{2}}$. Since this upper bound depends entirely on $\left\|\frac{1}{\pi}\int_\Omega \frac{dA(z)}{z-\zeta}\right\|_2$, and since in the case when Ω is a disk D we find that $\left\|\frac{1}{\pi}\int_D \frac{dA(z)}{z-\zeta}\right\|_2 = \frac{\text{Area}(D)}{\sqrt{2\pi}}$, we conjecture, in the spirit of the Ahlfors-Beurling inequality (cf. [1] and [20]), that

$$\left\| \frac{1}{\pi} \int_\Omega \frac{dA(z)}{z-\zeta} \right\|_2 \leq \frac{\text{Area}(\Omega)}{\sqrt{2\pi}}.$$

If true, this would provide an alternate proof to the upper bound for Bergman analytic content, as well as a more direct proof of the St. Venant inequality.

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