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On the Number of Colors in Quandle Knot Colorings

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On the Number of Colors in Quandle Knot Colorings

by

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Dedication

To my parents, who have always shown their love and support.

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I am very grateful to my mentor and advisor Dr. Mohamed Elhamdadi for putting me on the right track in my graduate studies. Through his guidance, I was able to experience the subject of Topology in a new light. I am also grateful to the members of my thesis committee Dr. Brian Curtin and Dr. Masahiko Saito for their time and advice.

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Abstract

A major question in Knot Theory concerns the process of trying to determine when two knots are different. A knot invariant is a quantity (number, polynomial, group, etc.) that does not change by continuous deformation of the knot. One of the simplest invariant of knots is colorability. In this thesis, we study Fox colorings of knots and knots that are colored by linear Alexander quandles. In recent years, there has been an interest in reducing Fox colorings to a minimum number of colors. We prove that any Fox coloring of a 13-colorable knot has a diagram that uses exactly five colors. The ideas behind the reduction of colors in a Fox coloring is extended to knots colored by linear Alexander quandles. Thus, we prove that any knot colored by either the linear Alexander quandle $\mathbb{Z}_5[t]/(t-2)$ or $\mathbb{Z}_5[t]/(t-3)$ has a diagram using only four colors.

Chapter 1
Fox Colorings of Knots

1.1 Introduction

One of the main questions in Knot Theory is how to tell knots apart. Thus, knot invariants are constructed to distinguish two knots. One of the earliest knot invariants is called colorability of knots. Fox introduced a diagrammatic definition of colorability of a knot K by $\mathbb{Z}_m$ (the integers modulo m) [13]. This notion of colorability is clearly one of the simplest invariant of knots. In this chapter we investigate Fox colorings of knots that are 13-colorable. We first recall a few definitions.

Definition 1.1.1 *A knot is an embedding of the circle in three dimensional space. In other words, it is a simple closed curve in* $\mathbb{R}^3$.

Two knots K_1 and K_2 are equivalent if K_1 can be deformed continuously to obtain K_2. We say that K_1 and K_2 are ambient isotopic. A diagram of a knot K is a *regular* projection $p : K \subset \mathbb{R}^3 \to \mathbb{R}^2$ on a plane $\mathbb{R}^2$ where there are only finitely many transverse *double* points P_i ($p^{-1}(P_i)$ contains only two points) called *crossings* such that at each crossing the information on over-strands and under-stands is specified. Given two strands, in a diagram of a knot, it is indicated which strand passes over and which passes under at each crossing by drawing the under strand broken. A continuous curve in a diagram is called an *arc*, thus there are finitely many arcs and each arc has its endpoints at a crossing. Two diagrams D_1 and D_2 represent the same knot K if and only if one can transform diagram D_1 to diagram D_2 by a finite number of Reidemeister moves I, II, and III. The Reidemeister moves are given in the following Figure 1.

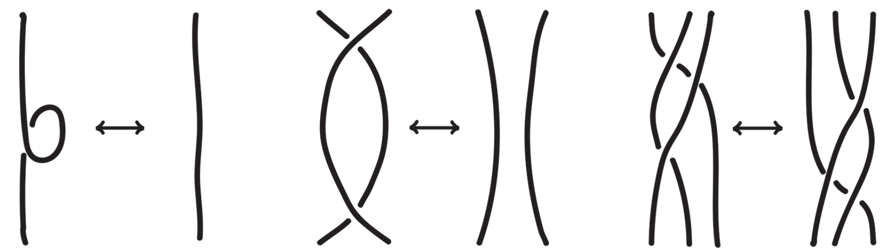

Figure 1.: Reidemeister moves I, II, and III, respectfully.

Definition 1.1.2 *Let* m *be a natural number greater than* 1*. A Fox coloring or* m*-coloring of the diagram* D *of a knot* K *is a map from the set of the arcs of* D *to the set of the integers* $\{0, 1, \ldots, m-1\}$ *such that at every crossing, the sum of the two integers assigned to the under-arcs is twice the integer assigned to the over-arc mod* m*.*

In the following Figure 2 we have a Fox coloring of an arbitrary crossing.

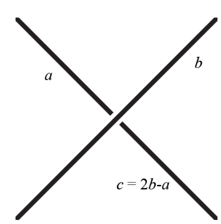

Figure 2.: Fox coloring: $a + c \equiv 2b \bmod m$

The integer assigned on an arc is called a *color*. A trivial coloring is a coloring of D that uses only one color. For example the trefoil knot has a non-trivial 3-coloring.

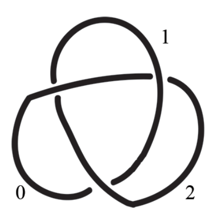

Figure 3.: 3-coloring of the trefoil knot

Definition 1.1.3 *A knot* K *is* p*-colorable if it has a diagram* D *that is non-trivially colored.*

It is well known [12] that for a prime p, a knot K is p-colorable if and only if p divides the determinant of K. The problem of finding the minimum number of colors for p-colorable knots with p prime less than or equal to 11 was studied in [2, 8, 11, 16, 20, 21]. For example Satoh proved in [16] that any 5-colorable knot admits a non-trivially 5-colored diagram where the coloring assignment uses only 4 of the 5 available colors. For a prime p, let K be a p-colorable knot and let $C_p(K)$ denotes the minimum number of colors among all diagrams of the knot K. When p is prime it was proved in [18] that $C_p(K) \geq \lfloor \log_2(p) \rfloor + 2$. This implies that in our case, $p = 13$, the minimum number of colors of 13-colorable knots is greater than or equal to 5. In fact, one of the goals of this thesis is to prove equality, that is $C_{13}(K) = 5$. A list of known results for small primes is given below:

- $C_3(K) = 3$ for any 3-colorable knot K,
- $C_5(K) = 4$ for any 5-colorable knot K [16],
- $C_7(K) = 4$ for any 7-colorable knot K [8],
- $C_{11}(K) = 5$ for any 11-colorable knot K [20].

1.2 Frequently used knot diagram transformations

In this section, we list a few transformations of knot diagrams that are used frequently in this thesis. Moreover, it is important to note that all transformations used in this thesis are sequences of compound Reidemeister moves, and therefore do not change the knot. Throughout this chapter we will use the notation $\{a|b|c\}$ in $\mathbb{Z}_m$ as seen in [16] to denote the crossing from Figure 2, where a and c are the colors of the under-arcs, b is the color of the over arc and $a + c \equiv 2b$, mod 13. When the crossing is of the type $\{c|c|c\}$, we will omit over- and under- arcs and draw the arcs crossing each other.

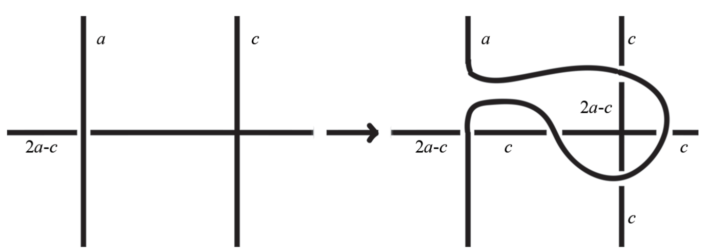

Figure 4.: Transformation of the crossing $\{c|c|c\}$ with a as an over-arc.

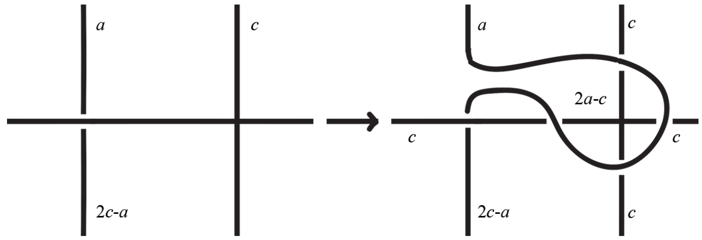

Figure 5.: Transformation of the crossing $\{c|c|c\}$ with a as an under-arc.

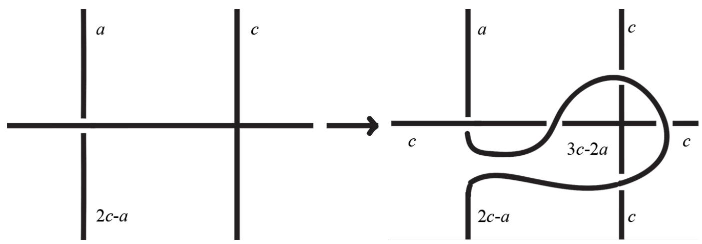

Figure 6.: Inverse transformation of the crossing $\{c|c|c\}$ with a as an under-arc.

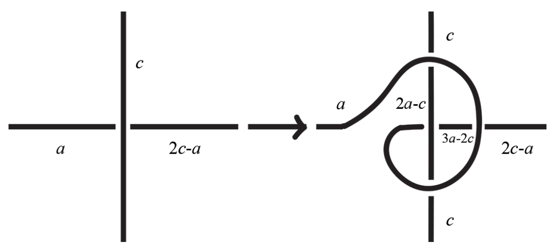

Figure 7.: Transformation of the crossing $\{a|c|2c-a\}$.

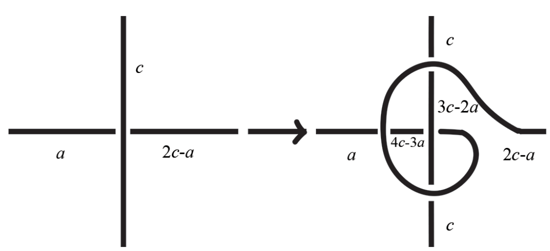

Figure 8.: Inverse transformation of the crossing $\{a|c|2c-a\}$.

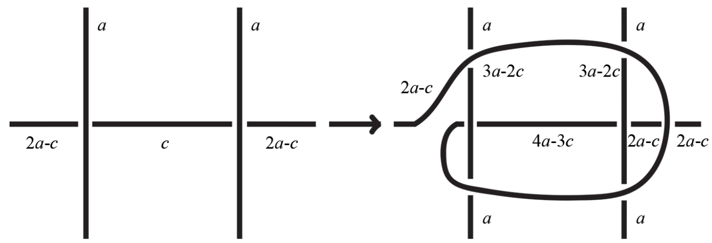

Figure 9.: Transformation of $\mathfrak{c}$ between two crossings $\{2\mathfrak{a}-\mathfrak{c}|\mathfrak{a}|\mathfrak{c}\}$.

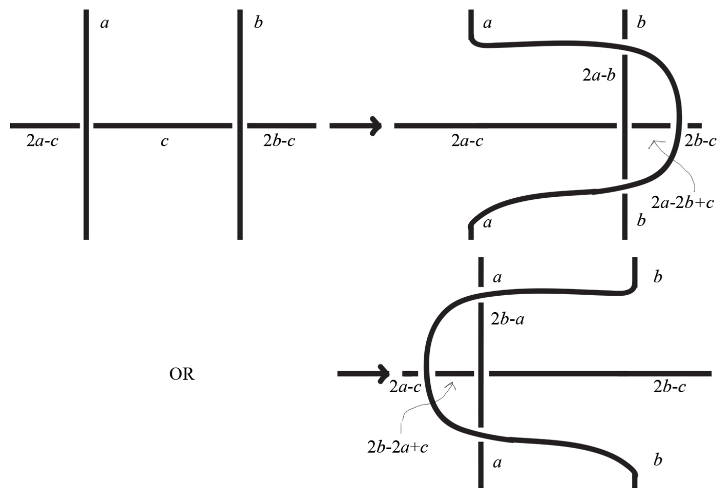

Figure 10.: Transformation of $\mathfrak{c}$ between the crossings $\{2\mathfrak{a}-\mathfrak{c}|\mathfrak{a}|\mathfrak{c}\}$ and $\{\mathfrak{c}|\mathfrak{b}|2\mathfrak{b}-\mathfrak{c}\}$.

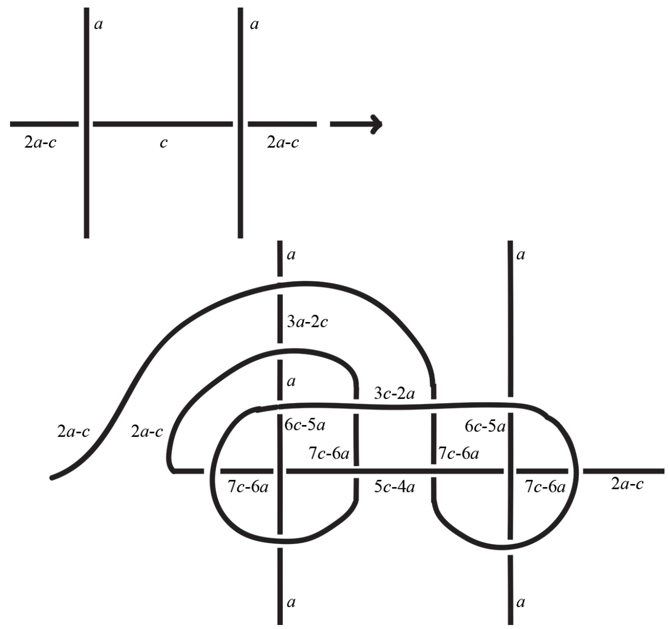

Figure 11.: Transformation of c between the crossings $\{2a-c|a|c\}$ and $\{c|a|2a-c\}$.

1.3 Fox coloring and the minimum number of colors of 13-colorable knots

Our first main result is describe in the following theorem.

Theorem 1.3.1 *Any* 13*-colorable knot has a* 13*-colored diagram with exactly five colors. Thus,* $C_{13}(K) = 5$ *for any* 13*-colorable knot* K*.*

Proof. We prove this theorem using eight lemmas. In each of the following lemmas we decrease the coloring scheme of the diagram by one color c, where c is in $\mathbb{Z}_{13}$. To accomplish this we first remove any crossings of the form $\{c|c|c\}$, that is, when c is both an over-arc and an under-arc. Then, we remove c as an over-arc by transforming any crossings of the form $\{a|c|2c-a\}$ where a is in $\mathbb{Z}_{13} \setminus \{c\}$. Finally, we complete each lemma by removing c as an under-arc in a case by case method. In these under-arc cases we must consider when c connects two crossings of the same color and when c connects two crossings of different colors. □

1.3.1 Eliminating the color 12

Lemma 1.3.2 *Any* 13*-colorable knot has a* 13*-colored diagram* D *with no arc colored by* 12*.*

Proof. We first transform any crossing of the form $\{12|12|12\}$. If there is any crossing of the form $\{12|12|12\}$, there is an adjacent crossing of the form $\{12|a|2a+1\}$ or $\{a|12|11-a\}$ where a is in $\mathbb{Z}_{13} \setminus \{12\}$. In either case, since $11-a \neq 12$ and $2a+1 \neq 12$ for any a in $\mathbb{Z}_{13} \setminus \{12\}$, we transform the diagram as seen in Figure 4 or Figure 5.

Next, we remove 12 as an over-arc by transforming any crossings of the form $\{a|12|11-a\}$. Since $2a+1 \neq 12$ and $3a+2 \neq 12$ for any a in $\mathbb{Z}_{13} \setminus \{12\}$ we transform the diagram as seen in Figure 7. We complete the proof of the lemma by removing 12 as an under-arc in a case by case method.

We first consider the case where 12 is an under-arc connecting two crossings of the form $\{12|a|2a+1\}$. Since $2a+1 \neq 12$, $3a+2 \neq 12$, and $4a+3 \neq 12$ for any a in $\mathbb{Z}_{13} \setminus \{12\}$, we transform the diagram as seen in Figure 9. Now we consider the case where 12 is an under-arc connecting two crossings of the form $\{2a+1|a|12\}$ and $\{12|2a+1|4a+3\}$. Since $2a+1 \neq 12$ and $3a+2 \neq 12$ for any a in $\mathbb{Z}_{13} \setminus \{12\}$ we transform the diagram as seen in the following Figure 12.

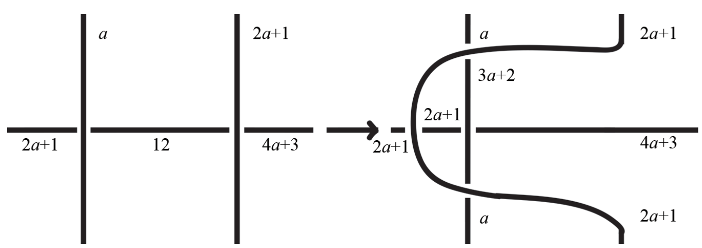

Figure 12.: Transformation of 12 between the crossings $\{2a+1|a|12\}$ and $\{12|2a+1|4a+3\}$.

Finally we consider the case where 12 is an under-arc connecting two crossings of the from $\{2a+1|a|12\}$ and $\{12|b|2b+1\}$ where $a \neq b$ and $b \neq 2a+1$ for any a and b in $\mathbb{Z}_{13} \setminus \{12\}$. Since $2a-2b-1 \neq 12$ and $2a-b \neq 12$ for any a and b in $\mathbb{Z}_{13} \setminus \{12\}$ (from $a \neq b$ and $b \neq 2a+1$ respectively) we transform the diagram as seen in the following Figure 13.

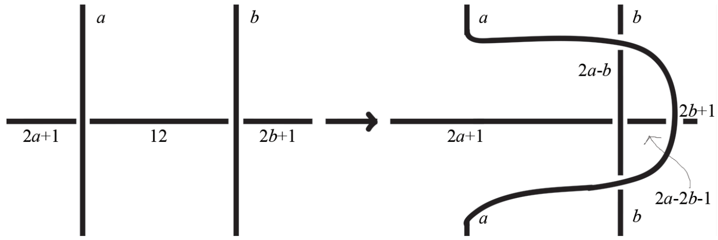

Figure 13.: Transformation of 12 between the crossings $\{2a+1|a|12\}$ and $\{12|b|2b+1\}$.

$\square$

1.3.2 Eliminating the color 11

Lemma 1.3.3 *Any* 13*-colorable knot has a* 13*-colored diagram* D *with no arc colored by* 11 *or* 12*.*

Proof. By the previous lemma we may assume that no arc in D is colored by 12. We first transform any crossing of the form $\{11|11|11\}$. If there is any crossing of the form $\{11|11|11\}$, there is an adjacent crossing of the form $\{11|a|2a+2\}$ or $\{a|11|9-a\}$ where a is in $\mathbb{Z}_{13}\backslash\{11,12\}$. If $a \neq 5, 10$, then $9-a \neq 11, 12$ and $2a+2 \neq 11, 12$ for any a in $\mathbb{Z}_{13} \setminus \{5, 10, 11, 12\}$, so we transform the diagram as seen in Figure 4 or Figure 5.

If $a = 5$ as an under-arc, we transform the diagram as seen in Figure 6. Now, a cannot equal 5 as an over-arc, otherwise $2a+2 = 12$ contradicting our assumption that no arc is colored by 12.

If $a = 10$ as an over-arc, we transform the diagram as seen in Figure 4. Similarly a cannot equal 10 as an under-arc, otherwise $9-a = 12$ which is a contradiction. Next, we remove 11 as an over-arc by transforming any crossings of the form $\{a|11|9-a\}$. Since $9-a \neq 11, 12$ we have that $a \neq 10$. Therefore if $a \neq 5, 7$, then $2a+2 \neq 11, 12$ and $3a+4 \neq 11, 12$ for any a in $\mathbb{Z}_{13} \setminus \{5, 7, 10, 11, 12\}$ we transform the diagram as seen in Figure 7.

If $a = 5$ or $a = 7$, we transform the diagram as seen in Figure 8.

We complete the proof of the lemma by removing 11 as an under-arc in a case by case method. We first consider the case where 11 is an under-arc connecting two crossings of the form $\{11|a|2a+2\}$. Since $2a+2 \neq 11, 12$ we have that $a \neq 5$. If $a \neq 7, 8,$, then $3a+4 \neq 11, 12$

and $4a+6 \neq 11, 12$ for any a in $\mathbb{Z}_{13} \setminus \{5, 7, 8, 11, 12\}$, we transform the diagram as seen in Figure 9. If $a = 7$, we transform the diagram as seen in the following Figure 14.

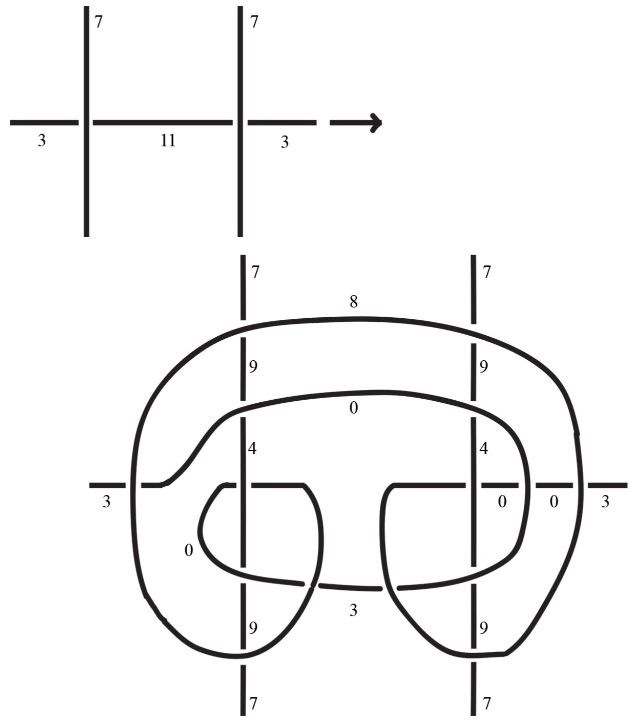

Figure 14.: Transformation of 11 between two crossings $\{3|7|11\}$.

If $a = 8$, we transform the diagram as seen in Figure 11. Now we consider the case where 11 is an under-arc connecting two crossings of the from $\{2a+2|a|11\}$ and $\{11|b|2b+2\}$ where $a \neq b$ for any a and b in $\mathbb{Z}_{13} \setminus \{5, 11, 12\}$. (Note that $a, b \neq 5$ otherwise $2a + 2 = 12$ or $2b + 2 = 12$.)

If $(a, b) \neq (0, 6), (6, 0), (3, 7)$, or $(7, 3)$, then either $2a-2b-2 \neq 11, 12$ and $2a-b \neq 11, 12$ or $2b-2a-2 \neq 11, 12$ and $2b-a \neq 11, 12$ for or any a and b in $\mathbb{Z}_{13} \setminus \{5, 11, 12\}$ we transform the diagram as seen in Figure 10.

If $(a, b) = (0, 6)$, we transform the diagram as seen in the following Figure 15. Similarly for the case of $(6, 0)$.

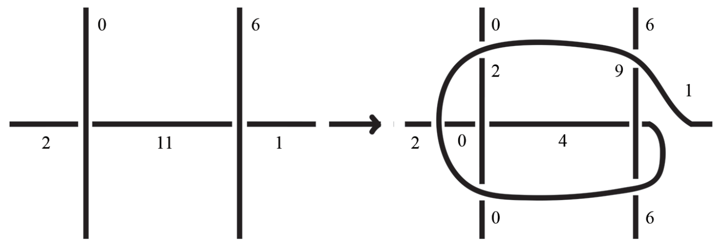

Figure 15.: Transformation of 11 between the crossings $\{2|0|11\}$ and $\{11|6|1\}$.

If $(a, b) = (3, 7)$, we transform the diagram as seen in the following Figure 16. Similarly for the case of $(7, 3)$.

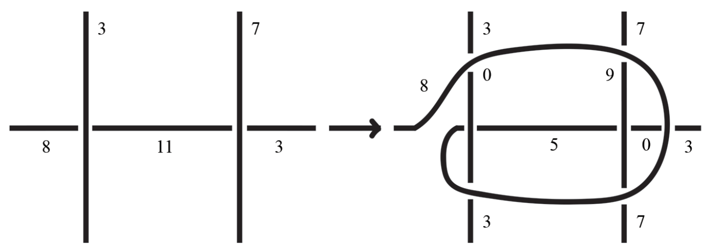

Figure 16.: Transformation of 11 between the crossings $\{8|3|11\}$ and $\{11|7|3\}$.

$\square$

1.3.3 Eliminating the color 7

Lemma 1.3.4 *Any 13-colorable knot has a 13-colored diagram D with no arc colored by 7, 11, or 12.*

Proof. By the previous lemmas we may assume that no arc in D is colored by 11 or 12. We first transform any crossing of the form $\{7|7|7\}$. If there is any crossing of the form $\{7|7|7\}$, there is an adjacent crossing of the form $\{7|a|2a+6\}$ or $\{a|7|1-a\}$ where a is in $\mathbb{Z}_{13} \setminus \{7, 11, 12\}$. If $a \neq 2, 3, 9$, then $1 - a \neq 7, 11, 12$ and $2a + 6 \neq 7, 11, 12$ for any a in $\mathbb{Z}_{13} \setminus \{2, 3, 7, 9, 11, 12\}$, so we transform the diagram as seen in Figure 4 or Figure 5.

If $a = 2$ as an over-arc, we transform the diagram as seen in Figure 4. Note that a cannot equal 2 as an under-arc, otherwise $1 - a = 12$ contradicting our assumption that no arc is colored by 12.

As an over-arc or an under-arc a cannot be 3, otherwise $1-a=11$ and $2a+6=12$, contradicting our assumption that no arc is colored by 11 or 12. If $a=9$ as an under-arc, we transform the diagram as seen in Figure 6. Note that a cannot equal 9 as an over-arc otherwise $2a+6=11$ contradicting our assumption that no arc is colored by 11. Therefore any crossings of the form $\{7|7|7\}$ are removed.

Next, we remove 7 as an over-arc by transforming any crossings of the form $\{a|7|1-a\}$. Since $1-a \neq 7, 11, 12$ we have that $a \neq 2, 3$. Therefore if $a \neq 0, 4, 9$, then $2a+6 \neq 7, 11, 12$ and $3a+12 \neq 7, 11, 12$ for any a in $\mathbb{Z}_{13} \setminus \{0, 2, 3, 4, 7, 9, 11, 12\}$ we transform the diagram as seen in Figure 7. If $a = 0, 4, 9$, we transform the diagram as seen in Figure 8. We complete the proof of the lemma by removing 7 as an under-arc in a case by case method. We first consider the case where 7 is an under-arc connecting two crossings of the form $\{7|a|2a+6\}$. Since $2a+6 \neq 7, 11, 12$ we have that $a \neq 3, 9$. If $a \neq 0, 4, 5, 8$, then $3a+12 \neq 7, 11, 12$ and $4a+5 \neq 7, 11, 12$ for any a in $\mathbb{Z}_{13} \setminus \{0, 3, 4, 5, 7, 8, 9, 11, 12\}$ we transform the diagram as seen in Figure 9. If $a=0$, we transform the diagram as seen in the following Figure 17.

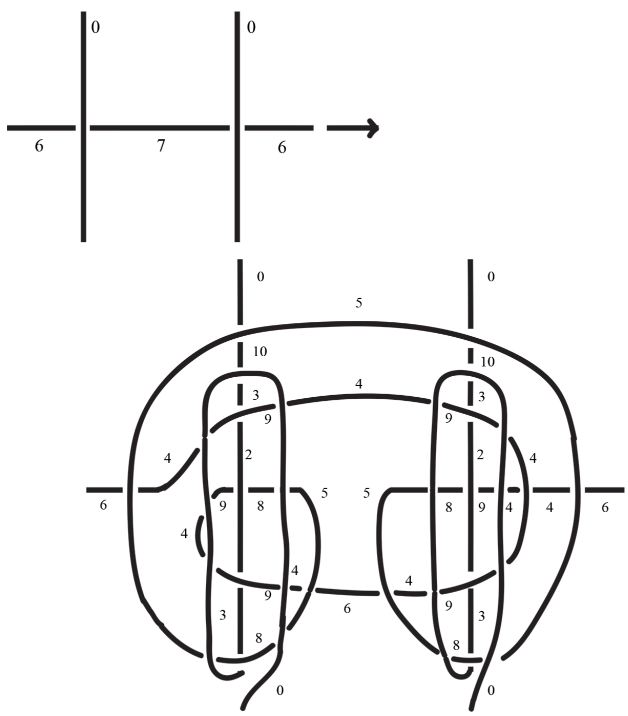

Figure 17.: Transformation of 7 between two crossings $\{6|0|7\}$.

If $\mathfrak{a} = 4$, we transform the diagram as seen in the following Figure 18.

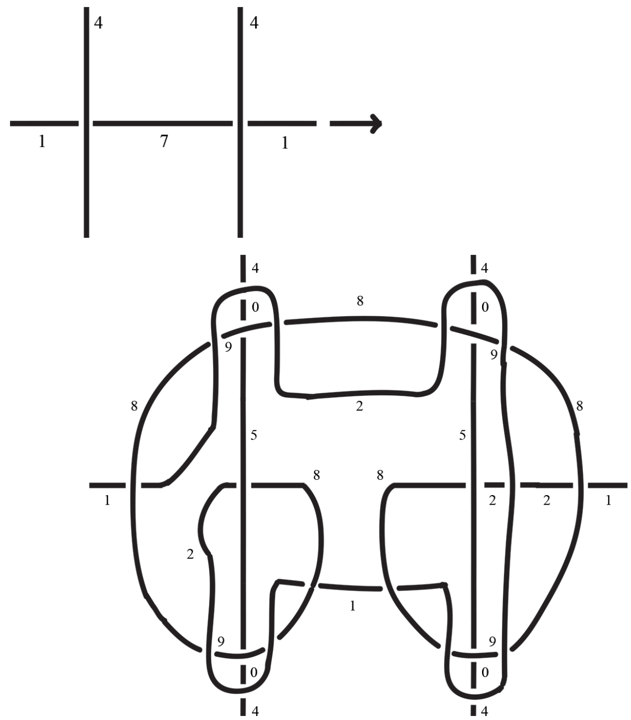

Figure 18.: Transformation of 7 between two crossings $\{1|4|7\}$.

If $\mathfrak{a} = 5$, we transform the diagram as seen in the following Figure 19.

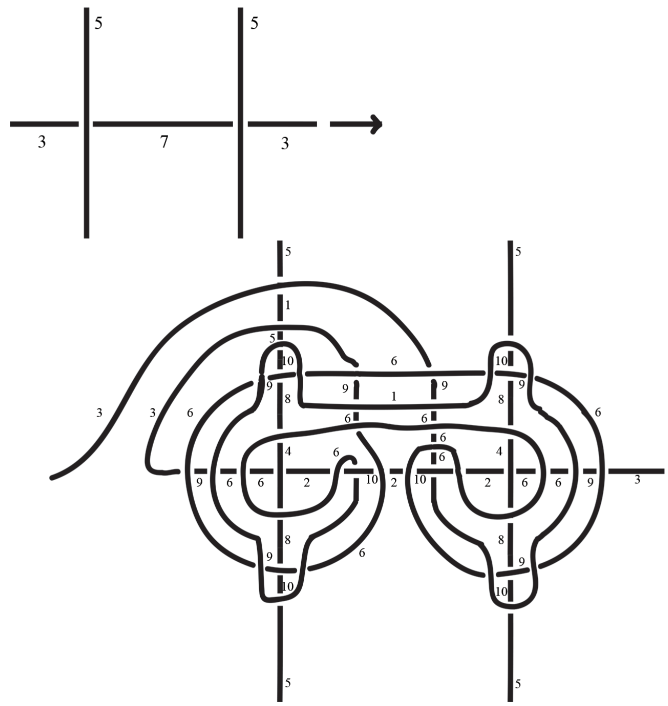

Figure 19.: Transformation of 7 between two crossings $\{3|5|7\}$.

If $a = 8$, we transform the diagram as seen in the Figure 11. Now we consider the case where 7 is an under-arc connecting two crossings of the from $\{2a{+}6|a|7\}$ and $\{7|b|2b{+}6\}$ where $a \neq b$ for any a and b in $\mathbb{Z}_{13} \backslash \{3, 7, 9, 11, 12\}$. (Note that $a, b \neq 3, 9$, otherwise $2a{+}6 = 11, 12$ or $2b + 6 = 11, 12$.) If $(a, b) \neq (0, 2), (2, 0), (0, 6), (6, 0), (1, 4), (4, 1), (4, 8), (8, 4)$, then either $2a - 2b - 6 \neq 7, 11, 12$ and $2a - b \neq 7, 11, 12$ or $2b - 2a - 6 \neq 7, 11, 12$ and $2b - a \neq 7, 11, 12$ for any a and b in $\mathbb{Z}_{13} \setminus \{3, 7, 9, 11, 12\}$ we transform the diagram as seen in Figure 10. If $(a, b) = (0, 2)$, we transform the diagram as seen in the following Figure 20. Similarly for the case of $(2, 0)$.

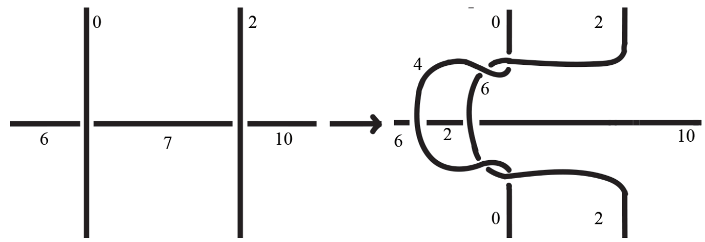

Figure 20.: Transformation of 7 between the crossings $\{6|0|7\}$ and $\{7|2|10\}$.

If $(a, b) = (0, 6)$, we transform the diagram as seen in the following Figure 21. Similarly for the case of $(6, 0)$.

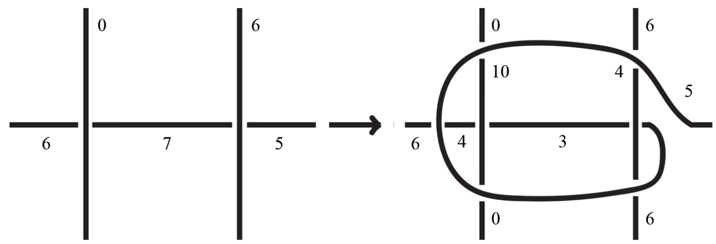

Figure 21.: Transformation of 7 between the crossings $\{6|0|7\}$ and $\{7|6|5\}$.

If $(a, b) = (1, 4)$, we transform the diagram as seen in the following Figure 22. Similarly for the case of $(4, 1)$.

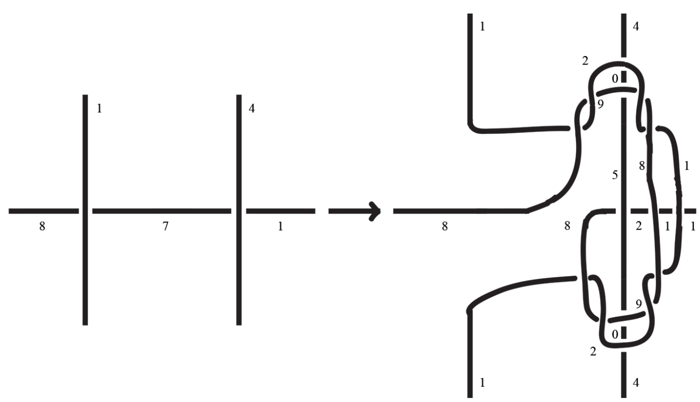

Figure 22.: Transformation of 7 between the crossings $\{8|1|7\}$ and $\{7|4|1\}$.

If $(a, b) = (4, 8)$, we transform the diagram as seen in the following Figure 23. Similarly for the case of $(8,4)$.

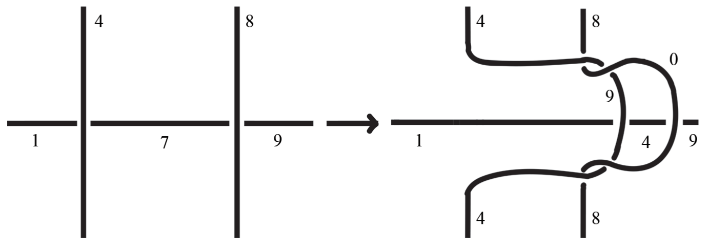

Figure 23.: Transformation of 7 between the crossings $\{1|4|7\}$ and $\{7|8|9\}$.

$\square$

1.3.4 Eliminating the color 8

Lemma 1.3.5 *Any* 13*-colorable knot has a* 13*-colored diagram* D *with no arc colored by* 7*,* 8*,* 11*, or* 12*.*

Proof. By the previous lemmas we may assume that no arc in D is colored by $7, 11$, or 12. We first transform any crossing of the form $\{8|8|8\}$. If there is any crossing of the form $\{8|8|8\}$, there is an adjacent crossing of the form $\{8|a|2a+5\}$ or $\{a|8|3-a\}$ where a is in $\mathbb{Z}_{13} \setminus \{7, 8, 11, 12\}$. If $a \neq 1, 3, 4, 5, 9, 10$, then $3-a \neq 7, 8, 11, 12$ and $2a+5 \neq 7, 8, 11, 12$ for any a in $\mathbb{Z}_{13} \setminus \{1, 3, 4, 5, 7, 8, 9, 10, 11, 12\}$, so we transform the diagram as seen in Figure 4 or Figure 5. If $a = 4, 5, 9$ as an over-arc, we transform the diagram as seen in Figure 4. Note that a cannot be $4, 5$, or 9 as an under-arc otherwise $3 - a = 7, 11, 12$ contradicting our assumption that no arc is colored by $7, 11$, or 12. If $a = 1$ as an under-arc, we transform the diagram as seen in Figure 6. Note that a cannot be 1 as an over-arc otherwise $2a+5 = 7$ contradicting our assumption that no arc is colored by 7. If $a = 3$ as an under-arc, we transform the diagram as seen in Figure 6. Note that a cannot be 3 as an over-arc otherwise $2a+5 = 11$ contradicting our assumption that no arc is colored by 11. If $a = 10$ as an under-arc, we transform the diagram as seen in Figure 6. Note that a cannot be 10 as an over-arc otherwise $2a+5 = 12$ contradicting our assumption that no arc is colored by 12. Therefore any crossings of the form $\{8|8|8\}$ are removed. Next, we remove 8 as an over-arc by transforming any crossings of the form $\{a|8|3-a\}$. Since $3-a \neq 7, 8, 11, 12$ we have that

$a \neq 4, 5, 9$. Therefore if $a \neq 1, 3, 10$, then $2a + 5 \neq 7, 8, 11, 12$ and $3a + 10 \neq 7, 8, 11, 12$ for any a in $\mathbb{Z}_{13} \setminus \{1, 3, 4, 5, 7, 8, 9, 10, 11, 12\}$ we transform the diagram as seen in Figure 7. If $a = 1, 3, 10$, we transform the diagram as seen in Figure 8. We complete the proof of the lemma by removing 8 as an under-arc in a case by case method. We first consider the case where 8 is an under-arc connecting two crossings of the form $\{8|a|2a + 5\}$. Since $2a + 5 \neq 7, 8, 11, 12$ we have that $a \neq 1, 3, 10$. If $a \neq 5, 9$, then $3a + 10 \neq 7, 8, 11, 12$ and $4a + 2 \neq 7, 8, 11, 12$ for any a in $\mathbb{Z}_{13} \setminus \{1, 3, 5, 7, 8, 9, 10, 11, 12\}$ we transform the diagram as seen in Figure 9.

If $a = 5$, we transform the diagram as seen in the following Figure 24.

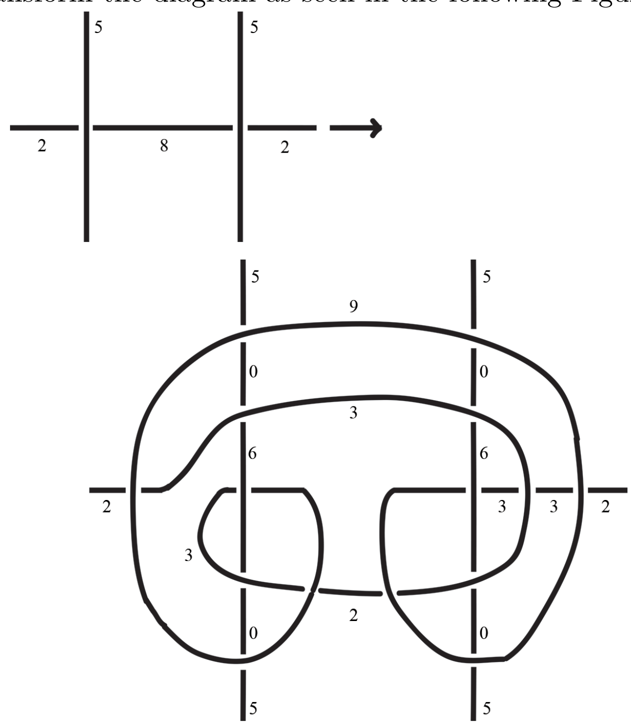

Figure 24.: Transformation of 8 between two crossings $\{2|5|8\}$.

If $a = 9$, we transform the diagram as seen in the following Figure 25.

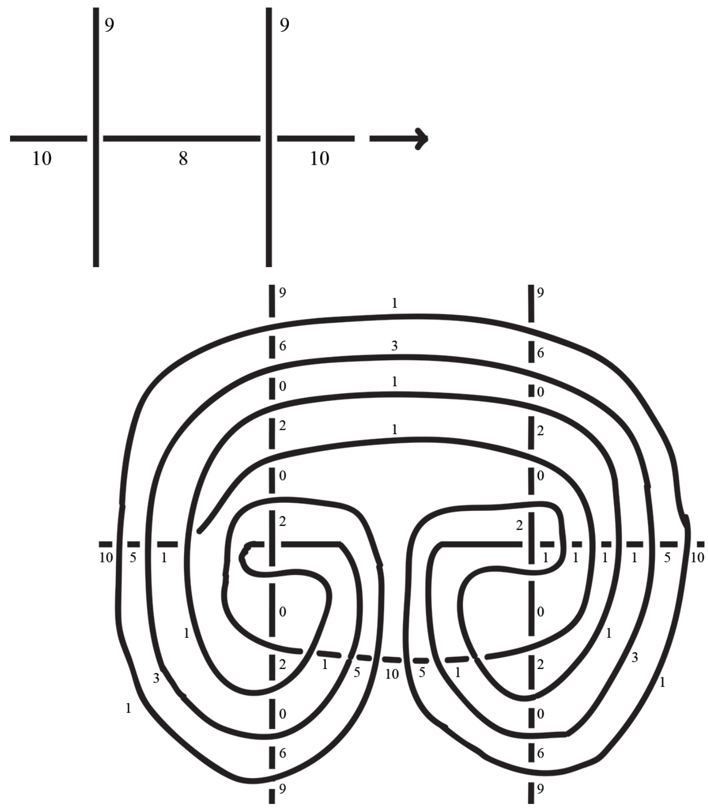

Figure 25.: Transformation of 8 between two crossings $\{10|9|8\}$.

Now we consider the case where 8 is an under-arc connecting two crossings of the from $\{2a+5|a|8\}$ and $\{8|b|2b+5\}$ where $a \neq b$ for any a and b in $\mathbb{Z}_{13} \setminus \{1,3,7,8,10,11,12\}$. (Note that $a, b \neq 1, 3, 10$ otherwise $2a+5 = 7, 11, 12$ or $2b+5 = 7, 11, 12$.). If $(a, b) \neq (0,2), (2,0), (0,6), (6,0), (2,5), (5,2)$ then either $2a-2b-5 \neq 7, 8, 11, 12$ and $2a-b \neq 7, 8, 11, 12$ or $2b-2a-5 \neq 7, 8, 11, 12$ and $2b-a \neq 7, 8, 11, 12$ for any a and b in $\mathbb{Z}_{13} \setminus \{1,3,7,8,10,11,12\}$ we transform the diagram as seen in Figure 10. If $(a, b) = (0, 2)$, we transform the diagram as seen in the following Figure 26. Similarly for the case of $(2, 0)$.

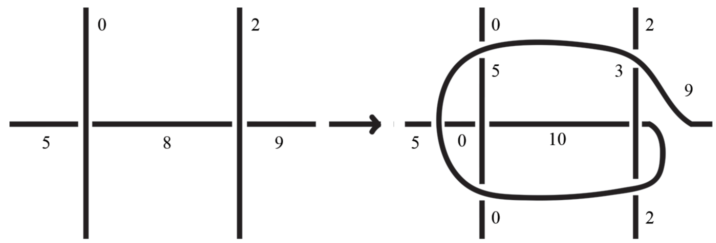

Figure 26.: Transformation of 8 between the crossings $\{5|0|8\}$ and $\{8|2|9\}$.

If $(a, b) = (0, 6)$, we transform the diagram as seen in the following Figure 27. Similarly for the case of $(6, 0)$.

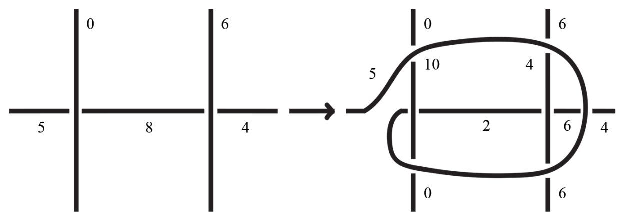

Figure 27.: Transformation of 8 between the crossings $\{5|0|8\}$ and $\{8|6|4\}$.

If $(a, b) = (2, 5)$, we transform the diagram as seen in the following Figure 28. Similarly for the case of $(5, 2)$.

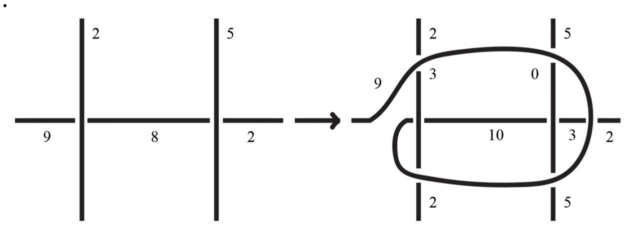

Figure 28.: Transformation of 8 between the crossings $\{9|2|8\}$ and $\{8|5|2\}$.

□

1.3.5 Eliminating the color 6

Lemma 1.3.6 *Any* 13*-colorable knot has a* 13*-colored diagram* D *with no arc colored by* 6*,* 7*,* 8*,* 11*, or* 12*.*

Proof. By the previous lemmas we may assume that no arc in D is colored by $7, 8, 11$, or 12. We first transform any crossing of the form $\{6|6|6\}$. If there is any crossing of the form $\{6|6|6\}$, there is an adjacent crossing of the form $\{6|a|2a + 7\}$ or $\{a|6|12 - a\}$ where a is in $\mathbb{Z}_{13} \setminus \{6, 7, 8, 11, 12\}$. With the exceptions of $a = 0, 2, 9$ as an over-arc (when $2a + 7 = 7, 8, 11, 12$) and $a = 0, 1, 4, 5$ as an under-arc (when $12 - a = 7, 8, 11, 12$) we transform the diagram as seen in Figure 4 or Figure 5. Now we must check when $a = 0, 2, 9$ as an under-arc. First and foremost a cannot equal 0 as an under-arc otherwise $12 - a = 12$ contradicting our assumption that no arc is colored by 12. If $a = 2, 9$ as an under-arc, we transform the diagram as seen in Figure 6. Therefore any crossings of the form $\{6|6|6\}$ are removed. Next, we remove 6 as an over-arc by transforming any crossings of the form $\{a|6|12 - a\}$. Since $12 - a \neq 6, 7, 8, 11, 12$ we have that $a \neq 0, 1, 4, 5$. With the exceptions of $a = 2, 9$ (when $2a + 7 = 6, 7, 8, 11, 12$ and $3a + 1 = 6, 7, 8, 11, 12$) we transform the diagram as seen in Figure 7. If $a = 2$ or $a = 9$, we transform the diagram as seen in Figure 8. We complete the proof of the lemma by removing 6 as an under-arc in a case by case method. We first consider the case where 6 is an under-arc connecting two crossings of the form $\{6|a|2a + 7\}$. Since $2a + 7 \neq 6, 7, 8, 11, 12$ we have that $a \neq 0, 2, 9$. If $a \neq 1, 3, 4$, then $3a + 1 \neq 6, 7, 8, 11, 12$ and $4a + 8 \neq 6, 7, 8, 11, 12$, so we transform the diagram as seen in Figure 9. If $a = 1$, we transform the diagram as seen in Figure 11. If $a = 3$, we transform the diagram as seen in the following Figures 29, 30, and 31.

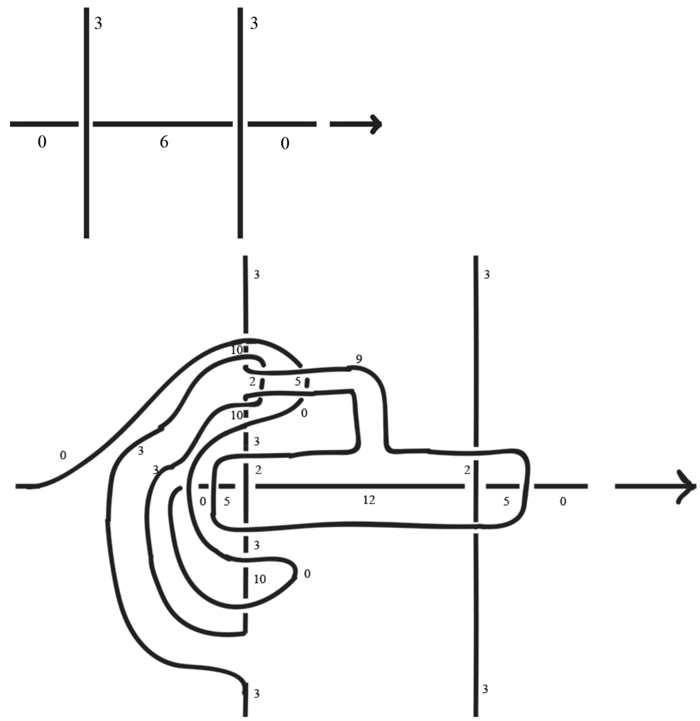

Figure 29.: Starting transformation of 6 between two crossings {0|3|6}.

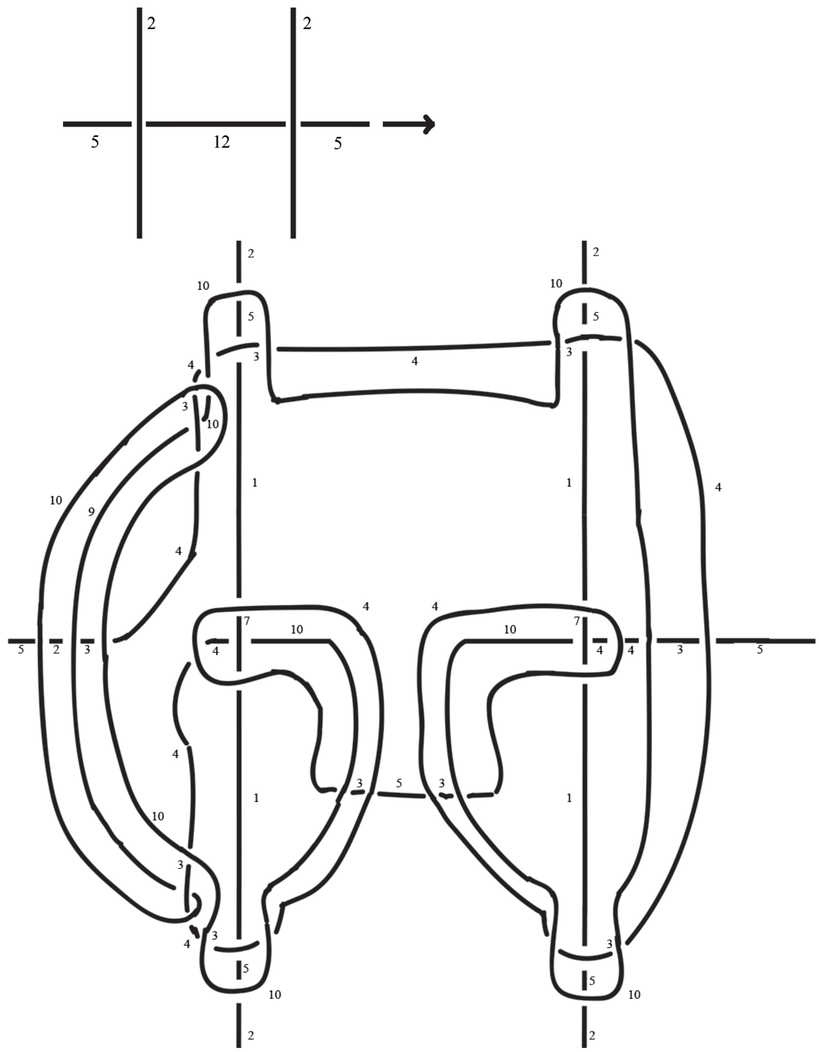

Figure 30.: Intermediate transformation of 6 between two crossings {0|3|6}.

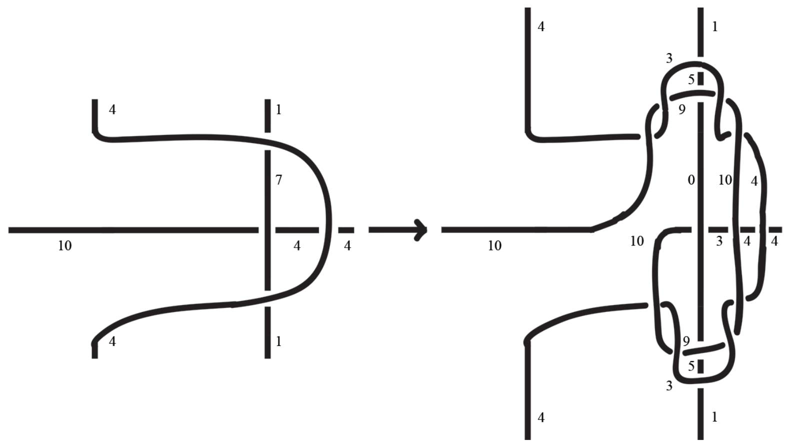

Figure 31.: Ending transformation of 6 between two crossings $\{0|3|6\}$.

If $a = 4$, we transform the diagram as seen in Figure 11. Now we consider the case where 6 is an under-arc connecting two crossings of the from $\{2a+7|a|6\}$ and $\{6|b|2b+7\}$ where $a \neq b$ for any a and b in $\mathbb{Z}_{13} \setminus \{0, 2, 6, 7, 8, 9, 11, 12\}$. (Note that $a, b \neq 0, 2, 9$ otherwise $2a + 7 = 7, 8, 11, 12$ or $2b + 7 = 7, 8, 11, 12$.)

If $(a, b) \neq (1, 4), (4, 1)$, then either $2a - 2b - 7 \neq 6, 7, 8, 11, 12$ and $2a - b \neq 6, 7, 8, 11, 12$ or $2b - 2a - 7 \neq 6, 7, 8, 11, 12$ and $2b - a \neq 6, 7, 8, 11, 12$ for or any a and b in $\mathbb{Z}_{13} \setminus \{0, 2, 6, 7, 8, 9, 11, 12\}$ we transform the diagram as seen in Figure 10. If $(a, b) = (1, 4)$, we transform the diagram as seen in the following Figure 32. Similarly for the case of $(4, 1)$.

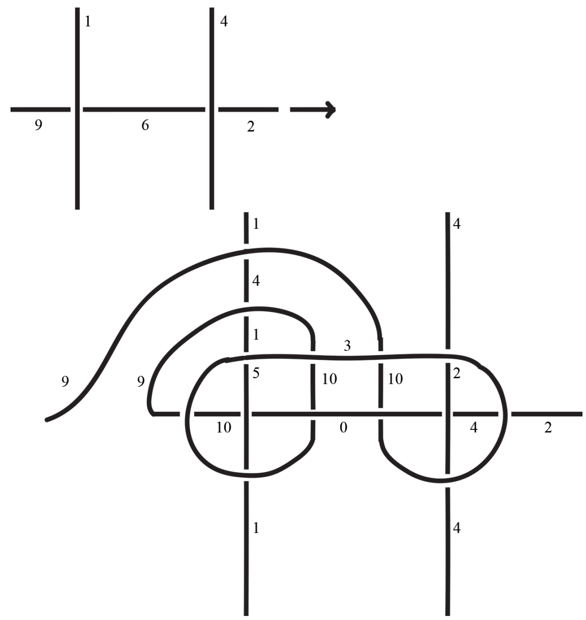

Figure 32.: Transformation of 6 between the crossings $\{9|1|6\}$ and $\{6|4|2\}$.

$\square$

1.3.6 Eliminating the color 1

Lemma 1.3.7 *Any 13-colorable knot has a 13-colored diagram D with no arc colored by 1, 6, 7, 8, 11, or 12.*

Proof. By the previous lemmas we may assume that no arc in D is colored by $6, 7, 8, 11$, or 12. We first transform any crossing of the form $\{1|1|1\}$. If there is a crossing of the form $\{1|1|1\}$, there is an adjacent crossing of the form $\{1|a|2a + 12\}$ or $\{a|1|2 - a\}$ where a is in $\mathbb{Z}_{13} \setminus \{1, 6, 7, 8, 11, 12\}$. With the exceptions of $a = 0, 4, 10$ as an over-arc (when $2a + 12 = 6, 7, 8, 11, 12$) and $a = 3, 4, 9$ as an under-arc (when $2 - a = 6, 7, 8, 11, 12$) we transform the diagram as seen in Figure 4 or Figure 5. Now we must check when $a = 0, 4, 10$ as an under-arc. First off, a cannot be 4 as an under-arc otherwise $2 - a = 11$ contradicting our assumption that no arc is colored by 11. If $a = 0$ or $a = 10$ as an under-arc, we transform the diagram as seen in Figure 6. Therefore any crossings of the form $\{1|1|1\}$ are removed.

Next, we remove 1 as an over-arc by transforming any crossings of the form $\{a|1|2-a\}$. Since $2-a \neq 1, 6, 7, 8, 11, 12$ we have that $a \neq 3, 4, 9$. With the exceptions of $a = 0, 10$ (when $2a+12 = 1, 6, 7, 8, 11, 12$ and $3a+11 = 1, 6, 7, 8, 11, 12$) we transform the diagram as seen in Figure 7. If $a = 0$ or $a = 10$, we transform the diagram as seen in Figure 8. We complete the proof by removing 1 as an under-arc in a case by case method. We first consider the case where 1 is an under-arc connecting two crossings of the form $\{1|a|2a+12\}$. Since $2a+12 \neq 1, 6, 7, 8, 11, 12$ we have that $a \neq 0, 4, 10$. If $a \neq 3, 9$, then $3a+11 \neq 1, 6, 7, 8, 11, 12$ and $4a+10 \neq 1, 6, 7, 8, 11, 12$, so we transform the diagram as seen in Figure 9. If $a = 3$, we transform the diagram as seen in the following Figure 33.

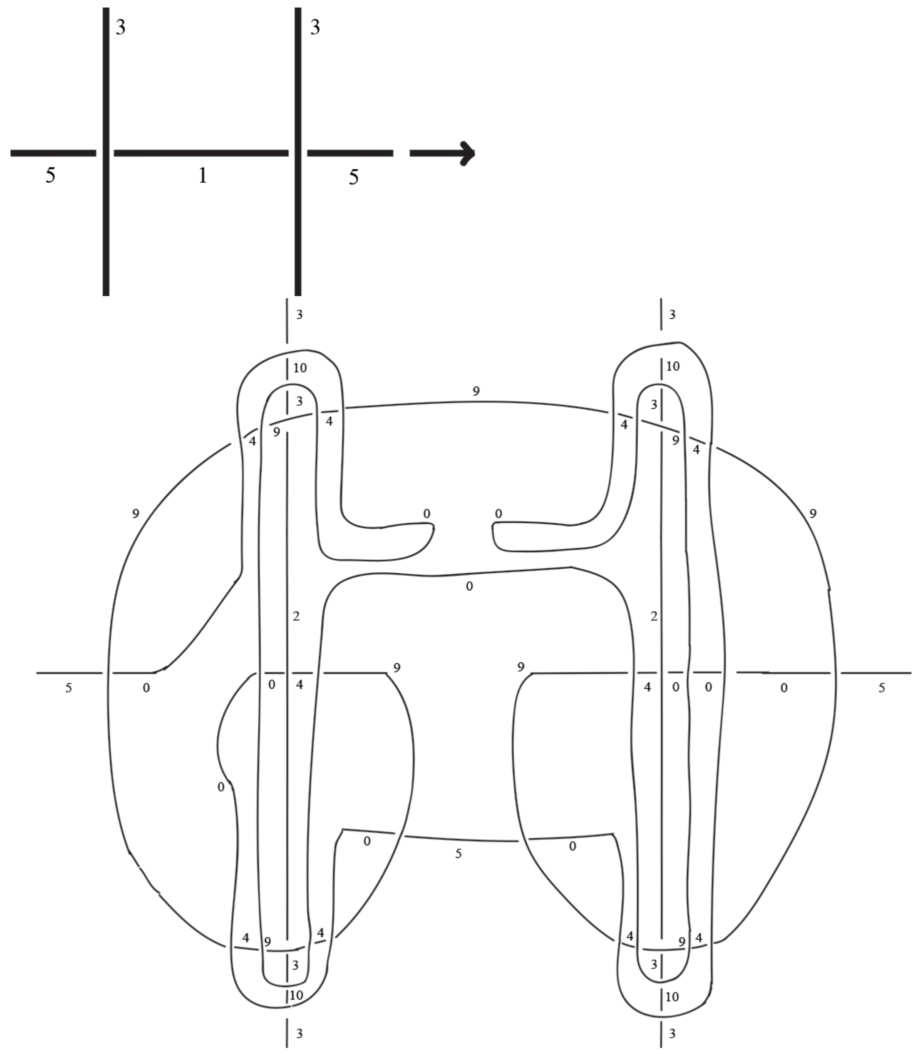

Figure 33.: Transformation of 1 between two crossings $\{5|3|1\}$.

If $\alpha = 9$, we transform the diagram as seen in the following Figure 34.

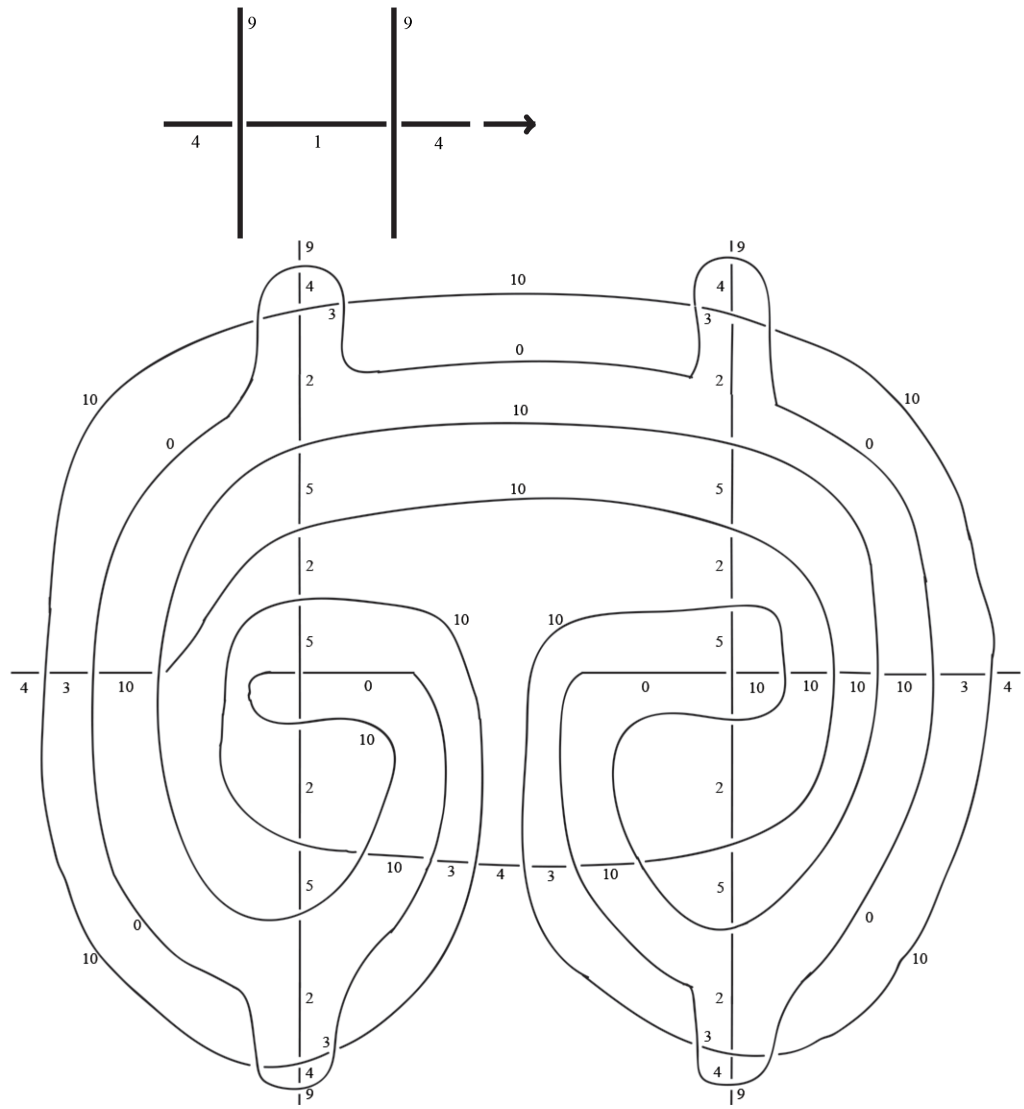

Figure 34.: Transformation of 1 between two crossings $\{4|9|1\}$.

Now we consider the case where 1 is an under-arc connecting two crossings of the from $\{2a+12|a|1\}$ and $\{1|b|2b+12\}$ where $a \neq b$ for any a and b in $\mathbb{Z}_{13} \setminus \{0,1,4,6,7,8,10,11,12\}$. (Note that $a, b \neq 0, 4, 10$ otherwise $2a+12 = 1,6,7,8,11,12$ or $2b+12 = 1,6,7,8,11,12$.) If $(a,b) \neq (2,5), (5,2), (3,5), (5,3)$, then either $2a-2b-12 \neq 1,6,7,8,11,12$ and $2a-b \neq 1,6,7,8,11,12$ or $2b-2a-12 \neq 1,6,7,8,11,12$ and $2b-a \neq 1,6,7,8,11,12$ for any a and b in $\mathbb{Z}_{13} \setminus \{0,1,4,6,7,8,10,11,12\}$ we transform the diagram as seen in Figure 10. If $(a,b) = (2,5)$, we transform the diagram as seen in the following Figure 35. Similarly for the case of $(5, 2)$.

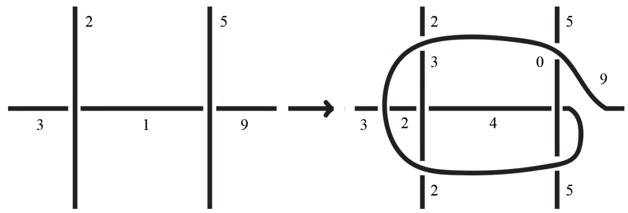

Figure 35.: Transformation of 1 between the crossings $\{3|2|1\}$ and $\{1|5|9\}$.

If $(a, b) = (3, 5)$, we transform the diagram as seen in the following Figure 36. Similarly for the case of $(5, 3)$.

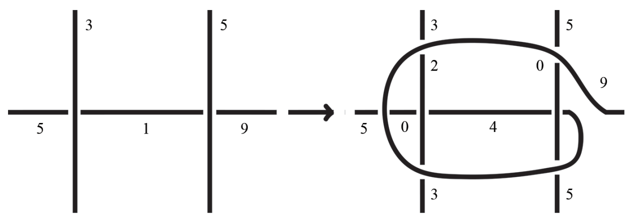

Figure 36.: Transformation of 1 between the crossings $\{5|3|1\}$ and $\{1|5|9\}$.

$\square$

1.3.7 Eliminating the color 10

Lemma 1.3.8 *Any* 13*-colorable knot has a* 13*-colored diagram* D *with no arc colored by* 1*,* 6*,* 7*,* 8*,* 10*,* 11*, or* 12*.*

Note during the proof of Lemma 1.3.8, we will use substitutions of the under-arc cases seen in Figures 38, 39, 40, and 41 above to complete the following transformations. Moreover, once the 10 arcs have been eliminated in Figures 42 and 43 we will use them as substitutions for Figures 44 and 45.

Proof. By the previous lemmas we may assume that no arc in D is colored by $1, 6, 7, 8, 11$, or 12. We first transform any crossing of the form $\{10|10|10\}$. If there is any crossing of the form $\{10|10|10\}$, there is an adjacent crossing of the form $\{10|a|2a+3\}$ or $\{a|10|7-a\}$ where a is in $\mathbb{Z}_{13} \setminus \{1, 6, 7, 8, 10, 11, 12\}$. With the exceptions of $a = 2, 4, 9$ as an over-arc (when $2a + 3 = 1, 6, 7, 8, 11, 12$) and $a = 0, 9$ as an under-arc (when $7 - a = 1, 6, 7, 8, 11, 12$) we transform the diagram as seen in Figure 4 or Figure 5. Now we must check when $a = 2, 4, 9$ as an under-arc. As an under-arc a cannot be 9 otherwise $7 - a = 11$ contradicting our assumption that no arc is colored by 11. If $a = 2$ or $a = 4$ as an under-arc, we transform the diagram as seen in Figure 6. Therefore any crossings of the form $\{10|10|10\}$ are removed. Next, we remove 10 as an over-arc by transforming any crossings of the form $\{a|10|7-a\}$. Since $7 - a \neq 1, 6, 7, 8, 10, 11, 12$ we have that $a \neq 0, 9$. With the exceptions of $a = 2, 4, 5$ (when $2a+3 = 1, 6, 7, 8, 10, 11, 12$ and $3a+6 = 1, 6, 7, 8, 10, 11, 12$) we transform the diagram as seen in Figure 7. If $a = 2$, we transform the diagram as seen in the following Figure 37.

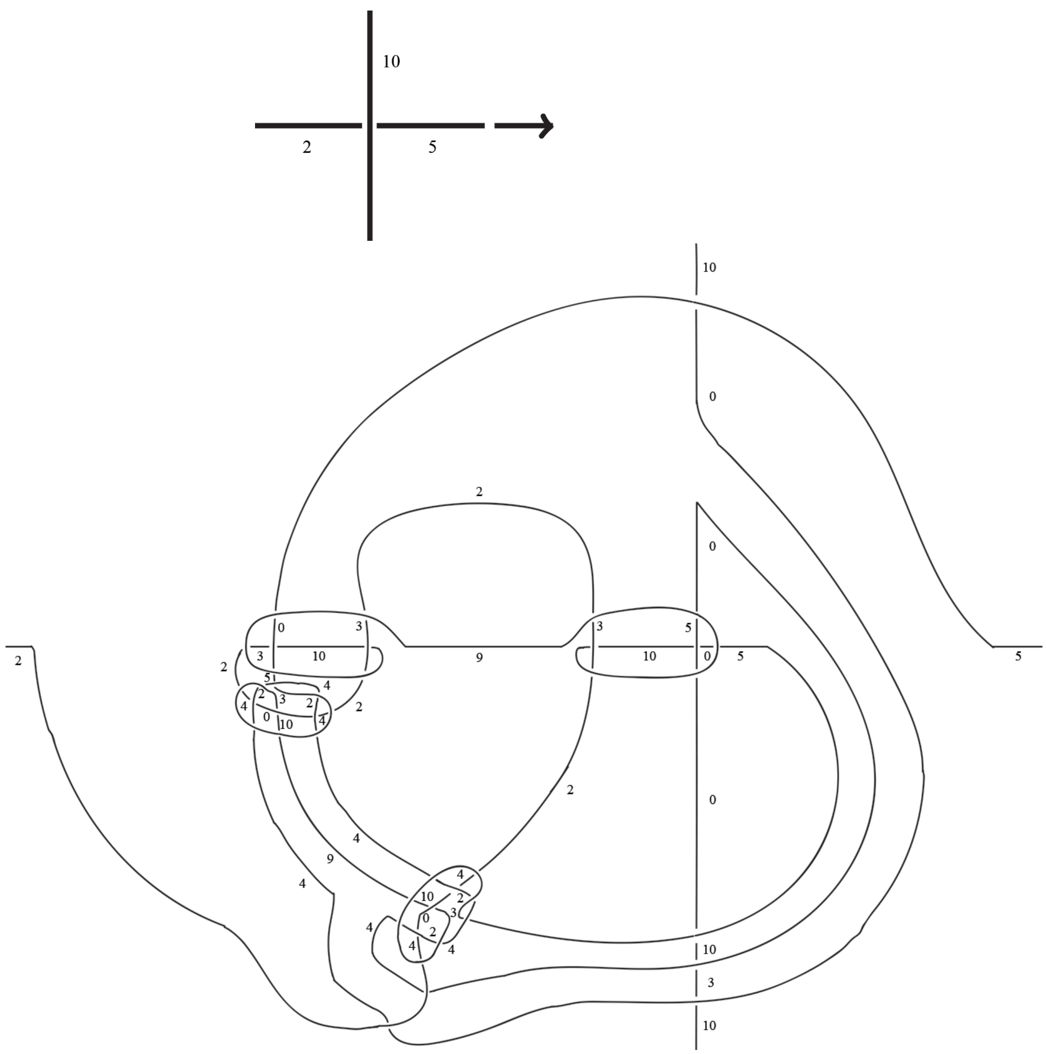

Figure 37.: Transformation of 10 as an over-arc $\{2|10|5\}$.

If $a = 4$, we transform the diagram as seen in Figure 8. If $a = 5$ since $7 - a = 2$, we transform the diagram similarly to the Figure 37, i.e. $a = 2$. We complete the proof by removing 10 as an under-arc in a case by case method. We first consider the case where 10 is an under-arc connecting two crossings of the form $\{10|a|2a+3\}$. Since $2a + 3 \neq 1, 6, 7, 8, 10, 11, 12$ we have that $a \neq 2, 4, 9$. So, we need to check $a = 0, 3, 5$. If $a = 0$, we transform the diagram as seen in the following Figure 38 where the six dashed boxes are given in Figure 39.

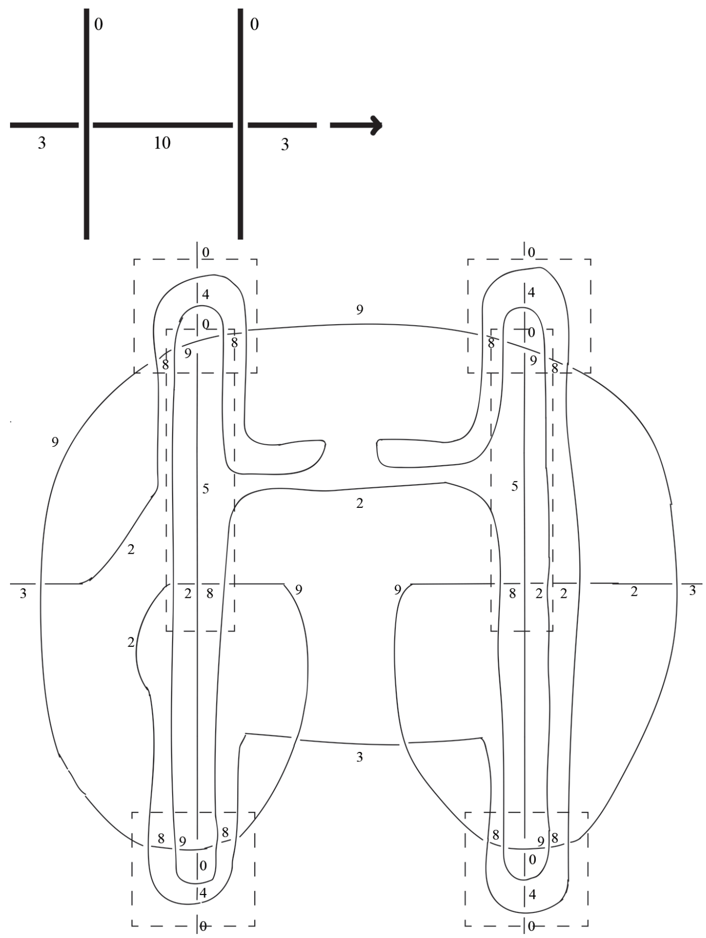

Figure 38.: Starting transformation of 10 between two crossings $\{3|0|10\}$.

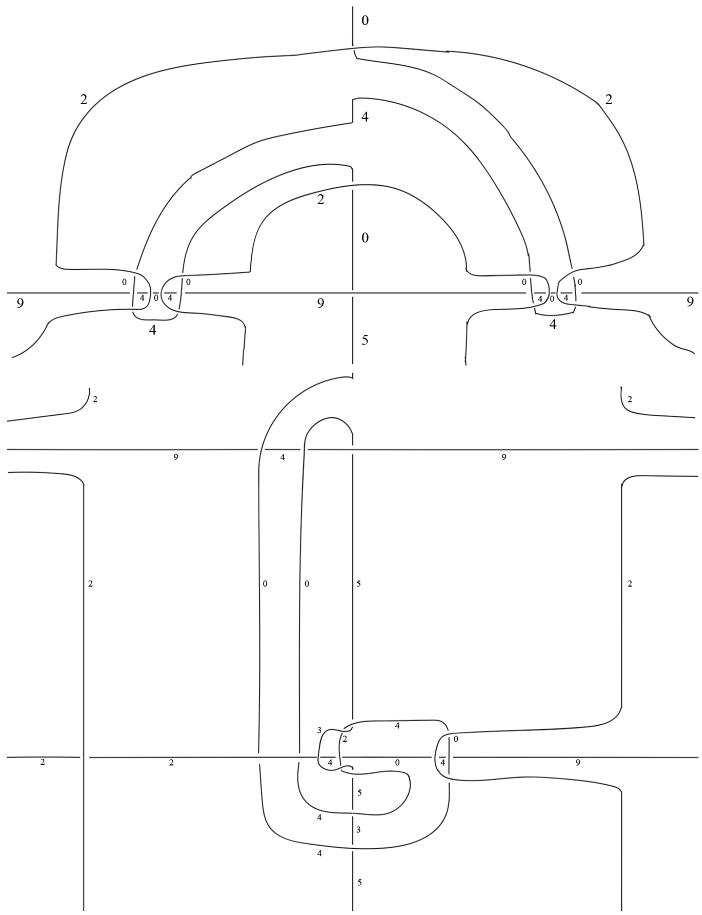

Figure 39.: Ending transformation of 10 between two crossings {3|0|10}.

We shall refer to the above transformations throughout Lemma 1.3.8. As such two variations of this transformation are given below in Figure 40.

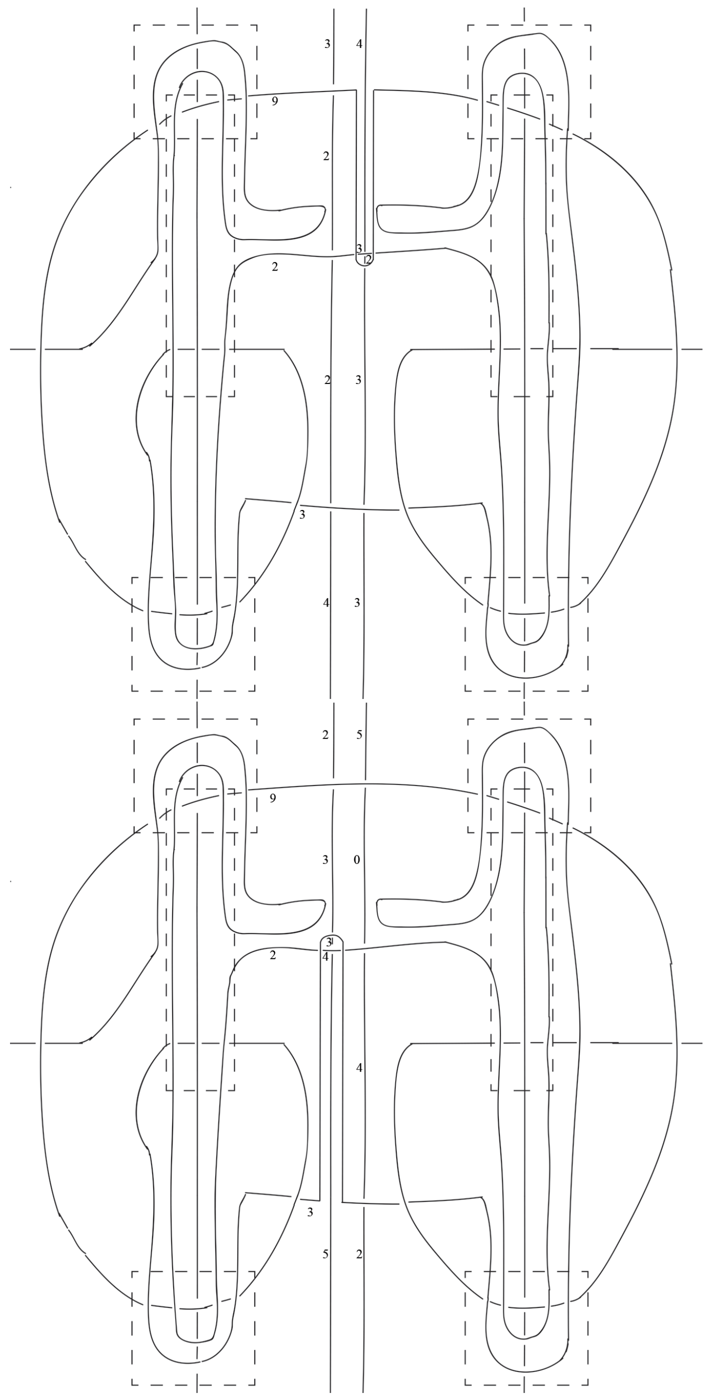

Figure 40.: Variations of the transformation of 10 between two crossings $\{3|0|10\}$.

If $\mathfrak{a} = 3$, we transform the diagram as seen in the following Figure 41.

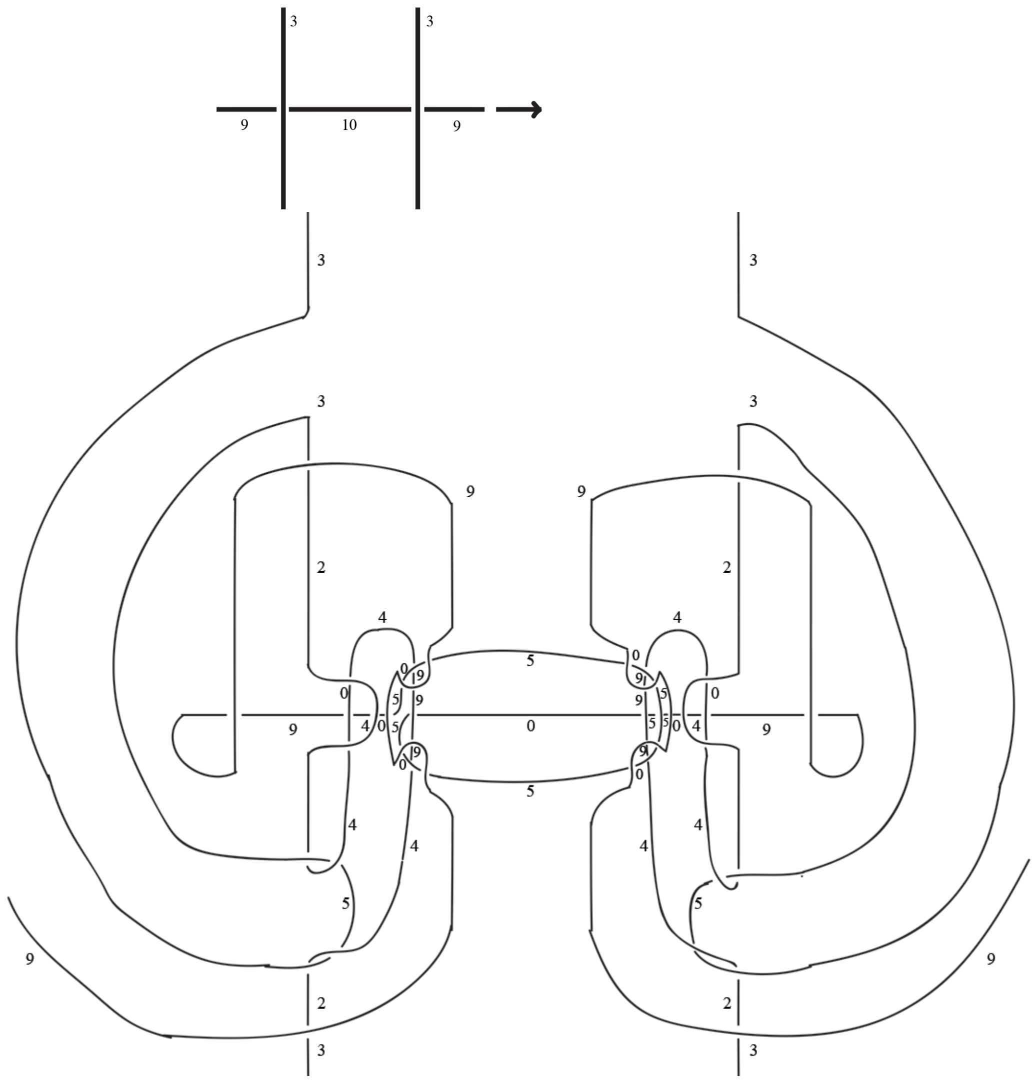

Figure 41.: Transformation of 10 between two crossings $\{9|3|10\}$.

If $a = 5$, we transform the diagram as seen in the following Figure 42. Note that the center of $a = 5$ as well as the six dashed boxes are the same transformations we used for $a = 0$ and its variations. Also, there are two arcs colored by 10 each of which are transformed by $a = 3$ as seen above in Figure 41.

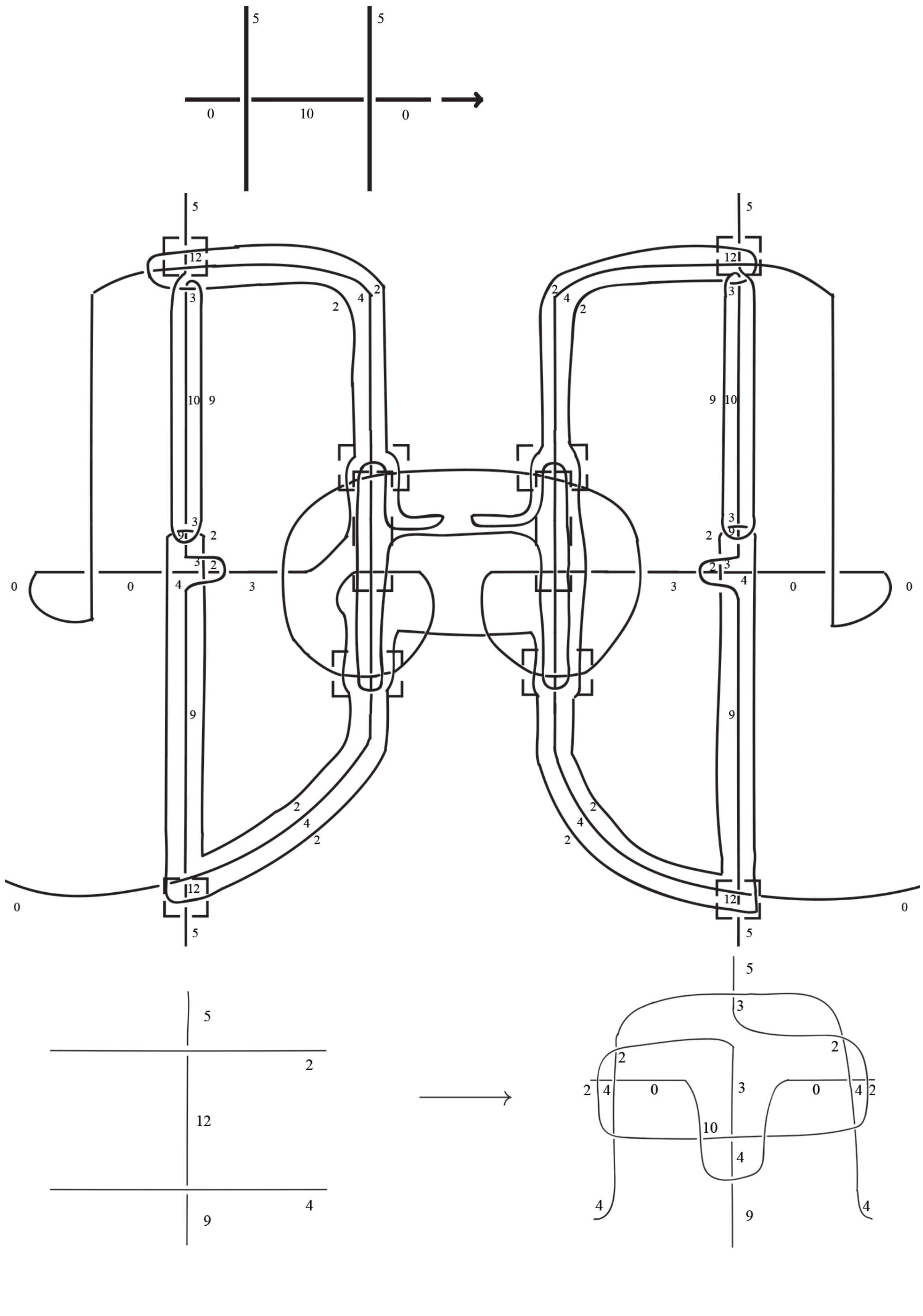

Figure 42.: Starting transformation of 10 between two crossings {0|5|10}.

Now we consider the case where 10 is an under-arc. There are six cases that we need to consider: $(a, b) = (0, 3), (3, 0), (0, 5), (5, 0), (3, 5), (5, 3)$. If $(a, b) = (0, 3)$, we transform the diagram as seen in the following Figure 43. For eliminating the 10 arc see the variations of $a = 0$ in Figure 40 above. Similarly for the case of $(3, 0)$.

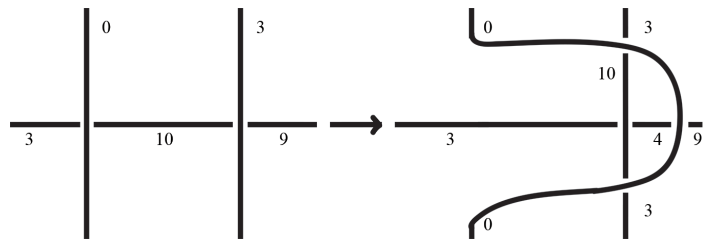

Figure 43.: Starting transformation of 10 between the crossings $\{3|0|10\}$ and $\{10|3|9\}$.

If $(a, b) = (0, 5)$, we transform the diagram as seen in the following Figure 44. For eliminating the 10 arc see $a = 5$ in Figure 42, however we will be using the variations of $a = 0$ in Figure 40 for the center. Similarly for the case of $(5, 0)$.

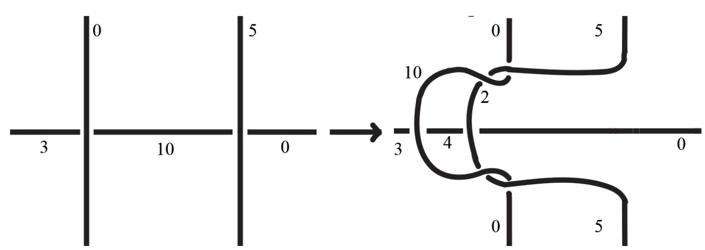

Figure 44.: Starting transformation of 10 between the crossings $\{3|0|10\}$ and $\{10|5|0\}$.

If $(a, b) = (3, 5)$, we transform the diagram as seen in the following Figure 45. For eliminating the arcs colored by 10 see the $(a, b) = (0, 3)$ case in Figure 43 and the $a = 5$ case in Figure 42 using the variations in Figure 40 for the center. Similarly for the case of $(5, 3)$.

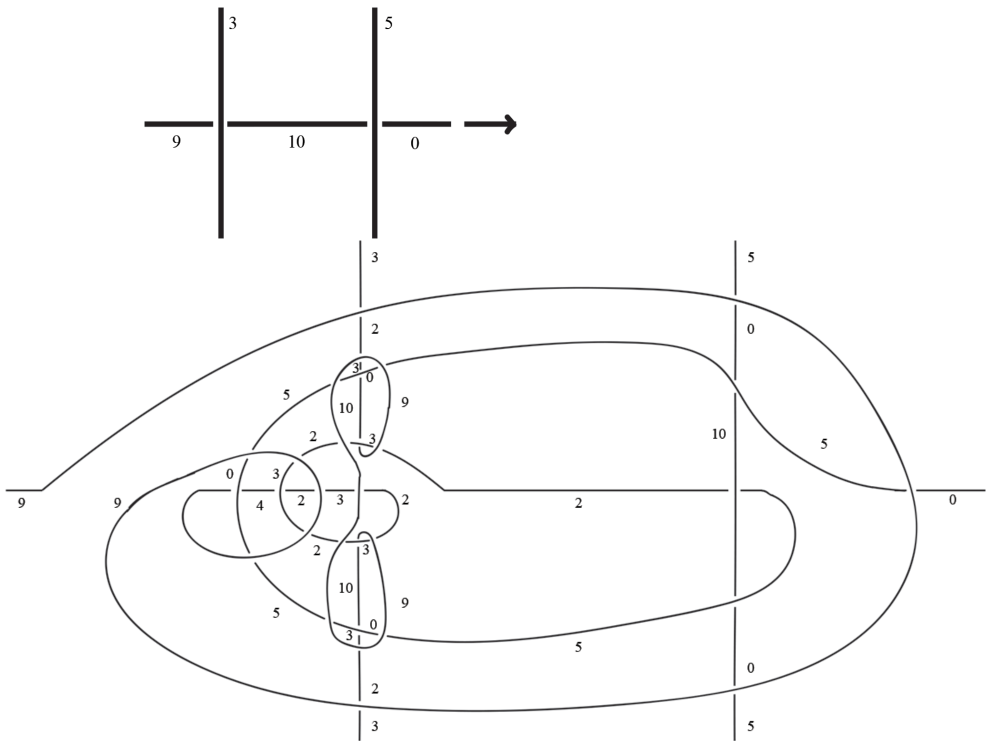

Figure 45.: Starting transformation of 10 between the crossings $\{9|3|10\}$ and $\{10|5|0\}$.

□

1.3.8 Eliminating the color 5

Lemma 1.3.9 *Any* 13*-colorable knot has a* 13*-colored diagram* D *with no arc colored by* 1*,* 5*,* 6*,* 7*,* 8*,* 10*,* 11*, or* 12*.*

Proof. By the previous lemmas we may assume that no arc in D is colored by $1, 6, 7, 8, 10, 11$, or 12. We first transform any crossing of the form $\{5|5|5\}$. If there is any crossing of the form $\{5|5|5\}$, there is an adjacent crossing of the form $\{5|a|2a+8\}$ where a is in $\mathbb{Z}_{13} \setminus \{1, 5, 6, 7, 8, 10, 11, 12\}$. Since $10 - a = 1, 6, 7, 8, 10, 11, 12$ when $a = 0, 2, 3, 4, 9$ a cannot be an under-arc. Therefore, with the exceptions of $a = 0, 2, 3$ as an over-arc (when $2a + 8 = 1, 5, 6, 7, 8, 10, 11, 12$) we transform the diagram as seen in Figure 4. Therefore any crossings of the form $\{5|5|5\}$ are removed. Next, we remove 5 as an over-arc by transforming any crossings of the form $\{a|5|10-a\}$. Since $10 - a \neq 1, 5, 6, 7, 8, 10, 11, 12$ we have that

$a \neq 0, 2, 3, 4, 9$. Therefore, 5 cannot be an over-arc. We complete the proof of Lemma 1.3.9 by removing 5 as an under-arc in a case by case method. We first consider the case where 5 is an under-arc connecting two crossings of the form $\{5|a|2a+8\}$. Since $2a+8 \neq 1, 5, 6, 7, 8, 10, 11, 12$ we have that $a \neq 0, 2, 3$. So, we need to check $a = 4, 9$. If $a = 4$, we transform the diagram as seen in the following Figure 46.

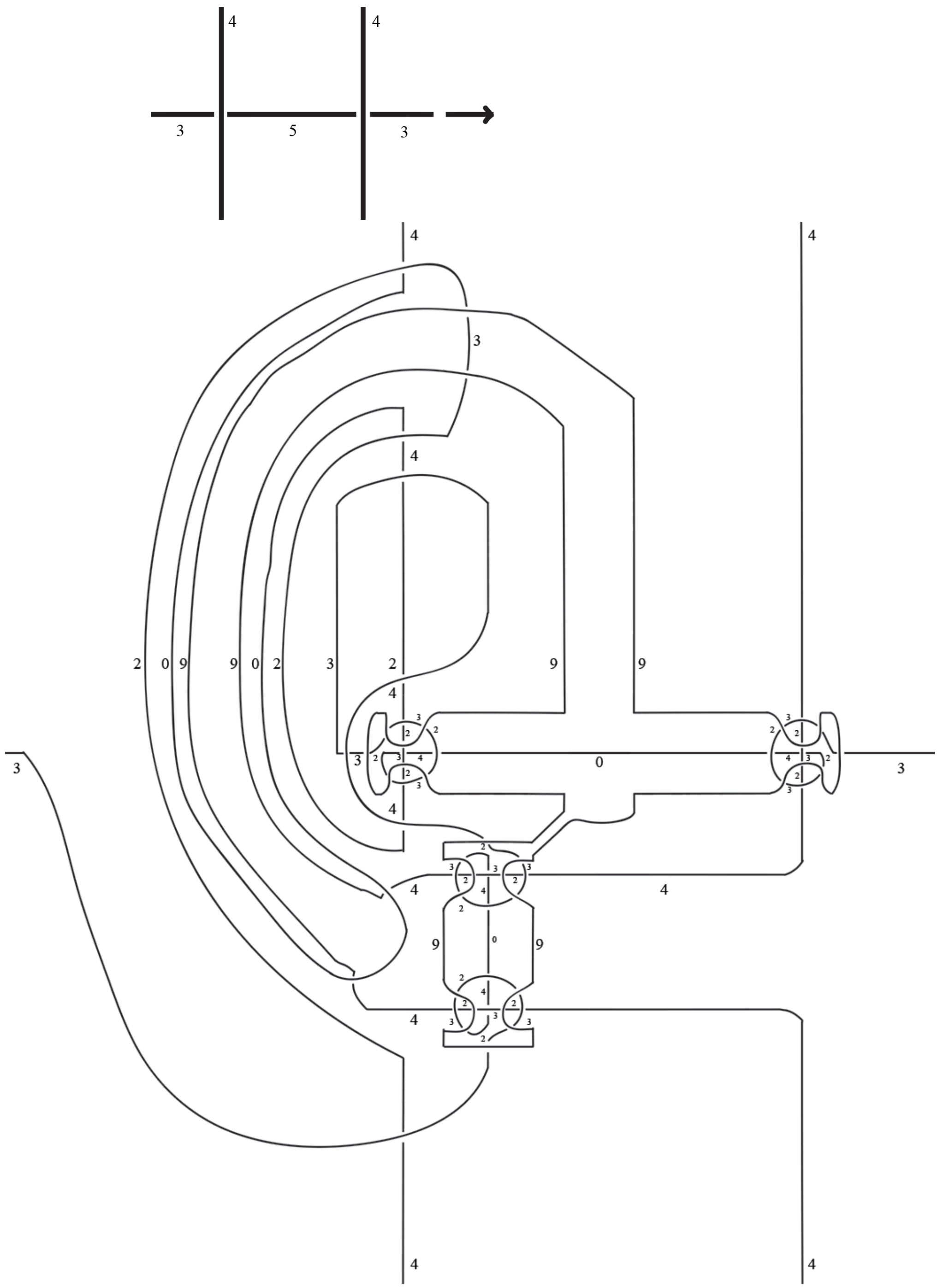

Figure 46.: Transformation of 5 between two crossings {3|4|5}.

If $\mathfrak{a} = 9$, we transform the diagram as seen in the following Figure 47.

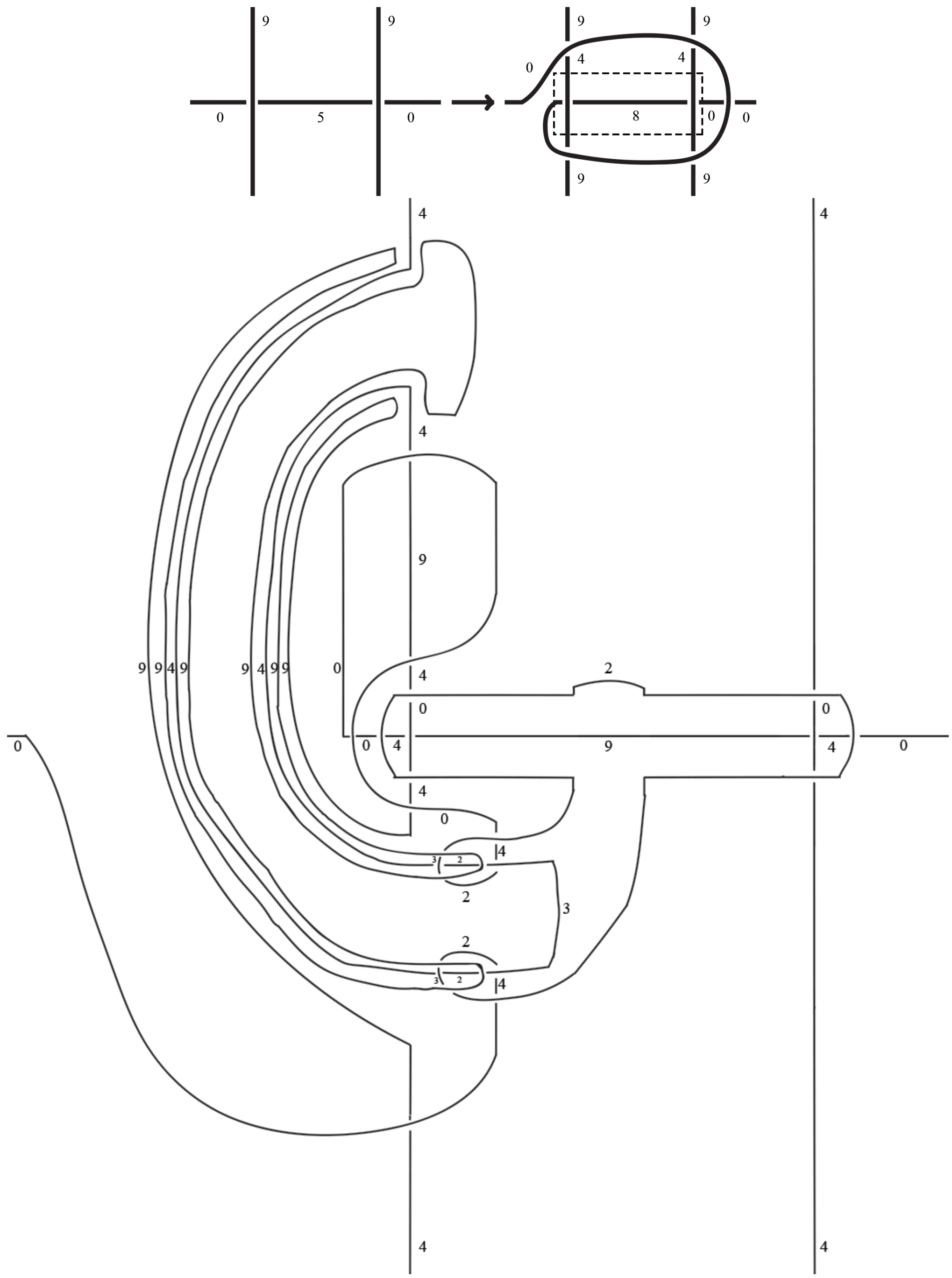

Figure 47.: Transformation of 5 between two crossings $\{0|9|5\}$.

Now we consider the case where 5 is an under-arc connecting two crossings of the forms $\{5|a|2a+8\}$ and $\{5|b|2b+8\}$. Since $2a+8, 2b+8 \neq 1, 5, 6, 7, 8, 10, 11, 12$ there are two cases that we need to consider: $(a, b) = (4, 9), (9, 4)$ If $(a, b) = (4, 9)$, we transform the diagram as seen in the following Figure 48. Similarly for the case of $(9, 4)$.

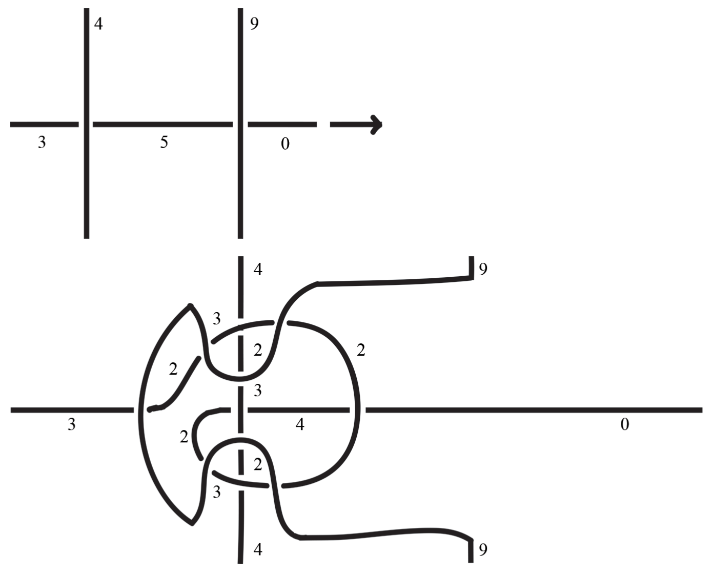

Figure 48.: Transformation of 5 between the crossings $\{3|4|5\}$ and $\{5|9|0\}$.

□

1.3.9 Summary

The above lemmas, in total, reduced the number of colors of an arbitrary 13-colorable knot by eight colors. Therefore, the minimal number of colors using Fox coloring for any 13-colorable knot is exactly five colors.

Chapter 2

Linear Alexander Quandle Colorings of Knots

2.1 Introduction

Quandles are algebraic structures whose axioms correspond to the algebraic distillation of the three Reidemeister moves in Knot Theory. Quandles appeared in mathematics with many different names. In 1942 Mituhisa Takasaki [17] introduced the notion of kei (involutive quandle in Joyce's terminology [4]) as an abstraction of the notion of symmetric transformation. Around 1982, Joyce [4] (used the term quandle) and Matveev [15] (who call them distributive groupoids) introduced independently the notion of a quandle. Joyce and Matveev associated to each oriented knot K a quandle $Q(K)$ called the knot quandle. Since then quandles have been investigated by topologists in order to construct knot invariants and their higher dimensional analogues (see for example [14] and references therein). We recall the definition of a quandle and give a few examples.

Definition 2.1.1 [4] A *quandle*, X, is a set with a binary operation $(x, y) \mapsto x * y$ such that

(1) For any $x \in X$, $x * x = x$;

(2) For any $x, y \in X$, there is a unique $z \in X$ such that $x = z * y$;

(3) For any $x, y, z \in X$, we have $(x * y) * z = (x * z) * (y * z)$.

The axioms for a quandle correspond respectively to the Reidemeister moves I, II, and III.

- Any set X with the operation $x * y = x$ for all $x, y \in X$, is a quandle called the *trivial* quandle.

- Let m be a positive integer. For elements $x, y \in \mathbb{Z}_m$, define $x * y \equiv 2y - x \pmod{m}$. Then $*$ defines a quandle structure called the *dihedral quandle*.

- For any abelian group M, an automorphism t of M defines a quandle structure on M by $x * y = t(x - y) + y$. This is called an *Alexander quandle*.

In [2] Hayashi, Hayashi and Oshiro considered a generalization of Fox colorings that corresponds to a class of Alexander quandle colorings. They also investigated an upper bound for the quandle coloring in relation to the Alexander polynomial of the knot. This gave us the idea to apply the reduction in colors studied in Fox coloring to that of knots colored by linear Alexander quandles. Therefore, another goal of this thesis is to prove that all knots colored by the linear Alexander quandles $\mathbb{Z}_5[t]/(t-2)$ or $\mathbb{Z}_5[t]/(t-3)$ have a diagram using only 4 of the 5 available colors.

2.2 Colorings of knots by linear Alexander quandles

In this section we consider the rules of coloring knots by linear Alexander quandles. It is important to note that we are only considering finite quandles and non-trivial colorings. Here, the quandle operation is given by $x * y = tx + (1 - t)y$, where the case $t = -1$ corresponds to colorings by dihedral quandles called Fox colorings. So, we must consider the rules of coloring for the cases of $t = 2$ and $t = 3$. The second axiom of quandles implies that there exists a inverse quandle operation, we denote this operation as $x \bar{*} y$. Therefore, we assign $x * y$ to positive crossings and $x \bar{*} y$ to negative crossings. Since the positive and negative crossings have different coloring outcomes, we must consider the orientation of the knot. This also creates an issue in the previous notation, because $\{a|b|c\}$ is not necessarily $\{c|b|a\}$ unlike in Fox coloring. Therefore, we introduce the notation $\{a|b|c\}_{\pm}$ where a is the under-arc with orientation entering the crossing, b is the over-arc, c is the under-arc with orientation leaving the crossing, and $\pm$ gives the sign of the crossing. Moreover, as the diagram of the knot can be viewed from any direction, in most figures the under-arc orientation will be pointing to the right.

For $t = 2$ we have the positive and negative crossings in the following Figure 49 and Figure 50.

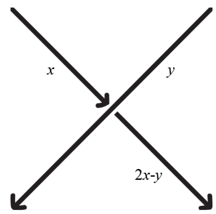

Figure 49.: $t = 2$ positive crossing $\{x|y|x * y\}_+$.

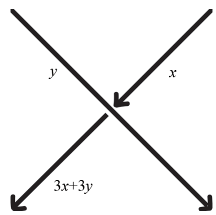

Figure 50.: $t = 2$ negative crossing $\{x|y|x \bar{*} y\}_-$.

For $t = 3$ we have the positive and negative crossings in the following Figure 51 and Figure 52.

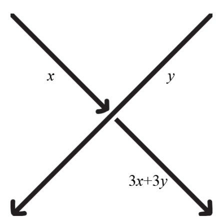

Figure 51.: $t = 3$ positive crossing $\{x|y|x * y\}_+$.

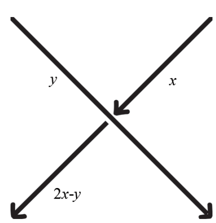

Figure 52.: $t = 3$ negative crossing $\{x|y|x \bar{*} y\}_-$.

2.3 Reducing the diagrams of knots that are colored by either the linear Alexander quandle $\mathbb{Z}_5[t]/(t-2)$ or $\mathbb{Z}_5[t]/(t-3)$

THEOREM 2.1 *Any knot colored by either the linear Alexander quandle* $\mathbb{Z}_5[t]/(t-2)$ *or* $\mathbb{Z}_5[t]/(t-3)$ *has a diagram using only four colors.*

Proof. Given any knot colored by either the linear Alexander quandle $\mathbb{Z}_5[t]/(t-2)$ or $\mathbb{Z}_5[t]/(t-3)$, every positive crossing has the operation $x * y = tx + (1-t)y$ where $t = 2, 3$ respectively. Thus, there are two colorings we must consider. So, we need to show that the colorings for $t = 2$ and $t = 3$ have a diagram using only four colors. We prove this using two lemmas. In Lemma 2.1 and Lemma 2.2 we decrease the number of colors in the respective colorings by one color c where c is in $\mathbb{Z}_5$. To accomplish this we first transform any crossings of the form $\{c|c|c\}$, that is, when c is the color of both an over-arc and an under-arc. Then, we remove c as an over-arc. Lastly, we complete each lemma by removing c as an under-arc. Since orientation must be considered in both colorings, each color removed will be in a case by case method. $\square$

2.3.1 Eliminating the color 4 for t=2

LEMMA 2.1 *Any knot colored by the linear Alexander quandle* $\mathbb{Z}_5[t]/(t-2)$ *has a diagram* D *with no arc colored by* 4*.*

Proof. We first transform all crossings of the form $\{4|4|4\}$ in four different cases. This eliminates the color 4 when it is both an over-arc and an under-arc.

Case 1: a and 4 are both over-arcs

If there is any crossing of the form $\{4|4|4\}_+$, there is either an adjacent crossing of the form $\{3a+2|a|4\}_+$ or $\{3-a|a|4\}_-$ where a is in $\mathbb{Z}_5 \setminus \{4\}$. Also, if there is any crossing of the form $\{4|4|4\}_-$, there is either an adjacent crossing of the form $\{3-a|a|4\}_-$ or $\{3a+2|a|4\}_+$ where a is in $\mathbb{Z}_5 \setminus \{4\}$. In each situation, $3-a \neq 4$ and $3a+2 \neq 4$ for any a in $\mathbb{Z}_5 \setminus \{4\}$, so we can transform the diagram as seen in the following Figures 53-56.

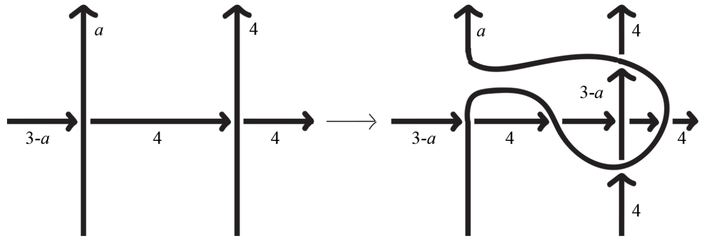

Figure 53.: Transformation of the crossing $\{4|4|4\}_-$ with the crossing $\{3-\mathfrak{a}|\mathfrak{a}|4\}_-$.

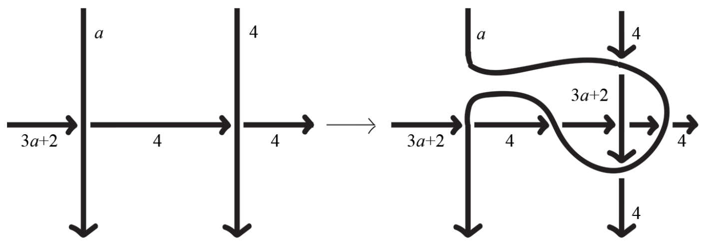

Figure 54.: Transformation of the crossing $\{4|4|4\}_+$ with the crossing $\{3\mathfrak{a}+2|\mathfrak{a}|4\}_+$.

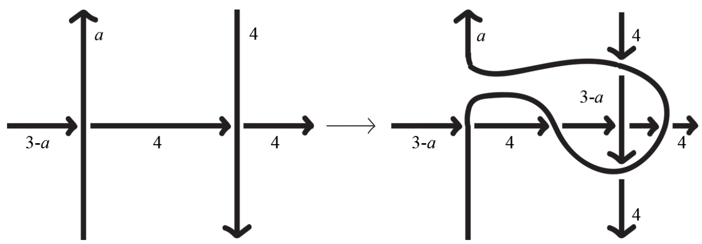

Figure 55.: Transformation of the crossing $\{4|4|4\}_+$ with the crossing $\{3-\mathfrak{a}|\mathfrak{a}|4\}_-$.

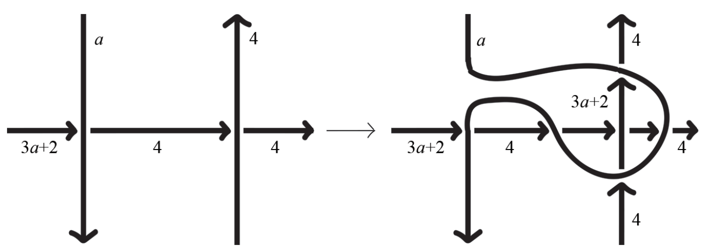

Figure 56.: Transformation of the crossing $\{4|4|4\}_-$ with the crossing $\{3\mathfrak{a}+2|\mathfrak{a}|4\}_+$.

Case 2: $\mathfrak{a}$ is an over-arc and 4 is an under-arc

If there is any crossing of the form $\{4|4|4\}_+$, there is either an adjacent crossing of the form $\{3\mathfrak{a}+2|\mathfrak{a}|4\}_+$ or $\{3-\mathfrak{a}|\mathfrak{a}|4\}_-$ where $\mathfrak{a}$ is in $\mathbb{Z}_5 \setminus \{4\}$. Also, if there is any crossing of the form $\{4|4|4\}_-$, there is either an adjacent crossing of the form $\{3-\mathfrak{a}|\mathfrak{a}|4\}_-$ or $\{3\mathfrak{a}+2|\mathfrak{a}|4\}_+$ where $\mathfrak{a}$ is in $\mathbb{Z}_5 \setminus \{4\}$. In each situation, $3-\mathfrak{a} \neq 4$ and $3\mathfrak{a}+2 \neq 4$ for any $\mathfrak{a}$ in $\mathbb{Z}_5 \setminus \{4\}$, so we can transform the diagram as seen in the following Figures 57-60.

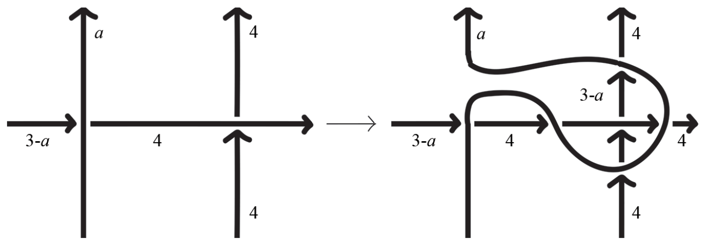

Figure 57.: Transformation of the crossing $\{4|4|4\}_+$ with the crossing $\{3-\mathfrak{a}|\mathfrak{a}|4\}_-$.

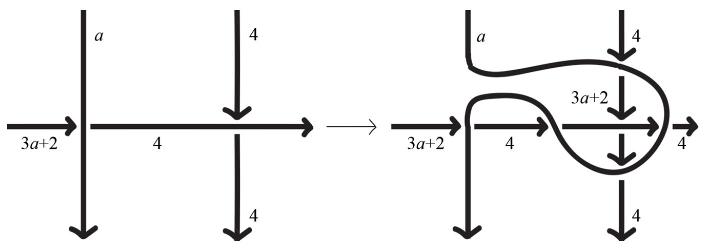

Figure 58.: Transformation of the crossing $\{4|4|4\}_-$ with the crossing $\{3\mathfrak{a}+2|\mathfrak{a}|4\}_+$.

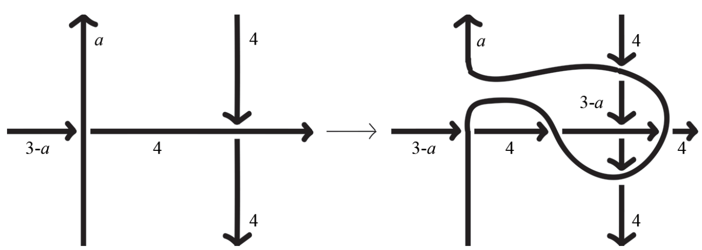

Figure 59.: Transformation of the crossing $\{4|4|4\}_-$ with the crossing $\{3-\mathfrak{a}|\mathfrak{a}|4\}_-$.

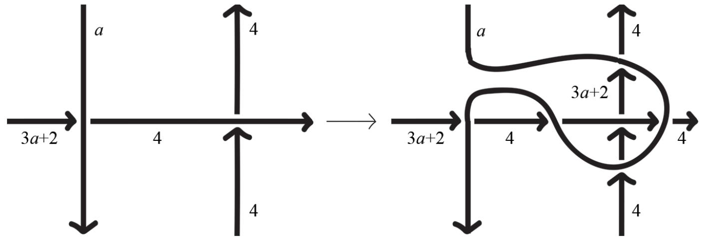

Figure 60.: Transformation of the crossing $\{4|4|4\}_+$ with the crossing $\{3\mathfrak{a}+2|\mathfrak{a}|4\}_+$.

Case 3: $\mathfrak{a}$ is an under-arc and 4 is an over-arc

If there is any crossing of the form $\{4|4|4\}_+$, there is either an adjacent crossing of the form $\{3\mathfrak{a}+2|4|\mathfrak{a}\}_+$ or $\{\mathfrak{a}|4|3\mathfrak{a}+2\}_-$ where $\mathfrak{a}$ is in $\mathbb{Z}_5\setminus\{4\}$. Also, if there is any crossing of the form $\{4|4|4\}_-$, there is either an adjacent crossing of the form $\{\mathfrak{a}|4|3\mathfrak{a}+2\}_-$ or $\{3\mathfrak{a}+2|4|\mathfrak{a}\}_+$ where $\mathfrak{a}$ is in $\mathbb{Z}_5\setminus\{4\}$. In each situation, $3-\mathfrak{a}\neq 4$ and $3\mathfrak{a}+2\neq 4$ for any $\mathfrak{a}$ in $\mathbb{Z}_5\setminus\{4\}$, so we can transform the diagram as seen in the following Figures 61-64. Note that we need not consider the case of $\mathfrak{a}$ switching from orientation in to orientation out (or vice versa) because $3\mathfrak{a}+2$ is in $\{0,1,2,3\}$ when $\mathfrak{a}$ is in $\mathbb{Z}_5\setminus\{4\}$, thus every value of $\mathfrak{a}$ is attained.

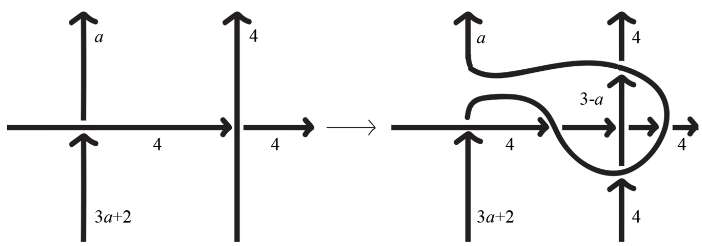

Figure 61.: Transformation of the crossing $\{4|4|4\}_-$ with the crossing $\{3\mathfrak{a}+2|4|\mathfrak{a}\}_+$.

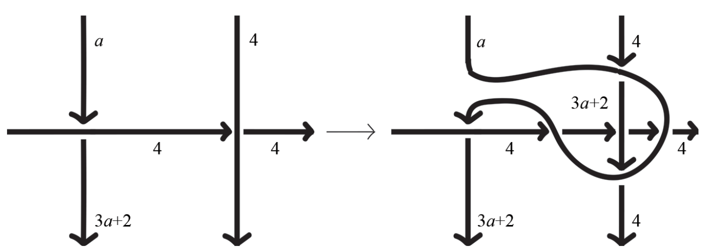

Figure 62.: Transformation of the crossing $\{4|4|4\}_+$ with the crossing $\{\mathfrak{a}|4|3\mathfrak{a}+2\}_-$.

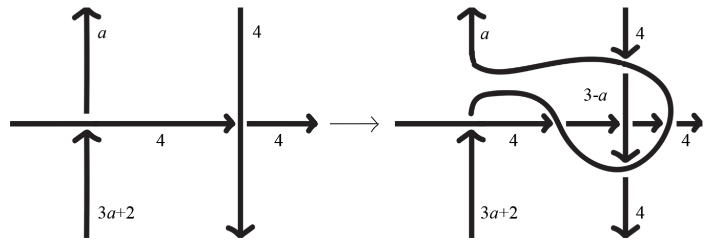

Figure 63.: Transformation of the crossing $\{4|4|4\}_+$ with the crossing $\{3\mathfrak{a}+2|4|\mathfrak{a}\}_+$.

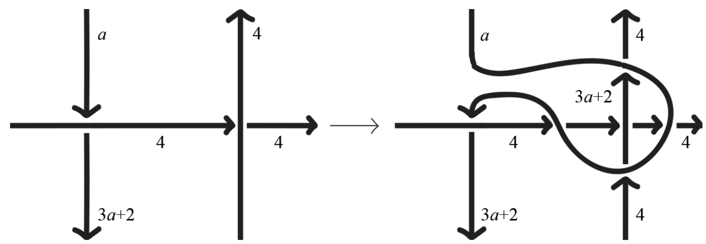

Figure 64.: Transformation of the crossing $\{4|4|4\}_-$ with the crossing $\{\mathfrak{a}|4|3\mathfrak{a}+2\}_-$.

Case 4: $\mathfrak{a}$ and 4 are both under-arcs

If there is any crossing of the form $\{4|4|4\}_+$, there is either an adjacent crossing of the form $\{3\mathfrak{a}+2|4|\mathfrak{a}\}_+$ or $\{\mathfrak{a}|4|3\mathfrak{a}+2\}_-$ where $\mathfrak{a}$ is in $\mathbb{Z}_5\setminus\{4\}$. Also, if there is any crossing of the form $\{4|4|4\}_-$, there is either an adjacent crossing of the form $\{\mathfrak{a}|4|3\mathfrak{a}+2\}_-$ or $\{3\mathfrak{a}+2|4|\mathfrak{a}\}_+$ where $\mathfrak{a}$ is in $\mathbb{Z}_5\setminus\{4\}$. In each situation, $3-\mathfrak{a}\neq 4$ and $3\mathfrak{a}+2\neq 4$ for any $\mathfrak{a}$ in $\mathbb{Z}_5\setminus\{4\}$, so we can transform the diagram as seen in the following Figures 65-68. Note that we need not consider the case of $\mathfrak{a}$ switching from orientation in to orientation out (or vice versa) because $3\mathfrak{a}+2$ is in $\{0,1,2,3\}$ when $\mathfrak{a}$ is in $\mathbb{Z}_5\setminus\{4\}$, thus every value of $\mathfrak{a}$ is attained.

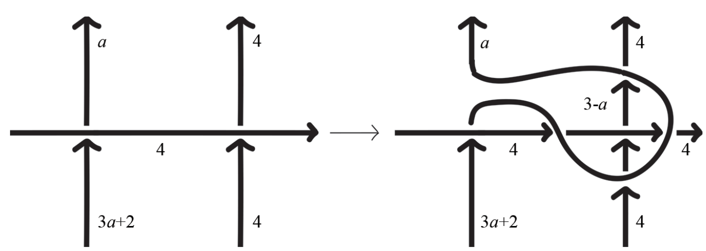

Figure 65.: Transformation of the crossing $\{4|4|4\}_+$ with the crossing $\{3\mathfrak{a}+2|4|\mathfrak{a}\}_+$.

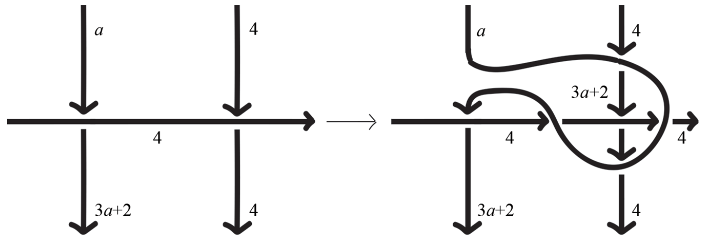

Figure 66.: Transformation of the crossing $\{4|4|4\}_{-}$ with the crossing $\{a|4|3a+2\}_{-}$.

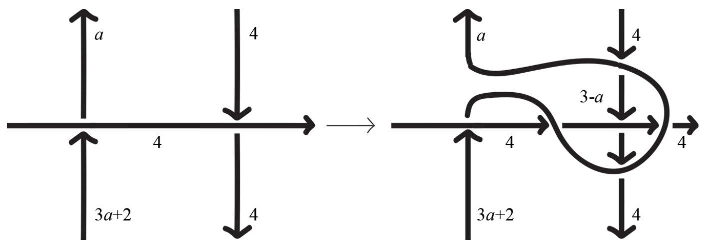

Figure 67.: Transformation of the crossing $\{4|4|4\}_{-}$ with the crossing $\{3a+2|4|a\}_{+}$.

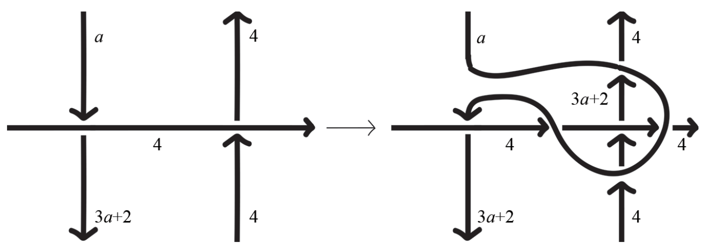

Figure 68.: Transformation of the crossing $\{4|4|4\}_{+}$ with the crossing $\{a|4|3a+2\}_{-}$.

Next, we remove 4 as an over-arc by transforming any crossings of the form $\{a|4|2a+1\}_{+}$ or $\{a|4|3a+2\}_{-}$. Note that we need not consider the case of a switching from orientation in to orientation out (or vice versa) because $3a+2$ is in $\{0,1,2,3\}$ and $2a+1$ is in $\{0,1,2,3\}$ when a is in $\mathbb{Z}_5 \setminus \{4\}$, thus every value of a is attained. In each situation, $2a+1 \neq 4, 3a+2 \neq 4$, and $4a+3 \neq 4$ for any a in $\mathbb{Z}_5 \setminus \{4\}$, so we can transform the diagram as seen in the following Figure 69 and Figure 70.

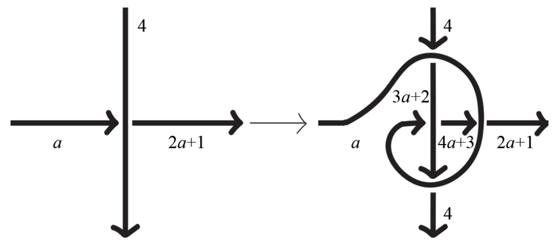

Figure 69.: Transformation of the crossing $\{\mathfrak{a}|4|2\mathfrak{a}+1\}_{+}$.

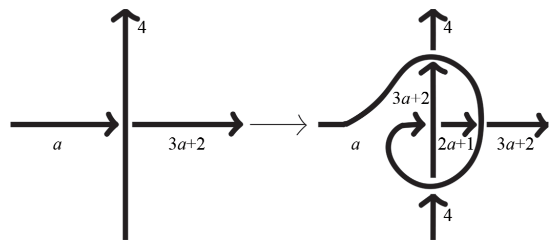

Figure 70.: Transformation of the crossing $\{\mathfrak{a}|4|3\mathfrak{a}+2\}_{-}$.

We complete the proof of Lemma 2.1 by eliminating 4 as an under-arc in a case by case method. We must first consider the cases when 4 is an under-arc adjacent to two arcs of the same color. Then, we will go through a few special cases that arise when 4 is an under-arc adjacent to two arcs of different colors. Finally, we consider the cases when 4 is an under-arc adjacent to two arcs of different colors.

If 4 is an under-arc adjacent to two arcs of the same color, then counting each possible orientation there are four cases we must consider. In each situation, $3-\mathfrak{a} \neq 4, 2\mathfrak{a}+1 \neq 4, 3\mathfrak{a}+2 \neq 4$, and $4\mathfrak{a}+3 \neq 4$ for any $\mathfrak{a}$ in $\mathbb{Z}_5 \setminus \{4\}$, so we can transform the diagram as seen in the following Figures 71-74.

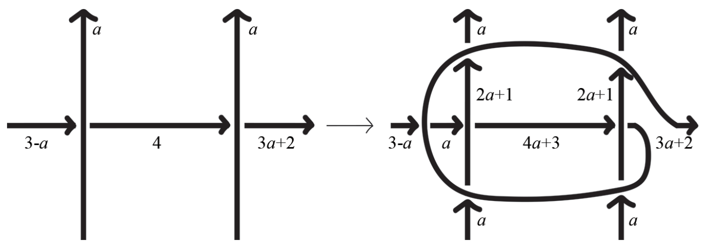

Figure 71.: Transformation of 4 between the crossings $\{3-a|a|4\}_-$ and $\{4|a|3a+2\}_-$.

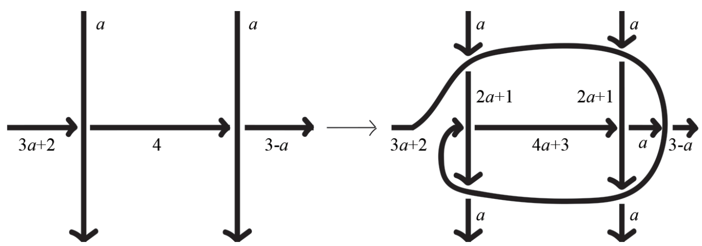

Figure 72.: Transformation of 4 between the crossings $\{3a+2|a|4\}_+$ and $\{4|a|3-a\}_+$.

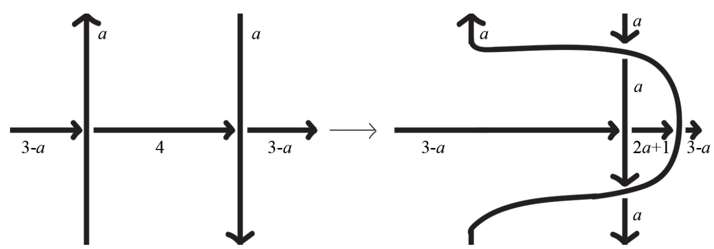

Figure 73.: Transformation of 4 between the crossings $\{3-a|a|4\}_-$ and $\{4|a|3-a\}_+$.

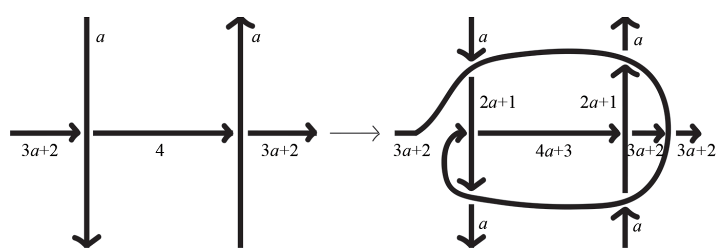

Figure 74.: Transformation of 4 between the crossings $\{3a+2|a|4\}_+$ and $\{4|a|3a+2\}_-$.

Now, we consider a few special cases that arise when 4 is an under-arc adjacent to two arcs of different colors. (Note that $\mathtt{a} \neq \mathtt{b}$.) In each situation, $3 - \mathtt{a} \neq 4, 2\mathtt{a}+1 \neq 4, 3\mathtt{a}+2 \neq 4, 4\mathtt{a}+3 \neq 4, 3-\mathtt{b} \neq 4, 2\mathtt{b}+1 \neq 4, 3\mathtt{b}+2 \neq 4$, and $4\mathtt{b}+3 \neq 4$ for any $\mathtt{a}, \mathtt{b}$ in $\mathbb{Z}_5 \setminus \{4\}$, so we can transform the diagram as seen in the following Figures 75-78.

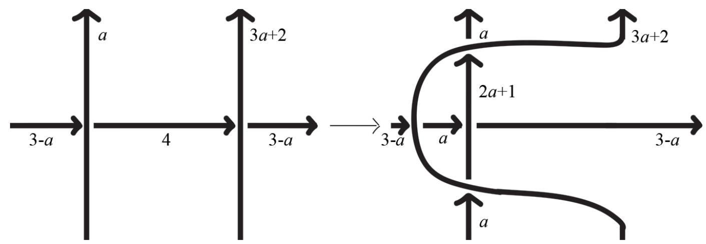

Figure 75.: Transformation of 4 between the crossings $\{3-\mathtt{a}|\mathtt{a}|4\}_-$ and $\{4|3\mathtt{a}+2|3-\mathtt{a}\}_-$.

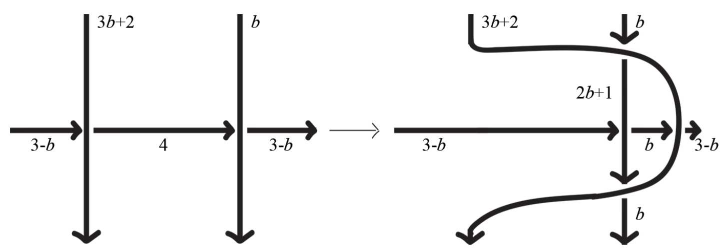

Figure 76.: Transformation of 4 between the crossings $\{3-\mathtt{b}|3\mathtt{b}+2|4\}_+$ and $\{4|\mathtt{b}|3-\mathtt{b}\}_+$.

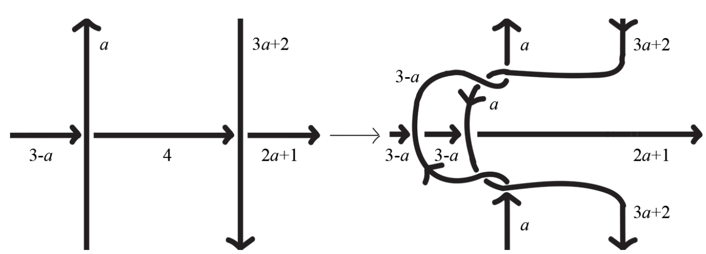

Figure 77.: Transformation of 4 between the crossings $\{3-\mathtt{a}|\mathtt{a}|4\}_-$ and $\{4|3\mathtt{a}+2|2\mathtt{a}+1\}_+$.

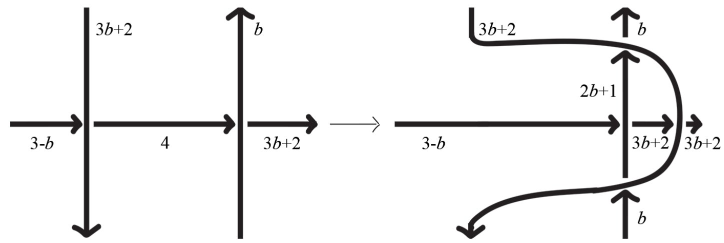

Figure 78.: Transformation of 4 between the crossings $\{3-\mathfrak{b}|3\mathfrak{b}+2|4\}_+$ and $\{4|\mathfrak{b}|3\mathfrak{b}+2\}_-$.

Finally, we consider the cases when 4 is an under-arc adjacent to two arcs of different colors. (Note that $\mathfrak{a} \neq \mathfrak{b}$.) Counting each possible orientation there are four cases we must consider. In each situation, we have that $3-\mathfrak{a} \neq 4, 2\mathfrak{a}+1 \neq 4, 3\mathfrak{a}+2 \neq 4, 4\mathfrak{a}+3 \neq 4, 3-\mathfrak{b} \neq 4, 2\mathfrak{b}+1 \neq 4, 3\mathfrak{b}+2 \neq 4$, and $4\mathfrak{b}+3 \neq 4$ for any $\mathfrak{a}, \mathfrak{b}$ in $\mathbb{Z}_5 \setminus \{4\}$. Also, since $\mathfrak{a} \neq \mathfrak{b}$, we have that $\mathfrak{b}-\mathfrak{a}+4 \neq 4, \mathfrak{a}-\mathfrak{b}+4 \neq 4, 3\mathfrak{b}-3\mathfrak{a}+4 \neq 4$ for any $\mathfrak{a}, \mathfrak{b}$ in $\mathbb{Z}_5 \setminus \{4\}$. Furthermore, we assume that $2\mathfrak{a}-\mathfrak{b} \neq 4$ and $2\mathfrak{b}-\mathfrak{a} \neq 4$ for any $\mathfrak{a}, \mathfrak{b}$ in $\mathbb{Z}_5 \setminus \{4\}$, otherwise we transform the diagram as seen in the special cases above. So, we can transform the diagram as seen in the following Figures 79-82.

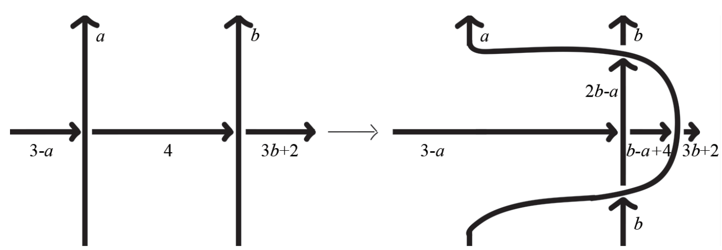

Figure 79.: Transformation of 4 between the crossings $\{3-\mathfrak{a}|\mathfrak{a}|4\}_-$ and $\{4|\mathfrak{b}|3\mathfrak{b}+2\}_-$.

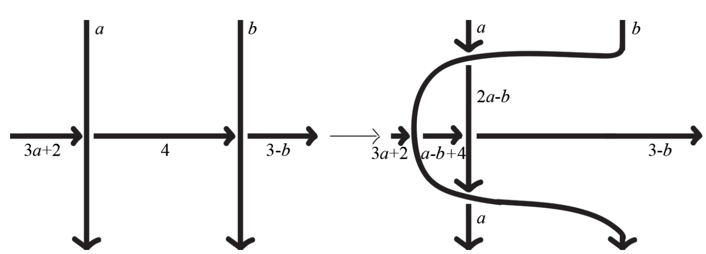

Figure 80.: Transformation of 4 between the crossings $\{3\mathfrak{a}+2|\mathfrak{a}|4\}_+$ and $\{4|\mathfrak{b}|3-\mathfrak{b}\}_+$.

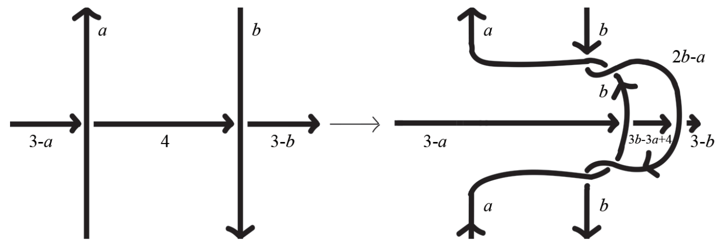

Figure 81.: Transformation of 4 between the crossings $\{3-a|a|4\}_-$ and $\{4|b|3-b\}_+$.

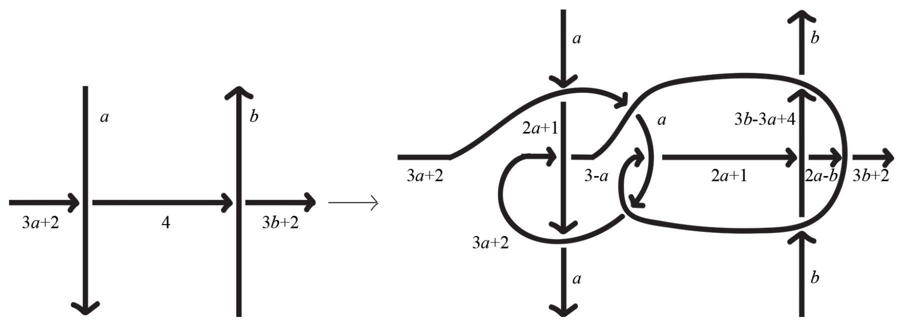

Figure 82.: Transformation of 4 between the crossings $\{3a+2|a|4\}_+$ and $\{4|b|3b+2\}_-$.

□

2.3.2 Eliminating the color 4 for t=3

LEMMA 2.2 *Any knot colored by the linear Alexander quandle* $\mathbb{Z}_5[t]/(t-3)$ *has a diagram* D *with no arc colored by* 4.

Proof. We first transform all crossings of the form $\{4|4|4\}$ in four different cases. This eliminates the color 4 when it is both an over-arc and an under-arc.

Case 1: a and 4 are both over-arcs

If there is any crossing of the form $\{4|4|4\}_+$, there is either an adjacent crossing of the form $\{3-a|a|4\}_+$ or $\{3a+2|a|4\}_-$ where a is in $\mathbb{Z}_5 \setminus \{4\}$. Also, if there is any crossing of the form $\{4|4|4\}_-$, there is either an adjacent crossing of the form $\{3a+2|a|4\}_-$ or $\{3-a|a|4\}_+$ where a is in $\mathbb{Z}_5 \setminus \{4\}$. In each situation, $3-a \neq 4$ and $3a+2 \neq 4$ for any a in $\mathbb{Z}_5 \setminus \{4\}$, so we can transform the diagram as seen in the following Figures 83-86.

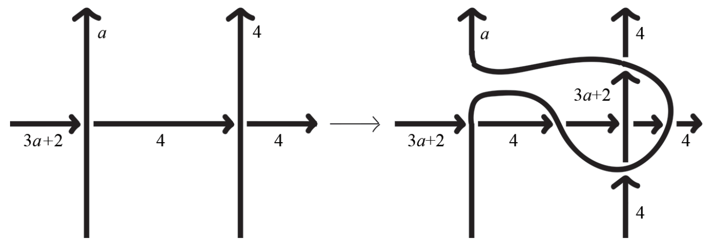

Figure 83.: Transformation of the crossing $\{4|4|4\}_-$ with the crossing $\{3\mathfrak{a}+2|\mathfrak{a}|4\}_-$.

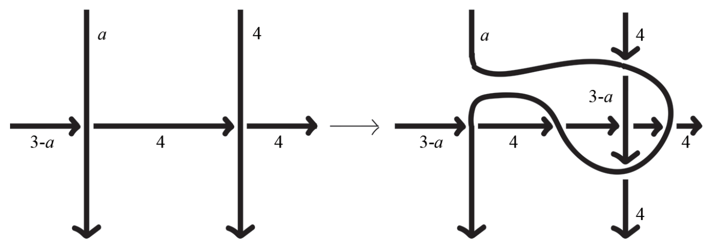

Figure 84.: Transformation of the crossing $\{4|4|4\}_+$ with the crossing $\{3-\mathfrak{a}|\mathfrak{a}|4\}_+$.

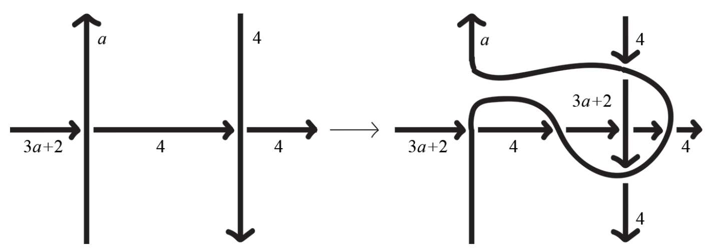

Figure 85.: Transformation of the crossing $\{4|4|4\}_+$ with the crossing $\{3\mathfrak{a}+2|\mathfrak{a}|4\}_-$.

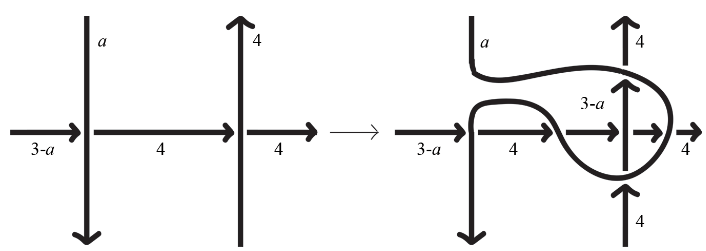

Figure 86.: Transformation of the crossing $\{4|4|4\}_-$ with the crossing $\{3-\mathfrak{a}|\mathfrak{a}|4\}_+$.

Case 2: $\mathfrak{a}$ is an over-arc and 4 is an under-arc

If there is any crossing of the form $\{4|4|4\}_+$, there is either an adjacent crossing of the form $\{3-\mathfrak{a}|\mathfrak{a}|4\}_+$ or $\{3\mathfrak{a}+2|\mathfrak{a}|4\}_-$ where $\mathfrak{a}$ is in $\mathbb{Z}_5 \setminus \{4\}$. Also, if there is any crossing of the form $\{4|4|4\}_-$, there is either an adjacent crossing of the form $\{3\mathfrak{a}+2|\mathfrak{a}|4\}_-$ or $\{3-\mathfrak{a}|\mathfrak{a}|4\}_+$ where $\mathfrak{a}$ is in $\mathbb{Z}_5 \setminus \{4\}$. In each situation, $3-\mathfrak{a} \neq 4$ and $3\mathfrak{a}+2 \neq 4$ for any $\mathfrak{a}$ in $\mathbb{Z}_5 \setminus \{4\}$, so we can transform the diagram as seen in the following Figures 87-90.

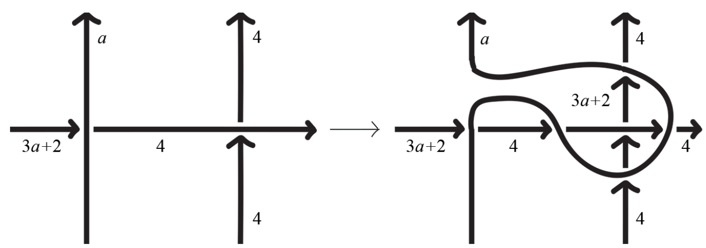

Figure 87.: Transformation of the crossing $\{4|4|4\}_+$ with the crossing $\{3\mathfrak{a}+2|\mathfrak{a}|4\}_-$.

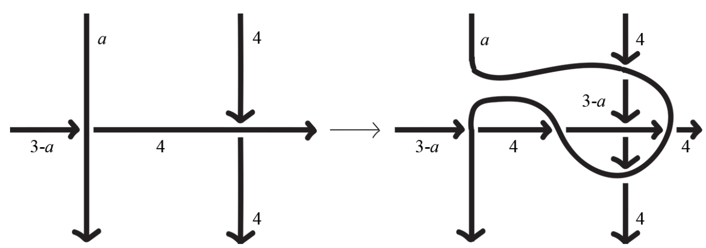

Figure 88.: Transformation of the crossing $\{4|4|4\}_-$ with the crossing $\{3-\mathfrak{a}|\mathfrak{a}|4\}_+$.

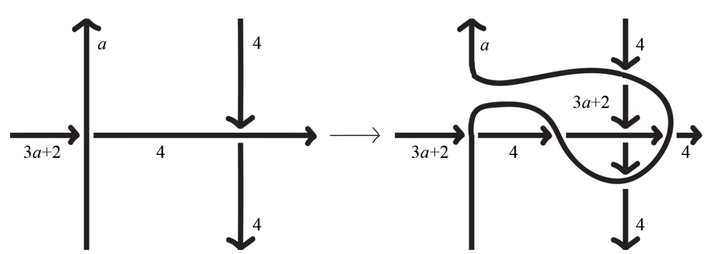

Figure 89.: Transformation of the crossing $\{4|4|4\}_-$ with the crossing $\{3\mathfrak{a}+2|\mathfrak{a}|4\}_-$.

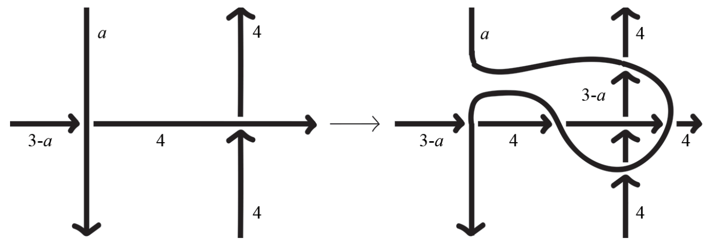

Figure 90.: Transformation of the crossing $\{4|4|4\}_+$ with the crossing $\{3-a|a|4\}_+$.

Case 3: a is an under-arc and 4 is an over-arc

If there is any crossing of the form $\{4|4|4\}_+$, there is either an adjacent crossing of the form $\{a|4|2a+1\}_+$ or $\{2a+1|4|a\}_-$ where a is in $\mathbb{Z}_5 \setminus \{4\}$. Also, if there is any crossing of the form $\{4|4|4\}_-$, there is either an adjacent crossing of the form $\{2a+1|4|a\}_-$ or $\{a|4|2a+1\}_+$ where a is in $\mathbb{Z}_5 \setminus \{4\}$. In each situation, $3-a \neq 4, 3a+2 \neq 4$, and $2a+1 \neq 4$ for any a in $\mathbb{Z}_5 \setminus \{4\}$, so we can transform the diagram as seen in the following Figures 91-94. Note that we need not consider the case of a switching from orientation in to orientation out (or vice versa) because $2a+1$ is in $\{0,1,2,3\}$ when a is in $\mathbb{Z}_5 \setminus \{4\}$, thus every value of a is attained.

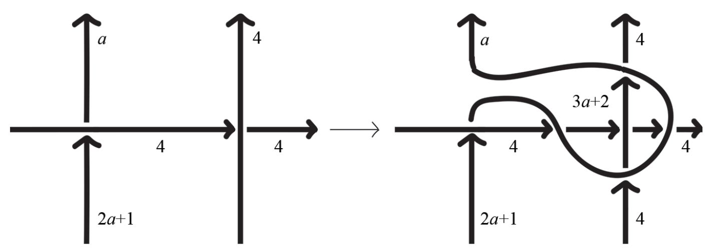

Figure 91.: Transformation of the crossing $\{4|4|4\}_-$ with the crossing $\{2a+1|4|a\}_+$.

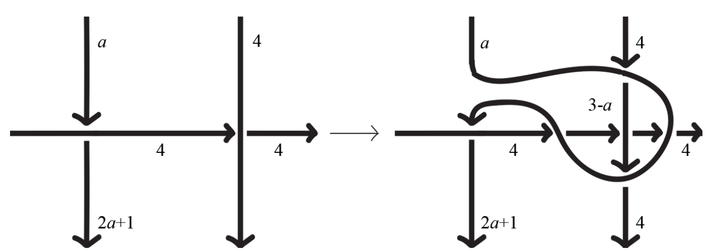

Figure 92.: Transformation of the crossing $\{4|4|4\}_+$ with the crossing $\{a|4|2a+1\}_-$.

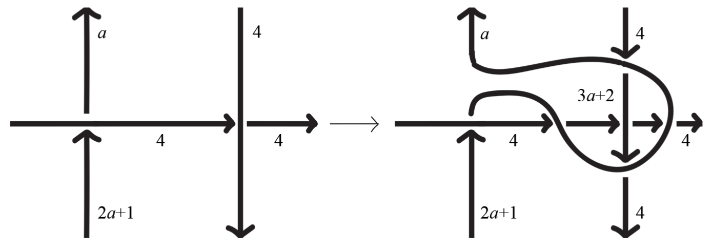

Figure 93.: Transformation of the crossing $\{4|4|4\}_+$ with the crossing $\{2a+1|4|a\}_+$.

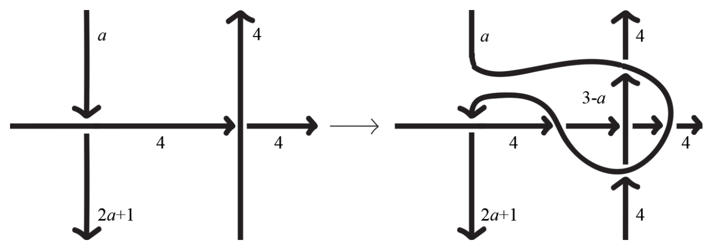

Figure 94.: Transformation of the crossing $\{4|4|4\}_-$ with the crossing $\{a|4|2a+1\}_-$.

Case 4: a and 4 are both under-arcs

If there is any crossing of the form $\{4|4|4\}_+$, there is either an adjacent crossing of the form $\{a|4|2a+1\}_+$ or $\{2a+1|4|a\}_-$ where a is in $\mathbb{Z}_5 \setminus \{4\}$. Also, if there is any crossing of the form $\{4|4|4\}_-$, there is either an adjacent crossing of the form $\{2a+1|4|a\}_-$ or $\{a|4|2a+1\}_+$ where a is in $\mathbb{Z}_5 \setminus \{4\}$. In each situation, $3-a \neq 4, 3a+2 \neq 4$, and $2a+1 \neq 4$ for any a in $\mathbb{Z}_5 \setminus \{4\}$, so we can transform the diagram as seen in the following Figures 95-98. Note that we need not consider the case of a switching from orientation in to orientation out (or vice versa) because $2a+1$ is in $\{0,1,2,3\}$ when a is in $\mathbb{Z}_5 \setminus \{4\}$, thus every value of a is attained.

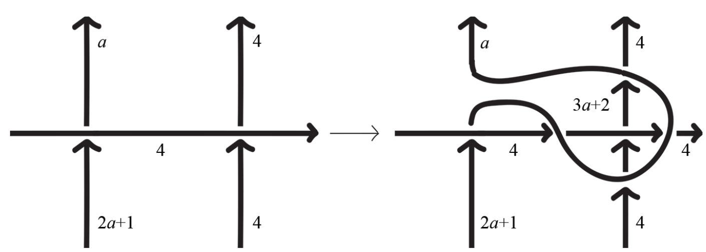

Figure 95.: Transformation of the crossing $\{4|4|4\}_+$ with the crossing $\{2a+1|4|a\}_+$.

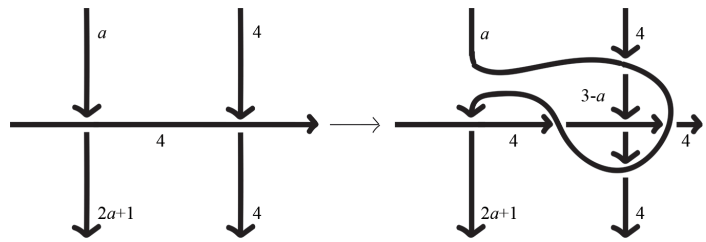

Figure 96.: Transformation of the crossing $\{4|4|4\}_{-}$ with the crossing $\{a|4|2a+1\}_{-}$.

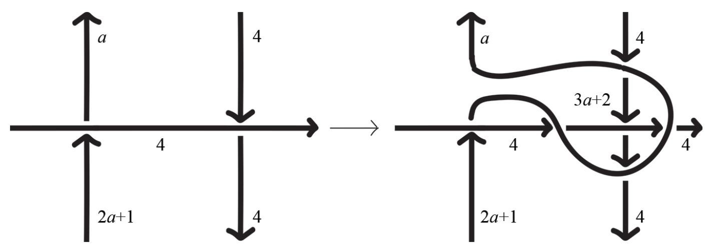

Figure 97.: Transformation of the crossing $\{4|4|4\}_{-}$ with the crossing $\{2a+1|4|a\}_{+}$.

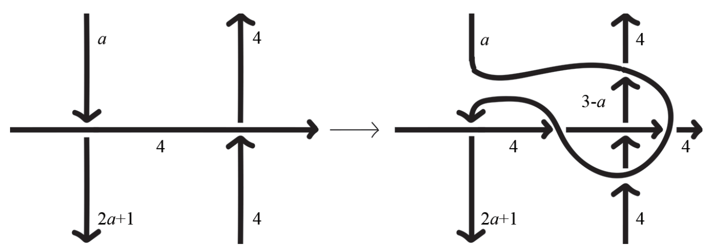

Figure 98.: Transformation of the crossing $\{4|4|4\}_{+}$ with the crossing $\{a|4|2a+1\}_{-}$.

Next, we remove 4 as an over-arc by transforming any crossings of the form $\{a|4|3a+2\}_{+}$ or $\{a|4|2a+1\}_{-}$. Note that we need not consider the case of a switching from orientation in to orientation out (or vice versa) because $3a+2$ is in $\{0,1,2,3\}$ and $2a+1$ is in $\{0,1,2,3\}$ when a is in $\mathbb{Z}_5\backslash\{4\}$, thus every value of a is attained. In each situation, $3-a \neq 4, 2a+1 \neq 4$, and $3a+2 \neq 4$ for any a in $\mathbb{Z}_5\backslash\{4\}$, so we can transform the diagram as seen in the following Figure 99 and Figure 100.

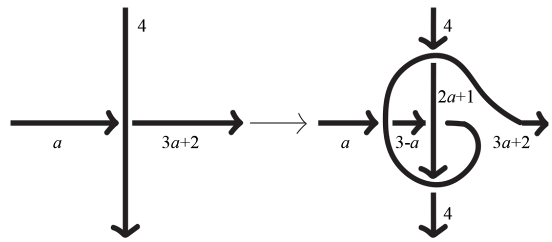

Figure 99.: Transformation of the crossing $\{\mathfrak{a}|4|3\mathfrak{a}+2\}_{+}$.

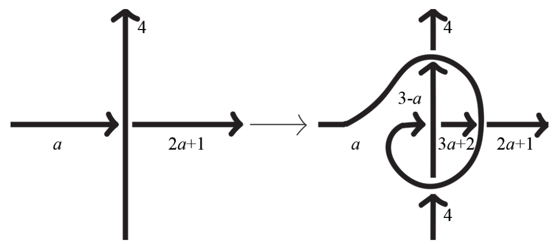

Figure 100.: Transformation of the crossing $\{\mathfrak{a}|4|2\mathfrak{a}+1\}_{-}$.

We complete the proof of the Lemma 2.2 by eliminating 4 as an under-arc in a case by case method. We must first consider the cases when 4 is an under-arc adjacent to two arcs of the same color. Then, we will go through a few special cases that arise when 4 is an under-arc adjacent to two arcs of different colors. Finally, we consider the cases when 4 is an under-arc adjacent to two arcs of different colors.

If 4 is an under-arc adjacent to two arcs of the same color, then counting each possible orientation there are four cases we must consider. In each situation, $3-\mathfrak{a} \neq 4, 2\mathfrak{a}+1 \neq 4$, and $3\mathfrak{a}+2 \neq 4$ for any $\mathfrak{a}$ in $\mathbb{Z}_5\setminus\{4\}$, so we can transform the diagram as seen in the following Figures 101-104.

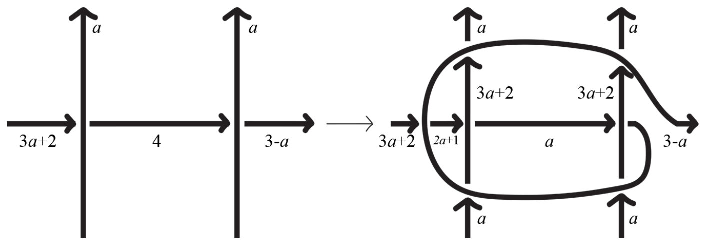

Figure 101.: Transformation of 4 between the crossings $\{3a+2|a|4\}_-$ and $\{4|a|3-a\}_-$.

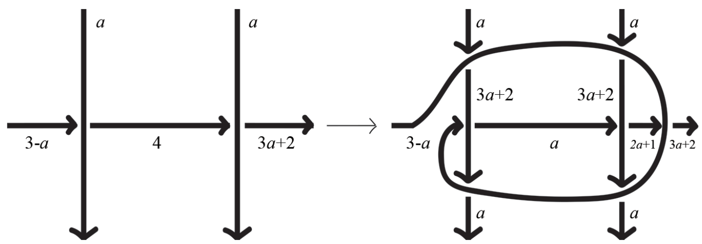

Figure 102.: Transformation of 4 between the crossings $\{3-a|a|4\}_+$ and $\{4|a|3a+2\}_+$.

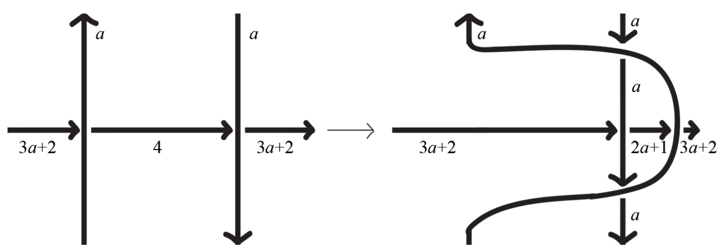

Figure 103.: Transformation of 4 between the crossings $\{3a+2|a|4\}_-$ and $\{4|a|3a+2\}_+$.

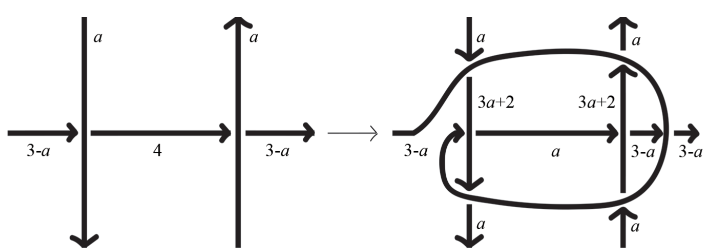

Figure 104.: Transformation of 4 between the crossings $\{3-a|a|4\}_+$ and $\{4|a|3-a\}_-$.

Now, we consider a few special cases that arise when 4 is an under-arc adjacent to two arcs of different colors. (Note that $\mathfrak{a} \neq \mathfrak{b}$.) In each situation, $3-\mathfrak{a} \neq 4, 2\mathfrak{a}+1 \neq 4, 3\mathfrak{a}+2 \neq 4, 3-\mathfrak{b} \neq 4, 2\mathfrak{b}+1 \neq 4$, and $3\mathfrak{b}+2 \neq 4$ for any $\mathfrak{a}, \mathfrak{b}$ in $\mathbb{Z}_5 \setminus \{4\}$, so we can transform the diagram as seen in the following Figures 105-109.

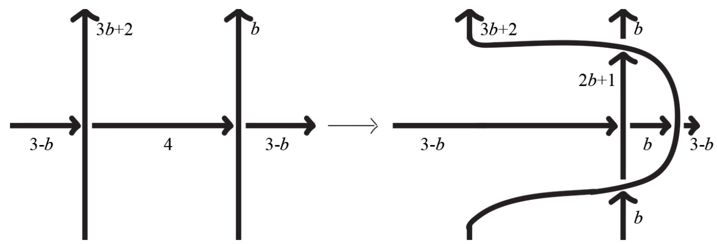

Figure 105.: Transformation of 4 between the crossings $\{3-\mathfrak{b}|3\mathfrak{b}+2|4\}_{-}$ and $\{4|\mathfrak{b}|3-\mathfrak{b}\}_{-}$.

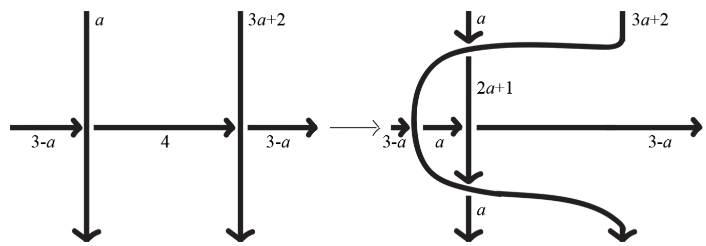

Figure 106.: Transformation of 4 between the crossings $\{3-\mathfrak{a}|\mathfrak{a}|4\}_{+}$ and $\{4|3\mathfrak{a}+2|3-\mathfrak{a}\}_{+}$.

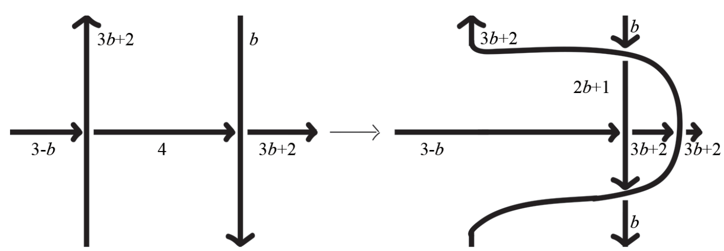

Figure 107.: Transformation of 4 between the crossings $\{3-\mathfrak{b}|3\mathfrak{b}+2|4\}_{-}$ and $\{4|\mathfrak{b}|3\mathfrak{b}+2\}_{+}$.

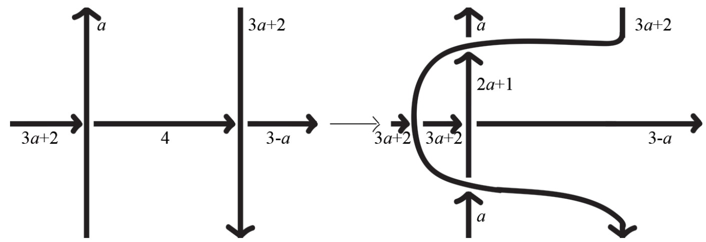

Figure 108.: Transformation of 4 between the crossings $\{3a+2|a|4\}_-$ and $\{4|3a+2|3-a\}_+$.

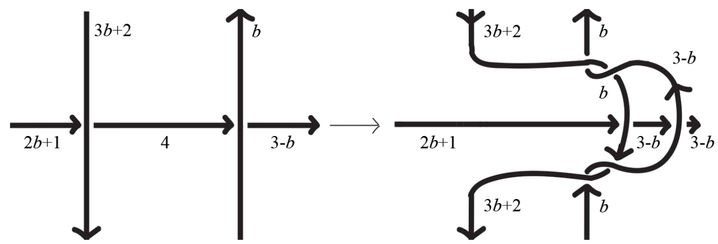

Figure 109.: Transformation of 4 between the crossings $\{2b+1|3b+2|4\}_+$ and $\{4|b|3-b\}_-$.

Finally, we consider the cases when 4 is an under-arc adjacent to two arcs of different colors. (Note that $a \neq b$.) Counting each possible orientation there are four cases we must consider. In each situation, we have that $3-a \neq 4, 2a+1 \neq 4, 3a+2 \neq 4, 3-b \neq 4$, and $3b+2 \neq 4$ for any a, b in $\mathbb{Z}_5 \setminus \{4\}$. Also, since $a \neq b$, we have that $b-a+4 \neq 4, a-b+4 \neq 4, 3a-3b+4 \neq 4$, and $2b-2a+4 \neq 4$ for any a, b in $\mathbb{Z}_5 \setminus \{4\}$. Furthermore, we assume that $2a-b \neq 4$ and $2b-a \neq 4$ for any a, b in $\mathbb{Z}_5 \setminus \{4\}$, otherwise we transform the diagram as seen in the special cases above. So, we can transform the diagram as seen in the following Figures 110-113. (Note that for Figure 112, since $a \neq b$, we have that $2a-b \neq 2b-a$.)

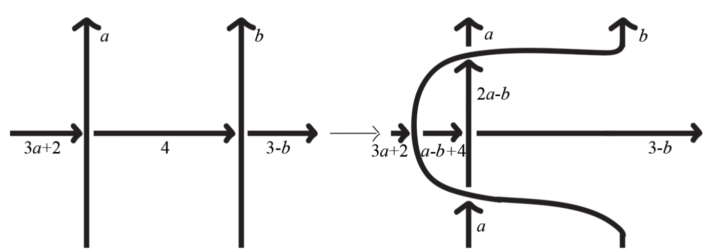

Figure 110.: Transformation of 4 between the crossings $\{3a+2|a|4\}_-$ and $\{4|b|3-b\}_-$.

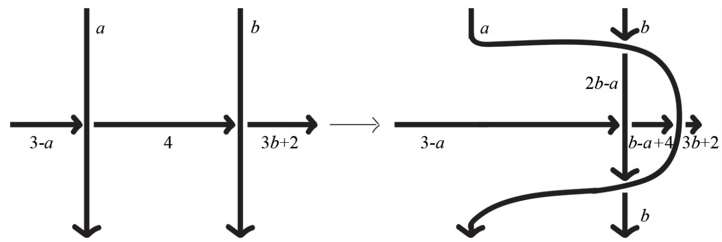

Figure 111.: Transformation of 4 between the crossings $\{3-a|a|4\}_+$ and $\{4|b|3b+2\}_+$.

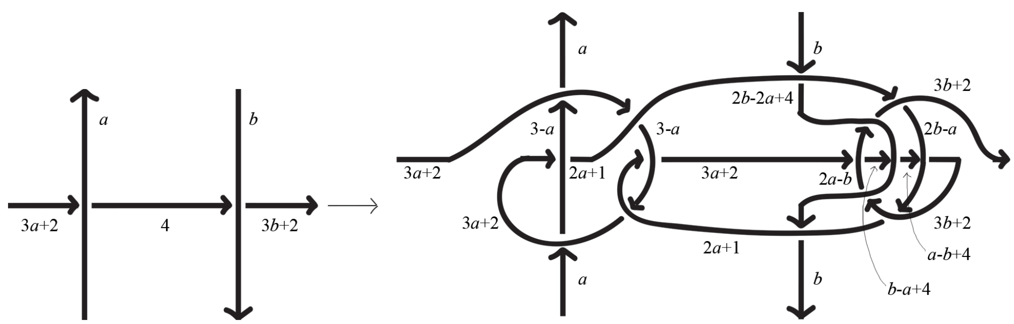

Figure 112.: Transformation of 4 between the crossings $\{3a+2|a|4\}_-$ and $\{4|b|3b+2\}_+$.

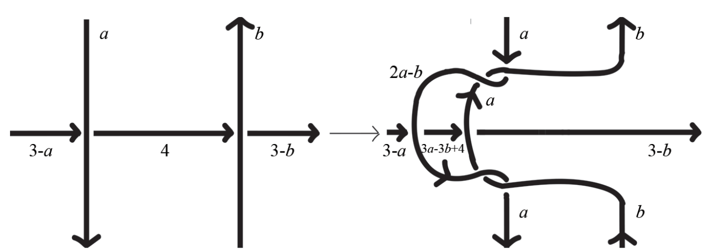

Figure 113.: Transformation of 4 between the crossings $\{3-a|a|4\}_+$ and $\{4|b|3-b\}_-$.

□

2.3.3 Summary

The above lemmas, in total, reduced the number of colors of an arbitrary knot colored by the linear Alexander quandle $\mathbb{Z}_5[t]/(t-2)$ and $\mathbb{Z}_5[t]/(t-3)$ by a single color. Therefore, any knot colored by either the linear Alexander quandle $\mathbb{Z}_5[t]/(t-2)$ or $\mathbb{Z}_5[t]/(t-3)$ has a diagram using only 4 of the 5 available colors.

Remark

An alternative proof of Lemma 2.2 for the case $t = 3$ can be obtained by symmetry from the case $t = 2$ using the Figures 49-52. Since the mirror image interchanges positive and negative crossings, we can take a diagram D colored by $t = 3$ and consider its mirror image $m(D)$. The mirror image $m(D)$ will be colored by $t = 2$. We have already shown that any diagram colored by $t = 2$ can be reduced to four colors. Therefore, we reduce the diagram $m(D)$ to a diagram D', where D' uses only four colors. Now the mirror image $m(D')$ will be a diagram colored by $t = 3$ using only four of the five available colors. (The proof of Lemma 2.2 from the subsection 2.3.2 looks a bit different than exact symmetry is due to the assumption that all under-arcs are pointing to the right.)

Chapter 3

Conclusion

Using Fox coloring modulo 13 we have shown that colorings of knots can be reduced to a minimal number of 5 colors. We have also investigated the coloring knots by linear Alexander quandles. Precisely, we have shown that any knot that is colorable by either the linear Alexander quandle $\mathbb{Z}_5[t]/(t-2)$ or $\mathbb{Z}_5[t]/(t-3)$ has a diagram using only 4 of the 5 available colors. Some areas for further investigation are whether or not knots colored by linear Alexander quandles of integers modulo 5 have a minimal number of colors. As well as, whether or not colorings of knots by linear Alexander quandles of integers modulo 7 can be reduced.

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