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A Maximum Principle in the Engel Group

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by

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Dedication

This work is dedicated to my son, Blaise. May his discoveries be even better.

Table of Contents

Abstract

In this thesis, we will examine the properties of subelliptic jets in the Engel group of step 3. Step-2 groups, such as the Heisenberg group, do not provide insight into the general abstract calculations. This thesis then, is the first explicit non-trivial computation of the abstract results.

Chapter 1

Background on the Engel Environment

One of the properties of Riemannian manifolds is that at each point, the dimension of the tangent space is equal to the topological dimension of the manifold. For example, if the manifold is $\mathbb{R}^n$, the tangent space is also $\mathbb{R}^n$. If the manifold is a surface in $\mathbb{R}^3$, then the tangent space at a point is $\mathbb{R}^2$. Thus, there are no restricted directions in the tangent space. However, many real-world models require restricted directions, because movement is limited. An example is driving a car [1]. This is because a car cannot move laterally, restricting the direction of motion. One other model is how the human brain processes visual images [5].

We then use sub-Riemannian spaces, which are spaces having points where the dimension of the tangent space is strictly less than the topological dimension of the manifold. We will consider sub-Riemannian manifolds with an algebraic group law, called Carnot groups.

In Bieske's paper [3], a sub-Riemannian maximum principle is proved for Heisenberg groups. Later, in [2], this maximum principle is proved for general Carnot groups. In the time between these two papers, it was discovered that the Heisenberg group and so-called "step-2 groups" do not have a sufficiently mellifluous geometry, resulting in these groups being inadequate concrete examples. In this thesis, we explore the step-3 Engel group, which allows us a more concrete understanding of the abstraction found in [2].

We start in $\mathbb{R}^4$ with coordinates (x_1, x_2, x_3, x_4) and for $\alpha \in \mathbb{R}$, let

$$\begin{aligned} u = u(x_1, x_2, x_3, \alpha) &= \left(\frac{1}{2}x_3 + \frac{1}{12}x_2(x_1 + \alpha x_2)\right) \\ m = m(x_1, x_2, x_3, \alpha) &= \left(-\frac{1}{2}\alpha x_3 + \frac{1}{12}x_1(x_1 + \alpha x_2)\right) \\ n = n(x_1, x_2, \alpha) &= \left(\frac{1}{2}x_1 + \frac{1}{2}\alpha x_2\right). \end{aligned}$$

Consider the linearly independent vector fields

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x_1} - \frac{x_2}{2}\frac{\partial}{\partial x_3} - u\frac{\partial}{\partial x_4} \\ X_2 &= \frac{\partial}{\partial x_2} + \frac{x_1}{2}\frac{\partial}{\partial x_3} + m\frac{\partial}{\partial x_4} \\ X_3 &= \frac{\partial}{\partial x_3} + n\frac{\partial}{\partial x_4} \\ \text{and} \quad X_4 &= \frac{\partial}{\partial x_4}. \end{aligned}$$

These vector fields obey the relations

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = X_4, \quad [X_2, X_3] = \alpha X_4, \text{ and} \quad [X_i, X_4] = 0 \text{ for } i = 1, 2, 3.$$

The vectors and these brackets form a Lie algebra, represented by g, that has a corresponding decomposition given by

$$g = V_1 \oplus V_2 \oplus V_3$$

where the vector space V_i is given by

$$\begin{aligned} V_1 &= \text{span}\,\{X_1, X_2\} \\ V_2 &= \text{span}\,\{X_3\} \\ \text{and } V_3 &= \text{span}\,\{X_4\}. \end{aligned}$$

Note that $[V_1, V_1] = V_2,\ [V_1, V_2] = V_3,$ and $[V_1, V_3] = 0.$

This Lie algebra, has an inner product that orthonomalizes the basis $\{X_1, X_2, X_3, X_4\}$ denoted by $\langle \cdot, \cdot \rangle$. It is given by the symmetric matrix

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{12} & k_{22} & k_{23} & k_{24} \\ k_{13} & k_{23} & k_{33} & k_{34} \\ k_{14} & k_{24} & k_{34} & k_{44} \end{bmatrix}$$

where

$$
\begin{aligned}
k_{11} &= 1 + u(u - nx_2) + \frac{x_2^2(1+n^2)}{4} \\
k_{12} &= -mu - \frac{(1+n^2)x_1x_2}{4} + \frac{1}{2}n(ux_1 + mx_2) \\
k_{13} &= -nu + \frac{(1+n^2)x_2}{2} \\
k_{14} &= u - \frac{nx_2}{2} \\
k_{22} &= 1 + m^2 - mnx_1 + \frac{(1+n^2)x_1^2}{4} \\
k_{23} &= mn - \frac{(1+n^2)x_1}{2} \\
k_{24} &= -m + \frac{nx_1}{2}, \quad k_{33} = 1 + n^2, \quad k_{34} = -n \\
k_{44} &= 1.
\end{aligned}
$$

The exponential map, which allows us to relate this Lie algebra to a Lie group called the step-3 Engel group, takes a vector from the Lie algebra at point p, say X_p and relates it to a unique integral curve given by $\gamma(t)$. The relationship is defined as exp $(X_p) = \gamma(1)$ where $\gamma'(0) = X_p$ and $\gamma(0) = p$.

THEOREM 1.1 *The exponential map exp: $g \to G$ is the identity map.*

Proof. Let $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t), \gamma_4(t))$ be a curve. Suppressing the parameter t, we compute:

$$
\begin{aligned}
\sum_{i=1}^{4} a_i X_i = \quad & a_1\left(\frac{\partial}{\partial x_1} - \frac{\gamma_2}{2}\frac{\partial}{\partial x_3} - u(\gamma_1,\gamma_2,\gamma_3,\alpha)\frac{\partial}{\partial x_4}\right) \\
& + a_2\left(\frac{\partial}{\partial x_2} + \frac{\gamma_1}{2}\frac{\partial}{\partial x_3} + m(\gamma_1,\gamma_2,\gamma_3,\alpha)\frac{\partial}{\partial x_4}\right) \\
& + a_3\left(\frac{\partial}{\partial x_3} + n(\gamma_1,\gamma_2,\alpha)\frac{\partial}{\partial x_4}\right) + a_4\frac{\partial}{\partial x_4} \\
= & \, a_1\frac{\partial}{\partial x_1} + a_2\frac{\partial}{\partial x_2} + \left(\frac{-a_1\gamma_2}{2} + \frac{\gamma_1 a_2}{2} + a_3\right)\frac{\partial}{\partial x_3} \\
& + \left(-\frac{a_1\gamma_3}{2} - \frac{a_1\gamma_1\gamma_2}{12} - \frac{\alpha a_1\gamma_2^2}{12} - \frac{\alpha a_2\gamma_3}{2} + \frac{a_2\gamma_1^2}{12}\right. \\
& \left. + \frac{\alpha a_2\gamma_1\gamma_2}{12} + \frac{a_3\gamma_1}{2} + \frac{\alpha a_3\gamma_2}{2} + a_4\right)\frac{\partial}{\partial x_4}.
\end{aligned}
$$

Thus the initial value problem

$$\begin{cases} \gamma'(t) = \sum_{i=1}^{4} a_i X_i(\gamma(t)) \\ \gamma(0) = 0 \end{cases}$$

is equivalent to the initial value problem

$$\begin{cases} \gamma_1'(t) = a_1 \\ \gamma_2'(t) = a_2 \\ \gamma_3'(t) = \frac{-a_1\gamma_2}{2} + \frac{\gamma_1 a_2}{2} + a_3 \\ \gamma_4'(t) = -\frac{a_1\gamma_3}{2} - \frac{a_1\gamma_1\gamma_2}{12} - \frac{\alpha a_1\gamma_2^2}{12} - \frac{\alpha a_2\gamma_3}{2} + \frac{a_2\gamma_1^2}{12} + \frac{\alpha a_2\gamma_1\gamma_2}{12} + \frac{a_3\gamma_1}{2} + \frac{\alpha a_3\gamma_2}{2} + x_4 \\ \gamma_i(0) = 0 \ \text{ for } \ i = 1,2,3,4. \end{cases}$$

Through integration,

$$\gamma_1(t) = a_1 t \ \text{ and } \ \gamma_2(t) = a_2 t.$$

Substituting, we get $\gamma_3'(t) = a_3$ and so

$$\gamma_3(t) = a_3 t$$

and $\gamma_4'(t) = a_4$, giving

$$\gamma_4(t) = a_4 t.$$

Thus, $\gamma(t) = (a_1 t, a_2 t, a_3 t, a_4 t)$ and so $\gamma(1) = (a_1, a_2, a_3, a_4)$. So

$$\exp(a_1 X_1 + a_2 X_2 + a_3 X_3 + a_4 X_4) = (a_1, a_2, a_3, a_4).$$

$\square$

The non-abelian algebraic group law is supplied by the Baker-Campbell-Hausdorff formula [10], which is given by

$$\exp X \star \exp Y = \exp(X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]]).$$

The higher order brackets are zero because they will all involve bracketing with X_4.

PROPOSITION 1 For $p = (x_1, x_2, x_3, x_4), q = (y_1, y_2, y_3, y_4) \in G$, the group law is

$$p \star q = \left(x_1 + y_1, x_2 + y_2, x_3 + y_3 + \frac{1}{2}\mu, x_4 + y_4 + \frac{1}{2}\nu + \frac{1}{12}\mu(x_1 + \alpha x_2 - y_1 - \alpha y_2) \right)$$

where $\mu = (x_1y_2 - x_2y_1)$ and $\nu = \big(x_1y_3 - x_3y_1 + \alpha(x_2y_3 - x_3y_2)\big)$ in the embedding space $\mathbb{R}^4$.

Proof.

$$\begin{aligned}
p \star q &= (x_1, x_2, x_3, x_4) \star (y_1, y_2, y_3, y_4) \\
&= \exp\bigg(x_1X_1 + x_2X_2 + x_3X_3 + x_4X_4\bigg)\exp\bigg(y_1X_1 + y_2X_2 + y_3X_3 + y_4X_4\bigg) \\
&= \exp\bigg((x_1X_1 + x_2X_2 + x_3X_3 + x_4X_4) + (y_1X_1 + y_2X_2 + y_3X_3 + y_4X_4) \\
&\quad + \frac{1}{2}[x_1X_1 + x_2X_2 + x_3X_3 + x_4X_4, y_1X_1 + y_2X_2 + y_3X_3 + y_4X_4] \\
&\quad + \frac{1}{12}[x_1X_1 + x_2X_2 + x_3X_3 + x_4X_4, \\
&\quad [x_1X_1 + x_2X_2 + x_3X_3 + x_4X_4, y_1X_1 + y_2X_2 + y_3X_3 + y_4X_4]] \\
&\quad - \frac{1}{12}[y_1X_1 + y_2X_2 + y_3X_3 + y_4X_4, \\
&\quad [x_1X_1 + x_2X_2 + x_3X_3 + x_4X_4, y_1X_1 + y_2X_2 + y_3X_3 + y_4X_4]]\bigg).
\end{aligned}$$

The only non-zero Lie brackets for $[x_1X_1 + x_2X_2 + x_3X_3 + x_4X_4, y_1X_1 + y_2X_2 + y_3X_3 + y_4X_4]$ are given by

$$\begin{aligned}
&x_1y_2[X_1, X_2] + x_1y_3[X_1, X_3] + x_2y_1[X_2, X_1] + x_2y_3[X_2, X_3] + x_3y_1[X_3, X_1] + x_3y_2[X_3, X_2] \\
&= x_1y_2X_3 + x_1y_3X_4 - x_2y_1X_3 + \alpha x_2y_3X_4 - x_3y_1X_4 - \alpha x_3y_2X_4 \\
&= \big(x_1y_2 - x_2y_1\big)X_3 + \big(x_1y_3 - x_3y_1 + \alpha(x_2y_3 - x_3y_2)\big)X_4 \\
&= \mu X_3 + \nu X_4.
\end{aligned}$$

The only non-zero Lie brackets for $[x_1X_1 + x_2X_2 + x_3X_3 + x_4X_4, [x_1X_1 + x_2X_2 + x_3X_3 + x_4X_4, y_1X_1 + y_2X_2 + y_3X_3 + y_4X_4]]$ are

$$x_1\mu[X_1, X_3] + x_2\mu[X_2, X_3] = \big(x_1\mu + \alpha x_2\mu\big)X_4.$$

Finally, the non-zero brackets for $[y_1X_1 + y_2X_2 + y_3X_3 + y_4X_4, [x_1X_1 + x_2X_2 + x_3X_3 + x_4X_4, y_1X_1 + y_2X_2 + y_3X_3 + y_4X_4]]$ are

$$y_1\mu[X_1, X_3] + y_2\mu[X_2, X_3] = \big(y_1\mu + \alpha y_2\mu\big)X_4.$$

Plugging these values into the formula, we get $\exp\Bigg((x_1X_1 + x_2X_2 + x_3X_3 + x_4X_4) + (y_1X_1 + y_2X_2 + y_3X_3 + y_4X_4) + \frac{1}{2}(\mu X_3 + \nu X_4) + \frac{1}{12}\big(x_1\mu + \alpha x_2\mu\big)X_4 - \frac{1}{12}\big(y_1\mu + \alpha y_2\mu\big)X_4\Bigg)$.

Combining like terms, we get

$\exp\Bigg((x_1 + y_1)X_1 + (x_2 + y_2)X_2 + \big(x_3 + y_3 + \frac{1}{2}\mu\big)X_3 + \big(x_4 + y_4 + \frac{1}{2}\nu + \frac{1}{12}\mu(x_1 + \alpha x_2) - \frac{1}{12}\mu(y_1 + \alpha y_2)\big)X_4\Bigg)$.

The proposition then follows since the exponential is the identity. □

COROLLARY 1.0.1 *The identity element under group multiplication is* $(0, 0, 0, 0)$ *and the inverse element is* $(-x_1, -x_2, -x_3, -x_4)$.

The next theorem tells us how the group law interacts with our vector fields.

THEOREM 1.2 *Let* $p = (x_1, x_2, x_3, x_4) \in G$ *be any point and let* 0 *be the identity element. The map* $L_p : G \to G$ *is left-multiplication by* p *and* DL_p *its derivative matrix. Then,* $X_i(p) = DL_pX_i(0)$. *This means the vector fields* $\{X_1, X_2, X_3, X_4\}$ *are left-invariant.*

Proof. Using the group law above, we compute for points $p = (x_1, x_2, x_3, x_4)$ and $q = (y_1, y_2, y_3, y_4)$:

$$DL_p = \begin{bmatrix} 1 & 0 & -\frac{x_2}{2} & -\frac{x_3}{2} + \frac{1}{12}\Bigg(-x_2(x_1 + \alpha x_2 - y_1 - \alpha y_2) - \mu\Bigg) \\ 0 & 1 & \frac{x_1}{2} & -\frac{x_3}{2}\alpha + \frac{1}{12}\Bigg(x_1(x_1 + \alpha x_2 - y_1 - \alpha y_2) - \alpha\mu\Bigg) \\ 0 & 0 & 1 & \frac{x_1}{2} + \alpha\frac{x_2}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and so

$$DL_p(0) = \begin{bmatrix} 1 & 0 & -\frac{x_2}{2} & -\frac{x_3}{2} - \frac{1}{12}x_2(x_1 + \alpha x_2) \\ 0 & 1 & \frac{x_1}{2} & -\frac{x_3}{2}\alpha + \frac{1}{12}x_1(x_1 + \alpha x_2) \\ 0 & 0 & 1 & \frac{x_1}{2} + \alpha\frac{x_2}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\frac{x_2}{2} & u(x_1, x_2, x_3, \alpha) \\ 0 & 1 & \frac{x_1}{2} & m(x_1, x_2, x_3, \alpha) \\ 0 & 0 & 1 & n(x_1, x_2, \alpha) \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Because $X_i(0) = \frac{\partial}{\partial x_i}$, the theorem follows. □

The tangent space to our Engel group, having topological dimension 4, is V_1, because those are the generating vector fields. It has topological dimension 2, so we have a sub-Riemannian space.

There exists a natural metric on G, given by the Carnot-Carathéodory distance. For the points p and q,

$$d_C(p,q) = \inf_{\Gamma} \int_0^1 \|\gamma'(t)\| \mathrm{d}t$$

where Γ is the set of all curves γ satisfying $\gamma(0) = p$, $\gamma(1) = q$ and $\gamma'(t) \in V_1$. Chow's theorem [1] tells us that $d_C(p,q)$ is a metric. By the previous theorem, this metric is invariant under left multiplication.

We now turn to calculus. Define a smooth function $f : G \to \mathbb{R}$. Because of the Lie brackets, vectors in V_i are i$^{\text{th}}$ order derivatives, with respect to the parameter of the curve ([1, Prop. 5.16]), where $i \in \{1,2,3\}$. The horizontal gradient, consisting of first order derivatives, uses only X_1 and X_2, so

$$\nabla_0 f = \big(X_1 f, X_2 f\big).$$

We note that this agrees with having V_1 as the tangent space for horizontal curves.

We use a symmetrized horizontal second derivative matrix, $(D^2 f)^\star$, with entries given by

$$\big((D^2 f)^\star\big)_{ij} = \frac{1}{2}\Big(X_i X_j f + X_j X_i f\Big)$$

for $i, j = 1, 2$. In our Engel group,

$$(D^2 f)^\star = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \tag{1.0.1}$$

where

$$D_{11} = \frac{\partial^2 f}{\partial x_1^2} - x_2 \frac{\partial^2 f}{\partial x_1 \partial x_3} - 2u \frac{\partial^2 f}{\partial x_1 \partial x_4} + x_2 u \frac{\partial^2 f}{\partial x_3 \partial x_4} + \frac{x_2}{6}\frac{\partial f}{\partial x_4} + \frac{x_2^2}{4}\frac{\partial^2 f}{\partial x_3^2} + u^2 \frac{\partial^2 f}{\partial x_4^2},$$

$$\begin{aligned} D_{12} = D_{21} = &\frac{\partial^2 f}{\partial x_1 \partial x_2} + \frac{x_1}{2}\frac{\partial^2 f}{\partial x_1 \partial x_3} + m \frac{\partial^2 f}{\partial x_1 \partial x_4} - \frac{x_2}{2}\frac{\partial^2 f}{\partial x_2 \partial x_3} - u \frac{\partial^2 f}{\partial x_2 \partial x_4} \\ &- \frac{x_1 u + m x_2}{2}\frac{\partial^2 f}{\partial x_3 \partial x_4} - \frac{x_1 x_2}{4}\frac{\partial^2 f}{\partial x_3^2} - um \frac{\partial^2 f}{\partial x_4^2} + \frac{\alpha x_2 - x_1}{12}\frac{\partial f}{\partial x_4} \end{aligned}$$

and

$$D_{22} = \frac{\partial^2 f}{\partial x_2^2} + x_1 \frac{\partial^2 f}{\partial x_2 \partial x_3} + 2m \frac{\partial^2 f}{\partial x_2 \partial x_4} + x_1 m \frac{\partial^2 f}{\partial x_3 \partial x_4} + \frac{-\alpha x_1}{6}\frac{\partial f}{\partial x_4} + \frac{x_1^2}{4}\frac{\partial^2 f}{\partial x_3^2} + m^2 \frac{\partial^2 f}{\partial x_4^2}.$$

Further, let the semi-horizontal derivative be given as

$$\nabla_1 f = (X_1 f, X_2 f, X_3 f)$$

and note this involves vectors in V_1 and V_2.

DEFINITION 1.0.1 *A function* $f : G \to \mathbb{R}$ *is* C^2_{sub} *if* $\nabla_1 f$ *and* $X_i X_j f$ *are continuous for* $i, j = 1, 2$.

A function that is C^2_{sub} is not necessarily C^2 in the Euclidean sense. For instance, the function $f(x_1, x_2, x_3, x_4) = (x_3)^{\frac{3}{2}}$ is not C^2 in the Euclidean sense at the origin, since the (Euclidean) second derivative is undefined. But we have the following proposition:

PROPOSITION 2 The function $f(x_1, x_2, x_3, x_4) = (x_3)^{\frac{3}{2}}$ is C^2_{sub} in the Engel group.

Proof. Using Theorem 1.1 and Proposition 1, get

$$X_1 f(0,0,0,0) = \frac{d}{dt} f((0,0,0,0) \star \exp(tX_1))\bigg|_{t=0} = \frac{d}{dt} f(t,0,0,0)\bigg|_{t=0} = \frac{d}{dt} 0\bigg|_{t=0} = 0.$$

Similarly, $X_2 f(0,0,0,0) = 0$ and

$$X_3 f(0,0,0,0) = \frac{3}{2}(t)^{\frac{1}{2}}\bigg|_{t=0} = 0.$$

We now need to check $X_i X_j f(0,0,0,0)$ for $i, j = 1, 2$. Using the Baker-Campbell-Hausdorff formula, we get

$$\begin{aligned}
X_1 X_2 f(0,0,0,0) &= \frac{\partial^2}{\partial t \partial s} f((0,0,0,0) \star \exp(tX_1) \star \exp(sX_2)\bigg|_{s,t=0} \\
&= \frac{\partial^2}{\partial t \partial s} f(\exp(tX_1 + sX_2 + \frac{1}{2} st X_3))\bigg|_{s,t=0} = \frac{\partial^2}{\partial t \partial s}\left(\frac{1}{2}st\right)^{\frac{3}{2}}\bigg|_{s,t=0} \\
&= \frac{\partial}{\partial t}\left(\frac{3t}{4}\left(\frac{1}{2}st\right)^{\frac{1}{2}}\bigg|_{s=0}\right)\bigg|_{t=0} = 0.
\end{aligned}$$

Similarly, $X_2 X_1 f(0,0,0,0) = 0$. For $X_1 X_1 f(0,0,0,0)$ we have

$$\begin{aligned}
X_1 X_1 f(0,0,0,0) &= \frac{\partial^2}{\partial t \partial s} f((0,0,0,0) \star \exp(tX_1) \star \exp(sX_1)\bigg|_{s,t=0} \\
&= \frac{\partial^2}{\partial t \partial s} f(\exp((s+t)X_1))\bigg|_{s,t=0} = \frac{\partial^2}{\partial t \partial s} 0\bigg|_{s,t=0} = 0
\end{aligned}$$

with the same for $X_2 X_2 f(0,0,0,0)$. The proposition is proved. $\square$

With the above derivatives and using the Engel divergence, which is the sum of our vectors spaces X_1 and X_2, we define the horizontal p-Laplacian of a smooth function f for $1 < \mathrm{p} < \infty$ by

$$\begin{aligned}
\Delta_{\mathrm{p}} f &= \text{div}(\|\nabla_0 f\|^{\mathrm{p}-2} \nabla_0 f) = X_1(\|\nabla_0 f\|^{\mathrm{p}-2} \nabla_0 f) + X_2(\|\nabla_0 f\|^{\mathrm{p}-2} \nabla_0 f) \\
&= X_1(\|\nabla_0 f\|^{\mathrm{p}-2} X_1 f) + X_2(\|\nabla_0 f\|^{\mathrm{p}-2} X_2 f) \\
&= \|\nabla_0 f\|^{\mathrm{p}-2}(X_1 X_1 f + X_2 X_2 f) + (\mathrm{p}-2)\|\nabla_0 f\|^{\mathrm{p}-4}\big\langle (D^2 f)^{\star} \nabla_0 f, \nabla_0 f \big\rangle.
\end{aligned}$$

Letting p run to infinity gives us the infinite Laplacian, defined as

$$\Delta_\infty f = \left\langle (D^2 f)^\star \nabla_0 f, \nabla_0 f \right\rangle.$$

Chapter 2

Carnot Jets and Viscosity Solutions

Following [2], we have the Taylor Theorem:

THEOREM 2.1 *For a smooth function $f : G \to \mathbb{R}$, the Taylor formula at the point p_0 is:*

$$f(p) = f(p_0) + \langle \nabla_1 f(p_0), \widehat{p_0^{-1}p} \rangle + \frac{1}{2}\langle (D^2 f(p_0))^{\star}\overline{p_0^{-1}p}, \overline{p_0^{-1}p} \rangle + o((d(p_0,p))^2) \tag{2.0.1}$$

where $\overline{p_0^{-1}p}$ is $p_0^{-1}p$ projected onto V_1 and $\widehat{p_0^{-1}p}$ is $p_0^{-1}p$ projected onto $V_1 \oplus V_2$. We remind ourselves that the exponential mapping is the identity.

Proof. Let $p = (x_1, x_2, x_3, x_4)$ and $p_0 = (x_1^0, x_2^0, x_3^0, x_4^0)$. Following [3], we will rewrite the Taylor polynomial as:

$$\begin{aligned} f(p) + o((d(p_0,p))^2) = & f(p_0) + (x_1 - x_1^0)X_1 f(p_0) + (x_2 - x_2^0)X_2 f(p_0) + \\ & \left(x_3 - x_3^0 + \frac{1}{2}(x_1 x_2^0 - x_1^0 x_2)\right) X_3 f(p_0) + \frac{1}{2}(x_1 - x_1^0)^2 X_1 X_1 f(p_0) \\ & + \frac{1}{2}(x_2 - x_2^0)^2 X_2 X_2 f(p_0) + \frac{1}{2}(x_1 - x_1^0)(x_2 - x_2^0) X_1 X_2 f(p_0) \\ & + \frac{1}{2}(x_1 - x_1^0)(x_2 - x_2^0) X_2 X_1 f(p_0) \end{aligned}$$

and we will call the right side polynomial $P(p)$. We then have

$$\begin{aligned}
X_1P(p) &= X_1f(p_0)+\frac{1}{2}x_2^0X_3f(p_0)-\frac{1}{2}x_2X_3f(p_0)\\
&\quad+(x_1-x_1^0)X_1X_1f(p_0)+\frac{1}{2}(x_2-x_2^0)X_1X_2f(p_0)+\frac{1}{2}(x_2-x_2^0)X_2X_1f(p_0)\\
X_2P(p) &= X_2f(p_0)-\frac{1}{2}x_1^0X_3f(p_0)+\frac{1}{2}x_1X_3f(p_0)\\
&\quad+(x_2-x_2^0)X_2X_2f(p_0)+\frac{1}{2}(x_1-x_1^0)X_1X_2f(p_0)+\frac{1}{2}(x_1-x_1^0)X_2X_1f(p_0)\\
X_3P(p) &= X_3f(p_0)\\
X_1X_1P(p) &= X_1X_1f(p_0)\\
X_2X_2P(p) &= X_2X_2f(p_0)\\
X_2X_1P(p) &= -\frac{1}{2}X_3f(p_0)+\frac{1}{2}X_1X_2f(p_0)+\frac{1}{2}X_2X_1f(p_0)\\
\text{and }\ X_1X_2P(p) &= \frac{1}{2}X_3f(p_0)+\frac{1}{2}X_1X_2f(p_0)+\frac{1}{2}X_2X_1f(p_0).
\end{aligned}$$

We then have $X_1P(p_0)=X_1f(p_0), X_2P(p_0)=X_2f(p_0)$, and $X_3P(p_0)=X_3f(p_0)$. And we have $X_1X_1P(p_0)=X_1X_1f(p_0), X_2X_2P(p_0)=X_2X_2f(p_0)$, and $\frac{1}{2}(X_2X_1P(p_0)+X_1X_2P(p_0))=\frac{1}{2}(X_1X_2f(p_0)+X_2X_1f(p_0))$. Note also that since $X_3=[X_1,X_2]$, we have $X_1X_2P(p_0)=X_1X_2f(p_0)$ and $X_2X_1P(p_0)=X_2X_1f(p_0)$. By definition, Equation (2.0.1) gives the second-order Taylor polynomial.

□

Because smoothness is too restrictive of a requirement we use the following definition to introduce some flexibility. This definition will invoke Taylor polynomials. Let us recall a function f is defined to be upper semicontinuous if

$$\limsup_{x\to x_0} f(x)\leq f(x_0)$$

and a function g is defined to be lower semicontinuous if

$$\liminf_{x\to x_0} g(x)\geq g(x_0)$$

Using these functions, we are able to define the superjets of our space.

DEFINITION 2.0.2 *Let f be an upper semicontinuous function $f:G\to\mathbb{R}$ and let S^2 be the set of all 2×2 symmetric matrices. For $\eta\in V_1\oplus V_2$ and $X\in\mathcal{S}^2$, then the following inequality motivates the definition of the second-order superjet.*

$$f(p)\leq f(p_0)+\langle\eta,\widehat{p_0^{-1}p}\rangle+\frac{1}{2}\langle X,\overline{p_0^{-1}p}\rangle+o((d(p_0,p))^2)\text{ as }p\to p_0. \tag{2.0.2}$$

The second order superjet of f at p_0, denoted $J^{2,+}f(p_0)$, is defined as

$$J^{2,+}f(p_0) = \{(\eta, X) \subset (V_1 \oplus V_2) \times S^2 : \textit{Equation } (2.0.2) \textit{ holds}\}.$$

There is also the second-order subjet of the lower semicontinuous function g at p_0, which is represented by $J^{2,-}g(p_0)$, and is defined by

$$J^{2,-}g(p_0) = -J^{2,+}(-g)(p_0).$$

Note that in (2.0.2) the inequality will flip to $\geq$. The set-theoretic closure of $J^{2,+}f(p_0)$, which will be denoted $\overline{J}^{2,+}f(p_0)$, is given by $(\eta, X) \in \overline{J}^{2,+}f(p_0)$ if there is a sequence $\{(p_i, f(p_i), \eta_i, X_i)\} \in G \times \mathbb{R} \times g \times S^2$ so that as $i \to \infty$, then $\{(p_i, f(p_i), \eta_i, X_i)\} \to (p_0, f(p_0), \eta, X)$ with $(\eta_i, X_i) \in J^{2,+}f(p_i)$. Given an upper semicontinuous function f, we define a set of test functions that touch f from above at p_0, denoted by $\mathcal{TA}(f, p_0)$ and given a lower semicontinuous function g, we can also define a set of test functions that touch g from below at p_0, denoted $\mathcal{TB}(g, p_0)$. Thus:

$$\mathcal{TA}(f, p_0) = \{\phi : G \to \mathbb{R} : \phi \in C^2_{\text{sub}}(p_0), \phi(p_0) = f(p_0) \text{ and } \phi(p) > f(p) \text{ for } p \text{ near } p_0\}$$

and

$$\mathcal{TB}(g, p_0) = \{\phi : G \to \mathbb{R} : \phi \in C^2_{\text{sub}}(p_0), \phi(p_0) = g(p_0) \text{ and } \phi(p) < g(p) \text{ for } p \text{ near } p_0\}.$$

Thus we can define another pair of sets, $K^{2,+}f(p_0)$ and $K^{2,-}g(p_0)$, where

$$\begin{aligned} K^{2,+}f(p_0) &= \{(\nabla_1\phi(p_0), (D^2\phi(p_0))^\star) : \phi \in \mathcal{TA}(f, p_0)\} \\ \text{and } \; K^{2,-}g(p_0) &= \{(\nabla_1\phi(p_0), (D^2\phi(p_0))^\star) : \phi \in \mathcal{TB}(g, p_0)\}. \end{aligned}$$

These definitions motivate the following lemma, from [2]. The proof is excluded.

LEMMA 2.1 *[2, Lemma 2.2] Also see [6]. Given an upper semicontinuous function f and lower semicontinuous function g, then*

$$J^{2,+}f(p_0) = K^{2,+}f(p_0) \quad \textit{and} \quad J^{2,-}g(p_0) = K^{2,-}g(p_0).$$

These jets will prove to be very useful in creating a type of solution to partial differential equations. In general, a k-order partial differential equation is solved by a classic solution if there is a k-times differentiable function on an interval that satisfies the conditions of the equation. For instance, the first order differential equation with $u : \mathbb{R}^2 \to \mathbb{R}$

$$\frac{\partial u}{\partial x} + u = 0, \text{ where } u = u(x, y)$$

has a solution of

$$u = e^{-x} f(y), \text{ where } f(y) \text{ is an arbitray function of } y$$

This is a classical solution over the region $\mathbb{R}^2$ [9]. Unfortunately, there are a great deal many partial differential equations for which the classic solution does not exist. For instance, the Eikonal equation from [8], given by

$$|u'(x)| = 1, \quad \text{for } x \in (-1, 1),$$

with initial conditions $u(-1) = u(1) = 0$. This equation does not have a differentiable function which satisfies the equation over the interval $(-1, 1)$[8]. However, if we relax the requirement of differentiability to only requiring continuity, we discover two possible solutions: $u(x) = -|x| + 1$ and $v(x) = |x| - 1 = -u(x)$. As discussed in [7], because the functions are merely continuous and not necessarily differentiable, we must "push" differentiation onto appropriate test functions. This is the purpose of the jets discussed above: they are the result of this process.

We consider a class of equations given by:

$$F(p, f(p), \nabla_1 f(p), (D^2 f(p))^\star) = 0,$$

where the function F, defined as

$$F : G \times \mathbb{R} \times g \times S^2 \to \mathbb{R},$$

fulfills the inequality

$$F(p, r, \eta, X) \leq F(p, s, \eta, Y)$$

when $r \leq s$ and $Y \leq X$. This function F is proper as defined in [7]. The $\mathtt{p}$-Laplace equation, defined for $1 < \mathtt{p} < \infty$ by

$$-\left(\|\nabla_0 f\|^{\mathtt{p}-2} \mathrm{tr}((D^2 f)^\star) + (p-2) \|\nabla_0 f\|^{\mathtt{p}-4} \langle (D^2 f)^\star \nabla_0 f, \nabla_0 f \rangle \right) = 0$$

and the infinite Laplace equation

$$-\langle (D^2 f)^\star \nabla_0 f, \nabla_0 f \rangle = 0$$

are examples of such equations.

Chapter 3

Sub-Riemannian Maximum Principle

We now state Lemma 3 from [2] which allows us to express our semi-horizontal derivative and our symmetrized second derivative matrix in terms of their Euclidean counterparts.

LEMMA 3.1 *Recalling the definition of u, m and n in Chapter 2, we define the matrix $\mathbb{A}$ as*

$$\mathbb{A} = \begin{bmatrix} 1 & 0 & -\frac{x_2}{2} & -u \\ 0 & 1 & \frac{x_1}{2} & m \end{bmatrix}$$

and the matrix $\mathbb{B}$ as

$$\mathbb{B} = \begin{bmatrix} 0 & 0 & 1 & n \end{bmatrix}.$$

Let f be a smooth function with $\nabla_{eucl} f$ its Euclidean gradient and let $D^2_{eucl} f$ be the Euclidean second-order derivative matrix of f. Let $\mathbb{A}^T$ denote the transpose of the matrix $\mathbb{A}$. Then

$$\nabla_1 f = \mathbb{A} \cdot \nabla_{eucl} f \oplus \mathbb{B} \cdot \nabla_{eucl} f.$$

Further, for all $t \in \mathbb{R}^n$,

$$\langle (D^2 f)^\star \cdot t, t \rangle = \langle (\mathbb{A} \cdot D^2_{eucl} f \cdot \mathbb{A}^T + \mathbb{M} \cdot t, t \rangle$$

where the matrix $\mathbb{M}$ is given by

$$\mathbb{M} = \begin{bmatrix} \frac{1}{6} x_2 \frac{\partial f}{\partial x_4} & \frac{1}{12}(\alpha x_2 - x_1) \frac{\partial f}{\partial x_4} \\ \frac{1}{12}(\alpha x_2 - x_1) \frac{\partial f}{\partial x_4} & -\frac{1}{6} \alpha x_1 \frac{\partial f}{\partial x_4} \end{bmatrix}$$

Proof.

A quick computation shows that

$$\mathbb{A} \cdot \nabla_{\text{eucl}} f = \begin{bmatrix} \frac{\partial f}{\partial x_1} - \frac{x_2}{2} \frac{\partial f}{\partial x_3} - u \frac{\partial f}{\partial x_4} \\ \frac{\partial f}{\partial x_2} + \frac{x_1}{2} \frac{\partial f}{\partial x_3} + m \frac{\partial f}{\partial x_4} \end{bmatrix}$$

and

$$\mathbb{B} \cdot \nabla_{\text{eucl}} f = \left[\frac{\partial f}{\partial x_3} + n \frac{\partial f}{\partial x_4} \right]$$

so the direct sum gives us

$$\begin{bmatrix} \frac{\partial f}{\partial x_1} - \frac{x_2}{2} \frac{\partial f}{\partial x_3} - u \frac{\partial f}{\partial x_4} \\ \frac{\partial f}{\partial x_2} + \frac{x_1}{2} \frac{\partial f}{\partial x_3} + m \frac{\partial f}{\partial x_4} \\ \frac{\partial f}{\partial x_3} + n \frac{\partial f}{\partial x_4} \end{bmatrix} = \nabla_1 f$$

For the second derivative matrix, matrix multiplication gives

$$\mathbb{A} \cdot D^2_{\text{eucl}} f \cdot \mathbb{A}^T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$

where

$$T_{11} = \frac{\partial^2 f}{\partial x_1^2} - x_2 \frac{\partial^2 f}{\partial x_1 \partial x_3} - 2u \frac{\partial^2 f}{\partial x_1 \partial x_4} + x_2 u \frac{\partial^2 f}{\partial x_3 \partial x_4} + \frac{x_2^2}{4} \frac{\partial^2 f}{\partial x_3^2} + u^2 \frac{\partial^2 f}{\partial x_4^2},$$

$$\begin{aligned} T_{12} = T_{21} = \frac{\partial^2 f}{\partial x_1 \partial x_2} + \frac{x_1}{2} \frac{\partial^2 f}{\partial x_1 \partial x_3} + m \frac{\partial^2 f}{\partial x_1 \partial x_4} - \frac{x_2}{2} \frac{\partial^2 f}{\partial x_2 \partial x_3} - u \frac{\partial^2 f}{\partial x_2 \partial x_4} \\ - \frac{x_1 u + m x_2}{2} \frac{\partial^2 f}{\partial x_3 \partial x_4} - \frac{x_1 x_2}{4} \frac{\partial^2 f}{\partial x_3^2} - um \frac{\partial^2 f}{\partial x_4^2}, \end{aligned}$$

and

$$T_{22} = \frac{\partial^2 f}{\partial x_2^2} + x_1 \frac{\partial^2 f}{\partial x_2 \partial x_3} + 2m \frac{\partial^2 f}{\partial x_2 \partial x_4} + x_1 m \frac{\partial^2 f}{\partial x_3 \partial x_4} + \frac{x_1^2}{4} \frac{\partial^2 f}{\partial x_3^2} + m^2 \frac{\partial^2 f}{\partial x_4^2}$$

Our choice of $\mathbb{M}$ provides the missing $\frac{\partial f}{\partial x_4}$ terms from Equation (1.0.1). We then have $\mathbb{A} \cdot D^2_{\text{eucl}} f \cdot \mathbb{A}^T + \mathbb{M} = (D^2 f)^\star$. □

By relating our derivatives to their Euclidean counterparts, we now have a way to relate our Euclidean superjets to the jets we created for our Engel group.

COROLLARY 3.1.1 *Let* $(\eta, X) \in \overline{J}^{2,+}_{eucl} f(p)$, *where* $(\eta, X) \in \mathbb{R}^4 \times S^4$. *Then*

$$(\mathbb{A} \cdot \eta \oplus \mathbb{B} \cdot \eta, \mathbb{A} X \mathbb{A}^T + \mathbb{M}(\eta, p)) \in \overline{J}^{2,+} f(p).$$

The matrix $\mathbb{M}(\eta, p)$ *in this case is*

$$\mathbb{M}(\eta, p) = \begin{bmatrix} \frac{1}{6} x_2 \eta_4 & \frac{1}{12}(\alpha x_2 - x_1)\eta_4 \\ \frac{1}{12}(\alpha x_2 - x_1)\eta_4 & -\frac{1}{6} \alpha x_1 \eta_4 \end{bmatrix}.$$

Proof.

Our goal is to convert the Euclidean Taylor polynomial for the upper semicontinuous function u

$$u(p) \leq u(p_0) + \langle \eta, p - p_0 \rangle_E + \frac{1}{2}\langle X(p - p_0), p - p_0 \rangle_E + o(|p - p_0|^2)$$

to

$$u(p) \leq u(p_0) + \langle \mathbb{A}\eta \oplus \mathbb{B}\eta, \widehat{p_0^{-1}p} \rangle + \frac{1}{2}\langle \mathbb{A}X\mathbb{A}^T\overline{(p_0^{-1}p)}, \overline{(p_0^{-1}p)} \rangle + \frac{1}{2}\langle \mathbb{M}\overline{p_0^{-1}p}, \overline{p_0^{-1}p} \rangle + o(d(p, p_0)^2)$$

where $\langle \cdot, \cdot \rangle_E$ is the Euclidean inner product and $\langle \cdot, \cdot \rangle$ is the Engel inner product. Further, $p = (y_1, y_2, y_3, y_4)$ and $p_0 = (x_1, x_2, x_3, x_4)$.

First, we will look at the error term. Suppose W is $o(|p - p_0|^2)$. Then,

$$\frac{W}{d(p, p_0)^2} = \frac{W}{|p - p_0|^2} \frac{|p - p_0|^2}{d(p, p_0)^2}$$

The first term goes to zero, and the second term is bounded by Prop 1.1 of [11]. Thus, the right hand side goes to zero as $p \to p_0$, and thus W is $o(d(p, p_0)^2)$. The Taylor theorem, thus, now can be made to read:

$$u(p) \leq u(p_0) + \langle \eta, p - p_0 \rangle_E + \frac{1}{2}\langle X(p - p_0), p - p_0 \rangle_E + o(d(p, p_0)^2).$$

Before proceeding, let us define the matrix A as

$$A = \begin{bmatrix} 1 & 0 & -\frac{x_2}{2} & -u \\ 0 & 1 & \frac{x_1}{2} & m \\ 0 & 0 & 1 & n \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Let the η vector be defined as

$$\eta = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{bmatrix}$$

and the symmetric matrix X is given as

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ X_{12} & X_{22} & X_{23} & X_{24} \\ X_{13} & X_{23} & X_{33} & X_{34} \\ X_{14} & X_{24} & X_{34} & X_{44} \end{bmatrix}.$$

Then

$$\begin{aligned}
\langle A\eta, p_0^{-1}p\rangle &= \left(\eta_1 - \frac{x_2}{2}\eta_3 + \left(-\frac{x_1x_2}{12} - \frac{\alpha x_2^2}{12} - \frac{x_3}{2}\right)\eta_4\right)(y_1 - x_1) \\
&\quad + \left(\eta_2 + \frac{x_1}{2}\eta_3 + \left(\frac{x_1^2}{12} + \frac{\alpha x_1 x_2}{12} - \frac{\alpha x_3}{2}\right)\eta_4\right)(y_2 - x_2) \\
&\quad + \left(y_3 - x_3 + \frac{1}{2}\big(x_2y_1 - x_1y_2\big)\right)\left(\eta_3 + (\frac{x_1}{2} + \frac{\alpha x_2}{2})\eta_4\right) \\
&\quad + \bigg(y_4 - x_4 + \frac{1}{2}\big(x_3y_1 - x_1y_3\big) + \frac{1}{2}\alpha\big(x_3y_2 - x_2y_3\big) \\
&\quad - \frac{1}{12}(x_1 + y_1)(x_2y_1 - x_1y_2) - \frac{1}{12}\alpha(x_2 + y_2)(x_2y_1 - x_1y_2)\bigg)\eta_4.
\end{aligned}$$

Distributing and collecting like terms, we get

$$\begin{aligned}
\langle A\eta, p_0^{-1}p\rangle &= (y_1 - x_1)\eta_1 - \frac{x_2}{2}(y_1 - x_1)\eta_3 + (y_1 - x_1)(-\frac{x_1x_2}{12} - \frac{\alpha x_2^2}{12} - \frac{x_3}{2})\eta_4 \\
&\quad + (y_2 - x_2)\eta_2 + \frac{x_1}{2}(y_2 - x_2)\eta_3 + (y_2 - x_2)(\frac{x_1^2}{12} + \frac{\alpha x_1x_2}{12} - \frac{\alpha x_3}{2})\eta_4 \\
&\quad + (y_3 - x_3)\eta_3 + \frac{1}{2}(x_2y_1 - x_1y_2)\eta_3 + (y_3 - x_3)\left(\frac{x_1}{2} + \frac{\alpha x_2}{2}\right)\eta_4 \\
&\quad + \frac{1}{2}(x_2y_1 - x_1y_2)(\frac{x_1}{2} + \frac{\alpha x_2}{2})\eta_4 \\
&\quad + \bigg(\frac{1}{12}(x_1 + y_1)(x_2y_1 - x_1y_2) - \frac{1}{12}\alpha(x_2 + y_2)(x_2y_1 - x_1y_2) \\
&\quad + (y_2 - x_2)(\frac{x_1^2}{12} + \frac{\alpha x_1x_2}{12} - \frac{\alpha x_3}{2})\bigg)\eta_4 \\
&= (y_1 - x_1)\eta_1 + (y_2 - x_2)\eta_2 + (y_3 - x_3)\eta_3 \\
&\quad + \frac{1}{2}(x_1x_2 - y_1x_2 + -x_2x_1 + y_2x_1 + x_2y_1 - x_1y_2)\eta_3 \\
&\quad + \bigg((y_3 - x_3)\left(\frac{x_1}{2} + \frac{\alpha x_2}{2}\right) + \left(\frac{1}{2}\big(x_2y_1 - x_1y_2\big)\right)\left(\frac{x_1}{2} + \frac{\alpha x_2}{2}\right) \\
&\quad + y_4 - x_4 + \frac{1}{2}\big(x_3y_1 - x_1y_3\big) \\
&\quad + \frac{1}{2}\alpha\big(x_3y_2 - x_2y_3\big) + \frac{1}{12}(x_1 + y_1)(x_2y_1 - x_1y_2) \\
&\quad - \frac{1}{12}\alpha(x_2 + y_2)(x_2y_1 - x_1y_2) + (y_2 - x_2)\left(\frac{x_1^2}{12} + \frac{\alpha x_1x_2}{12} - \frac{\alpha x_3}{2}\right)\bigg)\eta_4 \\
&= (y_1 - x_1)\eta_1 + (y_2 - x_2)\eta_2 + (y_3 - x_3)\eta_3 \\
&\quad + \left(y_4 - x_4 + \frac{1}{12}\big(x_1 - y_1 + \alpha(x_2 - y_2)\big)\big(x_1y_2 - y_1x_2\big)\right)\eta_4.
\end{aligned}$$

Further, $\langle \eta, p_0 - p\rangle_E$ is $(y_1 - x_1)\eta_1 + (y_2 - x_2)\eta_2 + (y_3 - x_3)\eta_3 + (y_4 - x_4)\eta_4$ and so $\langle \eta, p_0 - p\rangle_E - \langle A\eta, p_0^{-1}p\rangle$ is

$$\frac{1}{12}\big(x_1 - y_1 + \alpha(x_2 - y_2)\big)(x_1y_2 - x_2y_1)\eta_4.$$

Thus, the Euclidean Taylor polynomial $f(p) \leq f(p_0) + \langle \eta, p_0 - p\rangle_E + \langle X(p_0 - p), p_0 - p\rangle_E + o(|p_0 - p|^2)$ can be rewritten as

$$\begin{aligned} u(p) \leq & u(p_0) + \langle A\eta, p_0^{-1}p\rangle + \frac{1}{12}\big(x_1 - y_1 + \alpha(x_2 - y_2)\big)(x_1y_2 - x_2y_1)\eta_4 \\ & + \langle X(p_0 - p), p_0 - p\rangle_E + o(d(p, p_0)^2). \end{aligned}$$

Similarly, $\langle AXA^T p_0^{-1}p,\ p_0^{-1}p\rangle - \langle X(p - p_0), p - p_0\rangle_E$ is

$$\begin{aligned} \frac{1}{144}&(x_1 - y_1 + \alpha(x_2 - y_2))(x_1y_2 - x_2y_1) \\ & \times \Big(x_1^2y_2X_{44} - \alpha x_2^2y_1X_{44} + x_2(24X_{24} + y_1(y_1 + \alpha y_2)X_{44}) \\ & + x_1(24X_{14} - X_{44}(x_2(y_1 - \alpha y_2) + y_2(y_1 + \alpha y_2))) \\ & + 24(x_3X_{34} + x_4X_{44} - y_1X_{14} - y_2X_{24} - y_3X_{34} + y_4X_{44})\Big). \end{aligned}$$

Let us then look at what is left. After simplifying, the coefficient to X_{14} is given by

$$\frac{1}{144}(24x_1 - 24y_1)(x_1 - y_1 + \alpha(x_2 - y_2))(x_1y_2 - x_2y_1).$$

However, $x_1y_2 - x_2y_1 = y_2(x_1 - y_1) + y_1(y_2 - x_2)$ is $O(d(p, p_0))$ since we are in a bounded domain, and thus y_2 and y_1 are bounded numbers. Also, $24(x_1 - y_1)$ is $O(d(p, p_0))$. So this coefficient is $o(d^2(p_0, p))$, and is part of the error term. Similarly, the coefficient to X_{24} is given by

$$\frac{1}{144}(24x_2 - 24y_2)\big(x_1 - y_1 + \alpha(x_2 - y_2)\big)(x_1y_2 - x_2y_1)$$

and is also in the error term.

Further, we have the coefficient of X_{34} given by

$$\frac{1}{144}(24x_3 - 24y_3)(x_1y_2 - x_2y_1)\big(x_1 - y_1 + \alpha(x_2 - y_2)\big).$$

Since $24x_3 - 24y_3$ approaches 0 as $p \to p_0$, this term overall is $o(d^2(p_0, p))$ and thus part of the error.

Finally, the X_{44} coefficient is

$$\frac{1}{144}(x_1y_2 - x_2y_1)\big(x_1 - y_1 + \alpha(x_2 - y_2)\big)$$
$$\times \quad \big(24(x_4 - y_4) - x_1x_2y_1 - \alpha x_2^2y_1 + x_2y_1^2 + x_1^2y_2 + \alpha x_1x_2y_2 - x_1y_1y_2 + \alpha x_2y_1y_2 - \alpha x_1y_2^2\big).$$

Since we again have a $(x_1y_2 - x_2y_1)(x_1 - y_1 + \alpha(x_2 - y_2))$ term, we know this will be $O(d^2(p_0, p))$. Again, through a similarity of terms, we see that the rest of the multiplication tends to 0 as $p \to p_0$. That is, this term is $o(d^2(p_0, p))$ and part of the error. Thus, $\langle Xp - p_0, p - p_0\rangle_E = \langle AXA^T p_0^{-1}p, p_0^{-1}\rangle$ plus elements that are $o(d^2(p, p_0))$.

Because the first and second coordinates of $p_0^{-1}p$ are $O(d(p, p_0))$, the third coordinate of $p_0^{-1}p$ is $O(d^2(p, p_0))$ and the fourth coordinate of $p_0^{-1}p$ is $o(d^2(p, p_0))$, the Taylor polynomial

$$\begin{aligned} u(p) \leq u(p_0) + \langle A\eta, p_0^{-1}p\rangle + \frac{1}{12}\big(x_1 - y_1 + \alpha(x_2 - y_2)\big)(x_1y_2 - x_2y_1)\eta_4 \\ + \frac{1}{2}\langle AXA^T p_0^{-1}p, p_0^{-1}p\rangle + o(d^2(p, p_0)) \end{aligned}$$

can be rewritten as

$$\begin{aligned} u(p) \leq u(p_0) + \langle \mathbb{A}\cdot\eta \oplus \mathbb{B}\cdot\eta, \widehat{p_0^{-1}p}\rangle + \frac{1}{12}\big(x_1 - y_1 + \alpha(x_2 - y_2)\big)(x_1y_2 - x_2y_1)\eta_4 \\ + \frac{1}{2}\langle \mathbb{A}X\mathbb{A}^T\overline{p_0^{-1}p}, \overline{p_0^{-1}p}\rangle + o(d^2(p, p_0)) \end{aligned}$$

Recall that $\overline{p_0^{-1}p}$ is $p_0^{-1}p$ projected onto V_1 and $\widehat{p_0^{-1}p}$ is $p_0^{-1}p$ projected onto $V_1 \oplus V_2$.
Thus, we are just left with $\frac{1}{12}\big(x_1 - y_1 + \alpha(x_2 - y_2)\big)(x_1y_2 - x_2y_1)\eta_4$.
But we know matrix $\mathbb{M}$ is

$$\mathbb{M}(\eta, p) = \begin{bmatrix} \frac{1}{6}x_2\eta_4 & \frac{1}{12}(\alpha x_2 - x_1)\eta_4 \\ \frac{1}{12}(\alpha x_2 - x_1)\eta_4 & -\frac{1}{6}\alpha x_1\eta_4 \end{bmatrix}.$$

and the first two components of the Engel multiplication are $(y_1 - x_1, y_2 - x_2)$. So

$$\begin{bmatrix} y_1 - x_1 & y_2 - x_2 \end{bmatrix} \times \frac{1}{2}\mathbb{M} \times \begin{bmatrix} y_1 - x_1 \\ y_2 - x_2 \end{bmatrix}$$

is our left over term. Thus, $\frac{1}{12}\big(x_1 - y_1 + \alpha(x_2 - y_2)\big)(x_1y_2 - x_2y_1)\eta_4$ equals $\frac{1}{2}\langle \mathbb{M}\,\overline{p_0^{-1}p},\ \overline{p_0^{-1}p}\rangle$ and so our result holds. $\square$

With this corollary, we can make use of the Euclidean results from [2]. Specifically, we use Theorem 3.2 and Remark 3.8.

THEOREM 3.1 *Let $\epsilon \in \mathbb{R}^+$. Let f be an upper semicontinuous function in $\mathbb{R}^4$, g a lower semicontinuous function in $\mathbb{R}^4$ and ϕ a C^2 function in $\mathbb{R}^8$. Let $\mathcal{O}$ be a locally compact subset of $\mathbb{R}^4$ and let $(\hat{p}, \hat{q})$ be a maximum point of $f(p) - g(q) - \phi(p, q)$ over $\mathcal{O} \times \mathcal{O}$ and let the matrix $\mathcal{M} \in S^8$ be given by*

$$\mathcal{M} = \begin{bmatrix} D^2_{pp}\phi(\hat{p}, \hat{q}) & D^2_{pq}\phi(\hat{p}, \hat{q}) \\ D^2_{qp}\phi(\hat{p}, \hat{q}) & D^2_{qq}\phi(\hat{p}, \hat{q}) \end{bmatrix}$$

Where $p = (x_1, x_2, x_3, x_4)$ and $q = (y_1, y_2, y_3, y_4)$. This leads us to the three 4×4 matrices where the $(ij)^{th}$ terms are given by

$$(D^2_{pp}\phi(\hat{p}, \hat{q}))_{ij} = \frac{\partial^2 \phi(\hat{p}, \hat{q})}{\partial x_i \partial x_j}$$

$$(D^2_{pq}\phi(\hat{p}, \hat{q}))_{ij} = (D^2_{qp}\phi(\hat{p}, \hat{q}))_{ji} = \frac{\partial^2 \phi(\hat{p}, \hat{q})}{\partial x_i \partial y_j}$$

and

$$(D^2_{qq}\phi(\hat{p}, \hat{q}))_{ij} = \frac{\partial^2 \phi(\hat{p}, \hat{q})}{\partial y_i \partial y_j}.$$

Then there exist matrices $X, Y \in S^4$ such that

$$(D_p\phi(\hat{p}, \hat{q}), X) \in \overline{J}^{2,+}_{eucl} f(\hat{p}) \text{ and } (-D_q\phi(\hat{p}, \hat{q}), Y) \in \overline{J}^{2,-}_{eucl} g(\hat{q}).$$

In addition, for all vectors $\vec{a}, \vec{b} \in \mathbb{R}^4$,

$$\langle X\vec{a}, \vec{a}\rangle - \langle Y\vec{b}, \vec{b}\rangle \leq \langle (\epsilon\mathcal{M}^2 + \mathcal{M})(\vec{a} \oplus \vec{b}), (\vec{a} \oplus \vec{b}\rangle.$$

This inequality allows us to generate an upper bound for the matrix difference in our Engel group.

THEOREM 3.2 *Given a (Euclidean) C^2 function $\phi : G \times G \to \mathbb{R}$, let the semi-horizontal gradient at the point $r \in G$ be denoted $\nabla_{1,r}\phi$ and the symmetrized second derivative matrix at r be denoted $(D^2_r\phi)^\star$. Let $\epsilon \in \mathbb{R}^+$. Let $f, g, p, q, \hat{p}, \hat{q}, \mathcal{O}$, and $\mathcal{M}$ be as in Theorem 3.1. Then there are matrices $\mathcal{X}, \mathcal{Y} \in S^2$ so that*

$$(\nabla_{1,p}\phi(\hat{p}, \hat{q}), \mathcal{X}) \in \overline{J}^{2,+} f(\hat{p}) \text{ and } (-\nabla_{1,q}\phi(\hat{p}, \hat{q}), \mathcal{Y}) \in \overline{J}^{2,-} g(\hat{p})).$$

Furthermore, for all vectors $\xi \in V_1$

$$\begin{aligned} &\langle \mathcal{X}\xi, \xi\rangle - \langle \mathcal{Y}\xi, \xi\rangle \\ &\leq \langle \epsilon\mathcal{M}^2(\mathbb{A}(\hat{p})^T\xi \oplus \mathbb{A}(\hat{q})^T\xi), (\mathbb{A}(\hat{p})^T\xi \oplus \mathbb{A}(\hat{q})^T\xi)\rangle \end{aligned}$$

Proof. We omit the proof, as it is the same as Theorem 3.4 in [2]. □

With our Carnot group law, and using Theorem 3.2, we get the Carnot group maximum principle for our Engel group.

LEMMA 3.2 *Let $\Omega \subset G$ be a bounded domain. Let $\tau \in \mathbf{R}^+$ and let u be an upper semicontinuous function and v a lower semicontinuous function. Let $p = (x_1, x_2, x_3, x_4)$ and $q = (y_1, y_2, y_3, y_4)$ and $k \in 2 \cdot \mathbb{N}$ be an even whole number. Define $\phi(p, q)$ by*

$$\begin{aligned}
\phi(p,q) &= \frac{1}{k}(x_1-y_1)^k + \frac{1}{k}(x_2-y_2)^k + \frac{1}{k}(x_3-y_3+\frac{1}{2}(x_2y_1-x_1y_2))^k \\
&\quad + \frac{1}{k}\Bigg(x_4-y_4+\frac{1}{2}(x_3y_1-x_1y_3)+\frac{\alpha}{2}(x_3y_2-y_3x_2) \\
&\quad + \frac{1}{12}(x_1+y_1)(x_2y_1-x_1y_2)+\frac{\alpha}{12}(x_2+y_2)(x_2y_1-x_1y_2)\Bigg)^k \\
&= \frac{1}{k}\sum_{i=1}^{4}(\phi_i(p,q))^k,
\end{aligned}$$

where

$$\begin{aligned}
\phi_1(p,q) =&(x_1-y_1) \\
\phi_2(p,q) =&(x_2-y_2) \\
\phi_3(p,q) =&(x_3-y_3+\frac{1}{2}(x_2y_1-x_1y_2)) \\
\phi_4(p,q) =&(x_4-y_4+\frac{1}{2}(x_3y_1-x_1y_3)+\frac{\alpha}{2}(x_3y_2-y_3x_2) \\
&+\frac{1}{12}(x_1+y_1)(x_2y_1-x_1y_2)+\frac{\alpha}{12}(x_2+y_2)(x_2y_1-x_1y_2))
\end{aligned}$$

Let the points $p_\tau, q_\tau \in G$ be the local maximum in $\Omega \times \Omega$ of $u(p) - v(q) - \tau\phi(p, q)$ and let $u - v$ have a positive interior local maximum such that

$$\sup_{\Omega}(u-v) > 0.$$

Then the following hold:

1.

$$\lim_{\tau\to\infty} \tau\phi(p_\tau, q_\tau) = 0.$$

2. *There exists a point $\hat{p} \in \Omega$ such that $p_\tau \to \hat{p}$ (and so does q_τ by part 1) and*

$$\sup_{\Omega}(u-v) = u(\hat{p}) - v(\hat{p}) > 0.$$

3. *There exist symmetric matrices $\mathcal{X}_\tau, \mathcal{Y}_\tau$ and vector $\eta_\tau \in V_1 \oplus V_2$, namely $\eta_\tau = \nabla_{1,p}\phi(p, q_\tau)$, so that*

$$(\tau\eta_\tau, \mathcal{X}_\tau) \in \overline{J}^{2,+}u(p_\tau) \text{ and } (\tau\eta_\tau, \mathcal{Y}_\tau) \in \overline{J}^{2,-}v(q_\tau).$$

4. *For any vector $\xi \in V_1$, we have*

$$\begin{aligned}
\langle \mathcal{X}_\tau \xi, \xi\rangle - \langle \mathcal{Y}_\tau \xi, \xi\rangle &\leq \tau\langle \mathcal{M}^2(\mathbb{A}(p_\tau)^T\xi \oplus \mathbb{A}(q_\tau)^T\xi), (\mathbb{A}(p_\tau)^T\xi \oplus \mathbb{A}(q_\tau)^T\xi)\rangle \\
&\lesssim \tau\|\mathcal{M}\|^2\|\xi\|^2 \sim \tau\phi(p_\tau, q_\tau)^{\frac{2k-4}{k}}\|\xi\|^2
\end{aligned}$$

where

$$\|\mathcal{M}\| = \sup\{|\lambda| : \lambda \text{ is an eigenvalue of } \mathcal{M}\}.$$

In particular, if $\xi \sim \tau\phi(p_\tau, q_\tau)^{\frac{k-1}{k}}$, we have

$$\langle \mathcal{X}_\tau \xi, \xi\rangle - \langle \mathcal{Y}_\tau \xi, \xi\rangle \lesssim \tau^3\phi(p_\tau, q_\tau)^{\frac{4k-6}{k}}.$$

Proof. The proof of (1) and (2) is the same as [3]. Next, we have for the Euclidean derivatives with respect to p:

$$\begin{aligned}
\frac{\partial}{\partial x_1}\phi(p_\tau, q_\tau) &= (x_1^\tau - y_1^\tau)^{k-1} - \frac{1}{2}y_2^\tau\big(\phi_3(p_\tau, q_\tau)\big)^{k-1} \\
&\quad + \frac{1}{12}\Big(-2x_1^\tau y_2^\tau - y_1^\tau y_2^\tau - \alpha(y_2^\tau)^2 + x_2^\tau(y_1^\tau - \alpha y_2^\tau) - 6y_3^\tau\Big)\big(\phi_4(p_\tau, q_\tau)\big)^{k-1} \\
\frac{\partial}{\partial x_2}\phi(p_\tau, q_\tau) &= (x_2^\tau - y_2^\tau)^{k-1} + \frac{1}{2}y_1^\tau\big(\phi_3(p_\tau, q_\tau)\big)^{k-1} \\
&\quad + \frac{1}{12}\Big((y_1^\tau)^2 + x_1^\tau(y_1^\tau - \alpha y_2^\tau) + \alpha(2x_2^\tau y_1^\tau + y_1^\tau y_2^\tau - 6y_3^\tau)\Big)\big(\phi_4(p_\tau, q_\tau)\big)^{k-1} \\
\frac{\partial}{\partial x_3}\phi(p_\tau, q_\tau) &= \big(\phi_3(p_\tau, q_\tau)\big)^{k-1} + \frac{1}{2}(y_1^\tau + \alpha y_2^\tau)\big(\phi_4(p_\tau, q_\tau)\big)^{k-1} \\
\frac{\partial}{\partial x_4}\phi(p_\tau, q_\tau) &= \big(\phi_4(p_\tau, q_\tau)\big)^{k-1}.
\end{aligned}$$

and for the Euclidean derivatives with respect to q:

$$
\begin{aligned}
-\frac{\partial}{\partial y_1}\phi(p_\tau,q_\tau) &= (x_1^\tau - y_1^\tau)^{k-1} - \frac{1}{2}x_2^\tau\big(\phi_3(p_\tau,q_\tau)\big)^{k-1} \\
&\quad + \frac{1}{12}\bigg(6x_3^\tau + 2x_2^\tau y_1^\tau + x_1^\tau(x_2^\tau - y_2^\tau) + \alpha x_2^\tau(x_2^\tau + y_2^\tau)\bigg)\big(\phi_4(p_\tau,q_\tau)\big)^{k-1} \\
-\frac{\partial}{\partial y_2}\phi(p_\tau,q_\tau) &= (x_2^\tau - y_2^\tau)^{k-1} + \frac{1}{2}x_1^\tau\big(\phi_3(p_\tau,q_\tau)\big)^{k-1} \\
&\quad + \frac{1}{12}\bigg((x_1^\tau)^2 - \alpha(6x_3^\tau + x_2^\tau y_1^\tau) + x_1^\tau(y_1^\tau + \alpha(x_2^\tau + 2y_2^\tau))\bigg)\big(\phi_4(p_\tau,q_\tau)\big)^{k-1} \\
-\frac{\partial}{\partial y_3}\phi(p_\tau,q_\tau) &= \big(\phi_3(p_\tau,q_\tau)\big)^{k-1} + \frac{1}{2}(x_1^\tau + \alpha x_2^\tau)\big(\phi_4(p_\tau,q_\tau)\big)^{k-1} \\
-\frac{\partial}{\partial y_4}\phi(p_\tau,q_\tau) &= \big(\phi_4(p_\tau,q_\tau)\big)^{k-1}.
\end{aligned}
$$

Claim 3.1 *We have the following relations:*

$$\mathbb{A}(p_\tau)\nabla_{eucl}\phi(p_\tau,q_\tau) = -\mathbb{A}(q_\tau)\nabla_{eucl}\phi(p_\tau,q_\tau)$$

and

$$\mathbb{B}(p_\tau)\nabla_{eucl}\phi(p_\tau,q_\tau) = -\mathbb{B}(q_\tau)\nabla_{eucl}\phi(p_\tau,q_\tau).$$

So that we may set

$$\eta_\tau = \mathbb{A}(p_\tau)\nabla_{eucl}\phi(p_\tau,q_\tau) \oplus \mathbb{B}(p_\tau)\nabla_{eucl}\phi(p_\tau,q_\tau)$$

Proof. We compute the first row of $\mathbb{A}(p_\tau)\nabla_{\text{eucl}}\phi(p_\tau,q_\tau)$:

$$
\begin{aligned}
&\frac{\partial}{\partial x_1}\phi(p_\tau,q_\tau) - \frac{x_2^\tau}{2}\frac{\partial}{\partial x_3}\phi(p_\tau,q_\tau) - u(x_1^\tau,x_2^\tau,x_3^\tau,\alpha)\frac{\partial}{\partial x_4}\phi(p_\tau,q_\tau) = \\
&\quad (x_1^\tau - y_1^\tau)^{k-1} - \frac{1}{2}(y_2^\tau + x_2^\tau)\big(\phi_3(p_\tau,q_\tau)\big)^{k-1} \\
&\quad + \frac{1}{12}\bigg(- 2x_1^\tau y_2^\tau - y_1^\tau y_2^\tau - \alpha(y_2^\tau)^2 + x_2^\tau(y_1^\tau - \alpha y_2^\tau) - 6y_3^\tau\bigg)\big(\phi_4(p_\tau,q_\tau)\big)^{k-1} \\
&\quad - \bigg(\frac{x_2^\tau}{4}(y_1^\tau + \alpha y_2^\tau) + \Big(\frac{1}{2}x_3^\tau + \frac{1}{12}x_2^\tau(x_1^\tau + \alpha x_2^\tau)\Big)\bigg)\big(\phi_4(p_\tau,q_\tau)\big)^{k-1}
\end{aligned}
$$

We compute the first row of $-\mathbb{A}(q_\tau)\nabla_{\text{eucl}}\phi(p_\tau,q_\tau)$:

$$
\begin{aligned}
&-\frac{\partial}{\partial y_1}\phi(p_\tau,q_\tau) + \frac{y_2^\tau}{2}\frac{\partial}{\partial y_3}\phi(p_\tau,q_\tau) + u(y_1^\tau,y_2^\tau,y_3^\tau,\alpha)\frac{\partial}{\partial y_4}\phi(p_\tau,q_\tau) = \\
&\quad (x_1^\tau - y_1^\tau)^{k-1} - \frac{1}{2}(x_2^\tau + y_2^\tau)\big(\phi_3(p_\tau,q_\tau)\big)^{k-1} \\
&\quad - \frac{1}{12}\bigg(6x_3^\tau + 2x_2^\tau y_1^\tau + x_1^\tau(x_2^\tau - y_2^\tau) + \alpha x_2^\tau(x_2^\tau + y_2^\tau)\bigg)\big(\phi_4(p_\tau,q_\tau)\big)^{k-1} \\
&\quad - \bigg(\frac{y_2^\tau}{4}(x_1^\tau + \alpha x_2^\tau) + \big(\frac{1}{2}y_3^\tau + \frac{1}{12}y_2^\tau(y_1^\tau + \alpha y_2^\tau)\big)\bigg)\big(\phi_4(p_\tau,q_\tau)\big)^{k-1}.
\end{aligned}
$$

Routine calculations show these terms are equal. We now compute the second row of $\mathbb{A}(p_\tau)\nabla_{\text{eucl}}\phi(p_\tau, q_\tau)$:

$$
\begin{aligned}
&\frac{\partial}{\partial x_2}\phi(p_\tau,q_\tau) + \frac{x_1^\tau}{2}\frac{\partial}{\partial x_3}\phi(p_\tau,q_\tau) + m(x_1^\tau,x_2^\tau,x_3^\tau,\alpha)\frac{\partial}{\partial x_4}\phi(p_\tau,q_\tau) = \\
&\quad (x_2^\tau - y_2^\tau)^{k-1} + \frac{1}{2}(x_1^\tau + y_1^\tau)\big(\phi_3(p_\tau,q_\tau)\big)^{k-1} \\
&\quad + \frac{1}{12}\Big(-6\alpha y_3^\tau + y_1^\tau(x_1^\tau + y_1^\tau) + \alpha(x_2^\tau y_1^\tau - x_1^\tau y_2^\tau) + \alpha y_1^\tau(x_2^\tau + y_2^\tau)\Big)\big(\phi_4(p_\tau,q_\tau)\big)^{k-1} \\
&\quad + \Big(\frac{x_1^\tau}{4}(y_1^\tau + \alpha y_2^\tau) + \big(-\frac{\alpha}{2}x_3^\tau + \frac{1}{12}x_1^\tau(x_1^\tau + \alpha x_2^\tau)\big)\Big)\big(\phi_4(p_\tau,q_\tau)\big)^{k-1}.
\end{aligned}
$$

We compute the second row of $-\mathbb{A}(q_\tau)\nabla_{\text{eucl}}\phi(p_\tau, q_\tau)$:

$$
\begin{aligned}
&-\frac{\partial}{\partial y_2}\phi(p_\tau,q_\tau) - \frac{y_1^\tau}{2}\frac{\partial}{\partial y_3}\phi(p_\tau,q_\tau) - m(y_1^\tau,y_2^\tau,y_3^\tau,\alpha)\frac{\partial}{\partial y_4}\phi(p_\tau,q_\tau) = \\
&\quad (x_2^\tau - y_2^\tau)^{k-1} + \frac{1}{2}(x_1^\tau + y_1^\tau)\big(\phi_3(p_\tau,q_\tau)\big)^{k-1} \\
&\quad + \frac{1}{12}\Big(-6\alpha x_3^\tau + x_1^\tau(x_1^\tau + y_1^\tau) - \alpha(x_2^\tau y_1^\tau - x_1^\tau y_2^\tau) + \alpha x_1^\tau(x_2^\tau + y_2^\tau)\Big)\big(\phi_4(p_\tau,q_\tau)\big)^{k-1} \\
&\quad + \Big(\frac{y_1^\tau}{4}(x_1^\tau + \alpha x_2^\tau) + \big(-\frac{\alpha}{2}y_3^\tau + \frac{1}{12}y_1^\tau(y_1^\tau + \alpha y_2^\tau)\big)\Big)\big(\phi_4(p_\tau,q_\tau)\big)^{k-1}.
\end{aligned}
$$

Again, routine calculations show these terms are equal. We compute $\mathbb{B}(p_\tau)\nabla_{\text{eucl}}\phi(p_\tau, q_\tau)$:

$$
\begin{aligned}
&\frac{\partial}{\partial x_3}\phi(p_\tau,q_\tau) + k(x_1^\tau, x_2^\tau\alpha)\frac{\partial}{\partial x_4}\phi(p_\tau,q_\tau) = \\
&\quad \big(\phi_3(p_\tau,q_\tau)\big)^{k-1} + \frac{1}{2}(y_1^\tau + \alpha y_2^\tau)\big(\phi_4(p_\tau,q_\tau)\big)^{k-1} + \frac{1}{2}\big(x_1^\tau + \alpha x_2^\tau\big)\big(\phi_4(p_\tau,q_\tau)\big)^{k-1}.
\end{aligned}
$$

We compute $-\mathbb{B}(q_\tau)\nabla_{\text{eucl}}\phi(p_\tau, q_\tau)$:

$$
\begin{aligned}
&-\frac{\partial}{\partial x_3}\phi(p_\tau,q_\tau) - k(x_1^\tau, x_2^\tau\alpha)\frac{\partial}{\partial x_4}\phi(p_\tau,q_\tau) = \\
&\quad \big(\phi_3(p_\tau,q_\tau)\big)^{k-1} + \frac{1}{2}(x_1^\tau + \alpha x_2^\tau)\big(\phi_4(p_\tau,q_\tau)\big)^{k-1} + \frac{1}{2}(y_1^\tau + \alpha y_2^\tau)\big(\phi_4(p_\tau,q_\tau)\big)^{k-1}.
\end{aligned}
$$

It is easy to see these two terms are equal. $\square$

Thus, using Corollary 3.1.1, we have matrices $X, Y \in S^4$, where $(\eta, X) \in \overline{J}^{2,+}_{\text{eucl}}\phi(p_\tau)$ and $(\eta, Y) \in \overline{J}^{2,-}_{\text{eucl}}\phi(q_\tau)$. We then have

$$\mathcal{X}_\tau = \mathbb{A}(p_\tau)X\mathbb{A}^T(p_\tau) + M(\nabla_{\text{eucl}}\phi, p_\tau)$$

and

$$\mathcal{Y}_\tau = \mathbb{A}(q_\tau)Y\mathbb{A}^T(q_\tau) + M(\nabla_{\text{eucl}}\phi, q_\tau)$$

Thus, we have proved Part (3) of our Lemma 3.2. Part (4) follows from the fact that $\mathcal{M}$ is based on the Euclidean second derivatives of $\phi(p, q)$ and by definition, $\|\mathbb{A}\|$ is bounded by a constant since we are in a bounded domain. $\square$

References

[1] Bellaïche, André.: The tangent space in sub-Riemannian geometry. In Sub-Riemannian Geometry; Progress in Mathematics; Birkhäuser: Basel, Switzerland. 1996, Vol. 144, 1-78.

[2] Bieske, T.: A sub-Riemannian maximum principle and its application to the p-Laplacian in Carnot groups. Ann. Acad. Sci. Fenn. Math. 37 (2012), 119-134.

[3] Bieske, Thomas.: On Infinite Harmonic Functions on the Heisenberg Group. Comm. in PDE. 2002, 27 (3&4), 727762.

[4] Bieske, T.: Lipschitz extensions on generalized Grushin spaces. - Michigan Math J. (2005), 53 (1), 3-31.

[5] Citti, G., M. Manfredini, and A. Sarti: Neuronal oscillation in the visual cortex: Gamma-convergence to the Riemannian Mumford-Shah functional. - SIAM J. Math. Anal. 35:6, 2004, 1394-1419.

[6] Crandall, M.: Viscosity solutions: a primer. - Lecture Notes in Math. 1660, Springer-Verlag, Berlin, 1997.

[7] Crandall, M., H. Ishii, and P.-L. Lions: User's guide to viscosity solutions of second order partial differential equations. - Bull. Amer. Math. Soc. 27:1, 1992, 1-67.

[8] Dragoni, F.: Introduction to Viscosity Solutions for Nonlinear PDEs. http://www.maths.bris.ac.uk/ maxfd/viscnote.pdf.

[9] Heinbockel, J.H.: Mathematical methods for partial differential equations. Trafford, Victoria, 2003.

[10] Jacobson, N.: Lie algebras. Dover Publishing, New York, 1962.

[11] Nagel, A., E. Stein, and S. Wainger: Balls and metrics defined by vector fields I: Basic Properties. - Acta Math. (1985), 155 (1), 103-147.

[12] Rudin, W.: Real and complex analysis, 3rd ed., McGraw-Hill, New York, 1987.