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Nonlinear Techniques for Stochastic Systems of Differential Equations

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by

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Abstract

Two of the most well-known nonlinear methods for investigating nonlinear dynamic processes in sciences and engineering are nonlinear variation of constants parameters and comparison method. Knowing the existence of solution process, these methods provide a very powerful tools for investigating variety of problems, for example, qualitative and quantitative properties of solutions, finding error estimates between solution processes of stochastic system and the corresponding nominal system, and inputs for the designing engineering and industrial problems. The aim of this work is to systematically develop mathematical tools to undertake the mathematical frame-work to investigate a complex nonlinear nonstationary stochastic systems of differential equations.

A complex nonlinear nonstationary stochastic system of differential equations are decomposed into nonlinear systems of stochastic perturbed and unperturbed differential equations. Using this type of decomposition, the fundamental properties of solutions of nonlinear stochastic unperturbed systems of differential equations are investigated(1). The fundamental properties are used to find the representation of solution process of nonlinear stochastic complex perturbed system in terms of solution process of nonlinear stochastic unperturbed system(2).

Employing energy function method and the fundamental properties of Itô-Doob type stochastic auxiliary system of differential equations, we establish generalized variation of constants formula for solution process of perturbed stochastic system of differential equations(3). Results regarding deviation of solution of perturbed system with respect to solution of nominal system of stochastic differential equations are developed(4).

The obtained results are used to study the qualitative properties of perturbed stochas-

tic system of differential equations(5). Examples are given to illustrate the usefulness of the results.

Employing energy function method and the fundamental properties of Itô-Doob type stochastic auxiliary system of differential equations, we establish generalized variational comparison theorems in the context of stochastic and deterministic differential for solution processes of perturbed stochastic system of differential equations(6). Results regarding deviation of solutions with respect to nominal stochastic system are also developed(7). The obtained results are used to study the qualitative properties of perturbed stochastic system(8). Examples are given to illustrate the usefulness of the results.

A simple dynamical model of the effect of radiant flux density and CO_2 concentration on the rate of photosynthesis in light, dark and enzyme reactions are analyzed(9). The coupled system of dynamic equations are solved numerically for some values of rate constant and radiant flux density. We used Matlab to solve the system numerically. Moreover, with the assumption that dynamic model of CO_2 concentration is studied.

1 Preliminary Concepts and Results

This chapter deals with a basic existing preliminary concepts and tools needed to undertake the study of nonlinear techniques for stochastic systems of differential equations. Two of the most well known nonlinear methods for investigating nonlinear dynamic processes in engineering and sciences, are nonlinear variation of constant parameters and comparison methods[1-4,6-27,30]. Moreover, knowing the existence of solution process, these methods provide very powerful tools for investigating qualitative properties of solution process[14, 15, 16, 19, 36]. Moreover, the qualitative properties are also used for designing plants. In this chapter, definitions and some results are outlined.

1.1 Basic Properties of Stochastic Differential Equations

Let us consider a mathematical description of a nonlinear phenomenon under a random environmental perturbation described by a complex system of nonlinear nonstationary Itô-Doob-type stochastic differential equations:

$$dx = f(t,x)dt + \sigma(t,x)dw(t), \qquad x(t_0) = x_0, \tag{1.1.1}$$

where $x \in R^n$; f and column vectors of $\sigma \in C[J \times R^n, R^n]$; $C[J \times R^n, R^n]$ stands for the class of continuous functions defined on $J \times R^n$ into R^n for a positive integer n, and $J = [t_0, t_0 + a)$ for some positive real number $a > 0$; $\sigma = [\sigma^1, \sigma^2, ...\sigma^j..., \sigma^m]$ is $n \times m$ matrix; x_0 is an n-dimensional random variable defined on a complete probability space $(\Omega, \mathfrak{F}, P)$; $\mathfrak{F}_t$ is an increasing family of sub-σ-algebras of $\mathfrak{F}$; $w(t) = (w_1(t), w_2(t), \dots, w_m(t))^T$ is an m-dimensional normalized Wiener process with independent increments; x_0 and $w(t)$ are mutually independent for each $t \geq t_0$.

Definition 1.1.1 The random process $x(t)$ is said to be a solution of (1.1.1) on J if it satisfies the following conditions:

1. denoting by $\mathfrak{F}_t$, $t \in J$ the minimal σ-algebra with respect to which the variables $x(s)$ for $s \leq t$ and $w(s)$ for $s \leq t$ are measurable, the process $w_t(s) = w(t+s) - w(t)$ does not depend on $\mathfrak{F}_t$;

2. denoting by $H_2[J]$ the space of measurable random functions $\varphi(t)$ which, for each $t \in J$ are $\mathfrak{F}_t$-measurable and for which the integral $\int_{t_0}^{t_0+a} \varphi^2(t)dt$ is w.p.1 finite, $\mid f(t, x(t)) \mid^{1/2}$ and $\sigma(t, x(t))$ belong to $H_2[J]$;

3. the process $x(t)$ has on J the stochastic differential $dx(t) = \overline{f}(t)dt + \overline{\sigma}(t)dw(t)$, also, for all $t \in J$ we have w.p.1 $\overline{f} = f(t, x(t))$, $\overline{\sigma}(t) = \sigma(t, x(t))$.

Let $x(t) = x(t, t_0, x_0)$ be the solution process of (1.1.1) existing for $t \geq t_0$. Let us modify the stochastic differential equation (1.1.1) as:

$$dx = f(t, x, \lambda)dt + \sigma(t, x, \lambda)dw(t), \qquad x(t_0, \lambda) = x_0(\lambda), \tag{1.1.2}$$

and the corresponding nominal system,

$$dx = f(t, x, \lambda_0)dt + \sigma(t, x, \lambda_0)dw(t), \qquad x(t_0) = x_0, \tag{1.1.3}$$

where $(t_0, x_0, \lambda_0) \in J \times R^n \times \Lambda$, with Λ being an open system parameter λ set in R^m.

In the following, we present a very simple result that exhibits the continuous dependence of solution process of (1.1.2) with respect to $(t_0, x_0, \lambda_0) \in J \times R^n \times \Lambda$. The proof is given in [15, 17, 19, 20, 21].

Theorem 1.1.2 *Assume that*

1. *f and the m column vectors of $\sigma \in C[J \times R^n \times \Lambda, R^n]$;*

2. *There exist positive number M and N such that*

$$\|f(t,x,\lambda)\| + \|\sigma(t,x,\lambda)\| \leq N + M\|X\|$$

   for $(t,x,\lambda) \in J \times R^n \times \Lambda$;

3. *There exist a positive number L such that*

$$\|f(t,x,\lambda) - f(t,y,\lambda)\| + \|\sigma(t,x,\lambda) - \sigma(t,y,\lambda)\| \leq L\|x-y\|$$

   $(t,x,\lambda),(t,y,\lambda) \in J \times R^n \times \Lambda$;

4. *$x(t_0,\lambda) = x_0(\lambda)$ is independent of $w(t)$ and*

$$\lim_{\lambda \to \lambda_0} E[\|x(t_0,\lambda) - x(t_0,\lambda_0)\|^2] = 0;$$

5. *$E[\|x(t_0,\lambda)\|^4 \leq c_1$ for some constant $c_1 > 0$;*

6. *$\epsilon > 0, p > 0$,*

$$\lim_{\lambda \to \lambda_0} P[sup_{\|x\|<p}(\|f(t,x,\lambda) - f(t,x,\lambda_0)\| + \|\sigma(t,x,\lambda) - \sigma(t,x,\lambda_0)\| > \epsilon)] = 0$$

Then the IVP (1.1.2) *admit unique solution processes $x(t,t_1,x_0(\lambda),\lambda)$ and $x(t,t_0,x_0,\lambda_0)$ through $(t_1,x_0(\lambda))$ and (t_0,x_0), respectively. Moreover, for given $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that*

$$(E[\|x(t,t_1,x_0(\lambda),\lambda) - x(t,t_0,x_0,\lambda_0)\|^2])^{1/2} < \epsilon, \qquad t \in J,$$

whenever

$$|t_1 - t_0| + (E[\|x_0(\lambda) - x_0\|^2])^{1/2} + \|\lambda - \lambda_0\| < \delta(\epsilon).$$

Theorem 1.1.3 *Assume that σ, f, x_0 in (1.1.1) satisfy the hypotheses (1),(2),(4), and (5) of Theorem 1.1.2. Furthermore, σ and f are continuously differentiable with respect to x for fixed t. Let $x(t, t_0, x_0)$ be the solution process of (1.1.1) existing for $t \geq t_0$. Then,*

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x} x(t, t_0, x_0)$$

exists and is the solution of

$$dy = H(t, t_0, x_0)ydt + \Gamma(t, t_0, x_0)ydw(t)$$

where $\Phi(t_0, t_0, x_0)$ is the $n \times n$ identify matrix, $n \times n$ matrices $f_x(t, x)$ and $\sigma_x^l(t, x)$ are continuous in (t, x) for $l = 1, 2, \ldots, m$; $\sigma_x(t, x)$ is the $n \times nm$ matrix $\sigma_x(t,x)$=$[\sigma_x^1(t,x)\sigma_x^2(t,x)...\sigma_x^j(t,x)...\sigma_x^m(t,x)]$; $H(t, t_0, x_0) = f_x(t, x(t, t_0, x_0))$ and $\Gamma(t, t_0, x_0)$=$\sigma_x(t, x(t, t_0, x_0))$.

The proof is given in [15, 17, 19, 20, 21].

Definition 1.1.4 The trivial solution process of (1.1.1) is said to be

i) (SM_1) stable in the p-th moment, if for each $\epsilon > 0$, $t_0 \in R_+$ and $p \geq 1$ there exist a positive function $\delta = \delta(t_0, \epsilon)$ such that the inequality $\|x_0\|_p \leq \delta$ implies

$$\|x(t)\|_p < \epsilon, \qquad t \geq t_0,$$

where $\|x(t)\|_p = (E[\|x(t)\|^p])^{1/p}$;

ii) (SM_2) asymptotically stable in the p-th moment, if it is stable in the p-th moment and if for any $\epsilon > 0$, $t_0 \in R_+$, there exists $\delta_0 = \delta_0(t_0)$ and $T = T(t_0, \epsilon)$ such that the inequality $\|x_0\|_p \le \delta_0$ implies

$$\|x(t)\|_p < \epsilon, \qquad t \ge t_0 + T.$$

1.2 Nominal System and Error Estimate

Let us consider a nominal system of Itô-Doob type stochastic differential equations

$$dy \;=\; G(t,y)dt + H(t,y)dw(t), \qquad y(t_0) = y_0. \tag{1.2.1}$$

where $G \in C[J \times R^n, R^n]$ and $H \in C[J \times R^n, R^{n\times m}]$.

Definition 1.2.1 The two differential systems (1.1.1) and (1.2.1) are said to be

i) (RM_1) relatively stable in p-th moment, if for each $\epsilon > 0$, $t_0 \in R_+$, and $p \ge 1$, there exists a positive function $\delta = \delta(t_0, \epsilon)$ such that the inequality $\|x_0 - y_0\|_p \le \delta$ implies

$$\|x(t) - y(t)\|_p < \epsilon, \qquad t \ge t_0;$$

ii) (RM_2) relatively asymptotically stable in the p-th moment, if it is relatively stable in the p-th moment and if for any $\epsilon > 0$, $t_0 \in R_+$, there exist $\delta_0 = \delta_0(t_0)$ and $T = T(t_0, \epsilon)$ such that the inequality $\|x_0 - y_0\| \le \delta_0$ implies

$$\|x(t) - y(t)\|_p < \epsilon, \qquad t \ge t_0 + T.$$

1.3 Comparison System and Qualitative Properties

Let us consider the following Itô-Doob type stochastic comparison and auxiliary systems differential equations

$$du \;=\; g(t,u)dt, \qquad u(t_0) = u_0, \tag{1.3.1}$$

and

$$dz = \alpha(t, z)dt, \qquad z(t_0) = x_0, \tag{1.3.2}$$

where $g \in C[R_+ \times R^N, R^N]$ and $\alpha \in C[J \times R^n, R^n]$

Definition 1.3.1 The function $g(t, u)$ is said to possess a quasi-monotone nondecreasing property if for $u, v \in R^N$ such that $u \leq v$ and $u_i = v_i$, then $g_i(t, u) \leq g_i(t, v)$ for any $i = 1, 2, \ldots, N$ and fixed t.

Let $V \in C[R_+ \times R^n, R^N]$ and its partial derivatives V_t, V_x and V_{xx} exists and are continuous on $R_+ \times R^n$.

Definition 1.3.2 The trivial solution processes $z \equiv 0$ and $u \equiv 0$ of (1.2.1) and (1.3.1) are said to be

i) (JM_1) jointly stable in the mean, if for $\epsilon > 0, t_0 \in R_+$, there exists $\delta_1 = \delta_1(t_0, \epsilon) > 0$ such that $\sum_{i=1}^{N} E[V_i(t_0, x_0)] \leq \delta_1$ implies

$$\sum_{i=1}^{N} E[u_i(t, t_0, V(t_0, z(t, t_0, x_0)))] < \epsilon, \qquad t \geq t_0;$$

ii) (JM_2) jointly asymptotically stable in the mean, if it is jointly stable in the mean and if for any $\epsilon > 0$, $t_0 \in R_+$, there exist $\delta_0 = \delta_0(t_0) > 0$ and $T = T(t_0, \epsilon) > 0$ such that the inequality $\sum_{i=1}^{N} E[V_i(t_0, x_0)] \leq \delta_0$ implies

$$\sum_{i=1}^{N} E[u_i(t, t_0, V(t_0, z(t, t_0, x_0)))] < \epsilon, \qquad t \geq t_0 + T.$$

Definition 1.3.3 The systems (1.2.1) and (1.3.1) are said to be

i) (JR_1) jointly relatively stable in the mean, if for each $\epsilon > 0, t_0 \in R_+$, there exists $\delta_1 = \delta_1(t_0, \epsilon) > 0$ such that the inequality $\sum_{i=1}^{N} E[V_i(t_0, x_0 - y_0)] \leq \delta_1$ implies

$$\sum_{i=1}^{N} E[u_i(t, t_0, V(t_0, z(t, t_0, x_0 - y_0)))] < \epsilon, \qquad t \geq t_0;$$

whenever $\|y_0\|$ is small enough.

ii) (JR_2) jointly relatively asymptotically stable in the mean, if it is jointly relatively stable in the mean and if for each $\epsilon > 0, t_0 \in R_+$, there exists $\delta_0 = \delta_0(t_0) > 0$ and $T = T(t_0, \epsilon) > 0$ such that $\sum_{i=1}^{N} E[V_i(t_0, x_0 - y_0)] < \delta_0$ implies

$$\sum_{i=1}^{N} E[u_i(t, t_0, V(t_0, z(t, t_0, x_0 - y_0)))] < \epsilon, \qquad t \geq t_0.$$

Definition 1.3.4 The differential system (1.1.1) has asymptotic equilibrium if every solution of the system (1.1.1) tends to a finite limit vector ξ as $t \to \infty$ and to every constant vector ξ there is a solution $x(t)$ of (1.1.1) on $t_0 \leq t < \infty$ such that $lim_{t\to\infty} x(t) = \xi$.

Definition 1.3.5 The differential systems (1.1.1) and (1.2.1) are said to be asymptotically equivalent if, for every solution $y(t)$ of (1.2.1), there is a solution $x(t)$ of (1.1.1) such that

$$x(t) - y(t) \to 0 \text{ as } t \to \infty.$$

Theorem 1.3.6 *If ϕ is a real-valued continuous and concave function defined on a convex domain $D \subseteq R^n$, then*

$$E[\phi(x)] \leq \phi(E(x)).$$

The proof is given in [17, 19, 20].

2 Fundamental Properties of Solutions of Nonlinear Stochastic Differential Equations and Method of Variation of Parameters

2.1 Introduction

One of the most well known methods for investigating the nonlinear dynamic processes in sciences and engineering is the method of nonlinear variation of constant parameters [14, 15, 16, 19, 36].

Knowing the existence of solution process, the method of variation of parameters provides a very powerful tool for finding the solution representation of system of differential equations [14, 15, 16, 19, 36]. The idea is to decompose a complex system of differential equations in to two parts in such a way that a system of differential equations corresponding to the simpler part is either easily solvable in a closed form or analytically analyzable. However, the over all complex system of differential equations are neither easily solvable in a closed form nor analytically analyzable [14, 15]. The method of variation of parameters provides a formula for a solution to the complex system in terms of the solution process of simpler system of differential equations.

In this chapter, an attempt is made to find a representation of solutions of nonlinear and nonstationary Itô-Doob type stochastic system of differential equations in terms of solutions processes of smoother system of Itô-Doob type stochastic differentials. The organization is as follows: In section 2.2, the problem is formulated. In section 2.3, several auxiliary results are established for unperturbed system of nonlinear Itô-Doob type stochastic differential equations. In section 2.4, a variation of constants formula is established. In section 2.5, examples are given to illustrate the usefulness of the methods. The Developed results are a convenient tool in discussing the properties of solutions of the perturbed system.

2.2 Problem Formulation

Let us formulate a problem. We consider a mathematical description of a nonlinear dynamic phenomenon under randomly varying environmental perturbations described by a complex system of nonlinear nonstationary Itô-Doob type system of stochastic differential equations:

$$dy = c(t,y)dt + \Sigma(t,y)dw(t), \qquad y(t_0) = x_0, \tag{2.2.1}$$

where $y \in R^n, c \in C[J \times R^n, R^n], \Sigma \in C[J \times R^n, R^{n\times m}]$; $C[J \times R^n, R^n]$ ($C[J \times R^n, R^{n\times m}]$) stands for a class of continuous functions defined on $J \times R^n$ into R^n ($R^{n\times m}$); n and m are positive integers ; $J = [t_0, t_0 + a)$ for some positive real number a; x_0 is an n-dimensional random variable defined on a complete probability space $(\Omega, \mathfrak{F}, P)$; $w(t) = (w_1(t), w_2(t), ..., w_m(t))^T$ is an m-dimensional normalized Wiener process with independent increments; x_0 and $w(t)$ are mutually independent for each $t \geq t_0$. We decompose complex system of stochastic differential equations (2.2.1) into two parts. The decomposition of its drift and diffusion rate functions are as follows:

$$c(t,y) = f(t,y) + F(t,y)$$

and

$$\Sigma(t,y) = \sigma(t,y) + \Upsilon(t,y)$$

where the rate functions $f(t,y)$ and $\sigma(t,y)$ are considered to be smooth and simpler form in the sense of better structure and conceptually smooth. Thus, (2.2.1) can be rewritten as

$$\begin{aligned} dy &= [f(t,y)+F(t,y)]dt+[\sigma(t,y)+\Upsilon(t,y)]dw(t) \\ &= [f(t,y)+F(t,y)]dt+\textstyle\sum_{l=1}^{m}[\sigma^l(t,y)+\Upsilon^l(t,y)]dw_l(t), \qquad y(t_0)=x_0. \end{aligned} \tag{2.2.2}$$

The smoother and simpler form of mathematical model of dynamic process corresponding to (2.2.2) is described by

$$\begin{aligned} dx &= f(t,x)dt+\sigma(t,x)dw(t) \\ &= f(t,x)dt+\textstyle\sum_{l=1}^{m}\sigma^l(t,x)dw_l(t), \qquad x(t_0)=x_0. \end{aligned} \tag{2.2.3}$$

Moreover, systems (2.2.2) and (2.2.3) are considered to be perturbed and unperturbed systems of stochastic differential equations, respectively.

Remark 2.2.1 *In the absence of any reasonable decomposition of the type* (2.2.2)*, it is always possible to consider the above decompositions with* $F(t,y)=c(t,y)-f(t,y)$ *and* $\Upsilon(t,y)=\Sigma(t,y)-\sigma(t,y)$ *for any suitable choice of smoother and simpler rate functions* $f(t,y)$ *and* $\sigma(t,y)$*.*

2.3 Auxiliary Results

Our main objective is to develop the variation of constants formula with respect to (2.2.3) and its perturbed system (2.2.2). For this purpose, first we investigate the Itô-Doob stochastic partial differentials of solution process $x(t,t_0,x_0)$ of unperturbed system (2.2.3) with respect to initial conditions (t_0,x_0).

In the following, under certain smoothness assumption on the rate functions of unperturbed stochastic system of differential equations (2.2.3), we establish the second order Itô-Doob type of differentials of the solution process of (2.2.3) with respect to (t_0,x_0). In this section, by recalling the existence of Itô-Doob type differential of solution process of unperturbed system of stochastic differential equations with respect to initial state, we first establish the existence of second order differential with respect to x_0. Moreover, as a byproduct, we show that the differentials satisfy Itô-Doob type of stochastic non homogeneous matrix differential equation. In the following Lemma, we assume that

i) σ is $\mathbf{B} \times \mathfrak{F}$ measurable, where $\mathbf{B}$ denotes the Borel σ -algebra on $[0, \infty)$ and $\mathfrak{F}$ is a σ -algebra such that for $t_1 < t_2$ $\mathfrak{F}_{t_1} \subset \mathfrak{F}_{t_2}$ such that w_t is a martingale with respect to $\mathfrak{F}_\mathrm{t}$

ii) f_t and σ_t are $\mathfrak{F}_\mathrm{t}$-adapted;

iii) $P[\int_0^t \sigma_{ij}^2(s,w) < \infty \text{ for all } t \geq 0] = 1$;

iv) $P[\int_0^t \mid f_i(s,w) \mid < \infty \text{ for all } t \geq 0] = 1$;

v) $w(t)$ is $\mathfrak{F}_t$ measurable and x is jointly measurable in (t, w).

Lemma 2.3.1 *Assume that σ and f in* (2.2.3) *are twice continuously differentiable with respect to x for fixed t, and f_{xx} , σ_{xx} are bounded with respect to x for fixed t. Further, assume that the initial value problem* (2.2.3) *has a unique solution process $x(t, t_0, x_0)$ existing for $t \geq t_0$.*
Then

$$\frac{\partial}{\partial x_0}\Phi(t, t_0, x_0) = \frac{\partial^2}{\partial x_0^2}x(t, t_0, x_0) \tag{2.3.1}$$

exists, and is the solution process of the following Itô-Doob type nonhomogeneous stochastic matrix differential equation:

$$dY = [H(t, t_0, x_0)Y + P(t)]dt + \sum_{l=1}^{m}[\Gamma^l(t, t_0, x_0)Y + Q(t)]dw_l(t), \qquad Y(t_0) = 0; \tag{2.3.2}$$

where the $n \times n$ matrices $H(t, t_0, x_0) = f_x(t, x(t, t_0, x_0))$ and $\Gamma^l(t, t_0, x_0) = \sigma_x^l(t, x(t, t_0, x_0))$ are continuous; $P(t) = (\frac{\partial^2}{\partial x^2}f(t, x(t)) \otimes \sum_{k=1}^n \Phi(t, t_0, x_0)e_k)\Phi(t, t_0, x_0)$;
$Q(t) = (\frac{\partial^2}{\partial x^2}\sigma^l(t, x(t)) \otimes \sum_{k=1}^n \Phi(t, t_0, x_0)e_k)\Phi(t, t_0, x_0)$, $\Phi(t_0, t_0, x_0)$ is the $n \times n$ identity matrix and $\otimes$ is the tensor product of two matrices.

Proof. From the assumptions of the lemma, we conclude that $\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0)$ exists and is the solution of the Itô-Doob type stochastic matrix differential equations along the solution process $x(t, t_0, x_0)$ of (2.2.3)[15, 27]:

$$dY = H(t, t_0, x_0)Y dt + \Gamma(t, t_0, x_0)Y dw(t), \qquad Y(t_0) = I_{n\times n}. \tag{2.3.3}$$

In the following, we show that $\frac{\partial^2}{\partial x_0^2} x(t, t_0, x_0)$ exists and it satisfies the stochastic differential equation (2.3.2). For this purpose, we consider the following: For small $\lambda > 0$, let $\Delta x_0 = \sum_{k=1}^{n} \lambda e_k$; where $e_k = (0, 0, ..., 1, ..., 0)^T$ whose k-th component is 1. Moreover, let $\Phi(t, \lambda) = \Phi(t, t_0, x_0 + \Delta x_0)$ and $\Phi(t) = \Phi(t, t_0, x_0)$ be solutions of (2.3.3) through $(t_0, x_0 + \Delta x_0)$ and (t_0, x_0), respectively, and $x(t, \lambda) = x(t, t_0, x_0 + \Delta x_0)$ and $x(t) = x(t, t_0, x_0)$ be solutions of (2.2.3) through $(t_0, x_0 + \Delta x_0)$ and (t_0, x_0) respectively. Under the assumptions of Lemma 2.3.1 and applying Lemma 6.1[15] , we conclude that

$$\lim_{\lambda\to 0} \Phi(t, \lambda) = \Phi(t) \text{ uniformly on J.} \tag{2.3.4}$$

We set

$$\Delta\Phi(t, \lambda) = \Phi(t, \lambda) - \Phi(t), \qquad \Delta\Phi(t_0, \lambda) = \Delta x_0. \tag{2.3.5}$$

Let $R(\theta) = \frac{\partial}{\partial x} f(t, x(t, t_0, x_0 + \theta\Delta x_0))$ with $0 \le \theta \le 1$. From the assumptions , we note that R is continuously differentiable with respect to θ, and hence

$$\frac{d}{d\theta} R(\theta) = f_{xx}(t, x(t, t_0, x_0 + \theta\Delta x_0)) \otimes (\Phi(t, t_0, x_0 + \theta\Delta x_0)\Delta x_0). \tag{2.3.6}$$

By Integrating both sides of (2.3.6) with respect to θ over an interval [0,1], we obtain

$$R(1) - R(0) = \int_0^1 f_{xx}(t, x(t, t_0, x_0 + \theta\Delta x_0)) \otimes (\Phi(t, t_0, x_0 + \theta\Delta x_0)\Delta x_0) d\theta.$$

This, together with the fact that $R(1) = \frac{\partial}{\partial x} f(t, x(t, \lambda))$ and $R(0) = \frac{\partial}{\partial x} f(t, x(t))$, yields

$$\frac{\partial}{\partial x} f(t, x(t,\lambda)) - \frac{\partial}{\partial x} f(t, x(t)) = J(t, x(t,\lambda), \Phi(t,\lambda)),$$

where

$$J(t, x(t,\lambda), \Phi(t,\lambda)) = \int_0^1 f_{xx}(t, x(t, t_0, x_0 + \theta\Delta x_0)) \otimes (\Phi(t, t_0, x_0 + \theta\Delta x_0)\Delta x_0) d\theta \tag{2.3.7}$$

Similarly, by setting

$$G(\theta) = \sum_{l=1}^{m} \sigma_x^l(t, x(t, t_0, x_0 + \theta\Delta x_0))$$

and using the continuous differentiability of G with respect to θ and chain rule, we have

$$\frac{d}{d\theta} G(\theta) = \sum_{l=1}^{m} \sigma_{xx}^l(t, x(t, t_0, x_0 + \theta\Delta x_0)) \otimes (\Phi(t, t_0, x_0 + \theta\Delta x_0)\Delta x_0). \tag{2.3.8}$$

By Integrating both sides of (2.3.8) with respect to θ over an interval [0,1], we get

$$G(1) - G(0) = \sum_{l=1}^{m} \int_0^1 \sigma_{xx}^l(t, x(t, t_0, x_0 + \theta\Delta x_0)) \otimes (\Phi(t, t_0, x_0 + \theta\Delta x_0)\Delta x_0) d\theta.$$

This, together with the fact that $G(1) = \sum_{l=1}^{m} \sigma_x^l(t, x(t,\lambda))$ and $G(0) = \sum_{l=1}^{m} \sigma_x^l(t, x(t))$, yields

$$\sum_{l=1}^{m} [\sigma_x^l(t, x(t,\lambda)) - \sigma_x^l(t, x(t))] = \sum_{l=1}^{m} \Lambda^l(t, x(t,\lambda), \Phi(t,\lambda)),$$

where

$$\Lambda^l(t, x(t), \Phi(t,\lambda)) = \int_0^1 \sigma_{xx}^l(t, x(t, t_0, x_0 + \theta\Delta x_0)) \otimes (\Phi(t, t_0, x_0 + \theta\Delta x_0)\Delta x_0) d\theta. \tag{2.3.9}$$

Note that the integrals in (2.3.7) and (2.3.9) are cauchy-Riemann integrals. Using the hypotheses of the Lemma, $n \times n$ matrices $J(t, x(t,\lambda), \Phi(t,\lambda))$ and $\Lambda^l(t, x(t,\lambda), \Phi(t,\lambda))$

are continuous in (t, x, λ) for $l = 1, 2, 3, \ldots, m$. Furthermore, from (2.3.7), (2.3.9) and applying the bounded convergence theorem[34], we obtain

$$\lim_{\lambda \to 0} \frac{J(t, x(t, \lambda), \Phi(t, \lambda))}{\lambda} = f_{xx}(t, x(t, t_0, x_0)) \otimes (\sum_{k=1}^{n} \Phi(t, t_0, x_0) e_k) \tag{2.3.10}$$

and

$$\lim_{\lambda \to 0} \frac{\Lambda^l(t, x(t, \lambda), \Phi(t, \lambda))}{\lambda} = \sigma^l_{xx}(t, x(t, t_0, x_0)) \otimes (\sum_{k=1}^{n} \Phi(t, t_0, x_0) e_k). \tag{2.3.11}$$

From (2.3.5), using the fact that $\Phi(t, \lambda)$ and $\Phi(t)$ are solutions of (2.3.3), we obtain

$$\begin{aligned}
d(\Phi(t, \lambda) - \Phi(t)) &= d\Phi(t, \lambda) - d\Phi(t) \\
&= f_x(t, x(t, \lambda))\Phi(t, \lambda)dt + \sum_{l=1}^{m} \sigma^l_x(t, x(t, \lambda))\Phi(t, \lambda)dw_l(t) \\
&\quad - [f_x(t, x(t))\Phi(t)dt + \sum_{l=1}^{m} \sigma^l_x(t, x(t))\Phi(t)dw_l(t)] \\
&= [f_x(t, x(t, \lambda))\Phi(t, \lambda) - f_x(t, x(t))\Phi(t)]dt \\
&\quad + \sum_{l=1}^{m} [\sigma^l_x(t, x(t, \lambda))\Phi(t, \lambda) - \sigma^l_x(t, x(t))\Phi(t)]dw_l(t).
\end{aligned} \tag{2.3.12}$$

By adding and subtracting $f_x(t, x(t))\Phi(t, \lambda)dt$ and $\sum_{l=1}^{m} \sigma^l_x(t, x(t))\Phi(t, \lambda)dw_l(t)$ in (2.3.12), we obtain

$$\begin{aligned}
d(\Phi(t, \lambda) - \Phi(t)) &= [f_x(t, x(t, \lambda))\Phi(t, \lambda) - f_x(t, x(t))\Phi(t, \lambda) \\
&\quad + f_x(t, x(t))\Phi(t, \lambda) - f_x(t, x(t))\Phi(t)]dt \\
&\quad + [\sum_{l=1}^{m} [\sigma^l_x(t, x(t, \lambda))\Phi(t, \lambda) - \sigma^l_x(t, x(t))\Phi(t, \lambda)]dw_l(t) \\
&\quad + \sum_{l=1}^{m} [\sigma^l_x(t, x(t))\Phi(t, \lambda) - \sigma^l_x(t, x(t))\Phi(t)]dw_l(t)]
\end{aligned}$$

$$
\begin{aligned}
&= [f_x(t,x(t))(\Phi(t,\lambda)-\Phi(t)) \\
&\quad +(f_x(t,x(t,\lambda))-f_x(t,x(t)))\Phi(t,\lambda)]dt \\
&\quad +\sum_{l=1}^{m}[\sigma_x^l(t,x(t))[\Phi(t,\lambda)-\Phi(t)] \\
&\quad +[\sigma_x^l(t,x(t,\lambda))-\sigma_x^l(t,x(t))]\Phi(t,\lambda)]dw_l(t).
\end{aligned}
\tag{2.3.13}
$$

This, together with (2.3.7), (2.3.9) and the definitions of $\Delta\Phi(t,\lambda)$ in (2.3.5), yields

$$
\begin{aligned}
d(\frac{\Delta\Phi(t,\lambda)}{\lambda}) &= [f_x(t,x(t))\frac{\Delta\Phi(t,\lambda)}{\lambda}+\frac{J(t,x(t,\lambda),\Phi(t,\lambda))}{\lambda}\Phi(t,\lambda)]dt \\
&\quad +\sum_{l=1}^{m}[\sigma_x^l(t,x(t))\frac{\Delta\Phi(t,\lambda)}{\lambda}+\frac{\Lambda^l(t,x(t,\lambda),\Phi(t,\lambda))}{\lambda}\Phi(t,\lambda)]dw_l(t).
\end{aligned}
\tag{2.3.14}
$$

From (2.3.10) and (2.3.11), system (2.3.2) can be considered as the nominal system corresponding to (2.3.14) with initial data $Y(t_0)=0$. It is obvious that the initial value problem (2.3.14) satisfies all the hypothesis of Lemma 6.1 [15], and hence by its application, we have

$$
\lim_{\lambda\to 0}\frac{\Delta\Phi(t,\lambda)}{\lambda}=Y(t) \text{ uniformly on } J,
\tag{2.3.15}
$$

where Y(t) is the solution process of (2.3.14). Because of (2.3.4) and (2.3.5), we note that the limit of $\frac{\Delta\Phi(t,\lambda)}{\lambda}$ in (2.3.14) is equivalent to $\frac{\partial}{\partial x_0}\Phi(t,t_0,x_0)$. Thus $\frac{\partial}{\partial x_0}\Phi(t,t_0,x_0)$ is the solution process of (2.3.2). Moreover, $\frac{\partial}{\partial x_0}\Phi(t,t_0,x_0)=\frac{\partial^2}{\partial x_0^2}x(t,t_0,x_0)$.

■

Example 2.3.2 Let us consider a scalar nonlinear unperturbed stochastic differential equation:

$$
dx=\alpha x(\rho-x)dt+\beta x dw(t), \qquad x(t_0)=x_0. \tag{2.3.16}
$$

where α, β and ρ are any constant. Find $\frac{\partial}{\partial x_0}x(t,t_0,x_0)$ and $\frac{\partial^2}{\partial x_0^2}x(t,t_0,x_0)$, if it exists.

Solution: We note that $f(t,x)=\alpha x(\rho-x)$ and $\sigma(t,x)=\beta x$ are twice continuously differentiable with respect to x. In fact, $\frac{\partial}{\partial x}f(t,x)=\alpha(\rho-2x)$, $\frac{\partial^2}{\partial x^2}f(t,x)=-2\alpha$, $\frac{\partial}{\partial x}\sigma(t,x)=\beta$, and $\frac{\partial^2}{\partial x^2}\sigma(t,x)=0$. The closed form solution of (2.3.16) is

$$x(t,t_0,x_0)=\Big[\Phi(t,t_0)x_0^{-1}+\alpha\int_{t_0}^{t}\Phi(t,s)ds)\Big]^{-1},$$

where $\Phi(t,t_0)=exp[-(\alpha\rho-\frac{1}{2}\beta^2)(t-t_0)-\beta(w(t)-w(t_0))]$. The partial derivative of solution process $x(t,t_0,x_0)$ with respect to x_0 is

$$\frac{\partial}{\partial x_0}x(t,t_0,x_0)=\frac{\Phi(t,t_0)}{(\Phi(t,t_0)+\alpha x_0\int_{t_0}^{t}\Phi(t,s)ds)^2} \tag{2.3.17}$$

and

$$\frac{\partial^2}{\partial x_0^2}x(t,t_0,x_0)=\frac{-2\alpha\Phi(t,t_0)\int_{t_0}^{t}\Phi(t,s)ds}{(\Phi(t,t_0)+\alpha x_0\int_{t_0}^{t}\Phi(t,s)ds)^3}. \tag{2.3.18}$$

Moreover, $\frac{\partial^2}{\partial x_0^2}x(t,t_0,x_0)$ satisfies the following differential equation:

$$\begin{aligned} dy &= \Big[\Big[\alpha\Big(\rho-2\Big[\Phi(t,t_0)x_0^{-1}+\alpha\int_{t_0}^{t}\Phi(t,s)ds\Big]^{-1}\Big)\Big]y-2\alpha\Phi^2(t,t_0)\Big]dt \\ &\quad +\beta y dw(t), \qquad y(t_0)=0. \end{aligned} \tag{2.3.19}$$

Example 2.3.3 Consider a scalar nonlinear autonomous differential equation:

$$dx=\Big[-a(t)\frac{1}{2}x^3+b(t)x\Big]dt+\sigma x dw(t), \qquad x(t_0)=x_0. \tag{2.3.20}$$

where a, b, and σ are continuous functions defined on R_+ into R. Find $\frac{\partial}{\partial x_0}x(t,t_0,x_0)$ and $\frac{\partial^2}{\partial x_0^2}x(t,t_0,x_0)$, if it exists.

Solution: We note that $f(t,x)=-a(t)\frac{1}{2}x^3+b(t)x$ and $\sigma(t,x)=\sigma x$ are continuously differentiable with respect to x. In fact $\frac{\partial}{\partial x}f(t,x)=-\frac{3}{2}a(t)x^2+b(t)$, $\frac{\partial^2}{\partial x^2}f(t,x)=$

$-3a(t)x$, $\frac{\partial}{\partial x}\sigma(t,x) = \sigma$, and $\frac{\partial^2}{\partial x^2}\sigma(t,x) = 0$. The closed form solution of (2.3.20) is

$$x(t,t_0,x_0) = \frac{\Phi(t,t_0)|x_0|}{\sqrt{1 + x_0^2 \int_{t_0}^t a(s)\Phi^2(s,t_0)ds}}$$

where $\Phi(t,t_0) = exp\Big[\int_{t_0}^t \Big[b(s) - \frac{1}{2}\sigma^2(s)\Big]ds + \int_{t_0}^t \sigma(s)dw(s)\Big]$. The solution to the IVP is differentiable with respect to x_0 except at $x_0 = 0$. In this case, by the uniqueness of the solution process of (2.3.20), $x(t,t_0,x_0) \equiv 0$. This process is always differentiable with respect to x_0. The partial derivative of solution processes $x(t,t_0,x_0)$ with respect to x_0 is

$$\frac{\partial}{\partial x_0} x(t,t_0,x_0) = \frac{\Phi(t,t_0)sgn(x_0)}{[1 + x_0^2 \int_{t_0}^t a(s)\Phi^2(s,t_0)ds]^{3/2}} \tag{2.3.21}$$

and

$$\frac{\partial^2}{\partial x_0^2} x(t,t_0,x_0) = \frac{-3|x_0|\Phi(t,t_0)\int_{t_0}^t a(s)\Phi^2(s,t_0)ds}{[1 + x_0^2 \int_{t_0}^t a(s)\Phi^2(s,t_0)ds]^{5/2}}. \tag{2.3.22}$$

Moreover, $\frac{\partial^2}{\partial x_0^2} x(t,t_0,x_0)$ satisfies the following differential equation:

$$\begin{aligned} dy &= \Big[\Big[\frac{-3a(t)\Phi^2(t,t_0)x_0^2}{2(1 + x_0^2 \int_{t_0}^t a(s)\Phi^2(s,t_0)ds)} + b(t)\Big]y - \frac{3a(t)\Phi^3(t,t_0)|x_0|}{\sqrt{1 + x_0^2 \int_{t_0}^t a(s)\Phi^2(s,t_0)ds}}\Big]dt \\ &\quad + \sigma y dw(t), \quad Y(t_0) = 0. \end{aligned} \tag{2.3.23}$$

The following result shows the existence of partial differential of solution process of (2.2.3) with respect to t_0.

Lemma 2.3.4 *Let us assume that all the hypothesis of Lemma 2.3.1 be satisfied. Let $x(t,t_0,x_0)$ be the solution process of (2.2.3) existing for $t \geq t_0$.*
Then

$$\partial_{t_0} x(t,t_0,x_0)$$

exists and:

$$\begin{aligned}\partial_{t_0}x(t,t_0,x_0) &= \frac{1}{2}[\Big(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,t_0,x_0)\sigma^l(t_0,x_0)\sigma^l_j(t_0,x_0)\Big)_{n\times 1} \\ &\quad +\Phi(t,t_0,x_0)[\sum_{l=1}^{m}\sigma^l_x(t_0,x_0)\sigma^l(t_0,x_0)-f(t_0,x_0)]]dt_0 \\ &\quad -\sum_{l=1}^{m}\Phi(t,t_0,x_0)\sigma^l(t_0,x_0)dw_l(t_0) \qquad (2.3.24)\end{aligned}$$

with

$$\partial_{t_0}x(t_0,t_0,x_0) = [\sum_{l=1}^{m}\sigma^l_x(t_0,x_0)\sigma^l(t_0,x_0) - f(t_0,x_0)]dt_0 - \sum_{l=1}^{m}\sigma^l(t_0,x_0)dw_l(t_0) \qquad (2.3.25)$$

Proof. Let $\Delta t_0 = \lambda > 0$ be a positive increment to t_0, and define

$$\Delta x(t,\lambda) = x(t,t_0+\lambda,x_0) - x(t,t_0,x_0) \qquad (2.3.26)$$

where $x(t,t_0+\lambda,x_0)$ and $x(t,t_0,x_0)$ are solution processes of (2.2.3) through $(t_0+\lambda,x_0)$ and (t_0,x_0), respectively.

Let

$$\Delta x(t_0) = x(t_0+\lambda,t_0,x_0) - x(t_0,t_0,x_0).$$

Set $R(\theta) = x(t,t_0+\lambda,x_0+\theta\Delta x(t_0))$. It is obvious that R is continuously differentiable with respect to θ, and hence

$$\frac{d}{d\theta}R(\theta) = \frac{\partial}{\partial x_0}x(t,t_0+\lambda,x_0+\theta\Delta x(t_0))\Delta x(t_0) = \Phi(t,t_0+\lambda,x_0+\theta\Delta x(t_0))\Delta x(t_0). \qquad (2.3.27)$$

By Integrating both sides of (2.3.27) with respect to θ over an interval [0,1], we obtain

$$R(1) - R(0) = \int_0^1 \Phi(t, t_0 + \lambda, x_0 + \theta \Delta x(t_0)) \Delta x(t_0) d\theta.$$

This, together with the fact that $R(1) = x(t, t_0 + \lambda, x(t_0 + \lambda, t_0, x_0))$ and $R(0) = x(t, t_0 + \lambda, x_0)$, yields

$$x(t, t_0 + \lambda, x(t_0 + \lambda, t_0, x_0)) - x(t, t_0 + \lambda, x_0) = \int_0^1 \Phi(t, t_0 + \lambda, x_0 + \theta \Delta x(t_0)) \Delta x(t_0) d\theta. \tag{2.3.28}$$

Because of the uniqueness of solution of (2.2.3) we have $x(t, t_0, x_0) = x(t, t_0 + \lambda, x(t_0 + \lambda, t_0, x_0))$ and equation (2.3.28) can be written as

$$x(t, t_0 + \lambda, x_0) - x(t, t_0, x_0) = -\int_0^1 \Phi(t, t_0 + \lambda, x_0 + \theta \Delta x(t_0)) \Delta x(t_0) d\theta. \tag{2.3.29}$$

By adding and subtracting $\Phi(t, t_0 + \lambda, x_0) \Delta x(t_0)$, $\Phi(t, t_0 + \lambda, x(t_0 + \lambda, t_0, x_0)) \Delta x(t_0)$ and $\Phi(t, t_0, x_0) \Delta x(t_0)$ in (2.3.29) and using the fact that

$$\Phi(t, t_0, x_0) = \Phi(t, t_0 + \lambda, x(t_0 + \lambda, t_0, x_0)) \Phi(t_0 + \lambda, t_0, x_0), \tag{2.3.30}$$

we have

$$\begin{aligned}
\Delta x(t, \lambda) \quad = \quad & -\int_0^1 [\Phi(t, t_0 + \lambda, x_0 + \theta \Delta x(t_0)) - \Phi(t, t_0 + \lambda, x_0)] \Delta x(t_0) d\theta \\
& + [\Phi(t, t_0 + \lambda, x(t_0 + \lambda, t_0, x_0)) - \Phi(t, t_0 + \lambda, x_0)] \Delta x(t_0) \\
& + [\Phi(t, t_0, x_0) - \Phi(t, t_0 + \lambda, x(t_0 + \lambda, t_0, x_0))] \Delta x(t_0) - \Phi(t, t_0, x_0) \Delta x(t_0) \\
= \quad & -\int_0^1 [\Phi(t, t_0 + \lambda, x_0 + \theta \Delta x(t_0)) - \Phi(t, t_0 + \lambda, x_0)] \Delta x(t_0) d\theta \\
& + [\Phi(t, t_0 + \lambda, x(t_0 + \lambda, t_0, x_0)) - \Phi(t, t_0 + \lambda, x_0)] \Delta x(t_0) \\
& + [\Phi(t, t_0 + \lambda, x(t_0 + \lambda, t_0, x_0)) \Phi(t_0 + \lambda, t_0, x_0)
\end{aligned}$$

$$
\begin{aligned}
& -\Phi(t,t_0+\lambda,x(t_0+\lambda,t_0,x_0))]\Delta x(t_0)-\Phi(t,t_0,x_0)\Delta x(t_0) \\
= \quad & -\int_0^1[\Phi(t,t_0+\lambda,x_0+\theta\Delta x(t_0))-\Phi(t,t_0+\lambda,x_0)]\Delta x(t_0)d\theta \\
& +[\Phi(t,t_0+\lambda,x(t_0+\lambda,t_0,x_0))-\Phi(t,t_0+\lambda,x_0)]\Delta x(t_0) \\
& +[\Phi(t,t_0+\lambda,x(t_0+\lambda,t_0,x_0))(\Phi(t_0+\lambda,t_0,x_0)-\Phi(t_0,t_0,x_0))]\Delta x(t_0) \\
& -\Phi(t,t_0,x_0)\Delta x(t_0). \qquad (2.3.31)
\end{aligned}
$$

We set $G(\psi)=\Phi(t,t_0+\lambda,x_0+\psi\theta\Delta x(t_0))$ for $0\le\psi\le 1$. It is obvious that G is continuously differentiable with respect to ψ, and hence

$$\frac{d}{d\psi}G(\psi)=\frac{\partial}{\partial x_0}\Phi(t,t_0+\lambda,x_0+\psi\theta\Delta x(t_0))\otimes(\theta\Delta x(t_0)) \qquad (2.3.32)$$

By Integrating both sides of (2.3.32) with respect to ψ over an interval [0,1], we have

$$G(1)-G(0)=\int_0^1\frac{\partial}{\partial x_0}\Phi(t,t_0+\lambda,x_0+\psi\theta\Delta x(t_0))\otimes(\theta\Delta x(t_0))d\psi.$$

This, together with $G(1)=\Phi(t,t_0+\lambda,x_0+\theta\Delta x(t_0))$ and $G(0)=\Phi(t,t_0+\lambda,x_0)$, yields

$$\Phi(t,t_0+\lambda,x_0+\theta\Delta x(t_0))-\Phi(t,t_0+\lambda,x_0)=\int_0^1\frac{\partial}{\partial x_0}\Phi(t,t_0+\lambda,x_0+\psi\theta\Delta x(t_0))\otimes(\theta\Delta x(t_0))d\psi \qquad (2.3.33)$$

Similarly, by setting $g(\beta)=\Phi(t,t_0+\lambda,x_0+\beta\Delta x(t_0))$ for $0\le\beta\le 1$, and repeating the previous argument, we obtain

$$\frac{d}{d\beta}g(\beta)=\frac{\partial}{\partial x_0}\Phi(t,t_0+\lambda,x_0+\beta\Delta x(t_0))\otimes\Delta x(t_0). \qquad (2.3.34)$$

This, together with $g(1) = \Phi(t, t_0 + \lambda, x_0 + \Delta x(t_0))$ and $g(0) = \Phi(t, t_0 + \lambda, x_0)$, yields

$$\Phi(t, t_0+\lambda, x_0+\Delta x(t_0)) - \Phi(t, t_0+\lambda, x_0) = \int_0^1 \frac{\partial}{\partial x_0}\Phi(t, t_0+\lambda, x_0+\beta\Delta x(t_0)) \otimes \Delta x(t_0) d\beta. \tag{2.3.35}$$

Using (2.3.33) and (2.3.35), (2.3.31) reduces to

$$\begin{aligned}
\Delta x(t,\lambda) &= -\int_0^1 \int_0^1 \frac{\partial}{\partial x_0}\Phi(t, t_0 + \lambda, x_0 + \psi\theta\Delta x(t_0)) \\
&\quad \otimes(\theta\Delta x(t_0))\Delta x(t_0) d\psi d\theta \\
&\quad + \int_0^1 \frac{\partial}{\partial x_0}\Phi(t, t_0 + \lambda, x_0 + \beta\Delta x(t_0)) \otimes \Delta x(t_0)\Delta x(t_0) d\beta \\
&\quad +[\Phi(t, t_0 + \lambda, x(t_0 + \lambda, t_0, x_0))(\Phi(t_0 + \lambda, t_0, x_0) - \Phi(t_0, t_0, x_0))]\Delta x(t_0) \\
&\quad -\Phi(t, t_0, x_0)\Delta x(t_0).
\end{aligned} \tag{2.3.36}$$

Adding and subtracting $\frac{\partial}{\partial x_0}\Phi(t, t_0 + \lambda, x_0) \otimes (\theta\Delta x(t_0))\Delta x(t_0)$ and $\frac{\partial}{\partial x_0}\Phi(t, t_0 + \lambda, x_0) \otimes (\Delta x(t_0))\Delta x(t_0)$ in (2.3.36) yields,

$$\begin{aligned}
\Delta x(t,\lambda) &= -\int_0^1 \int_0^1 \frac{\partial}{\partial x_0}[\Phi(t, t_0 + \lambda, x_0 + \psi\theta\Delta x(t_0)) \\
&\quad -\Phi(t, t_0 + \lambda, x_0)] \otimes (\theta\Delta x(t_0))\Delta x(t_0) d\psi d\theta \\
&\quad + \int_0^1 \frac{\partial}{\partial x_0}[\Phi(t, t_0 + \lambda, x_0 + \beta\Delta x(t_0)) - \Phi(t, t_0 + \lambda, x_0)] \otimes \Delta x(t_0)\Delta x(t_0) d\beta \\
&\quad +[\Phi(t, t_0 + \lambda, x(t_0 + \lambda, t_0, x_0))(\Phi(t_0 + \lambda, t_0, x_0) - \Phi(t_0, t_0, x_0))]\Delta x(t_0) \\
&\quad +\frac{1}{2}\frac{\partial}{\partial x_0}\Phi(t, t_0 + \lambda, x_0) \otimes \Delta x(t_0)\Delta x(t_0) - \Phi(t, t_0, x_0)\Delta x(t_0).
\end{aligned} \tag{2.3.37}$$

Using the bounded convergence theorem[34], the concept of Itô-Doob type differential and sufficiently small increment Δt_0 to t_0, (2.3.37) reduces to

$$\begin{aligned}
\partial_{t_0} x(t, t_0, x_0) &= [\Phi(t, t_0, x_0) d\Phi(t_0)] dx(t_0) + \frac{1}{2}\frac{\partial}{\partial x_0}\Phi(t, t_0, x_0) \otimes dx(t_0) dx(t_0) \\
&\quad -\Phi(t, t_0, x_0) dx(t_0)
\end{aligned}$$

$$
\begin{aligned}
&= \Phi(t,t_0,x_0)\sum_{l=1}^{m}\sigma_x^l(t_0,x_0)dw_l(t_0)\sum_{l=1}^{m}\sigma^l(t_0,x_0)dw_l(t_0) \\
&\quad +\frac{1}{2}\frac{\partial}{\partial x_0}\Phi(t,t_0,x_0)\otimes\sum_{l=1}^{m}\sigma^l(t_0,x_0)dw_l(t_0)\sum_{l=1}^{m}\sigma^l(t_0,x_0)dw_l(t_0) \\
&\quad -\Phi(t,t_0,x_0)[f(t_0,x_0)dt_0+\sum_{l=1}^{m}\sigma^l(t_0,x_0)dw_l(t_0)] \\
&= \frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,t_0,x_0)\sigma^l(t_0,x_0)\sigma_j^l(t_0,x_0)\right)_{n\times 1} dt_0 \\
&\quad +\Phi(t,t_0,x_0)[\sum_{l=1}^{m}\sigma_x^l(t_0,x_0)\sigma^l(t_0,x_0)-f(t_0,x_0)]dt_0 \\
&\quad -\sum_{l=1}^{m}\Phi(t,t_0,x_0)\sigma^l(t_0,x_0)dw_l(t_0)
\end{aligned} \tag{2.3.38}
$$

This shows that $\partial_{t_0}x(t,t_0,x_0)$ exists and it is represented as in (2.3.24). This together with $t=t_0$ and (2.3.2) yields (2.3.25).

■

Example 2.3.5 Let us consider a scalar linear unperturbed stochastic differential equation:

$$dx = f(t)xdt + \sigma(t)xdw(t), \qquad x(t_0) = x_0, \tag{2.3.39}$$

where f and σ are any differentiable functions defined on $J=[t_0,t_0+a)$ into R, where $a>0$. Find $\partial_{t_0}x(t,t_0,x_0)$, if it exists.

Solution: Note that $f(t,x)=f(t)x$ and $\sigma(t,x)=\sigma(t)x$ are continuously differentiable with respect to x. Moreover, $\frac{\partial}{\partial x}f(t,x)=f(t)$ and $\frac{\partial}{\partial x}\sigma(t,x)=\sigma(t)$. The closed form solution of (2.3.39) is given by

$$x(t,t_0,x_0)=\Phi(t,t_0)x_0.$$

Let us consider the following:

$$
\begin{aligned}
&x(t, t_0 + \lambda, x_0) - x(t, t_0, x_0) \\
&= x(t, t_0 + \lambda, x_0) - x(t, t_0 + \lambda, x(t_0 + \lambda, t_0, x_0)) \\
&= \Phi(t, t_0 + \lambda)x_0 - \Phi(t, t_0 + \lambda)x(t_0 + \lambda, t_0, x_0) \\
&= -\Phi(t, t_0 + \lambda)[x(t_0 + \lambda, t_0, x_0) - x_0] \\
&= -\Phi(t, t_0 + \lambda)\Delta x(t_0) \\
&= -[\Phi(t, t_0 + \lambda) - \Phi(t, t_0) + \Phi(t, t_0)]\Delta x(t_0) \\
&= -[\Phi(t, t_0 + \lambda) - \Phi(t, t_0)]\Delta x(t_0) - \Phi(t, t_0)\Delta x(t_0) \\
&= [\Phi(t, t_0) - \Phi(t, t_0 + \lambda)]\Delta x(t_0) - \Phi(t, t_0)\Delta x(t_0) \\
&= [\Phi(t, t_0 + \lambda)\Phi(t_0 + \lambda, t_0) - \Phi(t, t_0 + \lambda)]\Delta x(t_0) - \Phi(t, t_0)\Delta x(t_0) \\
&= \Phi(t, t_0 + \lambda)[\Phi(t_0 + \lambda, t_0) - I]\Delta x(t_0) - \Phi(t, t_0)\Delta x(t_0) \\
&= \Phi(t, t_0 + \lambda)\Delta\Phi(t_0, t_0)\Delta x(t_0) - \Phi(t, t_0)\Delta x(t_0). \qquad (2.3.40)
\end{aligned}
$$

Using the bounded convergence theorem [34], the concept of Itô-Doob type differential and sufficiently small increment λ to t_0, (2.3.40) reduces to

$$
\begin{aligned}
\partial_{t_0} x(t, t_0, x_0) &= \Phi(t, t_0)d\Phi(t_0, t_0)dx(t_0) - \Phi(t, t_0)dx(t_0) \\
&= \Phi(t, t_0)[f(t_0)\Phi(t_0, t_0)dt_0 + \sigma(t_0)\Phi(t_0, t_0)dw(t_0)][f(t_0)x_0 dt_0 + \sigma(t_0)x_0 dw(t_0)] \\
&\quad -\Phi(t, t_0)[f(t_0)x_0 dt_0 + \sigma(t_0)x_0 dw(t_0)] \\
&= \Phi(t, t_0)\sigma(t_0)\sigma(t_0)x_0 dt_0 - \Phi(t, t_0)[f(t_0)x_0 dt_0 + \sigma(t_0)x_0 dw(t_0)] \\
&= \Phi(t, t_0)[\sigma^2(t_0) - f(t_0)]x_0 dt_0 - \Phi(t, t_0)\sigma(t_0)x_0 dw(t_0). \qquad (2.3.41)
\end{aligned}
$$

Example 2.3.6 Let us consider a scalar linear perturbed stochastic differential equation:

$$dx = [f(t)x + p(t)]dt + [\sigma(t)x + q(t)]dw(t), \qquad x(t_0) = x_0, \qquad (2.3.42)$$

where f, σ, p and q are any differentiable functions defined on $J = [t_0, t_0 + a)$ into R, where $a > 0$. Find $\partial_{t_0} x(t, t_0, x_0)$, if it exists.

Solution: Note that $f(t, x) = f(t)x + p(t)$ and $\sigma(t, x) = \sigma(t)x + q(t)$ are continuously differentiable with respect to x. Moreover, $\frac{\partial}{\partial x} f(t, x) = f(t)$ and $\frac{\partial}{\partial x}\sigma(t, x) = \sigma(t)$. Using the application of Lemma 2.3.4 we obtain

$$\begin{aligned}
\partial_{t_0} x(t, t_0, x_0) &= \Phi(t, t_0)[(\sigma^2(t_0) - f(t_0))x_0 + \sigma(t_0)q(t_0) - p(t_0)]dt_0 \\
&\quad -\Phi(t, t_0)[\sigma(t_0)x_0 + q(t_0)]dw(t_0). \qquad (2.3.43)
\end{aligned}$$

In the following, we state and prove the existence of Itô-Doob type mixed partial differentials of solution process of (2.2.3).

Lemma 2.3.7 *Assume that all the hypothesis of Lemma 2.3.1 hold. Let $x(t, t_0, x_0)$ be the solution process of* (2.2.3) *existing for $t \geq t_0$. Then the mixed Itô-Doob type partial differentials*
$\partial_{x_0}(\partial_{t_0} x(t, t_0, x_0))$ and $\partial_{t_0}(\partial_{x_0} x(t, t_0, x_0))$ exists and they are equal. Moreover,

$$\begin{aligned}
\partial_{x_0}(\partial_{t_0} x(t, t_0, x_0)) &= -\Big[\left(\sum_{j=1}^{n} \sum_{l=1}^{m} \frac{\partial}{\partial x_0} \Phi_{ij}(t, t_0, x_0) \sigma^l(t_0, x_0) \sigma^l_j(t_0, x_0) \right)_{n\times 1} \\
&\quad + \sum_{l=1}^{m} \Phi(t, t_0, x_0) \sigma^l_x(t_0, x_0) \sigma^l(t_0, x_0) \Big] dt_0 \qquad (2.3.44)
\end{aligned}$$

With initial condition:

$$\partial_{x_0}(\partial_{t_0} x(t_0, t_0, x_0)) = -\sum_{l=1}^{m} \sigma^l_x(t_0, x_0) \sigma^l(t_0, x_0) dt_0 \qquad (2.3.45)$$

Proof. Let $\Delta x(t_0) = x(t_0 + \lambda, t_0, x_0) - x(t_0, t_0, x_0)$. Using (2.3.30), Lemma 2.3.1 and the continuous dependence of solution process of (2.3.1), we examine the following differential:

$$\begin{aligned}
&\partial_{x_0}x(t,t_0+\lambda,x_0)-\partial_{x_0}x(t,t_0,x_0)\\
&= \frac{\partial}{\partial x_0}x(t,t_0+\lambda,x_0)dx_0+\frac{1}{2}(\frac{\partial}{\partial x_0}\Phi(t,t_0+\lambda,x_0)\otimes dx_0)dx_0\\
&\quad -[\frac{\partial}{\partial x_0}x(t,t_0,x_0)dx_0+\frac{1}{2}(\frac{\partial}{\partial x_0}\Phi(t,t_0,x_0)\otimes dx_0)dx_0]\\
&= [\Phi(t,t_0+\lambda,x_0)-\Phi(t,t_0,x_0)]dx_0+\frac{1}{2}[\frac{\partial}{\partial x_0}(\Phi(t,t_0+\lambda,x_0)-\Phi(t,t_0,x_0))\otimes dx_0]dx_0.
\end{aligned} \tag{2.3.46}$$

Since $\Phi(t,t_0,x_0)=\Phi(t,t_0+\lambda,x(t_0+\lambda,t_0,x_0))\Phi(t_0+\lambda,t_0,x_0)$, by adding and subtracting $\Phi(t,t_0+\lambda,x(t_0+\lambda,t_0,x_0))dx_0$ in (2.3.46), using generalized mean value theorem and algebraic manipulations, we get

$$\begin{aligned}
&\partial_{x_0}x(t,t_0+\lambda,x_0)-\partial_{x_0}x(t,t_0,x_0)\\
&= [\Phi(t,t_0+\lambda,x_0)-\Phi(t,t_0+\lambda,x(t_0+\lambda,t_0,x_0))\Phi(t_0+\lambda,t_0,x_0)]dx_0\\
&\quad +\frac{1}{2}[\frac{\partial}{\partial x_0}(\Phi(t,t_0+\lambda,x_0)-\Phi(t,t_0,x_0))\otimes dx_0]dx_0\\
&= [\Phi(t,t_0+\lambda,x_0)-\Phi(t,t_0+\lambda,x(t_0+\lambda,t_0,x_0))+\Phi(t,t_0+\lambda,x(t_0+\lambda,t_0,x_0))\\
&\quad -\Phi(t,t_0+\lambda,x(t_0+\lambda,t_0,x_0))\Phi(t_0+\lambda,t_0,x_0)]dx_0\\
&\quad +\frac{1}{2}[\frac{\partial}{\partial x_0}(\Phi(t,t_0+\lambda,x_0)-\Phi(t,t_0,x_0))\otimes dx_0]dx_0\\
&= -[\int_0^1\frac{\partial}{\partial x_0}\Phi(t,t_0+\lambda,x_0+\theta\Delta x(t_0))\otimes\Delta x(t_0)d\theta]dx_0\\
&\quad -\Phi(t,t_0+\lambda,x(t_0+\lambda,t_0,x_0))(\Phi(t_0+\lambda,t_0,x_0)-I_{n\times n})dx_0\\
&\quad +\frac{1}{2}[\frac{\partial}{\partial x_0}(\Phi(t,t_0+\lambda,x_0)-\Phi(t,t_0,x_0))\otimes dx_0]dx_0
\end{aligned} \tag{2.3.47}$$

Again by adding and subtracting $\frac{\partial}{\partial x_0}\Phi(t,t_0+\lambda,x_0)\otimes\Delta x(t_0)dx_0$ in (2.3.47), we get

$$\begin{aligned}
&\partial_{x_0}x(t,t_0+\lambda,x_0)-\partial_{x_0}x(t,t_0,x_0)\\
&= -[\int_0^1\frac{\partial}{\partial x_0}(\Phi(t,t_0+\lambda,x_0+\theta\Delta x(t_0))-\Phi(t,t_0+\lambda,x_0))\otimes\Delta x(t_0)d\theta]dx_0\\
&\quad -\Phi(t,t_0+\lambda,x(t_0+\lambda,t_0,x_0))\Delta\Phi(t_0)dx_0-\frac{\partial}{\partial x_0}\Phi(t,t_0+\lambda,x_0))\otimes\Delta x(t_0)dx_0\\
&\quad +\frac{1}{2}[\frac{\partial}{\partial x_0}(\Phi(t,t_0+\lambda,x_0)-\Phi(t,t_0,x_0))\otimes dx_0]dx_0 \qquad (2.3.48)
\end{aligned}$$

For sufficiently small $\Delta t_0=\lambda>0$, uniform convergence theorem, solution process of Itô-Doob type stochastic differential equations (2.2.3) and (2.3.3), Itô-Doob calculus and continuous dependence of solutions with respect to initial conditions, we obtain

$$\begin{aligned}
\partial_{t_0}(\partial_{x_0}x(t,t_0,x_0)) &= -\Big[\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,t_0,x_0)\sigma^l(t_0,x_0)\sigma_j^l(t_0,x_0)\right)_{n\times 1}\\
&\quad +\sum_{l=1}^{m}\Phi(t,t_0,x_0)\sigma_x^l(t_0,x_0)\sigma^l(t_0,x_0)\Big]dt_0. \qquad (2.3.49)
\end{aligned}$$

On the other hand, using (2.3.24) we examine the following differential

$$\begin{aligned}
&\partial_{t_0}x(t,t_0,x_0+\Delta x_0)-\partial_{t_0}x(t,t_0,x_0)\\
&= [\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,t_0,x_0+\Delta x_0)\sigma^l(t_0,x_0+\Delta x_0)\sigma_j^l(t_0,x_0+\Delta x_0)\right)_{n\times 1}\\
&\quad +\Phi(t,t_0,x_0+\Delta x_0)[\sum_{l=1}^{m}\sigma_x^l(t_0,x_0+\Delta x_0)\sigma^l(t_0,x_0+\Delta x_0)-f(t_0,x_0+\Delta x_0)]]dt_0\\
&\quad -\sum_{l=1}^{m}\Phi(t,t_0,x_0+\Delta x_0)\sigma^l(t_0,x_0+\Delta x_0)dw_l(t_0)\\
&\quad -[[\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,t_0,x_0)\sigma^l(t_0,x_0)\sigma_j^l(t_0,x_0)\right)_{n\times 1}\\
&\quad +\Phi(t,t_0,x_0)[\sum_{l=1}^{m}\sigma_x^l(t_0,x_0)\sigma^l(t_0,x_0)-f(t_0,x_0)]]dt_0-\sum_{l=1}^{m}\Phi(t,t_0,x_0)\sigma^l(t_0,x_0)dw_l(t_0)]
\end{aligned}$$

$$
\begin{aligned}
= \; & [\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,t_0,x_0+\Delta x_0)\sigma^l(t_0,x_0+\Delta x_0)\sigma_j^l(t_0,x_0+\Delta x_0)\right)_{n\times 1} \\
& -\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,t_0,x_0)\sigma^l(t_0,x_0)\sigma_j^l(t_0,x_0)\right)_{n\times 1}]dt_0 \\
& +\Phi(t,t_0,x_0+\Delta x_0)[\sum_{l=1}^{m}\sigma_x^l(t_0,x_0+\Delta x_0)\sigma^l(t_0,x_0+\Delta x_0)-f(t_0,x_0+\Delta x_0)]dt_0 \\
& -\Phi(t,t_0,x_0)[\sum_{l=1}^{m}\sigma_x^l(t_0,x_0)\sigma^l(t_0,x_0)-f(t_0,x_0)]dt_0 \\
& -\sum_{l=1}^{m}\Phi(t,t_0,x_0+\Delta x_0)\sigma^l(t_0,x_0+\Delta x_0)dw_l(t_0)+\sum_{l=1}^{m}\Phi(t,t_0,x_0)\sigma^l(t_0,x_0)dw_l(t_0).
\end{aligned}
\tag{2.3.50}
$$

By adding and subtracting $\Phi(t,t_0,x_0)\sum_{l=1}^{m}\sigma^l(t_0,x_0+\Delta x_0)dw_l(t_0)$ in (2.3.50) yields

$$
\begin{aligned}
& \partial_{t_0}x(t,t_0,x_0+\Delta x_0)-\partial_{t_0}x(t,t_0,x_0) \\
& = \; [\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,t_0,x_0+\Delta x_0)\sigma^l(t_0,x_0+\Delta x_0)\sigma_j^l(t_0,x_0+\Delta x_0)\right)_{n\times 1} \\
& \quad -\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,t_0,x_0)\sigma^l(t_0,x_0)\sigma_j^l(t_0,x_0)\right)_{n\times 1}]dt_0 \\
& \quad +\Phi(t,t_0,x_0+\Delta x_0)[\sum_{l=1}^{m}\sigma_x^l(t_0,x_0+\Delta x_0)\sigma^l(t_0,x_0+\Delta x_0)-f(t_0,x_0+\Delta x_0)]dt_0 \\
& \quad -\Phi(t,t_0,x_0)[\sum_{l=1}^{m}\sigma_x^l(t_0,x_0)\sigma^l(t_0,x_0)-f(t_0,x_0)]dt_0 \\
& \quad -[\Phi(t,t_0,x_0+\Delta x_0)-\Phi(t,t_0,x_0)]\sum_{l=1}^{m}\sigma^l(t_0,x_0+\Delta x_0)dw_l(t_0) \\
& \quad -\Phi(t,t_0,x_0)\sum_{l=1}^{m}(\sigma^l(t_0,x_0+\Delta x_0)-\sigma^l(t_0,x_0))dw_l(t_0)
\end{aligned}
\tag{2.3.51}
$$

From the continuity of rate coefficient matrices and the continuous dependence of solution process, we have

$$\partial_{x_0}(\partial_{t_0}x(t,t_0,x_0)) = -\Big[\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,t_0,x_0)\sigma^l(t_0,x_0)\sigma^l_j(t_0,x_0)\right)_{n\times 1} + \sum_{l=1}^{m}\Phi(t,t_0,x_0)\sigma^l_x(t_0,x_0)\sigma^l(t_0,x_0)\Big]dt_0 \tag{2.3.52}$$

This establishes the proof of (2.3.44). Since $\frac{\partial}{\partial x_0}\Phi(t_0,t_0,x_0) = 0$ and $\Phi(t_0,t_0,x_0) = I_{n\times n}$ at $t = t_0$, we have

$$\partial_{x_0}(\partial_{t_0}x(t_0)) = -\sum_{l=1}^{m}\sigma^l_x(t_0,x_0)\sigma^l(t_0,x_0)dt_0. \tag{2.3.53}$$

This completes the proof of the Lemma.

■

Example 2.3.8 Let us consider Example2.3.5. Find $\partial_{x_0}(\partial_{t_0}x(t,t_0,x_0))$.

Solution: Using (2.3.30), Lemma 2.3.1 and the continuous dependence of solution process of (2.3.1), we examine the following differential:

$$\begin{aligned}
&\partial_{x_0}x(t,t_0+\lambda,x_0) - \partial_{x_0}x(t,t_0,x_0)\\
&= [\frac{\partial}{\partial x_0}x(t,t_0+\lambda,x_0) - \frac{\partial}{\partial x_0}x(t,t_0+\lambda,x(t_0+\lambda,t_0,x_0))]dx_0\\
&= [\Phi(t,t_0+\lambda) - \Phi(t,t_0)]dx_0\\
&= [\Phi(t,t_0+\lambda) - \Phi(t,t_0+\lambda)\Phi(t_0+\lambda,t_0)]dx_0\\
&= -\Phi(t,t_0+\lambda)[\Phi(t_0+\lambda,t_0) - \Phi(t_0,t_0)]dx_0\\
&= -\Phi(t,t_0+\lambda)\Delta\Phi(t_0,t_0)dx_0.
\end{aligned} \tag{2.3.54}$$

Using the bounded convergence theorem[34], the concept of Itô-Doob type differential and sufficiently small increment λ to t_0, (2.3.54) reduces to

$$\begin{aligned}
&\partial_{t_0}(\partial_{x_0} x(t,t_0,x_0)) \\
&= -\Phi(t,t_0)d\Phi(t_0,t_0)dx_0 \\
&= -\Phi(t,t_0)[f(t_0)\Phi(t_0,t_0)dt_0+\sigma(t_0)\Phi(t_0,t_0)dw(t_0)][f(t_0)x_0dt_0+\sigma(t_0)x_0dw(t_0)] \\
&= -\Phi(t,t_0)\sigma^2(t_0)x_0dt_0. \qquad (2.3.55)
\end{aligned}$$

2.4 Method of Variation of Constants Formula

In this section we shall establish the method of variation of constants formula with respect to (2.2.3) and its perturbed system (2.2.2).

Theorem 2.4.1 *Let the assumption of Lemma 2.3.1 be satisfied. Let $y(t,t_0,x_0)$ and $x(t,t_0,x_0)$ be solution processes of* (2.2.2) *and* (2.2.3)*, respectively, through the same initial data (t_0,x_0), for all $t \geq t_0$.*
Then

$$\begin{aligned}
y(t,t_0,x_0) &= x(t,t_0,x_0)+\int_{t_0}^{t}\Big[\frac{1}{2}\Big(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))[\Upsilon^l(s,y(s))\sigma_j^l(s,y(s)) \\
&\quad +\Upsilon^l(s,y(s))\Upsilon_j^l(s,y(s))-\sigma^l(s,y(s))\Upsilon_j^l(s,y(s))]\Big)_{n\times 1} \\
&\quad +\Phi(t,s,y(s))[F(s,y(s))-\sum_{l=1}^{m}\sigma_x^l(s,y(s))\Upsilon^l(s,y(s))]\Big]ds \\
&\quad +\sum_{l=1}^{m}\int_{t_0}^{t}\Phi(t,s,y(s))\Upsilon^l(s,y(s))dw_l(s). \qquad (2.4.1)
\end{aligned}$$

Proof. From the application of Lemma 2.3.1, Lemma 2.3.4, Lemma 2.3.7 and the Itô-Doob differential formula with respect to s for $t_0 \le s \le t$, we have

$$\begin{aligned}
d_s x(t,s,y(s)) &= \partial_{t_0} x(t,s,y(s)) + \partial_{x_0} x(t,s,y(s)) + \partial_{x_0}(\partial_{t_0} x(t,s,y(s))) \\
&\quad + \frac{1}{2}(\frac{\partial}{\partial x_0}\Phi(t,s,y(s)) \otimes dy)dy \\
&= \frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\sigma^l(s,y(s))\sigma^l_j(s,y(s))\right)_{n\times 1} ds \\
&\quad + \Phi(t,s,y(s))(\sum_{l=1}^{m}\sigma^l_x(s,y(s))\sigma^l(s,y(s)) - f(s,y(s)))ds \\
&\quad - \Phi(t,s,y(s))\sum_{l=1}^{m}\sigma^l(s,y(s))dw_l(s) + \Phi(t,s,y(s))dy(s) \\
&\quad - \left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\sigma^l(s,y(s))(\sigma^l_j(s,y(s)) + \Upsilon^l_j(s,y(s)))\right)_{n\times 1} ds \\
&\quad - \Phi(t,s,y(s))\sum_{l=1}^{m}\sigma^l_x(s,y(s))(\sigma^l(s,y(s)) + \Upsilon^l(s,y(s)))ds \\
&\quad + \frac{1}{2}(\frac{\partial}{\partial x_0}\Phi(t,s,y(s)) \otimes \sum_{l=1}^{m}\sigma^l(s,y(s))dw_l(s)\sum_{l=1}^{m}\sigma^l(s,y(s))dw_l(s) \\
&\quad + \frac{1}{2}(\frac{\partial}{\partial x_0}\Phi(t,s,y(s)) \otimes \sum_{l=1}^{m}\Upsilon^l(s,y(s))dw_l(s)\sum_{l=1}^{m}\sigma^l(s,y(s))dw_l(s) \\
&\quad + \frac{1}{2}(\frac{\partial}{\partial x_0}\Phi(t,s,y(s)) \otimes \sum_{l=1}^{m}\sigma^l(s,y(s))dw_l(s)\sum_{l=1}^{m}\Upsilon^l(s,y(s))dw_l(s) \\
&\quad + \frac{1}{2}(\frac{\partial}{\partial x_0}\Phi(t,s,y(s)) \otimes \sum_{l=1}^{m}\Upsilon^l(s,y(s))dw_l(s)\sum_{l=1}^{m}\Upsilon^l(s,y(s))dw_l(s)
\end{aligned} \tag{2.4.2}$$

By simplifying (2.4.2), we get

$$\begin{aligned} d_s x(t,s,y(s)) \;=\; & \Big[-\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\sigma^l(s,y(s))\Upsilon^l_j(s,y(s))\right)_{n\times 1} \\ & +\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\Upsilon^l(s,y(s))\sigma^l_j(s,y(s))\right)_{n\times 1} \\ & +\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\Upsilon^l(s,y(s))\Upsilon^l_j(s,y(s))\right)_{n\times 1} \\ & +\Phi(t,s,y(s))[F(s,y(s)) - \sum_{l=1}^{m}\sigma^l_x(s,y(s))\Upsilon^l(s,y(s))]\Big]ds \\ & +\sum_{l=1}^{m}\Phi(t,s,y(s))\Upsilon^l(s,y(s))dw_l(s). \end{aligned} \tag{2.4.3}$$

Since the right hand side of (2.4.3) is continuous with respect to s, we integrate from t_0 to t, and obtain the variation of constant formula (2.4.1).

■

Corollary 2.4.2 *Let the assumption of Lemma 2.3.1 be satisfied, except that only $c(t,y)$ in* (2.2.2) *can be decomposed. Let $y(t,t_0,x_0)$ and $x(t,t_0,x_0)$ be solution processes of* (2.2.2) *and* (2.2.3), *respectively, through the same initial data (t_0,x_0), for all $t \geq t_0$. Then*

$$\begin{aligned} y(t,t_0,x_0) \;=\; & x(t,t_0,x_0) + \int_{t_0}^{t}\Big[\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\sigma^l(s,y(s))\sigma^l_j(s,y(s))\right)_{n\times 1} \\ & -\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\sigma^l(s,y(s))\Sigma^l_j(s,y(s))\right)_{n\times 1} \\ & +\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\Sigma^l(s,y(s))\Sigma^l_j(s,y(s))\right)_{n\times 1} \end{aligned}$$

$$+\Phi(t,s,y(s))(\sum_{l=1}^{m}[\sigma_x^l(s,y(s))\sigma^l(s,y(s))-\sigma_x^l(s,y(s))\Sigma^l(s,y(s))]$$
$$+F(s,y(s)))\Big]ds$$
$$+\sum_{l=1}^{m}\int_{t_0}^{t}\Phi(t,s,y(s))[\Sigma^l(s,y(s))-\sigma^l(s,y(s))]dw_l(s) \tag{2.4.4}$$

Proof. From the application of Lemma 2.3.1, Lemma 2.3.4, Lemma 2.3.7 and the Itô-Doob differential formula with respect to s for $t_0 \leq s \leq t$, we have

$$\begin{aligned}
d_s x(t,s,y(s)) &= \partial_{t_0}x(t,s,y(s)) + \partial_{x_0}x(t,s,y(s)) + \partial_{x_0}(\partial_{t_0}x(t,s,y(s)))\\
&\quad +\frac{1}{2}(\frac{\partial}{\partial x_0}\Phi(t,s,y(s))\otimes dy)dy\\
&= \frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\sigma^l(s,y(s))\sigma_j^l(s,y(s))\right)_{n\times 1} ds\\
&\quad +\Phi(t,s,y(s))(\sum_{l=1}^{m}\sigma_x^l(s,y(s))\sigma^l(s,y(s))-f(s,y(s)))ds\\
&\quad -\Phi(t,s,y(s))\sum_{l=1}^{m}\sigma^l(s,y(s))dw_l(s)\\
&\quad +\Phi(t,s,y(s))[(f(s,y(s))+F(s,y(s)))ds+\sum_{l=1}^{m}\Sigma^l(s,y(s))dw_l(s)]\\
&\quad -\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\sigma^l(s,y(s))\Sigma_j^l(s,y(s))\right)_{n\times 1} ds\\
&\quad -\Phi(t,s,y(s))\sum_{l=1}^{m}\sigma_x^l(s,y(s))\Sigma^l(s,y(s))ds\\
&\quad +\frac{1}{2}(\frac{\partial}{\partial x_0}\Phi(t,s,y(s))\otimes\sum_{l=1}^{m}\Sigma^l(s,y(s))dw_l(s)\sum_{l=1}^{m}\Sigma^l(s,y(s))dw_l(s)
\end{aligned} \tag{2.4.5}$$

By simplifying (2.4.5), we get

$$
\begin{aligned}
d_s x(t,s,y(s)) \quad = \quad & \Big[\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\sigma^l(s,y(s))\sigma^l_j(s,y(s))\right)_{n\times 1} \\
& -\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\sigma^l(s,y(s))\Sigma^l_j(s,y(s))\right)_{n\times 1} \\
& +\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\Sigma^l(s,y(s))\Sigma^l_j(s,y(s))\right)_{n\times 1} \\
& +\Phi(t,s,y(s))(\sum_{l=1}^{m}[\sigma^l_x(s,y(s))\sigma^l(s,y(s))-\sigma^l_x(s,y(s))\Sigma^l(s,y(s))] \\
& +F(s,y(s)))\Big]ds \\
& +\sum_{l=1}^{m}\Phi(t,s,y(s))[\Sigma^l(s,y(s))-\sigma^l(s,y(s))]dw_l(s)
\end{aligned}
\tag{2.4.6}
$$

Since the right hand side of (2.4.6) is continuous with respect to s, we integrate from t_0 to t, and obtain the variation of constant formula (2.4.4).

■

Corollary 2.4.3 *Let the assumption of Lemma 2.3.1 be satisfied, except that only $\Sigma(t,y)$ in (2.2.2) can be decomposed. Let $y(t,t_0,x_0)$ and $x(t,t_0,x_0)$ be solution processes of (2.2.2) and (2.2.2), respectively, through the same initial data (t_0,x_0), for all $t \geq t_0$. Then*

$$
\begin{aligned}
y(t,t_0,x_0) \quad = \quad & x(t,t_0,x_0)+\int_{t_0}^{t}\Big[\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\Upsilon^l(s,y(s))\sigma^l_j(s,y(s))\right)_{n\times 1} \\
& -\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\sigma^l(s,y(s))\Upsilon^l_j(s,y(s))\right)_{n\times 1} \\
& +\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\Upsilon^l(s,y(s))\Upsilon^l_j(s,y(s))\right)_{n\times 1}
\end{aligned}
$$

$$+\Phi(t,s,y(s))[c(s,y(s))-f(s,y(s))-\sum_{l=1}^{m}\sigma_x^l(s,y(s))\Upsilon^l(s,y(s))]\Big]ds$$

$$+\sum_{l=1}^{m}\int_{t_0}^{t}\Phi(t,s,y(s))\Upsilon^l(s,y(s))dw_l(s). \tag{2.4.7}$$

Proof. From the application of Lemma 2.3.1, Lemma 2.3.4, Lemma 2.3.7 and the Itô-Doob differential formula with respect to s for $t_0 \le s \le t$, we have

$$\begin{aligned}
d_s x(t,s,y(s)) &= \partial_{t_0}x(t,s,y(s)) + \partial_{x_0}x(t,s,y(s)) + \partial_{x_0}(\partial_{t_0}x(t,s,y(s)))\\
&\quad +\frac{1}{2}(\frac{\partial}{\partial x_0}\Phi(t,s,y(s))\otimes dy)dy\\
&= \frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\sigma^l(s,y(s))\sigma_j^l(s,y(s))\right)_{n\times 1} ds\\
&\quad +\Phi(t,s,y(s))(\sum_{l=1}^{m}\sigma_x^l(s,y(s))\sigma^l(s,y(s))-f(s,y(s)))ds\\
&\quad -\Phi(t,s,y(s))\sum_{l=1}^{m}\sigma^l(s,y(s))dw_l(s)\\
&\quad +\Phi(t,s,y(s))[c(s,y(s))ds+\sum_{l=1}^{m}(\sigma^l(s,y(s))+\Upsilon^l(s,y(s)))dw_l(s)]\\
&\quad -\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\sigma^l(s,y(s))(\sigma_j^l(s,y(s))+\Upsilon_j^l(s,y(s)))\right)_{n\times 1} ds\\
&\quad -\Phi(t,s,y(s))\sum_{l=1}^{m}\sigma_x^l(s,y(s))(\sigma^l(s,y(s))+\Upsilon^l(s,y(s)))ds\\
&\quad +\frac{1}{2}(\frac{\partial}{\partial x_0}\Phi(t,s,y(s))\otimes\sum_{l=1}^{m}\sigma^l(s,y(s))dw_l(s)\sum_{l=1}^{m}\sigma^l(s,y(s))dw_l(s)\\
&\quad +\frac{1}{2}(\frac{\partial}{\partial x_0}\Phi(t,s,y(s))\otimes\sum_{l=1}^{m}\Upsilon^l(s,y(s))dw_l(s)\sum_{l=1}^{m}\sigma^l(s,y(s))dw_l(s)\\
&\quad +\frac{1}{2}(\frac{\partial}{\partial x_0}\Phi(t,s,y(s))\otimes\sum_{l=1}^{m}\sigma^l(s,y(s))dw_l(s)\sum_{l=1}^{m}\Upsilon^l(s,y(s))dw_l(s)\\
&\quad +\frac{1}{2}(\frac{\partial}{\partial x_0}\Phi(t,s,y(s))\otimes\sum_{l=1}^{m}\Upsilon^l(s,y(s))dw_l(s)\sum_{l=1}^{m}\Upsilon^l(s,y(s))dw_l(s)
\end{aligned} \tag{2.4.8}$$

By simplifying (2.4.8), we get

$$
\begin{aligned}
d_s x(t,s,y(s)) \quad = \quad & \Big[\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\Upsilon^l(s,y(s))\sigma_j^l(s,y(s))\right)_{n\times 1} \\
& -\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\sigma^l(s,y(s))\Upsilon_j^l(s,y(s))\right)_{n\times 1} \\
& +\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\Upsilon^l(s,y(s))\Upsilon_j^l(s,y(s))\right)_{n\times 1} \\
& +\Phi(t,s,y(s))[c(s,y(s)) - f(s,y(s)) - \sum_{l=1}^{m}\sigma_x^l(s,y(s))\Upsilon^l(s,y(s))]\Big]ds \\
& +\sum_{l=1}^{m}\Phi(t,s,y(s))\Upsilon^l(s,y(s))dw_l(s). \qquad (2.4.9)
\end{aligned}
$$

Since the right hand side of (2.4.9) is continuous with respect to s, we integrate from t_0 to t, and obtain the variation of constant formula (2.4.7).

■

Corollary 2.4.4 *Let the assumption of Lemma 2.3.1 be satisfied, except that $c(t,y)$ in (2.2.2) and $\Sigma(t,y)$ in (2.2.3) cannot be decomposed. Let $y(t,t_0,x_0)$ and $x(t,t_0,x_0)$ be solution processes of (2.2.2) and (2.2.3), respectively, through the same initial data (t_0,x_0), for all $t \geq t_0$. Then*

$$
\begin{aligned}
y(t,t_0,x_0) \quad = \quad & x(t,t_0,x_0) + \int_{t_0}^{t}\Big[\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\sigma^l(s,y(s))\sigma_j^l(s,y(s))\right)_{n\times 1} \\
& +\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\Sigma^l(s,y(s))\Sigma_j^l(s,y(s))\right)_{n\times 1} \\
& -\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\sigma^l(s,y(s))\Sigma_j^l(s,y(s))\right)_{n\times 1}
\end{aligned}
$$

$$
\begin{aligned}
&+\Phi(t,s,y(s))[\sum_{l=1}^{m}\sigma_x^l(s,y(s))(\sigma^l(s,y(s))-\Sigma^l(s,y(s)))+c(s,y(s))\\
&-f(s,y(s))]\Big]ds+\sum_{l=1}^{m}\int_{t_0}^{t}\Phi(t,s,y(s))[\Sigma^l(s,y(s))-\sigma^l(s,y(s))]dw_l(s).
\end{aligned}
\tag{2.4.10}
$$

Proof. From the application of Lemma 2.3.1, Lemma 2.3.4, Lemma 2.3.7 and the Itô-Doob differential formula with respect to s for $t_0 \le s \le t$, we have

$$
\begin{aligned}
d_s x(t,s,y(s)) &= \partial_{t_0}x(t,s,y(s))+\partial_{x_0}x(t,s,y(s))+\partial_{x_0}(\partial_{t_0}x(t,s,y(s)))\\
&\quad+\frac{1}{2}(\frac{\partial}{\partial x_0}\Phi(t,s,y(s))\otimes dy)dy\\
&= \frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\sigma^l(s,y(s))\sigma_j^l(s,y(s))\right)_{n\times 1}ds\\
&\quad+\Phi(t,s,y(s))[(\sum_{l=1}^{m}\sigma_x^l(s,y(s))\sigma^l(s,y(s))-f(s,y(s)))ds\\
&\quad-\sum_{l=1}^{m}\sigma^l(s,y(s))dw_l(s)]\\
&\quad+\Phi(t,s,y(s))[c(s,y(s))ds+\sum_{l=1}^{m}\Sigma^l(s,y(s))dw_l(s)]\\
&\quad-\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\sigma^l(s,y(s))\Sigma_j^l(s,y(s))\right)_{n\times 1}ds\\
&\quad-\Phi(t,s,y(s))\sum_{l=1}^{m}\sigma_x^l(s,y(s))\Sigma^l(s,y(s))ds\\
&\quad+\frac{1}{2}(\frac{\partial}{\partial x_0}\Phi(t,s,y(s))\otimes\sum_{l=1}^{m}\Sigma^l(s,y(s))dw_l(s)\sum_{l=1}^{m}\Sigma^l(s,y(s))dw_l(s)
\end{aligned}
\tag{2.4.11}
$$

By simplifying (2.4.11), we get

$$
\begin{aligned}
d_s x(t,s,y(s)) \quad = \quad & \Big[\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\sigma^l(s,y(s))\sigma^l_j(s,y(s))\right)_{n\times 1} \\
& +\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\Sigma^l(s,y(s))\Sigma^l_j(s,y(s))\right)_{n\times 1} \\
& -\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\sigma^l(s,y(s))\Sigma^l_j(s,y(s))\right)_{n\times 1} \\
& +\Phi(t,s,y(s))[\sum_{l=1}^{m}\sigma^l_x(s,y(s))(\sigma^l(s,y(s))-\Sigma^l(s,y(s)))+c(s,y(s)) \\
& -f(s,y(s))]\Big]ds+\sum_{l=1}^{m}\Phi(t,s,y(s))[\Sigma^l(s,y(s))-\sigma^l(s,y(s))]dw_l(s) \qquad (2.4.12)
\end{aligned}
$$

Since the right hand side of (2.4.12) is continuous with respect to s, we integrate from t_0 to t, and obtain the variation of constant formula (2.4.10).

$\blacksquare$

Remark 2.4.5 *In Corollary* (2.4.4)*, if*

1. $\sigma(t,x)=0$ *and* $c(t,x)=f(t,x)$*, then the variation of constant formula reduces to*

$$
\begin{aligned}
y(t,t_0,x_0) \quad = \quad & x(t,t_0,x_0)+\int_{t_0}^{t}\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\Sigma^l(s,y(s))\Sigma^l_j(s,y(s))\right)_{n\times 1} ds \\
& +\sum_{l=1}^{m}\int_{t_0}^{t}\Phi(t,s,y(s))\Sigma^l(s,y(s))dw_l(s). \qquad (2.4.13)
\end{aligned}
$$

2. $f(t,x)=0$ *and* $\Sigma(t,x)=\sigma(t,x)$*, then the variation of constant formula reduces to*

$$
y(t,t_0,x_0) \quad = \quad x(t,t_0,x_0)+\int_{t_0}^{t}\Phi(t,s,y(s))c(s,y(s))d(s). \qquad (2.4.14)
$$

2.5 Examples

Example 2.5.1 Consider a scalar linear unperturbed and perturbed stochastic differential equations:

$$dx = f(t)xdt + \sigma(t)xdw(t), \qquad x(t_0) = x_0, \tag{2.5.1}$$

and

$$dy = [f(t)y + p(t)]dt + [\sigma(t)y + q(t)]dw(t), \qquad y(t_0) = x_0, \tag{2.5.2}$$

where f, σ, p and q are any differentiable functions defined on $J = [t_0, t_0 + a)$ into R, where $a > 0$. Then

$$y(t, t_0, x_0) = x(t, t_0, x_0) + \int_{t_0}^{t} \Phi(t, s)[p(s) - \sigma(s)q(s)]d(s) + \int_{t_0}^{t} \Phi(t, s)q(s)dw(s). \tag{2.5.3}$$

Solution: Let $x(t) = x(t, t_0, x_0)$ and $y(t) = y(t, t_0, x_0)$ be solution processes of (2.5.1) and (2.5.2) through (t_0, x_0), respectively. Since $x(t) = x(t, t_0, x_0) = \Phi(t, t_0)x_0$ [11], the partial derivative of $x(t, t_0, x_0)$ with respect to x_0 will be $\frac{\partial}{\partial x_0}x(t, t_0, x_0) = \Phi(t, t_0)$. Moreover, $\frac{\partial^2}{\partial x_0^2}x(t, t_0, x_0) = 0$. From Lemma 2.3.4, the partial differential of $x(t, t_0, x_0)$ with respect to t_0 is given by

$$\partial_{t_0}x(t, t_0, x_0) = \Phi(t, t_0)[\sigma^2(t_0) - f(t_0)]x_0dt_0 - \Phi(t, t_0)\sigma(t_0)x_0dw(t_0) \tag{2.5.4}$$

At $t = t_0$, we have

$$\partial_{t_0}x(t_0, t_0, x_0) = [\sigma^2(t_0) - f(t_0)]x_0dt_0 - \sigma(t_0)x_0dw(t_0) \tag{2.5.5}$$

Moreover, from Lemma 2.3.7, the corresponding Itô-Doob mixed partial differential of solution process $x(t_0, t_0, x_0)$ of (2.5.1) is given by

$$\partial_{t_0 x_0} x(t, t_0, x_0) = -\Phi(t, t_0)\sigma^2(t_0)x_0 dt_0 \tag{2.5.6}$$

Using the method of variational constants, Theorem 2.4.1, the solution of (2.5.2) is given by (2.5.3).

Example 2.5.2 Consider a scalar linear unperturbed and nonlinear perturbed stochastic differential equations:

$$dx = f(t)xdt + \sigma(t)xdw(t), \qquad x(t_0) = x_0, \tag{2.5.7}$$

and

$$dy = [f(t)y - p(t)\frac{1}{2}y^3]dt + \sigma(t)ydw(t), \qquad y(t_0) = x_0, \tag{2.5.8}$$

where f, σ and p are any differentiable functions defined on J into R. Then

$$y(t, t_0, x_0) = x(t, t_0, x_0) + \int_{t_0}^{t} \frac{1}{2}\Phi(t, s)p(t)y^3(s)ds. \tag{2.5.9}$$

Solution: Let $x(t) = x(t, t_0, x_0)$ and $y(t) = y(t, t_0, x_0)$ be solution processes of (2.5.7) and (2.5.8) through (t_0, x_0), respectively. The partial differential of $x(t, t_0, x_0)$ with respect to initial data is given in Example 2.5.1. The closed form solution of (2.5.8) is

$$y(t, t_0, x_0) = \frac{\Phi(t, t_0) \mid x_0 \mid}{\sqrt{1 + x_0^2 \int_{t_0}^{t} p(s)\Phi^2(t, s)ds}}. \tag{2.5.10}$$

The partial differential of $y(t, t_0, x_0)$ with respect to t_0 is

$$\partial_{t_0} y(t, t_0, x_0) = \Phi(t, t_0)[(f(t_0) + \sigma^2(t_0))x_0 - \frac{1}{2}p(t){x_0}^3]dt_0 - \Phi(t, t_0)\sigma(t_0)x_0 dw(t_0). \tag{2.5.11}$$

At $t = t_0$, we have

$$\partial_{t_0} y(t_0, t_0, x_0) = [(f(t_0) + \sigma^2(t_0))x_0 - \frac{1}{2}p(t){x_0}^3]dt_0 - \sigma(t_0)x_0 dw(t_0) \tag{2.5.12}$$

Moreover, the corresponding Itô-Doob mixed partial differential of solution process $y(t_0, t_0, x_0)$ of (2.5.7) is given by

$$\partial_{t_0 x_0} y(t_0, t_0, x_0) = -\Phi(t, t_0)\sigma^2(t_0)x_0 dt_0 \tag{2.5.13}$$

Using the method of variational constants, Theorem (2.4.1), the solution of (2.5.8) is given by (2.5.9).

Example 2.5.3 Consider a nonlinear unperturbed and perturbed stochastic differential equation:

$$dx = \alpha x(n - x)dt + \beta x dw(t), \qquad x(t_0) = x_0, \tag{2.5.14}$$

and

$$dy = [\alpha y(n - y) + g(t, y)]dt + [\beta y + \sigma(t, y)]dw(t), \qquad y(t_0) = x_0, \tag{2.5.15}$$

where α, β and n are any constant, g and σ are differentiable functions. Then

$$\begin{aligned} y(t, t_0, x_0) &= x(t, t_0, x_0) + \int_{t_0}^{t} [\frac{1}{2}\frac{\partial}{\partial x_0}\Phi(t, s, y(s))\sigma^2(s, y(s)) + \Phi(t, s, y(s))[g(s, y(s)) \\ &\quad -\beta\sigma(s, y(s))]]ds + \int_{t_0}^{t} \Phi(t, s)\sigma(s, y(s))dw(s). \end{aligned} \tag{2.5.16}$$

Solution: Let $x(t) = x(t,t_0,x_0)$ and $y(t) = y(t,t_0,x_0)$ be solution processes of (2.5.14) and (2.5.15) through (t_0,x_0), respectively. The partial derivative of solution processes $x(t,t_0,x_0)$ with respect to x_0 is

$$\frac{\partial}{\partial x_0}x(t,t_0,x_0) = \frac{\Phi(t,t_0)}{(\Phi(t,t_0)+\alpha x_0\int_{t_0}^{t}\Phi(t,s)ds)^2} \tag{2.5.17}$$

and

$$\frac{\partial^2}{\partial x_0^2}x(t,t_0,x_0) = \frac{-2\alpha\Phi(t,t_0)\int_{t_0}^{t}\Phi(t,s)ds}{(\Phi(t,t_0)+\alpha x_0\int_{t_0}^{t}\Phi(t,s)ds)^3}. \tag{2.5.18}$$

Moreover, the partial differential of $x(t,t_0,x_0)$ with respect to t_0 is

$$\begin{aligned}\partial_{t_0}x(t,t_0,x_0) \;=\; & \Big[\frac{-\alpha\beta^2x_0^2\Phi(t,t_0)\int_{t_0}^{t}\Phi(t,s)ds}{(\Phi(t,t_0)+\alpha x_0\int_{t_0}^{t}\Phi(t,s)ds)^3} \\ & +\beta^2\Phi(t,t_0)x_0-\Phi(t,t_0)\alpha x_0(n-x_0)dt_0\Big]-\Phi(t,t_0)\beta x_0dw(t_0).\end{aligned} \tag{2.5.19}$$

The corresponding Itô-Doob mixed partial differential of solution process $x(t,t_0,x_0)$ of (2.5.14) is given by

$$\partial_{t_0x_0}x(t,t_0,x_0) = \Big[\frac{2\alpha\beta^2x_0^2\Phi(t,t_0)\int_{t_0}^{t}\Phi(t,s)ds}{(\Phi(t,t_0)+\alpha x_0\int_{t_0}^{t}\Phi(t,s)ds)^3}+\Phi(t,t_0)\beta^2x_0\Big]dt_0. \tag{2.5.20}$$

Using the method of variational constants method, Theorem (2.4.1), with $f(t,x) = \alpha x(n-x)$ and $\sigma(t,x) = \beta x$, the solution of (2.5.15) is given by (2.5.16).

Example 2.5.4 Consider a scalar linear unperturbed and perturbed stochastic differential equation as:

$$dx = A(t)xdt + \sum_{l=1}^{m}\sigma^l(t)xdw_l(t), \qquad x(t_0) = x_0, \tag{2.5.21}$$

and

$$dy = [A(t)y + P(t)]dt + \sum_{l=1}^{m} [\sigma^l(t)y + \Upsilon^l(t)]dw_l(t), \qquad y(t_0) = x_0, \tag{2.5.22}$$

where A and σ^l are any differentiable functions defined on J into $R^{n\times n}$ and P, Υ^l are any differentiable functions defined on J into R^n.Then

$$y(t,t_0,x_0) = x(t,t_0,x_0) + \int_{t_0}^{t} \Phi(t,s)[P(s) - \sum_{l=1}^{m} \sigma^l(s)\Upsilon^l(s)]ds + \sum_{l=1}^{m} \int_{t_0}^{t} \Phi(t,s)\Upsilon^l(s)dw_l(s). \tag{2.5.23}$$

Solution: Let $x(t) = x(t,t_0,x_0)$ and $y(t) = y(t,t_0,x_0)$ be solution processes of (2.5.21) and (2.5.22) through (t_0,x_0), respectively. The partial differential of $x(t,t_0,x_0)$ with respect to initial data is given in Example 2.5.1. Using the method of variational constants, Theorem (2.4.1) with $f(t,x) = A(t)x$, $\sigma(t,x) = \sum_{l=1}^{m} \sigma^l(t)x$ the solution of (2.5.22) is given by (2.5.23)

3 Method of Generalized Variation of Constants Formula: Relative Stability

3.1 Introduction

A closed form representation of a dynamic process of a nonlinear nonstationary solution process is not always possible or we might not be interested in the closed form solution [15]. In the absence of this, qualitative or quantitative properties are investigated for both the ordinary and stochastic dynamic systems [15, 16, 17, 19, 20]. A well known nonlinear technique [2-30] is to measure a dynamic flow by means of a suitable auxiliary measurement device, and then to use this measured dynamic flow to determine the desired information about the original dynamic flow [15, 16, 17, 20]. Employing energy function method as a measurement device and the fundamental properties of Itô-Doob type stochastic auxiliary system of differential equations [40], we establish the relationship between the solution processes of stochastic perturbed, auxiliary and nominal systems of differential equations. In addition, several estimates are obtained with regard to the deviation of solution process of perturbed system with respect to the solution process of nominal system of differential equations. Moreover, stability and relative stability results are developed to illustrate the usefulness of the results. Presented results generalize the existing results in a systematic and unified way [2-4, 6-28, 31].

Let us consider the following Itô-Doob type stochastic perturbed and auxiliary systems of differential equations

$$\begin{aligned} dx &= f(t,x)dt + \sigma(t,x)dw(t) \\ &= f(t,x)dt + \textstyle\sum_{l=1}^{m} \sigma^l(t,x)dw_l(t), \qquad x(t_0) = x_0, \end{aligned} \tag{3.1.1}$$

and

$$\begin{aligned} dz &= \alpha(t,z)dt + \beta(t,z)dw(t) \\ &= \alpha(t,z)dt + \textstyle\sum_{l=1}^{m} \beta^l(t,z)dw_l(t), \qquad z(t_0) = x_0, \end{aligned} \tag{3.1.2}$$

respectively, where $x, z \in R^n$; f, α, column vectors of σ and $\beta \in C[J \times R^n, R^n]$; $J = [t_0, t_0 + a)$, $a > 0$; α and β are twice continuously differentiable with respect to z; $w(t) = (w_1(t), w_2(t), ..., w_m(t))^T$ is an m-dimensional normalized Wiener process with independent increments; x_0 and $w(t)$ are mutually independent for each $t \geq t_0$; let $z(t, t_0, x_0)$ be the solution process of (3.1.2) existing for $t \geq t_0$; furthermore assume that its second derivative, $\frac{\partial^2}{\partial x_0 \partial x_0} z(t, t_0, x_0)$ is locally Lipschitzian in x_0 for each t, t_0.

3.2 Energy Function Method and Generalized Variation of Constants Formula

In the following, by using energy function method as a measurement device and the fundamental properties of solution of Itô-Doob type stochastic auxiliary system of differential equations, we develop several basic results in terms of a measure of solution process of stochastic auxiliary differential equations. The presented results extend and generalize results [12, 13, 14, 15, 16] in a systematic and unified way. Examples are given to illustrate the results.

Theorem 3.2.1 *Assume that: α and β in* (3.1.2) *are twice continuously differentiable with respect to z for fixed $t \geq t_0$, α_{zz} and β_{zz} are bounded with respect to z. Moreover,*

a) *$z(t, t_0, x_0)$ is the solution process of the stochastic auxiliary system of differential equations* (3.1.2) *existing for $t \geq t_0$;*

b) *$V \in C[R_+ \times R^n, R^N]$ and its partial derivatives V_t, V_x and V_{xx} exists and are continuous on $R_+ \times R^n$;*

c) *$x(t)$ are solution processes of* (3.1.1).

Then,

$$
\begin{aligned}
V(t, x(t)) &= V(t_0, z(t)) + \int_{t_0}^{t} LV(s, z(t, s, x(s)))ds \\
&\quad + \sum_{l=1}^{m} \int_{t_0}^{t} V_x(s, z(t, s, x(s))) b^l(t, s, x(s)) dw_l(s), \qquad (3.2.1)
\end{aligned}
$$

where

$$
\begin{aligned}
LV(s, z(t, s, x(s))) &= V_s(s, z(t, s, x(s))) \\
&+ V_x(s, z(t, s, x(s)))\Big[\frac{1}{2}\Big(\sum_{j=1}^{n}\sum_{l=1}^{m} \frac{\partial}{\partial x_0}\Phi_{ij}(t, s, x(s))[\beta^l(s, x(s))\beta_j^l(s, x(s)) \\
&- 2\beta^l(s, x(s))\sigma_j^l(s, x(s)) + \sigma^l(s, x(s))\sigma_j^l(s, x(s))]\Big)_{n\times 1} \\
&+ \Phi(t, s, x(s))[\sum_{l=1}^{m} \beta_x^l(s, x(s))[\beta^l(s, x(s)) - \sigma^l(s, x(s))] \\
&+ f(s, x(s)) - \alpha(s, x(s))]\Big] \\
&+ \frac{1}{2}\Big(tr\Big(\sum_{l=1}^{m} \frac{\partial^2}{\partial x \partial x} V_i(s, z(t, s, x(s))) b^l(t, s, x(s)) b^{lT}(t, s, x(s))\Big)\Big)_{N\times 1},
\end{aligned}
\qquad (3.2.2)
$$

$b^l(t, s, x(s)) = \Phi(t, s, x(s))[\sigma^l(s, x(s)) - \beta^l(s, x(s))]$, and $\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} z(t, t_0, x_0)$ is the fundamental matrix solution process of the variational auxiliary system [31].

Proof. Let $x(t)$ and $z(t)$ be solution processes of (3.1.1) and (3.1.2) through (t_0, x_0), respectively. Under the assumptions on the rate coefficients of auxiliary system of stochastic differential equations (3.1.2), we recall [40] the fundamental properties of solution process $z(t, t_0, x_0)$. For $t_0 \le s \le t$, we apply Itô-Doob differential formula to

$z(t,s,x(s))$ and $V(s,z(t,s,x(s)))$ with respect to s for fixed t and t_0, and obtain

$$
\begin{aligned}
d_s z(t,s,x(s)) &= \partial_{t_0} z(t,s,x(s)) + \partial_{x_0} z(t,s,x(s)) + \partial_{x_0}(\partial_{t_0} z(t,s,x(s)))\\
&+ \frac{1}{2}[\frac{\partial^2}{\partial x_0 \partial x_0} z(t,s,x(s)) \otimes dx(s)]dx(s)\\
&= \frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,x(s))\beta^l(s,x(s))\beta^l_j(s,x(s))\right)_{n\times 1} ds\\
&+ \Phi(t,s,x(s))\big[\sum_{l=1}^{m}\beta^l_x(s,x(s))\beta^l(s,x(s)) - \alpha(s,x(s))\big]ds\\
&- \sum_{l=1}^{m}\Phi(t,s,x(s))\beta^l(s,x(s))dw_l(s)\\
&+ \Phi(t,s,x(s))dx(s)\\
&- \left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,x(s))\beta^l(s,x(s))\sigma^l_j(s,x(s))\right)_{n\times 1} ds\\
&- \Phi(t,s,x(s))\sum_{l=1}^{m}\beta^l_x(s,x(s))\sigma^l(s,x(s))ds\\
&+ \frac{1}{2}\big[\frac{\partial}{\partial x_0}\Phi(t,s,x(s)) \otimes \sum_{l=1}^{m}\sigma^l(s,x(s))dw_l(s)\big]\sum_{l=1}^{m}\sigma^l(s,x(s))dw_l(s).
\end{aligned}
\tag{3.2.3}
$$

By simplifying (3.2.3), we get

$$
\begin{aligned}
d_s z(t,s,x(s)) &= \Big[\frac{1}{2}\Big(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,x(s))[\beta^l(s,x(s))\beta^l_j(s,x(s))\\
&\quad -2\beta^l(s,x(s))\sigma^l_j(s,x(s)) + \sigma^l(s,x(s))\sigma^l_j(s,x(s))]\Big)_{n\times 1}\\
&\quad +\Phi(t,s,x(s))[\sum_{l=1}^{m}\beta^l_x(s,x(s))[\beta^l(s,x(s)) - \sigma^l(s,x(s))]\\
&\quad +f(s,x(s)) - \alpha(s,x(s))]\Big]ds + \sum_{l=1}^{m} b^l(t,s,x(s))dw_l(s).
\end{aligned}
\tag{3.2.4}
$$

Similarly, using (3.2.4), we have

$$
\begin{aligned}
& d_s V(s, z(t,s,x(s))) \\
&= \partial_s V(s, z(t,s,x(s))) + \partial_x V(s, z(t,s,x(s))) \\
&\quad + \frac{1}{2}(V_{xx}(s, z(t,s,x(s))) \otimes d_s z(t,s,x(s))) d_s z(t,s,x(s)) \\
&= V_s(s, z(t,s,x(s)))ds \\
&\quad + V_x(s, z(t,s,x(s)))\Big[\frac{1}{2}\Big(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,x(s))[\beta^l(s,x(s))\beta^l_j(s,x(s)) \\
&\quad -2\beta^l(s,x(s))\sigma^l_j(s,x(s)) + \sigma^l(s,x(s))\sigma^l_j(s,x(s))]\Big)_{n\times 1} \\
&\quad + \Phi(t,s,x(s))[\sum_{l=1}^{m}\beta^l_x(s,x(s))[\beta^l(s,x(s)) - \sigma^l(s,x(s))] + f(s,x(s)) - \alpha(s,x(s))]ds \\
&\quad + \sum_{l=1}^{m} b^l(t,s,x(s))dw_l(s)\Big] \\
&\quad + \frac{1}{2}\Big(V_{xx}(s, z(t,s,x(s))) \otimes \sum_{l=1}^{m} b^l(t,s,x(s))dw_l(s)\Big)\sum_{l=1}^{m} b^l(t,s,x(s))dw_l(s).
\end{aligned}
\tag{3.2.5}
$$

Simplifying (3.2.5) yields,

$$
\begin{aligned}
& d_s V(s, z(t,s,x(s))) \\
&= \Big[V_s(s, z(t,s,x(s))) \\
&\quad + V_x(s, z(t,s,x(s)))\Big[\frac{1}{2}\Big(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,x(s))[\beta^l(s,x(s))\beta^l_j(s,x(s)) \\
&\quad -2\beta^l(s,x(s))\sigma^l_j(s,x(s)) + \sigma^l(s,x(s))\sigma^l_j(s,x(s))]\Big)_{n\times 1} \\
&\quad + \Phi(t,s,x(s))[\sum_{l=1}^{m}\beta^l_x(s,x(s))[\beta^l(s,x(s)) - \sigma^l(s,x(s))] + f(s,x(s)) - \alpha(s,x(s))]\Big]
\end{aligned}
$$

$$+\frac{1}{2}\left(tr(\sum_{l=1}^{m}\frac{\partial^2}{\partial x\partial x}V_i(s,z(t,s,x(s)))b^l(t,s,x(s))b^{lT}(t,s,x(s)))\right)_{N\times 1}\Big]ds$$

$$+\sum_{l=1}^{m}V_x(s,z(t,s,x(s)))b^l(t,s,x(s))dw_l(s)$$

$$= LV(s,z(t,s,x(s)))ds+\sum_{l=1}^{m}V_x(s,z(t,s,x(s)))b^l(t,s,x(s))dw_l(s). \tag{3.2.6}$$

Now, using the uniqueness of solution process of (3.1.2) and integrating both sides of (3.2.6) with respect to s from t_0 to t, we get

$$V(t,z(t,t,x(t)))-V(t_0,z(t,t_0,x(t_0)))$$
$$= \int_{t_0}^{t}LV(s,z(t,s,x(s)))ds+\sum_{l=1}^{m}\int_{t_0}^{t}V_x(s,z(t,s,x(s)))b^l(t,s,x(s))dw_l(s),$$

where $LV(s,z(t,s,x(s)))$ is defined in (3.2.2). Hence the desired result (3.2.1) follows. This completes the proof of the theorem.

■

Example 3.2.2 Consider a stochastic perturbed and auxiliary system of differential equations

$$dx=[A(t)x+p(t,x)]dt+[B(t)x+q(t,x)]dw(t), \qquad x(t_0)=x_0, \tag{3.2.7}$$

$$dz=A(t)zdt+B(t)zdw(t), \qquad z(t_0)=x_0, \tag{3.2.8}$$

respectively, where $x,z\in R^n$; A and B are any $n\times n$ continuous matrix functions defined on J; $J=[t_0,t_0+a), a>0$; p and q are any n-dimensional smooth functions defined on $J\times R^n$ into R^n that insure the existence of the solution processes of (3.2.7) and (3.2.8); for each $t\in J$, $w(t)$ is a scalar normalized Wiener process independent of x_0. If $V(t,x)=\frac{1}{2}\|x\|^2$, then $V_t(t,x)=0$, $\frac{\partial}{\partial x}V(t,x)=x^T$; $\frac{\partial^2}{\partial x\partial x}V(t,x)=I$, $n\times n$ identity matrix; $\frac{\partial}{\partial x_0}z(t,s,x(s))=\Phi(t,s)$; and $\frac{\partial^2}{\partial x_0^2}z(t,s,x(s))=0$ [15]. The generalized variation of constants formula in Theorem 3.2.1, reduces to a well-known result [11]

$$\begin{aligned}\|x(t)\|^2 &= \|z(t)\|^2 \\ &+\int_{t_0}^{t}[2x^T(s)\Phi^T(t,s)\Phi(t,s)[p(s,x(s))-B(s)q(s,x(s))]+\|c(s,x(s))\|^2]ds \\ &+2\int_{t_0}^{t}x^T(s)\Phi^T(t,s)\Phi(t,s)q(s,x(s))dw(s),\end{aligned}$$

where $c(s,x(s))=\Phi(t,s)q(s,x(s))$.

In the following, we present a few special cases of Theorem 3.2.1. These special cases exhibit the significance of Theorem 3.2.1.

Corollary 3.2.3 *Let the assumptions of Theorem 3.2.1 be satisfied.*

a) *If* $\beta\equiv 0$ *in* (3.1.2) *[19], then Theorem 3.2.1 reduces to*

$$\begin{aligned}V(t,x(t)) &= V(t_0,z(t))+\int_{t_0}^{t}LV(s,z(t,s,x(s)))ds \\ &+\sum_{l=1}^{m}\int_{t_0}^{t}V_x(s,z(t,s,x(s)))b^l(t,s,x(s))dw_l(s), \qquad (3.2.9)\end{aligned}$$

where

$$\begin{aligned}&LV(s,z(t,s,x(s))) \\ &= V_s(s,z(t,s,x(s))) \\ &+ V_x(s,z(t,s,x(s)))\Big[\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,x(s))\sigma^l(s,x(s))\sigma_j^l(s,x(s))\right)_{n\times 1} \\ &+ \Phi(t,s,x(s))[f(s,x(s))-\alpha(s,x(s))]\Big] \\ &+ \frac{1}{2}\left(tr\Big(\sum_{l=1}^{m}\frac{\partial^2}{\partial x\partial x}V_i(s,z(t,s,x(s)))b^l(t,s,x(s))b^{lT}(t,s,x(s))\Big)\right)_{N\times 1},\end{aligned}$$

and

$$b^l(t,s,x(s)) = \Phi(t,s,x(s))\sigma^l(s,x(s)).$$

b) *Furthermore, if* $\beta \equiv 0$*, and* $\alpha \equiv f$ *in* (3.1.1) *and* (3.1.2)*, then* (3.2.9) *reduces to a well known result [16, 19]*

$$\begin{aligned} V(t,x(t)) &= V(t_0,z(t)) + \int_{t_0}^{t} LV(s,z(t,s,x(s)))ds \\ &\quad + \sum_{l=1}^{m}\int_{t_0}^{t} V_x(s,z(t,s,x(s)))b^l(t,s,x(s))dw_l(s), \end{aligned} \tag{3.2.10}$$

where

$$\begin{aligned} &LV(s,z(t,s,x(s))) \\ &= V_s(s,z(t,s,x(s))) \\ &+ \frac{1}{2}V_x(s,z(t,s,x(s)))\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,x(s))\sigma^l(s,x(s))\sigma^l_j(s,x(s))\right)_{n\times 1} \\ &+ \frac{1}{2}\left(tr\Big(\sum_{l=1}^{m}\frac{\partial^2}{\partial x\partial x}V_i(s,z(t,s,x(s)))b^l(t,s,x(s))b^{lT}(t,s,x(s))\Big)\right)_{N\times 1}, \end{aligned}$$

and

$$b^l(t,s,x(s)) = \Phi(t,s,x(s))\sigma^l(s,x(s)).$$

c) *Moreover, if* $\beta \equiv 0$*, and* $\alpha \equiv 0$*, then* (3.2.9) *reduces to a well known result [19, 20]*

$$\begin{aligned} V(t,x(t)) &= V(t_0,x_0) + \int_{t_0}^{t} LV(s,x(s))ds \\ &\quad + \sum_{l=1}^{m}\int_{t_0}^{t} V_x(s,x(s))\sigma^l(s,x(s))dw_l(s), \end{aligned} \tag{3.2.11}$$

where

$$
\begin{aligned}
LV(s,x(s)) &= V_s(s,x(s)) + V_x(s,x(s))f(s,x(s)) \\
&+ \frac{1}{2}\left(tr\Big(\sum_{l=1}^{m}\frac{\partial^2}{\partial x\partial x}V_i(s,x(s))\sigma^l(t,s,x(s))\sigma^{lT}(t,s,x(s))\Big)\right)_{N\times 1}.
\end{aligned}
$$

d) *If* $V(t,x) = x$, $x \in R^n$, *in (b) and (c), respectively, then equation* (3.2.10) *and* (3.2.11) *reduces to well known Alekseev type nonlinear variation of constants formula for system of stochastic differential equations*(3.1.1) *[16, 19, 20]*

$$
\begin{aligned}
x(t,t_0,x_0) &= z(t,t_0,x_0) \\
&+ \frac{1}{2}\int_{t_0}^{t}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,x(s))\sigma^l(s,x(s))\sigma^l_j(s,x(s))\right)_{n\times 1} ds \\
&+ \sum_{l=1}^{m}\int_{t_0}^{t}\Phi(t,s,x(s))\sigma^l(s,x(s))dw_l(s)
\end{aligned}
$$

and

$$
x(t,t_0,x_0) = x_0 + \int_{t_0}^{t} f(s,x(s))ds + \sum_{l=1}^{m}\int_{t_0}^{t}\sigma^l(s,x(s))dw_l(s),
$$

respectively.

The following presented examples show the scope of Theorem 3.2.1. In fact, Theorem 3.2.1 extends and generalizes the results [14, 19] in a systematic and unified way.

Example 3.2.4 Let us choose $V(t,x) = \frac{1}{2}\|x\|^2$ in Corollary 3.2.3 (c) and (d). We note that $V_t(t,x) \equiv 0$, $\frac{\partial}{\partial x}V(t,x) = x^T$ and $\frac{\partial}{\partial x\partial x}V(t,x) = I$. In this case, (3.2.10) and

(3.2.11) reduces to

$$\begin{aligned}\|x(t)\|^2 &= \|z(t)\|^2 \\ &+ \int_{t_0}^{t} \Big[z^T(t,s,x(s)) \left(\sum_{l=1}^{m} \sum_{j=1}^{n} \frac{\partial}{\partial x_0} \Phi_{ij}(t,s,x(s)) \sigma^l(s,x(s)) \sigma_j^l(s,x(s)) \right)_{n \times 1} \\ &+ \sum_{l=1}^{m} tr(\Phi(t,s,x(s)) \sigma^l(s,x(s)) \sigma^{lT}(s,x(s)) \Phi^T(t,s,x(s))) \Big] ds \\ &+ 2 \int_{t_0}^{t} z^T(t,s,x(s)) \Phi(t,s,x(s)) \sigma(s,x(s)) dw(s)\end{aligned}$$

and

$$\begin{aligned}\|x(t)\|^2 &= \|x_0\|^2 + \int_{t_0}^{t} \Big[2x^T(s) f(s,x(s)) + \sum_{l=1}^{m} \sigma^l(s,x(s)) \sigma^{lT}(s,x(s)) \Big] ds \\ &\quad + \int_{t_0}^{t} 2x^T(s) \sigma(s,x(s)) dw(s),\end{aligned}$$

respectively.

Example 3.2.5 Consider the following stochastic scalar differential equation

$$dx = -\frac{1}{2} x^3 dt + \sigma(t,x) dw(t), \qquad x(t_0) = x_0,$$

where $\sigma \in C[J \times R, R]$, $w(t)$ is defined in Example 3.2.2, and the corresponding auxiliary system

$$dz = -\frac{1}{2} z^3 dt \qquad z(t_0) = x_0.$$

Here, we assume (without loss in generality) that $x_0 > 0$, then $z(t,t_0,x_0) = \frac{x_0}{\sqrt{1+(t-t_0)x_0^2}}$, $\Phi(t,t_0,x_0) = \frac{1}{(1+(t-t_0)x_0^2)^{3/2}}$ and $\frac{\partial}{\partial x_0}\Phi(t,t_0,x_0) = \frac{-3(t-t_0)x_0}{(1+(t-t_0)x_0^2)^{5/2}}$. Let $V(t,x) = x$, then $V_t(t,x) = 0$, $\frac{\partial}{\partial x}V(t,x) = 1$ and $\frac{\partial^2}{\partial x^2}V(t,x) = 0$. Applying Corollary 3.2.3 (b), we have

$$
\begin{aligned}
x(t) &= z(t) - \frac{3}{2}\int_{t_0}^{t} \frac{(t-s)x(s)\sigma^2(s,x(s))}{(1+(t-s)x^2(s))^{5/2}}ds \\
&+ \int_{t_0}^{t} \frac{\sigma(s,x(s))}{(1+(t-s)x^2(s))^{3/2}}dw(s).
\end{aligned}
$$

Example 3.2.6 Consider a scalar stochastic differential equation

$$
dx = \frac{x}{1+t}dt + \sigma(t,x)dw(t), \qquad x(t_0) = x_0,
$$

where σ and $w(t)$ are as defined in Example 3.2.5, and an auxiliary system

$$
dz = \frac{z}{1+t}dt \qquad z(t_0) = x_0.
$$

Here $z(t,t_0,x_0) = \frac{1+t}{1+t_0}x_0$, $\Phi(t,t_0) = \frac{1+t}{1+t_0}$ and $\frac{\partial}{\partial x_0}\Phi(t,t_0) = 0$. Let $V(t,x) = x$, then $V_t(t,x) = 0$, $\frac{\partial}{\partial x}V(t,x) = 1$ and $\frac{\partial^2}{\partial x^2}V(t,x) = 0$. Applying Corollary 3.2.3 (b), we have

$$
\begin{aligned}
x(t) &= z(t) + \int_{t_0}^{t} \frac{1+t}{1+s}\sigma(s,x(s))dw(s) \\
&= \frac{1+t}{1+t_0}x_0 + \int_{t_0}^{t} \frac{1+t}{1+s}\sigma(s,x(s))dw(s).
\end{aligned}
$$

In the following, we present another special cases of Theorem 3.2.1. These special cases shows the importance of Theorem 3.2.1. Moreover, it extends the results presented in [15] in a systematic way.

Corollary 3.2.7 *Under the assumption of Theorem 3.2.1,*

a) *If $\alpha = 0$ in (3.1.2), then*

$$\begin{aligned} V(t,x(t)) &= V(t_0,z(t)) + \int_{t_0}^{t} LV(s,z(t,s,x(s)))ds \\ &\quad + \sum_{l=1}^{m}\int_{t_0}^{t} V_x(s,z(t,s,x(s)))b^l(t,s,x(s))dw_l(s), \end{aligned}$$

where

$$\begin{aligned} & LV(s,z(t,s,x(s))) \\ &= V_s(s,z(t,s,x(s))) \\ &\quad +V_x(s,z(t,s,x(s)))\Big[\frac{1}{2}\Big(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,x(s))[\beta^l(s,x(s))\beta^l_j(s,x(s)) \\ &\quad -2\beta^l(s,x(s))\sigma^l_j(s,x(s)) + \sigma^l(s,x(s))\sigma^l_j(s,x(s))]\Big)_{n\times 1} \\ &\quad +\Phi(t,s,x(s))[\sum_{l=1}^{m}\beta^l_x(s,x(s))[\beta^l(s,x(s)) - \sigma^l(s,x(s))] + f(s,x(s))]\Big] \\ &\quad +\frac{1}{2}\left(tr(\sum_{l=1}^{m}\frac{\partial^2}{\partial x\partial x}V_i(s,z(t,s,x(s)))b^l(t,s,x(s))b^{lT}(t,s,x(s)))\right)_{N\times 1}, \end{aligned}$$

and

$$b^l(t,s,x(s)) = \Phi(t,s,x(s))[\sigma^l(s,x(s)) - \beta^l(s,x(s))].$$

b) *If $\alpha = 0$ and $\beta = \sigma$ in (3.1.2), then Corollary 3.2.7 (a) reduces to*

$$\begin{aligned} V(t,x(t)) &= V(t_0,z(t)) \\ &\quad + \int_{t_0}^{t}[V_s(s,z(t,s,x(s))) + V_x(s,z(t,s,x(s)))\Phi(t,s,x(s))f(s,x(s))]ds. \end{aligned} \tag{3.2.12}$$

Moreover, if $V(t,x) = x$, then (3.2.12) reduces to

$$x(t, t_0, x_0) = z(t, t_0, x_0) + \int_{t_0}^{t} \Phi(t, s, x(s)) f(s, x(s)) ds.$$

Remark 3.2.8 *We note that the integral equations in Corollary 3.2.7 (b) and (c) are with stochastic process varying coefficient functions.*

3.3 Robustness of Solution Process

Let us consider a nominal system of Itô-Doob type stochastic differential equations

$$\begin{aligned} dy &= G(t, y)dt + H(t, y)dw(t) \\ &= G(t, y)dt + \textstyle\sum_{l=1}^{m} H^l(t, y)dw_l(t), \qquad y(t_0) = y_0, \end{aligned} \tag{3.3.1}$$

with respect to (3.1.1), where G and column vectors of $H \in C[J \times R^n, R^n]$, $J = [t_0, t_0 + a)$ for $a > 0$; $w(t) = (w_1(t), w_2(t), ..., w_m(t))^T$ is an m-dimensional normalized Wiener process with independent increments; y_0 and $w(t)$ are mutually independent for each $t \geq t_0$. Let $x(t, t_0, x_0)$ and $y(t, t_0, y_0)$ be the solution process of (3.1.1) and (3.3.1) existing for $t \geq t_0$, respectively.

The following theorem provides a deviation of the solution process of perturbed system of stochastic differential equations(3.1.1) with respect to the solution process of nominal system of stochastic differential equations (3.3.1) through (t_0, x_0) and (t_0, y_0), respectively.

Theorem 3.3.1 *Assume that all the hypothesis of Theorem 3.2.1 hold. Furthermore,* *let* $y(t) = y(t, t_0, y_0)$ *be a solution process of* (3.3.1) *through* (t_0, y_0). *Then,*

$$\begin{aligned} V(t, x(t) - y(t)) &= V(t_0, z(t, t_0, x_0) - z(t, t_0, y_0)) + \int_{t_0}^{t} LV(s, \Delta z) ds \\ &\quad + \sum_{l=1}^{m} \int_{t_0}^{t} V_x(s, \Delta z) b^l(t, s, x(s), y(s)) dw_l(s), \end{aligned} \tag{3.3.2}$$

where

$$\begin{aligned} LV(s, \Delta z) &= V_s(s, \Delta z) \\ &+ V_x(s, \Delta z)\Big[\frac{1}{2}\Big(\sum_{j=1}^{n}\sum_{l=1}^{m}[\frac{\partial}{\partial x_0}\Phi_{ij}(t, s, x(s))\beta^l(s, x(s))\beta^l_j(s, x(s)) \\ &- \frac{\partial}{\partial x_0}\Phi_{ij}(t, s, y(s))\beta^l(s, y(s))\beta^l_j(s, y(s))]\Big)_{n\times 1} \\ &+ \Big(\sum_{j=1}^{n}\sum_{l=1}^{m}[\frac{\partial}{\partial x_0}\Phi_{ij}(t, s, y(s))\beta^l(s, y(s))H^l_j(s, y(s)) \\ &- \frac{\partial}{\partial x_0}\Phi_{ij}(t, s, x(s))\beta^l(s, x(s))\sigma^l_j(s, x(s))]\Big)_{n\times 1} \\ &+ \frac{1}{2}\Big(\sum_{j=1}^{n}\sum_{l=1}^{m}[\frac{\partial}{\partial x_0}\Phi_{ij}(t, s, x(s))\sigma^l(s, x(s))\sigma^l_j(s, x(s)) \\ &- \frac{\partial}{\partial x_0}\Phi_{ij}(t, s, y(s))H^l(s, y(s))H^l_j(s, y(s))]\Big)_{n\times 1} \\ &+ \Phi(t, s, x(s))[\sum_{l=1}^{m}\beta^l_x(s, x(s))[\beta^l(s, x(s)) - \sigma^l(s, x(s))] + f(s, x(s)) - \alpha(s, x(s))] \\ &- \Phi(t, s, y(s))[\sum_{l=1}^{m}\beta^l_x(s, y(s))[\beta^l(s, y(s)) - H^l(s, y(s))] + G(s, y(s)) - \alpha(s, y(s))]\Big] \\ &+ \frac{1}{2}\Big(tr(\sum_{l=1}^{m}\frac{\partial^2}{\partial x\partial x}V_i(s, \Delta z)b^l(t, s, x(s), y(s))b^{lT}(t, s, x(s), y(s)))\Big)_{N\times 1}, \end{aligned}$$

$b^l(t, s, x(s), y(s)) = \Phi(t, s, x(s))[\sigma^l(s, x(s)) - \beta^l(s, x(s))] - \Phi(t, s, y(s))[H^l(s, y(s)) - \beta^l(s, y(s))]$ *and* $\Delta z = z(t, s, x(s)) - z(t, s, y(s))$.

Proof. By following the proof of Theorem 3.2.1 with $\Delta z = z(t,s,x(s)) - z(t,s,y(s))$, we have

$$
\begin{aligned}
& d_s V(s,\Delta z) \\
&= V_s(s,\Delta z)ds + V_x(s,\Delta z)d_s\Delta z + \frac{1}{2}[V_{xx}(s,\Delta z)\otimes d_s\Delta z]d_s\Delta z \\
&= V_s(s,\Delta z)ds + V_x(s,\Delta z)\Big[\frac{1}{2}\Big(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,x(s))[\beta^l(s,x(s))\beta^l_j(s,x(s)) \\
&\quad -2\beta^l(s,x(s))\sigma^l_j(s,x(s)) + \sigma^l(s,x(s))\sigma^l_j(s,x(s))]\Big)_{n\times 1} \\
&\quad -\frac{1}{2}\Big(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))[\beta^l(s,y(s))\beta^l_j(s,y(s)) \\
&\quad -2\beta^l(s,y(s))H^l_j(s,y(s)) + H^l(s,y(s))H^l_j(s,y(s))]\Big)_{n\times 1} \\
&\quad +\Phi(t,s,x(s))[\sum_{l=1}^{m}\beta^l_x(s,x(s))[\beta^l(s,x(s)) - \sigma^l(s,x(s))] + f(s,x(s)) - \alpha(s,x(s))] \\
&\quad -\Phi(t,s,y(s))[\sum_{l=1}^{m}\beta^l_x(s,y(s))[\beta^l(s,y(s)) - H^l(s,y(s))] + G(s,y(s)) - \alpha(s,y(s))]\Big]ds \\
&\quad +\sum_{l=1}^{m}V_x(s,\Delta z)b^l(t,s,x(s),y(s))dw_l(s) \\
&\quad +\frac{1}{2}[V_{xx}(s,\Delta z)\otimes\sum_{l=1}^{m}b^l(t,s,x(s),y(s))dw_l(s)]\sum_{l=1}^{m}b^l(t,s,x(s),y(s))dw_l(s)
\end{aligned}
$$

Hence,

$$
d_s V(s,\Delta z) = LV(s,\Delta z)ds + \sum_{l=1}^{m}V_x(s,\Delta z)b^l(t,s,x(s),y(s))dw_l(s). \tag{3.3.3}
$$

By integrating both sides of (3.3.3) with respect to s from t_0 to t, we establish the result (3.3.2).

∎

Example 3.3.2 We consider a stochastic perturbed, auxiliary and nominal system of differential equations

$$\begin{aligned} dx &= [(A(t)+\Delta A(t))x + a(t) + \Delta a(t) + p(t,x)]dt \\ &\quad +[(B(t)+\Delta B(t))x + b(t) + \Delta b(t) + q(t,x)]dw(t), \qquad x(t_0)=x_0, \end{aligned} \tag{3.3.4}$$

$$dz = [A(t)z + a(t)]dt + [B(t)z + b(t)]dw(t), \qquad z(t_0) = x_0, \tag{3.3.5}$$

and

$$dy = [A(t)y + a(t) + p(t,y)]dt + [B(t)y + b(t) + q(t,y)]dw(t), \qquad y(t_0) = y_0, \tag{3.3.6}$$

respectively, where $x, y, z \in R^n$; A, B, p and q are defined in Example 3.2.2; $a, b, \Delta a$ and Δb are any n-dimensional smooth functions defined on J into R^n; ΔA and ΔB are any $n \times n$ smooth matrix functions defined on J; $J, w(t), x_0$, and y_0 are defined in Example 3.2.2. We note that $\Phi(t,s,x(s)) = \Phi(t,s,y(s)) = \Phi(t,s)$. We apply Theorem 3.3.1 to (3.3.4) in the context of (3.3.5) and (3.3.6) with $N = 1$ (scalar function), we have

$$\begin{aligned} V(t, x(t) - y(t)) &= V(t_0, z(t,t_0,x_0) - z(t,t_0,y_0)) + \int_{t_0}^{t} LV(s,\Delta z)ds \\ &\quad + \int_{t_0}^{t} V_x(s,\Delta z)\Phi(t,s)[\Delta B(s)x(s) \\ &\quad + q(s,x(s)) - q(s,y(s)) + \Delta b(s)]dw(s), \end{aligned} \tag{3.3.7}$$

where

$$
\begin{aligned}
&LV(s,\Delta z)\\
&= V_s(s,\Delta z)+V_x(s,\Delta z)\Phi(t,s)[(\Delta A(s)-B(s)\Delta B(s))x(s)+p(s,x(s))-p(s,y(s))\\
&\quad -B(s)[q(s,x(s))-q(s,y(s))]+\Delta a(s)-B(s)\Delta b(s)]\\
&\quad +\frac{1}{2}(tr(\frac{\partial^2}{\partial x^2}V(s,\Delta z)c(t,s,x(s),y(s))c^T(t,s,x(s),y(s)))),
\end{aligned}
$$

$c(t,s,x(s),y(s)) = \Phi(t,s)[\Delta B(s)x(s)+q(s,x(s))-q(s,y(s))+\Delta b(s)]$ and $\Delta z = z(t,s,x(s))-z(t,s,y(s))$.

Moreover, in the absence of Δa and Δb, (3.3.7) reduces to

$$
\begin{aligned}
V(t,x(t)-y(t)) &= V(t_0,z(t,t_0,x_0)-z(t,t_0,y_0))+\int_{t_0}^{t}LV(s,\Delta z)ds\\
&\quad +\int_{t_0}^{t}V_x(s,\Delta z)\Phi(t,s)[\Delta B(s)x(s)+q(s,x(s))-q(s,y(s))]dw(s),
\end{aligned}
\tag{3.3.8}
$$

where

$$
\begin{aligned}
&LV(s,\Delta z)\\
&= V_s(s,\Delta z)+V_x(s,\Delta z)\Phi(t,s)[(\Delta A(s)-B(s)\Delta B(s))x(s)+p(s,x(s))-p(s,y(s))\\
&\quad +B(s)[q(s,y(s))-q(s,x(s))]]\\
&\quad +\frac{1}{2}(tr(\frac{\partial^2}{\partial x^2}V(s,\Delta z)c(t,s,x(s),y(s))c^T(t,s,x(s),y(s)))),
\end{aligned}
$$

$c(t,s,x(s),y(s)) = \Phi(t,s)[\Delta B(s)x(s)+q(s,x(s))-q(s,y(s))]$ and $\Delta z = z(t,s,x(s))-z(t,s,y(s))$.

Example 3.3.3 For $V(t,x) = \frac{1}{2}\|x\|^2$ and imitating the argument used in Example 3.2.2, (3.3.7) reduces to

$$
\begin{aligned}
\|x(t)-y(t)\|^2 &= \|z(t,t_0,x_0)-z(t,t_0,y_0)\|^2 + 2\int_{t_0}^{t} LV(s,\Delta z)ds \\
&\quad +2\int_{t_0}^{t}(x(s)-y(s))^T\Phi^T(t,s)\Phi(t,s)[\Delta B(s)x(s) \\
&\quad +q(s,x(s))-q(s,y(s))+\Delta b(s)]dw(s),
\end{aligned}
$$

where

$$
\begin{aligned}
&LV(s,\Delta z) \\
&= (x(s)-y(s))^T\Phi^T(t,s)\Phi(t,s)[(\Delta A(s)-B(s)\Delta B(s))x(s)+p(s,x(s))-p(s,y(s)) \\
&\quad -B(s)[q(s,x(s))-q(s,y(s))]+\Delta a(s)-B(s)\Delta b(s)] \\
&\quad +\frac{1}{2}c(t,s,x(s),y(s))c^T(t,s,x(s),y(s)),
\end{aligned}
$$

$c(t,s,x(s),y(s)) = \Phi(t,s)[\Delta B(s)x(s) + q(s,x(s)) - q(s,y(s)) + \Delta b(s)]$ and $\Delta z = z(t,s,x(s)) - z(t,s,y(s))$.

Moreover, for $\Delta a \equiv 0 \equiv \Delta b$, (3.3.8) reduces to

$$
\begin{aligned}
\|x(t)-y(t)\|^2 &= \|z(t,t_0,x_0)-z(t,t_0,y_0)\|^2 + 2\int_{t_0}^{t} LV(s,\Delta z)ds \\
&\quad +2\int_{t_0}^{t}(x(s)-y(s))^T\Phi^T(t,s)\Phi(t,s)[\Delta B(s)x(s) \\
&\quad +q(s,x(s))-q(s,y(s))]dw(s),
\end{aligned}
$$

where

$$
\begin{aligned}
&LV(s,\Delta z)\\
&= (x(s)-y(s))^T\Phi^T(t,s)\Phi(t,s)[(\Delta A(s)-B(s)\Delta B(s))x(s)+p(s,x(s))-p(s,y(s))\\
&\quad -B(s)[q(s,x(s))-q(s,y(s))]]+\frac{1}{2}c(t,s,x(s),y(s))c^T(t,s,x(s),y(s)),
\end{aligned}
$$

and c is defined in (3.3.8).

The following corollary provides a deviation of the solution process of the perturbed system(3.1.1) with respect to the solution process of the nominal system (3.3.1).

Corollary 3.3.4 *Under the hypotheses of Theorem 3.3.1 with* $\beta \equiv 0$ *in* (3.1.2)*,* $H \equiv 0$ *in* (3.3.1) *and* $G \equiv f$*, then Theorem 3.3.1 reduces to*

$$
\begin{aligned}
V(t,x(t)-y(t)) &= V(t_0, z(t,t_0,x_0)-z(t,t_0,y_0))+\int_{t_0}^{t} LV(s,\Delta z)ds\\
&\quad +\sum_{l=1}^{m}\int_{t_0}^{t} V_x(s,\Delta z)\Phi(t,s,x(s))\sigma^l(s,x(s))dw_l(s), \quad (3.3.9)
\end{aligned}
$$

where

$$
\begin{aligned}
&LV(s,\Delta z)\\
&= V_s(s,\Delta z)+V_x(s,\Delta z)\Big[\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,x(s))\sigma^l(s,x(s))\sigma_j^l(s,x(s))\right)_{n\times 1}\\
&\quad +\Phi(t,s,x(s))[f(s,x(s))-\alpha(s,x(s))]-\Phi(t,s,y(s))[f(s,y(s))-\alpha(s,y(s))]\Big]\\
&\quad +\frac{1}{2}\left(tr\Big(\sum_{l=1}^{m}\frac{\partial^2}{\partial x\partial x}V_i(s,\Delta z)b^{lT}(t,s,x(s))b^l(t,s,x(s))\Big)\right)_{N\times 1},
\end{aligned}
$$

$b^l(t,s,x(s)) = \Phi(t,s,x(s))\sigma^l(s,x(s))$*, and* $\Delta z = z(t,s,x(s)) - z(t,s,y(s))$*.*

Example 3.3.5 Under the assumptions of Corollary 3.3.4 and $V(t,x) = x$, (3.3.9) becomes

$$\begin{aligned}
&x(t) - y(t) \\
= \;& z(t, t_0, x_0) - z(t, t_0, y_0) + \int_{t_0}^{t} [\frac{1}{2} \left(\sum_{j=1}^{n} \sum_{l=1}^{m} \frac{\partial}{\partial x_0} \Phi_{ij}(t, s, x(s)) \sigma^l(s, x(s)) \sigma_j^l(s, x(s)) \right)_{n \times 1} \\
&+ \Phi(t, s, x(s))[f(s, x(s)) - \alpha(s, x(s))] - \Phi(t, s, y(s))[f(s, y(s)) - \alpha(s, y(s))]]ds \\
&+ \sum_{l=1}^{m} \int_{t_0}^{t} \Phi(t, s, x(s)) \sigma^l(s, x(s)) dw_l(s).
\end{aligned}$$

The following result provides an expression for the difference between the solution processes of (3.1.1) with solution processes of (3.3.1).

Corollary 3.3.6 *Suppose all the hypotheses of Theorem 3.3.1 hold. Assume that* $\alpha = G$ *and* $H = \sigma = \beta = 0$, *then*

$$\begin{aligned}
&V(t, x(t) - y(t)) - V(t_0, z(t, t_0, x_0) - z(t, t_0, y_0)) \\
= \;& \int_{t_0}^{t} V_s(s, z(t, s, x(s)) - z(t, s, y(s))) \\
&+ V_x(s, z(t, s, x(s)) - z(t, s, y(s))) \Phi(t, s, x(s))[f(s, x(s)) - \alpha(s, x(s))]ds.
\end{aligned} \tag{3.3.10}$$

Example 3.3.7 For $V(t, x) = \|x\|^2$, under the conditions of Corollary 3.3.6, (3.3.10) reduces to

$$\begin{aligned}
\|x(t) - y(t)\|^2 = \;& \|z(t, t_0, x_0) - z(t, t_0, y_0)\|^2 \\
&+ 2 \int_{t_0}^{t} (z(t, s, x(s)) - z(t, s, y(s)))^T \Phi(t, s, x(s))[f(s, x(s)) - \alpha(s, x(s))]ds.
\end{aligned}$$

The following theorem provides another version for a deviation of a solution process of the perturbed system(3.1.1) with respect to a solution process of nominal system (3.3.1).

Theorem 3.3.8 *Assume that all the hypothesis of Theorem 3.3.1 hold. Furthermore,*

$x(t) = x(t, t_0, x_0)$ and $y(t) = y(t, t_0, y_0)$ are the solution process of (3.1.1) *and* (3.3.1) *through (t_0, x_0) and (t_0, y_0), respectively. Then,*

$$\begin{aligned} V(t, n(t)) &= V(t_0, z(t, t_0, n_0)) + \int_{t_0}^{t} LV(s, z(t, s, n(s)))ds \\ &\quad + \sum_{l=1}^{m} \int_{t_0}^{t} V_x(s, z(t, s, n(s)))b^l(t, s, x(s), y(s))dw_l(s), \end{aligned} \tag{3.3.11}$$

where

$$\begin{aligned} &LV(s, z(t, s, n(s))) \\ &= V_s(s, z(t, s, n(s))) \\ &\quad +V_x(s, z(t, s, n(s)))\Big[\frac{1}{2}\Big(\sum_{j=1}^{n}\sum_{l=1}^{m} \frac{\partial}{\partial x_0}\Phi_{ij}(t, s, n(s))[\beta^l(s, n(s))\beta^l_j(s, n(s)) \\ &\quad -2\beta^l(s, n(s))(\sigma^l_j(s, x(s)) - H^l_j(s, y(s))) \\ &\quad +(\sigma^l(s, x(s)) - H^l(s, y(s)))(\sigma^l_j(s, x(s)) - H^l_j(s, y(s)))\Big)_{n\times 1} \\ &\quad +\Phi(t, s, n(s))[\sum_{l=1}^{m} \beta^l_x(s, n(s))[\beta^l(s, n(s)) - \sigma^l(s, x(s)) + H^l(s, y(s))] \\ &\quad +f(s, x(s)) - G(s, y(s)) - \alpha(s, n(s))]\Big] \\ &\quad +\frac{1}{2}\left(tr(\sum_{l=1}^{m} \frac{\partial^2}{\partial x \partial x} V_i(s, n(s))b^l(t, s, x(s), y(s))b^{lT}(t, s, x(s), y(s)))\right)_{N\times 1}, \end{aligned}$$

$b^l(t, s, x(s), y(s)) = \Phi(t, s, n(s))[\sigma^l(s, x(s)) - H^l(s, y(s)) - \beta^l(s, n(s))]$, and $n(s) = x(s) - y(s)$.

Proof. By following the proof of Theorem 3.2.1, we have

$$
\begin{aligned}
& d_s V(s, z(t,s,n(s))) \\
= \; & V_s(s, z(t,s,n(s)))ds + V_x(s, z(t,s,n(s)))d_s z(t,s,n(s)) \\
& + \frac{1}{2}[V_{xx}(s, z(t,s,n(s))) \otimes d_s(z(t,s,n(s)))]d_s(z(t,s,n(s))) \\
= \; & V_s(s, z(t,s,n(s)))ds \\
& + V_x(s, z(t,s,n(s)))\Big[\frac{1}{2}\Big(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,n(s))[\beta^l(s,n(s))\beta_j^l(s,n(s)) \\
& - 2\beta^l(s,n(s))(\sigma_j^l(s,x(s)) - H_j^l(s,y(s))) \\
& + (\sigma^l(s,x(s)) - H^l(s,y(s)))(\sigma_j^l(s,x(s)) - H_j^l(s,y(s)))\Big)_{n\times 1} \\
& + \Phi(t,s,n(s))[\sum_{l=1}^{m}\beta_x^l(s,n(s))[\beta^l(s,n(s)) - \sigma^l(s,x(s)) + H^l(s,y(s))] \\
& + f(s,x(s)) - G(s,y(s)) - \alpha(s,n(s))]\Big]ds \\
& + \sum_{l=1}^{m} V_x(s, z(t,s,n(s)))b^l(t,s,x(s),y(s))dw_l(s) \\
& + \frac{1}{2}V_{xx}(s, z(t,s,n(s))) \otimes \sum_{l=1}^{m} b^l(t,s,x(s),y(s))dw_l(s)\sum_{l=1}^{m} b^l(t,s,x(s),y(s))dw_l(s) \\
= \; & LV(s, z(t,s,n(s)))ds + \sum_{l=1}^{m} V_x(s, z(t,s,n(s)))b^l(t,s,x(s),y(s))dw_l(s).
\end{aligned}
\tag{3.3.12}
$$

Integrating both sides of (3.3.12) with respect to s from t_0 to t, we have(3.3.11). ■

The following corollary provides a deviation of the solution process of the perturbed system(3.1.1) with respect to the solution process of the nominal system(3.3.1).

Corollary 3.3.9 *Let the hypotheses of Theorem 3.3.8 be satisfied. If* $\beta \equiv 0$ *in* (3.1.2),

$H \equiv 0$ in (3.3.1) and $G \equiv \alpha$, then

$$
\begin{aligned}
& V(t, x(t)-y(t)) - V(t_0, z(t, t_0, x_0 - y_0)) \\
&\quad = \int_{t_0}^{t} LV(s, z(t, s, x(s) - y(s)))ds \\
&\qquad + \sum_{l=1}^{m} \int_{t_0}^{t} V_x(s, z(t, s, x(s) - y(s))) b^l(t, s, x(s), y(s)) dw_l(s),
\end{aligned}
\tag{3.3.13}
$$

where

$$
\begin{aligned}
& LV(s, z(t, s, x(s) - y(s))) \\
&\quad = V_s(s, z(t, s, x(s) - y(s))) + V_x(s, z(t, s, x(s) - y(s))) \\
&\qquad \Big[\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m} \frac{\partial}{\partial x_0}\Phi_{ij}(t, s, x(s) - y(s))\sigma^l(s, x(s))\sigma_j^l(s, x(s))\right)_{n\times 1} \\
&\qquad + \Phi(t, s, x(s) - y(s))[f(s, x(s)) - \alpha(s, y(s)) - \alpha(s, x(s) - y(s))]\Big] \\
&\qquad + \frac{1}{2}\left(tr(\sum_{l=1}^{m} \frac{\partial^2}{\partial x \partial x} V_i(s, \Delta z) b^l(t, s, x(s), y(s)) b^{lT}(t, s, x(s), y(s)))\right)_{N\times 1}
\end{aligned}
$$

and $b^l(t, s, x(s), y(s)) = \Phi(t, s, x(s) - y(s))\sigma^l(s, x(s))$.

Example 3.3.10 For $V(t, x) = \|x\|^2$, equation (3.3.13) in Corollary 3.3.9 becomes

$$
\begin{aligned}
& \|x(t) - y(t)\|^2 - \|z(t, t_0, x_0 - y_0)\|^2 \\
&\quad = \int_{t_0}^{t} LV(s, z(t, s, x(s) - y(s)))ds \\
&\qquad + \int_{t_0}^{t} 2z^T(t, s, x(s) - y(s)) b(t, s, x(s), y(s)) dw(s),
\end{aligned}
$$

where

$$
\begin{aligned}
& LV(s, z(t, s, x(s) - y(s))) \\
= \quad & 2z^T(t, s, x(s) - y(s))\Big[\frac{1}{2}\left(\sum_{j=1}^{n} \frac{\partial}{\partial x_0} \Phi_{ij}(t, s, x(s) - y(s))\sigma(s, x(s))\sigma_j(s, x(s))\right)_{n\times 1} \\
& + \Phi(t, s, x(s) - y(s))[f(s, x(s)) - \alpha(s, y(s)) - \alpha(s, x(s) - y(s))]\Big] \\
& + \frac{1}{2} b(t, s, x(s), y(s)) b^T(t, s, x(s), y(s))
\end{aligned}
$$

and $b(t, s, x(s), y(s)) = \Phi(t, s, x(s) - y(s))\sigma(s, x(s))$.

Example 3.3.11 We apply Theorem 3.3.8 to Example 3.3.2, and we have

$$
\begin{aligned}
V(t, n(t)) \quad = \quad & V(t_0, z(t, t_0, n_0)) + \int_{t_0}^{t} LV(s, z(t, s, n(s)))ds \\
& + \int_{t_0}^{t} V_x(s, z(t, s, n(s)))\Phi(t, s)[\Delta B(s)x(s) \\
& + q(s, x(s)) - q(s, y(s)) + \Delta b(s) - b(s)]dw(s), \qquad (3.3.14)
\end{aligned}
$$

where

$$
\begin{aligned}
LV(s, z(t, s, n(s))) \quad = \quad & V_s(s, z(t, s, n(s))) + V_x(s, z(t, s, n(s)))\Phi(t, s)\big[[\Delta A(s) \\
& - B(s)\Delta B(s)]x(s) \\
& + p(s, x(s)) - p(s, y(s)) - B(s)[q(s, x(s)) - q(s, y(s))] \\
& + B(s)(b(s) - \Delta b(s)) + \Delta a(s) - a(s)\big] \\
& + \frac{1}{2} tr(\sum_{k=1}^{n}\sum_{j=1}^{n} \frac{\partial^2}{\partial x \partial x} V(s, \Delta z) c(t, s, x(s), y(s)) c^T(t, s, x(s), y(s))),
\end{aligned}
$$

$c(t, s, x(s), y(s)) = \Phi(t, s)[\Delta B(s)x(s) + q(s, x(s)) - q(s, y(s)) + \Delta b(s) - b(s)]$, and $n(s) = x(s) - y(s)$.

Example 3.3.12 If $V(t,x) = \frac{1}{2}\|x\|^2$, then following the argument used in Example 3.2.2, Example 3.3.11 reduces to

$$
\begin{aligned}
\|x(t) - y(t)\|^2 &= \|z(t,t_0,x_0 - y_0)\|^2 + 2\int_{t_0}^{t} LV(s, z(t,s,x(s) - y(s)))ds \\
&\quad + 2\int_{t_0}^{t} (x(s) - y(s))^T \Phi^T(t,s)\Phi(t,s)[\Delta B(s)x(s) \\
&\quad + q(s,x(s)) - q(s,y(s)) + \Delta b(s) - b(s)]dw(s),
\end{aligned}
$$

where

$$
\begin{aligned}
& LV(s, z(t,s,x(s) - y(s))) \\
&= (x(s) - y(s))^T \Phi^T(t,s)\Phi(t,s)[(\Delta A(s) - B(s)\Delta B(s))x(s) + p(s,x(s)) - p(s,y(s)) \\
&\quad - B(s)[q(s,x(s)) - q(s,y(s))] + B(s)(b(s) - \Delta b(s)) + \Delta a(s) - a(s)] \\
&\quad + \frac{1}{2} c(t,s,x(s),y(s)) c^T(t,s,x(s),y(s)),
\end{aligned}
$$

$$
c(t,s,x(s),y(s)) = \Phi(t,s)[\Delta B(s)x(s) + q(s,x(s)) - q(s,y(s)) + \Delta b(s) - b(s)].
$$

3.4 Stability Analysis

In this section, stability results are constructed in the context of the method of generalized variation of constants parameters. For this purpose, it is assumed that f and σ in (3.1.1) satisfy desired conditions to insure that the IVP (3.1.1) has p-th order solution process.

Theorem 3.4.1 *Let the hypotheses of Theorem 3.2.1 be satisfied. Furthermore, we assume that*

a) $b(\|x\|^p) \leq \sum_{i=1}^{N} |V_i(t,x)| \leq a(\|x\|^p)$ *for all* $(t,x) \in R_+ \times R^n$ *where* $p \geq 1$, $b \in \mathcal{VK}$

and $a \in \mathcal{CK}$;

b) $f(t,0) \equiv 0$, $\sigma(t,0) \equiv 0$, $\alpha(t,0) \equiv 0$ *and* $\beta(t,0) \equiv 0$ *for* $t \in R_+$*;*

c) $E[\sum_{l=1}^{m} V_x(s, z(t,s,x(s)))\Phi(t,s,x(s))[\sigma^l(t,s,x(s)) - \beta^l(t,s,x(s))]]$ *exists for* $t \geq t_0$*;*

d) $E[\sum_{i=1}^{N} |LV_i(s, z(t,s,x(s)))|] \leq \lambda(s)\sum_{i=1}^{N} E[|V_i(s,x(s))|]$ *for* $t_0 \leq s \leq t$ *and* $E[\|x(s)\|^p] \leq \rho$*, where* $LV(s, z(t,s,x))$ *defined in* (3.2.2) *and* $z(t,s,x)$ *is the solution process of* (3.1.2) *through* (s,x)*,* $\rho > 0$ *and* $\lambda \in C[R_+, R_+] \cap L^1[R_+, R_+]$*;*

e) $E[\sum_{i=1}^{N} |V_i(t_0, z(t))|] \leq \mu(E[\|x_0\|^p])$*, whenever* $E[\|x_0\|^p] \leq \rho$ *for some* $\rho > 0$*, where* $\mu \in \mathcal{CK}$*.*

Then the trivial solution process of (3.1.1) *is stable in the p-th mean.*

Proof. Let $x(t,t_0,x_0)$ and $z(t,s,x(s))$ be a solution process of (3.1.1) and (3.1.2) through (t_0,x_0) and $(s,x(s))$, respectively, for $t_0 \leq s \leq t$ and $t_0 \in R_+$. From hypothesis (b) $x(t,t_0,0) \equiv 0$ and $z(t,t_0,0) \equiv 0$ be the trivial solution process of (3.1.1) and (3.1.2), respectively. From (3.2.1) and assumption (c), we obtain

$$E[|V_i(t,x(t))|] \leq E[|V_i(t_0,z(t))|] + \int_{t_0}^{t} E[|LV_i(s, z(t,s,x(s)))|]ds.$$

Hence

$$\sum_{i=1}^{N} E[|V_i(t,x(t))|] \leq E[\sum_{i=1}^{N} |V_i(t_0,z(t))|] + \int_{t_0}^{t} E[\sum_{i=1}^{N} |LV_i(s, z(t,s,x(s)))|]ds.$$

This together with hypotheses (d), (e), and setting

$$m(t) = \sum_{i=1}^{N} E[|V_i(t,x(t))|]$$

yields

$$m(t) \leq \mu(E[\|x_0\|^p]) + \int_{t_0}^{t} \lambda(s)m(s)ds, \tag{3.4.1}$$

as long as $E[\|x(s)\|^p] \leq \rho$ for $t_0 \leq s \leq t$. By applying Bellman-Gronwell-Reid Inequality [14,16], we get

$$m(t) \leq \mu(E[\|x_0\|^p])exp\Big[\int_{t_0}^{t} \lambda(s)ds\Big],$$

as long as $E[\|x(s)\|^p] \leq \rho$ for $t_0 \leq s \leq t$. This implies that

$$\sum_{i=1}^{N} E[|V_i(t, x(t))|] \leq \mu(E[\|x_0\|^p])exp\Big[\int_{t_0}^{t} \lambda(s)ds\Big], \qquad (3.4.2)$$

as long as $E[\|x(s)\|^p] \leq \rho$ for $t_0 \leq s \leq t$. From (3.4.2) and condition (a), we have

$$b(E[\|x(t)\|^p]) \leq E[b(\|x(t)\|^p)] \leq \sum_{i=1}^{N} E[|V_i(t, x(t))|] \leq \mu(E[\|x_0\|^p])exp\Big[\int_{t_0}^{t} \lambda(s)ds\Big], \qquad (3.4.3)$$

as long as $E[\|x(s)\|^p] \leq \rho$ for $t_0 \leq s \leq t$. First we show that (3.4.2) is valid for all $t \geq t_0$. For this purpose, we choose x_0 such that

$$\mu(E[\|x_0\|^p])exp\Big[\int_{t_0}^{\infty} \lambda(s)ds\Big] < b(\rho). \qquad (3.4.4)$$

We claim that $E[\|x(t)\|^p] < \rho$ for all $t \geq t_0$ whenever (3.4.4) holds. Assume that this claim is false, that is, there exists $t_1 \in R_+$ such that $t_1 > t_0$, $E[\|x(s)\|^p] < \rho$ for $t_0 \leq s < t_1$ and $E[\|x(t_1)\|^p] = \rho$. This implies that (3.4.2) is valid for $t \in [t_0, t_1]$. Hence, using (3.4.3) and (3.4.4) we get

$$b(\rho) = b(E[\|x(t_1)\|^p]) \leq \mu(E[\|x_0\|^p])exp\Big[\int_{t_0}^{\infty} \lambda(s)ds\Big] < b(\rho).$$

This contradiction establishes the impossibility of the existence of such a t_1. This proves the validity of the claim. Hence, (3.4.2) is true for all $t \geq t_0$. Finally, we need to conclude (SM_1) of (3.1.1). Let $0 < \epsilon < \rho$ and $t_0 \in R_+$. Choose x_0 such that

$$\mu(E[\|x_0\|^p])exp\Big[\int_{t_0}^{\infty} \lambda(s)ds\Big] < b(\epsilon^p) \tag{3.4.5}$$

Under this, we have

$$\|x_0\|_p = (E[\|x_0\|^p])^{1/p} < \delta_1,$$

where,

$$\delta_1 = \delta_1(\epsilon, t_0) = \Big(\mu^{-1}\Big(\frac{b(\epsilon^p)}{exp[\int_{t_0}^{\infty} \lambda(s)ds]}\Big)\Big)^{1/p}.$$

Moreover, from the continuity of $V(t,x)$ and the fact that $V(t_0, 0) \equiv 0$, we can find a $\delta_2 = \delta_2(\epsilon, t_0) > 0$ such that $|V(t_0, x_0)| < \delta_1$, whenever $\|x_0\|_p < \delta_2$. We define $\delta = Min\{\delta_1, \delta_2\}$ and $B(\sqrt[p]{\rho}) = \{x_0 \in R^n : \|x_0\|_p < \sqrt[p]{\rho}\}$. For $x_0 \in B(\sqrt[p]{\rho})$, $\|x(s)\|^p \leq \rho$ for all $t_0 \leq s \leq t$. Therefore, condition (d) is valid for all $t \geq t_0$. Moreover, using (3.4.3) and (3.4.5), we have

$$b(E[\|x(t)\|^p]) < b(\epsilon^p), \qquad t \geq t_0.$$

Hence $\|x(t)\|_p < \epsilon$ whenever $\|x_0\|_p < \delta, \quad t \geq t_0$. This completes the proof.

■

The next theorem provides sufficient conditions for asymptotic stability of the p-th moment of the trivial solution of (3.1.1) in the context of the method of generalized variation of parameters.

Theorem 3.4.2 *Assume that the hypotheses of Theorem 3.4.1 hold except (d) and (e) are replaced by*

f) $E[\sum_{i=1}^N |LV_i(s, z(t, s, x(s)))|] \leq \lambda(s)\eta(t-s)\sum_{i=1}^N E[|V_i(s,x)|]$ *for* $t_0 \leq s \leq t$, $\|x\|^p \leq \rho$;

g) $E[\sum_{i=1}^N |V_i(t_0, z(t))|] \leq \mu(\|x_0\|^p)\tau(t-t_0)$, *for* $E[\|x_0\|^p] \leq \rho$, *where* λ *and* μ *are as* *in (d) and (e), η, $\tau \in \mathcal{L}$, and η and τ satisfy*

$$\eta(t-s)\eta_1(s-t_0) \le k\eta_1(t-t_0) \qquad \textit{for some } \eta_1 \in \mathcal{L}, \qquad k > 0$$

and

$$\lim_{t\to\infty}(\eta_1(t-t_0)) \int_{t_0}^{t} \frac{k\lambda(s)\tau(s-t_0)}{\eta_1(s-t_0)} exp\Big[\int_s^t k\lambda(u)du\Big]ds = 0. \tag{3.4.6}$$

Then the trivial solution process of (3.1.1) *is asymptotically stable in the p-th mean.*

Proof. From the proof of Theorem 3.4.1 we have

$$\sum_{i=1}^{N} E[|V_i(t,x(t))|] \le E[\sum_{i=1}^{N} |V_i(t_0, z(t))|] + \int_{t_0}^{t} E[\sum_{i=1}^{N} |LV_i(s, z(t,s,x(s)))|]ds.$$

This together with hypotheses (f) and (g) yields

$$\sum_{i=1}^{N} E[|V_i(t,x(t))|] \le \mu(\|x_0\|^p)\tau(t-t_0) + \int_{t_0}^{t} \lambda(s)\eta(t-s) \sum_{i=1}^{N} E[|V_i(s,x(s))|]ds, \tag{3.4.7}$$

as long as $E[\|x(t)\|^p] \le \rho$. By setting

$$m(t) = \frac{\sum_{i=1}^{N} E[|V_i(t,x(t))|]}{\eta_1(t-t_0)}, \qquad n(t) = \frac{\mu(\|x_0\|^p)\tau(t-t_0)}{\eta_1(t-t_0)}$$

and $\nu(t) = k\lambda(t)$, (3.4.7) is rewritten as

$$m(t) \le n(t) + \int_{t_0}^{t} \nu(s)m(s)ds$$

as long as $E[\|x(t)\|^p] \le \rho$. Applying Theorem A.2.5 [19], we obtain

$$m(t) \le n(t) + \int_{t_0}^{t} \nu(s)n(s)exp\Big[\int_s^t \nu(u)du\Big]ds. \tag{3.4.8}$$

From the nature of functions η, λ, and the definitions of $\nu(t)$, $m(t)$ and $n(t)$, (3.4.8) becomes

$$\sum_{i=1}^{N} E[|V_i(t,x(t))|] \leq \mu(\|x_0\|^p)\tau(t-t_0)+\eta_1(t-t_0)\int_{t_0}^{t}\frac{\nu(s)\mu(\|x_0\|^p)\tau(s-t_0)}{\eta_1(s-t_0)}exp\Big[\int_{s}^{t}\nu(u)du\Big]ds \tag{3.4.9}$$

as long as $E[\|x(t)\|^p] \leq \rho$. From (3.4.9), condition (a) and properties of functions μ, λ, and the nature of x_0 and λ, we obtain

$$\begin{aligned} b(E[\|x(t)\|^p]) \;\; &\leq \;\; \mu(\|x_0\|^p)\Big[\tau(t-t_0) \\ &\qquad +\eta_1(t-t_0)\int_{t_0}^{t}\frac{k\lambda(s)\tau(s-t_0)}{\eta_1(s-t_0)}exp\Big[\int_{s}^{t}k\lambda(u)du\Big]ds\Big] \end{aligned} \tag{3.4.10}$$

as long as $E[\|x(t)\|^p] \leq \rho$. From (3.4.6), (3.4.10) and the fact that $\tau \in \mathcal{L}$, we can conclude that $E[\|x(t)\|^p] \leq \rho$ for all $t \geq t_0$. Moreover, from (3.4.8) and (3.4.10), relations (3.4.2) and (3.4.3) remain true. Moreover, the (SM_1) property of the trivial solution of (3.1.1) can be conclude by the following the argument of Theorem 3.4.1. To conclude the (SM_2), it is obvious from (3.4.10) and the nature of τ, $b(E[\|x(t)\|^p])$ tends to zero as $t \to \infty$. Hence, one can manipulate the technical details to validate the definition of (SM_2).

■

To appreciate the assumptions of Theorem 3.4.2, we present the following result which is applicable to many problems.

Corollary 3.4.3 *Let the hypotheses of Theorem 3.4.2 be satisfied except that* (3.4.6) *and the condition on η are replaced by*

$$\eta(t-s)\tau(s-t_0) \leq k\tau(t-t_0), \qquad t \geq t_0 \tag{3.4.11}$$

and

$$\lim_{t\to\infty}\Big[\tau(t-t_0)exp\Big[k\int_{t_0}^{t}\lambda(s)ds\Big]\Big]=0, \tag{3.4.12}$$

where k is some positive constant. Then the trivial solution process of (3.1.1) *is p-th mean asymptotically stable.*

Proof. By following the proof of Theorem 3.4.2, we arrive at (3.4.7). Now, by using (3.4.11), (3.4.7) can be rewritten as

$$m(t)\le\mu(\|x_0\|^p)+\int_{t_0}^{t}k\lambda(s)m(s)ds, \tag{3.4.13}$$

as long as $E[\|x(t)\|^p]\le\rho$, where

$$m(t)=\frac{\sum_{i=1}^{N}E[|V_i(t,x(t))|]}{\tau(t-t_0)}.$$

By applying Lemma A.2.4 [14] to (3.4.13), we get

$$m(t)\le\mu(\|x_0\|^p)exp\Big[k\int_{t_0}^{t}\lambda(s)ds\Big]$$

which implies

$$\sum_{i=1}^{N}E[|V_i(t,x(t))|]\ \le\ \mu(\|x_0\|^p)\tau(t-t_0)exp\Big[k\int_{t_0}^{t}\lambda(s)ds\Big].$$

Using the properties of b, μ and x_0, we have

$$b(E[\|x(t)\|^p])\ \le\ \mu(\|x_0\|^p)\tau(t-t_0)exp\Big[k\int_{t_0}^{\infty}\lambda(s)ds\Big]. \tag{3.4.14}$$

Hence, $b(E[\|x(t)\|^p])$ tends to zero as $t\to\infty$. By using the argument used in the proof of Theorem 3.4.2, we complete the proof of the corollary.

■

Example 3.4.4 We consider Example 3.2.2, and assume that $p(t,0) \equiv 0 \equiv q(t,0)$ and

$$E[x^T(s)\Phi^T(t,s)\Phi(t,s)(p(s,x(s))-B(s)q(s,x(s)))] \leq \eta(t-s)\lambda_1(s)E[\|x(s)\|^2] \quad (3.4.15)$$

and

$$E[q^T(s,x(s))\Phi^T(t,s)\Phi(t,s)q(s,x(s))] \leq \eta(t-s)\lambda_2(s)E[\|x(s)\|^2] \quad (3.4.16)$$

From (3.4.15) and (3.4.16), $LV(s,z(t,s,x(s)))$ satisfies Theorem 3.4.2,

$$\begin{aligned} LV(s,z(t,s,x(s))) &= 2x^T(s)\Phi^T(t,s)\Phi(t,s)[p(s,x(s)) - B(s)q(s,x(s))] \\ &\quad + q^T(s,x(s))\Phi^T(t,s)\Phi(t,s)q(s,x(s)) \\ &\leq \eta(t-s)\lambda(s)E[\|x(s)\|^2], \qquad t_0 \leq s \leq t, \end{aligned}$$

where $\lambda(s) = \lambda_1(s) + \lambda_2(s)$.

Further assume that

$$E[\|z(t,t_0,x_0)\|^2] \leq \mu(E[\|x(s)\|^2])\tau(t-t_0).$$

Therefore, by the application of Theorem 3.4.2, we conclude that the trivial solution of (3.2.7) is asymptotically mean square stable.

3.5 Relative Stability

The following result provides sufficient conditions for relative stability of Itô-Doob type systems of stochastic differential equations in the context of method of variation of parameters.

Theorem 3.5.1 *Let the assumption of Theorem 3.3.1 be satisfied. Further assume* *that*

a) $\alpha(t,0) \equiv 0$ *and* $\beta(t,0) \equiv 0$ *for* $t \in R_+$*;*

b) $b(\|x\|^p) \le \sum_{i=1}^N |V_i(t,x)| \le a(\|x\|^p)$ *for* $(t,x) \in R_+ \times R^n$ *where* $p \ge 1$*,* $b \in \mathcal{VK}$ *and* $a \in \mathcal{CK}$*;*

c) $E[\sum_{i=1}^N |LV_i(s, z(t,s,x(s)) - z(t,s,y(s)))|] \le \eta(t-s)\lambda(s) \sum_{i=1}^N E[|V_i(s, x(s) - y(s))|]$ *for* $t_0 \le s \le t$*,* $\|x(s) - y(s)\|^p < \rho$*,* $\|y(s)\|^p < \rho$ *for some* $\rho > 0$*,* λ *is locally integrable function and* $\eta \in \mathcal{L}$*;*

d) $\sum_{i=1}^N |V_i(t_0, z(t,t_0,x_0) - z(t,t_0,y_0))| \le \mu(\|x_0 - y_0\|^p)\tau(t-t_0)$*,* $t \ge t_0$*, whenever* $E[\|x_0 - y_0\|^p] < \rho$ *and* $E[\|y_0\|^p] < \rho$*, where* $\mu \in \mathcal{CK}$*,* $\tau \in \mathcal{L}$*;*

e) *there exists a positive number* k *such that*

$$\eta(t-s)\tau(s-t_0) \le k\tau(t-t_0);$$

f) $E\Big[\sum_{l=1}^N V_x(s,\Delta z)[\Phi(t,s,x(s))[\sigma^l(s,x(s)) - \beta^l(s,x(s))] - \Phi(t,s,y(s))[H^l(s,y(s)) - \beta^l(s,y(s))]]\Big]$ *exists for* $t \ge t_0$*, where* $\Delta z = z(t,s,x(s)) - z(t,s,y(s))$*.*

Then

(i) *the boundedness of* $\tau(t-t_0)exp\Big[k\int_{t_0}^t \lambda(s)ds\Big]$ *implies relative stability in the p-th moment,* (RM_1)*, of* (3.1.1) *and* (3.3.1)*;*

(ii) $\lim_{t\to\infty}\Big[\tau(t-t_0)exp\Big[k\int_{t_0}^t \lambda(s)ds\Big]\Big] = 0$ *implies relatively asymptotically stability in the p-th moment,* (RM_2)*, of* (3.1.1) *and* (3.3.1)*.*

Proof. Let $x(t) = x(t,t_0,x_0)$, $z(t) = z(t,t_0,x_0)$ and $y(t) = y(t,t_0,y_0)$ be the solution processes as defined in Theorem 3.3.1. From Theorem 3.3.1, hypotheses (c), (d) and (f), we have

$$\sum_{i=1}^{N} |V_i(t, x(t) - y(t))| \leq \mu(\|x_0 - y_0\|^p)\tau(t - t_0) + \int_{t_0}^{t} \eta(t - s)\lambda(s) \sum_{i=1}^{N} |V_i(s, x(s) - y(s))|ds. \tag{3.5.1}$$

as long as $E[\|x(t)\|^p] \leq \rho$ and $E[\|y(t)\|^p] \leq \rho$. By setting $m(t) = \frac{\sum_{i=1}^{N} E[|V_i(t,x(t)-y(t))|]}{\tau(t-t_0)}$ and using hypothesis (e), relation (3.5.1) reduces to

$$m(t) \leq R(t) \tag{3.5.2}$$

as long as $E[\|x(t)\|^p] \leq \rho$ and $E[\|y(t)\|^p] \leq \rho$, where

$$R(t) = \mu(E[\|x_0 - y_0\|^p]) + \int_{t_0}^{t} k\lambda(s)m(s)ds. \tag{3.5.3}$$

Therefore,

$$R'(t) = k\lambda(t)m(t) \leq k\lambda(t)R(t) \tag{3.5.4}$$

with

$$R(t_0) = \mu(E[\|x_0 - y_0\|^p]).$$

Solving this differential inequality, we have

$$R(t) \leq \mu(E[\|x_0 - y_0\|^p])exp\Big[k\int_{t_0}^{t} \lambda(s)ds\Big]. \tag{3.5.5}$$

This together with the definition of $m(t)$, (3.5.2) reduces to

$$\sum_{i=1}^{N} E[|V_i(t, x(t) - y(t))|] \leq \tau(t - t_0)\mu(E[\|x_0 - y_0\|^p])exp\Big[k\int_{t_0}^{t} \lambda(s)ds\Big] \tag{3.5.6}$$

From hypotheses (b), (3.5.6) and by using the argument used in the proof of Theorem 3.4.1, we get

$$\begin{aligned} b(E[\|x(t)-y(t)\|^p])| &\leq \tau(t-t_0)\mu(E[\|x_0-y_0\|^p])exp\Big[k\int_{t_0}^{t}\lambda(s)ds\Big] \\ &\leq K_1\mu(E[\|x_0-y_0\|^p]). \end{aligned} \tag{3.5.7}$$

for all $t \geq t_0$, where $K_1 > 0$ is the bound of (i).

Finally, we need to conclude (RM_1) property of (3.1.1) and (3.3.1). For given $0 < \epsilon < \rho$ and $t_0 \in R_+$, we choose x_0 and y_0 such that

$$K_1\mu(E[\|x_0-y_0\|^p]) < b(\epsilon^p) \tag{3.5.8}$$

Under these considerations, we have

$$\|x_0-y_0\|_p = (E[\|x_0-y_0\|^p])^{1/p} < \delta_1,$$

where,

$$\delta_1 = \delta_1(\epsilon, t_0) = \Big(\mu^{-1}\Big(\frac{b(\epsilon^p)}{K_1}\Big)\Big)^{1/p}.$$

From the continuity of $V(t,x)$ and the fact that $V(t_0,0) \equiv 0$, we can find a $\delta_2 = \delta_2(\epsilon, t_0) > 0$ such that $|V(t_0, x_0-y_0)| < \delta_1$ whenever $\|x_0-y_0\|_p < \delta_2$. We define $\delta = Min\{\delta_1, \delta_2\}$ and hence $\|x(t)-y(t)\|_p < \epsilon, t \geq t_0$ whenever $\|x_0-y_0\|_p < \delta$. Similarly, using (ii), we have $b(E[\|x(t)-y(t)\|^p])$ tends to zero as $t \to \infty$ and hence (RM_2). This completes the proof.

■

Example 3.5.2 Let us consider Example 3.3.2. We assume $\Delta A \equiv \Delta B \equiv \Delta a \equiv \Delta b \equiv 0$. Under these assumptions, (3.3.4), (3.3.5) and (3.3.6) reduces to

$$dx = [A(t)x + a(t) + p(t,x)]dt + [B(t)x + b(t) + q(t,x)]dw(t), \qquad x(t_0) = x_0, \tag{3.5.9}$$

$$dz = [A(t)z + a(t)]dt + [B(t)z + b(t)]dw(t), \qquad z(t_0) = x_0, \tag{3.5.10}$$

and

$$dy = [A(t)y + a(t) + p(t,y)]dt + [B(t)y + b(t) + q(t,y)]dw(t), \qquad y(t_0) = y_0, \tag{3.5.11}$$

respectively. We note that auxiliary system (3.5.10) acts like a nominal system corresponding to a system (3.5.9). Choosing $V(t,x) = \frac{1}{2}\|x(t)\|^2$ and following the argument used in Example 3.2.2, Example 3.3.2 reduces to

$$\begin{aligned} &\|x(t) - y(t)\|^2 \\ = \;& \|z(t,t_0,x_0) - z(t,t_0,y_0)\|^2 + \int_{t_0}^{t} 2LV(s,\Delta z)ds \\ & +2\int_{t_0}^{t} (x(s) - y(s))^T \Phi^T(t,s)\Phi(t,s)[q(s,x(s)) - q(s,y(s))]dw(s), \end{aligned} \tag{3.5.12}$$

where

$$\begin{aligned} &LV(s,\Delta z) \\ = \;& (x(s) - y(s))^T \Phi^T(t,s)\Phi(t,s)[p(s,x(s)) - p(s,y(s)) \\ & -B(s)[p(s,x(s)) - p(s,y(s))]] + \frac{1}{2}c(t,s,x(s),y(s))c^T(t,s,x(s),y(s)), \end{aligned}$$

$c(t,s,x(s),y(s)) = \Phi(t,s)[q(s,x(s)) - q(s,y(s))]$ and $\Delta z = z(t,s,x(s)) - z(t,s,y(s))$. Further assume that $E[LV(s,\Delta z)] \leq \eta(t-s)\lambda(s)E[\|x(s) - y(s)\|^2]$, where η and λ satisfies all conditions in Theorem 3.5.1. Thus, by the application of Theorem 3.5.1, systems (3.5.9) and (3.5.11) are relatively asymptotically stable in the mean square sense. In fact, the solution process (3.5.9) has asymptotic equilibrium property [16].

4 Variational Comparison Method: Relative Stability

4.1 Introduction

In general, a closed or exact form representation of time evolution flow described by a nonlinear nonstationary interconnected system is not always feasible. Having the knowledge about the existence and in the absence of a closed or exact form representation of dynamic flow, the time evolution of flow satisfying a stochastic dynamic inequality generated by silent or characteristic features of the dynamic system is estimated by the corresponding stochastic comparison dynamic flow [15, 16]. In particular, it is well known that an arbitrary measure of dynamic flow satisfying differential inequality is estimated by the extremal solution of the corresponding comparison system of differential equations [15, 16, 19, 35]. This technique is referred as the method of comparison. In the following, we generalize the comparison theorems based on the ideas of the classical Lyapunov's second method [15, 16] and its extensions to variety of systems of differential equations [2, 6, 7, 8, 9, 30]. By employing the concept of Lyapunov function and random differential inequalities, we present a generalized variational comparison theorems. These result connects solution processes of the auxiliary and perturbed systems of stochastic differential equations with the maximal solution of the corresponding system of comparison dynamic equations. Using the comparison results, the stochastic stability analysis of perturbed system of differential equations are investigated.

4.2 Energy Function Method and Variational Comparison Theorems

We consider the following Itô-Doob type stochastic perturbed and auxiliary systems of differential equations

$$\begin{aligned} dx &= f(t,x)dt + \sigma(t,x)dw(t) \\ &= f(t,x)dt + \textstyle\sum_{l=1}^{m} \sigma^l(t,x)dw_l(t), \qquad x(t_0) = x_0, \end{aligned} \tag{4.2.1}$$

and

$$\begin{aligned} dz &= \alpha(t,z)dt + \beta(t,z)dw(t) \\ &= \alpha(t,z)dt + \textstyle\sum_{l=1}^{m} \beta^l(t,z)dw_l(t), \qquad z(t_0) = x_0, \end{aligned} \tag{4.2.2}$$

respectively, where $x, z, x_0 \in R^n$; f, α, and column vectors of $n \times m$ matrices σ and β $\in C[R_+ \times R^n, R^n]$; $t_0 \in R_+$; α and β are twice continuously differentiable with respect to z; $w(t) = (w_1(t), w_2(t), ..., w_m(t))^T$ is an m-dimensional normalized Wiener process with independent increments; x_0 and $w(t)$ are mutually independent for each $t \geq t_0$; moreover, we assume that solutions of (4.2.1) and (4.2.2) exist for $t \geq t_0$.

By employing the concept of Lyapunov function and differential inequalities, we present a generalized variational comparison theorem. This result connects solution processes of stochastic dynamic system described by (4.2.1) with the maximal solution of corresponding stochastic /deterministic comparison systems of differential equations and solution of auxiliary stochastic dynamic system described by (4.2.2). As a byproduct of this, several auxiliary comparison results are formulated. Presented results generalize and extend the existing results [15, 16, 19, 20] in a systematic way.

Prior to the presentation of comparison results, we assume that $V \in C[R_+ \times R^n, R^N]$, and its partial derivatives V_t, V_x and V_{xx} exists and are continuous on $R_+ \times R^n$. We define the Itô-Doob differential of $V(s, z(t,s,x))$ with respect to (4.2.1) as

$$\begin{aligned} dV(s, z(t,s,x)) &= LV(s, z(t,s,x))ds \\ &+ V_x(s, z(t,s,x))\Phi(t,s,x)[\sigma(s,x) - \beta(s,x)]dw(s), \end{aligned} \tag{4.2.3}$$

where $z(t,s,x)$ is solution process of (4.2.2) with initial value $(s,x) \in R_+ \times R^n$; $t_0 \le s \le t$; $t_0, s, t \in R_+$;

$$
\begin{aligned}
& LV(s,z(t,s,x)) \\
= \quad & V_s(s,z(t,s,x)) + V_x(s,z(t,s,x))\Big[\frac{1}{2}\Big(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,x)[\beta^l(s,x)\beta^l_j(s,x) \\
& -2\beta^l(s,x)\sigma^l_j(s,x) + \sigma^l(s,x)\sigma^l_j(s,x)]\Big)_{n\times 1} \\
& +\Phi(t,s,x)[\sum_{l=1}^{m}\beta^l_x(s,x)[\beta^l(s,x) - \sigma^l(s,x)] + f(s,x) - \alpha(s,x)]\Big] \\
& +\frac{1}{2}\Big(tr\Big(\sum_{l=1}^{m}\frac{\partial^2}{\partial x\partial x}V_i(s,z(t,s,x))b^l(t,s,x)b^{lT}(t,s,x)\Big)\Big)_{N\times 1},
\end{aligned}
\tag{4.2.4}
$$

$b^l(t,s,x) = \Phi(t,s,x)[\sigma^l(s,x) - \beta^l(s,x)]$; and $\Phi(t,t_0,x_0) = \frac{\partial}{\partial x_0}z(t,t_0,x_0)$ is the fundamental matrix solution process of the variational auxiliary system [38].

Theorem 4.2.1 *Assume that*

(a) $V(s,z(t,s,x))$ *satisfies the following stochastic differential inequalities*

$$
\begin{cases}
LV(s,z(t,s,x)) \le g(s,V(s,z(t,s,x))), \\
V_x(s,z(t,s,x))\Phi(t,s,x)[\sigma(s,x) - \beta(s,x)] = \Sigma(s,V(s,z(t,s,x))),
\end{cases}
\tag{4.2.5}
$$

where $LV(s,z(t,s,x))$ *is defined in* (4.2.4)*;* g *and column vectors of* $N \times m$ *matrix* $\Sigma \in C[R_+ \times R^N, R^N]$ *and satisfy the following conditions*

(i) $g(t,u)$ *is concave and quasi-monotone nondecreasing in* u *for each* $t \in R_+$*;*

(ii) $\sum_{l=1}^{m}|\Sigma^l_i(t,u) - \Sigma^l_i(t,v)| \le h_i(\|u-v\|)$ *on* (t,u), $(t,v) \in R_+ \times R^N$, $h_i \in C[R_+,R_+]$, $h_i(0) = 0$, h_i *is non-decreasing and* $\int_{0^+}\frac{ds}{h_i(s)} = \infty$ *for each* $i = 1,2,...,N$*;*

(b) $r(t) = r(t,t_0,u_0)$ *is the maximal solution of the following stochastic comparison* *system of differential equations*

$$\begin{aligned} du &= g(t,u)dt + \Sigma(t,u)dw(t) \\ &= g(t,u)dt + \textstyle\sum_{l=1}^{m} \Sigma^l(t,u)dw_l(t), \qquad u(t_0) = u_0, \end{aligned} \tag{4.2.6}$$

existing for $t \geq t_0$*;*

(c) *Let* $x(t) = x(t, t_0, x_0)$ *and* $z(t) = z(t, t_0, x_0)$ *be solution processes of* (4.2.1) *and* (4.2.2) *existing for* $t \geq t_0$.

Then,

$$V(t, x(t, t_0, x_0)) \leq r(t, t_0, u_0), \qquad t \geq t_0, \tag{4.2.7}$$

provided that

$$V(t_0, z(t, t_0, x_0)) \leq u_0. \tag{4.2.8}$$

Proof. For $t_0 \leq s \leq t$ and $0 < \Delta s = ds$, using (4.2.3) and (4.2.5), we have

$$dV(s, z(t, s, x(s))) \leq g(s, V(s, z(t, s, x(s))))ds + \Sigma(s, V(s, z(t, s, x(s))))dw(s). \tag{4.2.9}$$

Set

$$m(s) = V(s, z(t, s, x(s))), \qquad m(t_0) = V(t_0, z(t, t_0, x_0)). \tag{4.2.10}$$

From (4.2.10), (4.2.9) reduces to

$$dm(s) \leq g(s, m(s))ds + \Sigma(s, m(s))dw(s), \qquad m(t_0) = V(t_0, z(t, t_0, x_0)). \tag{4.2.11}$$

From (4.2.11) and application of Theorem 6.4.1 [16], we obtain

$$m(t) \leq r(t, t_0, u_0), \qquad t \geq t_0,$$

provided that

$$V(t_0, z(t, t_0, x_0)) \le u_0.$$

Hence

$$V(t, z(t, t, x(t))) \le r(t, t_0, u_0), \qquad t \ge t_0.$$

This completes the proof.

■

A few special cases of Theorem 4.2.1 in literature [15, 16, 19] are exhibited in the following corollary.

Corollary 4.2.2 a) *For $g(t,u) \equiv 0$ and $\Sigma(t,u) \equiv 0$ in (4.2.5), the inequality (4.2.9) reduces to $dV(s, z(t,s,x)) \le 0$. The stochastic comparison differential equation (4.2.6) reduces to $du = 0$. Then the function $V(s, z(t, s, x(s)))$ is non-decreasing in t, and the relation (4.2.7) reduces to*

$$V(t, x(t)) \le V(t_0, z(t)), \qquad t \ge t_0.$$

b) *For $g(t,u) = \lambda(t)u$ and $\Sigma(t,u) = diag[G^1(t)u, G^2(t)u, ..., G^m(t)u]$ in (4.2.5), the comparison stochastic differential equation in (4.2.6) is*

$$du = \lambda(t)u + \sum_{l=1}^{m} G^l(t) u dw_l(t),$$

where λ and $G^l \in C[R, R]$. In this case, differential inequality (4.2.9) becomes

$$dV(s, z(t, s, x(s))) \le \lambda(s) V(t, z(t, s, x(s))) + \sum_{l=1}^{m} G^l(s) V(s, z(t, s, x(s))) dw_l(t).$$

Then relation (4.2.7) reduces to,

$$V(t, x(t)) \le V(t_0, z(t)) exp\Big[\int_{t_0}^{t} (\lambda(s) - \frac{1}{2}\sum_{l=1}^{m} (G^l)^2(s)) ds + \sum_{l=1}^{m} \int_{t_0}^{t} G^l(s) dw_l(s)\Big],$$

$t \geq t_0$.

Example 4.2.3 Let us consider stochastic perturbed and auxiliary systems of differential equations

$$dx = [A(t)x + p(t,x)]dt + [B(t)x + q(t,x)]dw(t), \qquad x(t_0) = x_0, \tag{4.2.12}$$

and

$$dz = A(t)zdt + B(t)zdw(t), \qquad z(t_0) = x_0, \tag{4.2.13}$$

respectively, where $x, z \in R^n$; A and B are any $n \times n$ continuous matrix functions defined on J; $J = [t_0, t_0 + a), a > 0$; p and q are any n-dimensional smooth functions defined on $J \times R^n$ into R^n that insures the existence of the solution processes of (4.2.12); for each $t \in J$, $w(t)$ is a scalar normalized Wiener process independent of x_0. For given $V(t,x) = \frac{1}{2}\|x\|^2$, we have $V_t(t,x) = 0$; $\frac{\partial}{\partial x}V(t,x) = x^T$; $\frac{\partial^2}{\partial x \partial x}V(t,x) = I$, $n \times n$ identity matrix; $\frac{\partial}{\partial x_0}z(t,s,x(s)) = \Phi(t,s)$; $\frac{\partial^2}{\partial x_0^2}z(t,s,x(s)) = 0$ [15] and

$$\begin{aligned} dV(s, z(t,s,x(s))) &= LV(s, z(t,s,x(s)))ds \\ &\quad + x^T(s)\Phi^T(t,s)\Phi(t,s)q(s,x(s))dw(s), \end{aligned}$$

where

$$LV(s, z(t,s,x(s))) = x^T(s)\Phi^T(t,s)\Phi(t,s)[p(s,x(s)) - B(s)q(s,x(s))] + \frac{1}{2}\|\Phi(t,s)q(s,x(s))\|^2.$$

We assume that

$$\begin{cases} x^T(s)\Phi^T(t,s)\Phi(t,s)[p(s,x(s)) - B(s)q(s,x(s))] \leq \lambda(s)\frac{1}{2}\|\Phi(t,s)x(s)\|^2, \\ \|\Phi(t,s)q(s,x(s))\|^2 \leq \gamma(s)\|\Phi(t,s)x(s)\|^2, \\ e(s) = \lambda(s) + \gamma(s); \end{cases}$$

and

$$x^T(s)\Phi^T(t,s)\Phi(t,s)q(s,x(s)) = \Sigma(s,V(s,z(t,s,x(s))))$$
$$= \frac{1}{2}\nu(s)\|\Phi(t,s)x(s)\|^2.$$

Under these considerations, we have (4.2.5) with $g(s,u) = e(s)u$ and $\Sigma(s,u) = \nu(s)u$, e and γ are defined above. Here, a stochastic comparison differential equation is

$$du = e(s)uds + \nu(s)udw(s). \tag{4.2.14}$$

We note that $q(s,x(s)) = \eta(s)x(s)$. Thus, Theorem 4.2.1 is applicable to the stochastic perturbed differential equation (4.2.12). In this case, estimate (4.2.7) on the solution process of (4.2.12) in the context of V is

$$V(t,x(t)) \le V(t_0,z(t))exp\Big[\int_{t_0}^t (e(s)-\nu^2(s))ds + \int_{t_0}^t \nu(s)dw(s)\Big], \text{ for } t \ge t_0.$$

From the definition of V, we have

$$\|x(t,t_0,x_0)\|^2 \le \|z(t,t_0,z_0)\|^2 exp\Big[\int_{t_0}^t (e(s)-\nu^2(s))ds + \int_{t_0}^t \nu(s)dw(s)\Big], \qquad t \ge t_0.$$

Moreover, we have the following mean square estimate:

$$E[\|x(t,t_0,x_0)\|^2] \le E[\|z(t,t_0,z_0)\|^2]exp\Big[\int_{t_0}^t e(s)ds\Big].$$

Corollary 4.2.4 a) *For* $\beta \equiv 0$ *in* (4.2.2) *[19] the conclusion of Theorem 4.2.1 re-*

mains true. In this case, we note that (4.2.4) *reduces to:*

$$\begin{aligned}
& LV(s,z(t,s,x(s))) \\
&= V_s(s,z(t,s,x(s))) \\
&+ V_x(s,z(t,s,x(s)))\Big[\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,x(s))\sigma^l(s,x(s))\sigma_j^l(s,x(s))\right)_{n\times 1} \\
&+ \Phi(t,s,x(s))[f(s,x(s))-\alpha(s,x(s))]\Big] \\
&+ \frac{1}{2}\left(tr(\sum_{l=1}^{m}\frac{\partial^2}{\partial x\partial x}V_i(s,z(t,s,x(s)))b^l(t,s,x(s))b^{lT}(t,s,x(s)))\right)_{N\times 1},
\end{aligned}$$

and

$$b^l(t,s,x(s)) = \Phi(t,s,x(s))\sigma^l(s,x(s)).$$

Thus, Theorem 4.2.1 includes a result [19] as a special case.

b) *For* $\beta \equiv 0$, *and* $\alpha \equiv f$ *in* (4.2.1) *and* (4.2.2), $LV(s,z(t,s,x(s)))$ *in* (4.2.4) *is replaced by*

$$\begin{aligned}
& LV(s,z(t,s,x(s))) \\
&= V_s(s,z(t,s,x(s))) \\
&+ \frac{1}{2}V_x(s,z(t,s,x(s)))\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,x(s))\sigma^l(s,x(s))\sigma_j^l(s,x(s))\right)_{n\times 1} \\
&+ \frac{1}{2}\left(tr(\sum_{l=1}^{m}\frac{\partial^2}{\partial x\partial x}V_i(s,z(t,s,x(s)))b^l(t,s,x(s))b^{lT}(t,s,x(s)))\right)_{N\times 1},
\end{aligned}$$

where

$$b^l(t,s,x(s)) = \Phi(t,s,x(s))\sigma^l(s,x(s)).$$

In this case, Theorem 4.2.1 yields a special case [19].

c) *For* $\beta \equiv 0$, *and* $\alpha \equiv 0$, $LV(s, z(t, s, x(s)))$ *in* (4.2.4) *becomes*

$$\begin{aligned} LV(s, x(s)) &= V_s(s, x(s)) + V_x(s, x(s)) f(s, x(s)) \\ &+ \frac{1}{2}\left(tr(\sum_{l=1}^{m} \frac{\partial^2}{\partial x \partial x} V_i(s, x(s)) \sigma^l(t, s, x(s)) \sigma^{lT}(t, s, x(s))) \right)_{N\times 1}. \end{aligned}$$

In view of this, Theorem 4.2.1 reduces to well known results in [16, 19].

Example 4.2.5 Let us choose $V(t, x) = \frac{1}{2}\|x\|^2$ in Corollary 4.2.4 (c). We note that $V_t(t, x) \equiv 0$, $\frac{\partial}{\partial x}V(t, x) = x^T$ and $\frac{\partial}{\partial x \partial x}V(t, x) = I$. In this case, LV, in (4.2.4) reduces to

$$\begin{aligned} LV(s, z(t, s, x(s))) &= z^T(t, s, x(s)) \left(\sum_{l=1}^{m} \sum_{j=1}^{n} \frac{\partial}{\partial x_0} \Phi_{ij}(t, s, x(s)) \sigma^l(s, x(s)) \sigma^l_j(s, x(s)) \right)_{n\times 1} \\ &+ \sum_{l=1}^{m} tr(\Phi(t, s, x(s)) \sigma^l(s, x(s)) \sigma^{lT}(s, x(s)) \Phi^T(t, s, x(s))). \end{aligned}$$

In this case, Theorem 4.2.1 reduces to a well known result [**?**, 14].

Remark 4.2.6 *Corollary 4.2.2 and several other comparison results play a very important role in the study of highly nonlinear and nonstationary system of stochastic differential equations. The use of the deterministic system of differential inequalities [14, 16, 19, 20, 35] and comparison theorems are attractive and computationally easy to verify. In particular, the condition* $V_x(s, z(t, s, x(s)))\Phi(t, s, x(s))[\sigma(s, x(s)) - \beta(s, x(s))] = \Sigma(s, V(s, z(t, s, x(s))))$ *in* (4.2.5) *is very restrictive. However, in the absence of this relation, the comparison results in the context of deterministic systems of differential inequalities have played a significant role in the theory of stochastic [15, 16, 19, 35] and deterministic [2,6-9, 15-28, 35, 38] systems of differential equations. In the following, we present a comparison theorem in the frame-work of deterministic system of differential inequalities [14, 16, 19, 20, 35].*

Theorem 4.2.7 *Assume that all the hypotheses of Theorem 4.2.1 remain true except* *that stochastic differential inequality* (4.2.5) *is replaced by the following:*

$$E[LV(s, z(t, s, x(s)))|\mathfrak{F}_{\mathfrak{s}}] \leq g(s, E[V(s, z(t, s, x(s)))|\mathfrak{F}_{\mathfrak{s}}]) \tag{4.2.15}$$

where g is defined in Theorem 4.2.1 and $E[V(s, z(t, s, x(s)))]$ exists for $t \geq s \geq t_0$. Then,

$$E[V(t, x(t))] \leq r(t), \qquad \text{for } t \geq t_0, \tag{4.2.16}$$

where $r(t) = r(t, t_0, u_0)$ is the maximal solution of system of nonlinear deterministic comparison differential equations

$$du = g(t, u)dt, \qquad u(t_0) = u_0. \tag{4.2.17}$$

Proof. Set

$$m(s) = E[V(s, z(t, s, x(s)))|\mathfrak{F}_{\mathfrak{s}}], \qquad m(t_0) = E[V(t_0, z(t))].$$

From (b) and assumptions on V implies that $m(s)$ is continuous for $t_0 \leq s \leq t$. Let $h > 0$ be sufficiently small so that $s + h \leq t$. Using (4.2.15), properties of solution process and sub - σ-algebra $\mathfrak{F}_{\mathfrak{t}}$, and concavity of $g(t, u)$, we have

$$\begin{aligned} m(s+h) - m(s) &= E[V(s+h, z(t, s+h, x(s+h)))|\mathfrak{F}_{\mathfrak{s}}] - E[V(s, z(t, s, x(s)))|\mathfrak{F}_{\mathfrak{s}}] \\ &= E\Big[\int_s^{s+h} LV(u, z(t, u, x(u)))du|\mathfrak{F}_{\mathfrak{s}}\Big] \\ &\leq \int_s^{s+h} E\Big[LV(u, z(t, u, x(u)))|\mathfrak{F}_{\mathfrak{s}}\Big]du \\ &\leq \int_s^{s+h} g(u, E[V(u, z(t, u, x(u)))|\mathfrak{F}_{\mathfrak{s}}])du. \end{aligned} \tag{4.2.18}$$

Therefor, it follows that

$$D^+m(s) \leq g(s, m(s)), \qquad t_0 \leq s < t.$$

An application of comparison theorem in [14, 15] establishes the result.

■

Example 4.2.8 We apply Theorem 4.2.7 to Example 4.2.3, and obtain

$$E[LV(s, z(t, s, x(s)))|\mathfrak{F}_\mathfrak{s}] \;\; \leq \;\; e(s)E[V(s, z(t, s, x(s)))|\mathfrak{F}_\mathfrak{s}], \tag{4.2.19}$$

whenever $E[\|\Phi(t, s)x(s)\||^2\mathfrak{F}_\mathfrak{s}]$ exists for $t \geq s \geq t_0$. Here the deterministic comparison differential equation is

$$du = e(s)uds, \qquad u(t_0) = u_0, \tag{4.2.20}$$

where e is defined in Example 4.2.3. From the definition of V and the application of Theorem 4.2.7, we arrive at

$$E[\|x(t, t_0, x_0)\|^2] \leq E[\|z(t, t_0, z_0)\|^2]exp\Big[\int_{t_0}^{t} e(s)ds\Big], \qquad t \geq t_0.$$

Remark 4.2.9 *Based on Example 4.2.8, Examples corresponding to Example 4.2.5 can be constructed, analogously.*

4.3 Comparison Theorems Robustness of Solution Process

Let us consider a nominal system of Itô-Doob type stochastic differential equations

$$\begin{aligned} dy &= G(t, y)dt + H(t, y)dw(t) \\ &= G(t, y)dt + \textstyle\sum_{l=1}^{m} H^l(t, y)dw_l(t), \qquad y(t_0) = y_0, \end{aligned} \tag{4.3.1}$$

with respect to (4.2.1), where G and column vectors of $n \times m$ matrix $H \in C[J \times R^n, R^n]$, $J = [t_0, t_0 + a)$ for $a > 0$; $w(t) = (w_1(t), w_2(t), ..., w_m(t))^T$ is an m-dimensional normalized Wiener process with independent increments; y_0 and $w(t)$ are mutually independent for each $t \geq t_0$. Let $x(t, t_0, x_0)$ and $y(t, t_0, y_0)$ be the solution process of (4.2.1) and (4.3.1) existing for $t \geq t_0$, respectively.

The following theorem provides an estimate on the deviation of the solution process of perturbed system of stochastic differential equations (4.2.1) with respect to the solution process of nominal system of stochastic differential equations (4.3.1) through (t_0, x_0) and (t_0, y_0), respectively.

Theorem 4.3.1 *Let the hypotheses of Theorem 4.2.1 be satisfied except the system of stochastic differential inequalities* (4.2.5) *is replaced by*

$$\begin{cases} LV(s, \Delta z) \leq g(s, V(s, \Delta z)) \\ V_x(s, \Delta z)[\Phi(t, s, x(s))[\sigma(s, x(s)) - \beta(s, x(s))] - \Phi(t, s, y(s))[H(s, y(s)) - \beta(s, y(s))] \\ = \Sigma(s, V(s, \Delta z)) \end{cases} \tag{4.3.2}$$

where

$$\begin{aligned} LV(s, \Delta z) &= V_s(s, \Delta z) + V_x(s, \Delta z)\Big[\frac{1}{2}\Big(\sum_{j=1}^{n}\sum_{l=1}^{m}[\frac{\partial}{\partial x_0}\Phi_{ij}(t, s, x(s))\beta^l(s, x(s))\beta^l_j(s, x(s)) \\ &- \frac{\partial}{\partial x_0}\Phi_{ij}(t, s, y(s))\beta^l(s, y(s))\beta^l_j(s, y(s))]\Big)_{n\times 1} \\ &+ \Big(\sum_{j=1}^{n}\sum_{l=1}^{m}[\frac{\partial}{\partial x_0}\Phi_{ij}(t, s, y(s))\beta^l(s, y(s))H^l_j(s, y(s)) \\ &- \frac{\partial}{\partial x_0}\Phi_{ij}(t, s, x(s))\beta^l(s, x(s))\sigma^l_j(s, x(s))]\Big)_{n\times 1} \\ &+ \frac{1}{2}\Big(\sum_{j=1}^{n}\sum_{l=1}^{m}[\frac{\partial}{\partial x_0}\Phi_{ij}(t, s, x(s))\sigma^l(s, x(s))\sigma^l_j(s, x(s)) \\ &- \frac{\partial}{\partial x_0}\Phi_{ij}(t, s, y(s))H^l(s, y(s))H^l_j(s, y(s))]\Big)_{n\times 1} \end{aligned}$$

$$
\begin{aligned}
&+ \quad \Phi(t,s,x(s))[\sum_{l=1}^{m} \beta_x^l(s,x(s))[\beta^l(s,x(s)) - \sigma^l(s,x(s))] + f(s,x(s)) - \alpha(s,x(s))] \\
&- \quad \Phi(t,s,y(s))[\sum_{l=1}^{m} \beta_x^l(s,y(s))[\beta^l(s,y(s)) - H^l(s,y(s))] + G(s,y(s)) - \alpha(s,y(s))]\Big] \\
&+ \quad \frac{1}{2}\Big(tr(\sum_{l=1}^{m} \frac{\partial^2}{\partial x \partial x} V_i(s,\Delta z) b^l(t,s,x(s),y(s)) b^{lT}(t,s,x(s),y(s)))\Big)_{N\times 1},
\end{aligned} \tag{4.3.3}
$$

$b^l(t,s,x(s),y(s)) = \Phi(t,s,x(s))[\sigma^l(s,x(s)) - \beta^l(s,x(s))] - \Phi(t,s,y(s))[H^l(s,y(s)) - \beta^l(s,y(s))]$ *and* $\Delta z = z(t,s,x(s)) - z(t,s,y(s))$.

Then

$$
V(t, x(t) - y(t)) \le r(t, t_0, u_0), \qquad t \ge t_0, \tag{4.3.4}
$$

whenever

$$
V(t_0, z(t,t_0,x_0) - z(t,t_0,y_0)) \le u_0, \tag{4.3.5}
$$

where $r(t,t_0,u_0)$ *is the maximal solution of* (4.2.6).

Proof. The proof of the theorem follows by imitating the proof of Theorem 4.2.1. The details are omitted [15, 16, 19].

■

In the following, we present an example to illustrate the usefulness of Theorem 4.3.1.

Example 4.3.2 We consider a stochastic perturbed, auxiliary and nominal system of differential equations

$$
\begin{aligned}
dx &= [(A(t) + \Delta A(t))x + a(t) + \Delta a(t) + p(t,x)]dt \\
&\quad + [(B(t) + \Delta B(t))x + b(t) + \Delta b(t) + q(t,x)]dw(t), \qquad x(t_0) = x_0,
\end{aligned} \tag{4.3.6}
$$

$$
dz = [A(t)z + a(t)]dt + [B(t)z + b(t)]dw(t), \qquad z(t_0) = x_0, \tag{4.3.7}
$$

and

$$dy = [A(t)y + a(t) + p(t,y)]dt + [B(t)y + b(t) + q(t,y)]dw(t), \qquad y(t_0) = y_0, \tag{4.3.8}$$

respectively, where $x, y, z \in R^n$; A and B are any $n \times n$ continuous matrix functions defined on J; $J = [t_0, t_0 + a), a > 0$; p and q are any n-dimensional smooth functions defined on $J \times R^n$ into R^n that insure the existence of solution processes of (4.3.6) and (4.3.8); for each $t \in J$, $w(t)$ is a scalar normalized Wiener process independent of x_0 and y_0. Note that $\Phi(t, s, x(s)) = \Phi(t, s, y(s)) = \Phi(t, s)$. We apply Theorem 4.3.1 to (4.3.6) in the context of (4.3.7) and (4.3.8) with $N = 1$ (scalar function), we have

$$\begin{aligned} &LV(s, \Delta z) \\ &= V_s(s, \Delta z) + V_x(s, \Delta z)\Phi(t, s)[(\Delta A(s) - B(s)\Delta B(s))x(s) + p(s, x(s)) - p(s, y(s)) \\ &\quad -B(s)[q(s, x(s)) - q(s, y(s))] + \Delta a(s) - B(s)\Delta b(s)] \\ &\quad +\frac{1}{2}(tr(\frac{\partial^2}{\partial x^2}V(s, \Delta z)c(t, s, x(s), y(s))c^T(t, s, x(s), y(s)))), \end{aligned} \tag{4.3.9}$$

$c(t, s, x(s), y(s)) = \Phi(t, s)[\Delta B(s)x(s) + q(s, x(s)) - q(s, y(s)) + \Delta b(s)]$ and $\Delta z = z(t, s, x(s)) - z(t, s, y(s))$. We assume that

$$\begin{cases} LV(s, \Delta z) \leq e(s)V(s, \Delta z) + \lambda(s) \\ V_x(s, \Delta z)[\Phi(t, s)[\Delta B(s))x(s) + q(s, x(s)) - q(s, y(s)) + \Delta b(s)] = \nu(s)V(s, \Delta z) + \gamma(s). \end{cases} \tag{4.3.10}$$

Under these assumptions we have $g(s, u) = e(s)u + \lambda(s)$ and $\Sigma(s, u) = \nu(s)u + \gamma(s)$. Here, the comparison differential equation is given by

$$du = [e(s)u + \lambda(s)]ds + [\nu(s)u + \gamma(s)]dw(s). \tag{4.3.11}$$

By the application of Theorem 4.3.1, we have

$$\begin{aligned} V(t, x(t) - y(t)) \quad &\leq \quad \Phi(t, t_0)V(t_0, z(t, t_0, x_0) - z(t, t_0, y_0)) + \int_{t_0}^{t} \Phi(t, s)[e(s) - \gamma(s)\nu(s)]ds \\ &\quad + \int_{t_0}^{t} \Phi(t, s)\gamma(s)dw(s)\Big], \qquad t \geq t_0, \end{aligned} \tag{4.3.12}$$

where

$$\Phi(t, t_0) = exp\Big[\int_{t_0}^{t}(e(s) - \frac{1}{2}\nu^2(s))ds + \int_{t_0}^{t} \nu(s)dw(s)\Big].$$

The following theorem provides another version for an estimate on the deviation of a solution process of the perturbed system (4.2.1) with respect to a solution process of nominal system (4.3.1).

Theorem 4.3.3 *Assume that all the hypothesis of Theorem 4.3.1 hold except inequality* (4.3.2) *is replaced by:*

$$E[LV(s, \Delta z)|\mathfrak{F}_{\mathfrak{s}}] \leq g(s, E[V(s, \Delta z)|\mathfrak{F}_{\mathfrak{s}}]) \tag{4.3.13}$$

where g is defined in Theorem 4.2.1 and $E[V(s, \Delta z)]$ exists for $t \geq s \geq t_0$. Then,

$$E[V(t, x(t) - y(t))] \leq r(t), \qquad t \geq t_0, \tag{4.3.14}$$

where $r(t) = r(t, t_0, u_0)$ is the maximal solution of system of nonlinear deterministic comparison differential equations (4.2.17).

Proof. To avoid the monotonicity, the proof is omitted [15, 16, 19].

■

Example 4.3.4 We consider Example 4.3.2 and replace (4.3.10) by

$$E[LV(s, \Delta z)|\mathfrak{F}_{\mathfrak{s}}] \leq e(s)V(s, \Delta z) + \lambda(s) \tag{4.3.15}$$

Under this consideration, we have

$$du = [e(s)u + \lambda(s)]ds, \tag{4.3.16}$$

Hence,

$$\begin{aligned} E[V(t, x(t) - y(t))|\mathfrak{F}_{\mathfrak{s}}] \;\; &\leq \;\; exp\Big[\int_{t_0}^{t} e(s)ds\Big] V(t_0, z(t, t_0, x_0) - z(t, t_0, y_0))) \\ &\quad + \int_{t_0}^{t} exp\Big[\int_{s}^{t} e(u)du\Big]\lambda(s)ds, \qquad t \geq t_0. \end{aligned} \tag{4.3.17}$$

We state without proofs other versions of Theorems 4.3.1, and 4.3.3 [15, 16, 19].

Theorem 4.3.5 *Let the hypotheses of Theorem 4.2.1 be satisfied except the system of stochastic differential inequalities* (4.2.5) *is replaced by*

$$\begin{cases} LV(s, z(t, s, n(s))) \leq g(s, V(s, z(t, s, n(s)))) \\ V_x(s, z(t, s, n(s)))[\Phi(t, s, n(s))[\sigma(s, x(s)) - H(s, y(s)) - \beta(s, n(s))] = \Sigma(s, V(s, z(t, s, n(s)))) \end{cases} \tag{4.3.18}$$

where

$$\begin{aligned} & LV(s, z(t, s, n(s))) \\ = \;\; & V_s(s, z(t, s, n(s))) + V_x(s, z(t, s, n(s)))\Big[\frac{1}{2}\Big(\sum_{j=1}^{n}\sum_{l=1}^{m} \frac{\partial}{\partial x_0}\Phi_{ij}(t, s, n(s))[\beta^l(s, n(s))\beta^l_j(s, n(s)) \\ & -2\beta^l(s, n(s))(\sigma^l_j(s, x(s)) - H^l_j(s, y(s))) \\ & +(\sigma^l(s, x(s)) - H^l(s, y(s)))(\sigma^l_j(s, x(s)) - H^l_j(s, y(s)))]\Big)_{n\times 1} \\ & +\Phi(t, s, n(s))[\sum_{l=1}^{m} \beta^l_x(s, n(s))[\beta^l(s, n(s)) - \sigma^l(s, x(s)) + H^l(s, y(s))] \\ & +f(s, x(s)) - G(s, y(s)) - \alpha(s, n(s))]\Big] \\ & +\frac{1}{2}\left(tr(\sum_{l=1}^{m} \frac{\partial^2}{\partial x \partial x} V_i(s, n(s)) b^{lT}(t, s, x(s), y(s)) b^l(t, s, x(s), y(s)))\right)_{N\times 1}, \end{aligned} \tag{4.3.19}$$

$b^l(t,s,x(s),y(s)) = \Phi(t,s,n(s))[\sigma^l(s,x(s)) - H^l(s,y(s)) - \beta^l(s,n(s))]$, *and* $n(s) = x(s) - y(s)$. *Then*

$$V(t, x(t) - y(t)) \leq r(t, t_0, u_0), \qquad t \geq t_0, \tag{4.3.20}$$

whenever

$$V(t_0, z(t, t_0, x_0 - y_0)) \leq u_0, \tag{4.3.21}$$

where $r(t, t_0, u_0)$ *is the maximal solution of* (4.2.6).

Theorem 4.3.6 *Assume that all the hypothesis of Theorem 4.3.5 hold except* (4.3.18) *is replaced by*

$$E[LV(s, z(t, s, n(s)))|\mathfrak{F}_\mathfrak{s}] \leq g(s, E[V(s, z(t, s, n(s)))|\mathfrak{F}_\mathfrak{s}]) \tag{4.3.22}$$

where g *is defined in Theorem 4.2.1 and* $E[V(s, z(t, s, n(s)))]$ *exists for* $t \geq s \geq t_0$. *Then,*

$$E[V(t, x(t) - y(t))] \leq r(t), \qquad \text{for } t \geq t_0, \tag{4.3.23}$$

where $r(t) = r(t, t_0, u_0)$ *is the maximal solution of system of nonlinear deterministic comparison differential equations* (4.2.17).

In the following we present a corollary that illustrates the significance of Theorems 4.3.1, 4.3.3, 4.3.5, and 4.3.6.

Corollary 4.3.7 *Let us assume that* $\beta \equiv 0$ *in* (4.2.2), $H \equiv 0$ *in* (4.3.1) *and* $G \equiv \alpha$.

Under this assumption, $LV(s, z(t, s, x(s) - y(s))$ *in* (4.3.3) *and* (4.3.19) *reduce to*

$$\begin{aligned} LV(s, \Delta z) &= V_s(s, \Delta z) + V_x(s, \Delta z)[\Phi(t, s, x(s))[f(s, x(s)) \\ &\quad -\alpha(s, x(s))]]\frac{1}{2}\Big(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t, s, x(s))\sigma^l(s, x(s))\sigma_j^l(s, x(s))\Big)_{n\times 1}, \\ &+ \frac{1}{2}\Big(tr(\sum_{l=1}^{m}\frac{\partial^2}{\partial x \partial x}V_i(s, \Delta z)b^l(t, s, x(s), y(s))b^{lT}(t, s, x(s), y(s)))\Big)_{N\times 1}, \end{aligned} \tag{4.3.24}$$

where $b^l(t, s, x(s), y(s)) = \Phi(t, s, x(s))\sigma^l(s, x(s))$, $\Delta z = z(t, s, x(s)) - z(t, s, y(s))$; *and*

$$\begin{aligned} &LV(s, z(t, s, n(s))) \\ &\quad = V_s(s, z(t, s, n(s))) \\ &\qquad +V_x(s, z(t, s, n(s)))\Big[\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t, s, n(s))\sigma^l(s, x(s))\sigma_j^l(s, x(s))\right)_{n\times 1} \\ &\qquad +\Phi(t, s, n(s))[f(s, x(s)) - \alpha(s, y(s)) - \alpha(s, n(s))]\Big] \\ &\qquad +\frac{1}{2}\left(tr(\sum_{l=1}^{m}\frac{\partial^2}{\partial x \partial x}V_i(s, n(s))b^{lT}(t, s, x(s), y(s))b^l(t, s, x(s), y(s)))\right)_{N\times 1}, \end{aligned} \tag{4.3.25}$$

where $b^l(t, s, x(s), y(s)) = \Phi(t, s, n(s))\sigma^l(s, x(s))$ *and* $n(s) = x(s) - y(s)$, *respectively. Under these simplifications, Theorem 4.3.1 , Theorem 4.3.3, Theorem 4.3.5, and Theorem 4.3.6 include respective results [19] as a special cases.*

Example 4.3.8 We apply Theorem 4.3.5 to Example 4.3.2, and we have

$$\begin{aligned} LV(s,z(t,s,n(s))) &= V_s(s,z(t,s,n(s))) + V_x(s,z(t,s,n(s)))\Phi(t,s)\big[[\Delta A(s) \\ &\quad -B(s)\Delta B(s)]x(s) + p(s,x(s)) - p(s,y(s)) \\ &\quad -B(s)[q(s,x(s)) - q(s,y(s))] + B(s)(b(s) - \Delta b(s)) \\ &\quad +\Delta a(s) - a(s)\big] \\ &\quad +\frac{1}{2}tr(\sum_{k=1}^{n}\sum_{j=1}^{n}\frac{\partial^2}{\partial x \partial x}V(s,\Delta z)c(t,s,x(s),y(s))c^T(t,s,x(s),y(s))), \end{aligned} \tag{4.3.26}$$

$c(t,s,x(s),y(s)) = \Phi(t,s)[\Delta B(s)x(s) + q(s,x(s)) - q(s,y(s)) + \Delta b(s) - b(s)]$, and $n(s) = x(s) - y(s)$. We assume that

$$\begin{cases} LV(s,z(t,s,n(s))) \leq e(s)V(s,z(t,s,n(s))) + \lambda(s) \\ V_x(s,z(t,s,n(s)))[\Phi(t,s)[\Delta B(s))x(s) + q(s,x(s)) - q(s,y(s)) + \Delta b(s)] \\ = \nu(s)V(s,z(t,s,n(s))) + \gamma(s). \end{cases} \tag{4.3.27}$$

Under these assumptions, we have $g(s,u) = e(s)u + \lambda(s)$ and $\Sigma(s,u) = \nu(s)u + \gamma(s)$. Here, the comparison differential equation is (4.3.11).

By the application of Theorem 4.3.5, we have

$$\begin{aligned} V(t,x(t)-y(t)) &\leq \Phi(t,t_0)V(t_0,z(t,t_0,x_0-y_0)) + \int_{t_0}^{t}\Phi(t,s)[e(s)-\gamma(s)\nu(s)]ds \\ &\quad + \int_{t_0}^{t}\Phi(t,s)\gamma(s)dw(s)\Big], \qquad t \geq t_0, \end{aligned} \tag{4.3.28}$$

where

$$\Phi(t,t_0) = exp\Big[\int_{t_0}^{t}(e(s) - \frac{1}{2}\nu^2(s))ds + \int_{t_0}^{t}\nu(s)dw(s)\Big].$$

Example 4.3.9 We consider Example 4.3.8 and replace (4.3.27) by

$$E[LV(s,z(t,s,n(s)))|\mathfrak{F}_s] \leq e(s)V(s,z(t,s,n(s))) + \lambda(s) \tag{4.3.29}$$

In this case, deterministic comparison equation is (4.3.16). By the application of Theorem 4.3.6, we have

$$\begin{aligned} E[V(t, x(t) - y(t))|\mathfrak{F}_{t_0}] \;\leq\; & exp\Big[\int_{t_0}^{t} e(s)ds\Big] V(t_0, z(t, t_0, x_0 - y_0))) \\ & + \int_{t_0}^{t} exp\Big[\int_{s}^{t} e(u)du\Big]\lambda(s)ds, \qquad t \geq t_0. \end{aligned} \tag{4.3.30}$$

4.4 Stability Analysis

In this section, we develop the qualitative properties of solution process of (4.2.1). In particular, depending on the mode of convergence, we present several results regarding the stability properties of solution process. This is achieved in the frame-work of Lypunov-type function, system of both deterministic and stochastic differential inequalities and variational comparison theorems. For this purpose, we need to modify the stability properties of comparison system of differential equations. The modified definitions are based on existing definitions[19], and are as follows.

Definition 4.4.1 The trivial solution process of (4.2.1) is said to be

i) (AS_1) almost sure stable, if for each $\epsilon > 0$, $t_0 \in R_+$, there exists a positive function $\delta = \delta(t_0, \epsilon)$ such that the inequality $\|x_0\| \leq \delta$ implies

$$\|x(t)\| < \epsilon, \qquad t \geq t_0, \text{ almost surely (a.s.) }.$$

ii) (AS_2) almost sure asymptotically stable, if it is almost sure stable and if for any $\epsilon > 0$, $t_0 \in R_+$, there exist a positive function $\delta_0 = \delta(t_0)$ and $T = T(t_0, \epsilon)$ such that the inequality $\|x_0\| \leq \delta_0$ implies

$$\|x(t)\| < \epsilon, \qquad t \geq t_0 + T, \quad \text{with a.s.}$$

Definition 4.4.2 The trivial solution process of (4.2.1) is said to be

i) (SM_1) stable in the p-th moment, if for each $\epsilon > 0$, $t_0 \in R_+$ and $p \geq 1$ there exists a positive function $\delta = \delta(t_0, \epsilon)$ such that the inequality $\|x_0\|_p \leq \delta$ implies

$$\|x(t)\|_p < \epsilon, \qquad t \geq t_0$$

where $\|x(t)\|_p = (E[\|x(t)\|^p])^{1/p}$

ii) (SM_2) asymptotically stable in the p-th moment, if it is stable in the p-th moment and if for any $\epsilon > 0$, $t_0 \in R_+$, there exist $\delta_0 = \delta(t_0)$ and $T = T(t_0, \epsilon)$ such that the inequality $\|x_0\|_p \leq \delta_0$ implies

$$\|x(t)\|_p < \epsilon, \qquad t \geq t_0 + T.$$

Definition 4.4.3 The trivial solution processes $u \equiv 0$ and $z \equiv 0$ of (4.2.6) and (4.2.2) are said to be

i) (JAS_1) jointly almost surely stable, if for $\epsilon > 0, t_0 \in R_+$, there exists a $\delta_1 = \delta_1(t_0, \epsilon) > 0$ such that $\sum_{i=1}^{N} V_i(t_0, x_0) \leq \delta_1$ implies

$$\sum_{i=1}^{N} r_i(t, t_0, V(t_0, z(t, t_0, x_0))) < \epsilon, \qquad t \geq t_0,$$

ii) (JAS_2) jointly almost surely asymptotically stable, if it is jointly almost surely stable and if for any $\epsilon > 0$, $t_0 \in R_+$, there exist $\delta_0 = \delta_0(t_0) > 0$ and $T = T(t_0, \epsilon) > 0$ such that the inequality $\sum_{i=1}^{N} V_i(t_0, x_0) \leq \delta_1$ implies

$$\sum_{i=1}^{N} r_i(t, t_0, V(t_0, z(t, t_0, x_0))) < \epsilon, \qquad t \geq t_0 + T.$$

Define

$$\nu(t, t_0, x_0) = u(t, t_0, V(t_0, z(t_0, x_0))$$

and note that $\nu(t_0, t_0, x_0) = V(t_0, x_0)$ and $V \in C[R_+ \times R^n, R^N]$ and its partial derivatives V_t, V_x and V_{xx} exists and are continuous on $R_+ \times R^n$.

Definition 4.4.4 The trivial solution processes $u \equiv 0$ and $z \equiv 0$ of (4.2.17) and (4.2.2) are said to be

i) (JSM_1) jointly stable in the mean, if for $\epsilon > 0, t_0 \in R_+$, there exists a $\delta_1 = \delta_1(t_0, \epsilon) > 0$ such that $\sum_{i=1}^{N} E[V_i(t_0, x_0)] \leq \delta_1$ implies

$$\sum_{i=1}^{N} E[r_i(t, t_0, V(t_0, z(t, t_0, x_0)))] < \epsilon, \qquad t \geq t_0;$$

ii) (JSM_2) jointly asymptotically stable in the mean, if it is jointly stable in the mean and if for any $\epsilon > 0$, $t_0 \in R_+$, there exist $\delta_0 = \delta_0(t_0) > 0$ and $T = T(t_0, \epsilon) > 0$ such that the inequality $\sum_{i=1}^{N} E[V_i(t_0, x_0)] \leq \delta_0$ implies

$$\sum_{i=1}^{N} E[r_i(t, t_0, V(t_0, z(t, t_0, x_0)))] < \epsilon, \qquad t \geq t_0 + T.$$

The following result provide sufficient conditions for stability properties of the trivial solution of (4.2.1) in the context of Theorem 4.2.1.

Theorem 4.4.5 *Let the hypotheses of Theorem 4.2.1 be satisfied. Further assume that*

(i) $\alpha(t, 0) \equiv 0$, $\beta(t, 0) \equiv 0$, $g(t, 0) \equiv 0$, $G(t, 0) \equiv 0$, *for* $t \in R_+$, *and for all* $(t, x) \in R_+ \times R^n$,

(ii) $b(\|x\|) \leq \sum_{i=1}^{N} V_i(t, x) \leq a(t, \|x\|)$, *where* $b \in \mathcal{VK}$, *and* $a \in \mathcal{CK}$.

Then

1. (JAS_1) *of* (4.2.6) *and* (4.2.2) *implies* (AS_1) *of* (4.2.1), *and*

2. (JAS_2) of (4.2.6) and (4.2.2) implies (AS_2) of (4.2.1)

Proof. Let $\epsilon > 0$, $t_0 \in R_+$ be given. Assume that (JAS_1) of (4.2.2) and (4.2.6) holds. From assumption (a), we have $x \equiv 0$ and $u \equiv 0$. Then for $b(\epsilon) > 0$ and $t_0 \in R_+$, there exists a $\delta_1 = \delta_1(\epsilon, t_0)$ such that $\sum_{i=1}^{N} r_i(t_0, t_0, V(t_0, z(t_0, t_0, x_0))) < \delta_1$, implies

$$\sum_{i=1}^{N} r_i(t, t_0, V(t_0, z(t, t_0, x_0))) < b(\epsilon), \qquad t \geq t_0 \tag{4.4.1}$$

where $r(t, t_0, u_0)$ is the maximal solution process of (4.2.6) and $z(t) = z(t, t_0, x_0)$ is the solution process of (4.2.2) through (t_0, x_0). Since $\sum_{i=1}^{N} r_i(t_0, t_0, V(t_0, z(t_0, t_0, x_0))) = \sum_{i=1}^{N} V_i(t_0, x_0)$, $\sum_{i=1}^{N} V_i(t_0, 0) \equiv 0$ and (ii), there exists $\delta(t_0, \epsilon) > 0$ such that $\sum_{i=1}^{N} V_i(t_0, x_0) < \delta_1$ whenever $\|x_0\| \leq \delta$. Now, we claim that if $\|x_0\| \leq \delta$ implies $\|x(t)\| < \epsilon$, $t \geq t_0$ with probability one(w.p.1). Assume that this claim is false, that is, there exists a solution process $x(t, t_0, x_0)$ with $\|x_0\| \leq \delta$, $t_1 > t_0$ and event $A \in \mathfrak{F}_{t_1}$ such that $p(A) > 0$,

$$\|x(t_1)\| = \epsilon \text{ and } \|x(t)\| \leq \epsilon, \qquad t \in [t_0, t_1]. \tag{4.4.2}$$

On the other hand, by Theorem 4.2.1, with $u_0 = V(t_0, z(t, t_0, x_0))$, we have

$$V(t, x(t)) \leq r(t, t_0, V(t_0, z(t, t_0, x_0))), \qquad t \geq t_0. \tag{4.4.3}$$

From (ii) and using the convexity of b, we obtain

$$\begin{aligned} b(\|x(t)\|) &\leq \sum_{i=1}^{m} V_i(t, x(t)) \\ &\leq \sum_{i=1}^{m} r_i(t, t_0, V(t_0, z(t, t_0, x_0))), \qquad t \geq t_0. \end{aligned} \tag{4.4.4}$$

Relations (4.4.1), (4.4.2), (4.4.3) and (4.4.4) lead to the contradiction

$$b(\epsilon) \leq \sum_{i=1}^{m} V_i(t_1, t_0, x_0) < b(\epsilon), \tag{4.4.5}$$

with $p(A) > 0$. This exhibits the almost sure property of the trivial solution of (4.2.1). To prove the second part, lets assume (JAS_2) of (4.2.2) and (4.2.6). We note that (JAS_2) implies, (JAS_1) of (4.2.2) and (4.2.6), and hence one can conclude that (AS_1) property of (4.2.1) is valid. Moreover, by imitating the above argument, one can prove the almost sure asymptotic stability property of the trivial solution process of (4.2.1).

∎

Example 4.4.6 We consider Example 4.2.3, and assume that $p(t, 0) \equiv 0 \equiv q(t, 0)$. Using the estimate on the solution process of (4.2.12) in Example 4.2.3, we obtain

$$\|x(t, t_0, x_0)\|^2 \leq \|x_0\|^2 \|\Phi(t, t_0)\|^2 exp[\int_{t_0}^{t} (e(s) - \frac{1}{2}\nu^2(s))ds + \int_{t_0}^{t} \nu(s)dw(s)],$$

where

$$\|z(t, t_0, x_0)\|^2 = \|\Phi(t, t_0)x_0\|^2 \leq \|x_0\|^2 \|\Phi(t, t_0)\|^2.$$

Depending on the nature of the real parts of the eigen values of $A(s)$ and $B(s)$, $e(s)$ and the magnitude of $\nu^2(s)$ in Example 4.2.3, the joint almost sure stability and joint almost sure asymptotic stability conditions can be imposed on (4.2.13) and (4.2.14) to conclude the corresponding almost sure stability and asymptotic stability of the trivial solution of (4.2.12).

The following result provides sufficient conditions for the p-th moment stability properties of the trivial solution of (4.2.1) in the context of Theorem 4.2.7.

Theorem 4.4.7 *Let the hypotheses of Theorem 4.2.7 be satisfied. Further assume that*

(i) $\alpha(t, 0) \equiv 0$, $\beta(t, 0) \equiv 0$, $g(t, 0) \equiv 0$, $G(t, 0) \equiv 0$, *for* $t \in R_+$, *and for all* $(t, x) \in R_+ \times R^n$,

(ii) $b(\|x\|^p) \le \sum_{i=1}^{N} V_i(t,x) \le a(t, \|x\|^p)$, *where* $p \ge 1$, $b \in \mathcal{VK}$, *and* $a \in \mathcal{CK}$.

Then

1. (JSM_1) *of* (4.2.17) *and* (4.2.2) *implies* (SM_1) *of* (4.2.1), *and*

2. (JSM_2) *of* (4.2.17) *and* (4.2.2) *implies* (SM_2) *of* (4.2.1)

Proof. Let $\epsilon > 0$, $t_0 \in R_+$ be given. Assume that (JSM_1) of (4.2.2) and (4.2.17) holds. From assumption (i), we have $x \equiv 0$ and $u \equiv 0$. Then for $b(\epsilon) > 0$ and $t_0 \in R_+$, there exists a $\delta_1 = \delta_1(\epsilon, t_0)$ such that $\sum_{i=1}^{N} E[r_i(t_0, t_0, V(t_0, z(t_0, t_0, x_0)))|\mathfrak{F}_{t_\circ}] < \delta_1$, implies

$$\sum_{i=1}^{N} E[r_i(t, t_0, V(t_0, z(t, t_0, x_0)))|\mathfrak{F}_{t_\circ}] < b(\epsilon^p), \qquad t \ge t_0 \tag{4.4.6}$$

where $r(t, t_0, u_0)$ is the maximal solution process of (4.2.16) and $z(t) = z(t, t_0, x_0)$ is the solution process of (4.2.2) through (t_0, x_0). Since $\sum_{i=1}^{N} E[r_i(t_0, t_0, V(t_0, z(t_0, t_0, x_0)))|\mathfrak{F}_{t_\circ}] = \sum_{i=1}^{N} E[V_i(t_0, x_0)]$, $\sum_{i=1}^{N} E[V_i(t_0, 0)] = 0$ and (ii), there exists $\delta(t_0, \epsilon) > 0$ such that $\sum_{i=1}^{N} E[V_i(t_0, x_0)] < \delta_1$ whenever $\|x_0\|_p \le \delta$. Now we claim that if $\|x_0\|_p \le \delta$ implies $\|x(t)\|_p < \epsilon$, $t \ge t_0$. Assume that this claim is false, that is, there exists a solution process $x(t, t_0, x_0)$ with $\|x_0\|_p \le \delta$ and a $t_1 > t_0$ such that

$$\|x(t_1)\|_p = \epsilon \text{ and } \|x(t)\|_p \le \epsilon, \qquad t \in [t_0, t_1]. \tag{4.4.7}$$

On the other hand, by Theorem 4.2.7, with $u_0 = E[V(t_0, z(t))|\mathfrak{F}_{t_\circ}]$, we have

$$E[V(t, x(t))|\mathfrak{F}_{t_\circ}] \le r(t, t_0, E[V(t_0, z(t))|\mathfrak{F}_{t_\circ}]), \qquad t \ge t_0. \tag{4.4.8}$$

From (ii) and using the convexity of b, we obtain

$$\begin{aligned} b(E[\|x(t)\|^p]) &\leq \sum_{i=1}^{N} E[V_i(t, x(t))] \\ &\leq \sum_{i=1}^{N} r_i(t, t_0, E[V(t_0, z(t))|\mathfrak{F}_{\mathfrak{t_0}}]), \qquad t \geq t_0. \end{aligned} \tag{4.4.9}$$

Relations (4.4.6), (4.4.7), (4.4.8) and (4.4.9) lead to the contradiction

$$b(\epsilon^p) \leq \sum_{i=1}^{N} E[V_i(t_1, t_0, x_0)] < b(\epsilon^p), \tag{4.4.10}$$

which proves (SM_1).

To prove the second part, lets assume (JSM_2) of (4.2.2) and (4.2.17), we note (JSM_2) implies, (JSM_1) of (4.2.2) and (4.2.17), and hence , one can use the same argument to conclude (SM_2) property of (4.2.1).

■

Example 4.4.8 We apply Theorem 4.4.7 to Example 4.2.8, and obtain

$$E[LV(s, z(t, s, x(s)))|\mathfrak{F}_{\mathfrak{s}}] \leq e(s)E[V(s, z(t, s, x(s)))|\mathfrak{F}_{\mathfrak{s}}], \tag{4.4.11}$$

whenever $E[\|\Phi(t,s)x(s)\||^2|\mathfrak{F}_{\mathfrak{s}}]$ exists for $t \geq s \geq t_0$. Here the deterministic comparison differential equation is (4.2.20). From Example 4.2.8, we have

$$E[\|x(t, t_0, x_0)\|^2] \leq E[\|z(t, t_0, z_0)\|^2] exp\Big[\int_{t_0}^{t} e(s)ds\Big], \qquad t \geq t_0.$$

Further assume that

$$E[\|z(t, t_0, x_0)\|^2] \leq \mu(E[\|x_0\|^2])\tau(t - t_0),$$

where $\tau(u) > 0$ and $\tau(u) \to 0$ as $u \to \infty$. Under this assumption, (JSM_2) of (4.2.20) and (4.2.13) follows immediately. Therefore, by the application of Theorem 4.4.7, we conclude that the trivial solution of (4.2.12) is asymptotically mean square stable.

4.5 Error Estimate and Relative Stability

In this section, we develop the qualitative properties of solution process of (4.2.1) relative to (4.3.1). In particular, depending on the mode of convergence, we present several results regarding error estimates and relative stability properties of solution process. This is achieved in the frame-work of Lypunov-type function, system of both deterministic and stochastic differential inequalities and variational comparison theorems. For this purpose, we need to modify the concept of relative stability properties of comparison system of differential equations. The modified definitions are based on existing definitions[19], and are as follows.

Definition 4.5.1 The two differential systems (4.2.1) and (4.3.1) are said to be

i) (ARS_1) relatively almost surely stable, if for each $\epsilon > 0$, $t_0 \in R_+$, there exists a positive function $\delta_0 = \delta(t_0, \epsilon)$ such that the inequality $\|x_0 - y_0\| \leq \delta_0$ implies

$$\|x(t) - y(t)\| < \epsilon, \qquad t \geq t_0;$$

ii) (AR_2) relatively asymptotically almost surely stable, if it is relatively almost surely stable and if for any $\epsilon > 0$, $t_0 \in R_+$, there exist $\delta = \delta(t_0)$ and $T = T(t_0, \epsilon)$ such that the inequality $\|x_0 - y_0\| \leq \delta$ implies

$$\|x(t) - y(t)\| < \epsilon, \qquad t \geq t_0 + T.$$

Definition 4.5.2 The two differential systems (4.2.1) and (4.3.1) are said to be

i) (RSM_1) relatively stable in p-th moment, if for each $\epsilon > 0$, $t_0 \in R_+$, and $p \geq 1$, there exists a positive function $\delta = \delta(t_0, \epsilon)$ such that the inequality $\|x_0 - y_0\|_p \leq \delta$

implies

$$\|x(t) - y(t)\|_p < \epsilon, \qquad t \geq t_0;$$

ii) (RSM_2) relatively asymptotically stable in the p-th moment, if it is relatively stable in the p-th moment and if for any $\epsilon > 0$, $t_0 \in R_+$, there exist $\delta_0 = \delta_0(t_0)$ and $T = T(t_0, \epsilon)$ such that the inequality $\|x_0 - y_0\| \leq \delta_0$ implies

$$\|x(t) - y(t)\|_p < \epsilon, \qquad t \geq t_0 + T.$$

Definition 4.5.3 The system (4.2.6) and (4.2.2) are said to be

i) (JAS_1) jointly relatively almost surely stable, if for each $\epsilon > 0, t_0 \in R_+$, there exists $\delta_1 = \delta_1(t_0, \epsilon) > 0$ such that the inequality $\sum_{i=1}^{N} V_i(t_0, x_0 - y_0)] \leq \delta_1$ implies

$$\sum_{i=1}^{N} r_i(t, t_0, V(t_0, z(t, t_0, x_0 - y_0))) < \epsilon, \qquad t \geq t_0;$$

ii) (JAS_2) jointly relatively asymptotically almost surely stable, if it is jointly relatively almost surely stable and if for each $\epsilon > 0, t_0 \in R_+$, there exists $\delta_0 = \delta_0(t_0) > 0$ and and $T = T(t_0, \epsilon) > 0$ such that $\sum_{i=1}^{N} V_i(t_0, x_0 - y_0)] \leq \delta_0$ implies

$$\sum_{i=1}^{N} r_i(t, t_0, V(t_0, z(t, t_0, x_0 - y_0))) < \epsilon, \qquad t \geq t_0 + T.$$

Definition 4.5.4 The system (4.2.6) and (4.2.2) are said to be

i) (JSM_1) jointly relatively stable in the mean, if for each $\epsilon > 0, t_0 \in R_+$, there exists $\delta_1 = \delta_1(t_0, \epsilon) > 0$ such that the inequality $\sum_{i=1}^{N} E[V_i(t_0, t_0, x_0 - y_0)] \leq \delta_1$ implies

$$\sum_{i=1}^{N} E[r_i(t, t_0, V(t_0, z(t, t_0, x_0 - y_0)))] < \epsilon, \qquad t \geq t_0;$$

ii) (JSM_2) jointly relatively asymptotically stable in the mean, if it is jointly relatively stable in the mean and if for each $\epsilon > 0, t_0 \in R_+$, there exists $\delta_0 = \delta_0(t_0) > 0$ and and $T = T(t_0, \epsilon) > 0$ such that $\sum_{i=1}^{N} E[V_i(t_0, t_0, x_0 - y_0)] < \delta_0$ implies

$$\sum_{i=1}^{N} E[r_i(t, t_0, V(t_0, z(t, t_0, x_0 - y_0)))] < \epsilon, \qquad t \geq t_0.$$

Definition 4.5.5 The differential systems (4.2.1) and (4.3.1) are said to be almost surely asymptotically equivalent if, for every solution $y(t)$ of (4.3.1), there is a solution $x(t)$ of (4.2.1) such that

$$x(t) - y(t) \to 0 \text{ as } t \to \infty.$$

Definition 4.5.6 The differential system (4.2.1) has asymptotic equilibrium if every solution of the system (4.2.1) tends to almost surely a finite limit vector ξ as $t \to \infty$ and to every constant vector ξ there is a solution $x(t)$ of (4.2.1) on $t_0 \leq t < \infty$ such that $lim_{t\to\infty} = \xi$.

In the following, we present an error estimate and relative stability results in the context of Theorems 4.3.1 and 4.3.3.

Theorem 4.5.7 *Let the hypotheses of Theorem 4.3.1 be satisfied. Further assume that*

$$b(\|x\|) \leq \sum_{i=1}^{N} V_i(t, x), \tag{4.5.1}$$

where $b \in \mathcal{VK}$. Then

1. (JAS_1) *of* (4.2.6) *and* (4.2.2) *implies* (ARS_1) *of* (4.2.1) *and* (4.3.1),
2. (JAS_2) *of* (4.2.6) *and* (4.2.2) *implies* (ARS_2) *of* (4.2.1) *and* (4.3.1).

Proof. By the choice of $u_0 = V(t_0, z(t, t_0, x_0) - z(t, t_0, y_0))$, Theorem 4.3.1 reduces to

$$\sum_{i=1}^{m} V_i(t, x(t) - y(t)) \leq \sum_{i=1}^{m} r_i(t, t_0, V(t_0, z(t, t_0, x_0) - z(t, t_0, y_0))).$$

This together with (4.5.1), we have

$$b(\|x(t) - y(t)\|) \leq \sum_{i=1}^{m} r_i(t, t_0, V(t_0, z(t, t_0, x_0) - z(t, t_0, y_0))).$$

The proofs of statements 1 and 2 follow by repeating the argument used in the proofs of Theorem 4.4.5. The details are omitted.

■

Theorem 4.5.8 *Let the hypotheses of Theorem 4.3.3 be satisfied. Further assume that*

$$b(\|x\|^p) \leq \sum_{i=1}^{N} V_i(t, x), \tag{4.5.2}$$

where $p \geq 1$, $b \in \mathcal{VK}$.

1. (JSM_1) *of* (4.2.17) *and* (4.2.2) *implies* (RSM_1) *of* (4.2.1) *and* (4.3.1),
2. (JSM_2) *of* (4.2.17) *and* (4.2.2) *implies* (RSM_2) *of* (4.2.1) *and* (4.3.1).

Proof. By the choice of $u_0 = V(t_0, z(t, t_0, x_0) - z(t, t_0, y_0))$, Theorem 4.3.3 reduces to

$$\sum_{i=1}^{m} E[V_i(t, x(t) - y(t))] \leq \sum_{i=1}^{m} r_i(t, t_0, E[V(t_0, z(t, t_0, x_0) - z(t, t_0, y_0))]).$$

This together with (4.5.2), we obtain

$$b(E[\|x(t) - y(t)\|^p]) \leq \sum_{i=1}^{m} r_i(t, t_0, E[V(t_0, z(t, t_0, x_0) - z(t, t_0, y_0))|\mathfrak{F}_{\mathfrak{s}}]).$$

The proofs of statements 1 and 2 follow by imitating the proofs of Theorem 4.4.7. The details are omitted.

■

Example 4.5.9 Let us consider Example 4.3.2. We assume $\Delta A \equiv \Delta B \equiv \Delta a \equiv \Delta b \equiv 0$. Under these assumptions, (4.3.6) and (4.3.8) reduces to

$$dx = [A(t)x + a(t) + p(t,x)]dt + [B(t)x + b(t) + q(t,x)]dw(t), \qquad x(t_0) = x_0, \tag{4.5.3}$$

and

$$dy = [A(t)y + a(t) + p(t,y)]dt + [B(t)y + b(t) + g(t,y)]dw(t), \qquad y(t_0) = y_0, \tag{4.5.4}$$

respectively. Here, we assume the auxiliary system (4.2.2) is as:

$$dz = A(t)zdt + B(t)zdw(t), \qquad z(t_0) = x_0. \tag{4.5.5}$$

We note that auxiliary system (4.5.5) acts like a nominal system corresponding to a system (4.5.3). Choosing $V(t,x) = \frac{1}{2}\|x(t)\|^2$ and following the argument used in Example 4.3.2, we have

$$\begin{aligned} &LV(s,\Delta z) \\ &\quad = (x(s)-y(s))^T \Phi^T(t,s)\Phi(t,s)[p(s,x(s)) - p(s,y(s)) \\ &\qquad -B(s)[p(s,x(s)) - p(s,y(s))]] + \frac{1}{2}c(t,s,x(s),y(s))c^T(t,s,x(s),y(s)), \end{aligned} \tag{4.5.6}$$

$c(t,s,x(s),y(s)) = \Phi(t,s)[q(s,x(s)) - q(s,y(s))]$ and $\Delta z = z(t,s,x(s)) - z(t,s,y(s))$.

In this case, (4.3.10) and the comparison differential equation are given by

$$\begin{cases} LV(s,z(t,s,x(s))) \leq e(s)\|z(t,s,x(s))\|^2 = e(s)V(s,z(t,s,x(s))) \\ (x(s)-y(s))^T\Phi^T(t,s)\Phi(t,s)[q(s,x(s))-q(s,y(s))] = \nu(s)V(s,z(t,s,x(s))). \end{cases} \tag{4.5.7}$$

and

$$du = e(s)uds + \nu(s)udw(s), \quad u(t_0) = u_0 = \|z(t,t_0,x_0)\|^2 = V(t_0,z(t,t_0,x_0)), \tag{4.5.8}$$

respectively. Moreover,

$$r(t,t_0,u_0) = u_0 exp\Big[\int_{t_0}^{t}(e(s)-\frac{1}{2}\nu^2(s))ds + \int_{t_0}^{t}\nu(s)dw(s)\Big].$$

From Theorem 2.1, we have $V(t,x(t)-y(t)) \leq r(t,t_0,u_0)$. The final conclusion of Theorem 4.5.7, 4.5.8 follows by assuming

$$limsup_{t\to\infty}\frac{1}{t-t_0}\Big[\int_{t_0}^{t}(e(s)-\frac{1}{2}\nu^2(s))ds + \int_{t_0}^{t}\nu(s)dw(s)\Big]$$

is finite and negative number, respectively. Then joint property of (4.5.7) and (4.5.5) are valid.

We present Theorems corresponding to Theorems 4.3.5 and 4.3.6 parallel to theorems 4.5.7 and 4.5.8. The proofs are omitted [15, 16, 19].

Theorem 4.5.10 *Let the hypotheses of Theorem 4.3.5 be satisfied. Further assume that*

$$b(\|z\|) \leq \sum_{i=1}^{N} V_i(t,z), \tag{4.5.9}$$

where $b \in \mathcal{CK}$. Then

1. (JAS_1) *of* (4.2.6) *and* (4.2.2) *implies* (ARS_1) *of* (4.2.1) *and* (4.3.1);

2. (JAS_2) *of* (4.2.6) *and* (4.2.2) *implies* (ARS_2) *of* (4.2.1) *and* (4.3.1).

Theorem 4.5.11 *Let the hypotheses of Theorem 4.3.5 be satisfied. Further assume that*

$$b(\|z\|^p) \leq \sum_{i=1}^{N} V_i(t,z), \tag{4.5.10}$$

where $p \geq 1$, $b \in \mathcal{CK}$. *Then*

1. (JSM_1) *of* (4.2.17) *and* (4.2.2) *implies* (RSM_1) *of* (4.2.1) *and* (4.2.2);

2. (JSM_2) *of* (4.2.17) *and* (4.2.2) *implies* (RSM_2) *of* (4.2.1) *and* (4.2.2).

Example 4.5.12 Let us consider Example 4.5.9. Choosing $V(t,x) = \frac{1}{2}\|x(t)\|^2$ and following the argument used in Example 4.5.9, we have

$$\begin{aligned} LV(s,z(t,s,n(s))) \;&=\; V_x(s,z(t,s,n(s)))\Phi(t,s)\big[p(s,x(s)) - p(s,y(s)) \\ &\quad -B(s)[q(s,x(s)) - q(s,y(s))] \\ &\quad +B(s)b(s) - a(s)\big] + \frac{1}{2}c(t,s,x(s),y(s))c^T(t,s,x(s),y(s))), \end{aligned} \tag{4.5.11}$$

$c(t,s,x(s),y(s)) = \Phi(t,s)[q(s,x(s)) - q(s,y(s)) - b(s)]$, and $n(s) = x(s) - y(s)$. We assume that

$$\begin{cases} LV(s,z(t,s,n(s))) \leq e(s)V(s,z(t,s,n(s))) \\ V_x(s,z(t,s,n(s)))[\Phi(t,s)[q(s,x(s)) - q(s,y(s))] \\ = \nu(s)V(s,z(t,s,n(s))). \end{cases} \tag{4.5.12}$$

Hence the conclusion of Theorem 4.5.10 follows.

Remark 4.5.13 *We note that conclusion 2 of Theorems 4.5.7 and 4.5.10 implies that stochastic differential systems of differential equations* (4.2.1) *and* (4.3.1) *are almost surely asymptotically equivalent. Similarly, conclusion 2 of Theorems 4.5.8 and 4.5.11* *implies that stochastic differential systems of differential equations* (4.2.1) *and* (4.3.1) *are p-th moment asymptotically equivalent. Moreover, these results also exhibit the asymptotic equilibrium properties in almost sure/p-th moment sense.*

5 Stochastic Dynamic Model for Photosynthesis

5.1 Introduction

All living beings require energy for their maintenance and normal activities. The activities include, reproduction, growth, or other activities. Photosynthetic organisms use light energy to produce glucose. The glucose is used at a latter time to supply energy.

Equations for photosynthesis and cellular respiration are [10]

$$6CO_2 + 6H_2O + Energy \rightarrow C_6H_{12}O_6 + 6O_2$$

and

$$C_6H_{12}O_6 + 6O_2 \rightarrow 6CO_2 + 6H_2O + Energy,$$

respectively. The photosynthetic mechanism is composed of receptors, X, that can be decomposed in two states. The radiant energy, I, transforms the receptors in the state X in to an excited state X^*. Pigments are receptors. When a photon of light strikes a photosynthetic pigment X, an electron in an atom contained in the molecule becomes excited, X^*. Energized electrons move further from the nucleus of the atom. The excited molecule can pass the energy to another molecule or release it in the form of light or heat. The reaction can be expressed as follows:

$$X + I \xrightarrow{k_1} X^*, \tag{5.1.1}$$

($X^* + Z \rightarrow X + Z^*, Z^* + X \rightarrow X^* + Z, X^* + CO_2 \rightarrow X^*$, where Z and X^* are light energy compound and $RUBP$-Ribulose-5-phosphate) where X^* is activated state with respect to carbon-dioxide CO_2, and k_1 is a rate constant. Being in the state X^*, the receptor can react with CO_2. This reaction transfers the receptor back to the state X and yields sugar CH_2O.

$$X^* + CO_2 \xrightarrow{k_2} X + CH_2O, \tag{5.1.2}$$

where k_2 is a rate constant. It is assumed that the capacity of photosynthesis is limited by

$$[X_0] = [X] + [X^*], \tag{5.1.3}$$

where $[X_0]$ is the constant total concentration of the receptors. Let P be the rate of photosynthesis, which is defined as

$$P = hk_2[X^*][CO_2], \tag{5.1.4}$$

where h is the thickness of the homogenous leaf and $[CO_2]$ is the concentration of CO_2.

From reactions (5.1.1), (5.1.2) and assumption (5.1.3), we have

$$\begin{cases} \frac{d[X^*]}{dt} = k_1[X_0]I - k_1[X^*]I - k_2[X^*][CO_2] \\ \frac{d[CO_2]}{dt} = -k_2[X^*][CO_2] \end{cases} \tag{5.1.5}$$

Under the assumptions on the photosynthesis process and using (5.1.3), (5.1.4), and (5.1.5), a dynamic equation for the rate of photosynthesis is as follows:

$$\begin{aligned} \frac{dP}{dt} &= hk_2[CO_2](k_1X_0I - k_1[X^*]I - k_2[X^*][CO_2]) - hk_2[X^*]k_2[CO_2][X^*] \\ &= hk_2k_1[CO_2][X_0]I - k_1Ihk_2[X^*][CO_2] - k_2[CO_2]hk_2[X^*][CO_2] - k_2[X^*]hk_2[X^*][CO_2] \\ &= hk_2k_1[CO_2][X_0]I - k_1IP - k_2([CO_2] - [X^*])P. \end{aligned}$$

Hence,

$$\frac{dP}{dt} = hk_2k_1[CO_2][X_0]I - k_1IP - k_2([CO_2] - [X^*])P. \tag{5.1.6}$$

5.2 A Dynamic Model for Photosynthesis

The photosynthetic process is divided into two complex reactions called light reaction and dark reaction. These reactions are briefly outlined below.

5.2.1 Light Dependent Reactions

The light dependent reactions require light. It produces ATP (Adenosine triphosphate) and $NADPH$ (nicotinamide adenine dinucleotide phosphate): these compounds are needed to produce glucose in the light independent reactions. Based on the following main chemical reactions, $NADP^+$ is the natural biological electron acceptor:

$$2H_2O + 2NADP^+ + I \to 2NADPH + 2H^+ + O_2.$$

The model of light reactions is based on the following assumptions[9].

L_1) The dynamics of the light reactions are determined by the dynamics of the concentration of $NADP^+$ (primary electron acceptor) and $NADPH$. This assumption is made because more ATP is produced in the light reactions than it is immediately needed in the reduction of CO_2 in dark reactions.

L_2) There is a maximum activity level of light reactions determined by some regulating mechanism. This level in weak radiant flux densities is proportional to the radiant flux density. Due to assumption (L_1), this activity level is supposed to determine the total concentrations of $NADP^+$ and $NADPH$ that take part in the process at each constant radiant flux of a longer period.

L_3) The rate of the formation of $NADPH$ is proportional to the product of the radiant flux density I, and the concentration of $NADP^+$.

From assumption(L_2), we have

$$N_{max}(I) = [NADP^+] + [NADPH]. \tag{5.2.1}$$

Here, light reactions are described by:

$$\begin{cases} NADP^+ + I \xrightarrow{k_1} NADPH, \\ NADPH + PGA \xrightarrow{k_2'} NADP^+, \end{cases} \tag{5.2.2}$$

where PGA is a 3-phosphoglyceric acid formed in dark reactions, and k_1 and k_2' are dynamic rate parameters. So the dynamics of light reaction is as follows:

$$\frac{d[NADPH]}{dt} = k_1 I[NADP^+] - k_2'[NADPH][PGA]. \tag{5.2.3}$$

Denoting $N = [NADPH]$, $M = [PGA]$, and $C = [CO_2]$, (5.2.3) reduces to

$$\frac{dN}{dt} = k_1 I(N_{max} - N) - k_2' NM. \tag{5.2.4}$$

This is a nonlinear differential equation for the dynamics of light reactions showing the rate of change in absorbing radiant energy and turning it into so called assimilatory power in the form of $NADPH$.

5.2.2 Dark (Light Independent) Reactions

The light independent reactions occur in light or dark conditions. The products of the light reactions, ATP and $NADPH$, are used to reduce CO_2 to glucose in the

Calvin cycle as

$$CO_2 + NADPH + ATP \to NADP^+ + C_6H_{12}O_6 + ADP.$$

The following assumptions are made for the dark reaction[9].

D_1) The reactions take place in constant environmental conditions (except with respect to the radiation and CO_2 concentration).

D_2) Of the photo-products needed in dark reactions only $NADPH$ is considered.

D_3) The reversible reactions in the calvin cycles are omitted.

D_4) The regeneration of $RUBP$ is in a quasi-stationary state, that is, the rate of the formation of $RUBP$ (Ribulose-5-phosphate) ($\equiv X^*$-excited receptor) is equal to the rate of formation of G3P (glyceraldehyde 3-phosphate).

D_5) There is a maximum constant rate of the regeneration of $RUBP$.

The description of the dark reaction are as follows:

$$\begin{cases} CO_2 + RUBP \xrightarrow{k_3} 2PGA, \\ 2PGA + 2NADPH \xrightarrow{k_4'} 2G3P + 2NADP^+, \end{cases} \tag{5.2.5}$$

with rate constants k_3 and k_4' respectively. The dynamic equations for $[RUBP]$, $[G3P]$ and $[PGA]$ are

$$\frac{d[RUBP]}{dt} = -k_3[CO_2][RUBP] \tag{5.2.6}$$

$$\frac{d[PGA]}{dt} = k_3[CO_2][RUBP] - k_4'[PGA][NADPH] \tag{5.2.7}$$

$$\frac{d[G3P]}{dt} = k_4'[PGA][NADPH]. \tag{5.2.8}$$

From (5.2.6), (5.2.8) and assumption (D_4), the net rate of change of $RUBP$ can be written in the form

$$\frac{d[RUBP]}{dt} = -k_3[CO_2][RUBP] + k_4'[PGA][NADPH]. \qquad (5.2.9)$$

From (5.2.7) and (5.2.9), it is obvious that

$$\frac{d[RUBP]}{dt} = -\frac{d[PGA]}{dt}.$$

From assumption (D_5), we have

$$M_{max} = [RUBP] + [PGA]. \qquad (5.2.10)$$

From (5.2.7), the model for dark reaction will be

$$\frac{d[M]}{dt} = k_3 C(M_{max} - M) - k_4' MN$$

From the light and dark reactions, the overall dynamic model for CO_2 assimilation controlled by the radiant flux density and CO_2 concentration is

$$\begin{cases} \frac{dN}{dt} = k_1 I(N_{max} - N) - k_2' NM \\ \frac{dM}{dt} = k_3 C(M_{max} - M) - k_4' NM, \end{cases} \qquad (5.2.11)$$

were k_1 in (5.2.2) is the rate of the radiant conversion to chemical energy by photo receptors; k_3 in (5.2.5)is the rate of CO_2 fixation by CO_2 receptors; k_2' in (5.2.2) and k_4' in (5.2.5) express the effects of the reaction of the receptors with each other. The rate of photosynthesis, P, is

$$P = k_3[CO_2][RUBP] = k_3 C(M_{max} - M). \qquad (5.2.12)$$

From (5.2.12), (5.1.6) reduces to

$$\frac{dP}{dt} = -k_3^2 C(M_{max} - M)^2 - k_3 C^2 (M_{max} - M). \tag{5.2.13}$$

The concentration of CO_2 in the closed measurement system changes according to the following dynamic equation [35]

$$\frac{dC}{dt} = -k_5 C(1 - \frac{M}{M_{max}}) + k_6, \tag{5.2.14}$$

where k_5 and k_6 are parameters.

5.2.3 Enzyme Reactions

For biochemical reactions, enzymes play very important role. The enzymes and proteins act as a catalyst. Enzymes react selectively on definite compounds called substrate. In the following, we use basic enzymatic reaction mechanisms initiated by Michaelis and Menten [1, 3]. Let S, E, and SE stand for substrate, enzyme, and enzyme-substrate complex that generates a product p, respectvely. The reaction mechanism is as follows:

$$\begin{cases} S + E \underset{k_8}{\overset{k_7}{\rightleftharpoons}} SE \\ SE \xrightarrow{k_9} p + E, \end{cases} \tag{5.2.15}$$

where k_7, k_8 and k_9 are parameters associated with the rates of reactions respectively. Note that, the rate of a reaction is proportional to the product of the concentrations of the reactants.

Let us denote $s = [S]$, $e = [E]$, $s_e = [SE]$, $p = [p]$

$$\begin{cases} \frac{ds}{dt} = -k_7 es + k_8 s_e \\ \frac{ds_e}{dt} = k_7 es - (k_8 + k_9) s_e \\ \frac{de}{dt} = -k_7 es + (k_8 + k_9) s_e \\ \frac{dp}{dt} = k_9 s_e. \end{cases} \tag{5.2.16}$$

We assume that $s(0) = s_0, e(0) = e_0, s_e(0) = 0, p(0) = 0$. From (5.2.16), we note that

$$\begin{cases} p(t) = k_9 \int_0^t s_e(r) dr, \\ \frac{de}{dt} + \frac{ds_e}{dt} = 0, \\ e(t) + s_e(t) = constant = e_0. \end{cases} \tag{5.2.17}$$

From (5.2.17), (5.2.16) reduces to

$$\begin{cases} \frac{ds}{dt} = -k_7 e_0 s + (k_7 s + k_8) s_e \\ \frac{ds_e}{dt} = k_7 e_0 s - (k_7 s + k_8 + k_9) s_e \\ \frac{dp}{dt} = k_9 s_e. \end{cases} \tag{5.2.18}$$

Using (5.2.11), (5.2.14) and (5.2.18), we obtain

$$\begin{cases} \frac{dN}{dt} = k_1 I (N_{max} - N) - k_2' NM \\ \frac{dM}{dt} = k_3 C (M_{max} - M) - k_4' NM \\ \frac{dC}{dt} = -k_5 C (1 - \frac{M}{M_{max}}) + k_6 \\ \frac{ds}{dt} = -k_7 e_0 s + (k_7 s + k_8) s_e \\ \frac{ds_e}{dt} = k_7 e_0 s - (k_7 s + k_8 + k_9) s_e \\ \frac{dp}{dt} = k_9 s_e. \end{cases} \tag{5.2.19}$$

Since,

$$\frac{ds}{dt} + \frac{ds_e}{dt} + \frac{dp}{dt} = 0,$$

we have $s(t) + s_e(t) + p(t) = s_0$. Thus, $s_e = s_0 - s - p$. Since, $s_e = s_0 - s - p$, we formulate the following initial value problem

$$\begin{cases} \frac{dN}{dt} = k_1 I(N_{max} - N) - k_2' NM \\ \frac{dM}{dt} = k_3 C(M_{max} - M) - k_4' NM \\ \frac{dC}{dt} = -k_5 C(1 - \frac{M}{M_{max}}) + k_6 \\ \frac{ds}{dt} = -k_7 e_0 s + (k_7 s + k_8)(s_0 - s - p) \\ \frac{dp}{dt} = k_9(s_0 - s - p), \end{cases} \tag{5.2.20}$$

where $N(0) = N_0, M(0) = M_0, C(0) = C_0, s(0) = s_0, p(0) = 0$. (5.2.20) is a deterministic mathematical dynamic model for photosynthetic process.

5.3 Normalizing and Equilibrium States of Deterministic Model

Let us use the following transformation to normalize the system of differential equations (5.2.20) as:

$$\begin{cases} n = \frac{N}{N_{max}} \\ m = \frac{M}{M_{max}}. \end{cases} \tag{5.3.1}$$

Using (5.3.1) and following the argument used in [10, 11] the effects of enzymatic reactions characterized by replacing k_4' with p , the system of dynamical differential equations (5.2.20) can be written as

$$\begin{cases} \frac{dn}{dt} = k_1 I(1 - n) - k_2 nm \\ \frac{dm}{dt} = k_3 C(1 - m) - N_{max} nmp \\ \frac{dC}{dt} = -k_5 C(1 - m) + k_6 \\ \frac{ds}{dt} = -k_7 e_0 s + (k_7 s + k_8)(s_0 - s - p) \\ \frac{dp}{dt} = k_9(s_0 - s - p), \end{cases} \tag{5.3.2}$$

where $k_2 = k_2' M_{max}$. The system (5.3.2) can be written as

$$dx = f(t,x)dt, \qquad x(t_0) = x_0, \tag{5.3.3}$$

where $x = (n, m, C, s, p)^T \equiv (x_1, x_2, x_3, x_4, x_5)^T \in R^5$;
$f(t,x) = (f_1(t,x), f_2(t,x), f_3(t,x), f_4(t,x), f_5(t,x))^T$; $f_1(t,x) = k_1 I(1-x_1) - k_2 x_1 x_2$, $f_2(t,x) = k_3 x_3 (1-x_2) - N_{max} x_1 x_2 x_5$, $f_3(t,x) = -k_5 x_3(1-x_2) + k_6$, $f_4(t,x) = -k_7 e_0 x_4 + (k_7 x_4 + k_8)(s_0 - x_4 - x_5)$, $f_5(t,x) = k_9(s_0 - x_4 - x_5)$; and $x_0 = (n_0, m_0, C_0, s_0, 0)^T$. The equilibrium states of the dynamic system (5.3.2), $x^* = (n^*, m^*, C^*, s^*, p^*)^T$, satisfies the following system of algebraic equations

$$\begin{cases} 0 = k_1 I(1-n^*) - k_2 n^* m^*, \\ 0 = k_3 C^*(1-m^*) - N_{max} n^* m^* p^*, \\ 0 = -k_5 C^*(1-m^*) + k_6, \\ 0 = -k_7 e_0 s^* + (k_7 s^* + k_8)(s_0 - s^* - p^*), \\ 0 = k_9(s_0 - s^* - p^*). \end{cases} \tag{5.3.4}$$

Solving the system of equations (5.3.4) for $I > 0$, yields

$$\begin{cases} n^* = \frac{k_1 I}{k_1 I + k_2 m^*} \\ (m^*)^2 + \frac{k_1 I N_{max} p^* + k_1 k_3 I C^* - k_2 k_3 C^*}{k_2 k_3 C^*} m^* - \frac{k_1 I}{k_2} = 0 \\ C^* = \frac{k_6}{k_5(1-m^*)} \\ s^* = 0 \\ p^* = s_0. \end{cases} \tag{5.3.5}$$

Now, let us find the linearized system with respect to system (5.3.3) at the equilibrium state. For this purpose, from $x = (n, m, C, s, p)^T$, $x^* = (n^*, m^*, C^*, s^*, p^*)^T$, we define $G(\theta) = f(t, \theta x + (1-\theta)x^*)$ for $0 \le \theta \le 1$. We note that G is continuously differentiable with respect to θ, and hence

$$\frac{d}{d\theta}G(\theta) = f_x(t, \theta x + (1-\theta)x^*)(x - x^*). \tag{5.3.6}$$

By Integrating both sides of (5.3.6) with respect to θ over an interval [0,1], we have

$$G(1) - G(0) = \int_0^1 f_x(t, \theta x + (1-\theta)x^*)(x - x^*)d\theta.$$

This together with the fact that $G(1) = f(t, x)$ and $G(0) = f(t, x^*) = 0$ yields

$$f(t, x) \quad = \quad \int_0^1 f_x(t, \theta x + (1-\theta)x^*)(x - x^*)d\theta. \tag{5.3.7}$$

Adding and subtracting $f_x(t, x^*)$ in (5.3.7) and applying again the generalized Mean Value Theorem[**?**], yields

$$\begin{aligned} f(t, x) &= \int_0^1 f_x(t, \theta x + (1-\theta)x^*)(x - x^*)d\theta \\ &= f_x(t, x^*)(x - x^*) + \int_0^1 [f_x(t, \theta x + (1-\theta)x^*) - f_x(t, x^*)](x - x^*)d\theta \\ &= f_x(t, x^*)(x - x^*) + R(t, x^*, x - x^*)(x - x^*) \end{aligned} \tag{5.3.8}$$

where

$$R(t, x^*, x - x^*)(x - x^*) = \int_0^1 [f_x(t, \theta x + (1-\theta)x^*) - f_x(t, x^*)](x - x^*)d\theta$$

and 5×5 matrices $f_x(t, x^*)$ is defined by:

$$\begin{bmatrix} -(k_1 I + k_2 m^*) & -k_2 n^* & 0 & 0 & 0 \\ -N_{max} s_0 m^* & -(k_3 C^* + N_{max} n^* s_0) & k_3(1-m^*) & 0 & -N_{max} n^* m^* \\ 0 & k_5 C^* & -k_5(1-m^*) & 0 & 0 \\ 0 & 0 & 0 & -(k_7 e_0 + k_8) & -k_8 \\ 0 & 0 & 0 & -k_9 & -k_9 \end{bmatrix} \tag{5.3.9}$$

System (5.3.3) at the equilibrium state is rewritten as

$$dx = [f_x(t, x^*)x + R(t, x^*, x)x]dt, \qquad x(t_0) = x_0, \tag{5.3.10}$$

where $x \equiv x - x^*$. Considering system (5.3.10) as perturbed system of the following unperturbed system

$$dz = f_x(t, x^*) z dt, \qquad z(t_0) = x_0. \tag{5.3.11}$$

Here, system (5.3.11) is assumed to be auxiliary system. We note that the diagonal elements of $f_x(t, x^*)$ are negative. We further assume that if $V(t, x) = \frac{1}{2}\|x\|^2$, then $V_t(t, x) = 0$, $\frac{\partial}{\partial x} V(t, x) = x^T$; $\frac{\partial^2}{\partial x \partial x} V(t, x) = I$, $n \times n$ identity matrix; $\frac{\partial}{\partial x_0} z(t, s, x(s)) = \Phi(t, s)$; and $\frac{\partial^2}{\partial x_0^2} z(t, s, x(s)) = 0$ [3]. The generalized variation of constants formula (3.2.1), reduces to

$$\begin{aligned} \|x(t)\|^2 &= \|z(t)\|^2 \\ &\quad + \int_{t_0}^{t} x^T(s) \Phi^T(t, s) \Phi(t, s) R(s, x^*, x(s)) x(s) ds. \end{aligned} \tag{5.3.12}$$

Remark 5.3.1 a) *Further assume that*

$$x^T(s) \Phi^T(t, s) \Phi(t, s) R(s, x^*, x(s)) x(s) \le \eta(t-s) \lambda \|x(s)\|^2 \tag{5.3.13}$$

and

$$\|z(t,t_0,x_0)\|^2 \le \mu(\|x(s)\|^2)\tau(t-t_0), \tag{5.3.14}$$

where λ is a positive constant. Now, applying Theorem 3.4.2, we conclude that the trivial solution of (5.3.10) *is asymptotically stable.*

b) *In the context of* (5.3.10), (5.3.11), (5.3.12), (5.3.13),(5.3.14), *considering auxiliary system* (5.3.11) *to be a nominal system corresponding to a system* (5.3.10) *and following the argument used in Example 3.3.3, we get*

$$\|x(t)-y(t)\|^2 = \|z(t,t_0,x_0)-z(t,t_0,y_0)\|^2 + 2\int_{t_0}^{t} LV(s,\Delta z)ds \tag{5.3.15}$$

where

$$\begin{aligned} & LV(s,\Delta z) \\ &= (x(s)-y(s))^T\Phi^T(t,s)\Phi(t,s)[R(t,x^*,x(s))x(s)-R(t,x^*,y(s))y(s)], \end{aligned}$$

and $\Delta z = z(t,s,x(s)) - z(t,s,y(s))$. *Further assume that* $LV(s,\Delta z) \le \eta(t-s)\lambda\|x(s)-y(s)\|^2$, *where η and λ are defined in* (5.3.13) *satisfy all conditions in Theorem 3.5.1. Thus, by the application of Theorem 3.5.1, systems* (5.2.10) *and* (5.3.11) *are relatively asymptotically stable. In fact, the solution process* (5.3.10) *has asymptotic equivalence property [6].*

c) *Imitating the argument used in Example 4.4.6, we apply Theorem 4.4.7 in the context of Example 4.2.8, and obtain*

$$LV(s,z(t,s,x(s))) \le e(s)V(s,z(t,s,x(s))). \tag{5.3.16}$$

Here the deterministic comparison differential equation is (4.2.20). *From Exam-*

ple 4.2.8, we have

$$\|x(t,t_0,x_0)\|^2 \leq \|z(t,t_0,z_0)\|^2 exp\Big[\int_{t_0}^{t} e(s)ds\Big], \qquad t \geq t_0.$$

Further assume that

$$\|z(t,t_0,x_0)\|^2 \leq \mu(\|x_0\|^2)\tau(t-t_0),$$

where $\tau(u) > 0$ *and* $\tau(u) \to 0$ *as* $u \to \infty$. *Under this assumption,* (JSM_2) *of* (4.2.20) *and* (4.2.13) *follows immediately. Therefore, by the application of Theorem 4.4.7, we conclude that the trivial solution of* (5.3.10) *is asymptotically stable.*

d) *From Example 4.5.9 and (b), we have*

$$LV(s,z(t,s,n(s))) = V_x(s,z(t,s,n(s)))\Phi(t,s)\big[R(t,x^*,x(s))x(s) - R(t,x^*,y(s))y(s)\big] \tag{5.3.17}$$

where $n(s) = x(s) - y(s)$. *We assume that*

$$LV(s,z(t,s,n(s))) \leq e(s)V(s,z(t,s,n(s))). \tag{5.3.18}$$

Hence the conclusion of Theorem 4.5.11 follows.

5.4 Stochastic Dynamical model for photosynthesis

In this section, we assume that the photosynthesis is under random perturbations. Following the argument used in the deterministic case, stochastic differential equation can be written as a perturbed system:

$$dx = f(t,x)dt + \sigma(t,x)dw(t), \qquad x(t_0) = x_0, \tag{5.4.1}$$

and its unperturbed as well as auxiliary system is as:

$$dz = f_x(t,x^*)zdt + \sigma_x(t,x^*)zdw(t), \qquad z(t_0) = x_0, \tag{5.4.2}$$

where,

$$f(t,x) = R(t,x^*,x-x^*)(x-x^*) + f_x(t,x^*)(x-x^*), \tag{5.4.3}$$

$$\sigma(t,x) = \gamma R(t,x^*,x-x^*)(x-x^*) + \gamma f_x(t,x^*)(x-x^*) \tag{5.4.4}$$

$f_x(t,x^*)$ is defined in (5.3.9).

Using the generalized variation of constants formula(Theorem 3.2.1) with $V(t,x) = \frac{1}{2}x^T x$, we obtain

$$\begin{aligned} V(t,x(t)) &= V(t_0,z(t)) + \int_{t_0}^{t} LV(s,z(t,s,x(s)))ds \\ &\quad + \int_{t_0}^{t} V_x(s,z(t,s,x(s)))\gamma R(t,x^*,x(s))x(s)dw(s), \end{aligned} \tag{5.4.5}$$

where

$$\begin{aligned} &LV(s,z(t,s,x(s))) \\ &\quad = z^T(t,s,x(s))\Phi^T(t,s)\Phi(t,s)(I-\gamma^2 f_x(s,x^*))R(s,x^*,x(s))x(s) \\ &\qquad + \frac{1}{2}c(t,s,x(s))c^T(t,s,x(s)), \end{aligned} \tag{5.4.6}$$

and $c(t,s,x(s)) = \gamma\Phi(t,s)R(s,x^*,x(s))x(s)$.

Remark 5.4.1 a) *From Example 3.4.4, we consider Example 3.2.4, and assume that*

$$E[x^T(s)\Phi^T(t,s)\Phi(t,s)(I-\gamma^2 f_x(s,x^*))R(s,x^*,x(s))x(s) \leq \eta(t-s)\lambda_1 E[\|x(s)\|^2] \tag{5.4.7}$$

and

$$\gamma^2 E[x^T(s)R^T(s,x^*,x(s))\Phi^T(t,s)\Phi(t,s)R(s,x^*,x(s))x(s))] \leq \eta(t-s)\lambda_2 E[\|x(s)\|^2] \tag{5.4.8}$$

From (5.4.7) *and* (5.4.8), $LV(s,z(t,s,x(s)))$ *satisfies Theorem 3.4.2,*

$$\begin{aligned} LV(s,z(t,s,x(s))) &= x^T(s)\Phi^T(t,s)\Phi(t,s)((I-\gamma^2 f_x(s,x^*))R(s,x^*,x(s))x(s) \\ &\quad +\gamma^2 x^T(s)R^T(s,x^*,x(s))\Phi^T(t,s)\Phi(t,s)R(s,x^*,x(s))x(s) \\ &\leq \eta(t-s)\lambda\frac{1}{2}E[\|x(s)\|^2], \qquad t_0 \leq s \leq t, \end{aligned}$$

where $\lambda = \lambda_1 + \lambda_2$.

Further assume that

$$E[\|z(t,t_0,x_0)\|^2] \leq \mu(E[\|x(s)\|^2])\tau(t-t_0). \tag{5.4.9}$$

Therefore, by the application of Theorem 3.4.2, we conclude that the trivial solution of (5.3.10) *is asymptotically mean square stable.*

b) *Imitating Example 3.5.2 and using* (5.4.1),(5.4.2),(5.4.6), (5.4.9), *considering auxiliary system* (5.4.2) *to be a nominal system corresponding to a system* (5.4.1) *and following the argument used in Example 3.2.2, Example 3.3.3, we obtain*

$$
\begin{aligned}
&\|x(t)-y(t)\|^2 \\
&= \|z(t,t_0,x_0)-z(t,t_0,y_0)\|^2 + 2\int_{t_0}^{t} LV(s,\Delta z)ds \\
&\quad +2\int_{t_0}^{t}(x(s)-y(s))^T\Phi^T(t,s)\Phi(t,s)\gamma[R(s,x^*,x(s))x(s)-R(s,x^*,y(s))y(s)]dw(s),
\end{aligned}
\tag{5.4.10}
$$

where

$$
\begin{aligned}
&LV(s,\Delta z) \\
&= (x(s)-y(s))^T\Phi^T(t,s)\Phi(t,s)(I-\gamma^2 f_x(s,x^*))(R(s,x^*,x(s))x(s)-R(s,x^*,y(s))y(s)) \\
&\quad +\frac{1}{2}c(t,s,x(s),y(s))c^T(t,s,x(s),y(s)),
\end{aligned}
$$

$c(t,s,x(s),y(s)) = \gamma\Phi(t,s)[R(s,x^*,x(s))x(s) - R(s,x^*,y(s))y(s)]$ *and* $\Delta z = z(t,s,x(s))-z(t,s,y(s))$. *Further assume that* $2E[LV(s,\Delta z)] \le \eta(t-s)\lambda E[\|x(s)-y(s)\|^2]$, *where* η *and* λ *are defined in* (5.3.12) *and satisfy all conditions in Theorem 3.5.1. Thus, by the application of Theorem 3.5.1, systems* (5.4.1) *and* (5.4.2) *are relatively asymptotically stable in the mean square sense. In fact, the solution process* (5.4.1) *has asymptotic equivalence property [19].*

c) *Considering Example 4.4.8, we apply Theorem 4.4.7 to Example 4.2.8, and obtain*

$$
E[LV(s,z(t,s,x(s)))|\mathfrak{F}_{\mathfrak{s}}] \le e(s)E[V(s,z(t,s,x(s)))|\mathfrak{F}_{\mathfrak{s}}], \tag{5.4.11}
$$

whenever $E[\|\Phi(t,s)x(s)\||^2|\mathfrak{F}_{\mathfrak{s}}]$ *exists for* $t \ge s \ge t_0$. *Here the deterministic* *comparison differential equation is* (4.2.20). *From Example 4.2.8, we have*

$$E[\|x(t,t_0,x_0)\|^2] \le E[\|z(t,t_0,z_0)\|^2]exp\Big[\int_{t_0}^{t} e(s)ds\Big], \qquad t \ge t_0.$$

Further assume that

$$E[\|z(t,t_0,x_0)\|^2] \le \mu(E[\|x_0\|^2])\tau(t-t_0),$$

where $\tau(u) > 0$ *and* $\tau(u) \to 0$ *as* $u \to \infty$. *Under this assumption,* (JSM_2) *of (4.2.20) and* (5.4.2) *follows immediately. Therefore, by the application of Theorem 4.4.7, we conclude that the trivial solution of* (5.4.1) *is asymptotically mean square stable.*

d) *From Example 4.5.9, and (b), we obtain*

$$\begin{aligned} &LV(s,z(t,s,n(s))) \\ &= V_x(s,z(t,s,n(s)))\Phi(t,s)(I-\gamma^2 f_x(s,x^*))(R(s,x^*,x(s))x(s) - R(s,x^*,y(s))y(s)) \\ &\quad +\frac{1}{2}c(t,s,x(s),y(s))c^T(t,s,x(s),y(s))), \end{aligned} \tag{5.4.12}$$

$c(t,s,x(s),y(s)) = \gamma\Phi(t,s)[R(s,x^*,x(s))x(s) - R(s,x^*,y(s))y(s)]$, *and* $n(s) = x(s) - y(s)$. *We assume that*

$$\begin{cases} LV(s,z(t,s,n(s))) \le e(s)V(s,z(t,s,n(s))) \\ V_x(s,z(t,s,n(s)))[\Phi(t,s)[R(s,x^*,x(s))x(s) - R(s,x^*,y(s))y(s)] \\ = \nu(s)V(s,z(t,s,n(s))). \end{cases} \tag{5.4.13}$$

Hence the conclusion of Theorem 4.5.10 follows.

5.5 Numerical Illustration

In this section, we conduct a numerical study of dynamic model of photosynthetic process. For this purpose, we consider the numerical solution of the system of differential equation (5.3.2), including s_e, using MATLAB and the following rate constants: $k_1 = 0.1$; $k_2 = 0.6$;$k_3 = 1.0$; $k_5 = 0.0036$; $k_6 = 0.000105$; $k_7 = 0.08$; $k_8 = 0.08$; $k_9 = 0.01$; $I = 424$. We used the parameters as suggested in [11]. Figures 5.1 to 5.6 show the numerical solution of the normalized differential equation(5.3.2). The model describes the property of light and dark reactions. The processes occurring in the light reactions are very rapid, Figure 2 compared with those occurring in the dark reactions. Thus, the concentration of photoreceptors in the model is almost constant. The dark reactions show more dynamic behaviour.

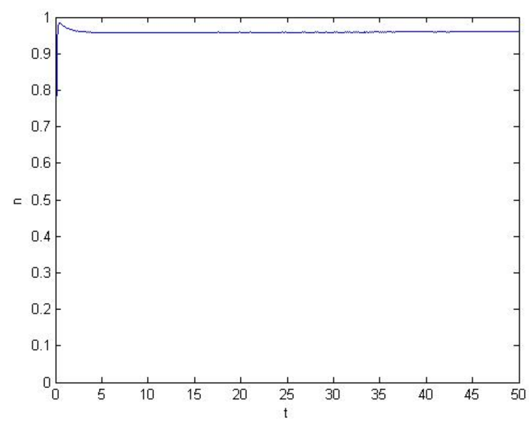

Figure 5.1: NADPH concentration as a function of time(min)

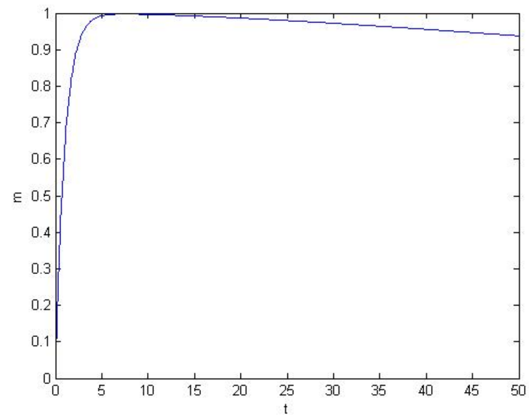

Figure 5.2: PGA concentration as a function of time(min)

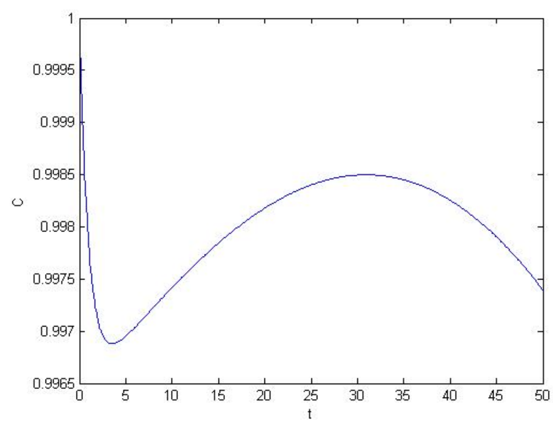

Figure 5.3: Carbon-dioxide concentration as a function of time(min)

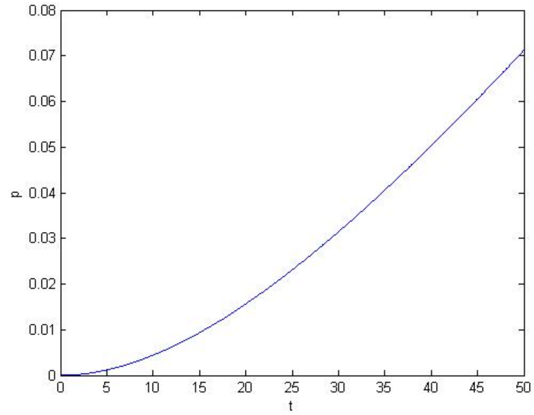

Figure 5.4: The product p as a function of time(min)

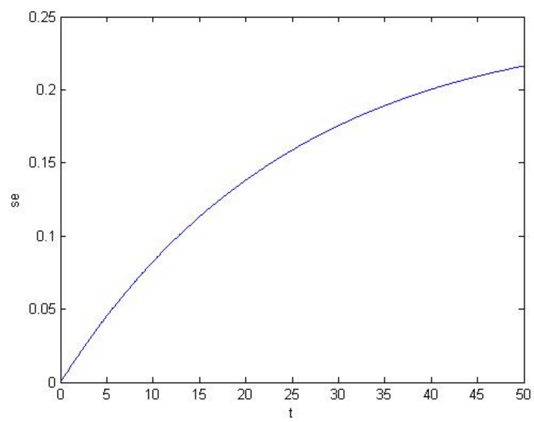

Figure 5.5: Concentration of the complex s_e as a function of time(min)

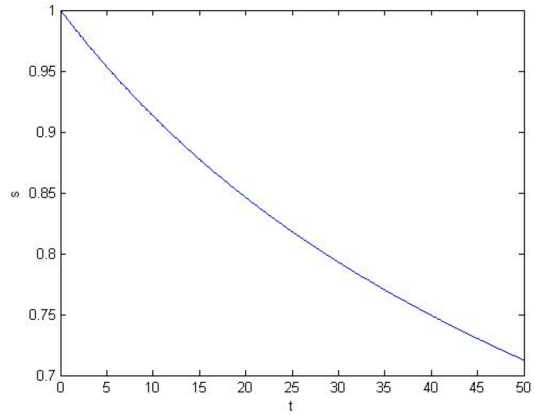

Figure 5.6: Substrate concentration s as a function of time(min)

Remark 5.5.1 *Figures 5.4 to 5.6 confirms the steady state exhibited in* (5.3.5) *in the context of* $e(t) + s_e(t) = e_0$ *and* $s(t) + s_I(t) + p(t) = s_0$ *where* e_0 *and* s_0 *are given constants. We further examine the qualitative properties of* (5.3.10) *and* (5.3.11). *For this purpose we compute* $f_x(t, x^*)$, *and it is the matrix whose entries are as follows*

$$\begin{bmatrix} -(42.4+0.6m^*) & -0.6n^* & 0 & 0 & 0 \\ -N_{max}s_0m^* & -(C^*+N_{max}n^*s_0) & (1-m^*) & 0 & -N_{max}n^*m^* \\ 0 & 0.0036C^* & -0.0036(1-m^*) & 0 & 0 \\ 0 & 0 & 0 & -(0.08e_0+0.08) & -0.08 \\ 0 & 0 & 0 & -0.01 & -0.01 \end{bmatrix}$$

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