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Analytic Functions with Real Boundary Values in Smirnov Classes E^p

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by

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Dedication

To my family who have supported my mathematical enterprise.

Acknowledgments

I wish to express my deepest gratitude to my advisor Dmitry Khavinson. When I first came to know him, I admired his breadth and depth of knowledge and the dedication to which he applied himself to teaching. Later I came to appreciate how he approached mathematical thought: much like an artist applies paint to canvas. As my advisor he has been a well of strength and support that has fortified me in times of need and he has always acted in my best interests.

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Table of Contents

List of Figures

Abstract

This thesis concerns the classes of analytic functions on bounded, n-connected domains known as the Smirnov classes E^p, where $p > 0$. Functions in these classes satisfy a certain growth condition and have a relationship to the more well known classes of functions known as the Hardy classes H^p. In this thesis I will show how the geometry of a given domain will determine the existence of non-constant analytic functions in Smirnov classes that possess real boundary values. This is a phenomenon that does not occur among functions in the Hardy classes.

The preliminary and background information is given in Chapters 1 and 3 while the main results of this thesis are presented in Chapters 2 and 4. In Chapter 2, I will consider the case of the simply connected domain and the boundary characteristics that allow non-constant analytic functions with real boundary values in certain Smirnov classes. Chapter 4 explores the case of an n-connected domain and the sufficient conditions for which the aforementioned functions exist. In Chapter 5, I will discuss how my results for simply connected domains extend Neuwirth-Newman's Theorem and finish with an open problem for n-connected domains.

Chapter 1

Preliminaries and background

1.1 Preliminaries

Let Ω be a bounded, simply connected domain in the complex plane $\mathbb{C}$, and suppose f is a real-valued, continuous function on $\partial\Omega$. A function u that is continuous on $\overline{\Omega}$ is a solution to the Dirichlet problem with data f if $u = f$ on $\partial\Omega$ and u is harmonic in Ω, i.e. $\Delta u = 0$ on Ω.

DEFINITION 1.1.1 *A domain Ω is called a Dirichlet Region if the Dirichlet Problem can be solved for each real-valued, continuous function f on $\partial\Omega$.*

A classic example of a Dirichlet Region is the unit circle. The solution to the Dirichlet Problem is

$$u(re^{it}) = \frac{1}{2\pi}\int_{\pi}^{\pi} P(r, t-\theta) f(e^{i\theta}) d\theta,$$

where

$$P(r,t) = \frac{1-r^2}{1-2r\cos(t)+r^2}$$

is the Poisson kernel.

The punctured disk, $\Omega = \{z : 0 < |z| < 1\}$ is an example of a domain that is not a Dirichlet Region. Choose f so that $f(0) = 1$ and $f(e^{it}) = 0$ for $0 \leq t \leq 2\pi$. Then if u is a solution to this problem, u would be harmonic in Ω and continuous in $\bar{\mathbb{D}}$, the closure of the unit disk. Thus u would extend to be harmonic in $\mathbb{D}$. But, $u(0) = 1$, which means that u attains a maximum value in $\mathbb{D}$. This contradicts the maximum principle for harmonic functions.

The sufficient criteria for a domain to be a Dirichlet Region in the plane is rather simple and is stated for reference in the following theorem, which will not be proven here (cf. [6], Ch. 1).

THEOREM 1.1 *Let Ω be a domain in $\mathbb{C}$. If no connected component of $\partial\Omega$ degenerates to a point then Ω is a Dirichlet Region.*

Domains that are classified as Dirichlet Regions necessarily possess Green functions (cf. [6], Ch. 1).

DEFINITION 1.1.2 *Let Ω be a domain in $\mathbb{C}$ and $a \in \Omega$. A function $g(z,a)$ with pole at a is called the Green function for Ω if*
(i) $g(z,a)$ is harmonic on $\Omega \setminus \{a\}$
(ii) $g(z,a) + \log|z-a|$ is harmonic near a
(iii) $\lim_{z\to\zeta} g(z,a) = 0$ for all $\zeta \in \partial\Omega$.

Given a point $a \in \Omega$, where Ω is a Dirichlet Region, one can also associate with Ω the harmonic measure, ω_a. This measure is determined by the positive linear functional $\Lambda u = \tilde{u}(a)$ which maps each real-valued continuous function u on $\partial\Omega$ to the value of its harmonic extension to Ω at the point a. By the F. Riesz Representation Theorem, ω_a is the unique measure on $\partial\Omega$ such that

$$\tilde{u}(a) = \int_{\partial\Omega} u d\omega_a, \ \ u \in C(\partial\Omega).$$

Its total mass is 1. This is because

$$1 = \tilde{1} = \int_{\partial\Omega} 1 d\omega_a = ||\omega_a||.$$

Furthermore, when the boundary of Ω is smooth,

$$d\omega_a = \frac{1}{2\pi}\frac{\partial g(\cdot,a)}{\partial n}ds,$$

where $g(\cdot,a)$ is the Green function for Ω with pole at a, $\frac{\partial}{\partial n}$ is the inner normal derivative and ds is arc length.

Let $\mathbb{D}$ be the unit disk. A class of functions that is noteworthy is the Nevanlinna class N. A function $f(z)$ analytic in the unit disk is said to be of class N if the integrals

$$\int_0^{2\pi} \log^+|f(re^{i\theta})|d\theta$$

are bounded for $0 < r < 1$ (cf. [2], Ch. 2). This is a broad class of functions that contains the class N^+ and the Hardy classes H^p which will be discussed in the next section. In fact, as we shall see, $H^p \subset N^+ \subset N$.

Let us recall the definitions of three types of functions: a Blaschke product, a singular inner function, and an outer function.

DEFINITION 1.1.3 *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of non-zero complex numbers in $\mathbb{D}$ such that $\sum_{n=1}^{\infty}(1 - |a_n|) < \infty$. Then the function*

$$B(z) = z^k \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \left(\frac{a_n - z}{1 - \bar{a}_n z} \right),$$

where k is a non-negative integer is called a Blaschke Product.

One property of a Blaschke product is that $|B(z)| = 1$ a.e on the unit circle $\mathbb{T}$.

DEFINITION 1.1.4 *Let $\mu(t)$ be a bounded nondecreasing function such that $\mu'(t) = 0$ a.e. A function of the form*

$$S(z) = \exp\left\{ -\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t) \right\},$$

is called a singular inner function.

A singular inner function is analytic in $\mathbb{D}$ and has the properties: $0 < |S(z)| \leq 1$ and $|S(e^{i\theta})| = 1$ a.e $\mathbb{T}$.

DEFINITION 1.1.5 *A function of the form*

$$F(z) = e^{ik} \exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log(\psi(t)) dt \right\},$$

where k is a real number, $\psi(t) \geq 0$, and $\log(\psi(t)) \in L^1$ is called an outer function.

Every function in the Nevalinna class can be factored using the three previously defined functions. The following theorem is from [2], Chapter 2.

THEOREM 1.2 *Every function $f(z) \not\equiv 0$ of the class N can be expressed in the form*

$$f(z) = B(z)[S_1(z)/S_2(z)]F(z)$$

where $B(z)$ is a Blaschke product, $S_1(z)$ and $S_2(z)$ are singular inner functions, and $F(z)$ is an outer function for the class N (where $\psi(t) = |f(e^{it})|$). Conversely every function of the form described above belongs to N.

The subclass of functions, N^+, will be mentioned later on in this thesis. The functions in this class are identified as those functions in the Nevalinna class for which $S_2(z) \equiv 1$.

1.2 The Hardy classes

Let G be a bounded, simply connected domain in $\mathbb{C}$ with Jordan boundary $\Gamma = \partial G$. In this thesis we will be concerned with two types of classes of analytic functions, namely the Hardy classes $H^p(G)$ and the Smirnov classes $E^p(G)$, $0 < p \leq \infty$. When $p = \infty$, these spaces coincide as the space of bounded, analytic functions in G. When $0 < p < \infty$, the Hardy classes can be defined in two different ways. The first definition is as follows (cf. [2], Ch. 10).

DEFINITION 1.2.1 *A function f that is analytic in G belongs to the class $H^p(G)$ for $0 < p < \infty$ if the subharmonic function $|f(z)|^p$ has a harmonic majorant in G, i.e. there is a harmonic function $v(z)$ in G such that*

$$|f(z)|^p \leq v(z), \quad z \in G.$$

The second way to describe the Hardy class $H^p(G)$ when $0 < p < \infty$ is by using a regular exhaustion of G (cf. [6], Ch. 2 and Ch. 3).

DEFINITION 1.2.2 *Let G be a domain. A regular exhaustion of G is a sequence $\{G_n\}$ of subdomains of G with the following three properties:*
(i) $\overline{G_n} \subset G_{n+1}$, $n = 1, 2, ...$
(ii) $\bigcup_{n=1}^{\infty} G_n = G$
(iii) each component of ∂G_n is a bounded Jordan curve.

PROPOSITION 1 A function f analytic in G is of class $H^p(G)$, $0 < p < \infty$, if and only if for each regular exhaustion $\{G_n\}$ of G

$$\overline{\lim_{n\to\infty}} \int_{\partial G_n} |f|^p d\omega_{n,a} < \infty,$$

where $\omega_{n,a}$ is the harmonic measure on ∂G_n for some point $a \in G_1$ (cf. [6], Ch.3).

From this proposition it is easy to see that $H^p(G) \subset H^q(G)$ whenever $p > q$.

One can define a norm for each of the Hardy classes. When $p = \infty$ the norm is simply

$$||f|| = \sup_{z\in G} |f(z)|.$$

When $0 < p < \infty$, the norm is defined as

$$||f|| = [v(z_0)]^{1/p},$$

where v is the least harmonic majorant of f and z_0 is a fixed point in G.

Let G^* be another simply connected domain. By the Riemann mapping theorem we know that there exists a conformal mapping φ from G^* onto G. It follows that if $f \in H^p(G)$ then $f(\varphi(w)) \in H^p(G^*)$. This means that the space $H^p(G)$ is conformally invariant. Furthermore, the correspondence $f \leftrightarrow f \circ \varphi$ is an isometric isomorphism between these two spaces.

PROPOSITION 2 Let G be a simply connected domain and $\varphi : \mathbb{D} \to G$ be a conformal mapping of the unit disk onto G. Then the correspondence $f \leftrightarrow f \circ \varphi$ is an isometric isomorphism between $H^p(G)$ and $H^p(\mathbb{D})$.

One consequence of this proposition is that when studying functions in $H^p(G)$, it suffices to study their counterparts in $H^p(\mathbb{D})$ via a conformal mapping $\varphi : \mathbb{D} \to G$. Therefore, we shall now turn our attention to properties of functions in the Hardy classes on the unit disk.

As was mentioned previously, $H^p(\mathbb{D}) \subset N^+$ for all p. Thus, each function has the factorization $f(z) = B(z)S(z)F(z)$. Furthermore, this factorization is unique (cf. [2], Ch. 2).

THEOREM 1.3 *Every function $f(z) \not\equiv 0$ of class $H^p(\mathbb{D})$ has a unique factorization (up to a unimodular constant) of the form $f(z) = B(z)S(z)F(z)$, where $B(z)$ is a Blaschke product, $S(z)$ is a singular inner function, and $F(z)$ is an outer function for the class H^p (where $\psi(t) \in L^p$ and $\psi(t) = |f(e^{it})|$). Conversely, every such product $B(z)S(z)F(z)$ belongs to $H^p(\mathbb{D})$.*

The following theorem is known as the Polubarinova-Kotchine Theorem (cf. [2], Ch. 2). It gives a way to identify those functions in N^+ that are in H^p.

THEOREM 1.4 (Polubarinova - Kotchine) *If $f \in N^+$ and $f(e^{i\theta}) \in L^p$ for some $p > 0$, then $f \in H^p$.*

When $p \geq 1$, functions in $H^p(\mathbb{D})$ can be represented by Poisson integrals of their boundary values (cf. [2], Ch. 3).

THEOREM 1.5 *A function f analytic in $\mathbb{D}$ is representable in the form*

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} P(r, t-\theta)g(t)dt,$$

as a Poisson integral of a function $g \in L^1$ if and only if $f \in H^1$. Here, $g(t) = f(e^{it})$ a.e.

COROLLARY 1.0.1 *A function $f(z)$ analytic in the unit disk is the Poisson integral of a function $g \in L^p$ ($1 \leq p \leq \infty$) if and only if $f \in H^p(\mathbb{D})$.*

COROLLARY 1.0.2 *Let $f(z) = B(z)[S_1(z)/S_2(z)]F(z) \in N$, where B is a Blaschke product, S_1 and S_2 are singular inner functions, and F is an outer function. If $\log[f(z)/B(z)] \in H^1(\mathbb{D})$, then $|S_1(z)/S_2(z)| \equiv 1$.*

When an analytic function maps the unit disk conformally onto the interior of a Jordan curve, we can determine if this function belongs to the class H^1 by knowing whether or not the curve is rectifiable (cf. [2], Ch. 3).

THEOREM 1.6 *Let $\varphi(z)$ map the unit disk $\mathbb{D}$ conformally onto the interior of a Jordan curve Γ. Then Γ is rectifiable if and only if $\varphi' \in H^1$.*

Interpreted geometrically, this theorem basically says that if

$$L_r = r\int_0^{2\pi} |\varphi'(re^{i\theta})|d\theta$$

is the length of the image of the circle $|w| = r$, then as $r \to 1$, the lengths L_r stay bounded if and only if Γ possesses finite length.

1.3 The Smirnov classes

The Smirnov class $E^p(G)$ for $0 < p < \infty$ has a definition somewhat similar to the statement of Proposition 1 (cf. [2], Ch. 10).

DEFINITION 1.3.1 *A function f analytic in G is said to be of class $E^p(G)$, $0 < p < \infty$, if there exists a sequence of rectifiable curves $\{\Gamma_n\}$ in G converging to Γ such that the interiors of the Γ_n exhaust G and*

$$\varlimsup_{n\to\infty}\int_{\Gamma_n} |f(z)|^p|dz| < \infty.$$

The classes E^p and H^p can coincide. For example, if $G = \mathbb{D}$ then $\frac{\partial g(\cdot,a)}{\partial n}$ is the Poisson Kernel. So if $a = 0$, then $d\omega_0 = ds$, and the sets $H^p(G)$ and $E^p(G)$ are equal. Among all of the choices of sequences of rectifiable curves $\{\Gamma_n\}$ that can be chosen, of particular interest are those curves that are obtained as the images of concentric circles under an arbitrary conformal mapping of the unit disk onto G. The following theorem and its proof can be found in [2], Ch. 10.

THEOREM 1.7 *Let $\varphi : \mathbb{D} \to G$ be a conformal mapping, and C_r be the image under φ of the circle $|w| = r$. Then if $f \in E^p(G)$,*

$$\sup_{r<1} \int_{C_r} |f(z)|^p |dz| < \infty.$$

Proof. Let $z_0 = \varphi(0)$ and suppose $\varphi'(0) > 0$. Let $f \in E^p(G)$ and $\{\Gamma_n\}$ be a sequence of Jordan rectifiable curves in G such that $\overline{\lim}_{n\to\infty} \int_{\Gamma_n} |f(z)|^p |dz| < M < \infty$. Let G_n denote the interior of C_n, and without loss of generality, suppose $z_0 \in G_n$ for all n. Define $\varphi_n(w)$ to be the conformal mapping of $\mathbb{D}$ onto G_n which is normalized by $\varphi_n(0) = z_0$ and $\varphi_n'(0) > 0$. Then Theorem 1.6 implies that $\varphi_n' \in H^1$. We also have that $f(\varphi_n(w))$ is continuous in $\mathbb{D}$. By the Carathéodory Convergence Theorem (cf. [3], Ch. 3), φ_n tends to φ uniformly in each disk $|w| \leq r < 1$. Therefore,

$$\begin{aligned}
\int_{C_r} |f(z)|^p |dz| &= \int_{|w|=r} |f(\varphi(w))|^p |\varphi'(w)| dw \\
&= \lim_{n\to\infty} \int_{|w|=r} |f(\varphi_n(w))|^p |\varphi_n'(w)| dw \\
&\leq \underline{\lim}_{n\to\infty} \int_{|w|=1} |f(\varphi_n(w))|^p |\varphi_n'(w)| dw \\
&= \underline{\lim}_{n\to\infty} \int_{\Gamma_n} |f(z)|^p |dz| < M < \infty.
\end{aligned}$$

$\square$

The theorem that follows from this one describes a key relationship that exists between functions in the Smirnov class $E^p(G)$ and those in the Hardy class $H^p(\mathbb{D})$. The following theorem is known as the Keldysh-Lavrentiev Theorem (cf. [2] Ch.10).

THEOREM 1.8 *Let $\varphi : \mathbb{D} \to G$ be a conformal mapping of $\mathbb{D}$ onto G. Then*

$$f(z) \in E^p(G) \text{ if and only if } f(\varphi(w))[\varphi'(w)]^{1/p} \in H^p(\mathbb{D}).$$

Note that since φ is a conformal mapping, then $\varphi' \neq 0$ in $\mathbb{D}$. Since $\mathbb{D}$ is simply connected, this implies that $\log(\varphi')$ is single-valued.

As was mentioned above, it is possible that the classes $E^p(G)$ and $H^p(G)$ coincide as sets. This occurs when G has certain properties, and can be stated as follows (cf. [14]).

THEOREM 1.9 *Let $\varphi(w)$ be an arbitrary conformal mapping of $\mathbb{D}$ onto G. Then $H^p(G) = E^p(G)$ if and only if there exist two positive constants a and b such that*

$$a \leq |\varphi'(w)| \leq b, \;\; w \in \mathbb{D}.$$

Thus, if all of the boundary curves of G are real-analytic, then $H^p(G) = E^p(G)$ as sets. The following example illustrates how the geometry of the domain determines the relationship between the sets of H^p and E^p.

EXAMPLE 1 Let G be a simply connected domain bounded by a curve that is analytic except at the point z_0, where there is a corner with interior angle α. Let $0 < p < \infty$. If $0 < \alpha < \pi$, then $E^p(G) \subsetneq H^p(G)$, and if $\pi < \alpha < 2\pi$ then $H^p(G) \subsetneq E^p(G)$.

To see this, let us first assume that $0 < \alpha < \pi$. Without loss of generality let $z_0 = 0$. Applying the mapping

$$g(z) = z^{\pi/\alpha}$$

to G produces an angle of π at 0. Since

$$g'(z) = \frac{\pi}{\alpha} z^{\pi/\alpha - 1}$$

and since $\pi/\alpha - 1 > 0$, $g'(z) \to 0$ as $z \to 0$. This implies that

$$[g^{-1}]'(\zeta) = \frac{1}{g'(g^{-1}(\zeta))} \to \infty \quad \text{as} \quad g^{-1}(\zeta) \to 0.$$

Let $h : \mathbb{D} \to g(G)$ be a conformal mapping of the unit disk onto $g(G)$. Then

$$\varphi = g^{-1} \circ h : \mathbb{D} \to G$$

is a conformal mapping of the unit disk onto G. Assume that $\varphi(1) = 0$. Since $\varphi'(w) \to \infty$ as $w \to 1$, then by Theorem 1.9, $E^p(G) \neq H^p(G)$. Let $f \in E^p(G)$. Then

$$F = f(\varphi)[\varphi']^{1/p} \in H^p(\mathbb{D}).$$

Since $\frac{1}{\varphi'} \in H^\infty(\mathbb{D})$, we have

$$f(\varphi) = \frac{F}{(\varphi')^{1/p}} \in H^p(\mathbb{D}).$$

This implies that $f \in H^p(G)$.

Next, assume that $0 < \alpha < 2\pi$. Let $g(z)$ be defined as above. Then $\frac{\pi}{\alpha} - 1 < 0$, and $g'(z) \to \infty$ as $z \to 0$. This implies that

$$[g^{-1}]'(\zeta) \to 0 \quad \text{as} \quad g^{-1}(\zeta) \to 0.$$

Let φ also be as defined above. Then $\varphi'(w) \to 0$ as $w \to 1$, so $H^p(G) \neq E^p(G)$. Now if $f \in H^p(G)$, then $f(\varphi) \in H^p(\mathbb{D})$. Since $\varphi' \in H^\infty(\mathbb{D})$, we have

$$f(\varphi)[\varphi']^{1/p} \in H^p(\mathbb{D}).$$

Therefore, $f \in E^p(G)$, as desired.

Functions in Smirnov classes possess boundary properties similar to functions in Hardy classes. Functions in the Smirnov class $E^p(G)$ have non-tangential boundary values a.e. with respect to ds. When $p \geq 1$, they can also be recovered by their boundary values. In this case, the values are obtained using the Cauchy integral (cf. [2] Ch.10).

THEOREM 1.10 *Each $f \in E^1(G)$ has a Cauchy representation*

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \in G,$$

and the integral vanishes when z is outside Γ. Conversely, if $g \in L^1(\Gamma)$ and

$$\int_\Gamma z^n g(z) dz = 0, \quad n = 0, 1, 2, ...,$$

then

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{g(\zeta) d\zeta}{\zeta - z} \in E^1(G),$$

and g coincides almost everywhere on Γ with the nontangential limit of f.

1.4 Smirnov domains

The example in the previous section illustrates how the geometry of the domain determines the classes of Smirnov and Hardy functions that exist on the domain. Simply connected domains can be either so-called Smirnov or non-Smirnov domains, depending on the domain.

Let G be a simply connected domain with Jordan rectifiable boundary Γ, let $\mathbb{D}$ be the unit disk, and let $\varphi(w)$ be a conformal mapping of $\mathbb{D}$ onto G. Since Γ is rectifiable, then by Theorem 1.6, $\varphi' \in H^1$ and $\varphi'(w) \neq 0$. This implies that $\varphi'(w) = S(w)F(w)$ where S is a singular inner function and F is an outer function. Note that φ' is an outer function when $S(w) \equiv 1$.

DEFINITION 1.4.1 *G is called a Smirnov domain if $\varphi'(w)$ is an outer function.*

Since $\varphi' \in H^1(\mathbb{D}) \subset N$, Corollary 1.0.2 implies that if $\log(\varphi') \in H^1(\mathbb{D})$, then G is Smirnov. Now $\log(\varphi') = log|\varphi'| + i\arg(\varphi')$, and $\log(\varphi') \in H^1(\mathbb{D})$ is equivalent to $\log|\varphi'| \in h^1(\mathbb{D})$ and $\arg(\varphi') \in h^1(\mathbb{D})$. (A harmonic function $u(z)$ is said to be of class $h^1(\mathbb{D})$ if the integral means of the absolute value of $u(z)$ over concentric circles tending to the boundary are bounded.) But since G has a rectifiable boundary,

$$\int_{\partial\mathbb{D}} log^+|\varphi'(w)|ds \leq \int_{\partial\mathbb{D}} |\varphi'(w)|ds \leq C.$$

This is equivalent to $\log|\varphi'| \in h^1(\mathbb{D})$ so, actually, a necessary and sufficient condition for G to be Smirnov is that $\arg(\varphi') \in h^1(\mathbb{D})$. In particular, if φ' is bounded either from above or below then G is said to be a Smirnov domain. This means that the boundary of G can either spiral in or out at a point but not both.

One may ask if a bounded domain being Smirnov implies that its complement is also Smirnov. In [9], Jones and S. K. Smirnov found that the answer to this question is in general "no". Moreover, when φ' is a singular inner function, the complement of a non-Smirnov domain that is bounded by Γ is Smirnov.

In the next chapter we will explore the boundary characteristics of simply connected Smirnov domains that allow analytic functions with real boundary values in Smirnov classes E^p.

Chapter 2

Simply connected Smirnov domains

2.1 S. Ya. Khavinson's Example

Since $H^p(G)$ functions can be recovered from their boundary values via harmonic measure, any $f \in H^p(G)$ with real boundary values is real-valued inside G and hence must be a constant. Unlike functions in the Hardy classes, it is not always true that functions in the Smirnov classes are constants when their boundary values are real. In [10] it was shown that if G is a non-Smirnov domain with Jordan rectifiable boundary, then there do exist functions in Smirnov classes $E^p(G)$ for $0 < p < \infty$ that are bounded a.e on Γ and possess real boundary values.

THEOREM 2.1 *Let G be a simply-connected domain with a rectifiable Jordan boundary. If G is a non-Smirnov domain, then there exists a non-constant function $f(z) \in E^1(G)$ such that $0 \leq f(z) \leq 1$ a.e. on Γ.*

COROLLARY 2.0.3 *Let G be the same as in Theorem 2.1. Then, for each p such that $0 < p < \infty$ there exists a non-constant function $f(z) \in E^p(G)$ such that $0 \leq f(z) \leq 1$ a.e. on Γ.*

Furthermore, in [10] it is shown that if G is Smirnov and $f(z) \in E^1(G)$ is real-valued and bounded on Γ then $f(z) \equiv const$. However, when the condition of boundedness is removed, an example by S. Ya. Khavinson [10] reveals that for a certain cardioid, a Smirnov domain, the function $f(z) = F(\varphi^{-1}(z))$ where $F(w) = i\frac{1+w}{1-w}$ and $\varphi(w) = (1-w)^2$ is the conformal map from the unit disk to the cardioid (shown below), furnishes a non-trivial E^1 function with real boundary values.

If we investigate this function, we notice that $f(z)$ has a pole of order 1 at the point zero on the boundary of the cardioid. The mapping $\frac{1+w}{1-w}$ is the Möbius transformation that takes the unit disk to the right half plane. It follows then that $F(w) = i\frac{1+w}{1-w}$ is the mapping of the unit disk to the upper half plane, so it is easy to see that $f(z)$ has real boundary values on the cardioid. Theorem 1.7

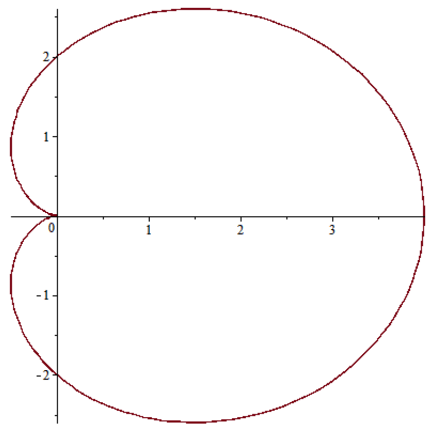

Figure 1.: Cardioid

states that if $f(z)$ is of Smirnov class E^1 in the cardioid then the integrals of the modulus of f over the images of the circles $|w| = r$ will remain bounded as r tends to 1. Looking at an image of the modulus of f over the cardioid this is not immediately clear. The image below shows the modulus of f and the images under φ of the circles of radius 1/4, 1/2, 3/4, and 1.

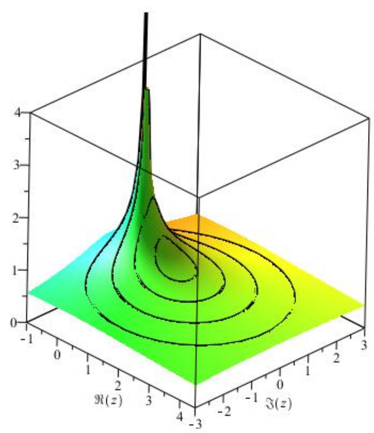

Figure 2.: E^1 function on the Cardioid

We know that by Theorem 1.8 that f is in E^1 if and only if $(f \circ \varphi)\varphi'$ is in H^1. On inspection of the graph of the modulus of this function it is clear that this must be true. Below we see a graph of the modulus of $(f \circ \varphi)\varphi'$ with the images of the circles of radius 1/4, 1/2, 3/4, and 1.

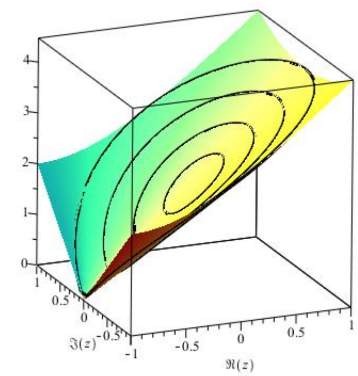

Figure 3.: H^1 function on the Circle

In fact, the example works for all p such that $0 < p < 2$. To see that this is true, recall that Theorem 1.8 states that $f(\varphi(w))[\varphi'(w)]^{1/p} \in H^p(\mathbb{D})$ if $f(z) \in E^p(G)$. Since

$$f(\varphi(w))[\varphi'(w)]^{1/p} = i\frac{1+w}{1-w} \cdot [-2(1-w)]^{1/p} = -2i(1+w)(1-w)^{1/p-1}$$

Then $f(z) \in E^p(G)$ if $p(1/p-1) > -1$ or $p < 2$.

S. Ya. Khavinson's example leads us to believe that the existence of E^p functions with real boundary values is dependent upon the local geometry of the domain. With this example in mind, one might ask the following questions: What are the boundary characteristics of a Smirnov domain that will admit functions with real boundary values in the Smirnov classes E^p? For what values of p will these functions exist? What can be said about analytic functions with real boundary values in the Smirnov classes E^p on finitely connected Smirnov domains? Let us note here that it is desirable to find analytic functions with real boundary values in classes E^p for $p \geq 1$ as it is these classes of functions that have the property that they can be recovered inside a domain by use of their boundary values.

2.2 Domains with interior angles less than π

After considering S. Ya. Khavinson's example, I first would like to consider a simply connected domain whose boundary is nice, say real analytic, except at one point were the boundary has an outward pointing corner. Let us suppose that the interior of this corner has an angle less than π. It happens that these functions do not admit non-constant analytic functions of Smirnov class E^p for

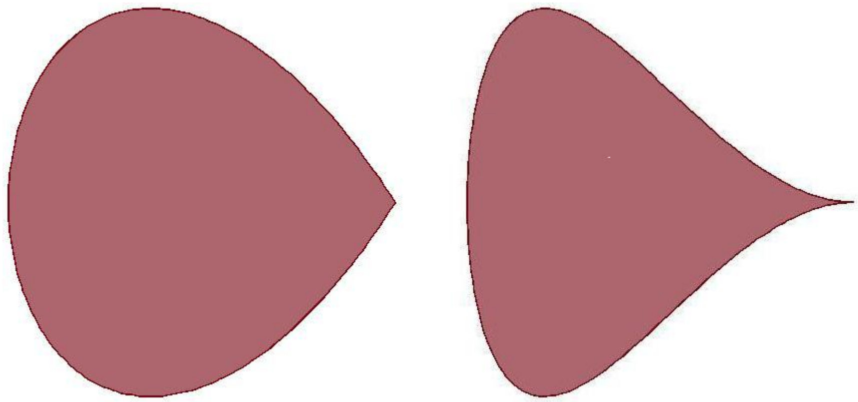

Figure 4.: Domains with interior angles less than π.

$p \geq 1$ with real boundary values on such domains [4].

THEOREM 2.2 *Let G be a domain bounded by a curve that is real analytic except at the point z_0 where it has a corner with interior angle α. If $0 \leq \alpha \leq \pi$, then for all $p \geq 1$ every $f(z) \in E^p(G)$ with real boundary values is a constant.*

Note. The angle $\alpha = 0$ corresponds to an outward cusp. The angle $\alpha = \pi$ corresponds to Γ being smooth and was discussed above.

Proof. Let $0 < \alpha < \pi$ and suppose that there exists an $f(z) \in E^1(G)$ with real boundary values a.e. Without loss of generality we may assume that $\varphi(1) = z_0$. According to [18], Ch. 3, Sec. 4,

$$\varphi'(w) = (1-w)^{\frac{\alpha}{\pi}-1} g(w), \tag{2.1}$$

where $g(w)$ is bounded away from 0 and ∞ in $\mathbb{D}$. Then by Theorem 1.8,

$$f(\varphi(w))\varphi'(w) = f(\varphi(w))(1-w)^{\frac{\alpha}{\pi}-1} g(w) \in H^1(\mathbb{D}).$$

Since $\frac{\alpha}{\pi} - 1 < 0$, multiplying by a bounded, analytic function in $\mathbb{D}$, $(1-w)^{1-\frac{\alpha}{\pi}}$, we conclude that $f(\varphi(w)) \in H^1(\mathbb{D})$. Since $f(\varphi(w))$ can be represented by the Poisson integral in $\mathbb{D}$ and has real boundary values then $f(\varphi(w))$ is real-valued in $\mathbb{D}$ and hence is a constant. For the case when $\alpha = 0$ we need a result from [18] Ch. 11, Sec. 5, which states that

$$\lim_{w \to 1} |\varphi(w) - \varphi(1)| = \frac{\frac{\pi}{a} + o(1)}{log \frac{1}{|w-1|}}, \tag{2.2}$$

where $a > 0$, is a constant. This means that

$$|\varphi'(1)| = \lim_{w\to 1} \frac{|\varphi(w) - \varphi(1)|}{|w-1|} = \infty. \tag{2.3}$$

So if $f(\varphi(w))\varphi'(w) \in H^1(\mathbb{D})$ then $f(\varphi(w)) \in H^1(\mathbb{D})$ which implies as before, that $f(\varphi(w)) \equiv const.$ $\square$

To show that for all p such that $0 < p < 1$ there exists in such G a $f(z) \in E^p(G)$ with real boundary values, consider the function $F(w) = i\frac{1-w}{1+w}$ which maps the unit disk onto the upper half plane. Suppose that $0 < \alpha < \pi$. Then $f(z) = F(\varphi^{-1}(z)) \in E^p(G)$ for $0 < p < 1$. Indeed by Theorem 1.8 and (2.1) we have:

$$f(\varphi(w))[\varphi'(w)]^{\frac{1}{p}} = F(w)(1-w)^{(\frac{\alpha}{\pi}-1)\frac{1}{p}} g^{\frac{1}{p}}(w) = i\frac{(1-w)^{1+\frac{\alpha}{\pi p}-\frac{1}{p}}}{(1+w)} g^{\frac{1}{p}}(w).$$

So $f(\varphi(w))[\varphi'(w)]^{\frac{1}{p}} \in H^p(\mathbb{D})$ for $0 < p < 1$, since $p\ (1 + \frac{\alpha}{\pi p} - \frac{1}{p}) = p + \frac{\alpha}{\pi} - 1 > -1$. When $\alpha = 0$ we use (2.2) and (2.3) to see that

$$|f(\varphi(w))|^p |\varphi'(w)| = \left| i\frac{1-w}{1+w} \right|^p |\varphi'(w)| \approx \frac{(\frac{\pi}{a} + o(1))|1-w|^{p-1}}{|1+w|^p \left| log \frac{1}{|w-1|} \right|},$$

which is integrable on $\mathbb{T}$ for $0 < p < 1$, while obviously is in class $N^+(\mathbb{D})$. Hence the Polubarinova-Kotchine Theorem applies and $f(z) \in E^p(G)$.

For $\alpha = \pi$, Γ is smooth and the existence of a non-trivial $f(z) \in \bigcap\limits_{p<1} H^p(G)$ with real boundary values is obvious. For example, in $\mathbb{D}$ it suffices to take $F(w)$ itself.

While this type of domain does admit non-constant analytic functions of class E^p with real boundary values, none of these functions can be recovered by their boundary values via the Cauchy integral since $p < 1$.

2.3 Domains with interior angle greater than π

Now we shall examine the case when G is a simply connected domain that has a real analytic boundary except at one point where there is an inward pointing corner with interior angle greater than π. As we shall see this is an interesting case when nontrivial functions with real boundary values can be found in E^p for $p \geq 1$ [4].

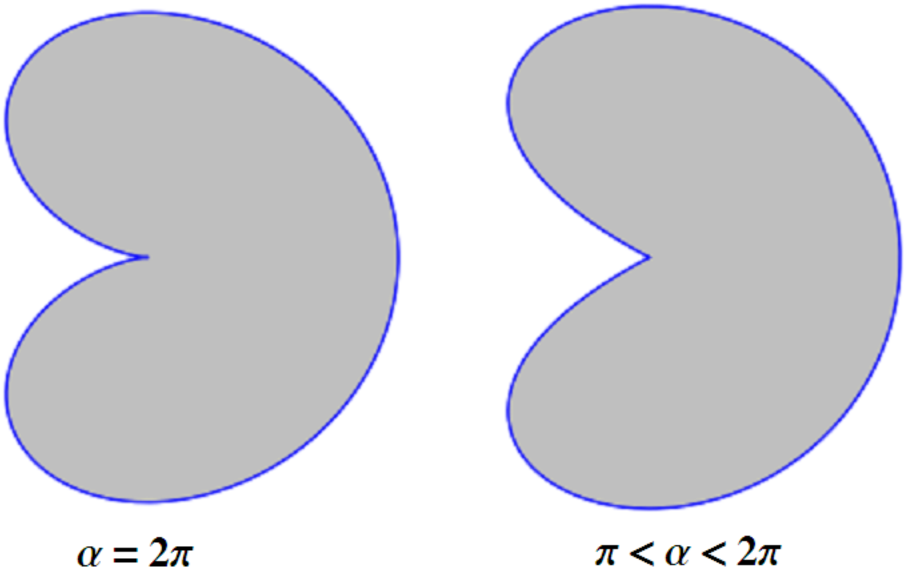

Figure 5.: Domains with interior angles greater than π.

THEOREM 2.3 *Let G be a domain bounded by a curve that is real analytic except at the point z_0 where it has a corner with interior angle α. If $\pi < \alpha \leq 2\pi$ then for all $p \geq \frac{\alpha}{\pi}$ every $f(z) \in E^p(G)$ with real boundary values is a constant. If $p < \frac{\alpha}{\pi}$ then there does exist a non-constant function in $E^p(G)$ with real boundary values.*

Note. The angle $\alpha = 2\pi$ corresponds to Γ having an inward cusp, e.g. a cardioid as in [10]. Thus G obtained from $\mathbb{D}$ via the conformal map $\varphi(w) = (1-w)^2$ doesn't have a non-constant $E^2(G)$ function with real boundary values.

Proof. Suppose that there exists such $f(z) \in E^{\frac{\alpha}{\pi}}(G)$. Without loss of generality assume that $\varphi(1) = z_0$. Then by the Keldysh-Laurentiev theorem and (2.1)

$$h(w) := f(\varphi(w))[\varphi'(w)]^{\frac{\pi}{\alpha}} = f(\varphi(w))(1-w)^{1-\frac{\pi}{\alpha}} g^{\frac{\pi}{\alpha}}(w) \in H^{\frac{\alpha}{\pi}}(\mathbb{D}).$$

Set

$$h_0(w) := \frac{h(w)(1-w)^{\frac{\pi}{\alpha}}}{g^{\frac{\pi}{\alpha}}(w)} = f(\varphi(w))(1-w) \in H^{\frac{\alpha}{\pi}}(\mathbb{D}).$$

Then

$$\frac{h_1(w)}{w} := h_0(w)(1-\overline{w}) = f(\varphi(w))|1-w|^2 \tag{2.4}$$

has real boundary values on $\mathbb{T}$. Now $h_1(w) \in H^{\frac{\alpha}{\pi}}(\mathbb{D}) \subset H^1(\mathbb{D})$ since $\frac{\alpha}{\pi} \geq 1$ and by the generalized Schwarz reflection principle, $\frac{h_1(w)}{w}$ extends across $\mathbb{T}$ to the Riemann sphere. So it is a rational function with a pole of order 1 at the origin and infinity.

Case 1. Suppose $h_1(w)$ does not have any poles on $\mathbb{T}$. Then

$$\frac{h_1(w)}{w} = C\frac{(w-\beta)(1-\beta w)}{w} \text{ where } \beta \in \overline{\mathbb{D}} \text{ and } C \in \mathbb{R}. \tag{2.5}$$

Since $f(\varphi(w))(1-w) \in H^{\frac{\alpha}{\pi}}(\mathbb{D}) \subset H^1(\mathbb{D})$, $f(\varphi(w))$, which is also a rational function since $h_0(w)$ and $h_1(w)$ are, cannot have a pole of order greater than 1. Then from (2.5) it follows at once that $\beta = 1$ since

$$\frac{h_1(w)}{w} = \frac{f(\varphi(w)(1-w)(w-1)}{w},$$

but then $f(\varphi(w)) \equiv C$, hence $f(z)$ is itself a constant.

Case 2. Suppose $h_1(w)$ does have poles on $\mathbb{T}$. From (2.4) it follows that it can only be at $w = 1$ and of order 1 (it is in $H^1(\mathbb{D})$!) Thus,

$$\frac{h_1(w)}{w} = C\frac{(w-\beta_1)(w-\beta_2)(1-\overline{\beta_2}w)}{(1-w)w}, \text{ where } \beta_1 \in \mathbb{T}\ , \beta_2 \in \overline{\mathbb{D}}$$

$$\text{and } C\frac{w-\beta_1}{1-w} \in \mathbb{R} \text{ on } \mathbb{T}.$$

We claim that $\beta_1 = -1$ and C is purely imaginary. Let $w = a + bi \in \mathbb{T}$ and $\beta_1 = c + di$. Then

$$C\frac{(w-\beta_1)}{(1-w)} = C\frac{(a+bi-c-di)}{(1-a)-bi} = C\frac{((a-c)+(b-d)i)}{(1-a)-bi}\frac{((1-a)+bi)}{((1-a)+bi)} \tag{2.6}$$

$$= C\frac{[(a-c)(1-a)+(1-a)(b-d)i+(a-c)bi-(b-d)b]}{(1-a)^2+b^2}.$$

Since (2.6) is real-valued, then either C is real and$(1-a)(b-d)+(a-c)b = 0$ or C is purely imaginary and $(a-c)(1-a)-(b-d)b = 0$. In the first instance $d = 0$ and $c = 1$ which implies that we have Case 1. In the second instance $a - a^2 - c + ca - b^2 + bd = 0$ and since $a^2 + b^2 = 1$ then $a - c + ca - 1 + bd = 0$ which implies that $c = -1$ and $d = 0$. Therefore $\beta_1 = -1$ and C is purely imaginary. Thus

$$\frac{h_1(w)}{w} = it\frac{(w+1)(w-\beta_2)(1-\overline{\beta_2}w)}{(1-w)w} = \frac{f(\varphi(w))(1-w)(w-1)}{w},$$

where $t \in \mathbb{R}$. Therefore, $\beta_2 = 1$ and

$$f(\varphi(w)) = it\frac{w+1}{1-w},$$

but,

$$f(\varphi(w))(1-w)^{1-\frac{\pi}{\alpha}} g^{\frac{\pi}{\alpha}}(w) = it\frac{w+1}{(1-w)^{\frac{\pi}{\alpha}}} g^{\frac{\pi}{\alpha}}(w) \notin H^{\frac{\alpha}{\pi}}(\mathbb{D}).$$

So Case 2 cannot occur and the proof is complete. □

For α such that $\pi < \alpha \leq 2\pi$ an argument similar to S. Ya. Khavinson's in [10] shows that for all p such that $0 < p < \frac{\alpha}{\pi}$ there exists a non-constant $f(z) \in E^p(G)$ with real boundary values. Let $F(w) = i\frac{1+w}{1-w}$. $F(w)$ maps the unit disk onto the upper half plane. Then $f(z) = F(\varphi^{-1}(z)) \in E^p(G)$. This follows from Theorem 1.8 and (2.1) since

$$f(\varphi(w))[\varphi'(w)]^{\frac{1}{p}} = F(w)(1-w)^{(\frac{\alpha}{\pi}-1)\frac{1}{p}} g^{\frac{1}{p}}(w)$$

$$= i(1+w)(1-w)^{\frac{\alpha}{\pi p}-\frac{1}{p}-1} g^{\frac{1}{p}}(w)$$

and $f(\varphi(w))[\varphi'(w)]^{\frac{1}{p}} \in H^p(\mathbb{D})$ when $p\,(\frac{\alpha}{\pi p} - \frac{1}{p} - 1) > -1$ or $p < \frac{\alpha}{\pi}$.

Interestingly, the proof of Theorem 2.3 also shows that the transplanted mappings $F(w)$ of $\mathbb{D}$ onto a half plane, e.g. $f(z) := F(\varphi^{-1}(z))$, where $F(w) = i\frac{1+w}{1-w}$ are essentially the only examples of functions in $E^p(G)$, $p \geq 1$ with real boundary values. Thus S. Ya. Khavinson's construction in [10] is, in this sense, unique.

If a domain is bounded by an analytic curve with finitely many corners (or, cusps) it follows that the existence of an E^p function with real boundary values is dependent upon the corner with the smallest interior angle.

Chapter 3

Finitely connected domains

3.1 Some background on n-connected domains

Let G be a finitely connected domain in the complex plane with boundary $\Gamma = \bigcup_{j=1}^{n} \gamma_j$. A bounded, finitely connected domain K is called a circular domain if all of the boundary components of K are all circles. The following theorem is due to Koebe and its proof can be found in [7], Ch. 5.

THEOREM 3.1 *Every n-connected Jordan domain G can be univalently mapped onto a circular domain K.*

When a conformal mapping from a circular domain K onto an n-connected domain G takes the outer boundary circle of K to the outer boundary curve of G, then the mapping preserves the orientation of all of the boundary curves. This implies that the logarithm of the derivative of the conformal mapping is single-valued on K. Namely, the following holds.

PROPOSITION 3 Let $\varphi : K \to G$ be the conformal map of K onto G. Assume that the outer circle of K is mapped onto the outer boundary curve of $\Gamma := \partial G$. Then $\log(\varphi')$ has a single-valued branch in K.

Proof. Let $w = \varphi(z)$. Then, the tangent vector to Γ is $dw = \varphi'(z)dz$.

Since

$$2\pi = \Delta_{l_n} \arg(dw) = \Delta_{\gamma_n} \arg(\varphi') + \Delta_{\gamma_n} arg(dz)$$

and

$$\Delta_{\gamma_n} \arg(dz) = 2\pi,$$

then $\Delta_{\gamma_n} \arg(\varphi') = 0$. Similarly,

$$-2\pi = \Delta_{l_j} \arg(dw) = \Delta_{\gamma_j} \arg(\varphi') + \Delta_{\gamma_j} \arg(dz)$$

and

$$\Delta_{\gamma_j} \arg(dz) = -2\pi \quad \text{for } j = 1, ..., n-1.$$

This implies that $\Delta_{\gamma_j} \arg(\varphi') = 0$ for all $j = 1, ..., n$. Therefore, $\log(\varphi')$ is single-valued on K. $\square$

In Section 1.1 the harmonic measure for a Dirichlet region was defined. One can also define the harmonic measures ω_j of γ_j, $j = 1, ..., n$ on ∂G. These are harmonic functions in G such that $\omega_j|_{\gamma_j} \equiv 1$ and $\omega_j|_{\gamma_k} \equiv 0$ when $j \neq k$.

If $u(z)$ is a harmonic function in G, then its harmonic conjugate $v(z)$ may be multivalued in G. Suppose that $u(z)$ can be represented by the Green-Stieltjes integral of a Borel measure μ supported on Γ. This means that

$$u(z) = \frac{1}{2\pi} \int_\Gamma \frac{\partial g(\zeta, z)}{\partial n_\zeta} d\mu(\zeta),$$

where $\frac{\partial}{\partial n_\zeta}$ is the inner normal derivative at the point ζ and $g(\zeta, z)$ is the Green funtion of G with pole at z. Then we can compute the period of $v(z)$ around γ_j, $j = 1, ..., n$ (cf. [12]) as

$$\Delta_{\gamma_j} v = -\int_\Gamma \frac{\partial \omega_j}{\partial n}(\zeta) d\mu(\zeta).$$

Lemma 1 in [12] provides a way to eliminate the periods of v and produce an analytic function by modifying μ. More precisely,

LEMMA 3.1 *Let $\mu > 0$ (or $\mu < 0$) be a Borel measure on Γ such that $\mu(\gamma_j) \neq 0$, $j = 1, ..., n-1$. Then, for arbitrary real numbers $\Lambda_1, ..., \Lambda_{n-1}$ there exists a unique vector $(\lambda_1, ..., \lambda_{n-1}) \in \mathbb{R}^{n-1}$ such that $\Lambda_1, ..., \Lambda_{n-1}$ are the periods around $\gamma_1, ..., \gamma_{n-1}$ respectively of the function conjugate to*

$$\bar{u}(z) = \frac{1}{2\pi} \int_\Gamma \frac{\partial g(\zeta, z)}{\partial n_\zeta} d\overline{\mu}(\zeta),$$

where $\overline{\mu}|_{\gamma_n} \equiv \mu|_{\gamma_n}$, $\overline{\mu}|_{\gamma_j} \equiv \lambda_j \mu|_{\gamma_j}$, $j = 1, ..., n-1$.

The proof of the lemma ([12]) reduces to showing that the system of linear equations

$$-\sum_{j=1}^{n-1} \lambda_j \int_{\gamma_j} \frac{\partial \omega_i(\zeta)}{\partial n_\zeta} d\mu(\zeta) = \Lambda_i + \int_{\gamma_n} \frac{\partial \omega_i(\zeta)}{\partial n_\zeta} d\mu(\zeta)\ ,\ i = 1, ..., n-1\ ,\ \lambda_j \in \mathbb{R}, \tag{3.1}$$

always has a unique solution.

Suppose G is an n-connected domain and $d\mu$ is a measure on ∂G that is defined as the sum of point masses. If u is the harmonic function represented by the Green-Stieltjes integral with measure $d\mu$ then it is often possible to modify the measure so that the resulting harmonic function has a single-valued conjugate. Hence, one can construct an analytic function on G that is real-valued almost everywhere on ∂G. The existence of such functions will be necessary for two theorems that will be presented later in this thesis. The next proposition is from [5].

PROPOSITION 4 Let $u(z)$ be a harmonic function in G such that $u(z)$ is represented by the Green-Stieltjes integral with $d\mu = \sum_{j=1}^{m} \delta_{\zeta_j}$, where δ_{ζ_j} is the unit point mass at ζ_j. If $m \geq n$ then there exist real numbers $\lambda_1, ..., \lambda_m$ that are not all equal to zero such that the function

$$\bar{u}(z) = \lambda_1 \frac{\partial g(z, \zeta_1)}{\partial n_{\zeta_1}} + ... + \lambda_m \frac{\partial g(z, \zeta_m)}{\partial n_{\zeta_m}}$$

has a single valued conjugate.

Proof. Consider the matrix A associated with the system of equations (3.1). Namely,

$$A = ||a_{ij}||, \text{ where } a_{ij} = \frac{\partial \omega_i(\zeta_j)}{\partial n_{\zeta_j}}. \tag{3.2}$$

Let $\vec{\lambda} = (\lambda_1, ..., \lambda_m) \neq 0$. Then $\bar{u}$ will have a single-valued conjugate if $A\vec{\lambda} = 0$. Since $A : \mathbb{R}^m \to \mathbb{R}^{n-1}$ is a linear map given by the matrix A, it suffices to show that $ker(A) \neq \{0\}$. Since $m = dim(ker(A)) + dim(rank(A))$ and $dim(rank(A)) \leq n - 1$ then $dim(ker(A)) \geq 1$. Therefore $ker(A) \neq \{0\}$. ☐

The following example illustrates a case when $rank(A) = n - 1$.

EXAMPLE 2 If $m \geq n$ and there exists j where $1 \leq j \leq m$ such that $\exists \zeta_j \in \gamma_i$ for each $i = 1, ..., n$ then there exist $\lambda_1, ..., \lambda_m$ not all equal to zero such that $\bar{u}$ has a single-valued conjugate. Without loss of generality we may assume that $\zeta_1 \in \gamma_1, ..., \zeta_{n-1} \in \gamma_{n-1}$. Then by Lemma 1 in [12] the minor of the matrix A in (3.2) formed by the first $(n-1)$ columns and first $(n-1)$ rows has rank $(n-1)$ and therefore, $A\vec{\lambda} = 0$ has a non-zero solution.

The converse of Proposition 4 is not true. It can happen that $m < n$ and yet $\bar{u}$ has a single-valued conjugate.

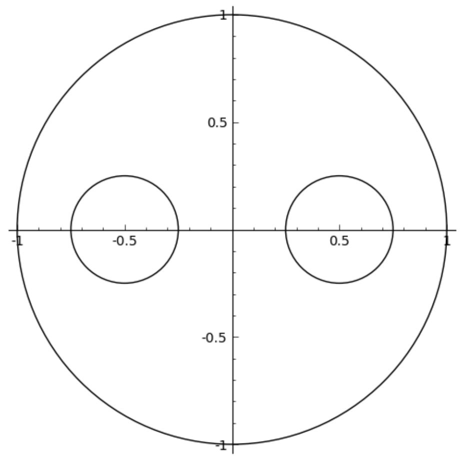

Figure 6.: S. Ya. Khavinson's example

EXAMPLE 3 (S. Ya. Khavinson, [13]): Let G be the domain bounded by the circles $\gamma_1 = \{z : |z| = 1\}$, $\gamma_2 = \{z : |z + 1/2| = 1/4\}$, and $\gamma_3 = \{z : |z - 1/2| = 1/4\}$. Let $\zeta_1 = -1/2 + 1/4i$ and $\zeta_2 = -1/2 - 1/4i$. Then $\{\zeta_1, \zeta_2\} \subset \gamma_2$ and ζ_1 is symmetric to ζ_2 with respect to the horizontal diameters of γ_i, $i = 1, 2, 3$. Thus by symmetry,

$$\frac{\partial \omega_i(\zeta_1)}{\partial n_{\zeta_1}} = \frac{\partial \omega_i(\zeta_2)}{\partial n_{\zeta_2}},$$

for $i = 1, 2$. So the two rows of A are linearly dependent, and the equations

$$\lambda_1 \frac{\partial \omega_i(\zeta_1)}{\partial n_{\zeta_1}} + \lambda_2 \frac{\partial \omega_i(\zeta_2)}{\partial n_{\zeta_2}} = 0, \quad i = 1, 2$$

have non-zero solutions $\lambda_1 = -\lambda_2$.

3.2 The Hardy and Smirnov classes revisited

The definition of the Hardy classes stays the same when G is a finitely connected domain. That is, $H^p(G)$ for $0 < p < \infty$ is the space of analytic functions f such that the subharmonic function $|f(z)|^p$ has a harmonic majorant in G. The definition of the norm of a function in $H^p(G)$ stays the same as in Section 1.2 however, the norm is only a true norm for all $p \geq 1$.

Before going further, let us recall a theorem from [7] Chapter VI, sec.1 that basically states that if a domain G possesses more than two boundary points then there exists an analytic function $T(w)$ that is locally univalent and maps the unit disk onto G. $T(w)$ is called the uniformization map of

G. Let Σ be the group of deck transformations of $\mathbb{D}$ associated with T (cf. [6] Ch. 2, Sec. 2). This means that if $g \in \Sigma$ then $g : \mathbb{D} \to \mathbb{D}$ is a Möbius automorphism of the unit disk onto itself and $T(g(w)) \equiv T(w)$. Given a point $z_0 \in G$, T can be chosen so that then $T(0) = z_0$ and $T'(0) > 0$. Given these normalizations, T is unique.

We now state the following theorem from [2], Thm. 10.11.

THEOREM 3.2 *The mapping*

$$f(z) \to F(w) = f(T(w))$$

is an isometric isomorphism of $H^p(G)$ onto the subspace of $H^p(\mathbb{D})$ invariant under Σ.

This theorem implies that if G is finitely connected then some properties of functions in $H^p(G)$ can be investigated using their counterparts in H^p in the unit disk. The proof of the following theorem follows from the Green formula and can be found in [2], Ch. 10, Sec. 5, also cf. [14], [15] and references therein.

THEOREM 3.3 *Let G be a finitely connected domain with Jordan boundary curves $\gamma_1, ..., \gamma_n$. For each $j = 1, ..., n$ let G_j be the simply connected domain with boundary γ_j which contains G. Then every $f \in H^p(G)$ can be represented in the form*

$$f(z) = f_1(z) + f_2(z) + ... + f_n(z),$$

where $f_j \in H^p(G_j)$ for $j = 1, ..., n$.

Note that if in the above theorem $f_j(\infty) = 0$ for $j = 1, ..., n-1$, then the f_j are all unique. It follows from this theorem and Theorem 1.5 that a function $f \in H^p(G)$ must have non-tangential boundary values a.e. with respect to harmonic measure, ω_a, on ∂G. That is $\omega_a = \omega(G, \Gamma, a)$ where $a \in G$ is fixed. It is also true that an analytic function in the Hardy class H^p, $p \geq 1$, of a finitely connected domain can be recovered by its boundary values just like in the simply connected case. The following theorem is from [14].

THEOREM 3.4 *Each $f \in H^p(G)$ for $1 \leq p \leq \infty$ has boundary values f^* almost everywhere with respect to $d\omega$ on Γ and $f^* \in L^p(\Gamma, d\omega)$. Furthermore,*

$$f(z) = \frac{1}{2\pi i} \int_\Gamma f^*(\zeta) d\omega_z(\zeta), \quad z \in G.$$

We can also speak of a factorization for functions in Hardy classes in finitely connected domains. First we need the notion of a Green-Schwarz kernel in G (cf. [1], [11], and [13]).

THEOREM 3.5 *There exists a unique function $\mathcal{P}(z,\zeta)$ continuous on $G \times \Gamma$ and satisfying the following properties:*

1. *If $\zeta \in \Gamma$ is fixed, then $\mathcal{P}(z,\zeta)$ is single-valued and analytic in G.*

2. *If $z \in G$ is fixed, then*
$$\int_{\gamma_j} \mathcal{P}(z,\zeta)d\omega_a = 0 \quad j = 1,..,n-1,$$
$$\int_{\gamma_n} \mathcal{P}(z,\zeta)d\omega_a = 1.$$

3. *If $f(z) = u(z) + iv(z)$ is a single-valued analytic function in G and $u(z)$ is continuous in $\overline{G}$, then*
$$f(z) = \int_{\Gamma} \mathcal{P}(z,\zeta)u(\zeta)d\omega_a + iv(a).$$

$\mathcal{P}(z,\zeta)$ is called the Green-Schwarz kernel in G.

Let $a \in \overline{G}$ and following [1] and [11], define the function
$$\mathcal{B}(z,a) = (z-a)exp\left(-\int_{\Gamma} \mathcal{P}(z,\zeta)ln|\zeta - a|d\omega\right).$$

Note that when $a \in G$, the above function maps G conformally onto the unit disk with slits along circular arcs centered at the origin. When $a \in \Gamma$, $\mathcal{B}(z,a)$ maps G conformally onto an annulus with slits along $n-2$ circular arcs centered at the origin (cf. [1]). This also means that $|\mathcal{B}(z,a)|$ is a constant on each boundary curve of G.

For the next theorem (cf. [11] and references there) let us recall the following notation. $g(a,z)$ is the Green function of G with pole at a and $\partial G = \Gamma = \bigcup_1^n \gamma_j$. Let G_j be the domain bounded by γ_j and containing G, $\omega_j(z)$ be the harmonic measure of γ_j, and $g_j(a,z)$ be the Green function of G_j.

THEOREM 3.6 *Let $f(z)$ be a single-valued analytic function in G. Let $\{z_k\}_1^\infty$ be the sequence of its zeros in G. The following statements are equivalent:*

1. *$\ln|f(z)|$ has a harmonic majorant in G.*

2. *The series $\sum_1^\infty g(z_k,z)$ converges uniformly on compact subsets of $G \setminus \{z_k\}_1^\infty$.*

3. *Let* $\{z_k^j\}_1^\infty$, $j = 1, ...n$, *be the subsequences of* $\{z_k\}_1^\infty$ *such that all cluster points of* $\{z_k^j\}$ *belong to* γ_j; $\{z_k\}_1^\infty = \bigcup_{j=1}^n \{z_k^j\}_{k=1}^\infty$, $\{z_k^i\}_{k=1}^\infty \cap \{z_k^j\}_{k=1}^\infty = \emptyset$ *when* $i \neq j$. *Then the series* $\sum_{k=1}^\infty g_j(z_k^j, z)$, $j = 1, ..., n$, *converge uniformly on compact subsets of* $G_j \setminus \{z_k^j\}_{k=1}^\infty$.

4. *The series* $\sum_{k=1}^\infty |\omega_i(z_k^j) - \delta_{ij}|$, $i = 1, ..., n$, $j = 1, ..., n$, *converge. (δ_{ij} is the Kronecker symbol.)*

If $\Gamma = \bigcup_1^n \gamma_j$ *is analytic and* ζ_j *denotes the closest point to* z_j *on* Γ, *then (1) through (4) are equivalent to the following:*

$$\sum_{j=1}^{\infty} |z_j - \zeta_j| < +\infty.$$

Let $\varphi : K \to G$ be the conformal map of the circular domain K onto G. Assume that $\gamma_1, ..., \gamma_{n-1}$ lie inside γ_n and introduce the periods of the conjugates of the harmonic functions ω_k

$$\pi_{kj} = \int_{\gamma_j'} \frac{\partial \tilde{\omega}_k}{\partial s} ds = -\int_{\gamma_j'} \frac{\partial \omega_k}{\partial n} ds,$$

$j, k = 1, ..., n-1$, where γ_j' is an analytic curve in G homologous to γ_j. The matrix $(\pi_{kj})_1^{n-1}$ is invertible (it follows from [12] Lemma 1; also see [1]), which implies that $(q_{kj})_1^{n-1} = \left((\pi_{kj})_1^{n-1}\right)^{-1}$ is well defined. The following theorem is from [11] and its proof there is similar to that in [1].

THEOREM 3.7 *Let* $\{z_k\}_{k=1}^\infty$ *satisfy (1)-(4) of Theorem 3.6. Define* $w_k^j = \varphi^{-1}(z_k^j)$ *and let* t_k^j *be the point closest to* w_k^j *on* ∂K. *Let* $\zeta_k^j = \varphi(t_k^j)$. *Then the product*

$$\mathcal{B}_0(z) = \prod_{j=1}^{n} \prod_{k=1}^{\infty} \frac{\mathcal{B}(z, z_k^j)}{\mathcal{B}(z, \zeta_k^j)},$$

converges absolutely and uniformly on compact subsets of G. $\mathcal{B}_0(z) = 0$ *if and only if* $z \in \{z_k\}_1^\infty$. *Moreover, if the sequence* $\{G^i\}_{i=1}^\infty$ *is a sequence of domains such that* $G^i \subset G^{i+1}$ *and* $\bigcup_{i=1}^\infty G^i = G$, *then (letting* $\Gamma^i = \partial G^i = \bigcup_{k=1}^n \gamma_k^i$*)*

$$\lim_{i \to \infty} \int_{\gamma_k'} |\ln |\mathcal{B}_0(z)| - C_k| d\omega^i(E, a, G^i) = 0$$

where

$$C_k = \sum_{i=1}^{\infty} \sum_{l=1}^{n} \sum_{j=1}^{n-1} q_{jk}(\delta_{jl} - \omega_j(z_i^l)), \quad k = 1, ..., n-1; \quad C_n = 0.$$

Finally, $|\mathcal{B}_0(\zeta)||_{\gamma_k} = \exp(C_k)$ *a.e. with respect to* $d\omega$, $k = 1, ..., n$.

$\mathcal{B}_0(z)$ is called the generalized Blaschke product.

We now present the factorization theorem for analytic functions in the Hardy classes of finitely connected domains as stated in [11] for Jordan domains, and [1] for domains with analytic boundaries.

THEOREM 3.8 *$f(z) \in H^p(G)$ if and only if*

$$f(z) = Q(z)\mathcal{B}_0(z)exp\left(\int_\Gamma \mathcal{P}(z,\zeta)d\mu(\zeta)\right), \tag{3.3}$$

where the zeros of $\mathcal{B}_0$ correspond to the zeros of $f(z)$; $d\mu = ln|f(\zeta)|d\omega + d\nu$ and $d\nu \leq 0$ is singular with respect to $d\omega$;

$$Q(z) = exp\left(\sum_1^{n-1} \lambda_j w_j(z)\right), \quad w_j(z) = \omega_j(z) + i\tilde{\omega}_j(z),$$

where the λ_j are real numbers such that the numbers $(1/2\pi)\Delta_{\gamma_k} arg(Q(z))$ are integers. Moreover,

$$\int_\Gamma |f(\zeta)|^p d\omega(E, a, G) \leq const < +\infty.$$

The definition of functions in the Smirnov classes in G is very similar to that of the simply connected case (cf.[2], Ch.10). A function f analytic in G is said to belong to the class $E^p(G)$ if there is a sequence of domains $\{\Omega_j\}$ in G with boundaries $\{\Gamma_j\}$ that consist of a finite number of rectifiable curves, such that Ω_j eventually contains each compact subset of G, and

$$\overline{\lim_{j\to\infty}} \int_{\Gamma_j} |f(z)|^p |dz| < \infty.$$

S. Ya. Khavinson and G. Tumarkin have extended the Keldysh-Lavrentiev theorem (theorem 1.8) to finitely connected domains [15].

THEOREM 3.9 *Let $\varphi : K \to G$ be the conformal mapping of the circular domain K onto G. Then*

$$f(z) \in E^p(G) \text{ if and only if } f(\varphi(w))[\varphi'(w)]^{1/p} \in H^p(K).$$

(Note that $[\varphi'(w)]^{1/p}$ is single-valued in view of Proposition 3.)

Let $\psi : G \to K$ be the conformal map from G onto K and assume that Γ is rectifiable. It follows from Theorem 3.9 that $\psi'(z) \in E^1(G)$. Also, since $\psi'(z) \neq 0$ in G, then $\psi'(z)$ can be written as

$$\psi'(z) = \exp\left\{\int_\Gamma \mathcal{P}(z,\zeta)d\mu_0\right\},$$

where $d\mu_0 = \ln|\psi'(\zeta)|d\omega + d\nu_0$ and $d\nu_0$ is a singular measure [11]. Moreover, $d\nu_0 \geq 0$ cf. [11].

Functions in Smirnov classes on finitely connected domains with rectifiable boundaries also admit a factorization that is similar to those of H^p function in finitely connected domains. The next theorem is due to D. Khavinson ([11], Thm.4.6).

THEOREM 3.10 $f(z) \in E^p(G)$, $p > 0$, *if and only if (3.3) holds with* $d\mu = \ln|f(\zeta)|d\omega + d\nu$, *where* $d\nu$ *is singular,* $d\nu \leq \frac{1}{p}d\nu_0$ *and* $\int_\Gamma |f(\zeta)|^p ds < +\infty$.

If we further require that all of the boundary curves of G are rectifiable, then a theorem analogous to Theorem 1.10 can be proven for functions of class $E^p(G)$. This means that if $f \in E^p(G)$ then f has non-tangential boundary values a.e., so if $p \geq 1$, then f can be recovered in G from its boundary values by a Cauchy integral over Γ (cf. [2], Ch. 10).

Remark. The theorems presented here are restricted to n-connected domains. However, the Smirnov classes can also be defined on infinitely connected domains. That is, domains that have an infinite number of boundary components. Let us note the following theorem from [19].

THEOREM 3.11 *Let G be a domain with countably many boundary components. Then G is conformally equivalent to a domain bounded by analytic Jordan curves and points.*

The implication of this theorem is that on such domains the Keldysh- Lavrentiev theorem holds and hence the Smirnov classes are linear spaces. Whether this stays true for Smirnov classes on arbitrary infinitely connected domains with uncountably many boundary components is an open question.

3.3 Finitely connected Smirnov domains

Definition 1.4.1 also applies to n-connected domains (cf. [15]). That is, G is called a Smirnov domain if the conformal mapping $\varphi : K \to G$ of an n-connected circular domain K onto G is an outer function. Let l_1,..., l_n be the circular boundary curves of K. Let K_i be the simply connected domain bounded by l_i containing K and G_i be the simply connected domain bounded by γ_i and containing G. Each function $\varphi_i : K_i \to G_i$, $i = 1$,...,n will denote the conformal mapping of K_i onto G_i. S. Ya. Khavinson and G. Tumarkin have shown that G being Smirnov is also equivalent to $\varphi_i(w)$ being outer functions for all $i = 1$,...,n ([15]).

The following useful observation from [15] connects the growth/decay of the conformal mapping $\varphi : K \to G$ with that for φ_i, $i = 1, ..., n$, near respective boundary components.

LEMMA 3.2 *There exist constants* c_1 *and* c_2 *such that the inequalities*

$$0 < c_1 \leq \left| \frac{\varphi'(w)}{\varphi_i{}'(w)} \right| \leq c_2 < \infty$$

holds near l_i *for all* $i = 1, ..., n$.

The proof follows at once after one observes that $\varphi \circ \varphi_i^{-1}$ preserves l_i, is one to one near it, and hence extends analytically across l_i. We emphasize here that in Lemma 3.2 the boundary of G is assumed to be merely Jordan and rectifiable.

It is convenient to single out a subclass of non-Smirnov domains that are called "completely non-Smirnov" (cf. [10]).

DEFINITION 3.3.1 *We shall call G a "completely non-Smirnov" domain if and only if all the simply connected domains* G_i, $i = 1, ..., n$ *do not belong to the Smirnov class.*

These domains do admit functions in Smirnov classes $E^p(G)$ for $0 < p < \infty$ which are bounded a.e on Γ and possess real boundary values. The following two theorems are from [10].

THEOREM 3.12 *Let* G *be a completely non-Smirnov domain. Then there exists a non-constant function* $f(z) \in E^1(G)$ *such that* $0 \leq f(z) \leq 1$ *a.e. on* Γ.

THEOREM 3.13 *Let G be a completely non-Smirnov domain. Then, for each p such that* $0 < p < \infty$ *there exists a non-constant function* $f(z) \in E^p(G)$ *such that* $0 \leq f(z) \leq 1$ *a.e. on* Γ.

At this point, we have resolved the problem of the existence of non-constant analytic functions of the Smirnov classes with real boundary values for simply connected Smirnov domains. D. Khavinson has discussed the case when the domain under consideration is simply connected and non-Smirnov. He also investigated the case when the domain is finitely connected and completely non-Smirnov. In the next chapter, I will consider the case when a given domain is a finitely connected Smirnov domain.

Chapter 4
Finitely connected Smirnov domains

4.1 The representation of analytic functions with certain boundary properties

Before the main theorems of this thesis are presented, we first need to investigate the structure of analytic functions in simply connected domains bounded by a real analytic curve that have real boundary values except at one point where the function is unbounded. The following lemma (cf. [5]) addresses the simple case when the domain is the unit disk. The corollary (cf. [5]) that follows shows that a similar result is obtained for arbitrary simply connected domains.

Let $\mathbb{D}$ be the unit disk and $\mathbb{T} = \partial\mathbb{D}$. Let Ω be a simply connected domain in the complex plane bounded by a real analytic curve. We shall define $R_\Omega(z,\zeta)$ to be the Schwarz kernel of Ω. This means that

$$R_\Omega(z,\zeta) = \frac{1}{2\pi}\frac{\partial P(z,\zeta)}{\partial n_\zeta}, \text{ where } P(z,\zeta) = g(z,\zeta) + i\widetilde{g}(z,\zeta),$$

$z \in \Omega, \zeta \in \partial\Omega$, $\frac{\partial}{\partial n_\zeta}$ is the inner normal derivative, and g and $\widetilde{g}$ are respectfully the Green function of Ω with pole at z and its harmonic conjugate. For example, when $\Omega = \mathbb{D}$ then $R_\mathbb{D}(z,\zeta) = \frac{1}{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}$. Let $\frac{\partial}{\partial\tau}$ denote the tangential derivative.

LEMMA 4.1 *Suppose $f = u + iv$ is analytic in $\mathbb{D}$, continuous in $\overline{\mathbb{D}} \setminus \{1\}$, and $v \equiv 0$ on $\mathbb{T} \setminus \{1\}$. If $|f| = O\left(\frac{1}{|1-z|^k}\right)$ near 1, then*

$$f(z) = \sum_{n=0}^{k-1} \lambda_n \frac{\partial^n}{\partial\tau^n} iR_\mathbb{D}(z,1) + c,$$

where $\lambda_{k-1},...,\lambda_0, c \in \mathbb{R}$.

Proof. We proceed by induction. Suppose $k = 1$. Then by the generalized Schwarz reflection principle f extends to be analytic in $\mathbb{C} \setminus [1,\infty)$. So $|f(z)| = O\left(\frac{1}{|1-z|}\right)$ in $\overline{\mathbb{D}}$. Consider $F = f \circ \tau$, where $\tau^{-1}(z) = \frac{1+z}{1-z}$ is the conformal mapping of $\mathbb{D}$ onto the right half plane. Then $|F(z)| = O(|z|)$

and $\Im F(z) = 0$ on $\{it : t \in \mathbb{R}\}$. By the Schwarz reflection principle F extends to be an entire function. Therefore $F(z) = Az + B$, where $A, B \in \mathbb{C}$. This implies that

$$f(z) = A\frac{1+z}{1-z} + B.$$

Since $\Im f(z) = 0$ on $\mathbb{T} \setminus \{1\}$, then $A = \lambda i$ and $\lambda, B \in \mathbb{R}$. Suppose the statement holds for some k. Again, looking at $f \circ \tau$, we conclude that f is a rational function with a pole at 1 of order k. Since

$$\left|\frac{\partial^k}{\partial\theta^k} R_{\mathbb{D}}(z,1)\right| = O\left(\frac{1}{|1-z|^{k+1}}\right),$$

then $\lambda_k \in \mathbb{R} \setminus \{0\}$ can be chosen so that

$$\left|f(z) - \lambda_k \frac{\partial^k}{\partial\theta^k} iR_{\mathbb{D}}(z,1)\right| = O\left(\frac{1}{|1-z|^{j}}\right),$$

where $j \leq k$. Since

$$\Im\left(f(z) - \lambda_k \frac{\partial^k}{\partial\theta^k} iR_{\mathbb{D}}(z,1)\right) = 0$$

on $\mathbb{T} \setminus \{1\}$, then

$$f(z) - \lambda_k \frac{\partial^k}{\partial\theta^k} iR_{\mathbb{D}}(z,1) = \sum_{n=0}^{k-1} \lambda_n \frac{\partial^n}{\partial\tau^n} iR_{\mathbb{D}}(z,1) + c,$$

where $\lambda_{k-1}, ..., \lambda_0, c \in \mathbb{R}$. Therefore $f(z) = \sum_{n=0}^{k} \lambda_n \frac{\partial^n}{\partial\tau^n} iR_{\mathbb{D}}(1,z) + c$. □

REMARK 1 If an analytic function $f(z)$ in $\mathbb{D}$ has real boundary values on $\mathbb{T} \setminus \{1\}$ and $|f(z)| = O\left(\frac{1}{|1-z|^{\beta}}\right)$ where $\beta \in \mathbb{R}_+$ then $|f(z)| = O\left(\frac{1}{|1-z|^{[\beta]}}\right)$. To see that this is true note that by the generalized Schwarz reflection principle f extends to be analytic in $\mathbb{C} \setminus [0,\infty)$ so $|f(z)| = O\left(\frac{1}{|1-z|^{\beta}}\right)$ in $\overline{\mathbb{D}}$. Letting $F = f \circ \tau$ where $\tau^{-1}(z) = \frac{1+z}{1-z}$ is the conformal mapping of $\mathbb{D}$ onto the right half plane, then $|F(z)| = O(|z|^{\beta})$ and $\Im F(z) = 0$ on $\{it : t \in \mathbb{R}\}$. By the Schwarz reflection principle F extends to be an entire function. Therefore, by an easy corollary of Liouville's Theorem, $F(z) = a_k z^k + ... + a_1 z + a_0$, where $k \leq [\beta]$. Thus

$$f(z) = a_k \left(\frac{1+z}{1-z}\right)^k + ... + a_1 \left(\frac{1+z}{1-z}\right)^{k-1} + a_0.$$

Therefore $|f(z)| = O\left(\frac{1}{|1-z|^{[\beta]}}\right)$.

The following Corollary (cf. [5]) follows immediately by a simple change of variables.

COROLLARY 4.1.1 *If $f = u + iv$ is analytic in Ω, where Ω is a simply connected domain with analytic boundary, $v \equiv 0$ on $\partial\Omega \setminus \{\zeta\}$, continuous in $\bar{\Omega} \setminus \{\zeta\}$, and $|f(z)| = O\left(\frac{1}{|\zeta - z|^k}\right)$ near ζ, then*

$$f(z) = \sum_{n=0}^{k-1} \lambda_n \frac{\partial^n}{\partial \tau^n} iR_\Omega(z, \zeta) + c,$$

where $\lambda_{k-1}, \ldots, \lambda_0, c \in \mathbb{R}$.

The next corollary (cf. [5]) shows that Lemma 4.1 and Corollary 4.1.1 are, in fact, local properties.

COROLLARY 4.1.2 *Let Ω be as above. Let u be harmonic in Ω and smooth in $\overline{\Omega} \setminus \{\zeta_0\}$. If on the arc $\gamma_\epsilon \subset \partial\Omega$ where $\zeta_0 \in \gamma_\epsilon$, $u(\zeta) \equiv 0$ on $\gamma_\epsilon \setminus \{\zeta_0\}$ and $|u(z)| = O\left(\frac{1}{|\zeta_0 - z|^k}\right)$ near ζ_0 then*

$$u(z) = \int_{\partial\Omega \setminus \gamma_\epsilon} \frac{\partial}{\partial n_\zeta} g(z, \zeta) u(\zeta) ds + \sum_{j=0}^{k-1} \lambda_j \frac{\partial^j}{\partial \tau^j} \left(\frac{\partial}{\partial n_{\zeta_0}} g(z, \zeta_0) \right),$$

where $\lambda_{k-1}, \ldots, \lambda_0 \in \mathbb{R}$.

Proof. Let

$$u_1(z) = \int_{\partial\Omega \setminus \gamma_\epsilon} \frac{\partial}{\partial n_\zeta} g(z, \zeta) u(\zeta) ds$$

and let v be the harmonic conjugate of $u - u_1$. Let $f = v + i(u - u_1)$. Then $|f| = O\left(\frac{1}{|\zeta_0 - z|^k}\right)$ and satisfies the hypothesis of Corollary 4.1.1. Therefore,

$$f(z) = \sum_{j=0}^{k-1} \lambda_j \frac{\partial^j}{\partial \tau^j} iR_\Omega(z, \zeta_0) + c,$$

where $c, \lambda_0, \ldots, \lambda_{k-1} \in \mathbb{R}$. Since $\Re\, R_\Omega(z, \zeta_0) = \frac{\partial}{\partial n_{\zeta_0}} g(z, \zeta_0)$,

$$(u - u_1)(z) = \sum_{j=0}^{k-1} \lambda_j \frac{\partial^j}{\partial \tau^j} \left(\frac{\partial}{\partial n_{\zeta_0}} g(z, \zeta_0) \right).$$

Therefore,

$$u(z) = \int_{\partial\Omega \setminus \gamma_\epsilon} \frac{\partial}{\partial n} g(z, \zeta) u(\zeta) ds + \sum_{j=0}^{k-1} \lambda_j \frac{\partial^j}{\partial \tau^j} \left(\frac{\partial}{\partial n_{\zeta_0}} g(z, \zeta_0) \right),$$

where $\lambda_{k-1}, \ldots, \lambda_0 \in \mathbb{R}$. ☐

REMARK 2 Note that if a harmonic function u satisfies the hypothesis of Corollary 4.1.2 with the exception that $|u(z)| = O\left(\frac{1}{|\zeta_0 - z|^\beta}\right)$ near ζ_0 where $\beta \in \mathbb{R}_+$, then $|u(z)| = O\left(\frac{1}{|\zeta_0 - z|^{[\beta]}}\right)$. This statement follows now from Remark 1 and the proof of Corollary 4.1.2.

4.2 A digression: the potential function of a fluid flow

Consider a two-dimensional incompressible fluid in a bounded simply connected domain Ω in the complex plane. Let $\Phi = \phi + i\psi$, be the fluid potential function of this system. This function describes the nature of a steady fluid flow.

The real part of Φ, ϕ, is the velocity potential of the fluid flow. It usually represents the irrotational component of the fluid potential. Its gradient, $\nabla\phi = \vec{v} = (u(x,y), v(x,y))$, is the velocity of the fluid. When the curl of the gradient is zero, the fluid is said to be irrotational. The divergence of the gradient, $\nabla^2\phi = \nabla\cdot\vec{v}$, is the divergence of the fluid flow. When Φ is analytic, then this divergence is zero indicating that the fluid is incompressible.

The imaginary part of Φ, is known as the stream function. The level curves of ψ are the stream lines of the fluid. The streamlines indicate the direction of travel of a component of the fluid. The difference in value of the steam function between any two points indicates the flux of the fluid across the curve connecting those two points. The gradient of ψ, $\nabla\psi = *\vec{v} = (-v(x,y), u(x,y))$ is the force on the rotational components (vortices) of the fluid (cf. [8]). The Laplacian of ψ, $\nabla^2\psi = \nabla\times\vec{v} = -\omega$ is the only component of vorticity of the fluid. If Φ is analytic in a domain, then $\nabla^2\psi = 0$ and no vortices exist in the domain.

For a general reference regarding the fluid potential function we direct the reader to [16] Ch. 3, Sec. 2. A function of the type described in Lemma 4.1 and Corollary 4.1.1 corresponds to a fluid flow in which 2^{k-1} sources and 2^{k-1} sinks of increasing strength approach each other from opposite sides of the boundary of Ω to form "multipoles". Let us take up the case (cf. [5]) when $k = 1$, Φ is analytic in the right half plane, and the source and sink meet at $\zeta = 0$ along the real axis. Here we will let $\psi_+(z) = \frac{-1}{2\epsilon}\log|z-\epsilon|$ represent the source located at the point $z = \epsilon$ and $\psi_-(z) = \frac{1}{2\epsilon}\log|z+\epsilon|$ represent the sink located at the point $z = -\epsilon$ (cf. [16] Ch. 3, Sec. 2). Define

$$\psi_\epsilon(z) := \psi_+(z) + \psi_-(z) = \frac{1}{2\epsilon}(\log|z+\epsilon| - \log|z-\epsilon|).$$

Then we have that

$$\begin{aligned}\psi_\epsilon(z) &= \frac{1}{4\epsilon}[\log(z+\epsilon) - \log(z-\epsilon) + \log(\bar{z}+\epsilon) - \log(\bar{z}-\epsilon)] \\ &= \frac{1}{4\epsilon}\left[\log\left(1+\frac{\epsilon}{z}\right) - \log\left(1-\frac{\epsilon}{z}\right) + \log\left(1+\frac{\epsilon}{\bar{z}}\right) - \log\left(1-\frac{\epsilon}{\bar{z}}\right)\right].\end{aligned}$$

Since

$$\log(1-z) = -\sum_{n=1}^{\infty} \frac{z^n}{n},$$

then

$$\psi_\epsilon(z) = \frac{1}{4\epsilon}\left[\frac{2\epsilon}{z} + O(\epsilon^3) + \frac{2\epsilon}{\bar{z}} + O(\epsilon^3)\right] = \frac{1}{2}\left[\frac{1}{z} + \frac{1}{\bar{z}} + O(\epsilon^2)\right].$$

Letting $\epsilon \to 0$ we have that

$$\psi(z) = \frac{1}{2}\left[\frac{1}{z} + \frac{1}{\bar{z}}\right] = \frac{\Re(z)}{|z|^2}.$$

This implies that

$$\phi(z) = \frac{\Im(z)}{|z|^2},$$

and thus

$$\Phi(z) = \frac{i}{z}.$$

Thus, $\Im\Phi(z) = 0$ on $\{it : t \in \mathbb{R} \setminus \{0\}\}$ and $|\Phi(z)| = \frac{1}{|z|}$. When $k = 1$, Ω is the unit disk, and $\zeta = 1$ then, using the conformal map $w = \tau(z) = \frac{1-z}{1+z}$, we see that $\Phi \circ \tau^{-1} = i\frac{1+w}{1-w}$ which is as to be expected by Lemma 4.1.

Essentially, when $k = 1$, the fluid potential function models a dipole which lies on the boundary of Ω. This situation is very similar to the case in electrodynamics in which an electric dipole is formed on a boundary by the meeting of two point charges of opposite and increasing strength. The difference lies in the assignment of the real and imaginary parts of the respective functions. Yet, similar to the electrodynamic case, if $k > 1$, then Φ models 2^k multipoles that are situated at the point ζ on the boundary of Ω.

4.3 Analytic functions with real boundary values in class E^p

Let G, K, $\varphi(w)$, and $\varphi_i(w)$ be defined as in the beginning of section 3.3. We shall now present the main theorems of this thesis (following [5]).

THEOREM 4.1 *Let G be an n-connected domain in $\mathbb{C}$ bounded by the curves $\gamma_1, ..., \gamma_n$ which are real analytic except at the points $z_1, ..., z_m \in \Gamma$, where there are corners with interior angles $\alpha_1, ..., \alpha_m$, $\pi < \alpha_j \leq 2\pi$ for all $j = 1, ..., m$. Then every $f(z) \in E^p(G)$ with real boundary values is a constant whenever $p \geq \frac{\alpha}{\pi}$, where*
$\alpha = min\{\alpha_q : f \text{ is unbounded near } z_q,\ q \in \{1, .., m\}\}$.

Note that an interior angle equal to 2π means that there is a cusp on the boundary pointing into the domain. It is readily seen (as noted in [4] for simply connected domains) that the case $\alpha_j \leq \pi$ for all j, yields no non-constant $E^1(G)$ functions with real boundary values. Hence we shall omit it.

Proof. Suppose that there exists such an $f(z) \in E^{\alpha/\pi}(G)$. Then by the Keldysh-Lavrentiev theorem for multiply connected domains

$$f(\varphi(w))[\varphi'(w)]^{\pi/\alpha} \in H^{\alpha/\pi}(K).$$

Let $w_j \in \partial K$ such that $\varphi(w_j) = z_j$ for each $j = 1, ..., m$. Assume that $\varphi_i(w_j) = z_j$ whenever $z_j \in \gamma_i$. Then, cf. [18], Ch. 3, Sec. 4, near each circular arc of l_i containing only w_j, we have

$$\varphi_i'(w) = (w - w_j)^{\alpha_j/\pi - 1} g_j(w),$$

where $g_j(w)$ is bounded away from 0 and ∞. Then

$$m_j |w - w_j|^{\alpha_j/\alpha - \pi/\alpha} \leq |\varphi_i'(w)|^{\pi/\alpha} \leq M_j |w - w_j|^{\alpha_j/\alpha - \pi/\alpha} \tag{4.1}$$

near each arc of l_i containing only w_j where $m_j = \min |g_j(w)|$ and $M_j = \max |g_j(w)|$. By Lemma 3.2 and (4.1) there exist constants C_{1j} and C_{2j} depending on j such that

$$C_{1j} |w - w_j|^{\alpha_j/\alpha - \pi/\alpha} \leq |\varphi'(w)|^{\pi/\alpha} \leq C_{2j} |w - w_j|^{\alpha_j/\alpha - \pi/\alpha}, \tag{4.2}$$

near each arc of l_i containing w_j. Since $0 < \alpha_j - \pi \leq \pi$, then $0 < \frac{\alpha_j - \pi}{\alpha} \leq \frac{\pi}{\alpha} < 1$. This implies that $f \circ \varphi$ is to be unbounded near some w_j (otherwise $f \equiv const.$, cf. [2], [6], [11]). Since $(f \circ \varphi)[\varphi']^{\pi/\alpha} \in H^{\alpha/\pi}(K)$ then repeating the argument in the proof of Theorem 2 in [4] (in particular, formula (4) and ff.) we obtain

$$|f \circ \varphi(w)| = O\left(\frac{1}{|w - w_j|^k}\right)$$

near each arc of l_i containing only w_j, where $0 < k < 2$. However, Remark 2 implies that k must equal 1. For each $q \in \{q_1, ..., q_s\} \subseteq \{1, ..., m\}$ such that f is unbounded near z_q, let $U_q \subset K$ be a simply connected domain such that the boundary of U_q contains the arc of l_i containing w_q and $w_r \notin \partial U_q \cap \partial K$ when $q \neq r$. Since $|f \circ \varphi(w)| = O\left(\frac{1}{|w - w_q|}\right)$ near w_q and $\Im(f \circ \varphi(w)) \equiv 0$ on $\partial K \setminus \{w_{q_1}, ..., w_{q_s}\}$ then, by Corollary 4.1.2 we have in U_q,

$$\Im(f \circ \varphi(w)) = v(w) = h_q(w) + \lambda_q \frac{\partial}{\partial n_{w_q}} g(w, w_q),$$

where h_q is a bounded harmonic function, $\lambda_q \in \mathbb{R}$ and $g_q(w, w_q)$ is the Green function of U_q. This implies that $|v|$ is integrable near each w_q and hence has a harmonic majorant in K. Since $v(w) \neq 0$ only at $\{w_{q_1}, ..., w_{q_s}\}$, where f is unbounded, then

$$v(w) = \frac{1}{2\pi} \int_{\partial K} \frac{\partial}{\partial n_\xi} g(w, \xi) d\mu(\xi),$$

where $g(w, \xi)$ is the Green function of K and $d\mu(\xi) = \lambda_1 \delta_{w_{q_1}} + ... + \lambda_m \delta_{w_{q_s}}$ for some $\lambda_{q_1}, ..., \lambda_{q_s} \in \mathbb{R}$. Therefore

$$v(w) = \lambda_{q_1} \frac{\partial g(w, w_{q_1})}{\partial n_{w_{q_1}}} + ... + \lambda_{q_s} \frac{\partial g(w, w_{q_s})}{\partial n_{w_{q_s}}}.$$

The coefficients $\lambda_{q_1}, ..., \lambda_{q_s}$ are not all equal to zero and $v(w)$ must have a single valued conjugate $u(w)$. Since $g(w, w_q)$ has a logarithmic pole at each w_q then near each arc l_i containing w_q, $|v(w)| \backsim \frac{|\lambda_q|}{|w_q - w|}$. Therefore $|f \circ \varphi(w)| \backsim \frac{|\lambda_q|}{|w_q - w|}$ near each arc of l_i containing w_q. In particular, when $\alpha = \alpha_{q_1}$, then near the arc of ∂K containing w_{q_1}, since $\lambda_{q_1} \neq 0$, we obtain from (4.2) that

$$|f \circ \varphi(w)||\varphi'(w)|^{\pi/\alpha} \geq C|w - w_{q_1}|^{1-\pi/\alpha - 1} = C|w - w_{q_1}|^{-\pi/\alpha}$$

for some constant C. This implies that $f(\varphi(w))[\varphi'(w)]^{\pi/\alpha} \notin H^{\alpha/\pi}(K)$ and therefore $f(z) \notin E^{\alpha/\pi}(G)$. This is a contradiction. $\square$

The following theorem shows that the latter result is essentially sharp.

THEOREM 4.2 *Let G be an n-connected domain in $\mathbb{C}$ bounded by the curves $\gamma_1, ..., \gamma_n$ which are real analytic except at the points $z_1, ..., z_m \in \Gamma$, $m \geq n$, where there are corners with interior angles $\alpha_1, ..., \alpha_m$. If $\pi < \alpha_j \leq 2\pi$ for all $j = 1, ..., m$ then for all $p < \frac{\alpha}{\pi}$ where $\alpha = min\{\alpha_1, ..., \alpha_m\}$, there exists a non-constant $f(z) \in E^p(G)$ with real boundary values a.e.*

Proof. Let $w_j \in \partial K$, $j = 1, ..., m$ be the same as in the proof of Theorem 4.1. Consider the harmonic function

$$v(w) = \frac{1}{2\pi} \int_{\partial K} \frac{\partial}{\partial n_\xi} g(w, \xi) d\mu(\xi),$$

where $g(w, \xi)$ is the Green function of K and $d\mu(\xi) = \lambda_1 \delta_{w_1} + ... + \lambda_m \delta_{w_m}$ for some $\lambda_1, ..., \lambda_m \in \mathbb{R}$. Then

$$v(w) = \lambda_1 \frac{\partial g(w, w_1)}{\partial n_{w_1}} + ... + \lambda_m \frac{\partial g(w, w_m)}{\partial n_{w_m}}$$

and $v(w) = 0$ on $\partial K \setminus \{w_1, ..., w_m\}$. Also, near each arc of ∂K containing w_j $|v(w)| \backsim \frac{\lambda_j}{|w_j - w|}$. By Proposition 4 $\lambda_1, ..., \lambda_m$ can be chosen so that $v \not\equiv 0$ and has a single valued harmonic conjugate u.

Let $F = u + iv$. Then F is an analytic function on K that is real valued a.e. and $|F(w)| \backsim \frac{\lambda_j}{|w_j - w|}$ near each arc of ∂K containing w_j. Let, as before, $\varphi : K \to G$ be the conformal mapping of the n-connected circular domain K onto G and $\varphi(w_j) = z_j$. Then as in the proof of Theorem 4.1,

$$|\varphi'(w)| \leq C_j |w - w_j|^{\alpha_j/\pi - 1}$$

near each arc of ∂K containing w_j. Let $f = F \circ \varphi^{-1}$. Then

$$|f \circ \varphi(w)||\varphi'(w)|^{1/p} \leq C_j |w - w_j|^{\frac{\alpha_j}{p\pi} - 1/p - 1}$$

near each arc of ∂K containing w_j. Then $f(\varphi(w))[\varphi'(w)]^{1/p} \in H^p(K)$ when $p(\frac{\alpha_j}{p\pi} - 1/p - 1) > -1$. (Indeed, it is immediately seen that F belongs to the Smirnov class $N^+(K)$, cf. [2], [6], [11], while $\varphi' \in H^1(K) \subset N^+(K)$ since ∂G is rectifiable. Thus, by the Polybarinova-Kotchine theorem, for $(f \circ \varphi)(\varphi')^{1/p}$ to be in $H^p(K)$ we only need to check that $|f \circ \varphi|^p |\varphi'| \in L^1(\partial K)$.) This means that $\alpha_j/\pi - 1 - p > -1$ for all j. That is $p < \alpha_j/\pi$ for all j. Therefore $f(z) \in E^p(G)$ for all $p < \alpha/\pi$. □

REMARK 3 The hypothesis of Theorem 5.4 is sufficient but not necessary. Under certain conditions of symmetry, it is possible to have fewer than n corners and still find a non-constant analytic function of class E^p with real boundary values.

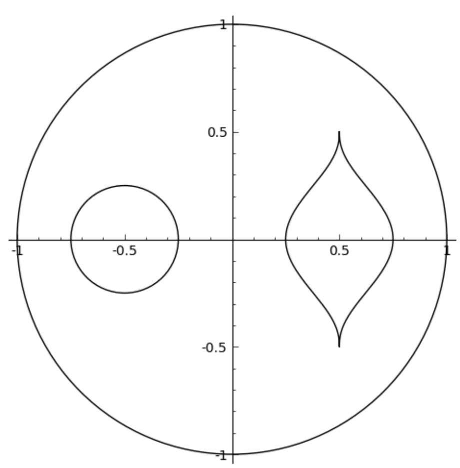

Figure 7.: Symmetry example.

EXAMPLE 4 Consider the triply connected domain bounded by the unit circle, a circle of radius $1/4$ centered at $-1/2$, and the closed curve parametrized by the function $(1/4\sin^5(t)+1/2, 1/2\cos(t))$, $0 \leq t \leq 2\pi$, cf. Fig. 6.

This domain has two cusps on the boundary that are symmetric about the real axis and there is an interior angle of 2π at each point of the cusps. There exists a conformal mapping from a circular domain similar to the one defined in Example 3 having the same symmetries with respect to the real and imaginary axes onto G. Using the same argument as in Example 3 and the proof of Theorem 4.2, one can construct an analytic function with real boundary values in G that is in the Smirnov class E^p for all $p < 2$.

Chapter 5

Neuwirth-Newman's Theorem

5.1 Neuwirth-Newman's Theorem

Recall that if $f \in H^1(\mathbb{D})$ and is real valued on the boundary of the unit disk then it is a constant function. In 1967 J. Neuwirth and D. Newman showed in [17] that any function in the class $H^{1/2}$ on the unit disk with non-negative boundary values a.e. is constant. They furnished the Koebe function as an example of a non-constant function that is in H^p for all $p < \frac{1}{2}$ and is positive valued on $\mathbb{T}$. The Koebe function is the function $\frac{w}{(1+w)^2}$ which is positive on the unit circle except at $w = -1$.

THEOREM 5.1 (Neuwirth - Newman) *If $f(w) \in H^{1/2}$ and $f(w) \geq 0$ a.e. on $|w| = 1$ then $f(w)$ is a constant.*

Proof. We may assume that $f(w) \not\equiv 0$. By theorem 1.3 f can be factored as

$$f(w) = B(w)S(w)F^2(w), \quad F(w) \in H^1.$$

Since $f(w) \geq 0$ on $|w| = 1$ then $f(w) = |f(w)|$ on $|w| = 1$ and we have

$$B(w)S(w)F^2(w) = |F(w)|^2 \quad \text{a.e. on} |w| = 1.$$

Since $f(w) \not\equiv 0$ then $F(w)$ is non-zero a.e. on $|w| = 1$, so dividing by $F(w)$ we get

$$B(w)S(w)F(w) = \overline{F(w)} \quad \text{a.e. on} |w| = 1. \tag{5.1}$$

The left side of (5.1) is in H^1 so all of the negative Fourier coefficients are zero. The right side is $\bar{H}^1$ so all of the positive Fourier coefficients are zero. Thus, only the constant terms remain. Therefore, f must be a constant function. □

Let us also recall that $f \in H^p(G)$ if and only if $f \circ T \in H^p(\mathbb{D})$, where $T : \mathbb{D} \to G$ is the uniformization map (cf. [7], Chapter VI Sec. 1 and [11], Sec. 1 Theorems 1.1 and 1.2 and references [2], [11], [18] there). Thus if $f \in H^{1/2}(G)$ and $f(\zeta) \geq 0$ a.e. with respect to the harmonic measure on Γ, then $f \circ T$ satisfies the hypothesis of the Neuwirth-Newman Theorem [17] and, therefore, is a constant.

If G is a simply connected Smirnov domain and for any p such that $p \geq p_0$, $f(z) \in E^p(G)$ and $f(z)$ having real boundary values imply that $f(z)$ is constant then the Neuwirth-Newman argument [17] extends essentially word for word to E^p classes in simply connected domains and renders the following [4]:

THEOREM 5.2 *Let G be a simply connected Smirnov domain with rectifiable boundary Γ. Let $p_0 \geq 1$ be defined as the smallest $p \geq 1$ such that $f \in E^p(G)$ and f has real boundary values a.e. on Γ imply that f is a constant. Then all $f \in E^{p_0/2}$ such that $f \geq 0$ on Γ are constants.*

Proof. Following the proof of the Neuwirth-Newman theorem, we have that

$$f(z) = B(z)S(z)F^2(z), \quad F(z) \in E^{p_0},$$

where $B(z)$ is a transplanted Blaschke product, $S(z)$ is a singular inner function, and $F(z)$ is an outer function, cf. [10] and [11] for details. Here we precisely use that G is Smirnov (see Theorem 3.10). But on Γ

$$B(z)S(z)F^2(z) = |f(z)| = F(z)\overline{F(z)} \quad \text{a.e.}$$

Thus dividing by $F(z)$ we have,

$$B(z)S(z)F(z) = \overline{F(z)} \in E^{p_0}(G),$$

which implies that

$$F(z) + \overline{F(z)} \in E^{p_0}(G),$$

and is real-valued hence a constant. Therefore, $f(z) = const \cdot B(z)S(z)$ is a bounded function in $E^{p_0}(G)$ with non-negative boundary values, and hence a constant. □

5.2 An open problem

Although it was conjectured in [5] that the statement of Theorem 5.2 holds for finitely connected domains, I have been unable to prove it. More precisely, the difficulty in directly extending the

Neuwirth-Newman argument to multiply connected domains is as follows. From Theorem 3.10 it follows that, $f = QBSF^2$, where B is the generalized Blaschke product, S a singular inner function, F^2 an outer factor, $F \in E^{p_0}$, Q is an invertible bounded analytic function, and $|Q|$ is a local constant on Γ. The problem is that $|B|$ and $|S|$ are local constants a.e. on the boundary of G. Hence, the assumption that $f \geq 0$ a.e. on Γ only yields that

$$f = QBSF^2 = |QBS|F\bar{F} \quad \text{a.e. on } \Gamma,$$

and accordingly,

$$QBSF = |QBS|\bar{F} \text{ a.e. on } \Gamma. \tag{5.2}$$

Unfortunately, (5.2) alone does not seem to imply that $\bar{F}$ and hence $F + \bar{F} = 2\Re(F)$ are analytic functions in $E^{p_0}(G)$. Indeed, already in the annulus $G = \{r < |z| < R\}$ the function $f(z) = \frac{1}{\bar{z}}$ differs only by constant multiples from an analytic function z:

$$\frac{1}{\bar{z}}\Big|_{|z|=r} = \frac{z}{r^2}\,,\ \frac{1}{\bar{z}}\Big|_{|z|=R} = \frac{z}{R^2}.$$

Thus, extending Theorem 5.2 to finitely connected Smirnov domains remains an open question. The following example from [5] illustrates why I believe Theorem 5.2 should extend to multiply connected domains. For the sake of clarity, it is presented for doubly connected domains although it readily extends to domains of higher connectivity.

THEOREM 5.3 *Let G be a doubly connected domain with boundary Γ that is real analytic except at the points $z_1, ..., z_m,\ m \geq 1$, where there are inward pointing cusps. Then every $f \in E^1(G)$ with non-negative boundary values is a constant.*

REMARK 4 In view of the main results, Theorems 4.1 and 4.2 ($\alpha = 2\pi$ for the inward cusps), $p_0 = 2$ in this case.

Proof. Suppose that there exists such an $f \in E^1(G)$. Let γ_1 denote the curve that contains z_1. Let $\varphi : A \to G$ be the conformal map of the annulus $A = \{z : 1 < |z| < R\}$ onto G and $\varphi_1(w)$ be the conformal mapping of the domain bounded by the unit circle and containing A onto the domain bounded by γ_1 and containing G. Assume that $\varphi(1) = z_1$ and that φ maps the unit circle onto γ_1.

Since there is an inward pointing cusp at z_1, the interior angle is 2π, so by [18], Ch. 3, Sec. 4, near 1, $\varphi_1'(w) = (w-1)g(w)$ where $g(w)$ is bounded away from 0 and ∞. This implies that by Lemma 3.2 there exist constants c_1 and c_2 such that near 1

$$c_1|w-1| \leq |\varphi'| \leq c_2|w-1|. \tag{5.3}$$

Let U be the intersection of a neighborhood of 1 and A such that U does not contain any of the pre-images of $z_2, ...z_m$. Since $f(\varphi(w))\varphi'(w) \in H^1(A)$ then by (5.3) $f(\varphi)(w-1) \in H^1(U)$. Let

$$h(w) = f(\varphi(w))(w-1)\left(\frac{1}{w} - 1\right) = f(\varphi(w))\frac{(1-w)^2}{-w}.$$

Then $h \in H^1(U)$ and, on $\eta = \partial U \cap \partial A$, we have $h(w) = f(\varphi(w))|1-w|^2 \geq 0$. So by the generalized Schwarz reflection principle h can be analytically continued across η. This implies that $|f(\varphi(w))| = O\left(\frac{1}{(w-1)^2}\right)$ in U. Then by Corollary 4.1.2

$$f(\varphi(w)) = \lambda_1 iR(w,1) + \lambda_2 iR'(w,1) + a(w),$$

where $R(w,1)$ is the Schwarz kernel of U and $a(w)$ is an analytic function that is real valued on η and analytic across. $\Re(\lambda_1 iR(w,1))$ takes arbitrarily large positive and negative values in a neighborhood of 1 on η. Thus,, since $f(\varphi(w)) \geq 0$ on η, $\lambda_2 \neq 0$. Therefore,

$$|f(\varphi(w))(1-w)| \geq \frac{const.}{|1-w|} \text{ near } 1.$$

Hence $f(\varphi(w))(1-w)$ is not in H^1. Hence, $\lambda_1 = \lambda_2 = 0$. The same argument applies to $z_2, ...z_m$. Hence $f(\varphi(w))$ is bounded on A and therefore, is a constant. $\square$

References

[1] Coifman, R., and Weiss, G., *A kernel associated with certain multiply-connected domains and its applications to factorization theorems*, Studia Math. 28.1, (1966), 31-68.

[2] Duren, P., *Theory of H^p Spaces*, Academic Press, New York, (1970).

[3] Duren, P., *Univalent Functions*, Springer-Verlag, New York, (2010).

[4] De Castro, L., and Khavinson, D., *Analytic functions in Smirnov classes E^p with real boundary values*, Complex Analysis and Operator Theory 7, (2013), 101-106.

[5] De Castro, L., and Khavinson, D., *Analytic functions in Smirnov classes E^p with real boundary values II*, Analysis and Math. Physics 2, (2013), 21-35.

[6] Fisher, S., *Function Theory on Planar Domains*, Dover Publications, New York, (1983).

[7] Goluzin, G. M., *Geometric Theory of Functions of a Complex Variable*, American Mathematical Society, Rhode Island, (1969), (translated from Russian).

[8] Gustafsson, B., *On the motion of a vortex in two-dimensional flow of an ideal fluid in simply and multiply connected domains*, TRITA-MAT-1979-7 (unpublished research bulletin).

[9] Jones, P. W., and Smirnov, S. K., *On V. I. Smirnov domains*, Anal. Acad. Sci. Fennicae Mathematica 24, (1999), 105-108 .

[10] Khavinson, D., *Remarks concerning boundary properties of analytic functions of E^p-classes*, Indiana Univ. Math. J. 31, (1982), 779-787.

[11] Khavinson, D., *Factorization theorems for different classes of analytic functions in multiply connected domains*, Pac. J. Math. 108, (1983), 295-318.

[12] Khavinson, D., *On the removal of periods of conjugate functions in multiply connected domains*, Mich. Math. J. 31, (1984), 371-379.

[13] Khavinson, S. Ya., *Factorization theory for single-valued analytic functions on compact Riemann surfaces with boundary*, Russ. Math. Surv. 44 (4), (1989), 113-156.

[14] Khavinson, S. Ya., and Tumarkin, G. C., *Classes of analytic functions in multiply-connected domains represented by the formulas of Cauchy and Green.* Uspehi Mat. Nauk 13 (1958), no. 2 (80), 215-221 (in Russian).

[15] Khavinson, S. Ya., and Tumarkin, G. C., *Classes of analytic functions in multiply-connected domains*, Contemporary Problems in the Theory of Functions of One Complex Variable (Russian), 45-77, Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, (1960). French translation: *Fonctions d'une variable complexe. Problèmes contemporains*, 37-71, Gauthers-Villars, Paris, (1962).

[16] Lavrentiev, M., and Chabat, B., *Methods of the Theory of Functions of One Complex Variable* (Russian), Nauka, Moscow, Russia, (1973). French translation: *Mèthodes de la thàeorie des fonctions d'une variable complexe*, Mir, Moscow, (1977).

[17] Neuwirth, J., and Newman, D., *Positive $H^{1/2}$-functions are constants*, Proc. Amer. Math. Soc. 18, (1967), 958.

[18] Pommerenke, Ch., *Boundary Behavior of Conformal Maps*, Springer-Verlag, Berlin, (1992).

[19] Sibner, R. J., *Domains bounded by analytic Jordan curves*, Bull. Amer. Math. Soc., Vol. 76, no. 1, (1970), 61-63.