January 2013

Boolean Partition Algebras

Joseph Anthony Van Name
University of South Florida, jvanname@mail.usf.edu

Boolean Partition Algebras

by

Joseph Van Name

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
Department of Mathematics
College of Arts and Sciences
University of South Florida

Major Professor: Vilmos Totik, Ph.D.
Xiang-Dong Hou, Ph.D.
Dmitry Khavinson, Ph.D.
Gregory McColm, Ph.D.

Date of Approval:
March 6, 2013

Keywords: Boolean algebra, uniform space, ultrafilter, Stone duality, inverse limit

Table of Contents

List of Tables

Abstract

A Boolean partition algebra is a pair (B, F) where B is a Boolean algebra and F is a filter on the semilattice of partitions of B where $\bigcup F = B \setminus \{0\}$. In this dissertation, we shall investigate the algebraic theory of Boolean partition algebras and their connection with uniform spaces. In particular, we shall show that the category of complete non-Archimedean uniform spaces is equivalent to a subcategory of the category of Boolean partition algebras, and notions such as supercompleteness of non-Archimedean uniform spaces can be formulated in terms of Boolean partition algebras.

1 Boolean partition algebras

The main objective of this chapter is to introduce and motivate the notion of a Boolean partition algebra. We shall motivate the notion of a Boolean partition algebra by generalizing Stone duality to a duality between uniform spaces and Boolean partition algebras.

1.1 Introduction

In this work, we shall investigate the notion of a Boolean partition algebra. More specifically, we shall focus on the algebraic theory of Boolean partition algebras and their relation to uniform spaces. It should be noted that Boolean partition algebras can be applied to mathematical logic, category theory, and point-free topology, but we shall not be able to cover the relation between Boolean partition algebras and these interesting topics here.

Marshall Stone proved in [20] a duality between Boolean algebras and compact totally disconnected spaces, and this duality is known as Stone duality. Stone duality transforms Boolean algebras into compact totally disconnected spaces and transforms compact totally disconnected spaces into Boolean algebras. Furthermore, Stone duality relates continuous functions with Boolean algebra homomorphisms, so there is an equivalence between the category of Boolean algebras and the category of compact totally disconnected spaces. This duality is the most essential result regarding Boolean algebras since it is often easier to work with compact totally disconnected spaces than with Boolean algebras. Furthermore, Stone duality is a significant topological result since it gives many examples of topological spaces derived from discrete structures rather than geometric or analytic objects. Also, since Stone discovered this duality, many people have discovered similar dualities and equivalence of categories such as Priestley duality [16], and the duality between sober spaces and spatial spaces [17].

A Boolean partition algebra is a Boolean algebra along with a filter on the meet-semilattice of partitions that contains all finite partitions. With this extra structure on Boolean algebras, we may extend Stone duality to a duality between a very broad category of complete uniform spaces called non-Archimedean complete uniform spaces and subcomplete and stable Boolean partition algebras. A special case of this duality gives Stone's original duality between Boolean algebras and compact totally disconnected spaces. With this general duality at hand, we will be able to go back and forth between Boolean algebraic concepts and topological and uniform concepts.

There are a few intuitive meanings behind the notion of a Boolean partition algebra. Intuitively, Boolean partition algebra is a Boolean algebra along with a certain collection of least upper bounds which we consider. For instance, if we have a complete Boolean algebra B, then there are Boolean partition algebras $(B,F),(B,G),(B,H)$ where in (B,F) we only consider the finite least upper bounds, in (B,G) we only consider the countable least upper bounds, and in (B,H) we consider all least upper bounds. We shall define the notions of admissibility, join-admissibility, and meet-admissibility in order to describe which least upper bounds we consider. We shall see that the Boolean partition algebra morphisms are precisely the mappings that preserve the least upper bounds of admissible sets and complementation. Furthermore, the Boolean partition algebra morphisms into a locally refinable Boolean partition algebra are precisely the maps that preserve join-admissible least upper bounds and complementation. Subcomplete Boolean partition algebras can also be thought of as inverse limits where all the transitional mappings are surjective or as point-free uniform spaces. In fact, the categories of subcomplete Boolean algebras, surjective inverse systems, and certain point-free uniform spaces are all equivalent.

In chapter 1, we shall introduce the notion of a Boolean partition algebra and enough of the theory of Boolean partition algebras in order to develop the duality between Boolean partition algebras and uniform spaces. In chapter 2, we shall develop much of the algebraic theory behind Boolean partition algebras. In section 1 of chapter 2, we briefly investigate certain types of subalgebras of Boolean partition algebras. In section 2 of chapter 2, we shall study special ideals and filters called F-ideals and F-filters and the quotient Boolean partition algebras by F-ideals and F-filters. We first prove the isomorphism theorems and correspondence theorem for Boolean partition algebras. We then show that the Boolean partition algebras where every F-filter is extendible to an F-ultrafilter correspond to the Boolean partition algebras that satisfy a generalized compactness property. We then give a one-to-one correspondence between closed sets in uniform spaces and certain filters in Boolean partition algebras. We conclude this section by representing all Boolean partition algebras as quotients of stable Boolean partition algebras. In section 3 of chapter 2, we investigate the notion of subcompleteness. In particular, we shall investigate the notion of a subcompletion and direct limits of Boolean partition algebras. In section 4 of chapter 2, we study products, along with the notions of precompleteness and local refinability.

In section 1 of chapter 3, I shall investigate the notion of admisssibility. Intuitively, admissible sets are the sets in Boolean partition algebras whose least upper bound and greatest lower bound is considered whenever it exists. For instance, the partition homomorphisms are the mappings that preserve these least upper bounds and greatest lower bounds. In section 3.2, we shall use basic facts on admissible sets and ideals in order to characterize the supercomplete uniform spaces as the uniform spaces where every F-filter is extendible to an F-ultrafilter. We shall also represent the F-ideals and F-filters in a Boolean partition algebra as threads in an inverse limit. In section 3.3, we shall study join-admissibility and its connection with ideals, and local refinability. In section 3.4, we conclude this dissertation by studying refinement properties of partitions and distributivity properties of Boolean partition algebras, and we shall characterize the Boolean partition algebras corresponding to uniform spaces and supercomplete uniform spaces in terms of these distributivity properties.

1.2 Preliminaries

In this dissertation, if $f : X \to Y$ is a function and $A \subseteq X, B \subseteq Y$, then we shall let $f[A]$ denote the image of A and $f^{-1}[B]$ denote the inverse image of B. Let $f'' : P(X) \to P(Y), f_{-1} : P(Y) \to P(X)$ denote the image and inverse image functions. In other words, $f''(A) = f[A] = \{f(a)|a \in A\}$ for $A \subseteq X$ and $f_{-1}(B) = f^{-1}[B] = \{a \in A|f(a) \in B\}$ for $B \subseteq X$.

If I is an index set, then let $\pi_i : \prod_{i \in I} X_i \to X_i$ be the projection mapping.

If P is a poset and $a \in P$, then define $\uparrow a := \{b \in P|b \geq a\}, \downarrow a := \{b \in P|b \leq a\}$. If $L \subseteq P$, then we say that L is a lower set if $x \in L, y \leq x \Rightarrow y \in L$. A poset A is a meet-semilattice if every pair of elements has a greatest lower bound. Similarly, a poset A is a join-semilattice if every pair of elements has a least upper bound. If A is a meet-semilattice, then we shall write $a \wedge b$ for the greatest lower bound of a and b. Likewise, if A is a join-semilattice, then we shall write $a \vee b$ for the least upper bound of a and b. A lattice is a poset that is both a meet-semilattice and a join-semilattice. In a join-semilattice, $a \leq b$ if and only if $a \vee b = b$, and in a meet-semilattice we have $a \leq b$ if and only if $a = a \wedge b$. Therefore we can recover the ordering from the suprema and infima of join-semilattices and meet-semilattices.

In a meet-semilattice or join-semilattice the operation $\wedge$ ($\vee$ respectively) are associative, commutative, and idempotent (i.e. $x = x * x$). Similarly, if $\vee$ is a binary operation on a set X that is commutative, associative, and idempotent, then let $x \leq y$ if and only if $x \vee y = y$. Then $\leq$ is a partial ordering on X where $x \vee y$ is the least upper bound of x, y for every pair x, y in X. Similarly, if $\wedge$ is a commutative, associative, and idempotent binary operation on a set X, then we may define a partial ordering $\leq$ on X by letting $x \leq y$ if and only if $x = x \wedge y$. Then $x \wedge y$ is the greatest lower bound of x and y for each pair of elements $x, y \in X$. Therefore one may define a semilattice to be a set X with a commutative, associative, and idempotent binary operation.

In a lattice, in addition to commutativity, associativity, and idempotence, we have the following identities called the absorption identities $x = x \wedge (x \vee y), x = x \vee (x \wedge y)$.

Furthermore, if $(X, \wedge, \vee)$ is a triple such that $(X, \wedge)$ is a meet-semilattice, $(X, \vee)$ is a join-semilattice, and $(X, \wedge, \vee)$ satisfies the absorption laws, then the partial order obtained from the join is the partial order obtained from the meet, and in particular $(X, \wedge, \vee)$ is a lattice. We can therefore define a lattice in terms of a partial ordering or in terms of algebraic operations.

If A is a meet-semilattice, then a nonempty subset $F \subseteq A$ is a filter if

1. $x \wedge y \in F$ whenever $x \in F$ and $y \in F$, and
2. if $x \leq y$ and $x \in F$, then $y \in F$.

If A is a meet-semilattice, then a nonempty subset $\mathfrak{F} \subseteq A$ is a filterbase if whenever $x, y \in \mathfrak{F}$, then there is a $z \in \mathfrak{F}$ with $z \leq x$ and $z \leq y$. Alternatively, a nonempty subset $\mathfrak{F} \subseteq A$ is a filterbase iff $\bigcup_{x \in \mathfrak{F}} \uparrow x$ is a filter. If $\mathfrak{F}$ is a filterbase, then the filter $\bigcup_{x \in \mathfrak{F}} \uparrow x$ is called the filter generated by $\mathfrak{F}$. If X is a set, then a filter on X is simply a filter on the powerset $P(X)$.

A lattice L is said to be distributive if it satisfies the identity $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. A lattice is distributive if and only if it satisfies the other distributive identity $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

A Boolean algebra is a distributive lattice B with least element 0 and greatest element 1 along with a unary operation $'$ that satisfies the identities $x \wedge x' = 0, x \vee x' = 1$. The main text for facts about Boolean algebras is [12].

If B is a Boolean algebra and $a \in B$, then let $B \upharpoonright a = \{b \in B | b \leq a\}$. Then $B \upharpoonright a$ is a sublattice of B. Furthermore, if we define a mapping $^* : B \upharpoonright a \rightarrow B \upharpoonright a$ by letting $b^* = a \wedge b'$ for $b \in B$, then the algebra $(B \upharpoonright a, \wedge, \vee, 0, a, *)$ is a Boolean algebra. Define a mapping $\alpha_a : B \rightarrow B \upharpoonright a$ by letting $\alpha_a(b) = a \wedge b$. Then α_a is a surjective Boolean algebra homomorphism.

A ring R with unity such that $x^2 = x$ for $x \in R$ is called a Boolean ring. Every Boolean ring has characteristic 2 since $x + x = (x + x)^2 = x^2 + x^2 + x^2 + x^2 = x + x + x + x$ implies $x + x = 0$. Furthermore, every Boolean ring is commutative since $x + y = (x + y)^2 = x^2 + xy + yx + y^2 = x + xy + yx + y$ implies $xy + yx = 0$ and hence $xy = yx$.

If R is a Boolean ring, then define operations

1. $x \wedge y = xy$.
2. $x \vee y = xy + x + y$
3. $x' = 1 + x$.

Then $(R, \wedge, \vee, 0, 1, ')$ is a Boolean algebra which we shall tentatively denote by $R^{\otimes}$.

If B is a Boolean algebra, then define operations

1. $x \cdot y = x \wedge y$
2. $x + y = (x \wedge y') \vee (y \wedge x')$.

Then $(B, \cdot, +, 0, 1)$ is a Boolean ring which we shall denote by $B^{\otimes}$. If B is a Boolean algebra, then $(B^{\otimes})^{\otimes} = B$. If R is a Boolean ring, then $(R^{\otimes})^{\otimes} = R$. Said differently, Boolean rings and Boolean algebras are essentially the same.

If B is a Boolean algebra, then for each ideal I, the set $\{x \in B | x' \in I\}$ is a filter. Furthermore, if $Z \subseteq B$ is a filter, then $\{x \in B | x' \in Z\}$ is an ideal. Therefore for a Boolean algebra B the ideals and filters are in a one-to-one correspondence.

If B is a Boolean algebra, then a subset $I \subseteq B$ is an ideal in the Boolean algebra B if and only if I is an ideal in the ring $B^{\otimes}$. Therefore one may take the quotient of a Boolean algebra modulo some ideal I. More specifically, if B is a Boolean algebra and I is an ideal on B, then let $\sim$ be the equivalence relation on B where $a \sim b$ if and only if $a - b = a + b \in I$. Then $\sim$ is a congruence on the Boolean algebra B. We shall let $B/I = B/\sim$ be the quotient Boolean algebra. If B is a Boolean algebra, and Z is a filter, then we shall let B/Z denote the quotient Boolean algebra $B/\{x \in B | x' \in Z\}$. If Z is a filter on a Boolean algebra B, then let $\pi_Z : B \rightarrow B/Z$ be the natural map where $\pi_Z(b) = b/Z$ for each $b \in B$. Similarly, if I is an ideal on B, then let $\pi_I : B \rightarrow B/I$ be the mapping where $\pi_I(b) = b/I$ for each $b \in B$.

A proper filter Z on a Boolean algebra B is an ultrafilter if Z cannot be extended to a larger proper filter. A filter $Z \subseteq B$ is an ultrafilter if and only if $\{x' | x \in B\}$ is a maximal ideal in the ring $B^{\otimes}$. Thus, the ultrafilters on B are in a one-to-one correspondence with maximal ideals. A filter Z on a Boolean algebra B is an ultrafilter if and only if $|B/Z| = 2$. Furthermore, a proper filter Z on a Boolean algebra B is an ultrafilter if and only if $b \in Z$ or $b' \in Z$ for each $b \in B$. Using Zorn's lemma, one may easily show that every proper filter on a Boolean algebra can be extended to an ultrafilter. If $\phi : A \to B$ is a Boolean algebra homomorphism and $\mathcal{U} \subseteq B$ is an ultrafilter, then $\phi^{-1}[\mathcal{U}]$ is an ultrafilter on A.

If X is a topological space, then call a subset $U \subseteq X$ *clopen* if U is both closed and open. The clopen subsets of X form an algebra of sets which we shall denote by $\mathfrak{B}(X)$. A Hausdorff space X is called *zero-dimensional* if the clopen sets in X form a basis. Clearly every zero-dimensional space is totally disconnected. Conversely, a totally disconnected compact space is zero-dimensional. A compact zero-dimensional space is also called a Boolean space.

We shall now outline the classical Stone duality between Boolean algebras and compact zero-dimensional spaces. If B is a Boolean algebra, then write $S(B)$ for the collection of all ultrafilters on the Boolean algebra B. If $b \in B$, then let $U_b = \{\mathcal{U} \in S(B) | b \in \mathcal{U}\}$. Then $\{U_b | b \in B\}$ is a basis for a compact zero-dimensional topology on $S(B)$. If B is a Boolean algebra, then the mapping $B \to \mathfrak{B}(S(B)), b \mapsto U_b$ is a Boolean algebra isomorphism. Furthermore, if X is a compact totally disconnected space, then let $\mathcal{C}(x) = \{R \in \mathfrak{B}(X) | x \in R\}$ for $x \in X$, then $\mathcal{C}(x)$ is an ultrafilter on $\mathfrak{B}(X)$. Furthermore, the mapping $X \to S(\mathfrak{B}(X)), x \mapsto \mathcal{C}(x)$ is a homeomorphism. Therefore one may regard Boolean algebras and compact totally disconnected spaces as equivalent. This equivalence between Boolean algebras and compact totally disconnected spaces also relates Boolean algebra homomorphisms with continuous function. For instance, if A, B are Boolean algebras, and $\phi : A \to B$ is a Boolean algebra homomorphism, then let $S(\phi) : S(B) \to S(A)$ be the mapping where $S(\phi)(\mathcal{U}) = \phi^{-1}[\mathcal{U}]$. Then $S(\phi)$ is a continuous function. One can clearly see that S gives a contravariant functor from the category of Boolean algebras to the category of compact totally disconnected spaces. On the other hand, if X, Y are topological spaces and $f : X \to Y$ is continuous, then the inverse image of a clopen set is clopen. Therefore let $\mathfrak{B}(f) : \mathfrak{B}(Y) \to \mathfrak{B}(X)$ be the mapping where $\mathfrak{B}(f)(C) = f^{-1}[C]$. Then $\mathfrak{B}$ is a contravariant functor from the category of compact totally disconnected spaces to the category of Boolean algebras. See [12][Ch. 3],[4][Ch. 2] for an exposition on the duality between compact totally disconnected spaces and Boolean algebras.

The following chart lists some of the relations between Boolean algebras and compact zero-dimensional spaces. Most of the correspondences in this chart are easy to prove, and a similar chart can be found in [15][p. 1237].

Table 1.1: Table of dualities for Boolean algebras

Boolean algebras	Compact zero-dimensional spaces
Elements	Clopen sets
Ultrafilters	Points
Filters	Closed sets
Ideals	Open sets
Homomorphisms	Continuous functions
Surjective homomorphisms	Injective continuous functions
Injective homomorphisms	Surjective continuous functions
Collection of ultrafilters $\mathcal{R}$ with $\bigcup\mathcal{R}=B^{+}$	Dense sets
Principal ultrafilters	Isolated points
Complete Boolean algebras	Extremally disconnected compact spaces
σ-complete Boolean algebras	Basically disconnected compact spaces
Atomic Boolean algebras	Isolated points are dense
Superatomic Boolean algebras	Scattered spaces
finitely additive finite measures on a Boolean algebra	finite Baire measures

A Boolean algebra B is said to be a κ-Boolean algebra if whenever $R\subseteq B$ and $|R|<\kappa$, then R has a least upper bound. By De Morgan's law, a Boolean algebra B is a κ-Boolean algebra if and only if every subset $R\subseteq B$ with $|R|<\kappa$ has a least upper bound. We shall also call a κ-Boolean algebra a κ-complete Boolean algebra. A filter Z on a κ-Boolean algebra B is a κ-filter if whenever $R\subseteq Z$ and $|R|<\kappa$, then $\bigwedge R\in Z$. An ideal I is a κ-ideal if the filter $\{x\in B|x'\in I\}$ is a κ-filter. We shall write $S_\kappa(B)$ for the collection of all κ-ultrafilters on B if B is a κ-Boolean algebra. An algebra of sets $(X,\mathcal{A})$ is a κ-algebra of sets if whenever $R\subseteq\mathcal{A}$ and $|R|<\kappa$, then $\bigcup R\in\mathcal{A}$ as well. We shall call a Boolean algebra B κ-representable if B is isomorphic to some κ-algebra of sets. One can easily show that a κ-Boolean algebra is κ-representable if $\bigcap S_\kappa(B)=\{1\}$. We shall call a Boolean algebra B strongly κ-representable if every κ-filter on B can be extended to a κ-ultrafilter. One can see that a Boolean algebra B is strongly κ-representable if and only if B/Z is representable for each κ-filter B [19].

We shall now give two constructions for the direct limit of Boolean algebras. It should be noted that these two constructions are universal algebraic and they work on general algebraic structures such as groups, rings, modules, lattices, etc.

An upwards directed set is a poset D such that if $x, y \in D$ then there is a $z \in D$ with $x \leq z, y \leq z$. A downwards directed set is a poset D such that if $x, y \in D$, then there is a $z \in D$ with $z \leq x, z \leq y$. If we say a poset is directed, then we shall mean that it is upwards directed.

We shall define the notion of a direct and an inverse limit first in the most general case of category theory. We shall then give special cases of direct and inverse limits in certain categories. Let $\mathcal{C}$ be a category and let D be a preordered set. For each $d \in D$ let X_d be an object in $\mathcal{C}$. If $d, e \in D$ and $d \leq e$ then assume that $\phi_{d,e} : X_d \rightarrow X_e$ is a morphism. We say that $((X_d)_{d\in D}, (\phi_{d,e})_{d,e\in D,d\leq e})$ is an inverse system if

1. $\phi_d : X_d \rightarrow X_d$ is the identity morphism for each $d \in D$ and
2. If $d, e, f \in D$ and $d \leq e \leq f$, then $\phi_{d,f} = \phi_{e,f} \circ \phi_{d,e}$.

If D is a downwards directed set, then the inverse limit of $((X_d)_{d\in D}, (\phi_{d,e})_{d,e\in D,d\leq e})$ consists of an object $\overset{\lim}{\longleftarrow}(X_d)_{d\in D}$ along with morphisms $\phi_d :\overset{\lim}{\longleftarrow} (X_d)_{d\in D} \rightarrow X_d$ for $d \in D$ such that:

1. If $d \leq e$, then $\phi_{d,e}\phi_d = \phi_e$ whenever $d \leq e$ and

2 If X is an object and $\alpha_d : X \rightarrow X_d$ is a morphism for $d \in D$ with $\phi_{d,e}\alpha_d = \alpha_e$ whenever $d \leq e$, then there is a unique morphism $\alpha : X \rightarrow \overset{\lim}{\longleftarrow} (X_d)_{d\in D}$ such that $\alpha_d = \phi_d\alpha$.

In a category one may easily show that the inverse limit is unique up to isomorphism. We shall now give an example of what an inverse limit looks like.

Let **Set** denote the category of sets. Let $((X_d)_{d\in D}, (\phi_{d,e})_{d,e\in D,d\leq e})$ be an inverse system of sets. Then the inverse limit $\overset{\lim}{\longleftarrow}(X_d)_{d\in D}$ in **Set** is the subset of the cartesian product $\prod_{d\in D} X_d$ where $(x_d)_{d\in D} \in \overset{\lim}{\longleftarrow} (X_d)_{d\in D}$ iff $\phi_{d,e}(x_d) = x_e$ whenever $d \leq e$. Furthermore, the inverse limit $\overset{\lim}{\longleftarrow}(X_d)_{d\in D}$ consists of mappings $\overline{\pi}_d :\overset{\lim}{\longleftarrow} (X_d)_{d\in D} \rightarrow X_d$ for $d \in D$ where $\overline{\pi}_d$ is the restriction of the projection mapping $\pi_d : \prod_{d\in D} X_d \rightarrow X_d$.

The inverse limit of a collection of sets may be empty. For example, if $(X_r)_{r\in\mathbb{Z}}$ is a collection of sets where $...X_{r-1} \subseteq X_r \subseteq X_{r+1}...$ and where $\bigcap_{r\in\mathbb{Z}} X_r = \emptyset$, then let $\iota_{r,s} : X_r \rightarrow X_s$ be the inclusion mapping. Then one can clearly see that $\overset{\lim}{\longleftarrow}X_r = \emptyset$. It turns out that the inverse limit of a collection of sets may be empty even if every transitional mapping is surjective. In fact, a very short six line journal article [22] gives an example of an empty inverse limit where every transitional mapping is surjective.

Let $\mathcal{C}$ be a category, and let D be a directed set. Let A_d be an object in $\mathcal{C}$ for $d \in D$. Furthermore, assume that whenever $d \leq e$, there is a morphism $\phi_{d,e} : A_d \rightarrow A_e$. Also assume that if $d \leq e \leq f$, then $\phi_{e,f}\phi_{d,e} = \phi_{d,f}$ and if $d \in D$, then $\phi_{d,d}$ is the identity morphism. Then we shall call the system $((A_d)_{d\in D}, (\phi_{d,e})_{d\leq e})$ a directed system. The direct limit of $((A_d)_{d\in D}, (\phi_{d,e})_{d\leq e})$ consists of an object $\varinjlim A_d$ along with morphisms $\phi_d : A_d \rightarrow \varinjlim A_d$ such that $\phi_d = \phi_e\phi_{d,e}$ whenever $d \leq e$ and direct limits must satisfy the following universal property: Whenever $\alpha_d : A_d \rightarrow B$ are morphisms for $d \in D$ and $\alpha_d = \alpha_e\phi_{d,e}$ for $d \leq e$, then there is a morphism $\alpha : \varinjlim A_d \rightarrow B$ where $\alpha_d = \alpha\phi_d$ for $d \in D$.

We shall now give constructions for a direct limit of Boolean algebras. It should be noted that these constructions work for any variety. Let $((B_d), (\phi_{d,e})_{d\leq e})$ be a directed system of Boolean algebras and assume that the collection $(B_d)_{d\in D}$ is pairwise disjoint for notational simplicity. Let $\sim$ be the relation on $\bigcup_{d\in D} B_d$ where if $a \in B_{d_1}, b \in B_{d_2}$, then $a \sim b$ iff $\phi_{d_1,e}(a) = \phi_{d_2,e}(b)$ for some $e \geq d_1, d_2$. Then $\bigcup_{d\in D} B_d/\sim$ is a Boolean algebra where if $a, b \in B_d$, then $[a] \wedge [b] = [a \wedge b], [a] \vee [b] = [a \vee b], [a'] = [a]'$. One can easily show that $\bigcup_{d\in D} B_d/\sim$ is the direct limit $\varinjlim B_d$ in the category of Boolean algebras.

If $B_d \subseteq B_e$ whenever $d \leq e$ and each $\phi_{d,e}$ is the inclusion mapping, then one can easily see that the direct limit $\varinjlim B_d$ is simply the union $\bigcup_{d\in D} B_d$.

Let D be a directed set and let $(B_d, \phi_{d,e})_{d\leq e}$ be a directed system of Boolean algebras. Let $B \subseteq \prod_{d\in D} B_d$ be the set where $(x_d)_{d\in D} \in B$ if and only if for some $d \in D$ we have $\phi_{d,e}(x_d) = x_e$ whenever $e \geq d$. One can easily show that B is a Boolean subalgebra of $\prod_{d\in D} B_d$. Let $Z \subseteq B$ be the collection of all $(x_d)_{d\in D} \in B$ where there is some $d \in D$ where $x_e = 1$ whenever $e \geq d$. One can clearly see that Z is a filter on B. One can show that the direct limit $\varinjlim B_d$ is the Boolean algebra B/Z with homomorphisms $\alpha_d : B_d \rightarrow B/Z$ defined by $\alpha_d(b) = (x_d)_{d\in D}/Z$ where $x_e = \phi_{d,e}(b)$ whenever $d \leq e$.

1.3 Boolean partition algebras

In this section, I shall develop some basic facts about Boolean partition algebras.

Definition 1.3.1 *If A is a Boolean algebra or more generally a partially ordered set with least element 0, and $R \subseteq A$, then we shall write R^+ for $R \setminus \{0\}$. Let P be a poset. Then $x, y \in P$ are said to be incompatible if there does not exist an $r \in P$ with $r \leq x, r \leq y$. A subset $A \subseteq P$ is said to be cellular if every pair of elements in A is incompatible. If I is an index set, then we say that a family $(c_i)_{i \in I} \in A^I$ is cellular if for $i \neq j$ there does not exist an $r \in P$ with $r \leq c_i, r \leq c_j$. We may generalize the notion of incompatibility to meet-semilattices. If $(A, \wedge, 0)$ is a meet-semilattice with least element 0, then we shall say $x, y \in A^+$ are incompatible if $x \wedge y = 0$, and we shall say $A' \subseteq A^+$ is cellular if $x \wedge y = 0$ for $x, y \in A', x \neq y$.*

Proposition 1.3.2 *Let P be a poset. Then every cellular family is contained in a maximal cellular family (ordered under $\subseteq$).*

Proof. This is a simple application of Zorn's lemma. If $(R_b)_{b \in B}$ is a chain of cellular families, then $\bigcup_{b \in B} R_b$ is cellular. □

Definition 1.3.3 *Let P be a poset (meet-semilattice). Write $c(P)$ for the collection of cellular families on P. If $A, B \in c(P)$, then A refines B (written $A \preceq B$) if for each $a \in A$, there is a $b \in B$ with $a \leq b$. Since B is cellular, there is a unique $b \in B$ with $a \leq b$. Therefore let $\phi_{A,B} : A \to B$ be the function with $a \leq \phi_{A,B}(a)$ for $a \in A$.*

Proposition 1.3.4 *$c(P)$ is a poset under the ordering $\preceq$, and $c(P)$ is an inverse system of sets with transition mappings $\phi_{A,B}$ whenever $A \preceq B$.*

Proof. If $A \in c(P)$, then $a \leq a$ for $a \in A$, so $A \preceq A$ and $\phi_{A,A}(a) = a$ for $a \in A$. If $A \preceq B$ and $B \preceq C$, then for $a \in A$ we have $a \leq \phi_{A,B}(a) \leq \phi_{B,C}\phi_{A,B}(a)$, thus $A \preceq C$ and $\phi_{A,C}(a) = \phi_{B,C}\phi_{A,B}(a)$ for $a \in A$. Therefore $\preceq$ is a preordering, and $c(P)$ is an inverse system with transition mappings $\phi_{A,B}$. To show that $c(P)$ is a partial ordering assume $A, B \in c(P), A \preceq B, B \preceq A$. Then $a \leq \phi_{A,B}(a) \leq \phi_{B,A}\phi_{A,B}(a) = \phi_{A,A}(a) = a$, so $a = \phi_{A,B}(a) \in B$. We therefore have $A \subseteq B$, and by an identical argument we have $B \subseteq A$. We therefore conclude that $A = B$. $\square$

Definition 1.3.5 *Let B be a Boolean algebra. Then a partition p of B is a subset of B^+ where $\bigvee p = 1$ and where $x \wedge y = 0$ for $x \neq y$. We shall write $\mathbb{P}(B)$ for the collection of all partitions of a Boolean algebra B.*

The partitions of a Boolean algebra are precisely the maximal elements of $c(B)$ with the inclusion ordering $\subseteq$. One can easily see that a cellular family $p \subseteq B$ is a partition iff $a = 0$ whenever $a \wedge b = 0$ for each $b \in p$. In other words, a cellular family $p \subseteq B$ is a partition iff whenever $a > 0$ there is a $b \in p$ with $a \wedge b > 0$.

Proposition 1.3.6 *If B is a Boolean algebra, then $\mathbb{P}(B)$ is a meet semilattice where $p \wedge q = \{a \wedge b | a \in p, b \in q\}^+$ for all $p, q \in F$.*

Proof. Let $s = \{a \wedge b | a \in p, b \in q\}^+$. We claim that s is the greatest lower bound of p and q. If $c_1, c_2 \in s$, then $c_1 = a_1 \wedge b_1$ and $c_2 = a_2 \wedge b_2$ for some $a_1, a_2 \in p$ and $b_1, b_2 \in q$. If $c_1 \neq c_2$, then $a_1 \neq a_2$ or $b_1 \neq b_2$. Therefore $a_1 \wedge a_2 = 0$ or $b_1 \wedge b_2 = 0$, so $c_1 \wedge c_2 = a_1 \wedge a_2 \wedge b_1 \wedge b_2 = 0$. Therefore s is cellular. Furthermore, we have $\bigvee s = \bigvee_{a \in p} \bigvee_{b \in q} a \wedge b = \bigvee_{a \in p}(a \wedge \bigvee_{b \in q} b) = \bigvee_{a \in p} a = 1$, thus s is a partition of B with $s \preceq p$ and $s \preceq q$. Now let $r \in \mathbb{P}(B)$ be a partition with $r \preceq p$ and $r \preceq q$. Then for $x \in r$ we have $x \leq \phi_{r,p}(x) \wedge \phi_{r,q}(x) \in s$, so $r \preceq s$. Therefore s is the greatest lower bound of p and q. $\square$

Definition 1.3.7 *An extended partition is a family $(a_i)_{i \in I} \in B^I$ such that $\bigvee_{i \in I} a_i = 1$ and if $i \neq j$, then $a_i \wedge a_j = 0$.*

Clearly, a family $(a_i)_{i \in I}$ is an extended partition iff $\{a_i | i \in I\}^+$ is a partition of B and $a_i \neq a_j$ whenever $a_i \neq 0$ and $i \neq j$.

Lemma 1.3.8 *Let B be a Boolean algebra and assume that $(a_i)_{i \in I}, (b_i)_{i \in I}$ are extended partitions with $a_i \leq b_i$ for $i \in I$. Then $a_i = b_i$ for $i \in I$.*

Proof. Assume that $a_k < b_k$ for some $k \in I$. Then $b_k \wedge a_k' \neq 0$. On the other hand, we have $a_k \wedge (b_k \wedge a_k') = 0$ and whenever $i \neq k$ we also have $a_i \wedge (b_k \wedge a_k') \leq a_i \wedge b_k \leq b_i \wedge b_k = 0$. This contradicts the fact that $(a_i)_{i \in I}$ is an extended partition. $\square$

Proposition 1.3.9 *Assume p, q are partitions of a Boolean algebra B. Then $p \preceq q$ if and only if $b = \bigvee\{a \in p | a \leq b\}$ for each $b \in q$.*

Proof. $\rightarrow$ Assume that $b^\sharp$ is an upper bound of $\{a \in p | a \leq b\}$ with $b^\sharp \leq b$ for $b \in q$. Then $\{b^\sharp | b \in q\}$ is a cellular family. For each $a \in p$ we have $a \leq \phi_{p,q}(a)^\sharp$, so $\bigvee_{b \in q} b^\sharp \geq \bigvee_{a \in p} a = 1$. Therefore $\{b^\sharp | b \in q\}$ is a partition of B, and $b^\sharp = b$ for $b \in q$ by Lemma 1.3.8.

$\leftarrow$ Let $R = \{a \in p | \exists b \in q, a \leq b\}$. Then $\bigvee R = \bigvee_{b \in q} \bigvee\{a \in p | a \leq b\} = \bigvee_{b \in q} b = 1$. Therefore R is a partition of B with $R \subseteq p$, so $R = p$, thus proving $p \preceq q$. □

Corollary 1.3.10 *If p_i is a partition of B for $i \in I$ and $p \preceq p_i$ for $i \in I$ and q is a partition with $q \subseteq \bigcup p_i$, then $p \preceq q$.*

Proof. If $b \in q$, then $b \in p_i$ for some $i \in I$, so $b = \bigvee\{a \in p | a \leq b\}$. Therefore $p \preceq q$. □

Proposition 1.3.11 *Let p be a partition of a Boolean algebra B. Then for $b \in B$ the following are equivalent.*

1. $b = \bigvee\{a \in p | a \leq b\}$
2. $b = 1$ *or* $p \preceq \{b, b'\}$
3. $b \in q$ *for some partition* q *with* $p \preceq q$.

Proof. $2 \Rightarrow 3$ If $b = 1$, then $b \in \{1\} \succeq p$. If $b < 1$, then $b \in \{b, b'\} \succeq p$.

$3 \Rightarrow 1$ This was proved in Proposition 1.3.9.

$1 \Rightarrow 2$ Assume $b \neq 1$, and $\bigvee\{a \in p | a \leq b\} = b$. If $c \in p, c \not\leq b$, then $c \notin \{a \in p | a \leq b\}$, so $c \wedge a = 0$ whenever $a \leq b, a \in p$. Therefore $c \wedge b = c \wedge \bigvee_{a \in p, a \leq b} a = \bigvee_{a \in p, a \leq b} (c \wedge a) = 0$, so $c \leq b'$. We therefore have $p \preceq \{b, b'\}$. □

Definition 1.3.12 *A partition p on a Boolean algebra B is said to be subcomplete if $\bigvee R$ exists whenever $R \subseteq p$. For example, every finite partition of a Boolean algebra is subcomplete.*

Proposition 1.3.13 *If a partition p of a Boolean algebra B is subcomplete, and $p \preceq q$, then q is subcomplete as well.*

Proof. Since $p \preceq q$, for each $a \in q$ there is a $P_a \subseteq p$ with $a = \bigvee P_a$ by Proposition 1.3.9. If $R \subseteq q$, then $\bigvee R = \bigvee_{a \in R}(\bigvee P_a) = \bigvee(\bigcup_{a \in R} P_a)$. $\square$

Definition 1.3.14 *A Boolean partition algebra is a pair (B, F) where B is a Boolean algebra and F is a (possibly improper) filter on $\mathbb{P}(B)$ with $\bigcup F = B^+$. A Boolean partition algebra (B, F) is subcomplete if each $p \in F$ is a subcomplete partition of B.*

It can be argued that subcompleteness is the most important property a Boolean partition algebra could have. Without subcompleteness, the theory of Boolean partition algebras is not nearly as interesting. For instance, subcompleteness is necessary in the ultrapower construction using Boolean partition algebras, in the dualities such as the duality between Boolean partition algebras and uniform spaces, and in the correspondence between based Boolean partition algebras and inverse systems where all bonding maps are surjective.

Proposition 1.3.15 *Let $F \subseteq \mathbb{P}(B)$ be a filter. Then the following are equivalent.*

1. *If $b \in B \setminus \{0, 1\}$, then $\{b, b'\} \in F$.*
2. *(B, F) is a Boolean partition algebra.*
3. *F contains all partitions of B into finitely many sets.*

Proof. $1 \to 2$ Let $b \in B^+$. If $b = 1$, then $b \in \{b\} \in F$. If $b \neq 1$, then $b \in \{b, b'\} \in F$.

$2 \to 3$ Let $\{b_1, ..., b_n\}$ be a partition of B. For $1 \leq i \leq n$ let $p_i \in F$ be a partition with $b_i \in p_i$. Then $\{b_1, ..., b_n\} \succeq p_1 \wedge ... \wedge p_n \in F$ by Corollary 1.3.10.

$3 \to 1$ This is trivial. $\square$

Example 1.3.16 *If B is a Boolean algebra, then $(B, \mathbb{P}(B))$ is a Boolean partition algebra. If B is a Boolean algebra, and λ is an infinite cardinal, then define $\mathbb{P}_\lambda(B) = \{p \in \mathbb{P}(B) : |p| < \lambda\}$. Then $(B, \mathbb{P}_\lambda(B))$ is a Boolean partition algebra. If B is λ-complete, then $(B, \mathbb{P}_\lambda(B))$ is a subcomplete Boolean partition algebra. If B is a complete Boolean algebra, then every Boolean partition algebra of the form (B, F) is subcomplete.*

Example 1.3.17 *Let B be a Boolean algebra and let μ be a finitely additive measure on B with $\mu(1) = 1$. Let F be the collection of all partitions p with $\sum_{a \in p} \mu(a) = 1$.*

Then one may easily show that (B, F) is a Boolean partition algebra. For example, let X be a compact totally disconnected space, let μ be a Baire measure on X with $\mu(X) = 1$, and let F be the collection of all partitions of P of the dual Boolean algebra $\mathfrak{B}(X)$ with $\sum_{R\in P} \mu(R) = 1$. Then $(\mathfrak{B}(X), F)$ is a Boolean partition algebra.

The following theorem gives many more examples of Boolean partition algebras.

Theorem 1.3.18 *If B is a Boolean algebra and $F \subseteq \mathbb{P}(B)$ is a filter, then $\{0\}\cup(\bigcup F)$ is a subalgebra of B, and $(\{0\}\cup(\bigcup F), F)$ is a Boolean partition algebra. If each $p \in F$ is subcomplete, then $(\{0\} \cup (\bigcup F), F)$ is a subcomplete Boolean partition algebra.*

Proof. Let $a, b \in \{0\} \cup (\bigcup F)$. If $a \wedge b = 0$, then $a \wedge b \in \{0\} \cup (\bigcup F)$. If $a \wedge b \neq 0$, then $a \neq 0$ and $b \neq 0$, so there are $p, q \in F$ with $a \in p, b \in q$. Therefore $a \wedge b \in p \wedge q \subseteq \{0\} \cup (\bigcup F)$ since $p \wedge q \in F$.

Now assume $a \in \{0\} \cup (\bigcup F)$. If $a \in \{0, 1\}$, then clearly $a' \in \{0\} \cup (\bigcup F)$. If $a \notin \{0, 1\}$, then $a \in p$ for some $p \in F$. But by Proposition 1.3.11 we have $p \preceq \{a, a'\}$. Since F is a filter we have $\{a, a'\} \in F$, so $a' \in \{0\} \cup (\bigcup F)$. We therefore conclude that $\{0\} \cup (\bigcup F)$ is a Boolean subalgebra.

Clearly F is a filterbase on $\mathbb{P}(\{0\} \cup (\bigcup F))$. Assume p and q are partitions of the Boolean algebra $\{0\} \cup (\bigcup F)$ with $p \in F$ and $p \preceq q$. Then we claim that q is a partition of B. Clearly q is a cellular family. If $x \in B$ and $x \geq b$ for each $b \in q$, then for each $a \in p$ we have $a \leq \phi_{p,q}(a) \leq x$, so $x = 1$. Therefore q is a partition of B, thus $q \in F$. Therefore F is a filter on $\mathbb{P}(\{0\} \cup (\bigcup F), F)$, so $(\{0\} \cup (\bigcup F), F)$ is a Boolean partition algebra.

If $p \in F$ is a subcomplete partition of B, then for $R \subseteq p$ we have $p \preceq \{\bigvee^B R, \bigvee^B(p\setminus R)\}^+$, thus $\{\bigvee^B R, \bigvee^B(p\setminus R)\}^+ \in F$. Therefore $\bigvee^B R \in \{0\}\cup(\bigcup F)$ is the least upper bound of R in $\{0\}\cup(\bigcup F)$. We conclude that $(\{0\}\cup(\bigcup F), F)$ is a subcomplete Boolean partition algebra. $\square$

Definition 1.3.19 *If B is a Boolean algebra and F is a filter on $\mathbb{P}(B)$, then write $\mathfrak{B}^*(B, F)$ for the Boolean partition algebra $(\{0\} \cup (\bigcup F), F)$. If $B = P(X)$ for some set X, then we shall write $\mathfrak{B}^*(X, F)$ for $\mathfrak{B}^*(P(X), F)$.*

Example 1.3.20 *Let X be a topological space. An open set U is a regular open set if $U = (\overline{U})^\circ$ (i.e. it is the interior of its closure). The collection of all regular open sets on a topological space forms a complete Boolean algebra denoted by $\mathcal{RO}(X)$. Let μ be a Borel probability measure on X, and let F be the collection of partitions of the Boolean algebra $\mathcal{RO}(X)$ where $\sum_{O \in F} \mu(O) = 1$. Then F is a filter on $\mathbb{P}(\mathcal{RO}(X))$ and $(\{\emptyset\} \cup \bigcup F, F)$ is a subcomplete Boolean partition algebra.*

Proposition 1.3.21 *Let $(A, \wedge, 0), (B, \wedge, 0)$ be meet semilattices and $f : (A, \wedge, 0) \to (B, \wedge, 0)$ be a semilattice homomorphism with $f(0) = 0$. Then $f[r]^+$ is cellular for each cellular $r \subseteq A$. Furthermore, if $(C, \wedge, 0)$ is a meet semilattice and $g : B \to C$ is a semilattice homomorphism with $g(0) = 0$, then $g[f[r]^+]^+ = g \circ f[r]^+$ for each cellular r.*

Proof. If $r \subseteq A$ is cellular, then for each $a, b \in r$ where $f(a) \neq f(b)$ we have $a \neq b$, so $f(a) \wedge f(b) = f(a \wedge b) = f(0) = 0$.

We shall now show that $g[f[r]^+]^+ = (g \circ f)[r]^+$. We have $g[f[r]^+]^+ \subseteq g[f[r]]^+ = (g \circ f)[r]^+$. For the converse, if $b \in (g \circ f)[r]^+$, then $b = g(f(a))$ for some $a \in r$. Then since $g(f(a)) = b \neq 0$ we have $f(a) \neq 0$ as well, so $b \in g[f[r]^+]^+$. □

Definition 1.3.22 *Let (A, F) be a Boolean partition algebra, and let B be a Boolean algebra. A function $f : A \to B$ is a partitional map from (A, F) to B if f is a Boolean algebra homomorphism, and $f[p]^+$ is a partition of B for each $p \in F$. A partition homomorphism f from (A, F) to (B, G) is a Boolean algebra homomorphism from A to B where $f[p]^+ \in G$ for each $p \in F$.*

A function $f : (A, F) \to B$ is partitional iff $f : (A, F) \to (B, \mathbb{P}(B))$ is a partition homomorphism. If $f : (A, F) \to B$ is an injective homomorphism, then f is partitional if and only if $f[p]$ is a partition of B for each $p \in F$. If $f : (A, F) \to (B, G)$ is an injective homomorphism, then f is a partition homomorphism iff $f[p] \in G$ for each $p \in F$.

Example 1.3.23 *Let (B, F) be a Boolean partition algebra, and let $a \in B$. Then the mapping $\alpha_a : (B, F) \to B \restriction a$ where $\alpha_a(x) = x \wedge a$ is partitional. For if $p \in F$, then*

$\bigvee \alpha_a[p] = \bigvee_{b \in p} a \wedge b = a \wedge \bigvee_{b \in p} b = a$, *so* $\alpha_a[p]^+$ *is a partition of* $B \upharpoonright a$.

Proposition 1.3.24 *1. Let* $f : (A, F) \to (B, G)$ *be a partition homomorphism, and let* $g : (B, G) \to C$ *be partitional. Then* $g \circ f : (A, F) \to C$ *is partitional as well.*

2. Let $f : (A, F) \to (B, G), g : (B, G) \to (C, H)$ *be partition homomorphism, then* $g \circ f$ *is also a partition homomorphism.*

Proof. Let $p \in F$.

1. We have $f[p]^+ \in G$, so $(g \circ f)[p]^+ = g[f[p]^+]^+$ is a partition of C.

2. Since $f[p]^+ \in G$, we have $(g \circ f)[p]^+ = g[f[p]^+]^+ \in H$, thus $g \circ f$ is a partition homomorphism. $\square$

The class of all Boolean partition algebras with partition homomorphisms forms a category.

Proposition 1.3.25 *If* $f : (A, F) \to B$ *is a partitional mapping, then for each* $p, q \in F$ *we have* $f[p]^+ \wedge f[q]^+ = f[p \wedge q]^+$.

Proof. If $x \in f[p]^+ \wedge f[q]^+$, then $x = f(a) \wedge f(b) = f(a \wedge b)$ for some $a \in p, b \in q$. Since $x \neq 0$ we have $a \wedge b \neq 0$ as well, therefore $a \wedge b \in p \wedge q$, hence $x \in f[p \wedge q]^+$ since $x \neq 0$.

Now assume $y \in f[p \wedge q]^+$. Then $y = f(x)$ for some $x \in p \wedge q$, so $x = a \wedge b$ for some $a \in p, b \in q$. Thus $y = f(x) = f(a \wedge b) = f(a) \wedge f(b)$. Therefore since $y \neq 0$ we have $f(a) \neq 0, f(b) \neq 0$ as well, hence $f(a) \in f[p]^+$ and $f(b) \in f[q]^+$, so $y \in (f[p]^+) \wedge (f[q]^+)$. $\square$

Proposition 1.3.26 *Let* A, B *be Boolean algebras. Then a function* $f : A \to B$ *is a Boolean algebra homomorphism iff whenever* (a, b, c) *is an extended partition of* A, *then* $(f(a), f(b), f(c))$ *is an extended partition of* B.

Proof. If f is a Boolean algebra homomorphism, then it is clear that $(f(a), f(b), f(c))$ is an extended partition of B for each extended partition (a, b, c) of A.

For the converse, we first take note that since $(0, 0, 1)$ is an extended partition of A, the triple $(f(0), f(0), f(1))$ is an extended partition of B, so $f(0) = f(0) \wedge f(0) = 0$.

If $a \in A$, then $(a, a', 0)$ is an extended partition of A, thus $(f(a), f(a'), f(0)) = (f(a), f(a'), 0)$ is an extended partition of B, therefore $f(a)' = f(a')$.

Assume $a, b \in A$ are incompatible. Then $(a, b, (a \vee b)')$ is an extended partition of A, thus $(f(a), f(b), f(a \vee b)')$ is an extended partition of B, so $f(a) \vee f(b) = f(a \vee b)$ and $f(a) \wedge f(b) = 0$.

Now assume $a \leq b$. Then $f(b) = f((b \wedge a') \vee a) = f(b \wedge a') \vee f(a)$, thus $f(a) \leq f(b)$.

Therefore for arbitrary $a, b \in B$ one has $f(a) \leq f(a \vee b), f(b) \leq f(a \vee b)$, so $f(a) \vee f(b) \leq f(a \vee b) = f((a \wedge b') \vee b) = f(a \wedge b') \vee f(b) \leq f(a) \vee f(b)$. We conclude that f is a Boolean algebra homomorphism. $\square$

Corollary 1.3.27 *Let (A, F) be a Boolean partition algebra, and let B be a Boolean algebra. Then a function(we do not assume beforehand that it is a Boolean algebra homomorphism) $f : (A, F) \to B$ is partitional if and only if $f(0) = 0$ and $(f(a))_{a \in p}$ is an extended partition of B for each $p \in F$.*

Proof. f satisfies the hypothesis of Proposition 1.3.26, so f is a Boolean algebra homomorphism. $\square$

In other words, a function $f : A \to B$ is a partitional mapping from (A, F) to B if and only if $f[p]^+$ is a partition of B for each $p \in F$, and $f(a) \wedge f(b) = 0$ whenever $a, b \in p, a \neq b$ and $f(0) = 0$.

Definition 1.3.28 *If X is a set, then $\mathbb{P}(P(X))$ is the lattice of partitions of X. We shall sometimes write $\mathbb{P}P(X)$ for $\mathbb{P}(P(X))$. Let $\mathcal{P}(X)$ denote the Boolean partition algebra $(P(X), \mathbb{P}P(X))$.*

Definition 1.3.29 *If B is a Boolean algebra, and p is a partition of B, then let $p^\sharp$ be the collection of all subsets of p where $R \in p^\sharp$ if and only if $\bigvee R$ exists. Define a function $\phi : p^\sharp \to B$ by $\phi(R) = \bigvee R$.*

Proposition 1.3.30 *Let p be a partition of a Boolean algebra B. Then*

1. *$(p, p^\sharp)$ is an algebra of sets.*
2. *The function $\phi : p^\sharp \to B$ is an injective Boolean algebra homomorphism that preserves all least upper bounds.*

3. If (B, F) is a Boolean partition algebra and p is subcomplete, then the mapping $\phi : \mathcal{P}(p) \to (B, F)$ is a partition homomorphism.

Proof. 1. If $R, S \in p^\sharp$, then $\bigvee(R \cup S) = (\bigvee R) \vee (\bigvee S)$, so the least upper bound $\bigvee(R \cup S)$ exists. Therefore $R \cup S \in p^\sharp$ as well. Now assume $R \in p^\sharp$. Then we claim that $(\bigvee R)'$ is the least upper bound of $p \setminus R$. If $a \in p \setminus R$, then $a \wedge \bigvee R = \bigvee_{b \in R}(a \wedge b) = 0$, so $a \leq (\bigvee R)'$. Therefore $(\bigvee R)'$ is an upper bound of $p \setminus R$. Now assume s is also an upper bound of $p \setminus R$. Then $s \vee (\bigvee R) \geq a$ for each $a \in p$. Therefore $s \vee (\bigvee R) = 1$, so $s \geq (\bigvee R)'$. Therefore $(\bigvee R)'$ is the least upper bound of $p \setminus R$, and $p^\sharp$ is an algebra of sets.

2. We have $\phi(R^c) = \bigvee R^c = (\bigvee R)' = \phi(R)'$. Assume $R_i \in p^\sharp$ for $i \in I$ and $\bigvee_{i \in I} R_i$ exists in $p^\sharp$. Then $\bigcup_{i \in I} R_i \subseteq \bigvee_{i \in I} R_i$ and in fact $\bigcup_{i \in I} R_i = \bigvee_{i \in I} R_i$ since if $a \in \bigvee_{i \in I} R_i \setminus \bigcup_{i \in I} R_i$, then $(\bigvee_{i \in I} R_i) \setminus \{a\}$ would be an upper bound of $(R_i)_{i \in I}$ smaller than $(\bigvee_{i \in I} R_i)$, a contradiction. Therefore $\phi(\bigvee_{i \in I} R_i) = \bigvee \bigcup_{i \in I} R_i = \bigvee_{i \in I}(\bigvee R_i) = \bigvee_{i \in I}(\phi(R_i))$. The mapping ϕ is injective since $\ker(\phi) = \{0\}$.

3. If $\mathcal{R} \in \mathbb{P}(P(p))$, then $\bigvee \phi[\mathcal{R}]^+ = \phi(\bigvee \mathcal{R}) = 1$. $\square$

Definition 1.3.31 *Let A be a Boolean algebra, and let p be a partition of A. Then let $p^* = \{\bigvee R | R \in p^\sharp\}$. Clearly p^* is a Boolean subalgebra of A.*

Theorem 1.3.32 *If (A, F) is subcomplete, and $f : (A, F) \to B$ is partitional, then $\{f[p]^+ | p \in F\}$ is a filter on $\mathbb{P}(B)$.*

Proof. $\{f[p]^+ | p \in F\}$ is a filterbase since the map $p \mapsto f[p]^+$ is order preserving.

We claim that if $p \in F$, then there is a partition q of B with $p \preceq q$ and where $f[p]^+ = f[q]$. Let $\mathcal{A} = \{a \in p | f(a) \neq 0\}$ and let $\mathcal{Z} : p \to \mathcal{A}$ be a mapping where $\mathcal{Z}|_{\mathcal{A}}$ is the identity function. Let $q = \{\bigvee \mathcal{Z}^{-1}[a] | a \in \mathcal{A}\}$. Then q is a partition of B with $p \preceq q$. We have $f[p]^+ = f[\mathcal{A}] = \{f(a) | a \in \mathcal{A}\}$, but $f[q] = \{f(\bigvee \mathcal{Z}^{-1}[a]) | a \in \mathcal{A}\}$, so since $f[p]^+$ and $f[q]$ are both partitions of B and $f(a) \leq f(\bigvee \mathcal{Z}^{-1}[a])$ for $a \in \mathcal{A}$, we have $f[p]^+ = f[q]$ by Lemma 1.3.8.

Now assume $p \in F$ and $f[p]^+ \preceq s$. Then there is a $q \in F$ with $p \preceq q$ and $f[q] = f[p]^+ \preceq s$. Let $L_b = \{a \in q | f(a) \leq b\}$ for $b \in s$. Then $(L_b)_{b \in s}$ is a partition of the set q, so if we set $r = \{\bigvee L_b | b \in s\}$, then r is a partition of B with $q \preceq r$. If $a \in L_b$, then $f(a) \leq f(\bigvee L_b)$, so $b = \bigvee\{f(a)|a \in q, f(a) \leq b\} = \bigvee\{f(a)|a \in L_b\} \leq f(\bigvee L_b)$. Therefore since $s, \{f(\bigvee L_b)|b \in s\} = f[r]$ are both partitions, we have $s = f[r]$ by Lemma 1.3.8. □

Lemma 1.3.33 *Let B be a Boolean algebra, let p be a partition of B, and assume $R \subseteq p$. Then x is the least upper bound of R iff $x \geq r$ for $r \in R$ and $x \wedge s = 0$ for $s \in p \setminus R$.*

Proof. $\rightarrow$ If $x = \bigvee R$, then for $s \in p \setminus R$ we have $s \wedge x = s \wedge (\bigvee_{r \in R} r) = \bigvee_{r \in R}(r \wedge s) = 0$, and $r \leq x$ for $r \in R$.

$\leftarrow$ Assume $x \geq r$ for $r \in R$ and $x \wedge s = 0$ for $s \in p \setminus R$. Then $x \wedge r = r$ for $r \in R$, so $x = x \wedge \bigvee p = \bigvee_{a \in p}(x \wedge a) = \bigvee_{r \in R} r$. □

Theorem 1.3.34 *Assume A, B are Boolean algebras, $f : A \rightarrow B$ is a Boolean algebra homomorphism, and p is a partition of A where $f[p]^+$ is a partition of B. Then for each $R \in p^\sharp$ we have $f(\bigvee R) = \bigvee(f[R]^+)$. In particular, if p is subcomplete, then $f[p]^+$ is also subcomplete.*

Proof. Clearly $f(\bigvee R)$ is an upper bound of $f[R]^+$. Now if $b \in (f[p]^+) \setminus (f[R]^+)$, then $b = f(a)$ for some $a \in p \setminus R$, so $a \wedge \bigvee R = 0$, and $b \wedge f(\bigvee R) = f(a) \wedge f(\bigvee R) = f(a \wedge (\bigvee R)) = f(0) = 0$. Therefore $f(\bigvee R)$ is the least upper bound of $f[R]^+$ by Lemma 1.3.33. If p is subcomplete, then every subset of $f[p]^+$ is of the form $f[R]^+$ for some $R \subseteq p$ and $\bigvee f[R]^+ = f(\bigvee R)$, so $f[p]^+$ is subcomplete. □

Corollary 1.3.35 *If A is a subalgebra of a Boolean algebra B, p is a partition of B with $p \subseteq A$, and $R \subseteq p$ for which $\bigvee^A R$ exists, then $\bigvee^B R$ exists and $\bigvee^B R = \bigvee^A R$.*

Proof. Let $\iota : A \rightarrow B$ be the inclusion mapping. Then, since p is a partition of A and $\iota[p] = p$ is a partition of B, we have $\bigvee^A R = \iota(\bigvee^A R) = \bigvee^B \iota[R] = \bigvee^B R$ by Theorem 1.3.34. □

Definition 1.3.36 *If (B, F) is a Boolean partition algebra, then an ultrafilter $\mathcal{U} \subseteq B$ is an F-ultrafilter if for each $p \in F$ there is an $a \in p \cap \mathcal{U}$. We shall write $S^*(B, F)$*

for the collection of all F-ultrafilters on B. A Boolean partition algebra (B, F) is said to be stable if $\bigcap S^*(B, F) = \{1\}$.

If there is an $a \in p \cap \mathcal{U}$, then a is unique since if $a, b \in p \cap \mathcal{U}, a \neq b$, then $0 = a \wedge b \in \mathcal{U}$ which is impossible.

Example 1.3.37 *If B is a Boolean algebra, then every ultrafilter is a $\mathbb{P}_\omega(B)$-ultrafilter. In other words, $S^*(B, \mathbb{P}_\omega(B)) = S(B)$. Therefore $(B, \mathbb{P}_\omega(B))$ is always stable.*

Every filter on a Boolean algebra can be extended to an ultrafilter. On the other hand, there are Boolean partition algebras (B, F) that do not contain any F-ultrafilters as the following examples illustrate.

Example 1.3.38 *If (B, F) is a Boolean partition algebra and $a \in B$ is an atom, then $\uparrow a$ is an ultrafilter. Furthermore, for each $p \in F$ there is a $b \in p$ with $a \wedge b \neq 0$. Therefore $a \leq b$, so $p \cap \uparrow a \neq 0$. Thus $\uparrow a$ is an F-ultrafilter. In particular, if B is atomic, then (B, F) is stable.*

Example 1.3.39 *If B is a Boolean algebra, then $S^*(B, \mathbb{P}(B))$ is the collection of all principal ultrafilters. If $\mathcal{U}$ is a nonprincipal ultrafilter, then $\bigvee(B \setminus \mathcal{U}) = 1$, so there is a partition $p \subseteq B \setminus \mathcal{U}$, and hence $\mathcal{U} \notin S^*(B, \mathbb{P}(B))$. In particular, if If B is atomless, then $S^*(B, \mathbb{P}(B)) = \emptyset$.*

If (B, F) is a Boolean partition algebra, then the set F becomes an inverse system with transition mappings $\phi_{p,q} : p \to q$ whenever $p \preceq q$. We may therefore refer to the inverse limit $\varprojlim F$ which may be empty.

Theorem 1.3.40 *Let (B, F) be a Boolean partition algebra, and let $(x_p)_{p \in F} \in \varprojlim F$. Then*

1. *If $p, q \in F$, then $x_{p \wedge q} = x_p \wedge x_q$.*
2. *$\{x_p | p \in F\}$ is an F-ultrafilter on B.*

Proof. 1. We have $x_{p \wedge q} \leq \phi_{p \wedge q, p}(x_{p \wedge q}) = x_p$ and $x_{p \wedge q} \leq x_q$, so $x_{p \wedge q} \leq x_p \wedge x_q$. Since $x_{p \wedge q} \in p \wedge q$ and $x_p \wedge x_q \in p \wedge q$ we have $x_{p \wedge q} = x_p \wedge x_q$.

2. Since $x_p \wedge x_q = x_{p\wedge q}$, the set $\{x_p|p \in F\}$ is a filterbase. Now assume $p \in F$ and $x_p \leq a$ and $a \notin \{x_p|p \in F\}$. Then $x_{\{a,a'\}} = a'$, so $x_{p\wedge\{a,a'\}} = x_p \wedge x_{\{a,a'\}} = x_p \wedge a' = x_p \wedge a \wedge a' = 0$. This is a contradiction. Therefore $\{x_p|p \in F\}$ is a filter. If $b \in B \setminus \{0,1\}$, then $x_{\{b,b'\}} = b$ or $x_{\{b,b'\}} = b'$, thus $\{x_p|p \in F\}$ is an ultrafilter. Furthermore $\{x_p|p \in F\}$ is an F-ultrafilter since $\{x_p|p \in F\} \cap p = \{x_p\}$ for each $p \in F$. $\square$

We shall now show that every F-ultrafilter is of the form $\{x_p|p \in F\}$ for some $(x_p)_{p\in F} \in \varprojlim F$.

Let $\mathcal{U}$ be an F-ultrafilter, and let $f : F \to B$ be the mapping where $f(p)$ is the unique element in $\mathcal{U} \cap p$. If $p \preceq q$, then $\phi_{p,q}(f(p)) \in q$ and $\phi_{p,q}(f(p)) \geq f(p) \in \mathcal{U}$, so $\phi_{p,q}(f(p)) = f(q)$. Therefore $f \in \varprojlim F$.

Definition 1.3.41 *Define maps $L : \varprojlim F \to S^*(B,F)$ and $M : S^*(B,F) \to \varprojlim F$ by letting $L(x_p)_{p\in F} = \{x_p|p \in F\}$ and $M(\mathcal{U})(p) \in p \cap \mathcal{U}$ for $p \in F$.*

Theorem 1.3.42 *The functions L and M are inverses.*

Proof. If $(x_p)_{p\in F} \in \varprojlim F$, then for $p \in F$ we have $M(L((x_p)_{p\in F}))(p) = M(\{x_p|p \in F\})(p) = x_p$ since $x_p \in \{x_p|p \in F\} \cap p$. This shows that $M \circ L$ is the identity function. Conversely, let $\mathcal{U}$ be an F-ultrafilter. If $a \in L(M(\mathcal{U}))$, then $a = M(\mathcal{U})(p)$ for some $p \in F$, so $a \in \mathcal{U} \cap p$, and particularly $a \in \mathcal{U}$. Therefore $L(M(\mathcal{U})) \subseteq \mathcal{U}$, so $L(M(\mathcal{U})) = \mathcal{U}$. $\square$

Proposition 1.3.43 *The following are equivalent.*

1. *(B,F) is stable.*
2. *$\bigcup S^*(B,F) = B^+$*
3. *For each $b \in B^+$ there is an $(x_p)_{p\in F} \in \varprojlim F$ with $b = x_p$ for some $p \in F$.*
4. *The canonical mappings $\phi_p : \varprojlim F \to p$ are all surjective.*

Proof. Clearly 1 and 2 are equivalent. Furthermore 2 and 3 are equivalent since the F-ultrafilters are precisely the sets of the form $\{x_p|p \in F\}$ for some $(x_p)_{p\in F} \in \varprojlim F$.

$4 \to 3$ Assume each $\phi_p : \varprojlim F \to p$ is surjective. For each $b \in B^+$, there is a $p \in F$ with $b \in p$, so there is an $(x_p)_{p\in F} \in \varprojlim F$ with $x_p = \phi_p((x_p)_{p\in F}) = b$.

$2 \to 4$ If we assume $\bigcup S^*(B,F) = B^+$, then for each $p \in F$ and $a \in p$ there is a $\mathcal{U} \in S^*(B,F)$ with $a \in \mathcal{U}$, so $M(\mathcal{U})(p) = a$. Therefore $\phi_p(M(\mathcal{U})) = a$, thus ϕ_p is surjective. $\square$

Definition 1.3.44 *If (B,F) is a Boolean partition algebra, then let $\iota : (B,F) \to P(S^*(B,F))$ be the mapping mapping where $\iota(a) = \{\mathcal{U} \in S^*(B,F) | a \in \mathcal{U}\}$.*

The map ι is a clearly a Boolean algebra homomorphism. If $p \in F$, then $\iota[p]^+$ is a partition of $S^*(B,F)$ for each $p \in F$, so ι is partitional. Clearly $\ker(\iota) = \{0\}$ if and only if (B,F) is stable, so (B,F) is stable if and only if ι is injective. If (B,F) is stable, then since ι is injective, for $p, q \in F, p \neq q$ we have $\iota[p] \neq \iota[q]$.

1.4 Uniform Spaces and Duality

In this section we shall generalize the Stone duality between Boolean algebras and compact totally disconnected spaces to a duality between Boolean partition algebras and uniform spaces. All the basic facts about uniform spaces can be found with proofs in the books [6], [7], [23].

Definition 1.4.1 *A uniform space is a pair (X,F) where X is a set and F is a filter on $X \times X$ that satisfies the following three properties.*

1. *If $R \in F$, then $1_X = \{(x,x) | x \in X\} \subseteq R$.*
2. *If $R \in F$, then $R^{-1} = \{(x,y) : (y,x) \in R\} \in F$.*
3. *If $R \in F$, then there is an $S \in F$ with $\{(x,z) | \exists y, (x,y),(y,z) \in S\} = S \circ S \subseteq R$.*

Elements of the filter F are called entourages. A uniform space is said to be separated if $\bigcap F = 1_X$.

Example 1.4.2 *Let (X,d) be a metric space. Then let $\mathcal{U} \subseteq P(X \times X)$ be the set where $R \in \mathcal{U}$ if and only if there is an $\epsilon > 0$ with $\{(x,y) | d(x,y) < \epsilon\} \subseteq R$. Then $(X,\mathcal{U})$ is a separated uniform space.*

Definition 1.4.3 *If $(X,\mathcal{U}), (Y,\mathcal{V})$ are uniform spaces, then a function $f : X \to Y$ is said to be uniformly continuous if for each $E \in \mathcal{V}$ there is a $D \in \mathcal{U}$ such that if $(x,y) \in D$, then $(f(x), f(y)) \in E$.*

This is a generalization of the ϵ, δ-definition of uniform continuity on metric spaces. Furthermore, if we define $f \times f : X \times X \to Y \times Y$ to be the function where $f \times f(x_1, x_2) = (f(x_1), f(x_2))$, then f is uniformly continuous if and only if $(f \times f)^{-1}[E] \in \mathcal{U}$ for each $E \in \mathcal{V}$.

Informally, a uniform space is a set X with a notion of whether a collection of pairs $\mathcal{A} \subseteq P(X \times X)$ gets arbitrarily close to each other. For instance, in a metric space X a collection $\mathcal{A} \subseteq P(X \times X)$ of pairs goes arbitrarily close to each other if $\inf\{d(x, y)|(x, y) \in \mathcal{A}\} = 0$. In a uniform space, a collection of pairs $\mathcal{A}$ of elements intuitively goes arbitrarily close to each other if $\mathcal{A} \cap E \neq \emptyset$ for each entourage E. However, one can easily see that $\mathcal{A} \cap E \neq \emptyset$ for each entourage E if and only if $E \not\subseteq \mathcal{A}^c$ for each entourage E and this happens if and only if $\mathcal{A}^c$ is not an entourage. Intuitively, a set $E \subseteq P(X \times X)$ is an entourage iff the elements in the pairs of E^c do not go arbitrarily close to each other.

Proposition 1.4.4 *A mapping $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is uniformly continuous if and only if whenever $\mathcal{A} \subseteq P(X \times X)$ and $\mathcal{A} \cap D \neq \emptyset$ for all $D \in \mathcal{U}$, then $(f \times f)[\mathcal{A}] \cap E \neq \emptyset$ for all $E \in \mathcal{V}$.*

Proof. $\to$ If f is uniformly continuous, then whenever $E \in \mathcal{V}$ there is a $D \in \mathcal{U}$ with $(f(x), f(y)) \in E$ whenever $(x, y) \in D$. Let $(x, y) \in D \cap \mathcal{A}$. Then $(f(x), f(y)) \in E \cap (f \times f)[\mathcal{A}]$.

$\leftarrow$ We shall prove this direction by contraposition. If f is not uniformly continuous, then there is an entourage E where $(f \times f)^{-1}[E]$ is not an entourage. Thus $(f \times f)^{-1}[E]^c \cap D \neq \emptyset$ for $D \in \mathcal{U}$, but $(f \times f)[(f \times f)^{-1}[E]^c] \cap E = \emptyset$. $\square$

Definition 1.4.5 *If $(X, \mathcal{U})$ is a uniform space, then we may put a topology on X. A set $U \subseteq X$ is open if for each $x \in U$ there is an $R \in \mathcal{U}$ with $\{y|(x, y) \in R\} = R[x] \subseteq U$.*

With this topology, we get $1_X \subseteq E^\circ$ for each entourage $E \in \mathcal{U}$. In other words, every entourage is a neighborhood of the diagonal.

Definition 1.4.6 *If $(X, \mathcal{U})$ is a uniform space and $Y \subseteq X$, then one may put a uniform structure on the subset Y. Let $\mathcal{U}_Y = \{R \cap (Y \times Y)|R \in \mathcal{U}\}$. Then $(Y, \mathcal{U}_Y)$ is* *a uniform space. Clearly the inclusion mapping from Y to X is uniformly continuous.*

Definition 1.4.7 *If X is a topological space, and $\mathfrak{F}$ is a filterbase on X, then we say the filterbase $\mathfrak{F}$ converges to a point $x \in X$ (written $\mathfrak{F} \to x$) if for each open neighborhood U of x there is an $R \in \mathfrak{F}$ with $R \subseteq U$.*

If F is a filter, then it is easy to see that F converges to x if and only if $U \in F$ for each neighborhood U of x. If $\mathfrak{F}$ is a filterbase, then let F be the filter generated by $\mathfrak{F}$. Then one can easily see that $F \to x$ if and only if $\mathfrak{F} \to x$. One can easily show that a space is Hausdorff if and only if every filterbase converges to at most one point.

Definition 1.4.8 *We say that a filterbase $\mathfrak{F}$ on a topological space X accumulates at a point $x \in X$ and we shall write $\mathfrak{F} \propto x$ if $U \cap R \neq \emptyset$ for each $R \in \mathfrak{F}$ and each open neighborhood U of x.*

It is easy to see that a filterbase $\mathfrak{F}$ accumulates at x if and only if $x \in \bigcap_{R \in \mathfrak{F}} \overline{R}$.

Definition 1.4.9 *If $(X,\mathcal{U})$ is a uniform space, then a filterbase $\mathfrak{F}$ is said to be Cauchy if for each $R \in \mathcal{U}$ there is an $S \in \mathfrak{F}$ with $S \times S \subseteq R$.*

If a Cauchy filterbase $\mathfrak{F}$ accumulates at some point x, then $\mathfrak{F}$ converges to x. If (X, d) is a metric space, then a filter $\mathfrak{F}$ on X is Cauchy if and only if

$$\inf\{\sup\{d(x,y)|(x,y) \in R\}|R \in \mathfrak{F}\} = 0.$$

Definition 1.4.10 *A separated uniform space $(X,\mathcal{U})$ is said to be complete if every Cauchy filter converges to some point.*

In a metric space every Cauchy filter converges if and only if every Cauchy sequence converges, so there is no discrepancy between the two notions of a complete metric space.

Every separated uniform space $(X,\mathcal{U})$ can be embedded into a complete uniform space $\hat{X}$ such that X is dense in $\hat{X}$. Furthermore, the space $\hat{X}$ is unique up to uniform homeomorphism preserving X, so we shall call $\hat{X}$ the completion of X. If $(X,\mathcal{U})$ is a uniform space and $Y \subseteq X$ is complete, then Y is closed. See [6][p. 18] or [7] for more details on completions of uniform spaces.

Definition 1.4.11 *A uniform space $(X,\mathcal{U})$ is said to be non-Archimedean if for each $R \in \mathcal{U}$ there is an equivalence relation E with $E \subseteq R$ (i.e. $\mathcal{U}$ is generated by equivalence relations).*

We shall extend Stone duality to a duality between non-Archimedean uniform spaces and stable subcomplete Boolean partition algebras.

If X is a set, then let $F = \{R \subseteq X^2 : 1_X \subseteq R\}$. Then (X, F) is a non-Archimedean uniform space and the uniformity F is called the discrete uniformity.

Example 1.4.12 *A metric d is called an ultrametric if it satisfies the strong triangle inequality $d(x,z) \leq Max(d(x,y), d(y,z))$. Every ultrametric induces a non-Archimedean uniformity since $\{(x,y) : d(x,y) < \epsilon\}$ is an equivalence relation for each $\epsilon > 0$. For instance, if p is a prime number, then define the p-adic norm on $\mathbb{Q}$ by $\|\frac{a}{b}p^r\|_p = p^{-r}$ where a, b integers that do not divide p. Define a metric d on $\mathbb{Q}$ by letting $d(x,y) = \|x-y\|_p$. Then d becomes an ultrametric on $\mathbb{Q}$ making $\mathbb{Q}$ a non-Archimedean uniform space. It is well known that the completion of $\mathbb{Z}$ in the p-adic norm is the inverse limit of the quotient rings $\mathbb{Z}/\mathbb{Z}p^r$. However, the uniformity on $\mathbb{Z}$ is generated by equivalence relations R_n where $(x,y) \in R_n$ if and only if $x = y (mod\, p^n)$. The reader is referred to [18] for an exposition on p-adic analysis. One may now conjecture that for each non-Archimedean uniform space (X, F) generated by a set E of equivalence relations, the completion of X is the inverse limit $\varprojlim (X/R)_{R \in E}$. We shall show that this conjecture holds after we prove the duality theorem.*

Example 1.4.13 *Let $(X,\mathcal{U})$ be a uniform space such that if $R_n \in \mathcal{U}$ for all $n \in \mathbb{N}$, then $\bigcap_n R_n \in \mathcal{U}$. Then $(X,\mathcal{U})$ is non-Archimedean. Let $R \in \mathcal{U}$. For all $n > 0$ let R_n be a symmetric entourage such that $R_n^n \subseteq R$, and let S be the equivalence relation generated by the set $\bigcap_n R_n$. Then for each m we have $(\bigcap_n R_n)^m \subseteq R_m^m \subseteq R$, so $S = \bigcup_m (\bigcap_n R_n)^m \subseteq R$.*

Definition 1.4.14 *A* partition space *is a pair (X, M) where M is a filter on the lattice $\mathbb{P}P(X)$. Elements in M shall sometimes be called* uniform partitions.

Since non-Archimedean uniform spaces are generated by equivalence relations, non-Archimedean uniform spaces are essentially filters on the lattice of equivalence relations on a set. Since the lattice of partitions on a set is isomorphic to the lattice of equivalence relations on a set, it is easy to see that partition spaces are essentially the non-Archimedean uniform spaces. We shall work with partition spaces rather than non-Archimedean uniform spaces since partitions are easier to work with than equivalence relations and because partition spaces are closely related to Boolean partition algebras. For if (X, M) is a partition space, then $(\{\emptyset\} \cup \bigcup M, M) = \mathfrak{B}^*(X, M)$ is a Boolean partition algebra. Furthermore, for partition spaces notions such as uniform continuity, the uniform space topology, and completeness can be described elegantly in terms of partition spaces.

Recall that a topological space is zero-dimensional if it has a basis consisting of clopen sets. If (X, M) is a partition space, then $\bigcup M$ is a basis for the topology on X. Since $\bigcup M$ consists of clopen sets, every partition space (X, M) is zero-dimensional. Likewise, every zero-dimensional space can be made into a partition space. For example, if X is a zero-dimensional space, then let M be the collection of all partitions of X into clopen sets. Then (X, M) is a partition space that induces the original topology on X and $\mathfrak{B}^*(X, M)$ is the algebra of all clopen subsets of X. Furthermore, if N is the collection of all partitions of X into finitely many clopen sets, then (X, N) is a partition space that induces the topology on X as well.

A partition space (X, M) is separated if $\bigwedge M = \{\{x\} | x \in X\}$. To put it differently, (X, M) is separated if for each pair x, y of distinct elements of X there is a $P \in M$ where x and y belong to distinct blocks of the partition P.

If (X, M) is a partition space, then a filterbase $\mathfrak{F}$ on X is Cauchy if and only if for each $P \in M$ there is an $R \in \mathfrak{F}$ and an $S \in P$ with $R \subseteq S$. A filter Z on X is Cauchy if and only if $Z \cap P \neq \emptyset$ for each partition $P \in M$.

If $(X, M), (Y, N)$ are partition spaces, then a function $f : X \rightarrow Y$ is uniformly continuous if and only if $\{f^{-1}[R] | R \in P\}^+ \in M$ for each $P \in N$.

If (X, M) is a partition space and $Y \subseteq X$, then we may put a partition space structure on Y. Let $M_Y = \{\{R \cap Y | R \in P\}^+ | P \in M\}$. Then (Y, M_Y) is a partition space called a partition subspace of X. The partition subspace structure on Y coincides with the uniform subspace structure on Y. If $(X, M), (Y, N)$ are partition spaces, then a mapping $f : (X, M) \to (Y, N)$ is an embedding if f is a uniform homeomorphism from (X, M) to $(f[X], N_{f[X]})$.

Remark 1.4.15 *We shall now describe the inverse limits in the category of partition spaces. It should be noted that the inverse limit of partition spaces is a special case of the inverse limit of the category of uniform spaces described in [6]. Let D be a downward directed set and let $((X_d, M_d), \phi_{d,e})_{d\in D}$ be an inverse system of partition spaces. Let $X = \overset{\lim}{\longleftarrow} X_d$ be the inverse limit of the system $(X_d, \phi_{d,e})_{d\in D}$ in the category of sets. In other words, let $X = \{(x_d)_{d\in D} \in \prod_{d\in D} X_d | \phi_{d,e}(x_d) = x_e \text{ whenever} d \leq e\}$. Let $\overline{\pi}_d : X \to X_d$ be the projection mapping with domain restricted to X for $d \in D$. Let $\mathfrak{F}$ be the collection of partitions of the form $((\overline{\pi}_d)_{-1})''(P)^+$ for $P \in M_d$. Then $\mathfrak{F}$ is a filter on the lattice $\mathbb{P}P(X)$ of partitions of X. Furthermore, if M is the filter generated by $\mathfrak{F}$, then (X, M) is the inverse limit of the system $((X_d, M_d), \phi_{d,e})_{d\in D}$ of partition spaces. The inverse limit of uniform spaces is defined similarly. In fact, the inverse limit of an inverse system of uniform spaces $(X_d)_{d\in D}$ is always a closed subspace of the product uniformity $\prod_{d\in D} X_d$ which is also defined in [6]. Since the product of uniform spaces is complete and a closed subspace of a complete uniform space is complete, the inverse limit of complete uniform spaces is complete.*

Theorem 1.4.16 *Let (X, N) be a separating partition space. Then the following are equivalent.*

1. *(X, N) is complete.*
2. *If $\mathcal{U} \subseteq \{\emptyset\} \cup (\bigcup N)$ is an N-ultrafilter, then $\mathcal{U} = \{R \in \bigcup N | x \in R\}$ for some $x \in X$.*
3. *If $\phi \in \overset{\lim}{\longleftarrow} N$, then there is an $x \in X$ with $x \in \phi(P)$ for each $P \in N$.*

Proof. $1 \to 2$ Assume (X, N) is complete. Let $\mathcal{U}$ be an N-ultrafilter. Then $\mathcal{U}$ is Cauchy since $\mathcal{U} \cap P$ is nonempty for each $P \in N$. Therefore since (X, N) is complete, the ultrafilter $\mathcal{U}$ accumulates at some point $x \in X$. In other words, if $R \in \mathcal{U}$, then $x \in \overline{R} = R$. Therefore $\mathcal{U} \subseteq \{R \in \bigcup N | x \in R\}$, so $\mathcal{U} = \{R \in \bigcup N | x \in R\}$ since $\mathcal{U}$ and $\{R \in \bigcup N | x \in R\}$ are both ultrafilters on $\{\emptyset\} \cup (\bigcup N)$.

$2 \rightarrow 3$ Every element of the inverse limit $\varprojlim N$ is of the form $M(\mathcal{U})$ for some N-ultrafilter $\mathcal{U}$ (see Theorem 1.3.42). However, $M(\mathcal{U}) = M(\{R \in \bigcup N | x \in R\})$. Therefore $x \in M(\mathcal{U})(P)$ for $P \in N$.

$3 \rightarrow 1$ Let F be a Cauchy filter. Then for each $P \in N$, there is a unique $X_P \in P$ with $X_P \in F$. If $P \preceq Q$, then $\phi_{P,Q}(X_P) \in Q$ and $\phi_{P,Q}(X_P) \supseteq X_P \in F$, so $\phi_{P,Q}(X_P) = X_Q$. Therefore $(X_P)_{P \in N} \in \varprojlim N$, so there is an $x \in X$ with $x \in X_P$ for each $P \in N$. We shall now show that $F \rightarrow x$. For each neighborhood U of x, there is a $P \in N$ and an $R \in P$ with $x \in R \subseteq U$. We must have $U \supseteq R = X_P \in F$. We therefore conclude that $F \rightarrow x$. $\square$

Let (B, F) be a Boolean partition algebra. Recall that $\iota : (B, F) \rightarrow P(S^*(B, F))$ is the mapping defined by $\iota(a) = \{\mathcal{U} \in S^*(B, F) | a \in \mathcal{U}\}$. If (B, F) is a Boolean partition algebra, then $\{\iota[p]^+ | p \in F\}$ is a filterbase on $\mathbb{P}P(S^*(B, F))$ since $(\iota[p]^+) \wedge (\iota[q]^+) = \iota[p \wedge q]^+$ for $p, q \in F$. Therefore $\{\iota[p]^+ | p \in F\}$ generates a partition space structure on $S^*(B, F)$. From now on, the set $S^*(B, F)$ shall be given the partition space structure generated by $\{\iota[p]^+ | p \in F\}$. Furthermore, if (B, F) is subcomplete, then $(S^*(B, F), \{\iota[p]^+ | p \in F\})$ is a partition space since $\{\iota[p]^+ | p \in F\}$ is a filter on $\mathbb{P}P(S^*(B, F))$.

Recall that if (X, M) is a partition space, then $\mathfrak{B}^*(X, M)$ denotes the Boolean partition algebra $(\{\emptyset\} \cup \bigcup M, M)$. The correspondences $(B, F) \mapsto S^*(B, F), (X, M) \mapsto \mathfrak{B}^*(X, M)$ allow us to travel between Boolean partition algebras and partition spaces, and we shall show that the functors $S^*, \mathfrak{B}^*$ are equivalences between the categories of complete partition spaces and subcomplete stable Boolean partition algebras.

Definition 1.4.17 *If (B, F) is a Boolean partition algebra, then let $\psi : (B, F) \rightarrow \mathfrak{B}^*(S^*(B, F))$ be the mapping where $\psi(b) = \iota(b)$ for $b \in B$.*

The functions ψ, ι shall be regarded as distinct because ψ and ι serve different purposes. If M is the partition structure on $S^*(B, F)$, then for $p \in F$ we have $\psi[p]^+ = \iota[p]^+ \in M$. Therefore ψ is a partition homomorphism. We may be more specific and write $\psi_{(B,F)}$ for the mapping from (B, F) to $\mathfrak{B}^*(S^*(B, F))$ if the domain of ψ is ambiguous.

Definition 1.4.18 *If (X, M) is a partition space and $x \in X$, then let $\mathcal{C}(x) = \{R \in \mathfrak{B}^*(X, M) | x \in R\}$. Then $\mathcal{C}(x)$ is an ultrafilter on $\mathfrak{B}^*(X, M)$. Also, for each $P \in M$ there is a unique $R \in P$ with $x \in R$, so $R \in \mathcal{C}(x)$. Therefore $\mathcal{C}(x) \in S^*(\mathfrak{B}^*(X, M))$ for each $x \in X$, so $\mathcal{C}$ is a mapping from (X, M) to $S^*(\mathfrak{B}^*(X, M))$.*

Proposition 1.4.19 *If $(A, F), (B, G)$ are Boolean partition algebras, and $\phi : A \to B$ is a partition homomorphism, then for each $\mathcal{U} \in S^*(B, G)$ we have $\phi^{-1}[\mathcal{U}] \in S^*(A, F)$*

Proof. Assume that $p \in F$. Then $\phi[p]^+ \in G$, so there is an $a \in p$ where $\phi(a) \in \mathcal{U}$, thus $a \in \phi^{-1}[\mathcal{U}]$. $\square$

We shall now prove the duality theorem between subcomplete stable Boolean partition algebras and partition spaces.

Theorem 1.4.20 *1. If (B, F) is a Boolean partition algebra, then $S^*(B, F)$ is a (possibly empty) complete partition space.*

2. If (X, M) is a partition space, then $\mathfrak{B}^*(X, M)$ is a subcomplete and stable Boolean partition algebra.

3. If (B, F) is stable, then the partition homomorphism ψ is injective. If (B, F) is both subcomplete and stable, the ψ is a Boolean partition algebra isomorphism.

4. Let (X, M) be a partition space. Then $\mathcal{C} : (X, M) \to S^*(\mathfrak{B}^*(X, M))$ is uniformly continuous, and $\mathcal{C}[X]$ is dense in $S^*(\mathfrak{B}^*(X, M))$. If (X, M) is separated, then $\mathcal{C}$ is a uniform embedding. If (X, M) is complete, then $\mathcal{C}$ is a uniform homeomorphism.

Proof. 1. If $\mathcal{U}, \mathcal{V} \in S^*(B, F)$ are distinct ultrafilters, then let $a \in \mathcal{U} \setminus \mathcal{V}$. Then $a \in \mathcal{U}, a' \in \mathcal{V}$, so $\mathcal{U} \in \iota(a), \mathcal{V} \in \iota(a')$. Therefore $\mathcal{U}, \mathcal{V}$ are in distinct blocks of the uniform partition $\{\iota(a), \iota(a')\}$ in the partition space $S^*(B, F)$. In other words, the partition space $S^*(B, F)$ is separating.

Let M be the partition space structure on $S^*(B, F)$. Then $\mathfrak{B}^*(S^*(B, F)) = (\{\emptyset\} \cup (\bigcup M), M)$. If $\mathcal{U} \subseteq \{\emptyset\} \cup \bigcup M$ is an M-ultrafilter, then since $\psi : (B, F) \to \mathfrak{B}^*(S^*(B, F)) = (\emptyset \cup \bigcup M, M)$ is a partition homomorphism, the set $\psi^{-1}[\mathcal{U}]$ is an F-ultrafilter, so $\psi^{-1}[\mathcal{U}] \in S^*(B, F)$. By Theorem 1.4.16, it suffices to show that if $R \in \mathcal{U}$, then $\psi^{-1}[\mathcal{U}] \in R$.

If $R \in \mathcal{U}$, then $R \in P$ for some $P \in M$, so there is a $p \in F$ with $\iota[p]^+ \preceq P$. If $a \in p \cap \psi^{-1}[\mathcal{U}]$, then $\psi(a) \in \mathcal{U} \cap \iota[p]^+$, and clearly $\psi(a) \subseteq R$. Since $a \in \psi^{-1}[\mathcal{U}]$, we have $\psi^{-1}[\mathcal{U}] \in \psi(a) \subseteq R$.

2. If $P \in M$ and $Z \subset P$ is non-empty, then $P \preceq \{\cup Z, \cup(P \setminus Z)\}$, so $\cup Z \in M$, thus $\mathfrak{B}^*(X, M)$ is subcomplete. If $R \in \bigcup M$, then for each $x \in R$, the set $\{S \in \bigcup M | x \in S\}$ is an M-ultrafilter that contains R. Therefore $\mathfrak{B}^*(X, M) = (\{\emptyset\} \cup \bigcup M, M)$ is stable.

3. If (B, F) is stable, then ψ is injective since ker $\psi =$ ker $\iota = \{0\}$.

If (B, F) is subcomplete and stable, then let $M = \{\iota[p] | p \in F\}$. Then $B^*(S^*(B, F)) = B^*(S^*(B, F), M) = (\{\emptyset\} \cup \bigcup M, M)$. Since $\{\emptyset\} \cup \bigcup M = \psi[B]$ and $M = \{\psi[p] | p \in F\}$, the mapping ψ is a partition isomorphism.

4. Take note that $\mathfrak{B}^*(X, M) = (\{\emptyset\} \cup \bigcup M, M)$, so $S^*(\mathfrak{B}^*(X, M)) = S^*(\{\emptyset\} \cup \bigcup M, M) = (S^*(\{\emptyset\} \cup \bigcup M, M), \{\iota[P] | P \in M\})$ since $\mathfrak{B}^*(X, M)$ is subcomplete. To show $\mathcal{C}$ that is uniformly continuous and with dense image, we shall take inverse images of the partitions $\iota[P]$ under $\mathcal{C}$. We have $\{\mathcal{C}^{-1}[R] | R \in \iota[P]\} = \{\mathcal{C}^{-1}[\iota(V)] | V \in P\}$. Now $x \in \mathcal{C}^{-1}[\iota(V)]$ iff $\mathcal{C}(x) \in \iota(V)$ iff $V \in \mathcal{C}(x)$ iff $x \in V$. Therefore $\mathcal{C}^{-1}[\iota(V)] = V$, so $\{\mathcal{C}^{-1}[R] | R \in \iota[P]\} = \{\mathcal{C}^{-1}[\iota(V)] | V \in P\} = \{V | V \in P\} = P$. We have shown that $\mathcal{C}$ is uniformly continuous, and since $\emptyset \notin \{\mathcal{C}^{-1}[R] | R \in \iota[P]\}$ for each partition $\iota[P]$, the set $\mathcal{C}[X] \subseteq S^*(\mathfrak{B}^*(X, M))$ is dense.

If (X, M) is separated, then for each pair x, y of distinct elements in X there is some $\{R, R^c\} \in M$ where $x \in R, y \in R^c$. In this case, we have $R \in \mathcal{C}(x)$, but $R \notin \mathcal{C}(y)$. Therefore $\mathcal{C}$ is injective. Since each $P \in M$ can be written as $\{\mathcal{C}^{-1}[R] | R \in \iota[P]\}$, the mapping $\mathcal{C}$ is a uniform embedding. If (X, M) is complete, then since $\mathcal{C}$ is a uniform embedding and $\mathcal{C}[X]$ is dense in $S^*(\mathfrak{B}^*(X, M))$, the mapping $\mathcal{C}$ is a uniform homeomorphism. $\square$

If (B, F) is a Boolean partition algebra, then since the partition space on $S^*(B, F)$ is generated by $\{\iota[p] | p \in F\}$, the topology on $S^*(B, F)$ is generated by the basis $\bigcup_{p \in F} \iota[p] = \iota[\bigcup F] = \iota[B^+]$.

The duality between partition spaces and Boolean partition algebras extends to an equivalence between categories.

Definition 1.4.21 *If* $f : (X, M) \to (Y, N)$ *is uniformly continuous, then define* *a mapping* $\mathfrak{B}^*(f) : \mathfrak{B}^*(Y,N) \to \mathfrak{B}^*(X,M)$ *by letting* $\mathfrak{B}^*(f)(R) = f^{-1}[R]$. *Then clearly* $\mathfrak{B}^*(f)$ *is a partition homomorphism. For each pair of Boolean partition spaces* $(A,F),(B,G)$ *and partition homomorphism* $\phi : (A,F) \to (B,G)$ *define a mapping* $S^*(\phi) : S^*(B,G) \to S^*(A,F)$ *by letting* $S^*(\phi)(\mathcal{U}) = \phi^{-1}[\mathcal{U}]$.

Theorem 1.4.22 *If* $(A,F),(B,G)$ *are Boolean partition algebras, and* $\phi : (A,F) \to (B,G)$ *is a partition homomorphism, then* $S^*(\phi)$ *is uniformly continuous.*

Proof. The set of partitions $\{\iota[p]^+ | p \in F\}$ generates the partition structure on $S^*(A,F)$. For $p \in F$, we shall take the inverse images of blocks of the partitions $\iota[p]^+$ to show $S^*(\phi)$ is uniformly continuous. Now for $\mathcal{U} \in S^*(B,G), a \in A$ we have $\mathcal{U} \in S^*(\phi)^{-1}[\iota(a)]$ iff $S^*(\phi)\mathcal{U} \in \iota(a)$ iff $a \in S^*(\phi)(\mathcal{U}) = \phi^{-1}[\mathcal{U}]$ iff $\phi(a) \in \mathcal{U}$ iff $\mathcal{U} \in \iota(\phi(a))$. Therefore $\{S^*(\phi)^{-1}[R] | R \in \iota[p]^+\}^+ = \{S^*(\phi)^{-1}[\iota(a)) | a \in p\}^+ = \{\iota(\phi(a)) | a \in p\}^+ = (\iota \circ \phi)[p]^+ = \iota[\phi[p]^+]^+$ is a uniform partition in $S^*(B,G)$. □

If $(X,L),(Y,M),(Z,N)$ are partition spaces, and $f : X \to Y, g : Y \to Z$ are uniformly continuous, then for $R \in \mathfrak{B}^*(Z,N)$ we have $\mathfrak{B}^*(g \circ f)(R) = (g \circ f)^{-1}[R] = f_{-1}[g_{-1}[R]] = \mathfrak{B}^*(f) \circ \mathfrak{B}^*(g)(R)$. Furthermore, if $(A,F),(B,G),(C,H)$ are Boolean partition algebras, and $f : (A,F) \to (B,G), g : (B,G) \to (C,H)$ are partition homomorphisms, then $S^*(g \circ f)(\mathcal{U}) = (g \circ f)^{-1}[\mathcal{U}] = f^{-1}[g^{-1}[\mathcal{U}]] = S^*(f) \circ S^*(g)(\mathcal{U})$. Therefore $\mathfrak{B}^*, S^*$ are contravariant functors since $\mathfrak{B}^*, S^*$ clearly map identity functions onto identity functions.

Theorem 1.4.23 *1. Let* $(X,M),(Y,N)$ *be partition spaces, and let* $f : X \to Y$ *be uniformly continuous, then* $S^*(\mathfrak{B}^*(f)) \circ \mathcal{C}_{(X,M)} = \mathcal{C}_{(Y,N)} \circ f$. *In other words, the following square commutes.*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathcal{C}_{(X,M)}"] & Y \arrow[d, "\mathcal{C}_{(Y,N)}"] \\ S^*(\mathfrak{B}^*(X,M)) \arrow[r, "S^*(\mathfrak{B}^*(f))"] & S^*(\mathfrak{B}^*(Y,N)) \end{tikzcd}$$

2. Let $(A,F),(B,G)$ *be stable Boolean partition algebras, and let* $f : A \to B$ *be a partition homomorphism, then* $\mathfrak{B}^*(S^*(f))\psi_{(A,F)} = \psi_{(B,G)}f$. *This means that the following diagram commutes.*

$$\begin{array}{ccc}(A,F) & \xrightarrow{f} & (B,G)\\ \Big\downarrow{\scriptstyle\psi_{(A,F)}} & & \Big\downarrow{\scriptstyle\psi_{(B,G)}}\\ \mathfrak{B}^*(S^*(A,F)) & \xrightarrow{\mathfrak{B}^*(S^*(f))} & \mathfrak{B}^*(S^*(B,G))\end{array}$$

3. If (X,M) is a partition space, then the functions $\mathfrak{B}^*(\mathcal{C}_X):\mathfrak{B}^*(S^*(\mathfrak{B}^*(X,M)))\rightarrow\mathfrak{B}^*(X,M)$ and $\psi:\mathfrak{B}^*(X,M)\rightarrow\mathfrak{B}^*(S^*(\mathfrak{B}^*(X,M)))$ are inverses.

4. If (B,F) is a stable Boolean partition algebra, then $S^*(\psi_B):S^*(\mathfrak{B}^*(S^*(B,F)))\rightarrow S^*(B,F)$ and $\mathcal{C}:S^*(B,F)\rightarrow S^*(\mathfrak{B}^*(S^*(B,F)))$ are inverses.

Proof. 1. For each $x\in X$ and $R\in\mathfrak{B}^*(Y,N)$ we have $R\in S^*(B^*(f))C_{(X,M)}(x)=B^*(f)^{-1}[C_{(X,M)}(x)]$ iff $\mathfrak{B}^*(f)(R)\in C_{(X,M)}(x)$ iff $x\in\mathfrak{B}^*(f)(R)=f^{-1}[R]$ iff $f(x)\in R$ iff $R\in C_{(Y,N)}f(x)$.

2. Assume $a\in A$ and $\mathcal{V}\in S^*(B,G)$. Then $\mathcal{V}\in\mathfrak{B}^*(S^*(f))\psi_A(a)=S^*(f)^{-1}[\psi_A(a)]$ iff $S^*(f)(\mathcal{V})\in\psi_A(a)$ iff $a\in S^*(f)(\mathcal{V})$ iff $f(a)\in\mathcal{V}$ iff $\mathcal{V}\in\psi_B f(a)$.

3. Since $\mathfrak{B}^*(X,M)$ is subcomplete and stable, the map $\psi:\mathfrak{B}^*(X,M)\rightarrow\mathfrak{B}^*S^*\mathfrak{B}^*(X,M)$ is a partition isomorphism by Theorem 1.4.20. We shall show that $\mathcal{B}^*(\mathcal{C})\psi:\mathfrak{B}^*(X,M)\rightarrow\mathcal{B}^*(X,M)$ is the identity function. If $R\in\mathfrak{B}^*(X,M)$, then $x\in\mathfrak{B}^*(\mathcal{C})\psi(R)=\mathcal{C}^{-1}[\psi(R)]$ iff $\mathcal{C}(x)\in\psi(R)$ iff $R\in\mathcal{C}(x)$ iff $x\in R$. Therefore $\mathfrak{B}^*(\mathcal{C})\psi(R)=R$, so $\mathfrak{B}^*(\mathcal{C})\psi$ is the identity function.

4. Since the partition space $S^*(B,F)$ is complete, the mapping $\mathcal{C}$ is a uniform homeomorphism by Theorem 1.4.20. It suffices to show that $S^*(\psi)\mathcal{C}:S^*(B,F)\rightarrow S^*(B,F)$ is the identity function. Let $\mathcal{U}\in S^*(B,F)$. Then $a\in S^*(\psi)\mathcal{C}(\mathcal{U})$ iff $\psi(a)\in\mathcal{C}(\mathcal{U})$ iff $\mathcal{U}\in\psi(a)$ iff $a\in\mathcal{U}$. Therefore $S^*(\psi)\mathcal{C}$ is the identity function. □

In light of the duality between Boolean partition algebras and uniform spaces, one should think of a subcomplete Boolean partition algebra as a point-free uniform space and the maps between subcomplete Boolean partition algebras should be thought of as uniformly continuous mappings between these point-free uniform spaces. In fact, there is an equivalence between the category of subcomplete Boolean partition algebras and certain point-free uniform spaces.

A uniform space $(X,\mathcal{U})$ is said to be totally bounded if for each $R\in\mathcal{U}$, there are $x_1,...,x_n\in X$ where $R[x_1]\cup...\cup R[x_n]=X$. If X is compact, then let F be the filter on $X \times X$ where $R \in F$ if and only if R is a neighborhood of the diagonal $1_X = \{(x,x)|x \in X\}$. Then F is the unique uniformity on X that induces the original topology on X. Furthermore, if $(X,\mathcal{U}),(Y,\mathcal{V})$ are uniform spaces and X is compact and $f : X \to Y$ is continuous, then f is also uniformly continuous. Moreover, if $(X,\mathcal{U})$ is a uniform space, then X is compact if and only if $(X,\mathcal{U})$ is complete and totally bounded. Therefore compact spaces may be regarded as complete and totally bounded uniform spaces, and the continuous mapping between compact spaces are precisely the uniformly continuous mappings. See [7] and [23] for proofs of these facts.

The totally bounded partition spaces are precisely the partition spaces (X, M) where each partition in M is finite. If X is a compact zero-dimensional space, then let M be the collection of all partitions of X into finitely many clopen sets. Then M is the unique partition space structure compatible with the topology on X. Therefore the compact zero-dimensional spaces are essentially the complete partition spaces (X, M) where each partition in M is finite. Stone's duality between compact totally disconnected spaces and Boolean algebras follows as a consequence of these facts when we relate each Boolean algebra B to the Boolean partition algebra $(B, \mathbb{P}_\omega(B))$. Specifically, if B is a Boolean algebra, then $S^*(B, \mathbb{P}_\omega(B))$ is compact and zero-dimensional, and if X is compact and zero dimensional, then $\mathfrak{B}^*(X, M) = (\{\emptyset\} \cup \bigcup M, M)$ is a Boolean partition algebra where M is precisely the finite partitions on the Boolean algebra $\{\emptyset\} \cup \bigcup M$.

Definition 1.4.24 *If A, B are Boolean algebras and $\phi : A \to B$ is a homomorphism, then let* $\ker^\uparrow \phi = \phi^{-1}[\{1\}]$.

Proposition 1.4.25 *1. If $(A, F), (B, G)$ are Boolean partition algebras, and $\phi : (A, F) \to (B, G)$ is a surjective partition homomorphism, then $S^*(\phi)$ is injective.*

2. If $(X, N), (Y, M)$ are partition spaces and $f : (X, N) \to (Y, M)$ is uniformly continuous with $f[X]$ dense in (Y, M), then $\mathfrak{B}^*(f) : \mathfrak{B}^*(Y, M) \to \mathfrak{B}^*(X, N)$ is injective.

3. If $(A, F), (B, G)$ are Boolean partition algberas and $\phi : A \to B$ is a partition homomorphism, and $S^*(\phi) : S^*(B, G) \to S^*(A, F)$ is surjective, then $\ker^\uparrow(\phi) \subseteq$

$\bigcap S^*(A,F)$. *In particular, if (A,F) is stable, then ϕ is injective.*

4. Let $f:(X,M)\to(Y,N)$ be uniformly continuous, and let (X,M) be separating. If $\mathfrak{B}^*(f):\mathfrak{B}^*(Y,N)\to\mathfrak{B}^*(X,M)$ is surjective, then f is injective.

Proof. 1. Let $\mathcal{U},\mathcal{V}\in S^*(B,G)$ be distinct G-ultrafilters. Then let $b\in\mathcal{U}\setminus\mathcal{V}$. Then since ϕ is surjective, there is an $a\in A$ with $\phi(a)=b\in\mathcal{U}\setminus\mathcal{V}$. Therefore $a\in\phi_{-1}[\mathcal{U}]=S^*(\phi)(\mathcal{U})$, and $a\not\in S^*(\phi)(\mathcal{V})$, so $S^*(\phi)(\mathcal{U})\neq S^*(\phi)(\mathcal{V})$. Therefore the mapping $S^*(\phi)$ is injective.

2. Assume that $R\in\bigcup M$. Then R is a nonempty open set, so $f^{-1}(R)=\mathfrak{B}^*(R)$ is nonempty since the image $f[X]$ is dense. Therefore $\ker(\mathfrak{B}^*(f))$ is trivial, thus f is injective.

3. If $\phi(a)=1$ and $\mathcal{U}\in S^*(A,F)$, then there is a $\mathcal{V}\in S^*(B,G)$ with $\mathcal{U}=S^*(\phi)(\mathcal{V})=\phi^{-1}[\mathcal{V}]$. Therefore since $\phi(a)=1\in\mathcal{V}$, we have $a\in S^*(\phi(\mathcal{V}))=\mathcal{U}$. Thus $\ker^{\uparrow}\phi\subseteq\mathcal{U}$ for $\mathcal{U}\in S^*(A,F)$. We conclude that $\ker^{\uparrow}\phi\subseteq\bigcap S^*(A,F)$. If (A,F) is stable, then $\ker^{\uparrow}\phi=\{1\}$, so ϕ is injective.

4. Let $x,y\in X$ be distinct points. Then there is an $R\in\mathfrak{B}^*(X,M)$ with $x\in R, y\in R^c$. Since $\mathfrak{B}^*(f)$ is surjective, there is an $S\in\mathfrak{B}^*(Y,N)$ with $\mathfrak{B}^*(f)(S)=R$. Therefore $x\in R=\mathfrak{B}^*(f)(S)$ and $y\in R^c=\mathfrak{B}^*(f)(S^c)$ so $f(x)\in S$ and $f(y)\in S^c$, thus $f(x)\neq f(y)$. □

Proposition 1.4.26 *Let (X,M) be a partition space, and let $C\subseteq X$ be a subspace. Let $\iota:(C,M_C)\to(X,M)$ be the inclusion mapping. Then $\mathfrak{B}^*(\iota):\mathfrak{B}^*(X,M)\to\mathfrak{B}^*(C,M\restriction_C)$ is a surjective partition homomorphism. Furthermore each partition in $M\restriction_C$ is of the form $\mathfrak{B}^*(\iota)[P]^+$ for some $P\in M$.*

Proof. We have $M_C=\{\{R\cap C|R\in P\}^+|P\in M\}=\{\{\iota^{-1}[R]|R\in P\}^+|P\in M\}=\{\mathfrak{B}^*(\iota)[P]^+|P\in M\}$. If $R\in\mathfrak{B}^*(C,M\restriction_C)^+$, then $R\in\mathfrak{B}^*(\iota)[P]^+$ for some $P\in M$, so $R=\mathfrak{B}^*(\iota)(S)$ for some $S\in P$. Therefore the mapping $\mathfrak{B}^*(\iota)$ is surjective. □

Example 1.4.27 *In general, then injectivity of a mapping $f:(X,M)\to(Y,N)$ does not imply the surjectivity of $\mathfrak{B}^*(f)$. For example, let X be an infinite compact totally disconnected space, and let N be the partition space structure on X inducing* *the topology on X. Let (X, M) be the discrete partition structure on X and let $i : (X, M) \to (X, N)$ be the identity mapping. Then both (X, M) and (X, N) are complete uniform spaces and i is uniformly continuous. The Boolean algebra $\mathfrak{B}^*(X, N)$ consists of only the clopen subsets of X while $\mathfrak{B}^*(X, M)$ consists of all subsets of X. The mapping $\mathfrak{B}^*(i) : \mathfrak{B}^*(X, N) \to \mathfrak{B}^*(X, M)$ is the inclusion mapping since i is the identity function. Therefore $\mathfrak{B}^*(i)$ is not surjective since not all subsets of X are clopen.*

Theorem 1.4.28 *Let (A, F) be a Boolean partition algebra, and give $\varprojlim F$ the inverse limit uniform space structure (see Remark 1.4.15 for the inverse limit of partition spaces) where each $p \in F$ is given the discrete uniformity. Then the mappings $L : \varprojlim F \to S^*(A, F)$ and $M : S^*(A, F) \to \varprojlim F$ in Theorem 1.3.42 are uniform homeomorphisms.*

Proof. We have already shown in Theorem 1.3.42 that L and M are inverses. The partition space structure on $S^*(A, F)$ is generated by the partitions of the form $\iota[p]^+$ for $p \in F$. Furthermore, the partition space structure on $\varprojlim F$ is generated by the partitions of the form $\{(\overline{\pi}_p)^{-1}[a] | a \in p\}^+$ where $\overline{\pi}_p : \varprojlim F \to p$ is the projection.

We have $\mathcal{U} \in M^{-1}[(\overline{\pi}_p)^{-1}[a]] = (\overline{\pi}_p \circ M)^{-1}[a]$ iff $\overline{\pi}_p(M(\mathcal{U})) = a$ iff $M(\mathcal{U})(p) = a$ iff $a \in \mathcal{U}$ iff $\mathcal{U} \in \iota(a)$. Hence $M^{-1}[(\overline{\pi}_p)^{-1}[a]] = \iota(a)$. Therefore M corresponds the partition $\{(\overline{\pi}_p)^{-1}[a] | a \in p\}^+$ to the partition $\iota[p]^+$. Therefore M is a uniform homeomorphism. $\square$

Proposition 1.4.29 *Let $(A, F), (B, G)$ be Boolean partition algebras, and let $\phi : (A, F) \to (B, G)$ be a partition homomorphism. Let $(y_q)_{q \in G} \in \varprojlim G$. Then for each $p \in F$ there is a unique $x_p \in p$ where $\phi(x_p) = y_{\phi[p]^+}$, and $(x_p)_{p \in F} \in \varprojlim F$.*

Proof. The existence of x_p is trivial. If $a, b \in p$ are distinct elements with $\phi(a) = y_{\phi[p]^+} = \phi(b)$, then $y_{\phi[p]^+} = \phi(a) \wedge \phi(b) = \phi(a \wedge b) = 0$, a contradiction, Therefore each x_p is unique.

Now assume $p \preceq q$, then $\phi[p]^+ \preceq \phi[q]^+$, so $y_{\phi[p]^+} \leq y_{\phi[q]^+}$, hence $\phi(x_p \wedge x_q) = \phi(x_p) \wedge \phi(x_q) = y_{\phi[p]^+} \wedge y_{\phi[q]^+} = y_{\phi[p]^+} > 0$. Therefore $\phi_{p,q}(x_p) \wedge x_q \geq x_p \wedge x_q > 0$, so since $\phi_{p,q}(x_p), x_q \in q$, we have $\phi_{p,q}(x_p) = x_q$. $\square$

Definition 1.4.30 *Under the assumptions of Proposition [?], let* $\overset{\lim}{\longleftarrow}\phi : \overset{\lim}{\longleftarrow} (B,G) \to \overset{\lim}{\longleftarrow} (A,F)$ *be the mapping where* $\overset{\lim}{\longleftarrow}\phi(y_q)_{q\in G} = (x_p)_{p\in F}$ *whenever* $\phi(x_p) = y_{\phi[p]^+}$ *for* $p \in F$*. In other words, if* $y \in \overset{\lim}{\longleftarrow} G$*, then* $\phi(\overset{\lim}{\longleftarrow}\phi(y)(p)) = y(\phi[p]^+)$ *for* $p \in F$*.*

Theorem 1.4.31 *If* $\phi : (A,F) \to (B,G)$ *is a partition homomorphism, then* $M \circ S^*(\phi) = \overset{\lim}{\longleftarrow} \phi \circ M$ *(here* M *is the mapping from Theorem 1.3.42).*

Proof. Let $\mathcal{U} \in S^*(B,G)$. Then $M \circ S^*(\phi)(\mathcal{U}) \in \overset{\lim}{\longleftarrow} F$ and for $p \in F$ we have $\{M \circ S^*(\phi)(\mathcal{U})(p)\} = p \cap S^*(\phi)(\mathcal{U})$. Going the other way, we have $\phi((\overset{\lim}{\longleftarrow}\phi \circ M(\mathcal{U}))(p)) = M(\mathcal{U})(\phi[p]^+) \in \mathcal{U}$, so $\overset{\lim}{\longleftarrow}\phi \circ M(\mathcal{U})(p) \in S^*(\phi)(\mathcal{U}) \cap p$. Therefore $\{M \circ S^*(\phi)(\mathcal{U})(p) = \overset{\lim}{\longleftarrow} \phi \circ M(\mathcal{U})(p)$. $\square$

In particular, since $M \circ S^*(\phi) = \overset{\lim}{\longleftarrow} \phi \circ M$, the mapping $\overset{\lim}{\longleftarrow}\phi : S^*(B,G) \to S^*(A,F)$ is uniformly continuous for each partition homomorphism $\phi : (A,F) \to (B,G)$.

Proposition 1.4.32 *Let* (B,F) *be a Boolean partition algebra. Then*

1. *Each principal ultrafilter is an isolated point of* $S^*(B,F)$
2. *If* (B,F) *is stable, then the principal ultrafilters of* (B,F) *are precisely the isolated points of* $S^*(B,F)$*.*

Proof. 1. Recall that $S^*(B,F)$ contains all the principal ultrafilters (see Example 1.3.38). If $\uparrow a$ is a principal ultrafilter, and $\mathcal{U} \in \iota(a)$, then $\uparrow a \subseteq \mathcal{U}$, so $\uparrow a = \mathcal{U}$. Therefore $\iota(a) = \{\uparrow a\}$ is an open set, so $\uparrow a$ is an isolated point.

2. We have already shown that the principal ultrafilters are isolated points in $S^*(B,F)$. If $\mathcal{U}$ is a nonprincipal ultrafilter and $\mathcal{U} \in \iota(a)$, then $a \in \mathcal{U}$, so a is not an atom. Therefore if $0 < b < a$, then $\emptyset \neq \iota(b) \subset \iota(a)$ since ι is injective. Therefore $\iota(a)$ contains more than one point. Therefore $\mathcal{U}$ is not an isolated point since $\{\iota(a) | a \in B^+\}$ is a basis for the topology on $S^*(B,F)$. $\square$

Definition 1.4.33 *A topological space* X *is called extremally disconnected if the closure of each open set is open.*

Clearly every regular extremally disconnected space is zero-dimensional, but extremal disconnectiveness is a far more restrictive condition than zero-dimensionality. In fact, by the following Lemma, a zero-dimensional space X is extremally disconnected if and only if the Boolean algebra $\mathfrak{B}(X)$ of clopen sets is complete.

Lemma 1.4.34 *Let $(X, \mathcal{A})$ be an algebra of sets, and assume that $\mathcal{A}$ is a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{A}$ is complete if and only if $(X, \mathcal{T})$ is extremally disconnected and $\mathcal{A}$ is the collection of all clopen subsets of X.*

Proof. See [21][p. 47] and [2][Lem 4.11]. $\square$

Corollary 1.4.35 *Let (X, M) be a partition space. The Boolean partition algebra $\mathfrak{B}^*(X, M)$ is complete if and only if (X, M) is extremally disconnected and $\mathfrak{B}^*(X, M)$ is the collection of all clopen subsets of X.*

2 Structure of Boolean Partition Algebras

In this chapter, we shall discuss the algebraic structure of Boolean partition algebras. This includes algebraic constructions common to many algebraic structures such as subalgebras, quotients, products, and direct limits. We shall also expound on algebraic constructions unique to Boolean partition algebras including subcompleteness and local refinability. Much of this theory developed in this chapter is useful for further study of Boolean partition algebras.

2.1 Subalgebras

In this section, we shall briefly discuss some simple results on subalgebras and some special types of subalgebras. In the upcoming sections, we shall see that subalgebras tend to preserve special properties of Boolean partition algebras such as stability, local refinability, and resplendence, and in some cases subcompleteness. Therefore subalgebras are useful in obtaining new Boolean partition algebras from old Boolean partition algebras.

Definition 2.1.1 *Assume (B, F) is a Boolean partition algebra, and A is a Boolean subalgebra of B. Then let $F|_A = \{p \in F | p \subseteq A\}$. In other words, $F|_A = F \cap P(A)$. Clearly each $p \in F|_A$ is a partition of A.*

Proposition 2.1.2 *1. $(A, F|_A)$ is a Boolean partition algebra.*

2. The inclusion map $\iota : (A, F|_A) \to (B, F)$ is a partition homomorphism.

3. If C is a subalgebra of A, then $F|_A|_C = F|_C$

Proof. 1. If $p, q \in F|_A$, then $p \wedge q = \{a \wedge b | a \in p, b \in q\}^+ \subseteq A$, so $p \wedge q \in F|_A$ as well. Moreover, if $p \in F|_A, q \in \mathbb{P}(A), p \preceq q$, then $q \subseteq A$ and q is a partition of B since $p \preceq q$. Therefore $q \in F$, so $q \in F|_A$. Furthermore, if $a \in A^+$, then $a \in \{a, a'\}^+ \in F|_A$, hence $(A, F|_A)$ is a Boolean partition algebra.

2. If $p \in F|_A$, then by definition $\iota[p] = p \in F$, so ι is a partition homomorphism.

3. We have $p \in F|_A|_C$ iff $p \subseteq C$ and $p \in F|_A$ iff $p \subseteq C$ and $p \subseteq A$ and $p \in F$ iff $p \subseteq C$ and $p \in F$ iff $p \in F|_C$. □

Proposition 2.1.3 *Let $(A, F), (B, G)$ be Boolean partition algebras, and let $f : A \to B$ be a partition homomorphism. Let $A_1 \subseteq A, B_1 \subseteq B$ be subalgebras such that $f[A_1] \subseteq B_1$, then the restriction mapping $\hat{f} : (A_1, F|_{A_1}) \to (B_1, F|_{B_1})$ is a partition homomorphism.*

Proof. Let $p \in F|_{A_1}$. Then $p \in F$ and $p \subseteq A_1$, so $\hat{f}[p]^+ \in G$ and $\hat{f}[p]^+ \subseteq B_1$, thus $\hat{f}[p]^+ \in G|_{B_1}$. □

Definition 2.1.4 *A Boolean subalgebra A of a Boolean partition algebra (B, F) is said to be a weak F-subalgebra if whenever $p \in F|_A, q \in F, p \preceq q$ we have $q \in F|_A$ as well. In other words, a Boolean subalgebra A of a Boolean partition algebra (B, F) is a weak F-subalgebra if and only if $F|_A$ is a filter on the meet-semilattice F.*

Proposition 2.1.5 *Let (B, F) be a Boolean partition algebra. Then a subalgebra $A \subseteq B$ is a weak F-subalgebra if and only if whenever $p \in F|_A, R \subseteq p$ and $\bigvee^B R$ exists, then $\bigvee^B R \in A$.*

Proof. $\to$ If $R = p$ or $R = \emptyset$, then clearly $\bigvee^B R \in A$. Otherwise, $\{\bigvee^B R, (\bigvee^B R)'\}$ is a partition of B refined by p. Since $\{\bigvee^B R, (\bigvee^B R)'\} \in F$, we have $\{\bigvee^B R, (\bigvee^B R)'\} \in F|_A$ as well. Therefore $\bigvee^B R \in A$.

$\leftarrow$ If $p \in F|_A, q \in F, p \preceq q$, then for each $b \in q$ we have $b = \bigvee^B \{a \in p | a \leq b\} \in A$. Therefore $q \subseteq A$, so $q \in F|_A$. □

Proposition 2.1.6 *Let A, B, C be Boolean algebras with $C \subseteq B \subseteq A$, and (A, F) a Boolean partition algebra.*

1. If C is a weak F-subalgebra, then C is also a weak $F|_B$-subalgebra.

2. Assume B is a weak F-subalgebra. Then C is a weak F-subalgebra iff C is a weak $F|_B$-subalgebra.

Proof. 1. Assume $p \in F|_C, q \in F|_B, p \preceq q$, then $q \in F$, so since C is a weak F-subalgebra we have $q \in F|_C$.

2. We have already shown $\rightarrow$. To prove $\leftarrow$ assume C is a weak $F|_B$-algebra. Now if $p \in F|_C, q \in F, p \preceq q$, then $p \subseteq C \subseteq B$, so $p \in F|_B$, and since B is an F-algebra, we have $q \in F|_B$ as well. Now since C is an $F|_B$-algebra, and $p \in F|_C, q \in F|_B, p \preceq q$ we have $q \in F|_C$ as well. □

Proposition 2.1.7 *Let (B, F) be a Boolean partition algebra and let A be a subalgebra of B. Then*

1. If $(A, F|_A)$ is subcomplete, then A is a weak F-subalgebra.

2. If (B, F) is subcomplete, then A is a weak F-subalgebra iff $(A, F|_A)$ is subcomplete.

Proof. 1. If $(A, F|_A)$ is subcomplete, then since the inclusion mapping $\iota : (A, F|_A) \rightarrow (B, F)$ is partitional, $\{\iota[p] | p \in F|_A\} = F|_A$ is a filter on $\mathbb{P}(B)$ by Theorem 1.3.32. Therefore A is a weak F-subalgebra.

2. We have already proven $\leftarrow$. To prove $\rightarrow$ assume $p \in F|_A$. Then for each $R \subseteq p$, the least upper bound $\bigvee^B R$ exists, and since A is a weak F-subalgebra, we have by Proposition 2.1.5 $\bigvee^B R \in A$, so $\bigvee^A R = \bigvee^B R$. Therefore $(A, F|_A)$ is subcomplete. □

Definition 2.1.8 *Let (B, F) be a Boolean partition algebra, then a subalgebra $A \subseteq B$ is said to be an F-extendible subalgebra if for each $p \in F$ there is a $q \in F|_A$ where $p \cap A \subseteq q$. In other words, a subalgebra $A \subseteq B$ is F-extendible if whenever $p \in F$ and $R \subseteq p, R \subseteq A$, then there is a $r \in F|_A$ with $R \subseteq r$.*

Example 2.1.9 *We give an example of a weak F-subalgebra that is not F-extendible. Let X be an infinite compact zero-dimensional space. Recall that $\mathfrak{B}(X)$ is the collection of all clopen subsets of X. Clearly $\mathfrak{B}(X)$ is a subalgebra of $\mathcal{P}(X)$. If $p \in$*

$\mathbb{P}P(X), p \subseteq \mathfrak{B}(X)$, then p is a partition of X into clopen sets, so since X is compact, p is finite. If $q \in \mathbb{P}P(X), p \preceq q$, then q is a partition of X into finitely many clopen sets, so $q \subseteq \mathfrak{B}(X)$. Therefore $\mathfrak{B}(X)$ is a weak $\mathbb{P}P(X)$-subalgebra of $P(X)$. On the other hand, since $\mathfrak{B}(X)$ is an infinite Boolean algebra, there is an infinite partition c of the Boolean algebra $\mathfrak{B}(X)$. In particular, $\bigcup c$ is a dense open subset of X. If we let d be a partition of $P(X)$ that extends c, then $d \cap \mathfrak{B}(X) = c$. However since $\mathbb{P}P(X)|_{\mathfrak{B}(X)}$ consists of only finite partitions, there is no $q \in \mathbb{P}P(X)|_{\mathfrak{B}(X)}$ with $c \subseteq q$. Therefore $\mathfrak{B}(X)$ is a weak $\mathbb{P}P(X)$-subalgebra, but $\mathfrak{B}(X)$ is not $\mathbb{P}P(X)$-extendible.

Example 2.1.10 *We shall give an example of an F-extendible Boolean partition algebra that is not a weak F-subalgebra. Let X be an infinite set, and let $A \subseteq P(X)$ be the subalgebra consisting of all finite and cofinite sets. Clearly A is not a weak $\mathbb{P}P(X)$-subalgebra of $\mathcal{P}(X)$. On the other hand, A is a $\mathbb{P}P(X)$-extendible subalgebra of $\mathcal{P}(X)$ since if $p \in \mathcal{P}(X)$ and $R \subseteq p \cap A$, then $R \cup \{\{x\}|x \in X \setminus R\}$ is a partition of the set X with $R \cup \{\{x\}|x \in X \setminus R\} \subseteq A$.*

Proposition 2.1.11 *Let A, B, C be Boolean algebras with $C \subseteq B \subseteq A$ and let (A, F) be a Boolean partition algebra. Then*

1. *If C is F-extendible, then C is $F|_B$-extendible.*
2. *If B is F-extendible, then C is F-extendible iff C is $F|_B$-extendible.*

Proof. 1. Assume $p \in F|_B$, then $p \in F$, so since C is F-extendible, there is a $q \in F|_C$ with $p \cap C \subseteq q$.

2. We have already shown $\rightarrow$. To prove $\leftarrow$ assume $p \in F$ and $R \subseteq p \cap C$. Then since $R \subseteq p \cap B$, there is a partition $q \in F|_B$ with $R \subseteq q$. Therefore since C is $F|_B$-extendible, there is an $r \in F|_C$ with $R \subseteq r$. Therefore C is F-extendible. $\square$

Definition 2.1.12 *Let (B, F) be a Boolean partition algebra. Then a subalgebra $A \subseteq B$ is an F-subalgebra if whenever $p \in F$ and $R \subseteq p \cap A$ and $\bigvee^B R$ exists, then $\bigvee^B R \in A$.*

One can easily see that the arbitrary intersection of F-subalgebras of a Boolean partition algebra is an F-subalgebra. Furthermore, it is easy to see that every F-subalgebra is a weak F-subalgebra.

Proposition 2.1.13 *Let (A, F) be a Boolean partition algebra, and let B, C be subalgebras of A with $C \subseteq B \subseteq A$. Then*

1. *If C is an F-subalgebra of A, then C is an $F|_B$-subalgebra*
2. *If B is an F-subalgebra, then C is an F-subalgebra iff C is an $F|_B$-subalgebra.*

Proof. 1. If $p \in F|_B$ and $R \subseteq p \cap C$ and $\bigvee^B R$ exists, then $\bigvee^A R = \bigvee^B R$, so $\bigvee^B R = \bigvee^A R \in C$ since C is an F-subalgebra.

2. We only need to show that if C is an $F|_B$-subalgebra, and B is an F-subalgebra, then C is an F-subalgebra. Let $p \in F, R \subseteq p \cap C$ and assume $\bigvee^A R$ exists. Then $R \subseteq p \cap B$, so since B is an F-subalgebra, we have $\bigvee^A R \in B$, thus $\bigvee^B R = \bigvee^A R$. Therefore, if $q = (R \cup \{(\bigvee^A R)'\})^+$, then $q \in F$ and $q \subseteq B$, so $q \in F|_B$. Therefore, since $R \subseteq q \cap C, q \in F|_B, \bigvee^B R$ exists, then since C is an $F|_B$-subalgebra, we have $\bigvee^B R = \bigvee^A R \in C$. □

Proposition 2.1.14 *If (B, F) is a Boolean partition algebra, then every F-extendible weak F-subalgebra is an F-subalgebra.*

Proof. Let $A \subseteq B$ be an F-extendible weak F-subalgebra. Then whenever $p \in F$ and $R \subseteq p \cap A$ and $\bigvee^B R$ exists, then there is a $r \in F|_A$ with $R \subseteq r$. Therefore since A is a weak F-subalgebra and $r \in F|_A, R \subseteq r$ and $\bigvee^B R$ exists, we have $\bigvee^B R \in A$. □

Proposition 2.1.15 *If (B, F) is subcomplete, then every F-subalgebra is F-extendible.*

Proof. Let A be an F-subalgebra of (B, F). Let $p \in F$ and let $R \subseteq p \cap A$. Then $\bigvee^B R$ exists, so $\bigvee^B R \in A$, and $(\bigvee^B R)' \in A$. Therefore $R \cup \{(\bigvee^B R)'\}$ is a partition and $R \cup \{(\bigvee^B R)'\} \succeq p$, hence $R \cup \{(\bigvee^B R)'\} \in F$ and $R \cup \{(\bigvee^B R)'\} \subseteq A$. □

Therefore if (B, F) is subcomplete, then the F-subalgebras are precisely the F-extendible weak F-subalgebras.

Theorem 2.1.16 *Let λ be a cardinal, and let B be a λ-Boolean algebra. Then a subalgebra $A \subseteq B$ is a λ-subalgebra if and only if A is a $\mathbb{P}_\lambda(B)$-subalgebra.*

Proof. $\rightarrow$ Assume A is a λ-subalgebra of B. Let $p \in \mathbb{P}_\lambda(B), R \subseteq p \cap A$ and assume $\bigvee^B R$ exists. Then since $|R| < \lambda$ we have $\bigvee^B R \in A$ as well.

$\leftarrow$ Assume A is a $\mathbb{P}_\lambda(B)$-subalgebra. Then we shall show that if $R \subseteq A, |R| < \lambda$, then $\bigvee R \in A$ by transfinite induction on the cardinality of R. Assume $|R| = \mu$ and assume that $\bigvee^B S \in A$ whenever $|S| < \mu, S \subseteq A$. Let $R = \{a_\alpha : \alpha < \mu\}$ be a well ordering. Then let $b_\alpha = a_\alpha \wedge (\bigvee_{\beta<\alpha} a_\beta)'$ for $\alpha < \mu$. Then $\bigvee_{\beta<\alpha} a_\beta \in A$ for each $\alpha < \mu$, so $b_\alpha \in A$ for $\alpha < \mu$. Furthermore $\{b_\alpha|\alpha < \mu\}^+$ is cellular. Let $p = (\{b_\alpha|\alpha < \mu\} \cup \{(\bigvee_{\alpha<\mu} b_\alpha)'\})^+$. Then $|p| \leq \mu < \lambda$, so $p \in \mathbb{P}_\lambda(B)$. Therefore since $\{b_\alpha|\alpha < \mu\}^+ \subseteq A \cap p$ and A is a $\mathbb{P}_\lambda(B)$-subalgebra, we have $\bigvee R = \bigvee_{\alpha<\mu} a_\alpha = \bigvee^B \{b_\alpha|\alpha < \mu\}^+ \in A$. $\square$

2.2 Filters, ultrafilters, and quotients.

Let (B, F) be a Boolean partition algebra. We shall now generalize the notion of an F-ultrafilter to filters so that we can take quotients of the Boolean partition algebra (B, F).

Recall that if Z is a filter on a Boolean algebra, then $\pi_Z : B \rightarrow B/Z$ is the canonical mapping where $\pi_Z(b) = b/Z$ for each $b \in B$. One can easily see that an ultrafilter $\mathcal{U}$ is an F-ultrafilter if and only if $1 \in \pi_\mathcal{U}[p]$. In other words, $\mathcal{U}$ is an F-ultrafilter if and only if $\pi_\mathcal{U}[p]^+$ is a partition of $B/\mathcal{U}$ for all $p \in F$. This perspective motivates the following definition.

Definition 2.2.1 *An F-filter is a filter Z such that the mapping $\pi_Z : B \rightarrow B/Z$ is partitional. Similarly, an ideal I is an F-ideal if $\pi_I : B \rightarrow B/I$ is partitional.*

If (B, F) is a Boolean partition algebra and Z is an F-filter, then let $F/Z \subseteq \mathbb{P}(B/Z)$ be the set where $q \in F/Z$ if and only if there is a $p \in F$ with $\pi_Z[p]^+ \preceq q$.

Proposition 2.2.2 *If (B, F) is a Boolean partition algebra, then $(B/Z, F/Z)$ is a Boolean partition algebra as well.*

Proof. Clearly $\{\pi_Z[p]^+|p \in F\}$ is a filterbase on $\mathbb{P}(B/Z)$. Therefore F/Z is the filter on $\mathbb{P}(B/Z)$ generated by $\{\pi_Z[p]^+|p \in F\}$. If $b \in B$ and $b/Z \in (B/Z) \setminus \{0/Z\}$, then

$b \neq 0$, so there is a $p \in F$ with $b \in p$. Therefore $b/Z \in \pi_Z[p]^+ \in F/Z$. Hence $(B/Z, F/Z)$ is a Boolean partition algebra. $\square$

Definition 2.2.3 *We shall also denote the Boolean partition algebra $(B/Z, F/Z)$ by $(B, F)/Z$. We shall call $(B, F)/Z$ the quotient of (B, F) by the filter Z. Quotients of Boolean partition algebras by F-ideals are defined analoguously.*

Example 2.2.4 *Let* **fin** *denote the collection of all finite subsets of $\mathbb{N}$. Then* **fin** *is clearly an ideal on $\mathbb{N}$. If P is an infinite partition of $\mathbb{N}$, then let $A \subseteq \mathbb{N}$ be a set with $|A \cap R| = 1$ for $R \in P$. Then $A/\mathbf{fin} \neq 0$, but $A/\mathbf{fin} \wedge R/\mathbf{fin} = 0$ for $R \in P$. Therefore $\{R/\mathbf{fin} | R \in P\} = \pi_{\mathbf{fin}}[P]^+$ is not a partition since it is not a maximal cellular family. We conclude that if $(P(\mathbb{N}), F)$ is a Boolean partition algebra and F contains an infinite partition, then* **fin** *is not an F-filter.*

Example 2.2.5 *Let B be a Boolean algebra. Then every filter $Z \subseteq B$ is a $\mathbb{P}_\omega(B)$-filter. Furthermore, if κ is a cardinal and B is a κ-Boolean algebra, then it is easy to see that every κ-filter is a $\mathbb{P}_\kappa(B)$-filter. In the next chapter, we shall prove that every $\mathbb{P}_\kappa(B)$-filter is a κ-filter.*

If (B, F) is subcomplete and Z is an F-filter, then $F/Z = \{\pi_Z[p]^+ | p \in F\}$ and each $\pi_Z[p]^+$ is subcomplete, so $(B, F)/Z$ is also subcomplete. In essense, the quotient of a subcomplete Boolean partition algebra is subcomplete.

Theorem 2.2.6 *Let (A, F) be a Boolean partition algebra and B be a Boolean algebra, and let $f : (A, F) \to B$ be a partitional mapping. Then*

1. $\ker(f)$ *is an F-ideal on A.*

2. Assume (B, G) is a Boolean partition algebra, and f is a partition homomorphism from (A, F) to (B, G) and $I \subseteq \ker(f)$ is an F-ideal. Let $\hat{f} : (A/I, F/I) \to (B, G)$ be the unique function with $f = \hat{f}\pi_I$. Then $\hat{f}$ is a partition homomorphism.

3. Let $f : (A, F) \to (B, G)$ be a partition homomorphism and let $\hat{f} : (A, F)/\ker(f) \to (B, G)$ be the unique mapping with $f = \hat{f}\pi_{\ker(f)}$. Then $\hat{f}$ is an isomorphism if and only if f is surjective and for each $q \in G$ there is a $p \in F$ with $q \succeq f[p]^+$.

Proof. 1. Let $\hat{f} : A/\ker(f) \to B$ be the unique mapping with $\hat{f}\pi_{\ker(f)} = f$. If $p \in F$, then $f[p]^+ = \hat{f}[\pi_{\ker(f)}[p]^+]^+$ is a partition of B. Therefore since $\hat{f}$ is injective, $\pi_{\ker(f)}[p]^+$ is a partition of $A/\ker(f)$.

2. If $p \in F/I$, then $\pi_I[q]^+ \preceq p$ for some $q \in F$, but $f[q]^+ = \hat{f}[\pi_I''[q]^+]^+ \preceq \hat{f}[p]^+$, and since $f[q]^+ \in G$, we have $\hat{f}[p]^+ \in G$ as well. Therefore $\hat{f}$ is a partition homomorphism.

3. $\to$ Assume $\hat{f}$ is a partition isomorphism. Then for $q \in G$ there is a $q_1 \in F/\ker(f)$ with $q = \hat{f}[q_1]$, so there is a $p \in F$ with $\pi_{\ker(f)}[p]^+ \preceq q_1$, thus $f[p]^+ = \hat{f}[\pi_{\ker(f)}[p]^+] \preceq \hat{f}[q_1] = q$. Furthermore, since $f = \hat{f}\pi_{\ker(f)}$, the function f is surjective being the composition of two surjective functions.

$\leftarrow$ Since f is surjective and $f = \hat{f}\pi_{\ker(f)}$, the function $\hat{f}$ is surjective as well, so $\hat{f}$ is bijective. Let $q \in G$. Then there is a $p \in F$ with $q \succeq f[p]^+ = \hat{f}[\pi_{\ker(f)}[p]^+]$, thus $(\hat{f}^{-1})[q] \succeq \pi_{\ker(f)}[p]^+ \in F/\ker(f)$. Therefore $\hat{f}^{-1}$ is a partition homomorphism, so $\hat{f}$ is a partition isomorphism. □

Example 2.2.7 *Let (B, F) be a Boolean partition algebra. Then the mapping $\alpha_a : (B, F) \to B \restriction a$ is a partition homomorphism, so $Ker(\alpha_a) = \downarrow a$ is an F-ideal.*

Theorem 2.2.8 (Correspondence Theorem) *Let (B, F) be a Boolean partition algebra, and let I be an F-ideal. Then an ideal $J \subseteq B/I$ is an F/I-ideal if and only if $\pi_I^{-1}[J]$ is an F-ideal.*

Proof. $\to$ If $J \subseteq B/I$ is an F/I-ideal, then $\pi_I^{-1}[J] = \pi_I^{-1}[\pi_J^{-1}[\{0\}]] = (\pi_J \circ \pi_I)^{-1}[\{0\}] = \ker(\pi_J \circ \pi_I)$ is an F-ideal since $\pi_J \circ \pi_I$ is partitional.

$\leftarrow$ Let $K = \pi_I^{-1}(J)$. Then I, K are F-ideals with $I \subseteq K$. Therefore since $I \subseteq \ker(\pi_K)$, there is a partition homomorphism $f : (B, F)/I \to (B, F)/K$ with $\pi_K = f\pi_I$. However $f(a+I) = 0$ iff $\pi_K(a) = f(\pi_I(a)) = 0$ iff $a \in K = \pi_I^{-1}(J)$ iff $a+I = \pi_I(a) \in J$. Therefore $J = \ker(f)$ is an F-ideal. □

We therefore have a one-to-one correspondence between the F/I-ideals and the F-ideals containing I.

Proposition 2.2.9 *Let (B, F) be a Boolean partition algebra. Then a proper F-filter Y is an F-ultrafilter if and only if whenever Z is a proper F-filter extending Y we* *have $Y = Z$.*

Proof. $\rightarrow$ This is because an F-ultrafilter is an ultrafilter.

$\leftarrow$ If Y is not an F-ultrafilter, then $(B, F)/Y$ is a Boolean partition algebra with more than two elements, so let $a \in (B/Y) \setminus \{0, 1\}$. Then $\uparrow a$ is a nontrivial proper F/Y-filter, so $\pi_Y^{-1}(\uparrow a)$ is a proper F-filter that properly extends Y. $\square$

Proposition 2.2.10 *If (B, F) is a Boolean partition algebra, then the intersection of F-ideals is an F-ideal.*

Proof. Let A be an index set, and let I_α be an F-ideal for $\alpha \in A$. Then for $\alpha \in A$ there is some Boolean algebra C_α and a partitional mapping $\phi_\alpha : (B, F) \rightarrow C_\alpha$ with kernel I_α. Let $C = \prod_{\alpha \in A} C_\alpha$ and let $\phi : B \rightarrow C$ be the mapping with $\phi(b) = (\phi_\alpha(b))_{\alpha \in A}$ for $\alpha \in A$. Then ϕ is a Boolean algebra homomorphism. Furthermore, if $p \in F$ and $(x_\alpha)_{\alpha \in A}$ is an upper bound of $\phi[p]$, then for each $a \in p, \alpha \in A$ we have $x_\alpha \geq \phi_\alpha(a)$. Therefore for $\alpha \in A$, the element x_α is an upper bound of $\phi_\alpha[p]^+$. Therefore $x_\alpha = 1$ for $\alpha \in A$. This implies $(x_\alpha)_{\alpha \in A} = 1$, so $\bigvee \phi[p]^+ = 1$. Thus $\phi[p]^+$ is a partition. Therefore ϕ is a partitional mapping, so $\ker(\phi) = \bigcap_{\alpha \in A} I_\alpha$ is an F-ideal. $\square$

We shall later discuss alternative characterizations of the notion of an F-ideal, and with such characterizations of F-ideals, it becomes trivial that the intersection of F-ideals is an F-ideal.

From the above proposition, one can see that the collection of F-ideals forms a complete lattice.

Definition 2.2.11 *We shall write Id(B, F) for the complete lattice of F-ideals on B and Fi(B, F) for the complete lattice of F-filters on B.*

The stability of a Boolean partition algebra (B, F) only depends on the lattice $\mathrm{Fi}(B, F)$. In particular, (B, F) is stable if and only if $\{1\}$ is the greatest lower bound of the maximal elements in $\mathrm{Fi}(B, F) \setminus \{B\}$. Therefore since the lattice $\mathrm{Fi}((B, F)/Z)$ is isomorphic to the lattice $\{V \in \mathrm{Fi}(B, F) | Z \subseteq V\}$, the Boolean partition algebra $(B, F)/Z$ is stable if and only if Z is the greatest lower bound of maximal elements in $\{V \in \mathrm{Fi}(B,F) | Z \subseteq V\} \setminus \{B\}$. In other words, $(B,F)/Z$ is stable if and only if Z is the intersection of F-ultrafilters.

Definition 2.2.12 *We shall call $(B,F)/\bigcap S^*(B,F)$ the stabilization of (B,F). An F-filter Z on a Boolean algebra B is said to be stabilizing if $(B,F)/Z$ is stable. In other words, Z is stabilizing if Z is the intersection of F-ultrafilters.*

Clearly the collection of stabilizing filters on (B,F) form a complete lattice.

Example 2.2.13 *Not every F-filter on a stable Boolean partition algebra (B,F) is stabilizing. Therefore the quotient of a stable Boolean partition algebra is not necessarily stable. For instance, consider the Boolean partition algebra $(P([0,1]), \mathbb{P}_{\aleph_1}(P([0,1])))$. Then the $\mathbb{P}_{\aleph_1}(P([0,1]))$-ultrafilters are the $\aleph_1$-complete ultrafilters on $P([0,1])$. On the other hand, since $[0,1]$ is compact, every ultrafilter must converge to a point in $[0,1]$. Therefore if $\mathcal{U}$ is an $\aleph_1$-complete ultrafilter on $[0,1]$, then $\mathcal{U} \to x$ for some $x \in [0,1]$, so $\{x\} = \bigcap_n [0,1] \cap (x - \frac{1}{n}, x + \frac{1}{n}) \in \mathcal{U}$. In other words, the $\aleph_1$-complete ultrafilters are all principal. Thus the only stabilizing filters are the principal filters, so there is an abundance of $\mathbb{P}_{\aleph_1}(P(\mathbb{R}))$-filters that are not stabilizing.*

Definition 2.2.14 *We shall call a Boolean partition algebra (B,F) superstable if $(B,F)/Z$ is stable for each F-filter Z. In other words, (B,F) is superstable if each F-filter is stabilizing.*

One should note that like stability, superstability only depends on the lattice $\mathrm{Fi}(B,F)$. A Boolean partition algebra (B,F) is superstable if and only if every element in $\mathrm{Fi}(B,F)$ is the greatest lower bound of a collection of some maximal elements in $\mathrm{Fi}(B,F) \setminus \{B\}$. Furthermore, the following result shows that (B,F) is superstable if and only if every element in $\mathrm{Fi}(B,F) \setminus \{B\}$ is bounded above by a maximal element in $\mathrm{Fi}(B,F) \setminus \{B\}$.

Proposition 2.2.15 *A Boolean partition algebra (B,F) is superstable iff every proper F-filter is extendible to an F-ultrafilter.*

Proof. $\rightarrow$ If (B,F) is superstable, then every proper F-filter is the intersection of F-ultrafilters, so every proper F-filter is extendible to an F-ultrafilter.

$\leftarrow$ If Z is a proper F-filter, then for every $a \in B \backslash Z$ the set $\uparrow (a'/Z)$ is a proper F/Z-filter, so $\pi_Z^{-1}[\uparrow (a'/Z)]$ is a proper F-filter that extends Z and where $a' \in \pi_Z^{-1}[\uparrow (a'/Z)]$. Therefore if $\mathcal{U}$ is an F-ultrafilter that extends $(\pi_Z)^{-1}[\uparrow (a'/Z)]$, then $a' \in \mathcal{U}$, but $Z \subseteq \mathcal{U}$. Hence Z is the intersection of F-ultrafilters. $\square$

Definition 2.2.16 *If $(X, \mathcal{A})$ is an algebra of sets, then for each $x \in X$ let $\mathcal{C}(x) = \{R \in \mathcal{A} | x \in R\}$. It is easy to see that each $\mathcal{C}(x)$ is an ultrafilter on $\mathcal{A}$.*

Proposition 2.2.17 *Let X be a zero-dimensional space and let $\mathcal{A}$ be an algebra of sets that is a basis for X.*

1. $\mathcal{C}(x)$ is the only proper filter on $\mathcal{A}$ that converges to x.

2. A filter $F \subseteq \mathcal{A}$ accumulates at $x \in X$ iff $F \subseteq \mathcal{C}(x)$.

3. Let $\mathcal{U} \subseteq \mathcal{A}$ be an ultrafilter. Then $\mathcal{U} \to x$ if and only if $\mathcal{U} \propto x$ (recall that $\mathcal{U} \propto x$ means that $\mathcal{U}$ accumulates at x).

Proof. 1. Clearly $\mathcal{C}(x) \to x$. Similarly, if $F \subseteq \mathcal{A}$ is a filter that converges to x, then for each $C \in \mathcal{C}(x)$, there is a $R \in F$ with $R \subseteq C$. Since F is a filter, we have $C \in F$ as well. Therefore $\mathcal{C}(x) \subseteq F$. Since $\mathcal{C}(x)$ is an ultrafilter we have $F = \mathcal{C}(x)$.

2. The filter F accumulates at x iff $x \in \bigcap_{R \in F} \overline{R} = \bigcap F$ iff $F \subseteq \mathcal{C}(x)$.

3. An ultrafilter $\mathcal{U} \subseteq \mathcal{A}$ accumulates at $x \in X$ iff $\mathcal{U} \subseteq \mathcal{C}(x)$ iff $\mathcal{U} = \mathcal{C}(x)$ iff $\mathcal{U}$ converges to x. $\square$

Theorem 2.2.18 *A separating partition space (X, M) is complete with $\mathfrak{B}^*(X, M)$ superstable if and only if whenever $Z \subseteq \mathfrak{B}^*(X, M)$ is an M-filter, then $Z \propto x$ for some $x \in X$.*

Proof. $\rightarrow$ If (X, M) is complete and $\mathfrak{B}^*(X, M)$ is superstable, then whenever $Z \subseteq \mathfrak{B}^*(X, M)$ is an M-filter, then there is an M-ultrafilter $\mathcal{U}$ with $Z \subseteq \mathcal{U}$, but since (X, M) is complete we have $Z \subseteq \mathcal{U} = \mathcal{C}(x)$ for some $x \in X$. Therefore $Z \propto x$.

$\leftarrow$ If $Z \subseteq \mathfrak{B}^*(X, M)$ is an M-filter, then $Z \propto x$ for some $x \in X$, so $Z \subseteq \mathcal{C}(x)$. Therefore Z is extendible to an M-ultrafilter, thus $\mathfrak{B}^*(X, M)$ is superstable. To prove completeness let $\mathcal{U}$ be an M-ultrafilter. Then $\mathcal{U} \propto x$ for some $x \in X$, so $\mathcal{U} \subseteq \mathcal{C}(x)$, hence $\mathcal{U} = \mathcal{C}(x)$. Therefore (X, M) is complete. $\square$

Remark 2.2.19 *As a consequence of the above result, a partition space (X, M) is complete and $\mathfrak{B}^*(X, M)$ is superstable if and only if whenever $Z \subseteq \mathfrak{B}^*(X, M)$ is a proper M-filter, then $\bigcap Z \neq \emptyset$. A partition space (X, M) complete and $\mathfrak{B}^*(X, M)$ is superstable if and only if whenever $I \subseteq \mathfrak{B}^*(X, M)$ is a proper M-ideal, then $\bigcup I \neq X$. In other words, (X, M) is complete and $\mathfrak{B}^*(X, M)$ is superstable if and only if whenever $\mathcal{O}$ is an open cover for X, then if I is an M-ideal containing $\bigcup_{O \in \mathcal{O}} \{R \in \mathfrak{B}^*(X, M) | R \subseteq O\}$, then $I = \mathfrak{B}^*(X, M)$.*

Theorem 2.2.20 *If (B, F) is a Boolean partition algebra, $Z \subseteq \bigcap S^*(B, F)$, and $\pi_Z : (B, F) \to (B, F)/Z$ is the canonical mapping, then $S^*(\pi_Z)$ is a uniform homeomorphism.*

Proof. By the correspondence theorem (Theorem 2.2.8), the mapping $S^*(\pi_Z)$ is bijective and we know that $S^*(\pi_Z)$ is uniformly continuous. Therefore we must show that $S^*(\pi_Z)$ maps uniform partitions to uniform partitions. Since $S^*((B, F)/Z)$ is generated by partitions of the form $\iota[\pi_Z[p]^+]^+ = \iota\pi_Z[p]^+$ where $p \in F$, it suffices to show that $(S^*(\pi_Z)'')''(\iota\pi_Z[p]^+) = [(S^*(\pi_Z)'')''(\iota\pi_Z[p])]^+$ is a uniform partition of $S^*(B, F)$ for $p \in F$. On the other hand, we have $\mathcal{U} \in S^*(\pi_Z)[\iota\pi_Z(a)]$ iff $S^*(\pi_Z)^{-1}(\mathcal{U}) \in \iota(\pi_Z(a))$ iff $\pi_Z(a) \in S^*(\pi_Z)^{-1}(\mathcal{U})$ iff $a \in S^*(\pi_Z)S^*(\pi_Z)^{-1}(\mathcal{U}) = \mathcal{U}$ iff $\mathcal{U} \in \iota(a)$. Therefore $S^*(\pi_Z)[\iota\pi_Z(a)] = \iota(a)$. Hence $[(S^*(\pi_Z)'')''(\iota\pi_Z[p])]^+ = \{S^*(\pi_Z)[\iota\pi_Z(a)] | a \in p\}^+ = \{\iota(a) | a \in p\}^+ = \iota[p]^+$ is a uniform partition of $S^*(B, F)$. Therefore $S^*(\pi_Z)$ is uniformly continuous. $\square$

We shall find a one-to-one correspondence between the closed subsets of $S^*(B, F)$ and the stabilizing filters in (B, F) using closure operators and Galois connections. The following basic results on closure operators and Galois connections are well known [8].

Definition 2.2.21 *Let X be a poset. Then a function $C : X \to X$ is a closure operator if*

1. *If $x \leq y$, then $C(x) \leq C(y)$.*
2. *$x \leq C(x) = C(C(x))$ for $x \in X$.*

A closure system on a poset X is a subset $C \subseteq X$ such that for each $x \in X$ there is a least $c \in C$ with $x \leq c$. If C is a closure operator on X, then let $C^* = C[X]$. One can easily see that $C^* = \{x \in X | x = C(x)\}$. If C is a closure system, then let $C^* : X \to X$ be the mapping where if $x \in X$, then $C^*(x)$ is the least element in C greater than x.

Proposition 2.2.22 *Let X be a poset. Then*

1. *If C is a closure operator, then C^* is a closure system.*
2. *If C is a closure system, then C^* is a closure operator.*
3. *If C is a closure operator, then $C = C^{**}$.*
4. *If C is a closure system, then $C = C^{**}$.*

Proof. 1. For each $x \in X$ we have $x \leq C(x) \in C^*$, but if $y \in C^*$ and $x \leq y$, then $C(x) \leq C(y) = y$. Therefore C^* is a closure system.

2. Assume C is a closure system. Clearly $x \leq C^*(x)$ for all $x \in X$ and if $x \leq y$, then $x \leq y \leq C^*(y)$ and since $C^*(y) \in C$ we have $C^*(x) \leq C^*(y)$. If $x \in X$, then $C^*(x) \in C$, so $C^*(C^*(x)) = C^*(x)$.

3. If C is a closure operator, then $x \leq C(x)$ and if $y \in C^*$ and $x \leq y$, then $C(x) \leq C(y) = y$. Therefore $C(x)$ is the least element in C^* with $x \leq C(x)$. Therefore $C(x) = C^{**}(x)$.

4. If C is a closure system, then $x \in C$ iff $C^*(x) = x$ iff $x \in C^{**}$. □

Proposition 2.2.23 *Let X be a complete lattice. Then a subset $C \subseteq X$ is a closure system if and only if C is closed under arbitrary intersection including empty intersection.*

Proof. $\leftarrow$ If C is closed under arbitrary intersection, then for each $x \in X$ we have that $\bigwedge\{y \in C | x \leq y\}$ is the least element in C greater than x.

$\rightarrow$ If C is a closure system and $x_i \in C$ for $i \in I$, then $C^*(\bigwedge_{i \in I} x_i) \leq C^*(x_i) = x_i$ for $i \in I$, so $C^*(\bigwedge_{i \in I} x_i) = \bigwedge_{i \in I} x_i$. Therefore $\bigwedge_{i \in I} x_i \in C$ as well. □

Definition 2.2.24 *Let X, Y be two partially ordered sets. Then a Galois connection is a pair of mappings $f : X \to Y, g : Y \to X$ such that $y \leq f(x)$ if and only if $x \leq g(y)$.*

Galois connections are used to construct one-to-one correspondences.

Proposition 2.2.25 *Let X, Y be posets with a Galois connection $f : X \to Y, g : Y \to X$.*

1. *$x \leq g(f(x))$ and $y \leq f(g(y))$ for $x \in X, y \in Y$.*
2. *If $x_1 \leq x_2$, then $f(x_2) \leq f(x_1)$. If $y_1 \leq y_2$, then $g(y_2) \leq g(y_1)$.*
3. *If $x \in X, y \in Y$, then $f(x) = f(g(f(x)))$ and $g(y) = g(f(g(y)))$.*
4. *The functions $f \circ g, g \circ f$ are closure operators.*

Proof. 1. Clearly $f(x) \leq f(x)$, so $x \leq g(f(x))$.

2. We have $x_1 \leq x_2 \leq g(f(x_2))$, thus $f(x_2) \leq f(x_1)$.

3. If $x \in X$, then $x \leq g(f(x))$, so $f(g(f(x))) \leq f(x)$ and clearly $f(x) \leq f(g(f(x)))$.

4. If $x \in X$, then we have already shown that $x \leq g(f(x))$. If $x_1 \leq x_2$, then $f(x_2) \leq f(x_1)$, so $g(f(x_1)) \leq g(f(x_2))$. If $x \in X$, then $f(x) = f(g(f(x)))$, hence $g(f(x)) = g(f(g(f(x))))$. Therefore $g \circ f$ is a closure operator. □

Given a Galois connection $f : X \to Y, g : Y \to X$, let $C = g \circ f$ and $D = f \circ g$. If $x \in X$, then $f(x) = f(g(f(x))) = D(f(x)) \in D^*$. Similarly, if $y \in Y$, then $g(y) \in C^*$. Therefore we candefine mappings $f^* : C^* \to D^*, g^* : D^* \to C^*$ by $f^*(x) = f(x), g^*(y) = g(y)$.

Proposition 2.2.26 *The functions f^* and g^* are inverses.*

Proof. If $x \in C^*$, then $x = C(x) = g(f(x)) = g^*(f^*(x))$. If $y \in D^*$, then $y = D(y) = f(g(y)) = f^*(g^*(y))$. □

Since f maps C^* bijectively onto D^* and g maps D^* bijectively onto C^*, we have $D^* = f[X]$ and $C^* = g[Y]$.

Let A, B be sets, and let $\mathcal{R} \subseteq A \times B$. Then for $R \subseteq A$, let $f(R) = \{b \in B | (a, b) \in \mathcal{R}\,\text{for all}\,a \in R\}$ and for $S \subseteq B$ let $g(S) = \{a \in A | (a, b) \in \mathcal{R}\,\text{for all}\,b \in S\}$. One can easily show that the mappings f and g form a Galois connection between the posets $P(A)$ and $P(B)$.

Definition 2.2.27 *Let (B,F) be a Boolean partition algebra. Let $f_T : P(B) \to P(S^*(B,F)), g_T : P(S^*(B,F)) \to P(B)$ be the mappings where $f_T(R) = \{\mathcal{U} \in S^*(B,F) | a \in \mathcal{U} \text{ for all } a \in R\}$ and $g_T(S) = \{a \in B | a \in \mathcal{U} \text{ for all } \mathcal{U} \in S\}$. In other words, $f_T(R) = \{\mathcal{U} \in S^*(B,F) | R \subseteq \mathcal{U}\}$ and $f_T(S) = \bigcap S$ for $R \subseteq B, S \subseteq S^*(B,F)$. The mappings f_T and g_T form a Galois connection between the posets $P(B)$ and $P(S^*(B,F))$. Let $C_T = g_T \circ f_T, D_T = f_T \circ g_T$, and let f_T^*, g_T^* be the restrictions of f_T and g_T to C_T^* and D_T^*.*

Take note that $C_T^* = g_T[P(S^*(B,F))]$ is the collection of all intersections $\bigcap S$ where $S \subseteq S^*(B,F)$, so C_T^* is the collection of all stabilizing filters on (B,F).

Theorem 2.2.28 *D_T is the closure operator for the topology on $S^*(B,F)$.*

Proof. Take note that D_T^* is the collection of all sets of the form $f_T(R)$ where $R \subseteq B$. We have $f_T(R) = \{\mathcal{U} \in S^*(B,F) | R \subseteq \mathcal{U}\} = \bigcap_{a \in R}\{\mathcal{U} \in S^*(B,F) | a \in \mathcal{U}\} = \bigcap_{a \in R} \iota(a)$, but the closed sets in the topology on $S^*(B,F)$ are precisely the sets of the form $\bigcap_{a \in R} \iota(a)$. □

Therefore the mappings $f_T^* : C_T^* \to D_T^*, g_T^* : D_T^* \to C_T^*$ are one-to-one correspondences between the closed subsets of $S^*(B,F)$ and the stabilizing filters on (B,F). Particularly, if (B,F) is superstable, then the closed subsets of $S^*(B,F)$ are in a one-to-one correspondence with the F-filters on B.

Proposition 2.2.29 *If (B,F) is a Boolean partition algebra, then $S \subseteq S^*(B,F)$ is dense if and only if $\bigcap S = \bigcap S^*(B,F)$.*

Proof. We have $\bigcap S = \bigcap S^*(B,F)$ iff $g(S) = \bigcap S^*(B,F)$ iff $f(g(S)) = X$ iff $\overline{S} = X$. □

In particular, if (B,F) is stable, then $S \subseteq S^*(B,F)$ is dense iff $\bigcap S = \{1\}$.

Proposition 2.2.30 *A stable Boolean partition algebra (B,F) is atomic if and only if the isolated points are dense in $S^*(B,F)$.*

Proof. For this proof, let V be the set of all atoms in B. If (B,F) is stable, then the principal ultrafilters are precisely the isolated points of $S^*(B,F)$ (see Proposition 1.4.32). Therefore (B,F) is atomic iff $\bigcup_{a\in V}\uparrow a = B^+$ iff $\bigcap_{a\in V}\uparrow a=\{1\}$ iff the isolated points are dense in $S^*(B,F)$. $\square$

We shall now show that every Boolean partition algebra is isomorphic to a quotient of a stable Boolean partition algebra.

Lemma 2.2.31 *Assume A is a Boolean algebra and (B,G) is a Boolean partition algebra. Let $\phi : A \to B$ be a Boolean algebra homomorphism, and let F be the collection of partitions p of the Boolean algebra A such that $\phi[p]^+ \in G$. Then (A,F) is a Boolean partition algebra.*

Proof. Clearly F contains every finite partition. If $p\in F$ and q is a partition with $p\preceq q$, then clearly $\phi[q]^+$ is cellular and $\phi[p]^+\preceq\phi[q]^+$, so $\phi[q]^+$ is a partition of B, so $q\in F$ as well. If $p,q\in F$, then $\bigvee\phi[p\wedge q]^+=\bigvee\phi[p\wedge q]=\bigvee_{a\in p,b\in q}\phi(a\wedge b)=\bigvee_{a\in p,b\in q}\phi(a)\wedge\phi(b)=1$. Therefore $\phi[p\wedge q]^+$ is a partition of B. Furthermore, $\phi[p\wedge q]^+=\phi[p]^+\wedge\phi[q]^+\in G$. Therefore $p\wedge q\in F$. We conclude that (A,F) is a Boolean partition algebra. $\square$

Theorem 2.2.32 *Let (A,F) be a Boolean partition algebra. Then there is an atomic Boolean partition algebra (B,G) and a surjective partition homomorphism $\phi:(B,G)\to(A,F)$ such that $F=\{\phi[p]^+|p\in G\}$. In particular, every Boolean partition algebra is isomorphic to a quotient of a stable Boolean partition algebra.*

Proof. Recall that if $p\in F$, then $p^*=\{\bigvee R|R\subseteq p,\text{ and }\bigvee R\text{ exists}\}$. Take note that A is the direct limit of the system $\{p^*|p\in F\}$ of Boolean algebras. Let $B\subseteq\prod_{p\in F}p^*$ be the collection of all $(x_p)_{p\in F}$ such that there is a $p\in F$ with $x_q=x_p$ whenever $q\preceq p$. In other words, B is the collection of all families that are eventually constant. Take note that if we give B the discrete topology, then B is the collection of all nets $(x_p)_{p\in F}\in\prod_{p\in F}p^*$ that converge. Therefore define a mapping $\phi:B\to A$ by letting $\phi(x_p)_{p\in F}=x$ if $x_p\to x$. Clearly ϕ is a surjective Boolean algebra homomorphism.

Let G be the collection of all partitions r of B such that $\phi[r]^+ \in F$. Then (B, G) is a Boolean partition algebra by Lemma 2.2.31 and ϕ is a partition homomorphism from (B, G) to (A, F). If $p \in F$, then for each $b \in p$ let $x_{b,q} = b$ whenever $q \preceq p$ and let $x_{b,q} = 0$ whenever $q \not\preceq p$. Then $(x_{b,q})_{q \in F} \in B$ and $\phi(x_{b,q})_{q \in F} = b$. Furthermore, if $b, c \in p$, then $(x_{b,q})_{q \in F} \wedge (x_{c,q})_{q \in F} = (x_{b,q} \wedge x_{c,q})_{q \in F} = 0$. In other words, the family $\{(x_{b,q})_{q \in F} | b \in p\}$ is cellular and $\phi[\{(x_{b,q})_{q \in F} | b \in p\}] = p$. Therefore if $\hat{p}$ is a partition of B that extends the cellular family $\{(x_{b,q})_{q \in F} | b \in p\}$, then $\phi[\hat{p}]^+ = p$, and $\hat{p} \in G$. Hence $F = \{\phi[p]^+ | p \in G\}$.

In order to show (B, G) is stable it suffices to show that B is atomic (see Example 1.3.38). Let $(x_p)_{p \in F} \in B^+$. Then for some $p \in F$ we have $x_p > 0$, so since $x_p \in (p^*)^+$ there is an $a \in p$ with $a \leq x_p$. Therefore if $y_p = a$ and $y_q = 0$ whenever $q \neq p$. Hence $(y_p)_{p \in F} \leq (x_p)_{p \in F}$ and clearly $(y_p)_{p \in F}$ is an atom. □

2.3 Subcompleteness

We begin this section by discussing the subcompletion of a Boolean partition algebra. We shall then give a relation between subcomplete Boolean partition algebras and point-free inverse limits where all transition mappings are surjective. Indeed, the category of subcomplete Boolean partition algebras is equivalent to the category of all inverse systems with surjective transitional mappings, but we shall not be able to prove this equivalence of categories here. We conclude this section with a discussion of basic properties of subcomplete Boolean partition algebras.

Definition 2.3.1 *Let (A, F) be a Boolean partition algebra. Then a subcompletion of (A, F) is a subcomplete Boolean partition algebra (B, G) along with a partition homomorphism $\iota : (A, F) \to (B, G)$ such that if (C, H) is a subcomplete Boolean partition algebra and $f : (A, F) \to (C, H)$, then there is a unique mapping $\overline{f} : (B, G) \to (C, H)$ such that $f = \overline{f}\iota$.*

The notion of the subcompletion is a special case of the more general category theoretic notion of a reflection. Let $\mathcal{C}$ be a category. Then a subcategory $\mathcal{D}$ is said to be a full subcategory if whenever X, Y are objects in $\mathcal{D}$ and $f : X \to Y$ is a morphism in $\mathcal{D}$, then f is a morphism in $\mathcal{C}$. We say that a full subcategory $\mathcal{D}$ is a reflective subcategory if for each $X \in \mathcal{C}$ there is a $Y \in \mathcal{D}$ and a morphism $\iota : X \to Y$ such that if $Z \in \mathcal{D}$ and $f : X \to Z$, then there is a unique $\hat{f} : Y \to Z$ with $f = \hat{f}\iota$. The object Y along with the morphism $\iota : X \to Y$ is called the $\mathcal{D}$-reflection of X. The $\mathcal{D}$-reflection of X is unique in the sense that if $\iota_1 : X \to Y_1, \iota_2 : X \to Y_2$ are $\mathcal{D}$-reflections of X, then there is an isomorphism $f : Y_1 \to Y_2$ such that $\iota_2 = f\iota_1$. In particular, the subcompletion of a Boolean algebra (A, F) is unique up to isomorphism. The following theorem shows that every Boolean partition algebra has a subcompletion.

Theorem 2.3.2 *Let $(A, F), (B, G)$ be Boolean partition algebras where (B, G) is subcomplete, and let $\iota : (A, F) \to (B, G)$ be a partition homomorphism. Then $\iota : (A, F) \to (B, G)$ is a subcompletion of (A, F) if and only if ι is injective and $\{\iota[p] | p \in F\}$ generates the filter G. Furthermore, if C is the completion of the Boolean algebra A and G is the filter on $\mathbb{P}(C)$ generated by the filterbase F, then $\mathfrak{B}^*(C, G)$ is a subcompletion of (A, F).*

Proof. $\leftarrow$ Assume ι is injective and $\{\iota[p] | p \in F\}$ generates G. Since ι is injective we may assume $A \subseteq B$ and ι is the inclusion. Let (C, H) be a subcomplete Boolean partition algebra, and let $f : (A, F) \to (C, H)$. Then for each $b \in B$ there is a $p \in G$ with $b \in p$. Therefore there is a $q \in F$ with $q \preceq p$. Hence, there is an $R \subseteq q$ with $\bigvee^B R = b$. As a result, if $\hat{f} : (B, G) \to (C, H)$ is a partition homomorphism extending the mapping f, then $\hat{f}(b) = \hat{f}(\bigvee^B R) = \bigvee^C_{a \in R} \hat{f}(a) = \bigvee^C_{a \in R} f(a)$. Therefore there is at most one extension $\hat{f}$ of f to a partition homomorphism from (B, G) to (C, H).

We also claim that the mapping $\hat{f}(B, G) \to (C, H)$ where $\hat{f}(b) = \bigvee^C_{a \in R} f(a)$ whenever $p \in F, R \subseteq p, \bigvee^B R$ is a well defined partition homomorphism.

Let $x \in B$ and assume $p, q \in F$ are partitions and $x = \bigvee^B P = \bigvee^B Q$ where $P \subseteq p, Q \subseteq q$. Let $r = p \wedge q$. Then there is an $R \subseteq r$ with $x = \bigvee^B R$. Clearly $R = \phi^{-1}_{r,p}[P]$. Therefore $\bigvee^C_{b \in P} f(b) = \bigvee^C_{b \in P} \bigvee^C_{a \in r, a \leq b} f(a) = \bigvee^C_{a \in R} f(a)$. Similarly $\bigvee^C_{b \in Q} f(b) = \bigvee^C_{a \in R} f(a)$. Hence $\bigvee^C_{b \in P} f(b) = \bigvee^C_{b \in Q} f(b)$. Thus $\hat{f}$ is a well defined mapping.

We now claim that $\hat{f}$ is a partition homomorphism. Clearly $\hat{f}(0) = 0$. Furthermore, if $a, b \in B$ and $a \wedge b = 0$, then there is an $r \in F$ and $R, S \subseteq r$ where $a = \bigvee^B R, b = \bigvee^B S$. Therefore $\hat{f}(a) \wedge \hat{f}(b) = \bigvee^C_{a \in R} f(a) \wedge \bigvee^C_{a \in S} f(a) = \bigvee^C_{a \in R, b \in S} f(a) \wedge f(b) = \bigvee^C_{a \in R, b \in S} f(a \wedge b) = 0$. Furthermore if $p \in G$, then there is a $q \in F$ with $q \preceq p$. Therefore $f[q]^+ \preceq \hat{f}[p]^+$, so $\hat{f}[p]^+$ is a partition and $\hat{f}[p]^+ \in H$. This shows that $\hat{f}$ is a partition homomorphism.

$\rightarrow$ Since all subcompletions are isomorphic, it suffices to show that there exists an injective partition homomorphism $\iota : (A, F) \rightarrow (B, G)$ where $\{\iota[p] | p \in F\}$ generates G. However if C is the completion of A and G is the filter on $\mathbb{P}(C)$ generated by F, then clearly $A \subseteq \mathfrak{B}^*(C, G)$ so if $\iota : (A, F) \rightarrow \mathfrak{B}^*(C, G)$ is the inclusion mapping, then ι is an injective partition homomorphism where $\{\iota[p] | p \in F\} = F$ generates G. $\square$

Theorem 2.3.3 *Let $(A, F), (B, G)$ be Boolean partition algebras and let $\iota : (A, F) \rightarrow (B, G)$ be an injective partition homomorphism. Then the following are equivalent.*

1. *$\{\iota[p]) | p \in F\}$ generates G.*

2. *(A, F) and (B, G) both have the same subcompletion. i.e. If $i : (B, G) \rightarrow (C, H)$ is a subcompletion, then $i\iota : (A, F) \rightarrow (C, H)$ is also a subcompletion.*

In either case, if $A \subseteq B$, then $F = G|_A$.

Proof. If $A \subseteq B$, then clearly $F \subseteq G|_A$. Furthermore, if $p \in G|_A$, then there is a $q \in F$ with $q \preceq p$. Therefore $p \in F$. Thus $G|_A = F$.

$1 \rightarrow 2$ Since ι and i are injective we may assume that $A \subseteq B \subseteq C$ and ι, i are the inclusion mappings. If $p \in H$, then since (C, H) is the subcompletion of (B, G) there is a $q \in G$ with $q \preceq p$. Since $\iota[F]$ generates G, there is an $r \in F$ with $r \preceq q$. Therefore $r \in F$ and $r \preceq p$, so F generates H. Thus the inclusion $i\iota$ is also a subcompletion.

$2 \rightarrow 1$ Let $i : (B, G) \rightarrow (C, H)$ be a subcompletion and assume $i\iota : (A, F) \rightarrow (C, H)$ is also a subcompletion. Since i and ι are injective we may assume $A \subseteq B \subseteq C$ and i, ι are the inclusion mappings. Then if $p \in G$, then $p \in H$ as well, so there is a $q \in F$ with $q \preceq p$ since $i\iota$ is a subcompletion. Therefore $\{\iota[p] | p \in F\}$ generates G. $\square$

Proposition 2.3.4 *Let A be a Boolean algebra, and let (B, G) be the subcompletion of $(A, \mathbb{P}(A))$. Then B is the completion of A.*

Proof. Let C be the completion of A. Then we may assume $G \subseteq \mathbb{P}(C)$ is the filter generated by $\mathbb{P}(A)$ and $B = \{0\} \cup \bigcup G$ (see Theorem 2.3.2). Since A is dense in C, for each $c \in C$ there is a cellular family $R \subseteq A$ with $c = \bigvee^C R$. Extend R to a partition $p \subseteq A$. Then since $p \in \mathbb{P}(A)$ we have $c = \bigvee^C R \in B$. Therefore $B = C$, so B is the completion of A. $\square$

Let $(X_d, \phi_{d,e})_{d,e \in D, d \leq e}$ be an inverse system of sets where each transition map $\phi_{d,e}$ is surjective. Then the inverse limit $\varprojlim X_d$ may be empty, but the inverse limit $\varprojlim X_d$ should be some sort of non-trivial point-free structure even if the inverse limit $\varprojlim X_d$ does not contain any points. We shall now show that subcomplete Boolean partition algebras are essentially point-free inverse limits of sets where all the transition mappings are surjective.

Example 2.3.5 *Let $(B_d, F_d)_{d \in D}$ be a directed system of Boolean partition algebras with a partition homomorphism $\phi_{d,e} : (B_d, F_d) \to (B_e, F_e)$ whenever $d \leq e$. Let $B = \varinjlim B_d$ be the direct limit of Boolean algebras, and let $\phi_d : B_d \to B$ be the canonical mappings. The canonical mappings ϕ_d are not necessarily partitional. For example, consider the Boolean partition algebras $\mathcal{P}[r, \infty)$ for $r \in \mathbb{R}$ and if $r \leq s$, then let $\phi_{r,s} : \mathcal{P}[r, \infty) \to \mathcal{P}[s, \infty)$ be the map where $\phi_{r,s}(R) = R \cap [s, \infty)$ for all $R \subseteq [r, \infty)$. Then each $\phi_{r,s}$ is a complete Boolean algebra homomorphism, so each $\phi_{r,s}$ is a partition homomorphism. However, if $p = \{\{x\} | x \in [0, \infty)\}$, then p is a partition of $[0, \infty)$, but for all $x \in [0, \infty)$ we have $\phi_{0,x+1}(\{x\}) = \emptyset$, thus $\phi_0(\{x\}) = \phi_{x+1}\phi_{0,x+1}(\{x\}) = 0$, so $\phi_0[p]^+ = \emptyset$. However $\varinjlim \mathbb{P}[r, \infty)$ is a nontrivial Boolean algebra.*

Theorem 2.3.6 *Let $((B_d, F_d)_{d \in D}, (\phi_{d,e})_{d \leq e})$ be a directed system of Boolean partition algebras, and let $B = \varinjlim B_d$ be the direct limit of the Boolean algebras B_d. Assume that each canonical mapping $\phi_d : B_d \to B$ is partitional. Then $\{\phi_d[p]^+ | p \in F_d, d \in D\}$ is a meet-subsemilattice of $\mathbb{P}(B)$. If F is the filter generated by $\{\phi_d[p]^+ | p \in F_d, d \in D\}$, then (B, F) is a Boolean partition algebra. Furthermore, each $\phi_d : (B_d, F_d) \to (B, F)$ is a partition homomorphism, and (B, F) is the direct limit of the system $(B_d, F_d)_{d \in D}$ of Boolean partition algebras in the category of Boolean partition algebras. If each (B_d, F_d) is subcomplete, then (B, F) is subcomplete as well.*

Proof. If $d_1, d_2 \in D$ and $p_1 \in F_{d_1}, p_2 \in F_{d_2}$, then let $e \in D$ be an element with $d_1 \leq e, d_2 \leq e$. Let $p = \phi_{d_1,e}[p_1]^+ \wedge \phi_{d_2,e}[p_2]^+$. Then $p \in F_e$ and $\phi_e[p]^+ = \phi_e[\phi_{d_1,e}[p_1]^+ \wedge \phi_{d_2,e}[p_2]^+]^+ = \phi_e[\phi_{d_1,e}[p_1]^+]^+ \wedge \phi_e[\phi_{d_2,e}[p_2]^+]^+ = \phi_e\phi_{d_1,e}[p_1]^+ \wedge \phi_e\phi_{d_2,e}[p_2]^+ = \phi_{d_1}[p_1]^+ \wedge \phi_{d_2}[p_2]^+$. This proves that $\{\phi_d[p]^+ | p \in F_d, d \in D\}$ is a meet-subsemilattice of $\mathbb{P}(B)$.

Let $b \in B^+$. Then there is a $d \in D$ and an $a \in B_d$ with $b = \phi_d(a)$. Clearly $a \neq 0$, so there is a $p \in F_d$ with $a \in p$. Therefore $b \in \phi_d[p]^+$, and hence (B, F) is a Boolean partition algebra. Clearly each ϕ_d is a partition homomorphism.

To show that (B, F) is the direct limit of the system $(B_d, F_d)_{d \in D}$, assume that (C, G) is a Boolean partition algebra, $\alpha_d : (B_d, F_d) \to (C, G)$ is a partition homomorphism for $d \in D$, and $\alpha_d = \alpha_e \circ \phi_{d,e}$ whenever $d \leq e$. Then, since B is the direct limit of the system $(B_d)_{d \in D}$ of Boolean algebras, there is a unique Boolean algebra homomorphism $\alpha : B \to C$ with $\alpha_d = \alpha \circ \phi_d$. We shall now show that α is a partition homomorphism. Let $p \in F$. Then there is a $d \in D$ and a $q \in F_d$ with $\phi_d[q]^+ \preceq p$, so $\alpha[p]^+ \succeq \alpha[\phi_d[q]^+]^+ = \alpha_d[q]^+ \in G$. Therefore $\alpha[p]^+ \in G$, thus α is a partition homomorphism.

Now assume each (B_d, F_d) is subcomplete. Then for each $p \in F$, there is a $d \in D$ and $q \in F_d$ with $\phi_d[q]^+ \preceq p$. Since q is subcomplete, p is subcomplete as well. $\square$

Theorem 2.3.7 *Let $(B_d, F_d)_{d \in D}$ be a directed system of Boolean partition algebras where each partition homomorphism $\phi_{d,e} : (B_d, F_d) \to (B_e, F_e)$ is injective. If B is the direct limit of the system of Boolean algebras $(B_d)_{d \in D}$, then each canonical mapping $\phi_d : B_d \to B$ is partitional.*

Proof. Let $d \in D$ and let $p \in F_d$. Let $x \in B$ be an upper bound of $\phi_d[p]$. Then $x = \phi_e(z)$ for some $z \in B_e$ and some $e \geq d$. Therefore for each $a \in p$ we have $\phi_e(\phi_{d,e}(a)) = \phi_d(a) \leq x = \phi_e(z)$, so $\phi_{d,e}(a) \leq z$ since ϕ_e is injective. Therefore z is an upper bound of the partition $\phi_{d,e}[p]$. Therefore $z = 1$, thus $x = \phi_e(z) = 1$. Therefore $\bigvee \phi_d[p] = 1$, so $\phi_d[p]$ is a partition of B, which means that each ϕ_d is partitional. $\square$

Definition 2.3.8 *If $(X, \mathcal{M}), (Y, \mathcal{N})$ are algebras of sets, then a measurable transformation from $(X, \mathcal{M})$ to $(Y, \mathcal{N})$ is a mapping $f : X \to Y$ such that $f^{-1}[R] \in \mathcal{M}$ whenever $R \in \mathcal{N}$.*

Theorem 2.3.9 *Let (B, F) be a Boolean partition algebra.*

1. Let $p, q \in F$ and $p \preceq q$. Then $\phi_{p,q}$ is a measurable transformation between the algebras of sets $(p, p^\sharp)$ and $(q, q^\sharp)$ (recall that $p^\sharp = \{R \subseteq p | \bigvee R$ exists$\}$).

2. Define a mapping $\phi_p^* : (p^\sharp, \mathbb{P}P(p)|_{p^\sharp}) \to B$ by $\phi_p^*(R) = \bigvee R$, then ϕ_p^* is an injective partitional mapping. Furthermore, if $p \in F$, then ϕ_p^* is a partition homomorphism.

3. Let $\phi_{p,q}^* : q^\sharp \to p^\sharp$ be the partition homomorphism defined by $\phi_{p,q}^*(R) = \phi_{p,q}^{-1}[R]$. Then $\phi_{p,q}^*$ is injective and $\phi_q^* = \phi_p^* \phi_{p,q}^*$.

4. Give the directed system $(p^\sharp, \mathbb{P}P(p)|_{p^\sharp})_{p \in F}$ the transition mappings $\phi_{p,q}^*$. Then (B, F) is the direct limit of the system $(p^\sharp, \mathbb{P}P(p)|_{p^\sharp})_{p \in F}$ of Boolean partition algebras.

Proof. 1. Assume $R \in q^\sharp$. Then $R \subseteq q$, and $\bigvee_{b \in R} b = \bigvee_{b \in R} \bigvee \{a \in p | \phi_{p,q}(a) = b\} = \bigvee \{a \in p | \phi_{p,q}(a) \in R\} = \bigvee \phi_{p,q}^{-1}[R]$, so $\phi_{p,q}^{-1}[R] \in p^\sharp$.

2. We already know ϕ_p^* is an injective Boolean algebra homomorphism that preserves all least upper bounds. Therefore if $P \in \mathbb{P}P(p)|_{p^\sharp}$, then $\bigvee \phi_p^*[P] = \phi_p^*(\bigvee P) = 1$, and so ϕ_p^* is a partition homomorphism.

3. If $R \in (q^\sharp)^+$, then $\phi_{p,q}^*(R) = \phi_{p,q}^{-1}[R] \neq \emptyset$ since $\phi_{p,q}$ is surjective, so $\phi_{p,q}^*$ is injective. Now assume that $R \in q^\sharp$. Then $\phi_q^*(R) = \bigvee R = \bigvee \phi_{p,q}^{-1}[R] = \phi_p^* \circ \phi_{p,q}^*(R)$, so $\phi_q^* = \phi_p^* \phi_{p,q}^*$.

4. We shall first show that B is the Boolean algebra direct limit of the system $(p^\sharp)_{p \in F}$. Since each $\phi_p^*, \phi_{p,q}^*$ is injective it suffices to show that each $a \in B$ is in the image of some ϕ_p^*. If $a = 0$, then $a = \phi_p^*(0)$ for each $p \in F$. If $a \neq 0$, then let $p \in F$ be a partition with $a \in p$. Then $\{a\} \in p^\sharp$ and $\phi_p^*(\{a\}) = \bigvee \{a\} = a$. Therefore B is the Boolean algebra direct limit of the system $(p^\sharp)_{p \in F}$ of Boolean algebras.

It now suffices to show that if p is a partition of B, then $p \in F$ if and only if there is a $q \in F$ and a $V \in \mathbb{P}P(q)|_{q^\sharp}$ with $(\phi_q^*)[V]^+ \preceq p$. If $p \in F$, then let $V = \{\{a\} | a \in p\}$. Then $V \subseteq p^\sharp$, and $(\phi_p^*)[V] = \{\phi_p^*(\{a\}) | a \in p\} = \{a | a \in p\} = p$. Now assume there is a $q \in F$ and a $V \in \mathbb{P}P(q)|_{q^\sharp}$ with $(\phi_q^*)[V]^+ \preceq p$. Then for each $a \in q$ there is an $R \in V$ with $a \in R$, so $a \leq \bigvee R = \phi_q^*(R) \in (\phi_q^*)[V]^+$, thus $q \preceq (\phi_q^*)[V]^+ \preceq p$, hence $p \in F$. □

Let (B,F) be a Boolean partition algebra, and let (A,G) be the subcompletion of (B,F). Then $B \subseteq A$ and $F \subseteq G$. Therefore by Theorem 2.3.9 and by the subcompleteness of (A,G) we have $(A,G) = \varinjlim (\mathcal{P}(q),(\phi_{q,r})_{-1}))_{q\in G}$. Since F is cofinal in G (under reverse refinement), we have $(A,G) = \varinjlim (\mathcal{P}(p),(\phi_{q,r})_{-1})_{p,q\in F, p\leq q}$. Therefore the subcompletion of a Boolean partition algebra can be represented as an inverse limit.

Proposition 2.3.10 *Let (A,F) be a Boolean partition algebra with subcompletion (B,G) and let $\iota : (A,F) \to (B,G)$ be the inclusion mapping. Then $S^*(\iota) : S^*(B,G) \to S^*(A,F)$ is a uniform homeomorphism.*

Proof. It suffices to show that $\varprojlim \iota$ is a homeomorphism. If $(y_p)_{p\in G} \in \varprojlim G$, then $\varprojlim \iota(y_p)_{p\in G} = (x_p)_{p\in F}$ where $x_p = \iota(x_p) = y_{\iota[p]^+} = y_p$ for $p \in F$. Therefore $\varprojlim \iota(y_p)_{p\in G} = (y_p)_{p\in F}$. Since F is cofinal in G, the mapping ι is a homeomorphism from $\varprojlim G$ to $\varprojlim F$. We conclude that $S^*(\iota)$ is an isomorphism as well. □

Theorem 2.3.11 *The subcompletion of a stable Boolean partition algebra is stable.*

Proof. Let (A,F) be a stable Boolean partition algebra with subcompletion (B,G) and let $b \in B^+$. Then let $q \in G$ be a partition with $b \in q$, and let $p \in F$ be a partition with $p \preceq q$. If $a \in p$ is an element with $\phi_{p,q}(a) = b$, then, since (A,F) is stable, there is an $(x_r)_{r\in F} \in \varprojlim F$ with $x_p = a$. Since $F \subseteq G$ is cofinal there is an extension $(x_r)_{r\in G}$ of the thread $(x_r)_{r\in F}$, so $x_q = \phi_{p,q}(x_p) = \phi_{p,q}(a) = b$. Therefore (B,G) is stable. □

Theorem 2.3.12 *Let (A,F) be a subcomplete Boolean partition algebra. Then there is a subcomplete stable Boolean partition algebra (C,H) and a surjective partition homomorphism $\phi : (C,H) \to (A,F)$ where $F = \{\phi[p]^+ | p \in H\}$.*

Proof. By Theorem 2.2.32 there is a stable Boolean partition algebra (B,G) and a surjective partition homomorphism $f : (B,G) \to (A,F)$ where $F = \{f[p]^+ | p \in G\}$. Let $\iota : (B,G) \to (C,H)$ be the subcompletion of (B,G). Then (C,H) is stable, and there is a partition homomorphism $\phi : (C,H) \to (A,F)$ with $f = \phi\iota$. Every element of G is of the form $f[p]^+$ for some $p \in G$, but $f[p]^+ = \phi[\iota[p]^+]^+$, so $G = \{\phi[q]^+ | q \in H\}$. In particular ϕ is surjective. $\square$

Theorem 2.3.13 *The following Boolean partition algebras are isomorphic.*

1. $\mathfrak{B}^*(S^*(B, F))$.
2. *The subcompletion of the stabilization of* (B, F).
3. *The stabilization of the subcompletion of* (B, F).

Proof. $2 \simeq 1$ The process of taking subcompletion and stabilization does not affect the dual partition space $S^*(B, F)$, therefore if (C, G) is the subcompletion of the stabilization of (B, F), then $S^*(C, G) \simeq S^*(B, F)$. Since (C, G) is subcomplete and stable, one has $(C, G) \simeq \mathfrak{B}^*(S^*(C, G)) \simeq \mathfrak{B}^*(S^*(B, F))$.

$3 \simeq 1$ This part of the proof is identical to the equivalence between 1 and 2. If (C, G) is the stabilization of the subcompletion of (B, F), then $S^*(C, G) \simeq S^*(B, F)$, but since (C, G) is subcomplete and stable one has $(C, G) \simeq \mathfrak{B}^*(S^*(C, G)) \simeq \mathfrak{B}^*(S^*(B, F))$. $\square$

In our next example, we shall show that the greatest lower bound of two subcomplete partitions is not necessarily subcomplete, but for our example we need to use some basic well known facts about semialgebras of sets (see for example [24][p. 358]).

Definition 2.3.14 *Let X be a set. Then a semialgebra of sets over X is a nonempty collection $\mathcal{A} \subseteq P(X)$ such that if $A, B \in \mathcal{A}$, then $A \cap B \in \mathcal{A}$ and if $A \in \mathcal{A}$, then $X \setminus A$ is a finite disjoint union of elements in $\mathcal{A}$.*

Lemma 2.3.15 *Let $\mathcal{A}$ be a semialgebra of sets over some set X. Then the collection of all finite pairwise disjoint unions of elements of $\mathcal{A}$ is an algebra of sets.*

Proof. Let $\mathcal{B}$ be the collection of all finite disjoint unions of elements in $\mathcal{A}$. If $R, S \in \mathcal{B}$, then there are $A_1, ..., A_n, B_1, ..., B_m \in \mathcal{A}$ such that $A_i \cap A_j = \emptyset$ for $i \neq j$ and $B_i \cap B_j = \emptyset$ for $i \neq j$ and $R = A_1 \cup ... \cup A_n, S = B_1 \cup ... \cup B_n$. Then clearly $R \cap S = (\bigcup_i A_i) \cap (\bigcup_j B_j) = \bigcup_{i,j} (A_i \cap B_j) \in \mathcal{B}$ being the finite disjoint union of elements of $\mathcal{A}$. Furthermore, since clearly $X \in \mathcal{B}$, $\mathcal{B}$ is closed under finite intersection including empty intersection. Now assume $R \in \mathcal{B}$. Then $R = A_1 \cup ... \cup A_n$ for some pairwise disjoint sequence $A_1, ..., A_n \in \mathcal{A}$. Since each A_i^c can be written as a finite disjoint union of elements in $\mathcal{A}$, we have $A_i^c \in \mathcal{B}$ for $1 \leq i \leq n$. Therefore $R^c = (A_1 \cup ... \cup A_n)^c = A_1^c \cap ... \cap A_n^c \in \mathcal{B}$ since $\mathcal{B}$ is closed under finite intersection. Hence $\mathcal{B}$ is an algebra of sets. $\square$

Example 2.3.16 *In this example, we shall show that $p \wedge q$ may not be subcomplete even though p and q are both subcomplete. If X is a set, then we call a set of the form $A \times B \subseteq X \times X$ a rectangle. The collection of all rectangles forms a semialgebra of sets. Let $\mathcal{M}$ be the algebra of sets generated by the collection of all rectangles. Now assume X is infinite. Let $1_X = \{(x, x)|x \in X\}$ denote the diagonal relation. Then $1_X \notin \mathcal{M}$ since 1_X is not a finite union of rectangles.*

Let $p = \{\{a\} \times X|a \in X\}$ and let $q = \{X \times \{a\}|a \in X\}$. Then p, q are both subcomplete partitions of $\mathcal{M}$, but $p \wedge q = \{\{r\}|r \in X^2\}$. The partition $p \wedge q$ is not subcomplete since $\{\{(x, x)\}|x \in X\} \subseteq p \wedge q$ does not have a least upper bound since $\{(x, x)|x \in X\} \notin \mathcal{M}$.

If p is a subcomplete partition of a Boolean algebra B, then since the mapping $\phi : P(p) \to B$ where $\phi(R) = \bigvee R$ is injective, we have $2^{|p|} \leq B$. In particular, if B is a Boolean partition algebra with $|B| < 2^{\aleph_0}$, and p is a subcomplete partition of B, then $2^{|p|} < 2^{\aleph_0}$, so p is finite. In particular, if $|B| < 2^{\aleph_0}$ and (B, F) is subcomplete, then $F = \mathbb{P}_\omega(B)$.

Theorem 2.3.17 *A Boolean algebra B is λ-complete if and only if $(B, \mathbb{P}_\lambda(B))$ is subcomplete and whenever $c \subseteq B$ is cellular with $|c| < \lambda$, then there is a partition p of B with $|p| < \lambda$.*

Proof. $\to$ Since B is λ-complete, clearly $(B, \mathbb{P}_\lambda(B))$ is subcomplete. Furthermore if $c \subseteq B$ is cellular with $|c| < \lambda$, then $(c \cup \{\bigvee c\})^+$ is a partition of B with cardinality less than λ.

$\leftarrow$ We shall show that if $R \subseteq B, |R| < \lambda$, then $\bigvee R$ exists by transfinite induction on the cardinality $|R|$. If $|R| = \gamma < \lambda$, then assume $R = \{x_\alpha|\alpha < \gamma\}$. Let $y_\alpha = x_\alpha \wedge$

$(\bigvee_{\beta<\alpha} x_\beta)'$ for $\alpha < \gamma$. Then $\{y_\alpha | \alpha < \gamma\}^+$ is a cellular family with $|\{y_\alpha | \alpha < \gamma\}^+| < \lambda$. Therefore there is a partition p of B with $|p| < \lambda$ and $\{y_\alpha | \alpha < \gamma\}^+ \subseteq p$. Since p is subcomplete, the least upper bound $\bigvee \{y_\alpha | \alpha < \gamma\}^+ = \bigvee_{\alpha<\gamma} x_\alpha = \bigvee R$ exists. $\square$

2.4 Products, complete partitions, and local refinability

Definition 2.4.1 *We shall call a partition p of a Boolean algebra complete if whenever $c_a \leq a$ for $a \in p$, then the least upper bound $\bigvee_{a\in p} c_a$ exists.*

For example, every finite partition is a complete partition. If B is a κ-Boolean algebra, and $p \in \mathbb{P}_\kappa(B)$, then p is a complete partition. The following Proposition motivates the notion of a complete partition. It can also be found in [12].

Proposition 2.4.2 *Let p be a partition of a Boolean algebra B. Then the mapping $\alpha : B \to \prod_{a\in p} B \restriction a$ given by $\alpha(b) = (\alpha_a(b))_{a\in p} = (a \wedge b)_{a\in p}$ is an injective Boolean algebra homomorphism, and α is an isomorphism iff p is complete.*

Proof. If $b \neq 0$, then since p is a maximal cellular family, $\alpha_a(b) = a \wedge b \neq 0$ for some $a \in p$. Therefore $\alpha(b) \neq 0$, so α is injective. If p is complete, then whenever $(c_a)_{a\in p} \in \prod_{a\in p} B \restriction a$ we have that $\bigvee_{a\in p} c_a$ exists, but $\alpha_a(\bigvee_{a\in p} c_a) = a \wedge \bigvee_{a\in p} c_a = \bigvee_{b\in p}(a \wedge c_b) = c_a$ for $a \in p$. Therefore $\alpha(\bigvee_{a\in p} c_a) = (c_a)_{a\in p}$, so α is an isomorphism. If α is an isomorphism, then clearly $\alpha[p]$ is a complete partition of $\prod_{a\in p} B \restriction a$, so since α is an isomorphism, the set $\alpha[p]$ is complete partition of B. $\square$

Definition 2.4.3 *We shall call a Boolean partition algebra (B, F) precomplete if each $p \in F$ is a complete partition. If B is a Boolean algebra, then write $\mathbb{P}_*(B)$ for the set of all complete partitions on B.*

Proposition 2.4.4 *If B is a Boolean algebra, then $(B, \mathbb{P}_*(B))$ is a Boolean partition algebra.*

Proof. Clearly each finite partition of B is a complete partition. To complete the proof, we must show that $\mathbb{P}_*(B)$ is a filter on $\mathbb{P}(B)$.

Assume that $p, q \in \mathbb{P}_*(B)$, and assume $p = (a_i)_{i\in I}, q = (b_j)_{j\in J}$. Then $(a_i \wedge b_j)_{i,j}$ is an extended partition of B. Moreover, if $c_{i,j} \leq a_i \wedge b_j$ for each $(i,j) \in I \times J$, then $c_{i,j} \leq b_j$ for $i \in I, j \in J$, so $\bigvee_{i\in I} c_{i,j}$ exists and $\bigvee_{i\in I} c_{i,j} \leq b_j$ for $j \in J$, thus the least upper bound $\bigvee_{j\in J}\bigvee_{i\in I} c_{i,j} = \bigvee_{(i,j)\in I\times J} c_{i,j}$ exists. Therefore $p \wedge q = \{a_i \wedge b_j | i \in I, j \in J\}^+$ is a complete partition.

Now assume p, q are partitions of B where p is complete and $p \preceq q$. Then assume that $c_a \leq a$ for all $a \in q$. Then for $a \in q$, we have $c_a = c_a \wedge a = c_a \wedge \bigvee_{b\in p, b\leq a} b = \bigvee_{b\in p, b\leq a}(c_a \wedge b)$, so since p is complete, we have $\bigvee_{a\in q} c_a = \bigvee_{a\in q}\bigvee_{b\in p,b\leq a}(c_a \wedge b) = \bigvee_{b\in p}(b \wedge c_{\phi_{p,q}(b)})$. Therefore q is a complete partition as well, and $(B, \mathbb{P}_*(B))$ is a Boolean partition algebra. $\square$

Proposition 2.4.5 *If B is a Boolean algebra, p is a complete partition of B, and q is a subcomplete partition of B, then $p \wedge q$ is also subcomplete.*

Proof. Let $S \subseteq p \wedge q$ and let $R = \{(a,b) \in p \times q | a \wedge b \in S\}$. Then $\bigvee_{b:(a,b)\in R} b$ exists and $a \wedge (\bigvee_{b:(a,b)\in R} b) = \bigvee_{b:(a,b)\in R}(a \wedge b) \leq a$, so since p is complete, the least upper bounds $\bigvee_{a\in p}\bigvee_{b:(a,b)\in R}(a \wedge b) = \bigvee_{(a,b)\in R} a \wedge b = \bigvee S$ exist. $\square$

If we compare the above result with example 2.3.16, then we see that there are subcomplete partitions that are not complete partitions.

Let B be a Boolean algebra. If $a \in B^+$, then we shall call a partition of $B \restriction a$ a partition of a. If p is a partition of B and p_a is a partition of a for each $a \in p$, then it is easy to see that $\bigcup_{a\in p} p_a$ is a partition of B as well.

We shall now discuss products of Boolean partition algebras. Let p be a partition of a Boolean algebra B. Let $(B \restriction a, F_a)$ be a Boolean partition algebra for each $a \in p$. Let $\mathfrak{F}$ be the collection of all partitions of B of the form $\bigcup_{a\in p} p_a$ where $p_a \in F_a$ for $a \in p$. Then $\mathfrak{F}$ is a filterbase on $\mathbb{P}(B)$; if $p_a, q_a \in F_a$ for $a \in p$, then there is a $r_a \in F_a$ for $a \in p$ with $r_a \preceq p_a, r_a \preceq q_a$, so $\bigcup_{a\in p} r_a \preceq \bigcup_{a\in p} p_a$ and $\bigcup_{a\in p} r_a \preceq \bigcup_{a\in p} q_a$.

Proposition 2.4.6 *Let F be the filter generated by $\mathfrak{F}$ described above. Then*

1. *(B, F) is a Boolean partition algebra.*
2. *The Boolean partition algebra (B, F) is subcomplete iff p is complete and each $(B \restriction a, F_a)$ is subcomplete.*

3. The Boolean partition algebra (B, F) is precomplete iff p is complete and each $(B \upharpoonright a, F_a)$ is precomplete.

Proof. 1. We already know F is a filter. Now assume $b \in B^+$. Then $b = \bigvee_{a \in p}(a \wedge b) = \bigvee_{a \in p}(a \wedge b)$. If $a \in p$, then there is some $p_a \in F_a$ and some $R_a \subseteq p_a$ with $\bigvee R_a = b$ since $(B \upharpoonright a, F_a)$ is a Boolean partition algebra. Therefore $\bigcup_{a \in p} p_a \in \mathfrak{F}$ and $\bigcup_{a \in p} R_a \subseteq \bigcup_{a \in p} p_a$. However $\bigvee \bigcup_{a \in p} R_a = \bigvee_{a \in p} \bigvee R_a = \bigvee_{a \in p}(a \wedge b) = b$. Therefore there is some partition q of B with $b \in q$ and $\bigcup_{a \in p} p_a \preceq q$, and clearly $q \in F$. Therefore (B, F) is a Boolean partition algebra.

2. $\leftarrow$ It suffices to show that each partition in $\mathfrak{F}$ is subcomplete. Let $p_a \in F_a$ for $a \in p$ and let $R_a \subseteq p_a$ for $a \in p$. Then we shall show that $\bigvee \bigcup_{a \in p} R_a$ exists. Since each p_a is subcomplete, the least upper bound $\bigvee R_a$ exists for $a \in p$ and $\bigvee R_a \leq a$ for $a \in p$. Therefore since p is precomplete, the least upper bound $\bigvee_{a \in p} \bigvee R_a$ exists, but $\bigvee_{a \in p} \bigvee R_a = \bigvee \bigcup_{a \in p} R_a$. Hence $\bigcup_{a \in p} p_a$ is subcomplete.

$\rightarrow$ Now assume that (B, F) is subcomplete. Let $p_a \in F_a$ for $a \in p$ and let $R_a \subseteq p_a$ for $a \in p$. Then $R_a \subseteq \bigcup_{a \in p} p_a \in F$, so $\bigvee R_a$ exists. Therefore each p_a is a subcomplete partition. Now assume that $c_a \leq a$ for $a \in p$. Then for $a \in p$ there is some $p_a \in F_a$ and some $R_a \subseteq p_a$ with $c_a = \bigvee R_a$. Therefore $\bigcup_{a \in p} p_a \in F$, and $\bigcup_{a \in p} R_a \subseteq \bigcup_{a \in p} p_a$, so the least upper bound $\bigvee \bigcup_{a \in p} R_a$ exists, but $\bigvee \bigcup_{a \in p} R_a = \bigvee_{a \in p} \bigvee R_a = \bigvee_{a \in p} c_a$. This shows that the partition p is a complete partition.

3. $\leftarrow$ We must show that each element of $\mathfrak{F}$ is a complete partition. Let $p_a \in F_a$ for $a \in p$ and assume that $c_b \leq b$ for $b \in \bigcup_{a \in p} p_a$. Then since each p_a is complete, the least upper bound $\bigvee_{b \in p_a} c_b$ exists and $\bigvee_{b \in p_a} c_b \leq a$ for $a \in p$. Therefore the least upper bound $\bigvee_{a \in p} \bigvee_{b \in p_a} c_b$ exists, and $\bigvee_{a \in p} \bigvee_{b \in p_a} c_b = \bigvee \bigcup_{a \in p} \{c_b | b \in p_a\}$, so each $\bigcup_{a \in p} p_a$ is a complete partition. We conclude that (B, F) is precomplete.

$\rightarrow$ If (B, F) is precomplete, then (B, F) is subcomplete, so p is a complete partition by 2. Let $a \in p$ and let $p_a \in F_a$. Let $c_b \leq b$ for $b \in p_a$. Then $p_a \cup (p \setminus \{a\}) \in F$, so since $p_a \cup (p \setminus \{a\})$ is a complete partition, the least upper bound $\bigvee_{b \in p_a} c_b$ exists. Therefore each $p_a \in F_a$ is complete. $\square$

We shall call the Boolean partition algebra (B, F) described in the previous proposition the interior direct product of the Boolean algebras $(B \restriction a, F_a)_{a \in p}$. The following theorem shows that if p is a complete partition, then the interior direct product (B, F) is a direct product in the category of Boolean partition algebras.

Theorem 2.4.7 *Let B be a Boolean algebra, and let p be a partition of B. Let $(B \restriction a, F_a)$ be a Boolean partition algebra for each $a \in p$, and let (B, F) be the interior direct product of the Boolean algebras $(B \restriction a, F_a)_{a \in p}$. Then each $\alpha_a : (B, F) \to (B \restriction a, F_a)$ is a partition homomorphism. Furthermore, if p is a complete partition, (A, H) is a Boolean partition algebra, and $f_a : (A, H) \to (B \restriction a, F_a)$ is a partition homomorphism for each $a \in p$, then there is a unique partition homomorphism $f : (A, H) \to (B, F)$ with $\alpha_a f = f_a$.*

Proof. Let $q \in F$. Then there is a system $(p_a)_{a \in p}$ with $\bigcup_{a \in p} p_a \preceq q$. Therefore $\alpha_a[\bigcup_{a \in p} p_a]^+ = p_a \in F_a$ and $\alpha_a[q]^+$ is a cellular family with $\alpha_a[\bigcup_{a \in p} p_a]^+ \preceq \alpha_a[q]^+$. Therefore $\alpha_a[q]^+$ is a partition contained in F_a. We conclude that α_a is a partition homomorphism.

Now assume p is a complete partition and each f_a is a partition homomorphism. Then define a mapping $f : A \to B$ by $f(b) = \bigvee_{a \in p} f_a(b)$. The mapping f is well defined since p is a complete partition, and clearly f is a Boolean algebra homomorphism. If $q \in H$, then we shall show that $f[q]^+ \in F$ by showing $\bigcup_{a \in p} f_a[q]^+ \preceq f[q]^+$. If $y \in \bigcup_{a \in p} f_a[q]^+$, then $y = f_a(x)$ for some $x \in q$. However we have $y = f_a(x) \leq f(x) \in f[q]^+$. Therefore f is a partition homomorphism. Furthermore, clearly $\alpha_a f(b) = a \wedge \bigvee_{a \in p} f_a(b) = f_a(b)$ for $a \in p$ and $b \in B$.

For uniqueness, take note that if $g : A \to B$ and $\alpha_a g = f_a$ for $a \in A$, then $g(b) = \bigvee_{a \in p} a \wedge g(b) = \bigvee_{a \in p} \alpha_a g(b) = \bigvee_{a \in p} f_a(b)$ for all b. $\square$

Proposition 2.4.8 *Assume (A, F) is a Boolean partition algebra, $a \in A^+$, and $p \in \mathbb{P}(A \restriction a)$. Then the following are equivalent.*

1. *$(p \cup \{a'\})^+ \in F$.*
2. *there is a $q \in F$ with $p \subseteq q$.*

3. There is some $q \in F$ with $\alpha_a[q]^+ = p$.

4. There is some $q \in F$ with $\alpha_a[q]^+ \preceq p$.

Proof. The directions $1 \to 2, 2 \to 3, 3 \to 4$ are trivial.

$4 \to 1$ If Assume $a \neq 1$. If $\alpha_a[q]^+ \preceq p$ and $q \in F$, then we claim that $q \wedge \{a, a'\} \preceq p \cup \{a'\}$. If $b \in q$, then $b \wedge a' \leq a'$ and $b \wedge a = \alpha_a(b) \in \alpha_a[q]$, so $b \wedge a \leq c$ for some $c \in p$. Therefore $q \wedge \{a, a'\} \preceq p \cup \{a'\}$, so $p \cup \{a'\} \in F$. □

Definition 2.4.9 *We shall write $F \restriction a$ for the collection of all partitions of $A \restriction a$ that satisfy the statements in Proposition* 2.4.8*. Clearly $(A \restriction a, F \restriction a)$ is a Boolean partition algebra for each $a \in A$. We shall write $(A, F) \restriction a$ for $(A \restriction a, F \restriction a)$.*

Clearly the mapping $\alpha_a : (A, F) \to (A, F) \restriction a$ is a quotient partition homomorphism, so $(A, F) \restriction a \simeq (A, F)/ \uparrow a$.

Proposition 2.4.10 *$\mathbb{P}_*(B) \restriction a = \mathbb{P}_*(B \restriction a)$.*

In particular, if (B, F) is precomplete, then $(B, F) \restriction a$ is precomplete as well.

Proof. If $a \in \{0, 1\}$, then the theorem is trivial. Assume $a \notin \{0, 1\}$, and let $p \in \mathbb{P}_*(B) \restriction a$. Then $p \cup \{a'\} \in \mathbb{P}_*(B)$. Therefore, since $p \cup \{a'\}$ is complete, whenever $c_b \leq b$ for $b \in p$, the least upper bound $\bigvee_{b \in p} c_b$ exists and is contained in $B \restriction a$. Therefore p is a complete partition of $B \restriction a$.

Now assume that $p \in \mathbb{P}_*(B \restriction a)$. Now let $c_b \leq b$ for each $b \in p \cup \{a'\}$. Then $\bigvee_{b \in p \cup \{a'\}} c_b = c_{a'} \vee \bigvee_{b \in p} c_b$, so the least upper bound $\bigvee_{b \in p \cup \{a'\}} c_b$ exists. Therefore $p \cup \{a'\} \in \mathbb{P}_*(B)$, hence $p \in \mathbb{P}_*(B) \restriction a$.

If (B, F) is precomplete, then $F \restriction a \subseteq \mathbb{P}_*(B) \restriction a = \mathbb{P}_*(B \restriction a)$, so $(B \restriction a, F \restriction a)$ is precomplete as well. □

Definition 2.4.11 *A Boolean partition algebra (A, F) is* locally refinable *if whenever $p \in F$ and $p_a \in F \restriction a$ for $a \in p$, then $\bigcup_{a \in p} p_a \in F$. In other words, a Boolean partition algebra (A, F) is locally refinable if (A, F) is the internal direct product of the system $(A \restriction a, F \restriction a)_{a \in p}$ for each $p \in F$.*

Example 2.4.12 *For every Boolean algebra A, the Boolean partition algebra $(A, \mathbb{P}(A))$ is locally refinable. Furthermore, if κ is a regular cardinal, then $(A, \mathbb{P}_\kappa(A))$ is locally refinable; if $p \in \mathbb{P}_\kappa(A)$ and $p_a \in \mathbb{P}_\kappa(A) \restriction a$ for $a \in A$, then $|p_a| < \kappa$ for $a \in p$, so $|\bigcup_{a \in p} p_a| = \sum_{a \in p} |p_a| < \kappa$, hence $\bigcup_{a \in p} p_a \in \mathbb{P}_\kappa(A)$.*

Example 2.4.13 *There are complete Boolean partition algebras that are not locally refinable. Let λ be a singular cardinal. Then the Boolean partition algebra $(P(\lambda), \mathbb{P}_\lambda(\lambda))$ is complete. On the other hand, there is a partition P of λ where $|P| < \lambda$ and $|R| < \lambda$ for $R \in P$. Since $|P| < \lambda$, we have $P \in \mathbb{P}_\lambda(\lambda)$, but $\{\{r\}|r \in R\} \in \mathbb{P}_\lambda(\lambda) \restriction R$ for $R \in P$, and $\bigcup_{R \in P}\{\{r\}|r \in R\} = \{\{r\}|r \in \lambda\} \not\in \mathbb{P}_\lambda(\lambda)$.*

Proposition 2.4.14 *If B is a Boolean algebra, then $(B, \mathbb{P}_*(B))$ is locally refinable.*

Proof. Let $p \in \mathbb{P}_*(B)$, and let $p_a \in \mathbb{P}_*(B) \restriction a = \mathbb{P}_*(B \restriction a)$ for $a \in p$. Now assume $c_b \leq b$ for each $b \in \bigcup_{a \in p} p_a$. Then since each p_a is a complete partition of $B \restriction a$, the least upper bound $\bigvee_{b \in p_a} c_b$ exists. Therefore since p is complete, the least upper bounds $\bigvee_{a \in p} \bigvee_{b \in p_a} c_b = \bigvee\{c_b | b \in \bigcup_{a \in p} p_a\}$ exist. Therefore $\bigcup_{a \in p} p_a \in \mathbb{P}_*(B)$, so $\mathbb{P}_*(B)$ is locally refinable. $\square$

If A is a Boolean algebra, and (A, F_i) is a locally refinable Boolean partition algebra for $i \in I$, then one can easily show that $(A, \bigcap_{i \in I} F_i)$ is locally refinable as well. In particular, if (A, F) is a Boolean partition algebra, then there is a smallest filter $G \subseteq \mathbb{P}(B)$ containing F such that (A, G) is locally refinable.

Definition 2.4.15 *If G is the smallest filter on the meet-semilattice $\mathbb{P}(B)$ such that $F \subseteq G$ and (A, G) is locally refinable, then we shall call the Boolean partition algebra (A, G) the total local refinement of (A, F).*

Proposition 2.4.16 *Let B be a Boolean algebra. Let p, q be partitions of B with $p \preceq q$. Let p_a be a partition of a for each $a \in p$, and let q_b be a partition of b for each $b \in q$. Furthermore, assume that $p_a \preceq q_{\phi_{p,q}(a)}$ for each $a \in p$. Then $\bigcup_{a \in p} p_a \preceq \bigcup_{b \in q} q_b$.*

Proof. Let $x \in \bigcup_{a \in p} p_a$. Then $x \in p_a$ for some $a \in p$, so $x \leq y$ for some $y \in q_{\phi_{p,q}(a)} \subseteq \bigcup_{b \in q} q_b$. Therefore $\bigcup_{a \in p} p_a \preceq \bigcup_{b \in q} q_b$. $\square$

If (A,F) is a Boolean partition algebra, then let $\mathfrak{F}$ be the collection of all partitions of the form $\bigcup_{a\in p} p_a$ where $p \in F$ and $p_a \in F \restriction a$ for $a \in p$.

Assume $p, q \in F$ and $p_a \in F \restriction a$ for $a \in p$ and $q_b \in F \restriction b$ for $b \in q$. Let $r \in F$ be a partition with $r \preceq p, r \preceq q$. Then for each $a \in r$ let $r_a = \alpha_a[p_{\phi_{r,p}(a)} \wedge q_{\phi_{r,q}(a)}]^+$. Then one can see that $r_a \preceq p_{\phi_{r,p}(a)}$ and $r_a \preceq q_{\phi_{r,q}(a)}$ for $a \in r$. Therefore $\bigcup_{a\in r} r_a \preceq \bigcup_{a\in p} p_a$ and $\bigcup_{a\in r} r_a \preceq \bigcup_{a\in q} q_a$ by Proposition 2.4.16. Therefore $\mathfrak{F}$ is a filterbase. If F is generated by a filterbase G, then by Proposition 2.4.16 one can easily show that $\mathfrak{F}$ is generated by partitions of the form $\bigcup_{a\in p} \alpha_a[p_a]^+$ where $p \in G$ and $p_a \in G$ for $a \in p$.

Definition 2.4.17 *If (A,F) is a Boolean partition algebra, then let F' be the filter on $\mathbb{P}(A)$ generated by the filterbase $\mathfrak{F}$ defined above. Then clearly (A,F') is a Boolean partition algebra. We also define a Boolean partition algebra (B,F^α) for all ordinals α by transfinite induction:*

1. *$F^0 = F$.*
2. *$F^{(\alpha+1)} = (F^{(\alpha)})'$.*
3. *If λ is a limit ordinal, then $F^{(\lambda)} = \bigcup_{\beta<\lambda} F^{(\beta)}$.*

This transfinite process must stop at some ordinal, so there is an ordinal α where $F^{(\alpha)} = F^{(\beta)}$ whenever $\beta \geq \alpha$. Therefore let $F^{(\infty)} = F^{(\alpha)}$ whenever $F^{(\alpha)} = F^{(\beta)}$ for $\beta \geq \alpha$.

Clearly $F^{(\infty)}$ is locally refinable, and it is easy to see that $(A, F^{(\infty)})$ is the total local refinement of (A,F). It should be noted that the notion of a locally refinable Boolean partition algebra is the Boolean partition algebra dual of the notion of a locally fine uniform space [6] and the transfinite process of constructing $F^{(\infty)}$ is analogous to the transfinite process of taking the locally fine coreflection of a uniform space.

Lemma 2.4.18 *Let A, B be Boolean algebras and let $\phi : A \to B$ be a Boolean algebra homomorphism, then define $f : A \restriction a \to B \restriction \phi(a)$ by letting $f(a) = \phi(a)$. Then f is a Boolean algebra homomorphism and $\alpha_{\phi(a)}\phi = f\alpha_a$.*

Proof. Since f is a bounded lattice homomorphism, f is also a Boolean algebra homomorphism. Furthermore, if $b \in A$, then $\alpha_{\phi(a)}\phi(b) = \phi(a) \wedge \phi(b) = \phi(a \wedge b) = f(a \wedge b) = f\alpha_a(b)$. □

Lemma 2.4.19 *Let $(A,F),(B,G)$ be Boolean partition algebras, and let $\phi:(A,F)\to(B,G)$ be a partition homomorphism. Then for each $a\in A$, the mapping $f:(A,F)\restriction a\to(B,G)\restriction\phi(a)$ defined by $f(b)=\phi(b)$ is a partition homomorphism.*

Proof. Let $p\in F\restriction a$. Then $p=\alpha_a[q]^+$ for some $q\in F$, so $f[p]^+=f[\alpha_a[q]^+]^+=(f\alpha_a)[q]^+=(\alpha_{\phi(a)}\phi)[q]^+=\alpha_{\phi(a)}[\phi[q]^+]^+\in G\restriction\phi(a)$. Therefore f is a partition homomorphism as well. □

Proposition 2.4.20 *Let $(A,F),(B,G)$ be Boolean partition algebras and let $\phi:(A,F)\to(B,G)$ be a partition homomorphism. Then ϕ is also a partition homomorphism from $(A,F^{(\alpha)})$ to $(B,G^{(\alpha)})$ for all ordinals α.*

Proof. We only need to show that if $\phi:(A,F)\to(B,G)$ is a partition homomorphism, then $\phi:(A,F')\to(B,G')$ is also a partition homomorphism. The Proposition will then follow immediately by transfinite induction since the limit ordinal case is obvious and if ϕ is a partition homomorphism from $(A,F^{(\alpha)})$ to $(B,G^{(\alpha)})$, then ϕ would also be a partition homomorphism from $(A,F^{(\alpha+1)})$ to $(B,G^{(\alpha+1)})$. Let $q\in F'$. Then there is $p\in F$ and $p_a\in F\restriction a$ for $a\in p$ where $\bigcup_{a\in p}p_a\preceq q$. If we let $r_{\phi(a)}=\phi[p_a]^+$ for $a\in p$, then $r_{\phi(a)}\in G\restriction\phi(a)$, so $\phi[\bigcup_{a\in p}p_a]^+=\bigcup_{a\in p}\phi[p_a]^+=\bigcup_{b\in\phi[p]^+}r_b\in G'$. Therefore, since $\bigcup_{a\in p}p_a\preceq q$, we have $\phi[q]^+\in G'$ as well. □

In particular, if $\phi:(A,F)\to(B,G)$ is a partition homomorphism, then ϕ is also a partition homomorphism from $(A,F^{(\infty)})$ to $(B,G^{(\infty)})$. Therefore a function $\phi:(A,F)\to B$ is a partitional mapping if and only if ϕ is a partitional mapping from $(A,F^{(\infty)})$ to B. Hence a filter $Z\subseteq A$ is an F-filter if and only if Z is an $F^{(\infty)}$-filter.

Theorem 2.4.21 *Let (B,F) be a Boolean partition algebra and let α be a non-zero ordinal. Then the following are equivalent.*

1. *(B,F) is precomplete.*
2. *$(B,F^{(\alpha)})$ is subcomplete.*
3. *$(B,F^{(\alpha)})$ is precomplete.*
4. *$(B,F^{(\infty)})$ is subcomplete.*
5. *$(B,F^{(\infty)})$ is precomplete.*

Proof. The directions $5 \to 3, 5 \to 4, 3 \to 2, 4 \to 2$ are all trivial.

$2 \to 1$ If $(B, F^{(\alpha)})$ is subcomplete, then (B, F') is subcomplete as well. Now assume $p \in F$ and $c_a \leq a$ for $a \in p$. Then $\{c_a, a \wedge c_a'\}^+ \in F \restriction a$ for $a \in p$, and if $q = \bigcup_{a \in p}\{c_a, a \wedge c_a'\}^+$, then $q \in F'$. Clearly $\{c_a | a \in p\}^+ \subseteq q$, so since $q \in F'$ is subcomplete, the least upper bound $\bigvee\{c_a | a \in p\}^+ = \bigvee_{a \in p} c_a$ exists. Therefore p is a complete partition, hence (B, F) is precomplete.

$1 \to 5$ If (B, F) is precomplete, then $F \subseteq \mathbb{P}_*(B)$, so $F^{(\infty)} \subseteq \mathbb{P}_*(B)$ since $\mathbb{P}_*(B)$ is locally refinable. Thus $(B, F^{(\infty)})$ is precomplete as well. □

In particular every subcomplete locally refinable Boolean partition algebra is precomplete. If (B, F) is locally refinable and subcomplete, then (B, F) is precomplete, so $F \subseteq \mathbb{P}_*(B)$. Therefore $\mathbb{P}_*(B)$ is the largest filter on $\mathbb{P}(B)$ subject to the condition that $(B, \mathbb{P}_*(B))$ is subcomplete and locally refinable.

Theorem 2.4.22 *The subcompletion of a locally refinable Boolean partition algebra is locally refinable.*

Proof. If (A, F) is a locally refinable Boolean partition algebra with subcompletion (B, G), then since G is generated by the filterbase F, the filter G' is generated by partitions of the form $\bigcup_{a \in q} \alpha_a[q_a]^+$ where $q \in F$ and $q_a \in F$ for $a \in F$. However each partition of the form $\bigcup_{a \in q} \alpha_a[q_a]^+$ is contained in F. Therefore G' is also generated by F, so $G = G'$. □

Furthermore, since a locally refinable subcomplete Boolean partition algebra is precomplete, the subcompletion of a locally refinable Boolean partition algebra is precomplete.

Theorem 2.4.23 *The category of subcomplete locally refinable subcategories is a reflective subcategory of the category of Boolean partition algebras. More specifically, if (A, F) is a Boolean partition algebra, and (B, G) is the subcompletion of $(A, F^{(\infty)})$, then the inclusion map $\iota : (A, F) \to (B, G)$ is the subcomplete locally refinable reflection of (A, F).*

Proof. Let (C, H) be a subcomplete locally refinable Boolean partition algebra. Let $f : (A, F) \to (C, H)$ be a partition homomorphism. Then f is also a partition homomorphism from $(A, F^{(\infty)})$ to $(C, H^{(\infty)}) = (C, H)$. Since (C, H) is subcomplete, the mapping f has a unique extension to a mapping $\overline{f} : (B, G) \to (C, H)$. In particular, $\overline{f}$ is the unique morphism with $\overline{f}\iota = f$. $\square$

Theorem 2.4.24 *The quotient of a locally refinable Boolean partition algebra is locally refinable. In particular, if (B, F) is a Boolean partition algebra and Z is an F-filter, then $(F/Z)^{(\alpha)} = F^{(\alpha)}/Z$ for all ordinals α.*

Proof. We shall first show that if (B, F) is a Boolean partition algebra, then $(F/Z)' = F'/Z$. First take note that F/Z is generated by the filterbase $\mathfrak{G} := \{\pi[p]^+ | p \in F\}$. Therefore $(F/Z)'$ is generated by $\bigcup_{a \in q} \alpha_a[q_a]^+$ where $q \in \mathfrak{G}$ and $q_a \in \mathfrak{G}$ for $a \in q$. In other words, $(F/Z)'$ is generated by partitions of the form $\bigcup_{a \in p, \pi(a) \neq 0} \alpha_{\pi(a)}[\pi[p_a]^+]^+$ where $p \in F$ and $p_a \in F$ for $a \in p$. However, we have $\bigcup_{a \in p, \pi(a) \neq 0} \alpha_{\pi(a)}[\pi[p_a]^+]^+ = \bigcup_{a \in p}(\alpha_{\pi(a)}\pi)[p_a]^+ = \bigcup_{a \in p}(\pi\alpha_a)[p_a]^+ = \bigcup_{a \in p} \pi[\alpha_a[p_a]^+]^+ = \pi[\bigcup_{a \in p} \alpha_a[p_a]^+]^+$. Now F'/Z is generated by the partitions of the form $\pi[\bigcup_{a \in p} \alpha_a[p_a]^+]^+$ where $p \in F$ and $p_a \in F$ for $a \in p$. Therefore $(F/Z)' = F'/Z$.

We shall now show that $(F/Z)^{(\alpha)} = F^{(\alpha)}/Z$ for all ordinals α. Clearly $(F/Z)^{(0)} = F^{(0)}/Z$. If $(F/Z)^{(\alpha)} = F^{(\alpha)}/Z$, then $(F/Z)^{(\alpha+1)} = ((F/Z)^{(\alpha)})' = (F^{(\alpha)}/Z)' = (F^{(\alpha)})'/Z = F^{(\alpha+1)}/Z$. Now if λ is a limit ordinal, then $(F/Z)^{\lambda} = \bigcup_{\alpha<\lambda}(F/Z)^{(\alpha)}$, so $(F/Z)^{\lambda}$ is generated by the filterbase $\bigcup_{\alpha<\lambda}\{\pi_Z[p] | p \in F^{(\alpha)}\}$. Similarly, $F^{(\lambda)}/Z$ is generated by $\{\pi_Z[p]^+ | p \in F^{(\lambda)}\} = \bigcup_{\alpha<\lambda}\{\pi_Z[p]^+ | p \in F^{(\alpha)}\}$. Therefore $F^{(\lambda)}/Z = (F/Z)^{\lambda}$. Therefore $F^{(\alpha)}/Z = (F/Z)^{(\alpha)}$ for all ordinals α. $\square$

Corollary 2.4.25 *If (B, F) is precomplete and Z is an F-filter, then $(B, F)/Z$ is also precomplete.*

Proof. If (B, F) is precomplete, then $(B, F^{(\infty)})$ is subcomplete by Theorem 2.4.21, so $(B/Z, F^{(\infty)}/Z) = (B/Z, (F/Z)^{(\infty)})$ is subcomplete. Therefore $(B/Z, F/Z)$ is precomplete. $\square$

Theorem 2.4.26 *Let (B, F) be a Boolean partition algebra and $A \subseteq B$ is a subalgebra, then $(F|_A)^{(\alpha)} \subseteq (F^{(\alpha)})|_A$ for all ordinals α. In particular, a subalgebra of a locally refinable Boolean partition algebra is locally refinable.*

Proof. We shall first show that $(F|_A)' \subseteq F'|_A$. The filter $(F|_A)'$ is generated by partitions of the form $\bigcup_{a\in p} \alpha_a[p_a]^+$ where $p \in F|_A$ and $p_a \in F|_A$ for $a \in A$. Then $\bigcup_{a\in p} \alpha_a[p_a]^+ \subseteq A$ and $\bigcup_{a\in p} \alpha_a[p_a]^+ \in F'$, so $\bigcup_{a\in p} \alpha_a[p_a]^+ \subseteq F'|_A$. Therefore $(F|_A)' \subseteq F'|_A$. We shall now prove that $(F|_A)^{(\alpha)} \subseteq (F^{(\alpha)})|_A$ for all ordinals α. Clearly $(F|_A)^{(0)} \subseteq (F^{(0)})|_A$. For the successor ordinal set, assume $(F|_A)^{(\alpha)} \subseteq (F^{(\alpha)})|_A$, then $(F|_A)^{(\alpha+1)} = (F|_A^{(\alpha)})' \subseteq (F^{(\alpha)}|_A)' \subseteq (F^{(\alpha)})'|_A = F^{(\alpha+1)}|_A$. Now assume that λ is a limit ordinal and $(F|_A)^{(\alpha)} \subseteq F^{(\alpha)}|_A$ for $\alpha < \lambda$. Then $F|_A^{(\lambda)} = \bigcup_{\alpha<\lambda} F|_A^{(\alpha)} \subseteq \bigcup_{\alpha<\lambda}(F^{(\alpha)}|_A) \subseteq F^{(\lambda)}|_A$. Therefore $(F|_A)^{(\alpha)} \subseteq (F^{(\alpha)})|_A$ for all ordinals α. $\square$

3 Admissibility

In this chapter, we shall study the sets whose least upper bounds and greatest lower bounds are preserved under partition homomorphisms, taking F-subalgebras, etc. Such sets are called admissible sets. In the previous chapter, we defined F-ideals on a Boolean partition algebra (B, F) so that we were able to take quotient algebras. We shall soon see that these are precisely the ideals closed under taking admissible least upper bounds. Using these facts, we shall show that the lattice of F-filters is the inverse limit of some inverse system and hence a complete partition space. In terms of partition spaces, given a complete partition space (X, M), the lattice of M-filters on $\mathfrak{B}^*(X, M)$ is the completion of the hyperspace of (X, M). With this characterization of the completion of the hyperspace of (X, M), we shall show that a complete partition space (X, M) has a complete hyperspace if and only if $\mathfrak{B}^*(X, M)$ is superstable. Later, we shall use admissibility to characterize the locally refinable Boolean partition algebras. In the final section of this chapter, we shall investigate the property of resplendence. With resplendence, we shall give a characterization of the stable Boolean partition algebras and the measurable cardinals. Furthermore, we shall also characterize superstability in terms of a strong distributivity property. As a consequence, strongly compact cardinals may be characterized in terms of such a distributivity property.

3.1 Admissibility

In this section, we shall characterize notions such as partition homomorphisms, F-ideals, F-subalgebras, products, and local refinability in terms of admissible sets and their least upper bounds.

Definition 3.1.1 *Let* (B,F) *be a Boolean partition algebra. Then a subset* $R \subseteq B$ *is said to be* admissible *if there is a* $p \in F$ *with* $R \subseteq p^*$.

Take note that if $p \preceq q$, then $q^* \subseteq p^*$. Therefore the collection of all admissible sets is an ideal on the set $P(B)$ generated by the collection $\{p^* | p \in F\}$. Clearly every finite subset of B is admissible. A set $R \subseteq B$ is admissible if and only if $\{a' | a \in R\}$ is admissible. It is easy to show that a Boolean partition algebra is subcomplete if and only if every admissible subset has a least upper bound. A partition $p \in \mathbb{P}(B)$ is admissible in (B,F) if and only if $p \in F$. In particular, one may recover the entire Boolean partition algebra (B,F) from the admissible sets.

Proposition 3.1.2 *If* (A,F) *is a Boolean partition algebra,* B *is a Boolean algebra, and* $\phi : A \to B$ *is a Boolean algebra homomorphism, Then the following are equivalent.*

1. ϕ *is a partitional.*
2. *For each admissible* $R \subseteq A$*, if* $\bigvee R$ *exists, then* $\phi(\bigvee R) = \bigvee \phi[R]$.
3. *For each admissible* $R \subseteq A$*, if* $\bigwedge R$ *exists, then* $\phi(\bigwedge R) = \bigwedge \phi[R]$.

Proof. It is easy to see the equivalence between 2 and 3, so we shall omit the proof of this equivalence.

$2 \to 1$ If $p \in F$, then p is admissible and $\bigvee p$ exists, so $\bigvee \phi[p] = \phi(\bigvee p) = 1$. Therefore $\phi[p]^+$ is a partition of B.

$1 \to 2$ Assume that $R \subseteq A$ is admissible and $\bigvee R$ exists. Then $R \subseteq p^*$ for some $p \in F$. Therefore each $r \in R$ is of the form $\bigvee R_r$ for some $R_r \in p^\sharp$, so $\bigvee R = \bigvee_{r \in R} \bigvee R_r = \bigvee(\bigcup_{r \in R} R_r)$. Thus by Theorem 1.3.34 and since $\bigcup_{r \in R} R_r \in p^\sharp$, we get

$$\phi(\bigvee R) = \phi(\bigvee(\bigcup_{r \in R} R_r)) = \bigvee \phi[\bigcup_{r \in R} R_r] = \bigvee \bigcup_{r \in R} \phi[R_r]$$

$$= \bigvee_{r \in R} \bigvee \phi[R_r] = \bigvee_{r \in R} \phi(\bigvee R_r) = \bigvee_{r \in R} \phi(r) = \bigvee \phi[R].$$

□

Proposition 3.1.3 *If $(A,F),(B,G)$ are Boolean partition algebras and $\phi:(A,F)\to(B,G)$ is a Boolean algebra homomorphism, then the following are equivalent.*

1. ϕ is a partition homomorphism.

2. Whenever $R\subseteq A$ is admissible, then $\phi[R]$ is admissible as well and $\phi(\bigvee R)=\bigvee\phi[R]$ whenever $\bigvee R$ exists.

3. Whenever $R\subseteq A$ is admissible, then $\phi[R]$ is admissible as well and $\phi(\bigwedge R)=\bigwedge\phi[R]$ whenever $\bigwedge R$ exists.

Proof. The equivalence between 2 and 3 should be obvious, so we shall omit the proof of this equivalence.

$2\to 1$ Let $p\in F$. Since ϕ is partitional by Proposition 3.1.2 , $\phi[p]^{+}$ is a partition of B. Since p is admissible, the set $\phi[p]$ is admissible as well, so $\phi[p]^{+}$ is an admissible partition. Therefore $\phi[p]^{+}\in G$, so ϕ is a partition homomorphism.

$1\to 2$ If $R\subseteq A$ is admissible, then $R\subseteq p^{*}$ for some $p\in F$. Since ϕ is a partition homomorphism, we get $\phi[p]^{+}\in G$. Each $r\in R$ is of the form $\bigvee R_r$ for some $R_r\subseteq p$, so

$$\phi(r)=\phi(\bigvee R_r)=\bigvee\phi[R_r]=\bigvee\phi[R_r]^{+}\in(\phi[p]^{+})^{*}.$$

Therefore $\phi[R]\subseteq(\phi[p]^{+})^{*}\in G$, so $\phi[R]$ is admissible as well. □

Theorem 3.1.4 *Let (B,F) be a Boolean partition algebra and let $I\subseteq B$ be a lower set. Then the following are equivalent.*

1. I is an F-ideal.

2. Whenever $R\subseteq I$ is admissible and $\bigvee R$ exists, then $\bigvee R\in I$.

3. Whenever $p\in F$ and $R\subseteq p\cap I$ and $\bigvee R$ exists, then $\bigvee R\in I$.

4. Whenever $p\in F$ there is an $S\subseteq p$ where if $R\subseteq p$ and $\bigvee R$ exists, then $\bigvee R\in I$ if and only if $R\subseteq S$.

Proof. $1\to 2$ If I is an F-ideal, then the quotient mapping $\pi_I:B\to B/I$ is partitional, so if $R\subseteq I$ is admissible and $\bigvee R$ exists, then $\pi_I(\bigvee R)=\bigvee\pi_I[R]=0$, thus $\bigvee R\in I$.

$2\to 3$ If $p\in F$, $R\subseteq p\cap I$, and $\bigvee R$ exists, then R is admissible, so $\bigvee R\in I$.

$3 \to 4$ This direction follows if we let $S = p \cap I$.

$4 \to 3$ Let $p \in F$. Then there is an $S \subseteq p$ where whenever $R \subseteq p$ and $\bigvee R$ exists, then $\bigvee R \in I$ if and only if $R \subseteq S$. Now assume $R \subseteq p \cap I$ and $\bigvee R$ exists. If $r \in R$, then $\bigvee\{r\} = r \in I$, so $\{r\} \subseteq S$. Therefore $R \subseteq S$, hence $\bigvee R \in I$.

$3 \to 1$ We shall first show that I is an ideal. Let $a, b \in I$. Then $b \wedge a' \in I$, and $\{a, b \wedge a', (a \vee b)'\}^+ \in F$. Therefore, $a \vee b = (a \vee b) \wedge (a \vee a') = a \vee (b \wedge a') = \bigvee\{a, b \wedge a'\}^+ \in I$. Thus I is an ideal.

Now assume that I is not an F-ideal. Then there is some $p \in F$ where $\pi_I[p]^+$ is not a partition of B/I. Therefore there is some $b \in B \backslash I$ where $\pi_I(a \wedge b) = \pi_I(a) \wedge \pi_I(b) = 0$ for $a \in p$. Thus $\{a \wedge b | a \in p\}^+ \subseteq I$, and $\{a \wedge b | a \in p\}^+ \subseteq p \wedge \{b, b'\}^+$, but $\bigvee\{a \wedge b | a \in p\}^+ = (\bigvee_{a \in p} a) \wedge b = b \notin I$. This is a contradiction. We conclude that I is an F-ideal. $\square$

Corollary 3.1.5 *Let (B, F) be a subcomplete Boolean partition algebra. Then a lower set $I \subseteq B$ is an F-ideal if and only if for each $p \in F$, the set $p^* \cap I$ has a largest element.*

We will now give an application of Theorem 3.1.4.

Theorem 3.1.6 *Let λ be an infinite cardinal, and let B be a λ-Boolean algebra. Then an ideal is a λ-ideal if and only if it is a $\mathbb{P}_\lambda(B)$-ideal.*

Proof. $\to$ If I is a λ-ideal, then whenever $p \in \mathbb{P}_\lambda(B)$ and $R \subseteq p \cap I$, we get $\bigvee R \in I$ as well.

$\leftarrow$ Assume I is a $\mathbb{P}_\lambda(B)$-ideal. Let γ be a cardinal with $\gamma < \lambda$ and assume $x_\alpha \in I$ for $\alpha < \gamma$. Let $y_\alpha = x_\alpha \wedge (\bigvee_{\beta < \alpha} x_\beta)'$ for $\alpha < \gamma$, and extend the set $\{y_\alpha | \alpha < \gamma\}^+$ to a partition $p \in \mathbb{P}_\lambda(B)$. Then clearly $\{y_\alpha | \alpha < \gamma\} \subseteq I$, so $\bigvee_{\alpha < \lambda} x_\alpha = \bigvee\{y_\alpha | \alpha < \lambda\}^+ \in I$. Therefore I is a λ-ideal. $\square$

Theorem 3.1.7 *Let B be a λ-complete Boolean algebra, then $\mathbb{P}_\lambda(B)/Z = \mathbb{P}_\lambda(B/Z)$ for each λ-filter Z.*

Proof. Let $q \in \mathbb{P}_\lambda(B)/Z$. Then there is a $p \in \mathbb{P}_\lambda(B)$ with $\pi_Z[p]^+ \preceq q$, so $|q| \leq |\pi_Z[p]^+| \leq |p| < \lambda$. Therefore $q \in \mathbb{P}_\lambda(B/Z)$.

Going the other direction, if $p \in \mathbb{P}_\lambda(B/Z)$, then there is a cardinal $\gamma < \lambda$ and a sequence $(a_\alpha)_{\alpha<\gamma}$ without repeating elements where $p = \{a_\alpha | \alpha < \gamma\}$. Now let $(b_\alpha)_{\alpha<\gamma}$ be a sequence in B with $\pi_Z(b_\alpha) = a_\alpha$ for all $\alpha < \gamma$. Let $c_\alpha = b_\alpha \wedge (\bigvee_{\beta<\alpha} b_\beta)'$ for each $\alpha < \gamma$. Then $\pi_Z(c_\alpha) = \pi_Z(b_\alpha \wedge (\bigvee_{\beta<\alpha} b_\beta)') = a_\alpha \wedge (\bigvee_{\beta<\alpha} a_\beta)' = a_\alpha$ for all $\alpha < \gamma$. Furthermore if we let $c_\gamma = (\bigvee_{\alpha<\gamma} c_\alpha)'$, then $\{c_\alpha | \alpha \leq \gamma\}^+ \in \mathbb{P}_\lambda(B)$, but $\pi_Z[\{c_\alpha|\alpha \leq \gamma\}^+]^+ = \pi_Z[\{c_\alpha|\alpha \leq \gamma\}]^+ = (\{a_\alpha|\alpha < \gamma\} \cup \{\bigvee_{\alpha<\gamma} a'_\alpha\})^+ = p$. Therefore $p \in \mathbb{P}_\lambda(B)/Z$. □

The admissible sets are well behaved when taking quotients of subcomplete Boolean partition algebras since the admissible sets in a quotient Boolean partition algebra $(B, F)/Z$ of a subcomplete Boolean partition algebra (B, F) are simply the images of the admissible sets in (B, F).

Proposition 3.1.8 *Let (B, F) be a subcomplete Boolean partition algebra and let Z be an F-filter. Then $S \subseteq B/Z$ is admissible in $(B, F)/Z$ if and only if S is of the form $\pi_Z[R]$ for some admissible $R \subseteq B$.*

Proof. $\leftarrow$ If $S = \pi_Z[R]$ for some admissible R, then since π_Z is a partition homomorphism, the set S is admissible as well (see Proposition 3.1.3).

$\rightarrow$ If $S \subseteq B/Z$ is admissible, then $S \subseteq q^*$ for some $q \in F/Z$, so there is a $p \in F$ where $\pi_Z[p]^+ \preceq q$, and in particular $S \subseteq (\pi_Z[p]^+)^*$. If $s \in S$, then there is an $T \subseteq p$ where $s = \bigvee \pi_Z[T] = \pi_Z(\bigvee T) \in \pi_Z[p^*]$. Therefore $S \subseteq \pi_Z[p^*]$, so there is an admissible $R \subseteq p^*$ where $\pi_Z[R] = S$. □

Proposition 3.1.9 *Let (B_i, F_i) be a Boolean partition algebra for $i \in I$, and let $(B, F) = \prod_{i\in I}(B_i, F_i)$. Then $R \subseteq B$ is admissible iff $\pi_i[R] \subseteq B_i$ is admissible for each $i \in I$.*

Proof. $\rightarrow$ Since each π_i is a partition homomorphism, each $\pi_i[R]$ is admissible.

$\leftarrow$ In order to simplify the notation, assume $(B_i, F_i) = (B \restriction a_i, F \restriction a_i)$ for $i \in I$. In this case, we have $\pi_i = \alpha_{a_i}$ for $i \in I$. Since $\alpha_{a_i}[R]$ is admissible for each $i \in I$, we have $\alpha_{a_i}[R] \subseteq p_i^*$ for some $p_i \in F \restriction a_i$. Thus $\bigcup_{i\in I} p_i \in F$. We now claim that $R \subseteq (\bigcup_{i\in I} p_i)^*$. If $r \in R$, then $a_i \wedge r = \alpha_{a_i}(r) \in p_i^*$ for $i \in I$, so $a_i \wedge r = \bigvee R_i$ for some $R_i \subseteq p_i$. Now $\bigcup_{i\in I} R_i \subseteq \bigcup_{i\in I} p_i$ and $\bigvee \bigcup_{i\in I} R_i = \bigvee_{i\in I} \bigvee R_i = \bigvee_{i\in I}(a_i \wedge r) = (\bigvee_{i\in I} a_i) \wedge r = r$, so $r \in (\bigcup_{i\in I} p_i)^*$. Therefore $R \subseteq (\bigcup_{i\in I} p_i)^*$, hence R is admissible. $\square$

Definition 3.1.10 *If (B, F) is a Boolean partition algebra, then a Boolean subalgebra $A \subseteq B$ is an* admissibly closed *subalgebra if whenever $R \subseteq B$ is admissible and $\bigvee^B R$ exists, then $\bigvee^B R \in A$.*

Example 3.1.11 *If (B, F) is a Boolean partition algebra and p is a partition of B (not necessarily in F), then p^* is always an admissibly closed subalgebra since p^* is closed under taking all least upper bounds.*

Example 3.1.12 *If $(A, F), (B, G)$ are Boolean partition algebras and $\phi : (A, F) \to (B, G)$ is a partition homomorphism, then whenever $C \subseteq B$ is admissibly closed, the subalgebra $\phi^{-1}[C]$ is admissibly closed as well. In particular, if I is an F-ideal. Let $Z = \{a \in B | a' \in I\}$. Then $I \cup Z = \pi_I^{-1}[\{0, 1\}]$ is an admissibly closed subalgebra of (A, F).*

Theorem 3.1.13 *Let (B, F) be a Boolean partition algebra. Then every admissibly closed subalgebra of (B, F) is an F-subalgebra. Conversely, if (B, F) is subcomplete, then every F-subalgebra of B is admissibly closed.*

Proof. The fact that every admissibly closed subalgebra of (B, F) is an F-subalgebra follows from the definitions. Now assume that (B, F) is subcomplete and $A \subseteq B$ is an F-subalgebra. We shall show that if $R \subseteq A$ is admissible in (B, F), then $\bigvee^B R \in A$ by transfinite induction on the cardinality $|R|$. Clearly, if $R \subseteq A$ is a finite set, then $\bigvee^B R \in A$. Now assume that whenever $S \subseteq A$ is admissible in (B, F) and $|S| < \lambda$, then $\bigvee^B S \in A$. Let $R \subseteq A$ be admissible in (B, F) and assume $|R| = \lambda$. Then $R \subseteq p^*$ for some $p \in F$, so there is a sequence $(A_\alpha)_{\alpha<\lambda}$ where $A_\alpha \subseteq p$ for $\alpha < \lambda$ and $\{\bigvee^B A_\alpha | \alpha < \lambda\} = R$. Now let $B_\alpha = A_\alpha \setminus (\bigcup_{\beta<\alpha} A_\beta)$ for $\alpha < \lambda$. Let

$$q = \{\bigvee^B B_\alpha | \alpha < \lambda\}^+ \cup p \setminus (\bigcup_{\alpha<\lambda} B_\alpha).$$

Then $p \preceq q$, so $q \in F$. Since $|\{\bigvee^B A_\beta | \beta < \alpha\}| < \lambda$, $\{\bigvee^B A_\beta | \beta < \alpha\}$ is admissible in (B, F), and $\{\bigvee^B A_\beta | \beta < \alpha\} \subseteq R \subseteq A$, we have

$$\bigvee^B \bigcup_{\beta<\alpha} A_\beta = \bigvee^B \{\bigvee^B A_\beta | \beta < \alpha\} \in A.$$

Therefore $\bigvee^B B_\alpha = (\bigvee^B A_\alpha) \wedge (\bigvee^B \bigcup_{\beta<\alpha} A_\beta)' \in A$ for $\alpha < \lambda$, so $\{\bigvee^B B_\alpha | \alpha < \lambda\}^+ \subseteq q \cap A$. Since A is an F-subalgebra, we have

$$\bigvee^B R = \bigvee^B (\bigcup_{\alpha<\lambda} A_\alpha) = \bigvee^B (\bigcup_{\alpha<\lambda} B_\alpha) = \bigvee^B \{\bigvee^B B_\alpha | \alpha < \lambda\}^+ \in A.$$

□

Proposition 3.1.14 *Let (B, F) be a Boolean partition algebra, and let $A \subseteq B$ be a subalgebra.*

1. *If $R \subseteq A$ is admissible in $(A, F|_A)$, then R is admissible in (B, F).*
2. *If (B, F) is a subcomplete, A is an F-subalgebra, and $R \subseteq A$ is admissible in (B, F), then R is admissible in $(A, F|_A)$.*

Proof. 1. Since the inclusion mapping $\iota : (A, F|_A) \to (B, F)$ is a partition homomorphism, if R is admissible in $(A, F|_A)$, then $R = \iota[R]$ is admissible in (B, F).

2. Assume that R is admissible in (B, F) and $R \subseteq A$. Then $R \subseteq p^*$ for some $p \in F$. Let V be the smallest admissibly closed (in (B, F)) subalgebra containing R. Clearly $V \subseteq A$ since A is an admissibly closed subalgebra of (B, F). Furthermore, $V \subseteq p^*$, and by Proposition 2.1.13, V is an $F|_{p^*}$-subalgebra of $(p^*, F|_{p^*}) = (p^*, \mathbb{P}(p^*))$. Therefore V is a complete subalgebra of p^* by Theorem 2.1.16. In other words, if $S \subseteq V$, then $\bigvee S \in V$ as well. Hence $V = q^*$ for some partition q with $p \preceq q$. Since $q \subseteq V \subseteq A$, we have $q \in F|_A$. Thus since $R \subseteq q^*$, the set R is admissible in $(A, F|_A)$.
□

The following result is essential in order to study the lattice of F-ideals in a Boolean partition algebra.

Theorem 3.1.15 *Let (B, F) be a locally refinable Boolean partition algebra and let*

$L \subseteq B$ be a lower set. Then $\{\bigvee R | \bigvee R \, exists \, and \, R \subseteq p \cap L \, for \, some \, p \in F\}$ is the F-ideal generated by L

Proof. Let $I = \{\bigvee R | \bigvee R \,\text{exists and}\,, R \subseteq p \cap L \,\text{for some}\, p \in F\}$. It suffices to show that I is an F-ideal. We shall first show that I is a lower set. Let $x \in B, y \in I$ and $x \leq y$. Then there is some $p \in F$ and some $R \subseteq p \cap L$ where $y = \bigvee R$. Let $q = p \wedge \{x, x'\}^+$. Then $\{r \wedge x | r \in R\}^+ \subseteq q \cap L$, and $\bigvee\{r \wedge x | r \in R\}^+ = \bigvee_{r \in R}(r \wedge x) = x \wedge \bigvee_{r \in R} r = x \wedge y = x$. Therefore $x \in I$ as well. We conclude that I is a lower set.

We now show that I is an F-ideal. Assume that $p \in F$, $R \subseteq p \cap I$ and $\bigvee R$ exists. If $r \in R$, then $r \in I$, so there is a $p_r \in F$ and some $q_r \subseteq p_r \cap L$ where $r = \bigvee q_r$. Also, $q_r \in F \restriction r$ for $r \in R$. Now let $q_s \in F \restriction s$ for each $s \in p \setminus R$. Let $q = \bigcup_{s \in p} q_s$. Then $q \in F$ since (B, F) is locally refinable. Moreover, $\bigcup_{r \in R} q_r \subseteq q \cap L$, and $\bigvee R = \bigvee_{r \in R} r = \bigvee_{r \in R} \bigvee q_r = \bigvee \bigcup_{r \in R} q_r \in I$. Thus I is an F-ideal. $\square$

As an easy consequence of the above theorem, if (B, F) is locally refinable and $L \subseteq B$ is a lower set then the F-ideal generated by L is the set of all least upper bounds of the form $\bigvee R$ where R is admissible and $R \subseteq L$.

Theorem 3.1.16 *Let (B, F) be a precomplete Boolean partition algebra, and let I, J be F-ideals. Then $\{a \vee b | a \in I, b \in J\}$ is also an F-ideal.*

Proof. Since (B, F) is precomplete, $(B, F^{(\infty)})$ is subcomplete and locally refinable and $\mathrm{Id}(B, F) = \mathrm{Id}(B, F^{(\infty)})$ by Theorem 2.4.21 and the remarks before Theorem 2.4.21. Thus by replacing (B, F) with $(B, F^{(\infty)})$, we may assume (B, F) is subcomplete and locally refinable. Let $\mathcal{I}$ be the F-ideal generated by $I \cup J$. Clearly $\{a \vee b | a \in I, b \in J\} \subseteq \mathcal{I}$. Now assume that $x \in \mathcal{I}$. Then there is an admissible $R \subseteq I \cup J$ where $x = \bigvee R$. Therefore the sets $R \cap I$ and $R \cap J$ are admissible, so $\bigvee(R \cap I) \in I$ and $\bigvee(R \cap J) \in J$, hence $x = \bigvee R = \bigvee(R \cap I) \vee \bigvee(R \cap J) \in \{a \vee b | a \in I, b \in J\}$. We conclude that $\mathcal{I} = \{a \vee b | a \in I, b \in J\}$. $\square$

3.2 Hyperspaces

In this section, we shall apply some of the results on admissibility developed in the last section to uniform spaces.

Definition 3.2.1 *If $(X,\mathcal{U})$ is a uniform space, then let $H(X)$ denote the collection of all closed subsets of X. For each $R \in \mathcal{U}$, let $\hat{R}$ be the relation on $H(X)$ where if $C, D \in H(X)$, then $(C,D) \in \hat{R}$ if and only if $D \subseteq R[C] = \{y \in X|(x,y) \in R \text{ for some } x \in C\}$ and $C \subseteq R[D]$. The collection $\{\hat{R}|R \in \mathcal{U}\}$ generates a uniformity on $H(X)$, so we may regard $H(X)$ as a uniform space, and if X is separated, then $H(X)$ is separated as well. We shall call $H(X)$ the* hyperspace *of the uniform space X. We shall call a separated uniform space $(X,\mathcal{U})$* supercomplete *if $H(X,\mathcal{U})$ is complete.*

Remark 3.2.2 *The closed set $\emptyset$ is an isolated point of $H(X)$, so $H(X)$ is complete if and only if $H(X) \setminus \{\emptyset\}$ is complete. It should be noted that some authors such as Isbell in [6] define the hyperspace $H(X)$ to be the collection of nonempty closed sets of X, but it is more practical for us to include the empty set in the hyperspace $H(X)$.*

Every supercomplete uniform space is complete since the mapping $x \mapsto \{x\}$ embeds X as a closed subspace of $H(X)$. On the other hand, supercompleteness is a much stronger property than completeness. In fact, if $(X,\mathcal{U})$ is a supercomplete uniform space, then X is paracompact, and if $(X,\mathcal{U})$ is a supercomplete non-Archimedean uniform space, then $(X,\mathcal{U})$ is ultraparacompact [6][p. 140]. We shall now give examples of hyperspaces and supercomplete spaces.

Example 3.2.3 *Recall that every compact space has a unique compatible uniform structure. If X is a compact space, then $H(X)$ is also a compact space [6][p. 31]. Hence, every compact space is supercomplete.*

Example 3.2.4 *Let X be a paracompact space, and let $\mathcal{U}$ be the collection of all sets E where $\{(x,x)|x \in X\} \subseteq E^{\circ}$. Then $\mathcal{U}$ is a uniformity on X and the uniform space $(X,\mathcal{U})$ is supercomplete by [23][p. 249] and [6][p. 140].*

Example 3.2.5 *Let (X, d) be a metric space. Then define a function $d^\sharp : H(X) \setminus \{\emptyset\} \to \mathbb{R}$ by*

$$d^\sharp(C, D) = \max(\sup\{d(x, D)|x \in C\}, \sup\{d(x, C)|x \in D\})$$

$$= \max(\sup\{\inf\{d(x, y)|y \in D\}|x \in C\}, \sup\{\inf\{d(x, y)|y \in C\}|x \in D\}).$$

Then $d^\sharp$ is a metric that induces the hyperspace uniformity on $H(X) \setminus \{\emptyset\}$. This metric is called the Hausdorff metric. The metric $d^\sharp$ is a complete metric if and only if d is a complete metric [6][p. 30]. In other words, every complete metric space is supercomplete.

In this section, we shall characterize the completion of a hyperspace of a non-Archimedean uniform space in terms of Boolean partition algebras and inverse limits. Given a Boolean partition algebra (B, F), we shall give the lattice of F-filters $\mathrm{Fi}(B, F)$ a complete partition space structure. If (X, M) is a complete partition space, then it turns out that the partition space $\mathrm{Fi}(\mathfrak{B}^*(X, M))$ is the completion of the hyperspace $H(X, M)$. We shall show that as a consequence, a complete partition space (X, M) is supercomplete if and only if $\mathfrak{B}^*(X, M)$ is superstable.

Definition 3.2.6 *Let (X, M) be a partition space. If $P \in M$, then let $\hat{P}$ be the collection of all sets of the form $\{C \in H(X)|\{A \in P|A \cap C \neq \emptyset\} = R\}$ where $R \subseteq P$. Then $\hat{P}$ is a partition of (X, M). In other words, $\hat{P}$ is the partition of $H(X)$ where if $C, D \in H(X)$, then $C = D(\hat{P})$ if and only if $\{A \in P|A \cap C \neq \emptyset\} = \{A \in P|A \cap D \neq \emptyset\}$. The hyperspace partition structure on $H(X, M)$ is the partition space on $H(X)$ generated by the set of partitions $\{\hat{P}|P \in M\}$.*

If $(X, \mathcal{U})$ is a non-Archimedean uniform space and M is the set of all uniform partitions of X, then the hyperspace uniformity $H(X, \mathcal{U})$ coincides with the hyperspace partition structure on $H(X, M)$. In particular, the hyperspace of a non-Archimedean uniform space is non-Archimedean. Assume that R is an equivalence relation in X which is simultaneously an entourage and P is the partition of X that corresponds with the equivalence relation R. Then it is easy to see that $\hat{R}$ is an equivalence relation on

$H(X)$ and the equivalence relation $\hat{R}$ corresponds to the partition $\hat{P}$. In other words, $(C, D) \in \hat{R}$ if and only if $C = D(\hat{P})$.

We shall now put a partition space structure on the lattice $\mathrm{Fi}(B, F)$ of F-filters on B. The lattice $\mathrm{Fi}(B, F)$ is essentially the "completion of the hyperspace" of the Boolean partition algebra (B, F).

Definition 3.2.7 *For each $p \in F$ and F-ideal I let $K_p(I) = \{a \in p | a \in I\} = p \cap I$, and for each F-filter Z let $J_p(Z) = \{a \in p | a' \notin Z\}$.*

If $I = \{a \in B | a' \in Z\}$, then $J_p(Z) = \{a \in p | a' \notin Z\} = \{a \in p | a \notin I\} = p \setminus K_p(I)$, so $J_p(Z) \cap K_p(I) = \emptyset$ and $J_p(Z) \cup K_p(I) = p$. One can easily show that if $R \subseteq p$ and $\bigvee R$ exists, then $\bigvee R \in I$ if and only if $R \subseteq K_p(I)$, and $\bigvee R \in Z$ if and only if $J_p(Z) \subseteq R$. Furthermore, $J_p(Z)$ and $K_p(I)$ are the unique sets with this property. In particular, if (B, F) is subcomplete, then $K_p(I)$ is the largest subset of p with $\bigvee K_p(I) \in I$, and $J_p(Z)$ is the smallest subset of p with $\bigvee J_p(Z) \in Z$.

The main motivation for studying the families $(K_p(I))_{p\in F}$ and $(J_p(Z))_{p\in F}$ is because one can recover the ideal I from the system $(K_p(I))_{p\in F}$ since $I = \{0\} \cup \bigcup_{p\in F} K_p(I)$ and because the families $(K_p(I))_{p\in F}$ are precisely the threads in an inverse system. Therefore, we may represent the F-ideals in a Boolean partition algebra (B, F) as an inverse limit. Similarly, one may also represent the F-filters as inverse limits. Since $\mathrm{Id}(B, F)$ and $\mathrm{Fi}(B, F)$ are inverse limits, the lattices $\mathrm{Id}(B, F)$ and $\mathrm{Fi}(B, F)$ are complete partition spaces.

Proposition 3.2.8 *Let (B, F) be a Boolean partition algebra. Let $p, q \in F$ be partitions with $p \preceq q$. Then*

1. $J_q(Z) = \{a \in q | \phi_{p,q}^{-1}[\{a\}] \cap J_p(Z) \neq \emptyset\} = \phi_{p,q}[J_p(Z)]$, *and*
2. $K_q(I) = \{a \in q | \phi_{p,q}^{-1}[\{a\}] \subseteq K_p(I)\} = \phi_{p,q}[K_p(I)^c]^c$.

Proof. If $f : X \to Y$ and $A \subseteq X$, then it is easy to show that $f[A] = \{b \in Y | f^{-1}[\{b\}] \cap A \neq \emptyset\}$ and $f[A^c]^c = \{b \in Y | f^{-1}[\{b\}] \subseteq A\}$.

1. Let $R \subseteq q$ be a set with a least upper bound. Then $\bigvee \phi_{p,q}^{-1}[R] = \bigvee R \in Z$ if and only if $J_p(Z) \subseteq \phi_{p,q}^{-1}[R]$ if and only if $\phi_{p,q}[J_p(Z)] \subseteq R$. Therefore, we have

$\phi_{p,q}[J_p(Z)] = J_q(Z)$.

2. We have $K_q(I) = J_q(Z)^c = \phi_{p,q}[J_p(Z)]^c = \phi_{p,q}[K_p(I)^c]^c$. $\square$

Definition 3.2.9 *If $p \in F$, then let $\hat{p}$ be the partition of Fi(B,F) where $Z_1 = Z_2(\hat{p})$ if and only if $J_p(Z_1) = J_p(Z_2)$. If $p \in F$, then let $\tilde{p}$ be the partition of Id(B,F) where $I_1 = I_2(\tilde{p})$ if and only if $K_p(I_1) = K_p(I_2)$. In other words, $I_1 = I_2(\tilde{p})$ if and only if $I_1 \cap p = I_2 \cap p$.*

Proposition 3.2.10 *Let (B,F) be a Boolean partition algebra, and let $p,q \in F$. Then the following are equivalent.*

1. $p \preceq q$.
2. $\hat{p} \preceq \hat{q}$.
3. $\tilde{p} \preceq \tilde{q}$.

Proof. $2 \rightarrow 3$. Assume $\hat{p} \preceq \hat{q}$. Then let I_1, I_2 be F-ideals with $I_1 = I_2(\tilde{p})$. Then let $Z_i = \{a \in B | a' \in I_i\}$ for $i \in \{1,2\}$. Then $K_p(I_1) = K_p(I_2)$, so $J_p(Z_1) = p \setminus K_p(I_1) = p \setminus K_p(I_2) = J_p(Z_2)$. Therefore $Z_1 = Z_2(\hat{p})$, hence $Z_1 = Z_2(\hat{q})$. In other words, $J_q(Z_1) = J_q(Z_2)$, so $K_q(I_1) = K_q(I_2)$ and thus $I_1 = I_2(\tilde{q})$. We conclude that $\tilde{p} \preceq \tilde{q}$.

$1 \rightarrow 2$. Let $Z_1, Z_2 \in \mathrm{Fi}(B,F)$ be F-filters with $Z_1 = Z_2(\hat{p})$. Then $J_p(Z_1) = J_p(Z_2)$, so $J_q(Z_1) = \phi_{p,q}[J_p(Z_1)] = \phi_{p,q}[J_p(Z_2)] = J_q(Z_2)$. Therefore $Z_1 = Z_2(\hat{q})$. We conclude that $\hat{p} \preceq \hat{q}$.

$3 \rightarrow 1$. We shall prove this direction by contrapositive. Assume that $p \not\preceq q$. Then there is some $b \in q$ where b is not the least upper bound of $p \cap \downarrow b$, so there is some $c \in B$ where $c < b$ and where c is an upper bound of $p \cap \downarrow b$. Thus $p \cap \downarrow c \subseteq p \cap \downarrow b \subseteq p \cap \downarrow c$, so $\downarrow b = \downarrow c(\tilde{p})$. On the other hand, $\downarrow b \cap q = \{b\}$, but $\downarrow c \cap q = \emptyset$. Therefore $\downarrow b \neq \downarrow c(\tilde{q})$. We conclude that $\tilde{p} \not\preceq \tilde{q}$. $\square$

We conclude from the above theorem that the partitions $\{\hat{p} | p \in F\}$ generate a partition space structure on $\mathrm{Fi}(B,F)$. Similarly, $\mathrm{Id}(B,F)$ is a partition space generated by the partitions $\{\tilde{p} | p \in F\}$. Furthermore, one can easily see that the mapping $I \mapsto \{a \in B | a' \in I\}$ is a uniform homeomorphism from $\mathrm{Id}(B,F)$ to $\mathrm{Fi}(B,F)$. We shall now show that the space $\mathrm{Id}(B,F)$ is complete by representing $\mathrm{Id}(B,F)$ as a inverse limit.

Definition 3.2.11 *Let (B,F) be a Boolean partition algebra. For each $p,q \in F$ with $p \preceq q$, let $\alpha_{p,q} : P(p) \to P(q)$ be the mapping where $\alpha_{p,q}(R) = \{a \in q | \phi_{p,q}^{-1}[\{a\}] \subseteq R\} = \phi_{p,q}[R^c]^c$.*

If $p \preceq q \preceq r$, then $\alpha_{q,r}\alpha_{p,q}(R) = \phi_{q,r}[\phi_{p,q}[R^c]]^c = \phi_{p,r}[R^c]^c = \alpha_{p,r}(R)$. Furthermore, $\alpha_{p,p}$ is the identity map. Therefore the system $(P(p))_{p\in F}$ becomes an inverse system with transitional mappings $\alpha_{p,q}$. Take note that $(P(p))_{p\in F}$ also becomes an inverse system with transitional mappings $\phi''_{p,q}$ and the inverse systems $(P(p))_{p\in F}, (\alpha_{p,q})_{p\preceq q})$ and $(P(p))_{p\in F}, (\phi_{p,q})_{p\preceq q})$ are isomorphic.

Lemma 3.2.12 *Let $(R_p)_{p\in F} \in \overset{\lim}{\longleftarrow} (P(p), \alpha_{p,q})_{p,q\in F}$. Let $b \in B$. Then the following are equivalent.*

1. *$b \in \{0\} \cup \bigcup_{p\in F} R_p$.*
2. *If $p \in F$ and $b \in p$, then $b \in R_p$.*
3. *There is some $p \in F$ where $b \in p^*$ and $\{a \in p | a \leq b\} \subseteq R_p$.*
4. *If $p \in F$ and $b \in p^*$, then $\{a \in p | a \leq b\} \subseteq R_p$.*

Proof. $2 \to 1$. This direction is obvious.

$4 \to 2$. If $p \in F$ and $b \in p$, then $b \in p^*$, so $\{b\} = \{a \in p | a \leq b\} \subseteq R_p$. Therefore $b \in R_p$.

$1 \to 3$. Assume $b \in \{0\} \cup \bigcup_{p\in F} R_p$. If $b = 0$, then $b \in p^*$ for $p \in F$, and clearly $\{a \in p | a \leq b\} = \emptyset \subseteq R_p$. If $b \in R_p$ for some $p \in F$, then $b \in p^*$, and $\{a \in p | a \leq b\} = \{b\} \subseteq R_p$.

$3 \to 4$. Assume $p \in F$, $b \in p^*$, and $\{a \in p | a \leq b\} \subseteq R_p$. Also assume $q \in F$ and $b \in q^*$. Let $r = p \wedge q$. Then $b \in r^*$.

We claim that $\{a \in r | a \leq b\} \subseteq R_r$. Take note that $R_p = \alpha_{r,p}(R_r) = \{a \in p | \phi_{r,p}^{-1}[\{a\}] \subseteq R_r\}$. Assume $a \in r$ and $a \leq b$. Then $\phi_{r,p}(a) \wedge b > 0$, so $\phi_{r,p}(a) \leq b$ since $b \in p^*$. Therefore $\phi_{r,p}(a) \in R_p$, so $\phi_{r,p}^{-1}[\{\phi_{r,p}(a)\}] \subseteq R_r$. However, since $a \in \phi_{r,p}^{-1}[\{\phi_{r,p}(a)\}]$, we have $a \in R_r$. We conclude that $\{a \in r | a \leq b\} \subseteq R_r$.

We shall now prove that $\{a \in q | a \leq b\} \subseteq R_q$. We take note that $R_q = \{a \in q | \phi_{r,q}^{-1}[\{a\}] \subseteq R_r\}$. Assume that $a \in q$ and $a \leq b$. If $c \in \phi_{r,q}^{-1}[\{a\}]$, then $\phi_{r,q}(c) = a \leq b$, so $c \leq b$, and hence $c \in R_r$. Thus $\phi_{r,q}^{-1}[\{a\}] \subseteq R_r$. Therefore $a \in R_q$. We conclude that $\{a \in q | a \leq b\} \subseteq R_q$. $\square$

Theorem 3.2.13 *Let (B, F) be a Boolean partition algebra, and let $(R_p)_{p \in F} \in \varprojlim (P(p), \alpha_{p,q})_{p \in F}$ Then $\{0\} \cup \bigcup_{p \in F} R_p$ is an F-ideal.*

Proof. We shall use Lemma 3.2.12 several times in this proof. We shall first show that $\{0\} \cup \bigcup_{p \in F} R_p$ is a lower set. Let $b \in \{0\} \cup \bigcup_{p \in F} R_p$ and let $a \leq b$. If $a = 0$, then $a \in \{0\} \cup \bigcup_{p \in F} R_p$. If $a \neq 0$, then let $p = \{a, b \wedge a', b'\}^+$. Then $p \in F$ and $b \in p^*$, so $\{c \in p | c \leq b\} \subseteq R_p$. Therefore $a \in R_p \subseteq \{0\} \cup \bigcup_{p \in F} R_p$.

To complete the proof, assume $p \in F$ and $S \subseteq p$ is a set where $\bigvee S$ exists and $S \subseteq \{0\} \cup \bigcup_{p \in F} R_p$. Then $\bigvee S \in p^*$. Furthermore, since $S \subseteq p$, we have $S \subseteq R_p$, and $\{a \in p | a \leq \bigvee S\} = S \subseteq R_p$, so $\bigvee S \in \{0\} \cup \bigcup_{p \in F} R_p$. Therefore $\{0\} \cup \bigcup_{p \in F} R_p$ is an F-ideal. $\square$

If $p \preceq q$, then $K_q(I) = \{a \in q | \phi_{p,q}^{-1}[\{a\}] \subseteq K_p(I)\} = \alpha_{p,q}(K_p(I))$. Therefore $(K_p(I))_{p \in F} \in \varprojlim (P(p))_{p \in F}$. We shall shortly show that every thread in $\varprojlim (P(p))_{p \in F}$ is of the form $\alpha_{p,q}(K_p(I))$ for some F-ideal I.

Definition 3.2.14 *Define $\mathcal{M} : Id(B, F) \to \varprojlim (P(p))_{p \in F}$ by letting $\mathcal{M}(I)(p) = K_p(I) = p \cap I$ for $p \in F$ and $I \in Id(B, F)$. Define a map $\mathcal{L} : \varprojlim (P(p))_{p \in F} \to Id(B, F)$ by $\mathcal{L}(R_p)_{p \in F} = \{0\} \cup \bigcup_{p \in F} R_p$.*

We shall give each set $P(p)$ the discrete partition space structure, and we shall give $\varprojlim (P(p))_{p \in F}$ the inverse limit partition space structure. Then $\varprojlim (P(p))_{p \in F}$ is a complete partition space. The following result can be considered a generalization of Theorem 1.3.42 except that we formulate this result in terms of ideals instead of filters.t/

Theorem 3.2.15 *The maps $\mathcal{L} : \varprojlim (P(p))_{p \in F} \to Id(B, F)$ and $\mathcal{M} : Id(B, F) \to \varprojlim (P(p))_{p \in F}$ are inverses. Furthermore, $\mathcal{M}$ and $\mathcal{L}$ are uniform homeomorphisms.*

Proof. Let $(R_p)_{p \in F} \in \varprojlim (P(p))_{p \in F}$, and let $q \in F$. Then $\mathcal{M}(\mathcal{L}(R_p)_{p \in F})(q) = \mathcal{M}(\{0\} \cup \bigcup_{p \in F} R_p)(q) = q \cap (\{0\} \cup \bigcup_{p \in F} R_p) = R_q$ by Lemma 3.2.12. Therefore $\mathcal{M} \circ \mathcal{L}(R_p)_{p \in F} = (R_p)_{p \in F}$.

Let I be an F-ideal. Then $\mathcal{L}(\mathcal{M}(I)) = \mathcal{L}((p \cap I)_{p\in F}) = \{0\} \cup \bigcup_{p\in F}(p \cap I) = \{0\} \cup (I \cap (\bigcup_{p\in F} p)) = \{0\} \cup (I \cap B^+) = I$. Therefore $\mathcal{L} \circ \mathcal{M}$ is the identity function as well. We conclude that the functions $\mathcal{L}$ and $\mathcal{M}$ are inverses.

For each $q \in F$, let $\exp(q)$ be the partition of $\overset{\lim}{\longleftarrow}(P(p))_{p\in F}$ where if $(R_p)_{p\in F}, (S_p)_{p\in F} \in \overset{\lim}{\longleftarrow} (P(p))_{p\in F}$, then $(R_p)_{p\in F} = (S_p)_{p\in F}(\exp(q))$ if and only if $R_q = S_q$. Then $\{\exp(p)|p \in F\}$ generates the partition space structure on $\overset{\lim}{\longleftarrow}(P(p))_{p\in F}$. We shall show that $\mathcal{L}, \mathcal{M}$ are uniform homeomorphisms by showing that $\mathcal{M}$ induces a one-to-one correspondence between $(\tilde{p})_{p\in F}$ and $(\exp(p))_{p\in F}$. Let $p \in F$ and let $I, J \in \mathrm{Id}(B, F)$. Then $I = J(\tilde{p})$ if and only if $\mathcal{M}(I)(p) = K_p(I) = K_p(J) = \mathcal{M}(J)(p)$ if and only if $\mathcal{M}(I) = \mathcal{M}(J)(\exp(p))$. $\square$

Theorem 3.2.16 *Let (X, M) be a separated uniform space. Let $\iota : H(X) \to Fi(\mathfrak{B}^*(X, M))$ be the mapping where $\iota(C) = \{R \in \mathfrak{B}^*(X, M) : C \subseteq R\}$. Then ι is a uniform embedding and $\iota[H(X)]$ is dense in the space $Fi(\mathfrak{B}^*(X, M))$. In other words, $Fi(\mathfrak{B}^*(X, M))$ is the completion of the hyperspace $H(X)$. In particular, (X, M) is supercomplete if and only if ι is a bijection.*

Proof. If $C \in H(X)$, then $C = \bigcap \iota(C)$ since C is closed. Thus, if $C, D \in H(X)$ and $\iota(C) = \iota(D)$, then $C = \bigcap \iota(C) = \bigcap \iota(D) = D$, so ι is injective.

If $P \in M, A \in P$, and $C \in H(X)$, then $C \cap A = \emptyset$ if and only if $C \subseteq A^c$ if and only if $A^c \in \iota(C)$. Thus

$$\{A \in P|C \cap A \neq \emptyset\} = \{A \in P|A^c \notin \iota(C)\} = J_P(\iota(C)).$$

Therefore, if $C, D \in H(X)$, then $C = D(\hat{P})$ if and only if

$$J_P(\iota(C)) = \{A \in P|C \cap A \neq \emptyset\} = \{A \in P|D \cap A \neq \emptyset\} = J_P(\iota(D))$$

if and only if $\iota(C) = \iota(D)(\hat{P})$. We conclude that ι is a uniform embedding.

If U is an open subset of $\mathrm{Fi}(\mathfrak{B}^*(X, M))$, then there is a $P \in M$ and a subset $R \subseteq P$ with $\{Z \in \mathrm{Fi}(\mathfrak{B}^*(X, M)) : J_P(Z) = R\} \subseteq U$. However $\bigcup R$ is closed in X and $J_P(\iota(\bigcup R)) = \{A \in P|A \cap \bigcup R \neq \emptyset\} = R$, so $\iota(\bigcup R) \in U$. Hence $\iota[H(X)]$ is dense in

$\mathrm{Fi}(\mathfrak{B}^*(X,M))$ $\square$

Theorem 3.2.17 *A partition space (X,M) is supercomplete if and only if (X,M) is complete and $\mathfrak{B}^*(X,M)$ is superstable.*

Proof. $\rightarrow$ If (X,M) is supercomplete, then whenever Z is a proper F-filter, $Z=\iota(C)$ for some non-empty closed set $C\subseteq X$, so $\bigcap Z=C\neq\emptyset$. We conclude that (X,M) is complete and $\mathfrak{B}^*(X,M)$ is superstable by Remark 2.2.19.

$\leftarrow$ If (X,M) is complete and $\mathfrak{B}^*(X,M)$ is superstable, then each $Z\in\mathrm{Fi}(\mathfrak{B}^*(X,M))$ is the intersection of M-ultrafilters. However, the M-ultrafilters are precisely the sets of the form $\{R\in\mathfrak{B}^*(X,M):x_0\subseteq R\}$ for some $x_0\in X$. Thus for each M-filter Z, there is a set $S\subseteq X$ where

$$Z=\{R\in\mathfrak{B}^*(X,M):S\subseteq R\}=\{R\in\mathfrak{B}^*(X,M):\overline{S}\subseteq R\}=\iota(\overline{S}).$$

Therefore the mapping ι is surjective, so $H(X)$ is complete. $\square$

Remark 3.2.18 *Theorem 3.2.17 may be used to characterize weakly compact cardinals and strongly compact cardinals in terms of supercompleteness without much effort. See [1] for such a characterization of weakly compact cardinals.*

Definition 3.2.19 *Let $(A,F),(B,G)$ be Boolean partition algebras, and let $\phi:(A,F)\rightarrow(B,G)$ be a partition homomorphism. Then define a mapping $Id(\phi):Id(B,G)\rightarrow Id(A,F)$ by letting $Id(\phi)(I)=\phi^{-1}[I]$ for each G-ideal I. Define a mapping $\phi^\sharp:\overset{\lim}{\longleftarrow}(P(p))_{p\in G}\rightarrow\overset{\lim}{\longleftarrow}(P(p))_{p\in F}$ by $\phi^\sharp=\mathcal{M}Id(\phi)\mathcal{L}$.*

Proposition 3.2.20 *The mapping $Id(\phi):Id(B,G)\rightarrow Id(A,F)$ is uniformly continuous.*

Proof. Let $p\in F$. Then let $q=\phi[p]^+$. Assume I,J are G-ideals, and $I=J(\tilde{q})$. Then $I\cap q=J\cap q$. Therefore $\mathrm{Id}(\phi)(I)\cap p=\phi^{-1}[I]\cap p=\{a\in p:\phi(a)\in I\}=\{a\in p:\phi(a)\in J\}=\mathrm{Id}(\phi)(J)\cap p$. Thus $\mathrm{Id}(\phi)(I)=\mathrm{Id}(\phi)(J)(\tilde{p})$. We conclude that $\mathrm{Id}(\phi)$ is uniformly continuous. $\square$

Proposition 3.2.21 *Let $\phi : (A, F) \to (B, G)$ be a partition homomorphism. Let $(S_p)_{p\in G} \in^{\lim}_{\longleftarrow} (P(p))_{p\in G}$ and assume that $(R_p)_{p\in F} = \phi^\sharp(S_p)_{p\in G}$. Then $R_p = \{a \in p|\phi(a) = 0\, or\, a \in S_{\phi[p]^+}\}$ for each $p \in F$.*

Proof. Let $I = \mathcal{L}((S_p)_{p\in G})$. Let $p \in F$ and let $a \in p$. Then $a \in R_p$ iff $a \in \mathrm{Id}(\phi)(\mathcal{L}(S_p)_{p\in F}) = \mathrm{Id}(\phi)(I)$ iff $\phi(a) \in I$ iff $\phi(a) = 0$ or $a \in S_{\phi[p]^+}$. Therefore $R_p = \{a \in p|\phi(a) = 0\,\mathrm{or}\, a \in S_{\phi[p]^+}\}$. □

Theorem 3.2.22 *Let $(A, F), (B, G)$ be Boolean partition algebras and let $\phi : (A, F) \to (B, G)$ be a partition homomorphism.*

1. *If $\phi : (A, F) \to (B, G)$ is surjective, then Id(ϕ) is injective.*
2. *If Id(ϕ) : Id$(B, G) \to$ Id(A, F) is surjective, then ϕ is injective.*
3. *If ϕ is injective, then Id$(\phi)[$Id$(B, G)]$ is dense in Id(A, F).*
4. *The mapping Id(ϕ) is a uniform embedding if and only if the filter G is generated by the filterbase $\{\phi[p]^+|p \in F\}$.*
5. *The mapping Id(ϕ) is a uniform homeomorphism if and only if whenever $\iota : (B, G) \to (C, H)$ is a subcompletion, then $\iota\phi : (A, F) \to (C, H)$ is also a subcompletion.*

Proof. 1. If I, J are distinct G-ideals. Without loss of generality, we may assume that there is a $b \in B$ with $b \in I$ and $b \notin J$. Then there is an $a \in A$ with $\phi(a) = b$. Therefore $a \in \mathrm{Id}(\phi)(I)$ and $a \notin \mathrm{Id}(\phi)(J)$. We conclude that $\mathrm{Id}(\phi)(I) \neq \mathrm{Id}(\phi)(J)$, so $\mathrm{Id}(\phi)$ is injective.

2. Suppose ϕ is not injective. Then $\ker(\phi) \neq \{0\}$, so if $I \subseteq B$ is a G-ideal, then $\mathrm{Id}(\phi)(I) = \phi^{-1}[I] \supseteq \ker(\phi) \neq \{0\}$. Therefore $\mathrm{Id}(\phi)(I) \neq \{0\}$, thus $\mathrm{Id}(\phi)$ is not surjective.

3. If $U \subseteq \mathrm{Id}(A, F)$ is a non-empty open set, then there is a $p \in F$ and an $R \subseteq p$ where $\{I \in \mathrm{Id}(A, F)|I \cap p = R\} \subseteq U$. Let J be the G-ideal generated by the set $\phi[R]$. We claim that $\mathrm{Id}(\phi)(J) \cap p = R$. If $a \in R$, then $a \in p$ and $\phi(a) \in \phi[R] \subseteq J$, so $a \in p \cap \mathrm{Id}(\phi)(J)$. Now assume $a \in p$, but $a \notin R$. Then $\phi(a) \wedge b = 0$ for $b \in \phi[R]$, so $b \leq \phi(a)'$ for $b \in \phi[R]$, thus $\phi[R] \subseteq\downarrow \phi(a)'$, and $J \subseteq\downarrow \phi(a)'$. Since $\phi(a) \neq 0$, we

have $\phi(a) \notin \downarrow \phi(a)'$, so $\phi(a) \notin J$, hence $a \notin \mathrm{Id}(\phi)(J)$. Therefore $R = p \cap \mathrm{Id}(\phi)(J)$, so $\mathrm{Id}(\phi)(J) \in U$. We conclude that $\mathrm{Id}(\phi)[\mathrm{Id}(B, G)]$ is dense in $\mathrm{Id}(A, F)$.

4. Before we prove this result, we take note that if $p \in F$, and $I, J \in \mathrm{Id}(B, G)$, then $\mathrm{Id}(\phi)(I) = \mathrm{Id}(\phi)(J)(\tilde{p})$ iff $p \cap \phi^{-1}[I] = p \cap \phi^{-1}[J]$ iff $\{a \in p | \phi(a) \in I\} = \{a \in p | \phi(a) \in J\}$ iff $I = J(\widetilde{\phi[p]^+})$.

$\longrightarrow$ Assume that $\mathrm{Id}(\phi)$ is a uniform embedding. Then for each $q \in G$ there is a $p \in F$ where if $\mathrm{Id}(\phi)(I) = \mathrm{Id}(\phi)(J)(\tilde{p})$, then $I = J(\tilde{q})$. In other words, if $I = J(\widetilde{\phi[p]^+})$, then $I = J(\tilde{q})$. Therefore, $\widetilde{\phi[p]^+} \preceq \widetilde{q}$. We conclude that $\phi[p]^+ \preceq q$.

$\longleftarrow$ We begin by noting that the mapping $\mathrm{Id}(\phi)$ is injective. If $I, J \in \mathrm{Id}(B, G)$ and $I \neq J$, then since $\{\phi[p]^+ | p \in F\}$ generates the filter G, there is some $p \in F$ where $I \neq J(\widetilde{\phi[p]^+})$, so $\mathrm{Id}(\phi)(I) \neq \mathrm{Id}(\phi)(J)(\tilde{p})$, and in particular $\mathrm{Id}(\phi)(I) \neq \mathrm{Id}(\phi)(J)$.

To show that $\mathrm{Id}(\phi)$ is a uniform embedding, assume that $q \in G$. Then there is a $p \in F$ where $\phi[p]^+ \preceq q$. In this case, for $I, J \in \mathrm{Id}(B, G)$, if $\mathrm{Id}(\phi)(I) = \mathrm{Id}(\phi)(J)(\tilde{p})$, then $I = J(\widetilde{\phi[p]^+})$, so $I = J(\widetilde{q})$.

5. $\rightarrow$ If $\mathrm{Id}(\phi)$ is a uniform homeomorphism, then since $\mathrm{Id}(\phi)$ is surjective, the mapping ϕ is injective. Furthermore, since $\mathrm{Id}(\phi)$ is a uniform embedding, G is generated by the filterbase $\{\phi[p]^+ | p \in F\}$. Therefore, whenever $\iota : (B, G) \to (C, H)$ is a subcompletion, the mapping $\iota\phi : (A, F) \to (C, H)$ is also a subcompletion.

$\leftarrow$ Assume that whenever $\iota : (B, G) \to (C, H)$ is a subcompletion, then $\iota\phi$ is also a subcompletion. Then ϕ is injective, and G is generated by the filterbase $\{\phi[p] | p \in F\}$. Therefore, $\mathrm{Id}(\phi)$ is a uniform embedding and $\mathrm{Id}(\phi)[\mathrm{Id}(B, G)]$ is dense in $\mathrm{Id}(A, F)$. Since $\mathrm{Id}(A, F)$ and $\mathrm{Id}(B, G)$ are complete, the mapping $\mathrm{Id}(\phi)$ is a uniform homeomorphism. $\square$

Remark 3.2.23 *By the above theorem, we can see that if (B, G) is the subcompletion of (A, F), and $\iota : (A, F) \to (B, G)$ is the inclusion mapping, then $Id(\iota) : Id(A, F) \to Id(B, G)$ is a lattice isomorphism. In particular, if (C, H) is the subcompletion of $(A, F^{(\infty)})$, then the lattices $Id(A, F)$ and $Id(C, H)$ are isomorphic.*

3.3 Join-Admissibility, Meet-Admissibililty, and Galois Connections

In this section, we shall dicuss the notions of join-admissibility and meet-admissibility and their relation between ideals and filters. We shall end this section by discussing the Galois correspondence between locally refinable Boolean partition algebras and lattices of ideals on a Boolean algebra. In the next section, we shall apply the notions of meet-admissibility and join-admissibility to characterize the notion of stability and some large cardinals.

Definition 3.3.1 *Let (A, F) be a Boolean partition algebra. If $R \subseteq A$, then we say that R is* join-admissible *if $\downarrow \bigvee R$ is the F-ideal generated by R. Similarly, we say that $R \subseteq A$ is* meet-admissible *if $\uparrow \bigwedge R$ is the F-filter generated by R. If $R \subseteq A$, then we shall write $\mathcal{L}(R)$ for the F-ideal generated by R. Clearly R is join-admissible if and only if $\bigvee R$ exists and $\bigvee R \in \mathcal{L}(R)$.*

In this section, we shall only study join-admissibility since all the results concerning meet-admissible sets are analogous. Intuitively, a set $R \subseteq A$ is join-admissible if the least upper bound $\bigvee R$ exists and we consider the least upper bound of the set R in our Boolean partition algebra.

Remark 3.3.2 *Take note that if $\mathcal{L}(R) = \downarrow x$, then x is the least upper bound of R. Clearly x is an upper bound of R. If y is also an upper bound of R, then $R \subseteq \downarrow y$, so $\downarrow x = \mathcal{L}(R) \subseteq \downarrow y$, hence $x \leq y$. We conclude that a set is join-admissible if and only if it generates a principal F-ideal.*

Example 3.3.3 *It is easy to see that if R is admissible and $\bigvee R$ exists, then R is join-admissible. In a subcomplete Boolean partition algebra every admissible set is both join-admissible and meet-admissible. From these basic facts, one can easily conclude that an ideal $I \subseteq B$ is an F-ideal if and only if I is closed under taking the least upper bound of join-admissible subsets.*

Example 3.3.4 *Let λ be a cardinal, and let B be a λ-Boolean algebra. If $R \subseteq B$ and $|R| < \lambda$, then R is join-admissible in the Boolean partition algebra $(B, \mathbb{P}_\lambda(B))$. If λ*

is a regular cardinal, then a subset $R \subseteq B$ is admissible in $(B, \mathbb{P}_\lambda(B))$ if and only if there is an $S \subseteq R$ with $|S| < \lambda$ and where $\bigvee S$ is an upper bound of R.

Proposition 3.3.5 *Let B be a Boolean algebra. Then a subset $R \subseteq B$ is join-admissible in $(B, \mathbb{P}(B))$ if and only if R has a least upper bound.*

Proof. $\rightarrow$. This should be trivial.

$\leftarrow$. Assume that R has a least upper bound. Now let c be a maximal cellular family (under the ordering $\subseteq$) subject to the condition that $c \preceq R$. Such a cellular family c exists by Zorn's lemma. We claim that $\bigvee c = \bigvee R$. Clearly $\bigvee R$ is an upper bound of c. If x is an upper bound of c with $x < \bigvee R$, then $\bigvee_{r \in R}(x' \wedge r) = x' \wedge \bigvee R > 0$. Therefore there is some $r \in R$ where $x' \wedge r > 0$. However, $c \cup \{x' \wedge r\}$ is a cellular family that properly extends c with $c \cup \{x' \wedge r\} \preceq R$. This is a contradiction. Thus $\bigvee R$ is the least upper bound of c. The set c is admissible since there is a partition p that extends c, and $c \subseteq \mathcal{L}(R)$, so $\bigvee R = \bigvee c \in \mathcal{L}(R)$ as well. We conclude that R is admissible. $\square$

We conclude that if B is a Boolean algebra, then the $\mathbb{P}(B)$-ideals are precisely the complete ideals.

The join-admissible sets are precisely the sets with least upper bounds preserved under partitional mappings.

Proposition 3.3.6 *Let (A, F) be a Boolean partition algebra. Then a subset $R \subseteq (A, F)$ is join-admissible if and only if $\bigvee R$ exists and whenever $\phi : (A, F) \rightarrow B$ is a partitional mapping, we have $\bigvee \phi[R]$ exist and $\phi(\bigvee R) = \bigvee \phi[R]$.*

Proof. $\leftarrow$ Let $I = \mathcal{L}(R)$. Then $\pi_I(\bigvee R) = \bigvee \pi_I[R] = 0$, so $\bigvee R \in I = \mathcal{L}(R)$. We conclude that R is join-admissible.

$\rightarrow$ Assume that R is join-admissible. Then since the F-ideals are precisely the $F^{(\infty)}$-ideals, the set R generates a principal $F^{(\infty)}$-ideal, so there is some $p \in F^{(\infty)}$ and some $S \subseteq p$ where $S \preceq R$ and where $x = \bigvee S$. Since S is admissible in $(A, F^{(\infty)})$, and ϕ is a partitional mapping from $(A, F^{(\infty)})$ to B, we have $\phi(\bigvee S) = \bigvee \phi[S]$. Clearly $\phi(\bigvee R) = \phi(\bigvee S) = \bigvee \phi[S] \leq \bigvee \phi[R] \leq \phi(\bigvee R)$ (here we take the least upper bound

$\bigvee \phi[R]$ in the completion of B, so we do not need to be concerned whether least upper bound $\bigvee \phi[R]$ exists in B). Therefore $\bigvee \phi[R] = \phi(\bigvee R)$. □

In the following result, recall that $\iota : (A, F) \to P(S^*(A, F))$ is the mapping defined by $\iota(a) = \{\mathcal{U} \in S^*(A, F) | a \in \mathcal{U}\}$.

Proposition 3.3.7 *Let (A, F) be a superstable Boolean partition algebra. Then $R \subseteq A$ is join-admissible with $\bigvee R = a$ if and only if $\bigcup \iota[R] = \iota(a)$. Similarly, a subset $R \subseteq A$ is meet-admissible with $\bigwedge R = a$ if and only if $\bigcap \iota[R] = \iota(a)$.*

Proof. $\to$ This is because $\iota : (A, F) \to P(S^*(A, F))$ is a partitional mapping.

$\leftarrow$ We shall just prove the case for meet-admissibility. Then whenever $\mathcal{U} \in S^*(A, F)$ we have $a \in \mathcal{U}$ iff $\mathcal{U} \in \iota(a) = \bigcap \iota[R]$ iff $\mathcal{U} \in \iota(b)$ for $b \in R$ iff $R \subseteq \mathcal{U}$. Therefore since every F-filter is the intersection of F-ultrafilters, $a \in Z$ if and only if $R \subseteq Z$ whenever Z is an F-filter. In other words, $\uparrow a \subseteq Z$ if and only if $R \subseteq Z$ whenever Z is an F-filter. Therefore $\uparrow a$ is the F-filter generated by R, so R is meet-admissible with $\bigwedge R = a$. □

Example 3.3.8 *The above result does not necessarily hold for stable Boolean partition algebras. For example, let X be an uncountable set of cardinality continuum. Then $(P(X), \mathbb{P}_{\aleph_1} P(X))$ is a stable Boolean partition algebra. If $I = \{R \subseteq X : |R| < \aleph_1\}$, then I is a non-principal σ-ideal, so I is not join-admissible. On the other hand, one can easily see that*

$$\bigcup \iota[I] = S^*(P(X), \mathbb{P}_{\aleph_1} P(X)) = \iota(X)$$

since the only σ-ultrafilters on X are the principal ultrafilters.

We shall now establish a Galois connection between Boolean partition algebras and lattices of ideals and a similar Galois connection between Boolean partition algebras and sets of ultrafilters. If we are given a Boolean algebra B and a collection R of ideals on B, then we may want to find the largest filter $F \subseteq \mathbb{P}(B)$ where each $Z \in R$ is an F-ideal. Similarly, if we are given a Boolean partition algebra (B, F), then we may want to extend F to the largest filter $G \subseteq \mathbb{P}(B)$ where each F-ideal is a G-ideal.

We shall show that this filter G is simply the total local refinement of F. A similar result holds for ultrafilters.

Definition 3.3.9 *Let $Id(B)$ denote the lattice of ideals in a Boolean algebra B. Define a relation $\mathcal{R}(B) \subseteq \mathbb{P}(B) \times Id(B)$ by letting $(p, I) \in \mathcal{R}(B)$ if and only if $\pi_I[p]^+$ is a partition of B/I.*

Now define maps $f : P(\mathbb{P}(B)) \to P(Id(B)), g : P(Id(B)) \to P(\mathbb{P}(B))$ by letting

$$f(P) = \{I \in Id(B) | \forall p \in P, (p, I) \in \mathcal{R}(B)\}$$

and

$$g(R) = \{p \in \mathbb{P}(B) | \forall I \in R, (p, I) \in \mathcal{R}(B)\}.$$

The functions f, g form a Galois connection. In other words, if $P \subseteq P(\mathbb{P}(B))$ and $R \subseteq P(Id(B))$, then $R \subseteq f(P)$ if and only if $P \subseteq g(R)$. Let $C = g \circ f, D = f \circ g$, and let $f^* : C^* \to D^*$ and $g^* : D^* \to C^*$ be the restriction mappings of f and g respectively.

Example 3.3.10 *Let B be a Boolean algebra. Then $\mathbb{P}(B)$ is the greatest element in C^*. Therefore, $f(\mathbb{P}(B))$ is the least element in D^*. On the other hand, an ideal is a $\mathbb{P}(B)$-ideal if and only if it is a complete ideal. We conclude that $f(\mathbb{P}(B))$ consists of all complete ideals.*

Theorem 3.3.11 *Let B be a Boolean algebra. Then $F \in C^*$ if and only if (B, F) is a locally refinable Boolean partition algebra. In particular, if (B, F) is a Boolean partition algebra, then $C(F) = F^{(\infty)}$.*

Proof. $\to$ Assume $F \in C^*$. Then $F = g(R)$ for some $R \subseteq \mathrm{Id}(B)$. In other words, $F = \{p \in \mathbb{P}(B) | \forall I \in R, \pi_I[p]^+ \text{ is a partition}\}$. Let $p \in F$ and assume $q \in \mathbb{P}(B)$ is a partition with $p \preceq q$. Let $I \in R$. Then $\pi_I[p]^+ \preceq \pi_I[q]^+$. Therefore since $\bigvee \pi_I[p]^+ = 1$, it follows that $\bigvee \pi_I[q]^+ = 1$ as well. We conclude that $q \in F$, so F is an upper set.

Now assume that $p, q \in F$. Then $\bigvee \pi_I[p]^+ = 1$ and $\bigvee \pi_I[q]^+ = 1$ for each $I \in R$. Thus, for $I \in R$

$$\bigvee \pi_I[p \wedge q]^+ = \bigvee \{\pi_I(a \wedge b) | a \in p, b \in q, a \wedge b \neq 0\}$$

$$= \bigvee_{a \in p, b \in q} \pi_I(a \wedge b) = \bigvee_{a \in p, b \in q} \pi_I(a) \wedge \pi_I(b) = \bigvee_{a \in p} \pi_I(a) \wedge \bigvee_{b \in q} \pi_I(b) = 1.$$

Thus $\pi_I[p \wedge q]^+$ is a partition for each $I \in R$. We conclude that $p \wedge q \in F$ as well.

If $\{a_1, ..., a_n\}$ is a finite partition of B, then $\pi_I(a_1) \vee .... \vee \pi_I(a_n) = \pi_I(a_1 \vee ... \vee a_n) = \pi_I(1) = 1$ for each $I \in R$. Therefore $\{a_1, ..., a_n\} \in F$. Thus (B, F) is a Boolean partition algebra.

To prove local refinability, we take note that each $I \in R$ is an F-ideal, so each $I \in R$ is also an $F^{(\infty)}$-ideal. If $p \in F^{(\infty)}$, then $\pi_I[p]^+$ is a partition of B/I for each $I \in R$, so $p \in F$.

$\leftarrow$ Let (B, F) be a locally refinable Boolean partition algebra. Assume $p \in \mathbb{P}(B) \setminus F$. Let L be the lower set generated by p and let I be the F-ideal generated by L. We claim that $1 \notin I$. If $1 \in I$, then there is some $q \in F$ and some $R \subseteq q \cap L$ with $1 = \bigvee R$. This is only possible if $R = q$, so $q \subseteq L$. Thus $q \preceq p$ since L is the lower set generated by p, so $p \in F$ after all. This is a contradiction. Hence I must be a proper F-ideal, hence $I \in f(F)$. On the other hand, $\bigvee \pi_I[p]^+ = 0 \neq 1$. Thus $p \notin g(f(F)) = C(F)$. We conclude that $C(F) \subseteq F$, so $F \in C^*$. $\square$

On the other side of the Galois correspondence, a satisfactory characterization of D^* requires frames. We shall now give an analogous galois connection between sets of ultrafilters and Boolean partition algebras.

Definition 3.3.12 *Let $\mathcal{R}_U(B) \subseteq \mathbb{P}(B) \times S(B)$ be the relation where $(p, \mathcal{U}) \in \mathcal{R}_U(B)$ if $p \cap \mathcal{U}$ is nonempty. In other words, $(p, \mathcal{U}) \subseteq \mathcal{R}_U(B)$ if $\pi_U[p]^+$ is a partition of $B/\mathcal{U}$. Therefore $\mathcal{R}_U(B)$ is in a sense the restriction of $\mathcal{R}(B)$ to $\mathbb{P}(B) \times S(B)$. Now define mappings $f_U : P(\mathbb{P}(B)) \rightarrow P(S(B)), g_U : P(S(B)) \rightarrow P(\mathbb{P}(B))$ by $f_U(P) = \{\mathcal{U} \in S(B) | \forall p \in P, (p, \mathcal{U}) \in \mathcal{R}_U(B)\}$ and $g_U(R) = \{p \in \mathbb{P}(B) : \forall \mathcal{U} \in R, (p, \mathcal{U}) \in \mathcal{R}_U(B)\}$. Then the mappings f_U, g_U forms a galois connection between*

$P(\mathbb{P}(B))$ and $P(S(B))$. Now define closure operators $C_U : P(\mathbb{P}(B)) \to P(\mathbb{P}(B))$ and $D_U : P(S(B)) \to P(S(B))$ by $C_U = g_U \circ f_U, D_U = f_U \circ g_U$. It is easy to see that $C_U^* \subseteq C^*$, so if $F \in C_U^*$, then (B, F) is a locally refinable Boolean partition algebra.

Example 3.3.13 *If (B, F) is a superstable Boolean partition algebra, then $C_U(F) = C(F)$. If Z is an F-filter, then Z is the intersection of F-ultrafilters, so Z is the intersection of $C_U(F)$-ultrafilters, hence Z is a $C_U(F)$-filter. Therefore $C_U(F) \subseteq C(F) \subseteq C_U(F)$. In particular, if (B, F) is superstable, then $F \in C_U^*$ if and only if $F \in C^*$.*

We shall now characterize the closure operator C_U.

Theorem 3.3.14 *$q \in C_U(\mathcal{P})$ if and only if whenever $x_p \in p$ for $p \in \mathcal{P}$ there are $p_1, ..., p_n \in \mathcal{P}$ and $a_1, ..., a_m \in q$ where $x_{p_1} \wedge ... \wedge x_{p_n} \leq a_1 \vee ... \vee a_m$.*

Proof. $\leftarrow$ Let $\mathcal{U} \in f_U(\mathcal{P})$. Then $\mathcal{U}$ is an ultrafilter where $\mathcal{U} \cap p \neq \emptyset$ for each $p \in \mathcal{P}$. Therefore let $x_p \in \mathcal{U} \cap p$ for each $p \in \mathcal{P}$. Then there are $p_1, ..., p_n \in \mathcal{P}$ and $a_1, ..., a_m \in q$ where $x_{p_1} \wedge ... \wedge x_{p_n} \leq a_1 \vee ... \vee a_m$. Since $x_{p_1} \wedge ... \wedge x_{p_n} \in \mathcal{U}$, $a_1 \vee ... \vee a_m \in \mathcal{U}$ as well, so there is some $i \in \{1, ..., m\}$ where $a_i \in \mathcal{U}$. Thus $a_i \in \mathcal{U} \cap q$, so $a_i \in \mathcal{U} \cap q$. Therefore $q \in g_U(f_U(\mathcal{P})) = C_U(\mathcal{P})$.

$\to$ Let $\iota : B \to P(S(B))$ be the mapping where $\iota(b) = \{\mathcal{U} \in S(B) | b \in \mathcal{U}\}$. We take note that $\mathcal{U} \cap p \neq \emptyset$ if and only if $\mathcal{U} \in \bigcup \iota[p]$. Assume that $q \in C_U(\mathcal{P}) = g_U(f_U(\mathcal{P}))$. Then if $\mathcal{U} \cap p \neq \emptyset$ for each $p \in \mathcal{P}$, then $\mathcal{U} \cap q \neq \emptyset$ as well. In other words, $\bigcap_{p \in \mathcal{P}} \bigcup \iota[p] \subseteq \bigcup \iota[q]$. However, $\bigcap_{p \in \mathcal{P}} \bigcup \iota[p]$ is the union of all the sets of the form $\bigcap_{p \in F} \iota(x_p)$ where $x_p \in p$ for $p \in \mathcal{P}$. Therefore $\bigcap_{p \in F} \iota(x_p) \subseteq \bigcup \iota[q]$ whenever $x_p \in p$ for $p \in \mathcal{P}$. Since the space $S(B)$ is compact and each $\iota(x_p)$ is closed and $\bigcup \iota[q]$ is open, there are $p_1, ..., p_n \in \mathcal{P}$ where $\iota(x_{p_1}) \cap ... \cap \iota(x_{p_n}) \subseteq \bigcup \iota[q]$. Again, by compactness, there are $a_1, ..., a_m \in q$ where $\iota(x_{p_1}) \cap ... \cap \iota(x_{p_n}) \subseteq \iota(a_1) \vee ... \vee \iota(a_m)$. We conclude that $x_{p_1} \wedge ... \wedge x_{p_n} \leq a_1 \vee ... \vee a_m$. $\square$

Corollary 3.3.15 *If B is a Boolean algebra and $\mathbb{P}_\omega(B) \subseteq \mathcal{P} \subseteq \mathbb{P}(B)$, then $q \in C_U(\mathcal{P})$ if and only if whenever $x_p \in p$ for $p \in \mathcal{P}$, there are $p_1, ..., p_n \in \mathcal{P}$ and an $a \in q$ where $x_{p_1} \wedge ... \wedge x_{p_n} \leq a$.*

Proof. $\leftarrow$ This follows from 3.3.14.

$\rightarrow$ Assume that $q \in C_U(\mathcal{P})$. Let $x_p \in p$ whenever $p \in \mathcal{P}$. Then there are $p_1, ..., p_n \in \mathcal{P}$ and $a_1, ..., a_m \in q$ where $x_{p_1} \wedge ... \wedge x_{p_n} \leq a_1 \vee ... \vee a_m$. Now let $p = \{a_1, ..., a_m, (a_1 \vee ... \vee a_m)'\}^+$. Then $p \in \mathcal{P}$. If $x_p = a_i$ for some i, then $x_p \wedge x_{p_1} \wedge ... \wedge x_{p_n} \leq a_i \in p$. If $x_p = (a_1 \vee ... \vee a_m)'$, then $x_p \wedge x_{p_1} \wedge ... \wedge x_{p_n} \leq (a_1 \vee ... \vee a_m) \wedge (a_1 \vee ... \vee a_m)' = 0$. In any case, there is some $a \in q$ where $x_p \wedge x_{p_1} \wedge ... \wedge x_{p_n} \leq a$. $\square$

We shall now characterize the set D_U^*. Take note the elements of D_U^* are the sets of the form $f_U(P)$ for some $P \subseteq \mathbb{P}(B)$, but

$$f_U(P) = \bigcap_{p \in P} \{\mathcal{U} \in S(B) | (p, \mathcal{U}) \in \mathcal{R}_U(B)\} = \bigcap_{p \in P} \bigcup_{a \in p} \{\mathcal{U} \in S(B) | a \in \mathcal{U}\},$$

and the sets of the form $\bigcup_{a \in p} \{\mathcal{U} \in S(B) | a \in \mathcal{U}\}$ are simply the dense open sets in $S(B)$ that are the unions of a pairwise disjoint collection of compact sets. Therefore the sets in D_U^* are the subsets of $S(B)$ of the form $\bigcap_{i \in I} U_i$ where each U_i is a dense open set that is the disjoint union of compact open sets.

Proposition 3.3.16 *A locally compact zero-dimensional space X is paracompact if and only if X is the union of a disjoint collection of compact open sets.*

Proof. $\leftarrow$ This is trivial.

$\rightarrow$ If X is locally compact zero-dimensional and paracompact, then by [5][p. 25] $X = \bigcup_{i \in I} X_i$ where each X_i is a σ-compact open set and the collection $(X_i)_{i \in I}$ is pairwise disjoint. Since each X_i is σ-compact, each set X_i is Lindelof. Since X_i is locally compact and zero-dimensional, each $r \in X_i$ has a compact open neighborhood $V_r \subseteq X_i$. Since X_i is Lindelof, there is a countable subcovering $\{V_n | n \in \mathbb{N}\}$ of $(V_r)_{r \in X_i}$. Let $R_n = V_n \setminus (V_0 \cup ... \cup V_{n-1})$ for all n. Then $(R_n)_{n \in \mathbb{N}}$ is a partition of X_i into compact clopen sets. Therefore we may partition the set $X = \bigcup_{i \in I} X_i$ into a collection of compact open sets. $\square$

As a consequence, a dense open subset U of a totally disconnected compact space is the disjoint union of compact sets iff U is paracompact. Therefore the sets in D_U^* are precisely the subsets of $S(B)$ that are the intersections of dense open paracompact sets. In particular, $D_U^* = P(S(B))$ if and only if $S(B)$ has no isolated points and $S(B) \setminus \{\mathcal{U}\}$ is paracompact for each $\mathcal{U} \in S(B)$.

If $R \subseteq S(B)$ is the intersection of dense open paracompact subspaces, then $R = S^*(B, F)$ for some Boolean partition algebra (B, F). Since the topology on R is the same as the topology $S^*(B, F)$, the space R is uniformizable by a complete non-Archimedean uniformity.

Example 3.3.17 *Let B be a countable atomless Boolean algebra. Then $S(B)$ has no isolated points and $S(B)$ is metrizable. Since every metrizable space is paracompact, the space $S(B) \setminus \{x\}$ is paracompact for each $x \in S(B)$. Therefore $D_U^* = P(S(B))$. In particular, $|C_U^*| = |D_U^*| = |P(S(B))| = 2^{\mathfrak{c}}$, so there are $2^{\mathfrak{c}}$ filters F on the meet-semilattice $\mathbb{P}(B)$ such that (B, F) is a Boolean partition algebra.*

3.4 Resplendence and Strong Distributivity Properties

In this section, we shall investigate a very strong distributivity property of Boolean partition algebras called resplendence. While resplendence is a distributivity property, the notion of resplendence is helpful for studying ultrafilters. In particular, resplendence can be used to characterize stability and some large cardinal axioms. Before we begin examine the notion of resplendence, we need to first look at some basic results concerning infinite meets in the meet-semilattice $\mathbb{P}(B)$.

Proposition 3.4.1 *Let B be a Boolean algebra, and let $p_i \in \mathbb{P}(B)$ for $i \in I$.*

1. Assume that whenever $a_i \in p_i$ for $i \in I$, the greatest lower bound $\bigwedge_{i\in I} a_i$ exists. If $\{p_i | i \in I\}$ has a lower bound in $\mathbb{P}(B)$, then $\bigwedge_{i\in I} p_i$ exists in $\mathbb{P}(B)$ and $\bigwedge_{i\in I} p_i = \{\bigwedge_{i\in I} a_i | (a_i)_{i\in I} \in \prod_{i\in I} p_i\}^+$.

2. Assume that the greatest lower bound $\bigwedge_{i\in I} a_i$ exists whenever $a_i \in p_i$ for $i \in I$. If $\{\bigwedge_{i\in I} a_i | (a_i)_{i\in I} \in \prod_{i\in I} p_i\}^+$ is a partition of B, then $\{\bigwedge_{i\in I} a_i | (a_i)_{i\in I} \in \prod_{i\in I} p_i\}^+$ is the greatest lower bound of $\{p_i | i \in I\}$.

3. If the greatest lower bound $\bigwedge_{i\in I} p_i$ exists, then $\bigwedge_{i\in I} p_i = \{\bigwedge_{i\in I} a_i | (a_i)_{i\in I} \in \prod_{i\in I} p_i\}^+$.

4. If $\{p_i | i \in I\}$ has a subcomplete lower bound, then $\{p_i | i \in I\}$ has a greatest lower bound.

Proof. 1. The family $\{\bigwedge_{i\in I} a_i | (a_i)_{i\in I} \in \prod_{i\in I} p_i\}^+$ is cellular since if $(a_i)_{i\in I}, (b_i)_{i\in I}$ are distinct elements in $\prod_{i\in I} p_i$, then $a_j \neq b_j$ for some $j \in I$, so $a_j \wedge b_j = 0$, hence $\bigwedge_{i\in I} a_i \wedge \bigwedge_{i\in I} b_i = \bigwedge_{i\in I}(a_i \wedge b_i) = 0$. Assume that q is a lower bound of $\{p_i | i \in I\}$, then for each $a \in q$, we have $a \leq \bigwedge_{i\in I} \phi_{q,p_i}(a) \in \{\bigwedge_{i\in I} a_i | (a_i)_{i\in I} \in \prod_{i\in I} p_i\}^+$. Therefore $q \preceq \{\bigwedge_{i\in I} a_i | (a_i)_{i\in I} \in \prod_{i\in I} p_i\}^+$. We conclude that $\{\bigwedge_{i\in I} a_i | (a_i)_{i\in I} \in \prod_{i\in I} p_i\}^+$ is a partition and $\{\bigwedge_{i\in I} a_i | (a_i)_{i\in I} \in \prod_{i\in I} p_i\}^+$ is the greatest lower bound of $\{p_i | i \in I\}$.

2. If $\{\bigwedge_{i\in I} a_i | (a_i)_{i\in I} \in \prod_{i\in I} p_i\}^+$ is a partition of B, then clearly $\{\bigwedge_{i\in I} a_i | (a_i)_{i\in I} \in \prod_{i\in I} p_i\}^+$ is a lower bound of $\{p_i | i \in I\}$, so by 1. $\{\bigwedge_{i\in I} a_i | (a_i)_{i\in I} \in \prod_{i\in I} p_i\}^+$ is the greatest lower bound of $\{p_i | i \in I\}$.

3. We only need to show that $\bigwedge_{i\in I} a_i$ exists whenever $a_i \in p_i$ for $i \in I$. Therefore, let $p = \bigwedge_{i\in I} p_i$, and let $R_i = \{a \in p | a \leq a_i\}$ for each $i \in I$. We claim that $\bigcap_{i\in I} R_i$ has at most one element. If $\bigcap_{i\in I} R_i$ contains distinct elements a, b and if $q = (p \setminus \{a, b\}) \cup \{a \vee b\}$, then q is a partition of B with $p \prec q$ and $q \preceq p_i$ for $i \in I$. This is a contradiction, so $\bigcap_{i\in I} R_i$ has at most one element. Thus, since $\bigcap_{i\in I} R_i$ is always finite, the least upper bound $\bigvee \bigcap_{i\in I} R_i$ exists, and $\bigvee \bigcap_{i\in I} R_i = \bigwedge_{i\in I} \bigvee R_i = \bigwedge_{i\in I} a_i$. Therefore the greatest lower bound $\bigwedge_{i\in I} a_i$ exists.

4. Assume that p is a subcomplete partition with $p \preceq p_i$ for $i \in I$. We need to show that if $a_i \in p_i$ for $i \in I$, then the greatest lower bound $\bigwedge_{i\in I} a_i$ exists. However, if $R_i = \{a \in p | a \leq a_i\}$ for $i \in I$, then the least upper bound $\bigvee \bigcap_{i\in I} R_i$ exists, and

$$\bigvee \bigcap_{i\in I} R_i = \bigwedge_{i\in I} \bigvee R_i = \bigwedge_{i\in I} a_i.$$

$\square$

Definition 3.4.2 *Let (B, F) be a Boolean partition algebra. A family of partitions $(p_i)_{i\in I}$ in F is called* admissible *if whenever $x_i \in p_i$ for $i \in I$, then $\{x_i | i \in I\}$ is meet-admissible. A Boolean partition algebra (B, F) is said to be* resplendent *if whenever $(p_i)_{i\in I}$ is an admissible family of partitions, then the partition $\bigwedge_{i\in I} p_i$ exists* *and* $\bigwedge_{i\in I} p_i \in F$. *If* (B,F) *is a Boolean partition algebra, then we say that* (B,F) *has* resplendence closure *if there is a resplendent Boolean partition algebra* (B,H) *with* $F \subseteq H$. *We shall say that a Boolean partition algebra* (B,F) *is* partially resplendent *if whenever* $(p_i)_{i\in I}$ *is an admissible family of partitions in* F *and the greatest lower bound* $\bigwedge_{i\in I} p_i$ *exists, then* $\bigwedge_{i\in I} p_i \in F$.

It is not difficult to see that if (B,F_i) is resplendent for $i \in I$, then $(B,\bigcap_{i\in I} F_i)$ is also resplendent. Similarly, if (B,F_i) is partially resplendent for $i \in I$, then $(B,\bigcap_{i\in I} F_i)$ is also partially resplendent. Therefore, if (B,F) is a Boolean algebra, then there is a minimal $H \supseteq F$ where (B,H) is partially resplendent, and (B,H) is resplendent if and only if (B,F) has resplendence closure.

Definition 3.4.3 *Let* $(B,F),(B,H)$ *be Boolean partition algebras such that* $H \supseteq F$ *is the smallest set such that* (B,H) *is partially resplendent. Then* (B,H) *is called the* partial resplendence closure *of* (B,F). *If* (B,H) *is resplendent, then* (B,H) *shall be called the* resplendence closure *of* (B,F).

Remark 3.4.4 *Resplendence is a distributivity property for Boolean partition algebras. A Boolean partition algebra is resplendent if and only if whenever* $p_i \in F$ *for* $i \in I$ *and* $\{x_i|i \in I\}$ *is meet-admissible whenever* $x_i \in p_i$ *for* $i \in I$, *then* $\bigwedge_{i\in I}(\bigvee p_i) = \bigvee\{\bigwedge_{i\in I} x_i | x_i \in p_i \, for \, i \in I\}$ *and* $\{\bigwedge_{i\in I} x_i | x_i \in p_i \, for \, i \in I\}$ *is admissible.*

Lemma 3.4.5 *Let* B *be a Boolean algebra. If* R *is a dense subset of* $S(B)$ *and* c *is a cellular family with* $c \cap \mathcal{U} \neq \emptyset$ *whenever* $\mathcal{U} \in R$, *then* c *is a partition of* B.

Proof. Take note that a set R is dense in $S(B)$ if and only if $\bigcup R = B^+$. Assume that R is a dense subset of $S(B)$. Let $a \neq 0$. Then there is some $\mathcal{U} \in R$ with $a \in \mathcal{U}$, so there is some $b \in c$ with $c \in \mathcal{U}$ as well. Therefore $a \wedge b \in \mathcal{U}$, thus $a \wedge b \neq 0$. We conclude that c is a maximal cellular family in B, so c is a partition of B. $\square$

Theorem 3.4.6 *Let* B *be a Boolean algebra and let* $R \subseteq S(B)$ *be a collection of ultrafilters. Then* $(B,g_U(R))$ *is partially resplendent (Recall that* $g_U(R) = \{p \in \mathbb{P}(B)|\forall \mathcal{U} \in R, p \cap \mathcal{U} \neq \emptyset\}$*). If* R *is dense in* $S(B)$, *then* $(B,g_U(R))$ *is resplendent.*

Proof. Assume that $(p_i)_{i\in I}$ is an admissible family of partitions where the greatest lower bound $\bigwedge_{i\in I} p_i$ exists. Let $\mathcal{U} \in R$. Then for $i \in I$ there is some a_i with $a_i \in p_i \cap \mathcal{U}$, so since $\{a_i|i \in I\}$ is meet-admissible and $\{a_i|i \in I\} \subseteq \mathcal{U}$, we have $\bigwedge_{i\in I} a_i \in \mathcal{U} \cap \bigwedge_{i\in I} p_i$. Therefore $\bigwedge_{i\in I} p_i \in g_U(R)$. We conclude that $(B, g_U(R))$ is partially resplendent.

Now assume that R is dense in $S(B)$. Let $(p_i)_{i\in I}$ be an admissible family of partitions. We must show that $\{\bigwedge_{i\in I} a_i|(a_i)_{i\in I} \in \prod_{i\in I} p_i\}^+$ is a partition in $g_U(R)$. If $\mathcal{U} \in R$, then for each $i \in I$ there is some $a_i \in p_i \cap \mathcal{U}$. The set $\{a_i|i \in I\}$ is meet-admissible, so $\bigwedge_{i\in I} a_i \in \mathcal{U}$. Thus the cellular family $\{\bigwedge_{i\in I} a_i|(a_i)_{i\in I} \in \prod_{i\in I} p_i\}^+$ intersects each $\mathcal{U} \in R$, and since R is dense in $S(B)$, the family $\{\bigwedge_{i\in I} a_i|(a_i)_{i\in I} \in \prod_{i\in I} p_i\}^+$ is a partition of B. Therefore the greatest lower bound $\bigwedge_{i\in I} p_i$ exists, and $\bigwedge_{i\in I} p_i = \{\bigwedge_{i\in I} a_i|(a_i)_{i\in I} \in \prod_{i\in I} p_i\}^+ \in g_U(R)$. $\square$

Corollary 3.4.7 *Every superstable locally refinable Boolean partition algebra is resplendent.*

Proof. Assume (B, F) is superstable and locally refinable. Then $F \in C^*$, so $F \in C_U^*$, thus since (B, F) is stable and $F \in C_U^*$, the Boolean partition algebra (B, F) is resplendent. $\square$

Theorem 3.4.8 *A Boolean partition algebra is stable if and only if it has resplendence closure.*

Proof. $\rightarrow$ If (B, F) is stable, then $f_U(F)$ is the collection of all F-ultrafilters. Therefore $\bigcap f_U(F) = \{1\}$, and $f_U(F)$ is dense in $S(B)$, so $(B, C_U(F)) = (B, g_U(f_U(F)))$ is resplendent. Hence (B, F) has resplendence closure.

$\leftarrow$ We shall prove this direction by contrapositive. Assume (B, F) is not stable. Then there is an $a \in B \setminus \{0, 1\}$ where there is no $F \restriction a$-ultrafilter in the Boolean partition algebra $(B \restriction a, F \restriction a)$. Let $\mathcal{P} = \{p \cup \{a'\}|p \in F \restriction a\}$. Assume $x_p \in p \cup \{a'\}$ for $p \in F \restriction a$. We shall show $\{x_p|p \in F \restriction a\}$ is meet-admissible by cases. If $x_p = a'$ for all $p \in F \restriction a$, then $\{x_p|p \in F \restriction a\} = \{a'\}$ is meet-admissible. If there are $p, q \in F \restriction a$ with $x_p = a', x_q \neq a'$, then $x_q \in q \cup \{a'\}$ and $x_q \neq a'$, so $x_p \wedge x_q = a' \wedge x_q = 0$, thus

$\{x_p | p \in F \restriction a\}$ is meet-admissible. Now assume that $x_p \neq a'$ for $p \in F \restriction a$. Then $x_p \in p$ for $p \in F \restriction a$. Since $(B \restriction a, F \restriction a)$ has no $F \restriction a$-ultrafilters, the inverse limit $\underleftarrow{\lim} F \restriction a$ is empty, so $(x_p)_{p \in F\restriction a} \notin \underleftarrow{\lim}\ F \restriction a$. Therefore there are $p, q \in F \restriction a$ with $p \preceq q$ and where $\phi_{p,q}(x_p) \neq x_q$, hence $x_p \wedge x_q \leq \phi_{p,q}(x_p) \wedge x_q = 0$. We conclude that $\{x_p | p \in F \restriction a\}$ is meet-admissible. □

Lemma 3.4.9 *If $(A, F), (B, G)$ are Boolean partition algebras and $\phi : (A, F) \to (B, G)$ is a partition homomorphism and $\mathcal{P} \subseteq F$ is admissible in (A, F), then $\{\phi[p]^+ | p \in \mathcal{P}\}$ is admissible in (B, G). Furthermore, if $\bigwedge \mathcal{P}$ exists, and $\bigwedge\{\phi[p]^+ | p \in \mathcal{P}\}$ exists, then $\phi[\bigwedge \mathcal{P}]^+ = \bigwedge\{\phi[p]^+ | p \in \mathcal{P}\}$.*

Proof. Assume that $y_p \in \phi[p]^+$ for $p \in \mathcal{P}$. Then for $p \in \mathcal{P}$ there is an $x_p \in p$ with $y_p = \phi(x_p)$. Therefore since $\{x_p | p \in \mathcal{P}\}$ is meet-admissible, the set $\phi[\{x_p | p \in \mathcal{P}\}] = \{y_p | p \in \mathcal{P}\}$ is meet-admissible as well. Hence $\{\phi[p]^+ | p \in \mathcal{P}\}$ is admissible in (B, G).

Now assume that the greatest lower bounds $\bigwedge \mathcal{P}$ and $\bigwedge\{\phi[p]^+ | p \in \mathcal{P}\}$ exist. Then

$$\phi[\bigwedge \mathcal{P}]^+ = \phi[\{\bigwedge_{p \in \mathcal{P}} x_p | x_p \in p \text{ for } p \in \mathcal{P}\}]^+$$

$$= \{\bigwedge_{p \in \mathcal{P}} \phi(x_p) | x_p \in p \text{ for } p \in \mathcal{P}\}^+ = \{\bigwedge_{p \in \mathcal{P}} y_p | y_p \in \phi[p]^+ \text{ for } p \in \mathcal{P}\}^+$$

$$= \bigwedge\{\phi[p]^+ | p \in \mathcal{P}\}.$$

□

Theorem 3.4.10 *Let (A, F) be a Boolean partition algebra, and assume that for each $i \in I$, (B_i, G_i) is a resplendent Boolean partition algebra and $\phi_i : (A, F) \to (B_i, G_i)$ is a partition homomorphism. Furthermore, assume $F = \{p \in \mathbb{P}(B) | \phi_i[p]^+ \in G_i \text{ for all } i \in I\}$. Then (A, F) is partially resplendent. If (A, F) is stable, then (A, F) is resplendent.*

Proof. Assume that $\mathcal{P}$ is an admissible family of partitions in F such that $\bigwedge \mathcal{P}$ exists. Then for each $i \in I$, the set $\{\phi_i[p]^+ | p \in \mathcal{P}\}$ is an admissible family of partition where

$\bigwedge\{\phi_i[p]^+|p \in \mathcal{P}\}$ exists. Therefore $\phi_i[\bigwedge \mathcal{P}]^+ = \bigwedge\{\phi_i[p]^+|p \in \mathcal{P}\} \in G_i$. We conclude that $\bigwedge \mathcal{P} \in F$, so$(A, F)$ is partially resplendent. $\square$

Corollary 3.4.11 *1. If (B, F) is resplendent and $A \subseteq B$ is a Boolean subalgebra, then $(A, F|_A)$ is resplendent as well.*

2. The product of resplendent Boolean partition algebras is resplendent.

Proof. 1. Since (B, F) is resplendent, the Boolean partition algebra (B, F) is stable, so $(A, F|_A)$ is stable as well. Furthermore, if $\iota : (A, F|_A) \to (B, F)$ is the canonical embedding, then $F|_A = \{p \in \mathbb{P}(A)|\iota[p] \in F\}$, so $(A, F|_A)$ is resplendent.

2. Assume that $(B, F) = \prod_{i \in I}(B_i, F_i)$, and let $\pi_i : (B, F) \to (B_i, F_i)$ be the canonical projection. Assume each (B_i, F_i) is resplendent. Then since the product of stable Boolean partition algebras is stable, the Boolean partition algebra (B, F) is stable as well. Furthermore, since $p \in F$ if and only if $\pi_i[p]^+ \in F_i$ for each $i \in I$, the Boolean partition algebra (B, F) is resplendent. $\square$

Example 3.4.12 *Assume that μ is a measurable cardinal that is not strongly compact. Then there is some set X where $(P(X), \mathbb{P}_\mu(P(X)))$ is resplendent and subcomplete, but not superstable. If (B, F) is a resplendent Boolean partition algebra that is not superstable, then there is some F-filter Z where there are no F/Z-ultrafilters. In this case, $(B, F)/Z$ is not resplendent.*

Proposition 3.4.13 *Every partially resplendent Boolean partition algebra is locally refinable.*

Proof. Let (B, F) be a partially resplendent Boolean partition algebra. Let $p \in F$ be a partition with $|p| > 1$, and let $p_a \in F \restriction a$ for $a \in p$. Let $q_a = p_a \cup \{a'\}$ for $a \in p$. We claim that $\{q_a|a \in p\}$ is an admissible family of partitions, and we shall prove that $\{x_a|a \in p\}$ is admissible whenever $x_a \in q_a$ for each $a \in p$ in three cases. Furthermore, we shall show that $\{\bigwedge_{a \in p} x_a|(x_a)_{a \in p} \in \prod_{a \in p} q_a\}^+ = \bigcup_{a \in p} p_a$, and from this fact, we conclude that $\bigcup_{a \in p} p_a = \{\bigwedge_{a \in p} x_a|(x_a)_{a \in p} \in \prod_{a \in p} q_a\}^+ = \bigwedge_{a \in q} q_a \in F$ since (B, F) is partially resplendent.

If there are distinct $a, b \in p$ where $x_a \in p_a$ and $x_b \in p_b$, then $x_a \wedge x_b \leq a \wedge b = 0$, so $\{x_a | a \in p\}$ is meet-admissible with $\bigwedge_{a \in p} x_a = 0$. If $x_a \in p_a$, but $x_b = b'$ for each $b \in p$ with $b \neq a$, then for $b \neq a$ we have $x_a \leq a \leq b' = x_b$. In this case, $\{x_a | a \in p\}$ is meet-admissible with $\bigwedge_{a \in p} x_a = x_a$. If $x_a = a'$ for $a \in p$, then $\{x_a | a \in p\} \subseteq p^*$, thus $\{x_a | a \in p\}$ is admissible, and $\bigwedge\{x_a | a \in p\} = (\bigvee_{a \in p} a)' = 0$. Therefore $\{x_a | a \in p\}$ is meet-admissible in this case. We have covered all the cases, so we conclude that $\{q_a | a \in p\}$ is an admissible family of partitions. We have also shown that $\{\bigwedge_{a \in p} x_a | (x_a)_{a \in p} \in \prod_{a \in p} q_a\}^+ = \bigcup_{a \in p} p_a$. $\square$

Theorem 3.4.14 *Let (B, F) be a Boolean partition algebra, and let $p \in \mathbb{P}(B)$ be a partition where each $F|_{p^*}$-ultrafilter is principal (The principal ultrafilters on p^* are the ultrafilters of the form $\{b \in p^* | a \leq b\}$ for some $a \in p$). Then there is some set $\mathcal{P} \subseteq F$ of partitions admissible in F where $p = \bigwedge \mathcal{P}$.*

Proof. We shall show that $F|_{p^*}$ is an admissible set of partitions with $\bigwedge F|_{p^*} = p$. Take note that $F|_{p^*} = \{q \in F | p \preceq q\}$. Assume that $x_q \in q$ for each $q \in F|_{p^*}$. We shall show $\{x_q | q \in F\}$ is meet-admissible. If $x_{q_1} \wedge x_{q_2} = 0$ for some pair $q_1, q_2 \in F|_{p^*}$, then clearly $\{x_q | q \in F|_{p^*}\}$ is meet-admissible with $\bigwedge_{q \in F, p \preceq q} x_q = 0$. Now assume that $x_{q_1} \wedge x_{q_2} \neq 0$ whenever $q_1, q_2 \in F|_{p^*}$. If $q_1, q_2 \in F|_{p^*}$ and $q_1 \preceq q_2$, then since $0 < x_{q_1} \wedge x_{q_2} < \phi_{q_1,q_2}(x_{q_1}) \wedge x_{q_2}$, we have $\phi_{q_1,q_2}(x_{q_1}) = x_{q_2}$. Therefore, the system $(x_q)_{q \in F, p \preceq q}$ is an inverse system, so $\{x_q | q \in F|_{p^*}\}$ is an $F|_{p^*}$-ultrafilter. By the hypotheses of the theorem, the set $\{x_q | q \in F|_{p^*}\}$ is a principal ultrafilter, so there is some $a \in p$ where $\{x_q | q \in F|_{p^*}\} = \{b \in p^* | a \leq b\}$. Therefore $\{x_q | q \in F|_{p^*}\}$ is meet-admissible with $\bigwedge\{x_q | q \in F|_{p^*}\} = a$. We conclude that $F|_{p^*}$ is an admissible family of partitions with $\bigwedge F|_{p^*} = \{\bigwedge x_q | x_q \in q \text{ for } q \in F|_{p^*}\}^+ = p$. $\square$

A cardinal κ is said to be measurable if there is a κ-complete ultrafilter $\mathcal{U}$ on κ with $\{\alpha\} \notin \mathcal{U}$ for $\alpha < \kappa$. We shall now characterize measurable cardinals in terms of resplendence.

Theorem 3.4.15 *Let κ be a cardinal. Then the following are equivalent.*

1. *κ is measurable.*

2. $(P(X), \mathbb{P}_\kappa(P(X)))$ *is resplendent for each set* X.

3. $(P(\kappa), \mathbb{P}_\kappa(P(\kappa)))$ *is resplendent.*

Proof. $1 \rightarrow 2$. Let X be a set. Let Z be the collection of all κ-ultrafilters on X. If $P \in \mathbb{P}_\kappa(P(X))$, then $P \cap \mathcal{U} \neq \emptyset$ for each $\mathcal{U} \in Z$. Similarly, if P is a partition of X with $P \notin \mathbb{P}_\kappa(P(X))$, then let $x_R \in R$ for each $R \in P$. Then since κ is measurable, there is some non-principal κ-ultrafilter $\mathcal{U} \in Z$ where $\{x_R | R \in P\} \in \mathcal{U}$. Therefore $P \cap \mathcal{U} = \emptyset$, since if $R \in P \cap \mathcal{U}$, then $\{x_R\} = R \cap \{x_R | R \in P\} \in \mathcal{U}$. Thus, $\mathbb{P}_\kappa(P(X))$ is the collection of all partitions that intersect each ultrafilter in Z. We conclude that $(P(X), \mathbb{P}_\kappa(P(X))) = (P(X), g_U(Z))$, so $(P(X), \mathbb{P}_\kappa(P(X)))$ is resplendent.

$2 \rightarrow 3$. This is trivial.

$3 \rightarrow 1$. The set $\mathbb{P}_\kappa(P(\kappa))$ of partitions is not admissible, so there is a system $(R_P)_{P \in \mathbb{P}_\kappa(P(\kappa))}$ where $\{R_P | P \in \mathbb{P}_\kappa(P(\kappa))\}$ is not meet-admissible. In particular, if $P \preceq Q$, then $R_P \cap R_Q \neq \emptyset$, so $R_P \subseteq R_Q$. Therefore $(R_P)_{P \in \mathbb{P}_\kappa(P(\kappa))} \in \overset{\lim}{\longleftarrow} \mathbb{P}_\kappa(P(\kappa))$. Thus $\{R_P | P \in \mathbb{P}_\kappa(P(\kappa))\}$ is a κ-ultrafilter on κ, and since $\{R_P | P \in \mathbb{P}_\kappa(P(\kappa))\}$ is not meet-admissible, the ultrafilter $\{R_P | P \in \mathbb{P}_\kappa(P(\kappa))\}$ is non-principal. □

Superstable Boolean partition algebras satisfy distributivity properties related to resplendence. In fact, superstable locally refinable Boolean partition algebras satisfy distributivity properties much stronger than resplendence.

Theorem 3.4.16 *Let* (B, F) *be a Boolean partition algebra. Then the following are equivalent.*

1. (B, F) *is superstable.*

2. Let I *be an index set. Let* $C_i \subseteq B$ *be a join-admissible set for* $i \in I$. *Assume that whenever* $x_i \in C_i$ *for* $i \in I$, *the set* $\{x_i | i \in I\}$ *is meet-admissible. Then* $\{\bigvee C_i | i \in I\}$ *is meet-admissible if and only if* $\{\bigwedge_{i \in I} x_i | x_i \in C_i \text{ for } i \in I\}$ *is join-admissible. Furthermore, if* $\{\bigvee C_i | i \in I\}$ *is meet-admissible, then*

$$\bigwedge_{i \in I} \bigvee C_i = \bigvee \{\bigwedge_{i \in I} x_i | x_i \in C_i \text{ for } i \in I\}.$$

3. Let I *be an index set. Let* $C_i \subseteq B$ *be a join-admissible set for* $i \in I$. *Assume* *that whenever $x_i \in C_i$ for $i \in I$, the set $\{x_i | i \in I\}$ is meet-admissible. If $\{\bigwedge_{i\in I} x_i | x_i \in C_i$ for $i \in I\}$ is join-admissible, then $\{\bigvee C_i | i \in I\}$ is meet-admissible and*

$$\bigwedge_{i\in I} \bigvee C_i = \bigvee\{\bigwedge_{i\in I} x_i | x_i \in C_i \textit{ for } i \in I\}.$$

Proof. $1 \to 2$. Assume that $C_i \subseteq B$ is join-admissible for $i \in I$ and whenever $x_i \in C_i$ for $i \in I$ then $\{x_i | i \in I\}$ is meet-admissible. Then $\iota(\bigvee C_i) = \bigcup \iota[C_i]$ and $\iota(\bigwedge_{i\in I} x_i) = \bigcap_{i\in I} \iota(x_i)$ whenever $x_i \in C_i$ for $i \in I$. We note that

$$\bigcap_{i\in I} \iota(\bigvee C_i) = \bigcap_{i\in I} \bigcup \iota[C_i]$$

$$= \bigcup\{\bigcap_{i\in I} \iota(x_i) | x_i \in C_i \text{ for } i \in I\} = \bigcup\{\iota(\bigwedge_{i\in I} x_i) | x_i \in C_i \text{ for } i \in I\}.$$

Therefore $\{\bigvee C_i | i \in I\}$ is meet-admissible if and only if $\bigcap_{i\in I} \iota(\bigvee C_i) \in \iota[B]$ if and only if $\bigcup\{\iota(\bigwedge_{i\in I} x_i) | x_i \in C_i \text{ for } i \in I\} \in \iota[B]$ if and only if $\{\bigwedge_{i\in I} x_i | x_i \in C_i \text{ for } i \in I\}$ is join-admissible. Furthermore, if $\{\bigvee C_i | i \in I\}$ is meet-admissible, then since $\bigcap_{i\in I} \iota(\bigvee C_i) = \bigcup\{\iota(\bigwedge_{i\in I} x_i) | x_i \in C_i \text{ for } i \in I\} \in \iota[B]$, we obtain the distributivity law

$$\bigwedge_{i\in I} \bigvee C_i = \bigvee\{\bigwedge_{i\in I} x_i | x_i \in C_i \text{ for } i \in I\}.$$

$2 \to 3$. This direction is trivial.

$3 \to 1$. We shall prove this direction by contrapositive. Assume that (B, F) is not superstable. Then there is a proper F-filter Z that is not extendible to an F-ultrafilter. Let $I = \{(p, R) | p \in F, R \in p^\sharp, \bigvee R \in Z\}$. Let $C_{p,R} = R$ whenever $(p, R) \in I$. Then each set $C_{p,R}$ is join-admissible.

Now assume that $x_{p,R} \in C_{p,R}$ whenever $(p, R) \in I$. We claim that there are $(p, R), (q, S) \in I$ where $x_{p,R} \wedge x_{q,S} = 0$. Assume to the contrary that $x_{p,R} \wedge x_{q,S} \neq 0$ whenever $(p, R), (q, S) \in I$. If $(p, R), (p, S) \in I$, then $x_{p,R} \wedge x_{p,S} \neq 0$, so since $x_{p,R}, x_{p,S} \in p$, we have $x_{p,R} = x_{p,S}$. Therefore for each $p \in F$ there is some $y_p \in p$ where $x_{p,R} = y_p$ whenever $(p, R) \in I$. Furthermore, because $x_{p,R} \wedge x_{q,S} \neq 0$ whenever $(p, R), (q, S) \in I$, we get $y_p \wedge y_q \neq 0$ whenever $p, q \in F$. If $p \preceq q$, then $0 < y_p \wedge y_q \leq$

$\phi_{p,q}(y_p) \wedge y_q$, so $\phi_{p,q}(y_p) = y_q$. Thus $(y_p)_{p\in F} \in \overset{\lim}{\longleftarrow} F$, so $\{y_p | p \in F\}$ is an F-ultrafilter. Now assume that $b \in Z$. Then $b \neq 0$, so there is some $p \in F$ with $b \in p$. Now $(p, \{b\}) \in I$, and $x_{p,\{b\}} \in \{b\}$, hence $x_{p,\{b\}} = b$, thus $y_p = b$, so $b \in \{y_p|p \in F\}$. We conclude that $Z \subseteq \{y_p|p \in F\}$. This contradicts the fact that Z is not extendible to an F-ultrafilter. Hence $x_{p,R} \wedge x_{q,S} = 0$ for some pair $(p, R), (q, S) \in I$, so $\{x_{p,R}|(p, R) \in I\}$ is meet-admissible with $\bigwedge_{(p,R)\in I} x_{p,R} = 0$. Therefore, the set

$$\{ \bigwedge_{(p,R)\in I} x_{p,R} | x_{p,R} \in C_{p,R} \text{ whenever } (p, R) \in I\}$$

is join-admissible with

$$\bigvee \{ \bigwedge_{(p,R)\in I} x_{p,R} | x_{p,R} \in C_{p,R} \text{ whenever } (p, R) \in I\} = 0.$$

Clearly, $\{\bigvee C_{p,R}|(p, R) \in I\} \subseteq Z$. Furthermore, if $b \in Z$, then there is some partition p with $b \in p$, so $(p, \{b\}) \in I$, and $b = \bigvee C_{p,\{b\}}$. Therefore $\{\bigvee C_{p,R}|(p, R) \in I\} = Z$. Now, for the sake of contradiction, assume that $\{\bigvee C_{p,R}|(p, R) \in I\}$ is meet-admissible and

$$\bigwedge_{(p,R)\in I} \bigvee C_{p,R} = \bigvee \{ \bigwedge_{(p,R)\in I} x_{p,R} | x_{p,R} \in C_{p,R} \text{ whenever } (p, R) \in I\}.$$

Then Z is meet-admissible and $\bigwedge Z = 0$, so $0 \in Z$. This is a contradiction since we assumed that Z was a proper F-ideal. □

Remark 3.4.17 *By examining the proof of Theorem 3.4.16, it is easy to see that there are several minor variants to the conditions 2. and 3. in Theorem 3.4.16 that are equivalent to superstability.*

Remark 3.4.18 *In the paper [3], Bruns gives similar conditions for when a Boolean algebra with specified least upper bounds is representable as an algebra of sets. However, the conditions in Theorem 3.4.16 are stronger.*

Lemma 3.4.19 *Let (X, M) be a complete partition space such that $\mathfrak{B}^*(X, M)$ is superstable and locally refinable. Then M contains all partition of X into clopen sets.*

In particular, $\mathfrak{B}^*(X,M)$ is the collection of all clopen subsets of X.

Proof. Since $\mathfrak{B}^*(X,M)$ is superstable and locally refinable, the space (X,M) is supercomplete. Therefore by [6][Ch 7. Thm 41, p. 140] the uniformity M is the fine uniformity, so M contains all partitions of X into clopen sets. $\square$

Theorem 3.4.20 *Let (B,F) be a Boolean partition algebra. Then the following are equivalent.*

1. (B,F) is precomplete and superstable.

2. Let I be an index set. Let $C_i \subseteq B$ be a join-admissible set for $i \in I$. Assume that whenever $x_i \in C_i$ for $i \in I$, then $\{x_i | i \in I\}$ is meet-admissible. Then

i. $\{\bigvee C_i | i \in I\}$ is meet-admissible,

ii. $\{\bigwedge_{i\in I} x_i | x_i \in C_i \text{ for } i \in I\}$ is join-admissible, and

iii.

$$\bigwedge_{i\in I} \bigvee C_i = \bigvee \{\bigwedge_{i\in I} x_i | x_i \in C_i \text{ for } i \in I\}.$$

Proof. $1 \to 2$ Since (B,F) is precomplete, the total local refinement $(B,F^{(\infty)})$ is subcomplete and superstable and locally refinable, and the join-admissible and meet-admissible sets are unchanged. Therefore, we may assume that (B,F) is subcomplete superstable and locally refinable. Since (B,F) is subcomplete superstable and locally refinable, we may assume that $(B,F) = \mathfrak{B}^*(X,M)$ for some complete partition space (X,M). In this case, by superstability, a subset $C \subseteq \mathfrak{B}^*(X,M)$ is join-admissible if and only if $\bigcup C \in \mathfrak{B}^*(X,M)$, and a subset $C \subseteq \mathfrak{B}^*(X,M)$ is meet-admissible if and only if $\bigcap C \in \mathfrak{B}^*(X,M)$. Now assume that I is an index set and $C_i \subseteq \mathfrak{B}^*(X,M)$ is join-admissible for $i \in I$, and $\{R_i | i \in I\}$ is meet-admissible whenever $R_i \in C_i$ for $i \in I$. Then $\bigcup C_i \in \mathfrak{B}^*(X,M)$ for $i \in I$ and $\bigcap_{i\in I} R_i \in \mathfrak{B}^*(X,M)$ whenever $R_i \in C_i$ for $i \in I$, and $\bigvee C_i = \bigcup C_i$ for $i \in I$ and $\bigwedge_{i\in I} R_i = \bigcap_{i\in I} R_i$. Furthermore, $\bigcup\{\bigcap_{i\in I} R_i | R_i \in C_i \text{ for } i \in I\} = \bigcap_{i\in I} \bigcup C_i$. Since $\bigcup C_i \in \mathfrak{B}^*(X,M)$, the set $\bigcup C_i$ is clopen for $i \in I$, so the set $\bigcap_{i\in I} \bigcup C_i$ is closed being the intersection of clopen sets. Meanwhile, if $R_i \in C_i$ for $i \in I$, then the set $\bigcap_{i\in I} R_i$ is clopen as well since

$\bigcap_{i\in I} R_i \in \mathfrak{B}^*(X,M)$. Therefore $\bigcup\{\bigcap_{i\in I} R_i | R_i \in C_i \text{ for } i \in I\}$ is open being the union of open sets. Since $\bigcup\{\bigcap_{i\in I} R_i | R_i \in C_i \text{ for } i \in I\} = \bigcap_{i\in I}\bigcup C_i$, the set $\bigcap_{i\in I}\bigcup C_i$ is clopen. Therefore, $\bigcap_{i\in I}\bigcup C_i = \bigcup\{\bigcap_{i\in I} R_i | R_i \in C_i \text{ for } i \in I\} \in \mathfrak{B}^*(X,M)$ by Lemma 3.4.19. We conclude that

i. $\{\bigcup C_i | i \in I\} = \{\bigvee C_i | i \in I\}$ is meet-admissible,

ii. $\{\bigcap_{i\in I} R_i | R_i \in C_i \text{ for } i \in I\} = \{\bigwedge_{i\in I} R_i | R_i \in C_i \text{ for } i \in I\}$ is join-admissible, and

iii.

$$\bigwedge_{i\in I}\bigvee C_i = \bigcap_{i\in I}\bigcup C_i$$

$$= \bigcup\{\bigcap_{i\in I} R_i | R_i \in C_i \text{ for } i \in I\} = \bigvee\{\bigwedge_{i\in I} R_i | R_i \in C_i \text{ for } i \in I\}$$

$2 \rightarrow 1$ We shall prove this direction by contrapositive. If we assume that (B,F) is not superstable, then 2 cannot hold by Theorem 3.4.16. Now assume that (B,F) is not precomplete. Then $(B,F^{(\infty)})$ has the same join-admissible sets and meet-admissible sets as (B,F), but $(B,F^{(\infty)})$ is not subcomplete. Therefore there is some $p \in F^{(\infty)}$ and some subset $R \subseteq p$ that does not have a least upper bound. Let $V = \{S | R \subseteq S \subseteq p, \bigvee S \text{ exists}\}$. Then each $S \in V$ is join-admissible, and whenever $x_S \in S$ for $S \in V$, then $\{x_S | S \in V\}$ is meet-admissible. However, the set $\{\bigvee S | S \in V\}$ is not meet-admissible, otherwise $\bigwedge\{\bigvee S | S \in V\} = \bigvee R$, a contradiction. □

Corollary 3.4.21 *Let κ be a regular cardinal. Let B be a κ-complete Boolean algebra. Then the following are equivalent.*

1. B is strongly κ-representable.

2. Let I be an index set. Let $C_i \subseteq B$ be a subset where $|C_i| < \kappa$ for $i \in I$. Assume that whenever $x_i \in C_i$ for $i \in I$, then there is a $J \subseteq I$ with $|J| < \kappa$ and where $\bigwedge_{j\in J} x_j$ is a lower bound of $\{x_i | i \in I\}$. Then

i. there is a $J \subseteq I$ where $|J| < \kappa$ and $\bigwedge_{j\in J}\bigvee C_j$ is a lower bound of $\{\bigvee C_i | i \in I\}$,

ii. there is a $K \subseteq \prod_{i\in I} C_i$ with $|K| < \kappa$ and where $\bigvee\{\bigwedge_{i\in I} a_i | (a_i)_{i\in I} \in K\}$ is an upper bound of $\{\bigwedge_{i\in I} a_i | (a_i)_{i\in I} \in \prod_{i\in I} C_i\}$, and

iii. $\bigwedge_{j\in J}\bigvee C_i = \bigvee\{\bigwedge_{i\in I} a_i | (a_i)_{i\in I} \in \prod_{i\in I} C_i\}$.

Proof. See the proof of Theorem 3.4.16. $\square$

Definition 3.4.22 *A cardinal κ is said to be strongly compact if every κ-complete filter on a set is extendible to a κ-complete ultrafilter.*

We have the following characterization of strongly compact cardinals. A similar characterization holds for weakly compact cardinals.

Corollary 3.4.23 *Let κ be a regular cardinal. Then the following are equivalent.*

1. κ is strongly compact.

2. Let I be an index set. Let $\mathcal{C}_i$ be a collection of sets with $|\mathcal{C}_i| < \kappa$ for $i \in I$. Assume that whenever $R_i \in \mathcal{C}_i$ for $i \in I$ there is a $J \subseteq I$ with $|J| < \kappa$ and where $\bigcap_{j\in J} R_j = \bigcap_{i\in I} R_i$. Then

i. There is a subset $J \subseteq I$ where $|J| < \kappa$ and where $\bigcap_{j\in J} \bigcup \mathcal{C}_j = \bigcap_{i\in I} \bigcup \mathcal{C}_i$, and

ii. There is a collection $\mathcal{F}$ of choice functions f with domain I and where $f(i) \in \mathcal{C}_i$ for $i \in I$ where $|\mathcal{F}| < \kappa$ and where

$$\bigcup_{f\in\mathcal{F}} \bigcap_{i\in I} f(i) = \bigcup \{ \bigcap_{i\in I} R_i | R_i \in \mathcal{C}_i \text{ for } i \in I \}.$$

Appendix A - Summary of further results

In this appendix, I will briefly go over results on Boolean partition algebras not covered in this dissertation along with notions related to Boolean partition algebras.

In section 2.3, it is mentioned that subcomplete Boolean partition algebras are essentially point-free surjective inverse systems. We shall now formalize the intuitive idea that subcomplete Boolean partition algebras are essentially surjective inverse limits of sets. Let **PF** denote the category of inverse systems of sets $((X_d)_{d\in D}, (f_{d,e})_{d,e\in D, d\leq e})$ where each $f_{d,e}$ is surjective. We define the set of homomorphisms by

$$Hom((X_d)_{d\in D}, (Y_e)_{e\in E}) = \underset{\longleftarrow}{\lim}{}_{,e\in E}\ \underset{\longrightarrow}{\lim}{}_{,d\in D} Hom(X_d, Y_e)$$

in which there are canonical notions of morphisms between the sets in the direct limits and inverse limits. The composition between morphisms in **PF** is defined in a natural way. The category **PF** is a full subcategory of the category of all pro-sets (the category of pro-sets it the category of all inverse systems of sets). See [14] for more information on pro-sets and the pro-completion of other categories.

Theorem 3.4.24 *The category* **PF** *is contravariantly equivalent to the category of subcomplete Boolean partition algebras.*

In this duality the correspondences are defined as follows. If (B, F) is a subcomplete Boolean partition algebra, then F is a surjective inverse limit of sets with transitional mappings $\phi_{p,q} : p \rightarrow q$ whenever $p \preceq q$. If $(X_d)_{d\in D} \in \mathbf{PF}$, then the direct limit $\underset{\longrightarrow}{\lim}{}_{,d\in D}\mathcal{P}(X_d)$ of Boolean partition algebras is the corresponding Boolean partition algebra. The rest of the proof of the equivalence between these two categories consists mainly of technical details.

This duality not only gives intuition behind the notion of a Boolean partition algebra, but it also makes objects in **PF** and more generally in all pro-sets much easier to handle. While the objects in **PF** are fairly easy to describe, the morphisms **PF** are difficult to grasp and handle being the inverse limit of a direct limit of sets of homomorphisms. This difficulty arises because unlike many categories, objects in **PF** are not structures built over sets but rather they are inverse systems. With this duality, one is able to translate the category **PF** into the category of subcomplete Boolean partition algebras where the morphisms between Boolean partition algebras are simply functions and the composition of morphisms in the category of Boolean partition algebras is simply the ordinary composition of functions.

Boolean partition algebras can also be interpreted as "point-free uniform spaces." In order to formalize the notion of a point-free topological space and a point-free uniform space, one will need concepts from lattice theory. We define a frame to be a complete lattice L that satisfies the following distributive law

$$x \wedge \bigvee_{i \in I} y_i = \bigvee_{i \in I} (x \wedge y_i).$$

Frames are the main object of study in point-free topology. Observe that the open sets in any topological space form a frame, and most of the information of a topological space is contained in the lattice of open sets. In fact, any Hausdorff space X can be completely reconstructed from the frame of open subsets of X. Therefore, the notion of a frame is a generalization of the notion of a topological space. Furthermore, many concepts in general topology can be formulated in terms of frames such as regularity, complete regularity, normality, paracompactness, compactness, connectedness, zero-dimensionality, etc. In fact, many results in point-free topology are much stronger than their point-set analogues, so in some sense point-free topology is more well behaved than point-set topology. Also, the notion of a uniform space has a point-free analogue called a uniform frame. See the books [9],[17] for information on point-free topology.

If (B, F) is a Boolean partition algebra, then the lattice $\mathrm{Id}(B, F)$ of F-ideals is an ultraparacompact frame. In fact, the category of subcomplete locally refinable Boolean partition algebras is equivalent to the category of all ultraparacompact frames. More generally, the category of subcomplete Boolean partition algebras is equivalent to the category of ultracomplete uniform frames. From this representation of ultracomplete uniform frames, we can conclude that every ultracomplete uniform frame is ultraparacompact.

The notion of a Boolean partition algebra can be generalized to the notion of a Boolean covering algebra. Essentially, a Boolean covering algebra is like a Boolean partition algebra except we replace the notion of a partition with the notion of a cover (i.e. a subset C of the Boolean algebra with $\bigvee C = 1$). A Boolean covering algebra is essentially a Boolean algebra along with a collection of distinguished least upper bounds. Some of the results on Boolean partition algebras can be generalized to Boolean covering algebras including Theorem 3.4.16 that shows that superstability is equivalent to a strong distributivity property. The equivalence between the category of ultraparacompact frames and Boolean partition algebras may be generalized to all zero-dimensional frames. The category of zero-dimensional frames is equivalent to the category of all subcomplete locally refinable Boolean covering algebras.

The notion of join-admissibility in section 3.3 can be generalized to structures called admissibility systems, LUB-systems, and LUB-based lattices. In these structures, we have an abstract notion of a join-admissible set over any poset. The notion of an admissiblity system is related to lattice theory, point-free topology, and even ordered topological spaces. Furthermore, it turns out that the category of Boolean covering systems is isomorphic to the category of all Boolean admissibility systems.

We shall now outline a generalized ultrapower construction using Boolean partition algebras. Assume (B, F) is a subcomplete Boolean partition algebra and $\mathcal{U}$ is an ultrafilter on the Boolean algebra B. Then for each $p \in F$, let $\mathcal{U}_p = \{R \subseteq p | \bigvee R \in \mathcal{U}\}$. Then each $\mathcal{U}_p$ is an ultrafilter on the set p. Therefore, we are able to speak of an ultrapower $\mathcal{A}^p/\mathcal{U}_p$ for each first order structure $\mathcal{A}$. Furthermore, the transitional mappings $\phi_{p,q} : p \rightarrow q$ induce elementary embeddings from $\mathcal{A}^q/\mathcal{U}_q$ to $\mathcal{A}^p/\mathcal{U}_p$. We conclude that systems $(\mathcal{A}^p/\mathcal{U}_p)_{p \in F}$ is a directed system of elementarily embedded structures. We therefore define the Boolean partition algebra ultrapower (or, more concisely BPA-ultrapower)$\mathcal{A}^{(B,F)}/\mathcal{U}$ to be the direct limit of ultrapowers $\underrightarrow{\lim}\mathcal{A}^p/\mathcal{U}_p$.

The Boolean partition algebra ultrapower satisfies the properties that one would like in an ultrapower. The canonical embeddings $\mathcal{A} \rightarrow \mathcal{A}^{(B,F)}/\mathcal{U}, \mathcal{A}^{p}/\mathcal{U}_{p} \rightarrow \mathcal{A}^{(B,F)}/\mathcal{U}$ are elementary embeddings. In a similar fashion to this ultrapower construction, one may use subcomplete Boolean partition algebras to construct reduced powers. We take note that subcompleteness is necessary for this ultrapower construction since $\mathcal{U}_{p}$ can only be an ultrafilter if the partition p is subcomplete.

The BPA-ultrapower construction is a generalization of most ultrapower constructions including the Boolean ultrapower construction [13] and the limit ultrapower [10],[11]. However, it should be noted that the correspondence between limit ultrapowers and BPA-ultrapowers requires the duality between Boolean partition algebras and uniform spaces. It should be noted that the limit ultrapower and BPA-ultrapower are in the following sense the most general ultrapower constructions. A *complete structure* is a first order structure $\mathcal{A}$ such that every function is a fundamental operation and every relation is a fundamental relation. Keisler showed in [11] that every elementary extension of a complete first order structure can be realized as a limit ultrapower (and hence as a BPA-ultrapower). In a similar sense, the BPA-reduced power is the most general reduced power construction.

Even though the BPA-ultrapower is very general, BPA-ultrapowers behave very algebraically and they are very versatile. For instance, partition homomorphisms between Boolean partition algebras induce elementary embeddings between their corresponding BPA-ultrapowers. More precisely, if $\phi : (B,F) \rightarrow (C,G)$ is a partition homomorphism between subcomplete Boolean partition algebras and $\mathcal{U} \subseteq C$ is an ultrafilter, then for each first order structure $\mathcal{A}$, the mapping ϕ induces an elementary embedding $\phi_{\circ} : \mathcal{A}^{(B,F)}/\phi^{-1}[\mathcal{U}] \rightarrow \mathcal{A}^{(C,G)}/\mathcal{U}$. Also, the direct limit of a system of BPA-ultrapowers can often be represented nicely as a single BPA-ultrapower. One can even represent an iterated BPA-ultrapower as a single BPA-ultrapower. More specifically, if $(B,F),(C,G)$ are subcomplete Boolean partition algebras and $\mathcal{U} \subseteq B, \mathcal{V} \subseteq C$ are ultrafilters, then there is a subcomplete Boolean partition algebra $(B,F)^{(C,G)}$ and an ultrafilter $\mathcal{U}^{\mathcal{V}} \subseteq (B,F)^{(C,G)}$ such that for every first order structure $\mathcal{A}$, we have

$$(\mathcal{A}^{(B,F)}/\mathcal{U})^{(C,G)}/\mathcal{V} \simeq \mathcal{A}^{(B,F)^{(C,G)}}/\mathcal{U}^{\mathcal{V}}.$$

A similar result holds for the ultraproduct of BPA-ultrapowers.

Theorem 3.4.25 *Let I be a set and let $\mathcal{U}$ be an ultrafilter on I. For each $i \in I$, let (B_i, F_i) be a Boolean partition algebra and let $\mathcal{U}_i \subseteq B_i$ be an ultrafilter. Let $\mathcal{V} \subseteq \prod_{i \in I} B_i$ be the ultrafilter where $(b_i)_{i\in I} \in \mathcal{V}$ if and only if $\{i \in I | b_i \in \mathcal{U}_i\} \in \mathcal{U}$. Then for each first order structure $\mathcal{A}$, we have*

$$\prod_{i \in I}(\mathcal{A}^{(B_i,F_i)}/\mathcal{U}_i)/\mathcal{U} \simeq \mathcal{A}^{\prod_{i\in I}(B_i,F_i)}/\mathcal{V}.$$

Appendix B - Table of preservation of properties

Here we summarize when a certain operation on Boolean partition algebras preserves a certain property of Boolean partition algebras. These facts are not difficult to prove or they have been proven already in this work.

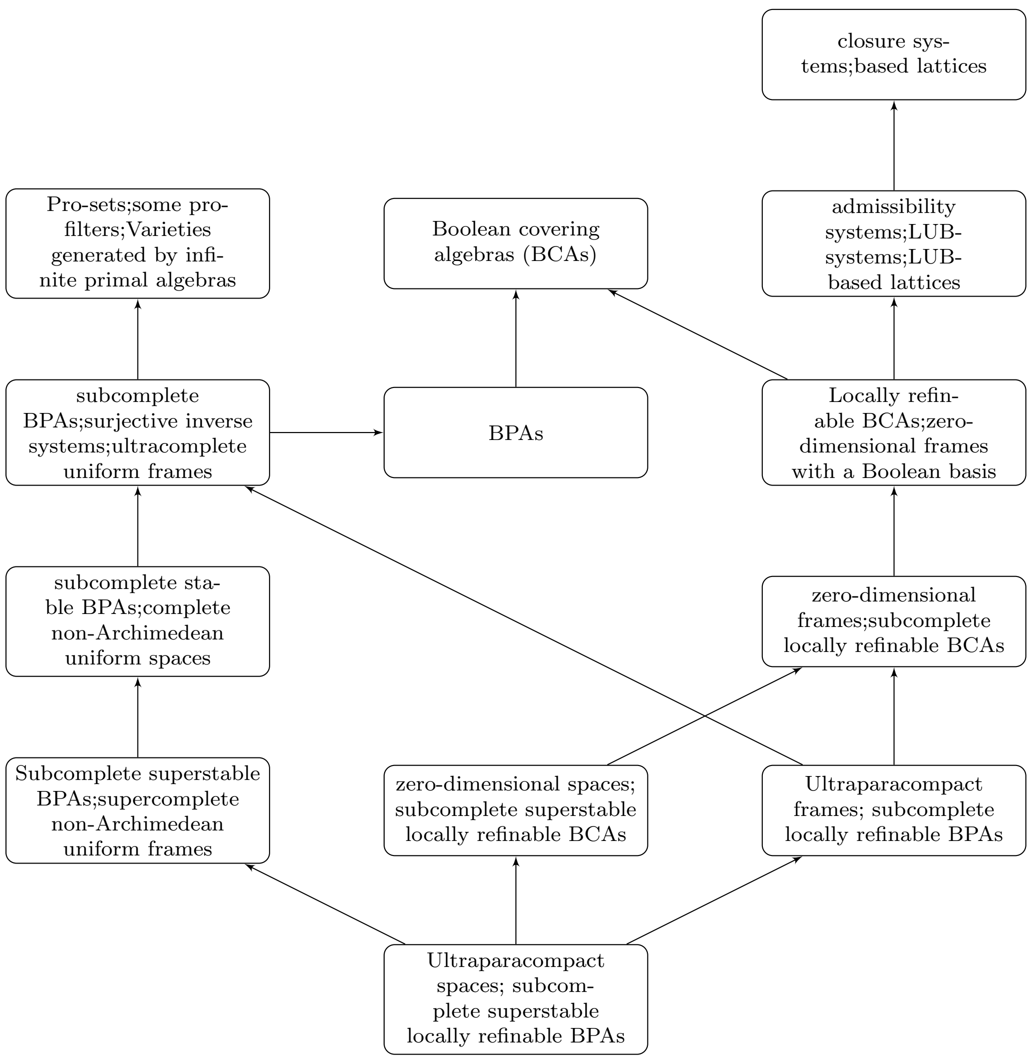
Table 3.1: Table of preservation of properties

$(B,F),(B_i,F_i)$	Stable	Subcomplete	Precomplete	Locally Refinable	Superstable
Quotients	No	Yes	Yes	Yes	Yes
Subalgebras	Yes	No	No	Yes	No
Products	Yes	Yes	Yes	Yes	Yes
Subcompletion	Yes	Yes	Yes	Yes	Yes
Stabilization	Yes	Yes	Yes	Yes	Yes
$(B,F^{(\infty)})$	Yes	No	Yes	Yes	Yes
$(B,F)\restriction a$	Yes	Yes	Yes	Yes	Yes
$\mathfrak{B}^*(S^*(B,F))$	Yes	Yes	Yes	Yes	Yes
$(B,G):G\subseteq F$	Yes	Yes	Yes	No	No
$(\{0\}\cup\bigcup G,G)$: G is a filter in F	Yes	Yes	No	No	No
$(B,G):F\subseteq G$	No	No	No	No	No
$(B,C_U(F))$	Yes	No	No	Yes	Yes

Appendix C - Chart of categories

The following is a table of categories related to Boolean partition algebras. Equivalent categories are put in the same box.

Table 3.2: Chart of categories



References

[1] Artico, Giuliano; Marconi, Umberto; Pelant, Jan On supercomplete ω_μ -metric spaces. Bull. Polish Acad. Sci. Math. 44 (1996), no. 3, 299-310.

[2] G. Bezhanishvili, Zero-dimensional Proximities and Zero-dimensional Compactifications, Topology Appl. 156 (2009), 1496–1504.

[3] Bruns, G.: On the representation of Boolean algebras, Canad. Math. Bull. 5 (1962), 37-41.

[4] Comfort, W. W., and S. Negrepontis. The Theory of Ultrafilters. Berlin: Springer-Verlag, 1974.

[5] J. Dugundji, *Topology.* Boston: Allyn and Bacon, 1966.

[6] Isbell, J. R.: Uniform Spaces, American Mathematical Society, Providence (1964)

[7] James, I. M.: Topological and Uniform Spaces, Springer-Verlag, New York (1987)

[8] Blyth, T. S., and M. F. Janowitz. Residuation Theory. Oxford: Pergamon, 1972.

[9] Johnstone, Peter T. Stone Spaces. Cambridge: Cambridge UP, 1982.

[10] Chang, Chen Chung, and H. Jerome Keisler. Model Theory. Amsterdam: North-Holland Pub., 1973.

[11] Keisler, H. Jerome. Limit ultrapowers. Trans. Amer. Math. Soc. 107 1963 382-408

[12] Koppelberg, Sabine. Handbook of Boolean Algebras. Ed. James Donald. Monk and Robert Bonnet. Amsterdam: North-Holland, 1989.

[13] Mansfield, Richard. The theory of Boolean ultrapowers. Ann. Math. Logic 2 (1970/71), no. 3, 297323.

[14] Kashiwara, Masaki, and Pierre Schapira. Categories and Sheaves. Berlin: Springer, 2006.

[15] Monk, J. Donald, and Robert Bonnet. Handbook of Boolean Algebras. Vol. 3. Amsterdam: North-Holland, 1989.

[16] Priestley, H. A. Representation of distributive lattices by means of ordered stone spaces. Bull. London Math. Soc. 2 1970 186-190.

[17] Picado, Jorge, and Pultr, Ales. Frames and Locales: Topology without Points. Basel: Birkha¨user, 2012.

[18] Robert, Alain. A Course in p-adic Analysis. New York: Springer, 2000.

[19] Sikorski, Roman. Boolean Algebras. Berlin: Springer, 1969.

[20] Stone, M. H.: Applications of the theory of Boolean rings to general topology, Trans. Amer. Math. Soc. 41 (1937), no. 3, 375–481.

[21] Walker, Russell C. The Stone-Cech Compactification. Berlin: Springer-Verlag, 1974

[22] An empty inverse limit William C. Waterhouse Proc. Amer. Math. Soc. 36 (1972), 618.

[23] Willard, Stephen. General Topology. Reading, MA: Addison-Wesley Pub., 1970.

[24] Yeh, J. Lectures on Real Analysis. Singapore: World Scientific, 2000.

INDEX