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A Study of Permutation Polynomials over Finite Fields

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A Study of Permutation Polynomials over Finite Fields

by

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A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy
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Dedication

This doctoral dissertation is dedicated to my mother, my father, my sister and my late grandmother.
I am heavily indebted to my advisor, Dr. Xiang-dong Hou for his inspiration and invaluable support throughout my graduate studies at the University of South Florida. Undeniably, this work would not have been a success without his directions, guidance and immeasurable contribution. It was really a great honor to have been his student.

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4.1 Desirable triples \((n,e;4)\), \(e \leq 6\), \(w_4(n) > 4\) \hspace{1cm} 68
Let $p$ be a prime and $q = p^k$. The polynomial $g_{n,q} \in \mathbb{F}_p[x]$ defined by the functional equation

$$\sum_{a \in \mathbb{F}_q} (x + a)^n = g_{n,q}(x^q - x)$$

gives rise to many permutation polynomials over finite fields. We are interested in triples $(n, e; q)$ for which $g_{n,q}$ is a permutation polynomial of $\mathbb{F}_{q^e}$. In Chapters 2, 3, and 4 of this dissertation, we present many new families of permutation polynomials in the form of $g_{n,q}$. The permutation behavior of $g_{n,q}$ is becoming increasingly more interesting and challenging. As we further explore the permutation behavior of $g_{n,q}$, there is a clear indication that $g_{n,q}$ is a plenteous source of permutation polynomials.

We also describe a piecewise construction of permutation polynomials over a finite field $\mathbb{F}_q$ which uses a subgroup of $\mathbb{F}_q^*$, a “selection” function, and several “case” functions. Chapter 5 of this dissertation is devoted to this piecewise construction which generalizes several recently discovered families of permutation polynomials.
Let $p$ be a prime and $q$ a power of $p$. Let $\mathbb{F}_q$ be the finite field with $q$ elements. A polynomial $f \in \mathbb{F}_q[x]$ is called a permutation polynomial of $\mathbb{F}_q$ if the mapping $x \mapsto f(x)$ is a permutation of $\mathbb{F}_q$. Every function from $\mathbb{F}_q$ to $\mathbb{F}_q$ can be represented by a polynomial in $\mathbb{F}_q[x]$. In fact, if $\phi : \mathbb{F}_q \to \mathbb{F}_q$ is an arbitrary function form $\mathbb{F}_q$ to $\mathbb{F}_q$, then there exists a unique polynomial $g \in \mathbb{F}_q[x]$ with $\deg(g) \leq q - 1$ representing $\phi$, that is $g(c) = \phi(c)$ for all $c \in \mathbb{F}_q$. The polynomial $g$ can be found by the Lagrange’s interpolation method for the function $\phi$. If $\phi$ is already given as a polynomial function, say $\phi : c \mapsto f(c)$ where $f \in \mathbb{F}_q[x]$, then $g$ can be obtained from $f$ by reduction modulo $x^q - x$. We call permutation polynomials of $\mathbb{F}_q$ PPs over $\mathbb{F}_q$. Search for PPs with nice algebraic structures is an important topic in the study of finite fields since they play a central role in both arithmetic and combinatorial aspects of finite fields. PPs have important applications in Coding Theory, Cryptography, Finite Geometry, Combinatorics and Computer Science, among other fields.

In history, the general study of PPs started with Hermite who considered PPs over finite prime fields. L.E. Dickson was the first person to study PPs of arbitrary finite fields; see [9].

Let $n \geq 0$ be an integer. Since the elementary symmetric polynomials $x_1 + x_2$ and $x_1 x_2$ generate the ring of symmetric polynomials in $\mathbb{Z}[x, y]$, there exists a polynomial $D_n(x, y) \in \mathbb{Z}[x, y]$ such that

$$x_1^n + x_2^n = D_n(x_1 + x_2, x_1 x_2);$$

see [31]. The explicit form of $D_n(x, y)$ is given by Waring’s formula [30, Theorem
1.76] 

\[ D_n(x, y) = \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{n}{n-i} \binom{n-i}{i} (-y)^{i} x^{n-2i}. \]

For fixed \( a \in \mathbb{F}_q \), \( D_n(x, a) \in \mathbb{F}_q[x] \) is the Dickson polynomial of degree \( n \) and parameter \( a \). Dickson polynomials are closely related to the well-known Chebyshev polynomials \( T_n(x) \) over the complex numbers by

\[ D_n(2xa, a^2) = 2a^n T_n(x). \]

The permutation property of the Dickson polynomial is completely known. When \( a = 0 \), \( D_n(x, a) = x^n \), which is a PP over \( \mathbb{F}_q \) if and only if \( (n, q - 1) = 1 \). When \( 0 \neq a \in \mathbb{F}_q \), \( D_n(x, a) \) is a PP over \( \mathbb{F}_q \) if and only if \( (n, q^2 - 1) = 1 \); see [30, Theorem 7.16] or [29, Theorem 3.2].

The concept of the reversed Dickson polynomial \( D_n(a, x) \) was first introduced by Hou, Mullen, Sellers and Yucas in [24] by reversing the roles of the variable and the parameter in the Dickson polynomial \( D_n(x, a) \). When \( a = 0 \), \( D_n(0, x) \) is a PP over \( \mathbb{F}_q \) if and only if \( n = 2k \) with \( (k, q - 1) = 1 \). When \( a \neq 0 \),

\[ D_n(a, x) = a^n D_n(1, \frac{x}{a^2}). \]

Hence \( D_n(a, x) \) is a PP on \( \mathbb{F}_q \) if and only if \( D_n(1, x) \) is a PP on \( \mathbb{F}_q \). The \( n \)th reversed Dickson polynomial \( D_n(1, x) \in \mathbb{Z}[x] \) is defined by

\[ D_n(1, x(1-x)) = x^n + (1-x)^n. \]

There is a connection between reversed Dickson polynomials and almost perfect non-linear (APN) functions which have very important applications in Cryptography [34]. Please refer [24] for more background of the reversed Dickson polynomial.

X. Hou showed in [21] that for each integer \( n \geq 0 \), there exists a unique
polynomial $g_{n,q} \in \mathbb{F}_p[x]$ such that

$$
\sum_{a \in \mathbb{F}_q} (x + a)^n = g_{n,q}(x^q - x). \tag{1.0.1}
$$

The explicit form of $g_{n,q}$ is given by Waring’s formula

$$
g_{n,q}(x) = \sum_{\frac{n}{q} \leq l \leq \frac{n}{q-1}} \frac{n}{l} \left( n - l(q - 1) \right) x^{n-l(q-1)}. \tag{1.0.2}
$$

The polynomial $g_{n,q}$ was introduced in [21] as a $q$-ary version of the reversed Dickson polynomial. We describe the context which led to the formation of the polynomial $g_{n,q}$ in Section 1.1. When $q = 2$, $g_{n,2}$ is the $n$th reversed Dickson polynomial over $\mathbb{F}_2$ since in characteristic 2

$$
g_{n,2}(x^2 - x) = x^n + (x + 1)^n = x^n + (1 - x)^n = D_n(1, x(1 - x)) = D_n(1, x^2 - x).
$$

Permutation properties of the polynomial $g_{n,q}$ were first studied by X. Hou in [22]. The results of this study indicated that the polynomial $g_{n,q}$ opens the door to many new classes of PPs in a new approach. In [22], several families of PPs were found, but there were still many instances in which there was no theoretic explanation. Chapters 2, 3, and 4 of this dissertation are an attempt to answer those unexplained cases that also deal with questions about $g_{n,q}$ that were not touched in [22].

Constructing PPs of finite fields piecewise has been in discussion in numerous recent articles on permutation polynomials. We also construct several families of PPs in this dissertation that generalize some existing results.

Hence this dissertation focuses on the following:

(i) When is $g_{n,q}$ a permutation polynomial of $\mathbb{F}_{q^e}$?

(ii) A piecewise construction of permutation polynomials over finite fields.

The main question concerning permutation polynomials is how to recognize them. The following two criteria for this purpose have been useful in our study.
(1) (Hermite’s Criterion). Let $\mathbb{F}_q$ be of characteristic $p$. Then $f \in \mathbb{F}_q[x]$ is a permutation polynomial of $\mathbb{F}_q$ if and only if the following two conditions hold:

(i) $f^{q-1} \pmod{x^q - x}$ has degree $q - 1$;

(ii) for each integer $s$ with $1 \leq s \leq q - 2$, $f^s \equiv f_s \pmod{x^q - x}$ for some $f_s \in \mathbb{F}_q[x]$ with deg $f_s \leq q - 2$.

(2) $f$ is a permutation polynomial of $\mathbb{F}_{p^n}$ if and only if

$$\sum_{x \in \mathbb{F}_{p^n}} \zeta_p^{\text{Tr}_{p^n/p}(cf(x))} = 0$$

for all $0 \neq c \in \mathbb{F}_{p^n}$, where $\zeta_p = e^{2\pi i/p}$ and $\text{Tr}_{p^n/p}(x) = x + x^p + \cdots + x^{p^{n-1}}$ is the absolute trace function from $\mathbb{F}_{p^n}$ to $\mathbb{F}_p$.

**Definition 1.0.1** (Desirable triple). If $g_{n,q}$ is a PP of $\mathbb{F}_{q^e}$, we say that the triple $(n,e;q)$ is desirable.

A desirable triple is considered categorized if an infinite class containing it has been found. Here is an overview of the dissertation.

In Chapter 2, we discuss the polynomial $g_{n,q}$ when $q = 2$ and list some known families of PPs of $\mathbb{F}_{2^e}$. The case $e = 1$ is completely explained in Chapter 2. Table 2.1, generated by a computer search contains all desirable triples $(n,e;3)$ with $e \leq 6$. We also explain two desirable families of the table. The desirable triple $(407,3;3)$ is explained in Chapter 2 as a sporadic case.

Chapter 3 discusses the permutation behavior of the polynomial $g_{n,q}$ where $n$ is of the form $n = q^a - q^b - 1$. Our computer results showed that this type of desirable triples seems to occur more frequently. The case $e = 2$ is of more interest since all known desirable triples when $e > 2$ are explained by Corollary 3.1.2 and Theorem 3.1.3, and Conjecture 3.1.4 states that there are no other cases. A table (Table 3.2), generated by a computer search, which contains desirable triples $(q^a - q^b - 1,2;q)$ for $q \leq 97$, is also presented. Some of the results listed in table are explained by several new classes discovered in this dissertation, but a theoretical explanation has not been found for many of them.
Chapter 4 primarily deals with desirable triples with even $q$. Numerous classes of desirable triples with $q = 4$ and $e \leq 6$ (see Table 4.1) are explained. Most of the results are also generalized for an even $q$.

Chapter 5 describes a piecewise construction of permutation polynomials over a finite field $\mathbb{F}_q$. Permutation polynomials obtained by this construction unify and generalize several recently discovered families of permutation polynomials.

There are two appendices. Appendix A contains some useful Mathematica codes written to identify the permutation behavior of the polynomial $g_{n,q}$. Appendix B contains computational results used in the proof of Theorem 2.4.1.

In our notation, letters in typewriter typeface, $x$, $y$, $t$, are reserved for indeterminates. The trace function $\text{Tr}_{q^e/q}$ and the norm function $N_{q^e/q}$ from $\mathbb{F}_{q^e}$ to $\mathbb{F}_q$ are also treated as polynomials, that is, $\text{Tr}_{q^e/q}(x) = x + x^q + \cdots + x^{q^{e-1}}$, $N_{q^e/q}(x) = x^{1+q+\cdots+q^{e-1}}$.

When $q$ is given, we define $S_a = x + x^q + \cdots + x^{q^{a-1}}$ for every integer $a \geq 0$. Note that $\text{Tr}_{q^e/q} = S_e$.

1.1 The Polynomial $g_{n,q}$

In this section, we derive the formula (1.0.1) and recall some basic properties of $g_{n,q}$ that will be used in later chapters. We refer the reader to [22] for proofs and further details of properties of $g_{n,q}$.

Let $p$ be a prime and $q$ a power of $p$.

In $\mathbb{F}_q[x]$ we have $x^q - x = \prod_{a \in \mathbb{F}_q} (x+a)$. Let $t$ be another indeterminate and substitute $t + x$ for $x$. Then we have

$$t^q - t + x^q - x = (t + x)^q - (t + x) = \prod_{a \in \mathbb{F}_q} (t + x + a) = \sum_{k=0}^{q} \sigma_k((x + a)_{a \in \mathbb{F}_q}) t^{q-k}, \quad (1.1.3)$$

where $\sigma_k$ is the $k$th elementary symmetric polynomial in $q$ variables. A comparison
of the coefficients of $t$ on both sides of (1.1.3) tells that

$$
\sigma_k((x + a)_{a \in \mathbb{F}_q}) = \begin{cases}
1 & \text{if } k = 0, \\
-1 & \text{if } k = q - 1, \\
x^q - x & \text{if } k = q, \\
0 & \text{otherwise}.
\end{cases} 
$$  
(1.1.4)

Let $n \geq 0$ be an integer. By Waring’s formula [30, Theorem 1.76] and (1.1.4), we have

$$
\sum_{a \in \mathbb{F}_q} (x + a)^n = \sum_{\alpha(q-1) + \beta = n} (-1)^\alpha \frac{(\alpha + \beta - 1)!n}{\alpha!\beta!} (-1)^\alpha (x^q - x)^\beta
$$

$$
= \sum_{\frac{n}{q} \leq l \leq \frac{n}{q-1}} \frac{(l - 1)!n}{(lq - n)!(n - l(q - 1))!} (x^q - x)^{n-l(q-1)} \quad (l = \alpha + \beta)
$$

$$
= \sum_{\frac{n}{q} \leq l \leq \frac{n}{q-1}} \frac{n}{l} \left( \frac{l}{n - l(q - 1)} \right) (x^q - x)^{n-l(q-1)}.
$$

Set

$$
g_{n,q}(x) = \sum_{\frac{n}{q} \leq l \leq \frac{n}{q-1}} \frac{n}{l} \left( \frac{l}{n - l(q - 1)} \right) x^{n-l(q-1)} \in \mathbb{Z}[x].
$$

(Note that the coefficients of $g_{n,q}(x)$ are integers since the coefficients in Waring’s formula are integers.) Then in $\mathbb{F}_q[x]$ we have

$$
\sum_{a \in \mathbb{F}_q} (x + a)^n = g_{n,q}(x^q - x).
$$

**Proposition 1.1.1 ([22]).** The polynomial $g_{n,q}$ satisfies the recurrence relation

$$
\begin{cases}
g_{0,q} = \cdots = g_{q-2,q} = 0, \\
g_{q-1,q} = -1, \\
g_{n,q} = xg_{n-q,q} + g_{n-q+1,q}, \quad n \geq q.
\end{cases} 
$$  
(1.1.5)
Using the above recurrence relation, \( g_{n,q} \) can be defined for \( n < 0 \):

\[
g_{n,q} = \frac{1}{x}(g_{n+q,q} - g_{n+1,q}).
\]

For \( n < 0 \), \( g_{n,q} \) belongs to \( \mathbb{F}_p[x,x^{-1}] \), the ring of Laurent polynomials in \( x \) over \( \mathbb{F}_p \).

Hence the functional equation \((1.0.1)\) holds for all \( n \in \mathbb{Z} \).

By \((1.1.5)\) we have the generating function of \( \{g_{n,q}\}_{n \geq 0} \):

\[
\sum_{n \geq 0} g_{n,q} t^n = \frac{-t^{q-1}}{1-t^{q-1}-xt^q}.
\]

**(Proposition 1.1.2)**

(i) We have \( g_{pm,q} = g_{n,q}^p \).

(ii) If \( n_1, n_2 > 0 \) are integers such that \( n_1 \equiv n_2 \pmod{q^{pe} - 1} \), then \( g_{n_1,q} \equiv g_{n_2,q} \pmod{x^{qe} - x} \).

**Proof.**

(i) We have

\[
g_{pm,q}(x^q - x) = \sum_{a \in \mathbb{F}_q} (x + a)^{pn} = \left( \sum_{a \in \mathbb{F}_q} (x + a)^n \right)^p = [g_{n,q}(x^q - x)]^p.
\]

(ii) For all \( x \in \mathbb{F}_{q^{pe}} \), we have

\[
g_{n_1,q}(x^q - x) = \sum_{a \in \mathbb{F}_q} (x + a)^{n_1} = \sum_{a \in \mathbb{F}_q} (x + a)^{n_2} = g_{n_2,q}(x^q - x).
\]

In particular, \( g_{n_1,q}(x) = g_{n_2,q}(x) \) for all \( x \in \mathbb{F}_{q^{pe}} \), i.e., \( g_{n_1,q} \equiv g_{n_2,q} \pmod{x^{qe} - x} \).

If two integers \( m, n > 0 \) belong to the same \( p \)-cyclotomic coset modulo \( q^{pe} - 1 \), the two triples \((m, e; q)\) and \((n, e; q)\) are called *equivalent*, and we write \((m, e; q) \sim (n, e; q)\) or
$m \sim_{(e,q)} n$. It follows from Proposition 1.1.2 that desirability of triples is preserved under the $\sim$ equivalence.

Given integers $d > 1$ and $a = a_0d^0 + \cdots + a_td^t$, $0 \leq a_i \leq d - 1$, the base $d$ weight of $a$ is $w_d(a) = a_0 + \cdots + a_t$.

Let $n \geq 0$ be any integer and $w_q(n)$ denote the base $q$ weight of $n$.

**Lemma 1.1.3** ([22]). Let $n = \alpha_0q^0 + \cdots + \alpha_tq^t$, $0 \leq \alpha_i \leq q - 1$ and $w_q(n)$ be the base $q$ weight of $n$,

$$g_{n,q} = \begin{cases} 0 & \text{if } w_q(n) < q - 1, \\ -1 & \text{if } w_q(n) = q - 1, \\ \alpha_0x^q + (\alpha_0 + \alpha_1)x^{q^1} + \cdots + (\alpha_0 + \cdots + \alpha_{t-1})x^{q^{t-1}} + \delta & \text{if } w_q(n) = q, \end{cases}$$

(1.1.7)

where

$$\delta = \begin{cases} 1 & \text{if } q = 2, \\ 0 & \text{if } q > 2. \end{cases}$$

**Definition 1.1.4** An $\mathbb{F}_q$-linearized polynomial (or a $q$-polynomial) over $\mathbb{F}_{q^e}$ is a polynomial of the form

$$L(x) = \sum_{i=0}^{k} a_ix^{q^i} \in \mathbb{F}_{q^e}[x].$$

It is well known that $L$ is a PP of $\mathbb{F}_{q^e}$ if and only if $L(x)$ only has the root 0 in $\mathbb{F}_{q^e}$, i.e., $L$ is a PP of $\mathbb{F}_{q^e}$ if and only if $\gcd(L(x), x^{q^e} - x) = 1$.

**Definition 1.1.5** The polynomials

$$l(x) = \sum_{i=0}^{k} a_ix^i \quad \text{and} \quad L(x) = \sum_{i=0}^{k} a_ix^{q^i}$$
over \( \mathbb{F}_{q^e} \) are called \( q \)-associates of each other. More precisely, \( l(x) \) is the conventional \( q \)-associate of \( L(x) \) and \( L(x) \) is the linearized \( q \)-associate of \( l(x) \).

Now by [30, Theorem 3.62], the above condition for \( L \) to be a PP of \( \mathbb{F}_{q^e} \) can be restated as follows. \( L \) is a PP of \( \mathbb{F}_{q^e} \) if and only if \( \gcd(l(x), x^e - 1) = 1 \).

So by (1.1.7) and the above fact, we have the following proposition when \( w_q(n) = q \).

**Proposition 1.1.6 ([22])**. Let \( n = \alpha_0 q^0 + \cdots + \alpha_t q^t \), \( 0 \leq \alpha_i \leq q - 1 \), with \( w_q(n) = q \). Then \( (n, e; q) \) is desirable if and only if

\[
\gcd(\alpha_0 + (\alpha_0 + \alpha_1)x + \cdots + (\alpha_0 + \cdots + \alpha_{t-1})x^{t-1}, x^e - 1) = 1.
\]

Next lemma considers triples \((n, e; p)\) where \( n \) is of the form \( n = \alpha(p^0e + p^1e + \cdots + p^{(p-1)e}) + \beta \), where \( \alpha, \beta \in \mathbb{Z} \).

**Lemma 1.1.7 ([22])**. Let \( n = \alpha(p^0e + p^1e + \cdots + p^{(p-1)e}) + \beta \), where \( \alpha, \beta \in \mathbb{Z} \). Then for \( x \in \mathbb{F}_{p^e} \),

\[
g_{n,p}(x) = \begin{cases} 
g_{\alpha p + \beta, p}(x) & \text{if } \Tr_{p^e/p}(x) = 0, \\
x^\alpha g_{\beta, p}(x) & \text{if } \Tr_{p^e/p}(x) \neq 0.
\end{cases} \tag{1.1.8}
\]

**Proposition 1.1.8 ([22])**. In the previous lemma, \((n, e; p)\) is desirable if the following two conditions are satisfied.

(i) Both \( g_{\alpha p + \beta, p} + \delta \) and \( x^\alpha g_{\beta, p} \) are \( \mathbb{F}_p \)-linear on \( \mathbb{F}_{p^e} \) and are \( 1 \)-1 on \( \Tr_{p^e/p}^{-1}(0) = \{x \in \mathbb{F}_{p^e} : \Tr_{p^e/p}(x) = 0\} \).

(ii) \( g_{\beta, p}(1) \neq e\delta \).

**Proposition 1.1.9 ([22])**. Assume that both \( g_{\alpha p + \beta, p} + \delta \) and \( x^\alpha g_{\beta, p} \) are \( \mathbb{F}_p \)-linear on
Then $g_{\alpha + \beta,p}$ is 1-1 on $\text{Tr}^{-1}_{\mathbb{F}_{p^e}/\mathbb{F}_p}(0)$ if and only if

$$\gcd\left( \sum_{i=0}^{e-1} a_i x^i, x^e - 1 \right) = x - 1;$$

$x^\alpha g_{\beta,p}$ is 1-1 on $\text{Tr}^{-1}_{\mathbb{F}_{p^e}/\mathbb{F}_p}(0)$ if and only if

$$\gcd\left( \sum_{i=0}^{e-1} b_i x^i, x^e - 1 \right) = x - 1.$$

**Lemma 1.1.10** ([22]). Let $l$ and $i > 0$ be integers. Then

$$g_{l+q^i,q} = g_{l+1,q} + S_i \cdot g_{l,q}, \quad (1.1.9)$$

where $S_i = x + x^q + \cdots + x^{q^{i-1}}$.

From (1.1.9), we have

$$(S_a - S_b) g_{n,q} = g_{n+q^a,q} - g_{n+q^b,q}, \quad (1.1.10)$$

where $a, b > 0$ are integers. Also note that

$$S_a - S_b \equiv S_{a-b} \pmod{x^{q^e} - x} \quad \text{if} \ b \equiv 0 \ \text{or} \ a \pmod{e}.$$
In this chapter, we consider some special cases of the polynomial \( g_{n,q} \). This chapter is organized as follows: Section 2.1 discusses the polynomial \( g_{n,q} \) when \( q = 2 \). Section 2.2 explains the case \( e = 1 \) completely. In Section 2.3, we explain two families of desirable triples when \( p = 3 \). The desirable triple \((407, 3; 3)\) is explained in Section 2.4 as a sporadic case. Table 2.1 contains all desirable triples \((n, e; 3)\) with \( e \leq 6 \).

### 2.1 The Polynomial \( g_{n,2} \)

When \( q = 2 \), \( g_{n,2} \) is the \( n \)th reversed Dickson polynomial \( D_n(1, x) \) over \( \mathbb{F}_2 \). Unlike its twin, Dickson polynomial \( D_n(x, a) \), reversed Dickson polynomial \( D_n(a, x) \) is difficult to describe. Reversed Dickson permutation polynomials (RDPPs) are connected to almost perfect nonlinear (APN) functions, a well-studied class of functions in cryptography [34].

A function \( f : \mathbb{F}_q \to \mathbb{F}_q \) is called almost perfect nonlinear (APN) if for each \( a \in \mathbb{F}_q^* \) and \( b \in \mathbb{F}_q \), the equation \( f(x + a) - f(x) = b \) has at most two solutions in \( \mathbb{F}_q \). APN functions were introduced by Nyberg [34].

Because of the connection between RDPPs and APN functions, some classes of reversed Dickson permutation polynomials were obtained from known APN functions. However, not all reversed Dickson permutation polynomials are obtainable from APN functions (see [24, Prop. 5.4]).

*Sections 2.2 and 2.4 of this chapter are taken from [14] which has been published in the journal “Finite Fields and Their Applications”.*
All known desirable triples \((n, e; 2)\) are covered by four classes listed below and an implicit conjecture states that there are no other classes.

1. \(n = 2^k + 1, (k, 2e) = 1.\)
2. \(n = 2^{2k} - 2^{k} + 1, (k, 2e) = 1.\)
3. \(n = 2^e + 2^k + 1, k > 0, e \text{ is even}, (k - 1, e) = 1.\)
4. \(n = 2^{8k} + 2^{6k} + 2^{4k} + 2^{2k} - 1, e = 5k.\)

Classes (i), (ii), and (iv) were obtained from known APN functions. Classes (i) and (ii) were due to Gold [15] and Kasami [26] respectively. Class (iii) appeared in [24] and it was shown that class (iii) is not obtainable from an APN function. In [12], Dobbertin proved that there is a sequence of APN functions when \(e\) is a multiple of 5. Class (iv) was obtained from that APN function. Even though Dobbertin’s class is known, it is still not well understood. We refer the reader to [24] for a connection between reversed Dickson permutation polynomials and APN functions.

### 2.2 The Case \(e = 1\)

In this section, we determine all desirable triples \((n, 1; q)\).

**Theorem 2.2.1** We have

\[
\sum_{n \geq 0} g_{n,q}(x) t^n \equiv \frac{-(xt)^{q-1}}{1 - (xt)^{q-1} - (xt)^q} + (1 - x^{q-1}) \frac{-t^{q-1}}{1 - t^{q-1}} \quad \text{(mod } x^q - x). \tag{2.2.1}
\]

Namely, modulo \(x^q - x,\)

\[
g_{n,q}(x) \equiv a_n x^n + \begin{cases} 
  x^{q-1} - 1 & \text{if } n > 0, \ n \equiv 0 \pmod{q - 1}, \\
  0 & \text{otherwise},
\end{cases} \tag{2.2.2}
\]

where

\[
\sum_{n \geq 0} a_n t^n = \frac{-t^{q-1}}{1 - t^{q-1} - t^q}. \tag{2.2.3}
\]
Proof. From (1.1.6),
\[ \sum_{n \geq 0} g_{n,q} t^n = \frac{-t^{q-1}}{1 - t^{q-1} - xt^q}. \]
Clearly,
\[ \frac{-t^{q-1}}{1 - t^{q-1} - xt^q} \equiv \frac{-(xt)^{q-1}}{1 - (xt)^{q-1} - (xt)^q} + (1 - x^{q-1}) \frac{-t^{q-1}}{1 - t^{q-1}} \quad (\text{mod } x^{q-1} - 1), \]
and
\[ \frac{-t^{q-1}}{1 - t^{q-1} - xt^q} \equiv \frac{-(xt)^{q-1}}{1 - (xt)^{q-1} - (xt)^q} + (1 - x^{q-1}) \frac{-t^{q-1}}{1 - t^{q-1}} \quad (\text{mod } x). \]
Thus (2.2.1) is proved. □

Corollary 2.2.2  
(i) Assume \( q > 2 \). Then \((n, 1; q)\) is desirable if and only if
\[ \gcd(n, q - 1) = 1 \text{ and } a_n \neq 0 \text{ (in } \mathbb{F}_p). \]

(ii) Assume \( q = 2 \). Then \((n, 1; 2)\) is desirable if and only if \( a_n = 0 \) (in \( \mathbb{F}_2 \)).

Proof. (i) By (2.2.2), \( g_{n,q}(x) = a_n x^n \) for all \( x \in \mathbb{F}_q^* \). If \( g_{n,q} \) is a PP of \( \mathbb{F}_q \), then \( a_n \neq 0 \) and \( \gcd(n, q - 1) = 1 \). On the other hand, assume \( a_n \neq 0 \) and \( \gcd(n, q - 1) = 1 \). By (5.2.6), we have \( g_{n,q} \equiv a_n x^n \pmod{x^q - x} \), which is a PP of \( \mathbb{F}_q \).

(ii) By (2.2.2), \( g_{n,2} \equiv a_n x + x - 1 \pmod{x^2 - x} \). If \( a_n = 0 \), then \( g_{n,2} = x - 1 \) which is clearly a PP of \( \mathbb{F}_2 \). Now assume \( g_{n,2} \) is a PP of \( \mathbb{F}_2 \). Then \( g_{n,2}(0) = 1 \) and \( g_{n,2}(1) = a_n \). Since \( g_{n,2} \) is a PP of \( \mathbb{F}_2 \), \( a_n = 0 \).

From (2.2.3) one can easily derive an explicit expression for \( a_n \). But that expression does not give any simple pattern of those \( n \) with \( a_n \neq 0 \) (in \( \mathbb{F}_p \)).

2.3 Two Families of Desirable Triples when \( p = 3 \)

Theorem 2.3.1  Let \( n = 26(3^0 + 3^e + 3^{2e}) + 7 \). Then \((n, e; 3)\) is desirable if and only if \( \gcd(1 + x + x^4, x^e - 1) = x - 1 \).
Proof. Since \( 26 \cdot 3 + 7 = 85 = 1 \cdot 3^0 + 1 \cdot 3^1 + 1 \cdot 3^4 \), by Lemma 1.1.3 we have
\[
g_{26 \cdot 3 + 7}(x) = g_{85,3}(x) = x^{3^0} - x^{3^1} - x^{3^2} - x^{3^4}.
\]
Also, \( g_{7,3}(x) = x \), so \( x^{26} g_{7,3}(x) = x^{27} \). Both \( g_{26 \cdot 3 + 7} \) and \( x^{26} g_{7,3} \) are \( \mathbb{F}_3 \)-linear on \( \mathbb{F}_{3^e} \).

Moreover, \( x^{26} g_{7,3} \) is 1-1 on \( \text{Tr}_{\mathbb{F}_{3^e}/\mathbb{F}_3}(0) \) and \( g_{7,3}(1) = 1 \neq 0 \). So by Proposition 1.1.8, \( g_{n,3} \) is a PP of \( \mathbb{F}_{3^e} \) if and only if \( g_{85,3} \) is 1-1 on \( \text{Tr}_{\mathbb{F}_{3^e}/\mathbb{F}_3}(0) \).

We have, by [22, Eq. 3.4], \( -g_{85,3}(x^3 - x) = x^{3^0} + x^{3^1} + x^{3^4} \). So by Proposition 1.1.9, \( g_{85,3} \) is 1-1 on \( \text{Tr}_{\mathbb{F}_{3^e}/\mathbb{F}_3}(0) \) if and only if \( \gcd(1 + x + x^4, x^{e-1}) = x - 1 \).

\[\Box\]

**Theorem 2.3.2** Let \( n = 163(3^0 + 3^e + 3^{2e}) - 162 \). Then \( (n, e; 3) \) is desirable if and only if \( \gcd(x + x^4 + x^5, x^e - 1) = x - 1 \).

Proof. Since \( 163 \cdot 3 - 162 = 327 = 0.3^0 + 1.3^1 + 0.3^2 + 0.3^3 + 1.3^4 + 1.3^5 \),
by Lemma 1.1.3 we have
\[
g_{327,3} = x^{3^1} + x^{3^2} + x^{3^3} - x^{3^4}.
\]
Also, \( g_{-162,3}(x) = \frac{1}{x^{162}} \), so \( x^{163} g_{-162,3}(x) = x \). Both \( g_{163,3-162} \) and \( x^{163} g_{-162,3} \) are
\( \mathbb{F}_3 \)-linear on \( \mathbb{F}_{3^e} \). Moreover, \( x^{163} g_{-162,3} \) is 1-1 on \( \text{Tr}_{\mathbb{F}_{3^e}/\mathbb{F}_3}(0) \) and \( g_{-162,3}(1) = 1 \neq 0 \). So by Proposition 1.1.8, \( g_{n,3} \) is a PP of \( \mathbb{F}_{3^e} \) if and only if \( g_{327,3} \) is 1-1 on \( \text{Tr}_{\mathbb{F}_{3^e}/\mathbb{F}_3}(0) \).

We have, by [22, Eq. 3.4], \( -g_{327,3}(x^3 - x) = x^{3^1} + x^{3^4} + x^{3^5} \). So by Proposition 1.1.9, \( g_{327,3} \) is 1-1 on \( \text{Tr}_{\mathbb{F}_{3^e}/\mathbb{F}_3}(0) \) if and only if \( \gcd(x + x^4 + x^5, x^e - 1) = x - 1 \).

\[\Box\]

**2.4 A Sporadic Case**

The second unexplained case of desirable triple in Table 3 of [22] is \( (407, 3; 3) \), where
\( 407 = 2 \cdot 3^0 + 2 \cdot 3^4 + 3^5 \). Theorem 2.4.1 suggests that this might be a sporadic case.
By (1.1.9) and Lemma 1.1.3, we have

\[ g_{407,3}(x) \]

\[ = g_{3+2^3,3} + S_5 \cdot g_{2+2^3,3} \]

\[ = g_{3+2^3,3} + S_5 \cdot (g_{3+2^3,3} + S_4 \cdot g_{2+2^3,3}) \]

\[ = x^3 + x^{2^3} + x^{3^3} + S_5 \cdot (1 + S_4 \cdot (-x - x^3 - x^{2^3} - x^{3^3})) \]

\[ \equiv \text{Tr}_{3^3/3}(x) - S_5(1 + S_4^2) \pmod{x^{3^3} - x} \]

\[ \equiv \text{Tr}_{3^3/3}(x) + S_4^2 (1 + S_4^2) \pmod{x^{3^3} - x} \]

\[ \equiv \text{Tr}_{3^3/3}(y) + y^{2^2} (1 + y^2) \pmod{x^{3^3} - x}, \]

where \( y = S_4(x) \), which is a PP of \( \mathbb{F}_{3^3} \). We can further write

\[ g_{407,3}(x) \equiv \text{Tr}_{3^3/3}(y) + y^8 \left( \text{Tr}_{3^3/3}(y) - y^{3^2} \right) \pmod{x^{3^3} - x} \]

\[ = (1 + y^8) \text{Tr}_{3^3/3}(y) - y^{17}. \]

For \( x' \in \mathbb{F}_{3^3}^* \), \( y = S_4(x') \), we have

\[ g_{407,3}(x') = (1 + y^8) \text{Tr}_{3^3/3}(y) - y^{17} = (1 + x^2) \text{Tr}_{3^3/3} \left( \frac{1}{x} \right) - x, \]

where \( x = y^{-9} = S_4(x')^{-9} \). So the fact that \( g_{407,3} \) is a PP of \( \mathbb{F}_{3^3} \) is equivalent to the fact that the function

\[ h(x) = (1 + x^2) \text{Tr}_{3^3/3} \left( \frac{1}{x} \right) - x \]

is a permutation of \( \mathbb{F}_{3^3}^* \). In the next theorem (and its proof), we investigate some peculiar properties of \( h \) in (2.4.4) as a function defined on \( \mathbb{F}_{q^*} \).

**Theorem 2.4.1** Let \( h \) be as in (2.4.4). \( h \) is a permutation of \( \mathbb{F}_{q^*} \) if and only if \( q = 3 \).

**Proof.**
We will show that for every $z \in \mathbb{F}_{3^3}^*$, there exists an $x \in \mathbb{F}_{3^3}^*$ such that

$$(1 + x^2)\text{Tr}_{3/3}(\frac{1}{x}) - x = z.$$ 

(2.4.5)

If $\text{Tr}_{3/3}(\frac{1}{x}) = 0$, $x = -z$ is the solution. If $\text{Tr}_{3/3}(\frac{1}{x}) \neq 0$, we may assume $\text{Tr}_{3/3}(\frac{1}{x}) = 1$. Then

$$z - 1 = az^2(z + b), \quad (a, b) = (1, 0), \ (1, 1), \ (-1, 1).$$

(2.4.6)

We show that one of the following systems has a solution $x \in \mathbb{F}_{3^3}^*$:

\[
\begin{cases}
x^2 - x + 1 - z = 0, \\
\text{Tr}_{3/3}(\frac{1}{x}) = 1; 
\end{cases}
\]

(2.4.7)

\[
\begin{cases}
x^2 + x + 1 + z = 0, \\
\text{Tr}_{3/3}(\frac{1}{x}) = -1.
\end{cases}
\]

(2.4.8)

The solutions of the quadratic equation in (2.4.7) are $x = -1 + w$, where $w^2 = z$; the solutions of the quadratic equation in (2.4.8) are $x = 1 + u$, where $u^2 = -z$.

**Case 1.** Assume $(a, b) = (1, 0)$. Then $z - 1 = z^3$, from which we have $-z = (\frac{z - 1}{z + 1})^2$. Let $u = \frac{z - 1}{z + 1}$. Then $x = 1 + u = -\frac{z}{z + 1}$ is a solution of the quadratic equation in (2.4.8), and $\text{Tr}_{3/3}(\frac{1}{x}) = \text{Tr}_{3/3}(-1 - \frac{1}{z}) = -1$.

**Case 2.** Assume $(a, b) = (1, 1)$. Then $z - 1 = z^2(z + 1)$, from which we have $(-z)^3 = (z + 1)^2$. Let $u^3 = -(z + 1)$. Then $x = 1 + u$ is a solution of the quadratic equation in (2.4.8), and

$$\text{Tr}_{3/3}(\frac{1}{x}) = \text{Tr}_{3/3}(\frac{1}{x^3}) = \text{Tr}_{3/3}(\frac{1}{1 - u^3}) = \text{Tr}_{3/3}(\frac{1}{-z}) = -1.$$

**Case 3.** Assume $(a, b) = (-1, 1)$. Then $z - 1 = -z^2(z + 1)$, from which we have $z = (\frac{1}{z - 1})^2$. Let $w = -\frac{1}{z - 1}$. Then $x = -1 + w = \frac{-z}{z - 1}$ is a solution of the quadratic
equation in (2.4.7), and \( \text{Tr}_{3^3/3}(\frac{1}{x}) = \text{Tr}_{3^3/3}(-1 + \frac{1}{x}) = 1. \)

\[
(\Rightarrow) \text{ We show that if } q \neq 3, \text{ then } h \text{ is a not a permutation of } \mathbb{F}_{q^3}^*.
\]

In general,

\[
h(x) = (1 + x^2)(x^{-1} + x^{-q} + x^{-q^2}) - x,
\]

\[
= x^{-1} + x^{-q} + x^{-q^2} - x^{-q} + x^{-q^2} + x^2 - q + x^2 - q^2
\]

\[
= y + y^q + y^{q^2} + y^{q-2} + y^{q^2-2}
\]

\[
= g(y),
\]

where \( y = x^{-1} \in \mathbb{F}_{q^3}^* \), and \( g(y) = y + y^q + y^{q^2} + y^{q-2} + y^{q^2-2} \).

First assume \( q = 2 \). We have

\[
g(y) = y^4 + y + 1, \quad y \in \mathbb{F}_{2^3}^*.
\]

It is obvious that \( g \) is not 1-1 on \( \mathbb{F}_{2^3}^* \).

Now assume \( q > 3 \). We show that \( g \) is not a PP of \( \mathbb{F}_{q^3} \). (Since \( g(0) = 0 \), it follows from (2.4.9) that \( h \) is not a permutation of \( \mathbb{F}_{q^3}^* \).)

**Case 1.** Assume \( q > 3 \) is odd. We have

\[
g(y)^{2q^2+2} \equiv 8y^{q^3-1} + \text{terms of lower degree} \pmod{y^{q^3} - y}.
\]

(The complete expression of \( g^{2q^2+2} \) (mod \( y^{q^3} - y \)) is given in Appendix B.) By Hermite’s criterion, \( g \) is not a PP of \( \mathbb{F}_{q^3} \).

**Case 2.** Assume \( q > 3 \) is even. We have

\[
g(y)^{2q^2+q+3} \equiv y^{q^3-1} + \text{terms of lower degree} \pmod{y^{q^3} - y}.
\]

(The complete expression of \( g^{2q^2+q+3} \) (mod \( y^{q^3} - y \)) is given in Appendix B.) By Hermite’s criterion, \( g \) is not a PP of \( \mathbb{F}_{q^3} \).
Table 2.1: Desirable triples \((n, e; 3)\), \(e \leq 6\), \(w_3(n) > 3\)

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3 Desirable Triples of the Form \((q^a - q^b - 1, e; q)\)†

In this chapter, we study desirable triples \((n, e; q)\), where \(n\) is of the form \(q^a - q^b - 1\). From our initial computer search we noticed that \(g_{n,q}\) is always a PP of \(\mathbb{F}_{q^2}\) when base \(q\) digits of \(n\) are \((q - 1, q - 1, q - 2, q - 1)\). These observations motivated us to discover all desirable triples \((n, 2; 5)\) where the base 5 digits of \(n\) are all 4 except only one being 3. Table 3.1 contains all such desirable triples when \(q = 5\) and \(e = 2\) with their corresponding \(a\) and \(b\) values.

Table 3.1: Desirable triples \((5^a - 5^b - 1, 2; q)\), \(a, b \geq 0\)

<table>
<thead>
<tr>
<th>(n)</th>
<th>base 5 digits of (n)</th>
<th>(a)</th>
<th>(b)</th>
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<tr>
<td>599</td>
<td>4 4 3 4</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>14999</td>
<td>4 4 4 4 3 4</td>
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<td>4</td>
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<td>4 4 3 4 4 4</td>
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<td>1</td>
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<td>74999</td>
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<td>7</td>
<td>5</td>
</tr>
<tr>
<td>374999</td>
<td>4 4 4 4 4 4 3 4</td>
<td>8</td>
<td>6</td>
</tr>
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<td>389999</td>
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<td>4</td>
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<tr>
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<td>8</td>
<td>2</td>
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<td>1</td>
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<td>5</td>
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<td>7812499</td>
<td>4 4 4 4 4 4 4 4 3</td>
<td>10</td>
<td>9</td>
</tr>
</tbody>
</table>

These results clearly indicated that the form \(n = q^a - q^b - 1\) is special. As a result, a separate computer search was conducted for this type of desirable triples only.

†Portions of this chapter are taken from [14] which has been published in the journal “Finite Fields and Their Applications”. 

23
This chapter is organized as follows: In Section 3.1, we discuss the case \( b = 0 \) and present results that explain desirable triples when \( e > 2 \). Section 3.2 focuses on desirable triples \( (q^a - q^b - 1, 2; q) \). All desirable triples \( (q^a - q^b - 1, 2; q), q \leq 97, 0 < b < a < 2p \), that are not covered by Corollary 3.1.2 and Theorems 3.1.3, 3.2.1, 3.2.2 are included in Table 3.2 that can be found at the end of this chapter. Theorem 3.2.7 explains some desirable triples in Table 3.2. But in many other cases, no theoretic explanation of the computer results is known.

Three conjectures are stated in this chapter. Conjecture 3.1.1 is related to Payne’s Theorem when \( q \) is even. Conjecture 3.1.4 states that there are no other cases when \( e > 2 \) except the cases explained by Corollary 3.1.2 and Theorem 3.1.3. Conjecture 3.2.6 predicts several classes of permutation binomials of \( \mathbb{F}_{q^2} \).

Recall that \( S_a = x + x^2 + \cdots + x^{q^a-1} \) for every integer \( a \geq 0 \).

### 3.1 The Polynomial \( g_{q^a-q^b-1/q} \)

Assume \( n > 0 \) and \( n \equiv q^a - q^b - 1 \pmod{q^{pe} - 1} \) for some integers \( a, b \geq 0 \). If \( a = 0 \) or \( b \), then \( n \sim_{(e, q)} q^{pe} - 2 \), where \( (q^{pe-2}, e; q) \) is desirable if and only if \( q > 2 \) [22, Proposition 3.2 (i)]. If \( b = 0 \) and \( a > 0 \), we have \( n \equiv q^a - 2 \pmod{q^{pe} - 1} \). By Proposition 1.1.1 and Lemma 1.1.3,

\[
\begin{align*}
g_{q^a-2,q} &= \frac{1}{x} \left( g_{q^a+q-2,q} - g_{q^a-1,q} \right) \\
&= \frac{1}{x} \left[ -1 - \frac{1}{x} \left( g_{q^a+q-1,q} - g_{q^a,q} \right) \right] \\
&= \frac{1}{x} \left( -1 + \frac{S_a}{x} \right) \\
&= \frac{S_a^q}{x^2} \\
&= x^{q-2} + x^{q^2-2} + \cdots + x^{q^a-1-2}.
\end{align*}
\]

For which \( a, e \) and \( q \) is \( g_{q^a-2,q} \) a PP of \( \mathbb{F}_{q^e} \)? The complete answer is not known. We have the following conjecture.
Conjecture 3.1.1 Let $e \geq 2$ and $2 \leq a < pe$. Then $(q^a - 2, e; q)$ is desirable if and only if

(i) $a = 3$ and $q = 2$, or

(ii) $a = 2$ and $\gcd(q - 2, q^e - 1) = 1$.

Note. When $q$ is even,

$$g_{q^a - 2,q} = \left(\frac{x^{\frac{1}{2}q}\!^{1} + x^{\frac{1}{2}q}\!^{2} + \cdots + x^{\frac{1}{2}q}\!^{a-1}}{x}\right)^{2},$$

and the claim of the conjecture follows from Payne’s Theorem which says that the linearized polynomials $f(x) \in \mathbb{F}_{2^n}[x]$ such that $f(x)$ and $f(x)/x$ are permutations of $\mathbb{F}_{2^n}$ and $\mathbb{F}_{2^n}^*$ respectively, are exactly of the form $f(x) = ax^{2^k}$ with $a \in \mathbb{F}_{2^n}^*$ and $\gcd(k, n) = 1$ [19, §8.5], [20, 35, 36].

For a general $q$, the “if” part is obvious. So for the conjecture, one only has to prove that if $q$ is odd, $e \geq 2$, and $a > 2$, then $(q^a - 2, e; q)$ is not desirable.

Now assume $n > 0$ and $n \equiv q^a - q^b - 1 \pmod{q^{pe} - 1}$, where $0 < a, b < pe$ and $a \neq b$. If $a < b$, we have

$$n \sim_{(e,q)} q^{pe-b-n} \equiv q^{pe-a-b} - q^{pe-b} - 1 \pmod{q^{pe} - 1},$$

where $0 < pe - b < pe + a - b < pe$. Therefore we may assume $0 < b < a < pe$.

By (1.1.9), we have

$$S_b g_{q^a-q^b-1,q} = g_{q^{a-1},q} - g_{q^a-q^b,q}$$

$$= g_{q^{a-1},q} - (g_{q^{a-b-1},q})^{q^b}$$

$$= -\frac{S_a}{x} + \left(\frac{S_{a-b}}{x}\right)^{q^b}$$

$$= -\frac{S_a}{x} - S_{a-b}^{q^b} + \left(1 - \frac{1}{x^{q^b}}\right)S_{a-b}^{q^b}$$

$$= -\frac{S_b}{x} - S_{b}^{q^b} - S_{b}^{q^b} S_{a-b}^{q^b},$$
So
\[
g_{q^a-q^b-1,q} = -\frac{1}{x} - \frac{(S_{q-1}^a - 1)S_{q-b}^b}{x^{q+1}}. \tag{3.1.2}
\]
(Note that (3.1.2) also holds for \(b = 0\); see (3.1.1).) Assume \(e \geq 2\). Write
\[
a - b = a_0 + a_1 e, \quad b = b_0 + b_1 e,
\]
where \(a_0, a_1, b_0, b_1 \in \mathbb{Z}\) and \(0 \leq a_0, b_0 < e\). Then from (3.1.2) we have
\[
g_{q^a-q^b-1,q} \equiv -x^{q^e-2} - x^{q^e-q^{b_0}-2}(a_1S_e + S_{a_0}^{q^{b_0}})((b_1S_e + S_{b_0})^{q-1} - 1) \pmod{x^{q^e} - x}. \tag{3.1.3}
\]

Corollary 3.1.2 We have
\[
g_{q^2-q-1,q} = -x^{q^2-2}.
\]
In particular, \((q^2-q-1,e;q)\) is desirable if and only if \(q > 2\) and \(\gcd(q-2,q^e-1) = 1\).

Proof. It follows from (3.1.2).

The following theorem is a generalization of [22, Proposition 3.2 (i)].

Theorem 3.1.3 Assume \(e \geq 2\). Let \(0 < b < a < pe\). Then
\[
g_{q^a-q^b-1,q} \equiv -x^{q^e-2} \pmod{x^{q^e} - x} \tag{3.1.4}
\]
if and only if \(a \equiv b \equiv 0 \pmod{e}\). In particular, if \(0 < b < a < pe\), and \(a \equiv b \equiv 0 \pmod{e}\), then \((q^a - q^b - 1,e;q)\) is a desirable triple.

Proof. \(\Leftarrow\) In the notation of (3.1.3), we have \(a_0 = b_0 = 0\) and \(0 < b_1 < p\). So
\[
g_{q^a-q^b-1,q} \equiv -x^{q^e-2} - x^{q^e-3}a_1S_e((b_1S_e)^{q-1} - 1) \pmod{x^{q^e} - x}
\]
\[
= -x^{q^e-2} - x^{q^e-3}a_1S_e(S_{q-1}^e - 1)
\]
\[
= -x^{q^e-2} - x^{q^e-3}a_1(S_q^e - S_e)
\]
\[
\equiv -x^{q^e-2} \pmod{x^{q^e} - x}. \tag{3.1.5}
\]
(⇒) Assume (3.1.4) holds. Then by (3.1.2),

\[ (x^{q^b} - x)S_{a-b}^{q^b} = (S_{b}^q - S_{b})S_{a-b}^{q^b} \equiv 0 \mod x^{q^e} - x. \]

For \( f \in \mathbb{F}_q[x] \), denote \( \{ x \in \mathbb{F}_q : f(x) = 0 \} \) by \( V(f) \), where \( \overline{\mathbb{F}}_q \) is the algebraic closure of \( \mathbb{F}_q \). Then \( V(x^{q^e} - x) \subset V(x^{q^b} - x) \cup V(S_{a-b}) \), i.e., \( \mathbb{F}_{q^e} \subset \mathbb{F}_{q^b} \cup V(S_{a-b}) \). Since \( V(S_{a-b}) \) is a vector space over \( \mathbb{F}_q \), we must have \( \mathbb{F}_{q^e} \subset \mathbb{F}_{q^b} \) or \( \mathbb{F}_{q^e} \subset V(S_{a-b}) \). However, since \( 0 < a < pe \),

\[ S_{a-b} = S_{a_1e+a_0} \equiv a_1S_{e} + S_{a_0} \not\equiv 0 \mod x^{q^e} - x. \]

So we must have \( \mathbb{F}_{q^e} \subset \mathbb{F}_{q^b} \). Hence \( b \equiv 0 \mod e \). Now by (3.1.3) and the calculation in (3.1.5), we have

\[ S_{a_0}(S_{e}^{q-1} - 1) \equiv 0 \mod x^{q^e} - x. \tag{3.1.6} \]

If \( a_0 > 0 \), then

\[ \deg S_{a_0}(S_{e}^{q-1} - 1) = (q-1)q^{e-1} + q^{a_0-1} = q^e - q^{e-1} + q^{a_0-1} < q^e, \]

which is a contradiction to (3.1.6). So we must have \( a_0 = 0 \), i.e., \( a \equiv 0 \mod e \).

\[ \blacksquare \]

**Remark.** If \( (q^a - q^b - 1, 2; q) \) is desirable, where \( 0 < b < a < 2p \) and \( b \equiv 0 \mod 2 \), then we must have \( a \equiv 0 \mod 2 \). Otherwise, with \( e = 2 \), \( a_0 = 1 \), \( b_0 = 0 \) in (3.1.3), we have

\[ g_{q^{e-1} - q^{b-1}} \equiv -x^{q^2-2} - x^{q^2-3}(a_1S_2 + x)((b_1S_2)^{q-1} - 1) \mod x^{q^2} - x. \]

Then \( g_{q^{a-1} - q^{b-1}}(x) = 0 \) for every \( x \in \mathbb{F}_{q^2} \) with \( \text{Tr}_{q^2/q}(x) = 0 \), which is a contradiction.

The results of our computer search suggest that when \( e \geq 3 \), the only desirable triples \( (q^a - q^b - 1, e; q) \), \( 0 < b < a < pe \), are those given by Corollary 3.1.2 and Theorem 3.1.3.

**Conjecture 3.1.4** Let \( e \geq 3 \) and \( n = q^a - q^b - 1 \), \( 0 < b < a < pe \). Then \( (n, e; q) \) is
desirable if and only if

(i) \( a = 2, b = 1, \text{ and } \gcd(q - 2, q^e - 1) = 1, \) or

(ii) \( a \equiv b \equiv 0 \pmod{e}. \)

3.2 Desirable Triples of the Form \((q^a - q^b - 1, 2; q)\)

While Corollary 3.1.2 and Theorem 3.1.3 cover all known desirable triples \((q^a - q^b - 1, e; q)\) when \( e \geq 3 \), Conjecture 3.1.4 states that there are no other cases. In contrast, the case \( e = 2 \) seems to be chaotic, and of course very interesting too; see Table 3.2.

For the rest of this chapter, we will focus on desirable triples of the form \((q^a - q^b - 1, 2; q), 0 < b < a < 2p.\)

3.2.1 The Case \( b = p \)

**Theorem 3.2.1** Let \( p \) be an odd prime and \( q \) a power of \( p.\)

(i) \( \mathbb{F}_{q^2} \setminus \mathbb{F}_q \) consists of the roots of \((x - x^q)^{q-1} + 1.\)

(ii) Let \( 0 < i \leq \frac{1}{2}(p - 1) \) and \( n = q^{p+2i} - q^p - 1. \) Then

\[
g_{n,q}(x) = \begin{cases} 
(2i - 1)x^{q-2} & \text{if } x \in \mathbb{F}_q, \\
\frac{2i - 1}{x} + \frac{2i}{x^q} & \text{if } x \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q.
\end{cases}
\]

(iii) For the \( n \) in (ii), \((n, 2; q)\) is desirable if and only if \( 4i \not\equiv 1 \pmod{p}. \)

**Proof.** (i) We have

\[
(x^q - x)[(x - x^q)^{q-1} + 1] = -(x - x^q)^q + x^q - x = x^{q^2} - x.
\]

Hence the claim.
(ii) Let \( e = 2, a = p + 2i, b = p \). In the notation of (3.1.3), \( a_0 = 0, a_1 = i, \) \( b_0 = 1, b_1 = \frac{p - 1}{2} \). Thus

\[
g_{n,q} \equiv -x^{q^2 - 2} - ix^{q^2 - q - 2} S_2 \left( \frac{1}{2} S_2 + x \right)^{q-1} - 1 \pmod{x^{q^2} - x}
\]

\[
= -x^{q^2 - 2} - ix^{q^2 - q - 2}(x + x^q)(x - x^q)^{q-1} - 1.
\]

When \( x \in \mathbb{F}_q, x - x^q = 0 \), so

\[
g_{n,q}(x) = -x^{q^2 - 2} + ix^{q^2 - q - 2}(x + x^q) = (2i - 1)x^{q-2}.
\]

When \( x \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q \), by (i), \((x - x^q)^{q-1} = -1\). Thus

\[
g_{n,q}(x) = -x^{-1} + 2ix^{q^2 - q - 2}(x + x^q)
\]

\[
= -x^{-1} + 2ix^{q^2 - q - 1} + 2ix^{q^2 - 2}
\]

\[
= (2i - 1)x^{-1} + 2ix^{-q}.
\]

(iii) Since \( 0 < 2i - 1 < p \), \((2i - 1)x^{-1} + 2ix^{-q}\) permutes \( \mathbb{F}_q \). We claim that \((2i - 1)x^{-1} + 2ix^{-q}\) maps \( \mathbb{F}_{q^2} \setminus \mathbb{F}_q \) to itself. In fact, for \( x \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q \),

\[
\left[ \frac{2i - 1}{x} + \frac{2i}{x^q} - \left( \frac{2i - 1}{x} + \frac{2i}{x^q} \right)^q \right]^{q-1} = \left( \frac{x - x^q}{x^{q+1}} \right)^{q-1} = -1
\]

since \((x - x^q)^{q-1} = -1\).

Therefore, \( g_{n,q} \) is a PP of \( \mathbb{F}_{q^2} \) if and only if \((2i - 1)x^{-1} + 2ix^{-q}\) is 1-1 on \( \mathbb{F}_{q^2} \setminus \mathbb{F}_q \), i.e., if and only if \((2i - 1)x + 2ix^q\) is 1-1 on \( \mathbb{F}_{q^2} \setminus \mathbb{F}_q \). So, it remains to show that \((2i - 1)x + 2ix^q\) is 1-1 on \( \mathbb{F}_{q^2} \setminus \mathbb{F}_q \) if and only if \( 4i \not\equiv 1 \pmod{p} \).

\( (\Leftarrow) \) Assume \( 4i \not\equiv 1 \pmod{p} \). We claim that \((2i - 1)x + 2ix^q\) is a PP of \( \mathbb{F}_{q^2} \). Otherwise, there exists \( 0 \neq x \in \mathbb{F}_{q^2} \) such that \((2i - 1)x + 2ix^q = 0\). Then \( x^{q-1} = -\frac{2i-1}{2i} \).

Hence

\[
1 = (x^{q-1})^{q+1} = \left( -\frac{2i - 1}{2i} \right)^{q+1} = \left( \frac{2i - 1}{2i} \right)^2.
\]

So \((2i - 1)^2 \equiv (2i)^2 \pmod{p}\), i.e., \( 4i - 1 \equiv 0 \pmod{p} \), which is a contradiction.
(⇒) Assume $4i \equiv 1 \pmod{p}$. Then $(2i - 1)x + 2ix^q = 2i(x^q - x)$, which is clearly not 1-1 on $\mathbb{F}_q^2 \setminus \mathbb{F}_q$.

\[\]

**Theorem 3.2.2** Let $p$ be an odd prime and $q$ a power of $p$.

(i) Let $0 < i \leq \frac{1}{2}(p - 1)$ and $n = q^{p+2i-1} - q^p - 1$. Then

\[
g_{n,q}(x) = \begin{cases} 2(i - 1)x^{q-2} & \text{if } x \in \mathbb{F}_q, \\ \frac{2i - 1}{x} + \frac{2i - 2}{x^q} & \text{if } x \in \mathbb{F}_q^2 \setminus \mathbb{F}_q. \end{cases}
\]

(ii) For the $n$ in (i), $(n, 2; q)$ is desirable if and only if $i > 1$ and $4i \not\equiv 3 \pmod{p}$.

**Proof.** (i) Let $e = 2$, $a = p + 2i - 1$, $b = p$. In the notation of (3.1.3), $a_0 = 1$, $a_1 = i - 1$, $b_0 = 1$, $b_1 = \frac{p-1}{2}$. Thus

\[
g_{n,q} = -x^{q^2-2} - x^{q^2-q^2}(i - 1)S_2 + x^q)\left[\left(-\frac{1}{2}S_2 + x\right)^{q-1} - 1\right] \pmod{x^{q^2} - x}
\]

\[
= -x^{q^2-2} - x^{q^2-q^2}(i(x + x^q) - x)\left[(x - x^q)^{q-1} - 1\right].
\]

When $x \in \mathbb{F}_q$, $x - x^q = 0$, so

\[
g_{n,q}(x) = -x^{q^2-2} + x^{q^2-q^2}(2i - 1) = 2(i - 1)x^{q-2}.
\]

When $x \in \mathbb{F}_q^2 \setminus \mathbb{F}_q$, by (i), $(x - x^q)^{q-1} = -1$. Thus

\[
g_{n,q}(x) = -x^{-1} + 2x^{q^2-q^2}((i - 1)x + ix^q)
\]

\[
= -x^{-1} + 2(i - 1)x^{q^2-q^2} + 2ix^{q^2-2}
\]

\[
= (2i - 1)x^{-1} + (2i - 2)x^{-q}.
\]

(ii) Since $0 < 2i - 2 < p$, $2(i - 1)x^{q^2-2}$ permutes $\mathbb{F}_q$. We claim that $(2i - 1)x^{-1} + (2i - 2)x^{-q}$ maps $\mathbb{F}_q^2 \setminus \mathbb{F}_q$ to itself. In fact, for $x \in \mathbb{F}_q^2 \setminus \mathbb{F}_q$,

\[
\left[\frac{2i - 1}{x} + \frac{2i - 2}{x^q} - \left(\frac{2i - 1}{x} + \frac{2i - 2}{x^q}\right)\frac{x}{x^q + 1}\right]^{q-1} = \left(\frac{1}{x} - \frac{1}{x^q}\right)^{q-1} = \left(\frac{x - x^q}{x^q + 1}\right)^{q-1} = -1
\]
since \((x - x^q)^{q-1} = -1\).

Therefore, \(g_{n,q}\) is a PP of \(\mathbb{F}_{q^2}\) if and only if \((2i - 1)x^{-1} + (2i - 2)x^{-q}\) is 1-1 on \(\mathbb{F}_{q^2} \setminus \mathbb{F}_q\), i.e., if and only if \((2i - 1)x + (2i - 2)x^q\) is 1-1 on \(\mathbb{F}_{q^2} \setminus \mathbb{F}_q\). So, it remains to show that \((2i - 1)x + (2i - 2)x^q\) is 1-1 on \(\mathbb{F}_{q^2} \setminus \mathbb{F}_q\) if and only if \(4i \not\equiv 3 \pmod{p}\).

\((\Leftarrow)\) Assume \(4i \not\equiv 3 \pmod{p}\). We claim that \((2i - 1)x + (2i - 2)x^q\) is a PP of \(\mathbb{F}_{q^2}\). Otherwise, there exists \(0 \neq x \in \mathbb{F}_{q^2}\) such that \((2i - 1)x + (2i - 2)x^q = 0\). Then \(x^{q-1} = -\frac{2i - 1}{2i - 2}\). Hence

\[
1 = (x^{q-1})^{q+1} = \left(-\frac{2i - 1}{2i - 2}\right)^{q+1} = \left(\frac{2i - 1}{2i - 2}\right)^2.
\]

So \((2i - 1)^2 \equiv (2i - 2)^2 \pmod{p}\), i.e., \(4i - 3 \equiv 0 \pmod{p}\), which is a contradiction.

\((\Rightarrow)\) Assume \(4i \equiv 3 \pmod{p}\). Then \((2i - 1)x + (2i - 2)x^q = (2i - 2)(x^q - x)\), which is clearly not 1-1 on \(\mathbb{F}_{q^2} \setminus \mathbb{F}_q\).

\[
\square
\]

**Proposition 3.2.3** Let \(p\) be an odd prime and \(q = p^k\). Let \(i > 0\). If \(i\) is even,

\[
g_{q^{p+i} - q^{p-1}, q} \equiv -x^{q^2-2} + ix^{q^2-2} \sum_{j=0}^{q-2} x^{(q-1)j} \pmod{x^{q^2} - x}.
\]

If \(i\) is odd,

\[
g_{q^{p+i} - q^{p-1}, q} \equiv -x^{q^2-q-1} + ix^{q^2-2} \sum_{j=0}^{q-2} x^{(q-1)j} \pmod{x^{q^2} - x}.
\]

**Proof.** Let \(n = q^{p+i} - q^p - 1\). Throughout the proof, “\(\equiv\)” means “\(\equiv \pmod{x^{q^2} - x}\)”.
Case 1. Assume that $i$ is even. Let $e = 2$, $a = p + i$, $b = p$. In the notation of (3.1.3), $a_0 = 0$, $a_1 = \frac{i}{2}$, $b_0 = 1$, $b_1 = \frac{p-1}{2}$. By (3.1.3), we have

\[
g_{n,q} \equiv -x^{q^2-2} - x^{q^2-q-2} - \frac{i}{2} S_2 \cdot \left[ \left( -\frac{1}{2} S_2 + S_1 \right)^{q-1} - 1 \right] \\
= -x^{q^2-2} - \frac{i}{2} x^{q^2-q-2}(x + x^q)(x - x^q)^{q-1} - 1 \\
= -x^{q^2-2} - \frac{i}{2} x^{q^2-q-1}(1 + x^{q-1})(x^{q-1}(1 - x^{q-1})^{q-1} - 1).
\]

Note that

\[
(1 - x^{q-1})^{q-1} = \frac{1 - x^{(q-1)q}}{1 - x^{q-1}} = \sum_{j=0}^{q-1} x^{(q-1)j}.
\]

So

\[
g_{n,q} \equiv -x^{q^2-2} - \frac{i}{2} x^{q^2-q-1}(1 + x^{q-1}) \left[ x^{q-1} \sum_{j=0}^{q-1} x^{(q-1)j} - 1 \right] \\
= -x^{q^2-2} - \frac{i}{2} x^{q^2-q-1} \left[ \sum_{j=1}^{q} x^{(q-1)j} + \sum_{j=2}^{q+1} x^{(q-1)j} - 1 - x^{q-1} \right] \\
= -x^{q^2-2} - \frac{i}{2} x^{q^2-q-1} \cdot 2 \sum_{j=2}^{q} x^{(q-1)j} \\
= -x^{q^2-2} - i x^{q-2} \sum_{j=0}^{q-2} x^{(q-1)j}.
\]

Case 2. Assume that $i$ is odd. In the notation of (3.1.3), $a_0 = 1$, $a_1 = \frac{i-1}{2}$, $b_0 = 1$, $b_1 = \frac{p-1}{2}$. By (3.1.3),

\[
g_{n,q} \equiv -x^{q^2-2} - x^{q^2-q-2} \left( \frac{i-1}{2} S_2 + S_1 \right) \left[ \left( -\frac{1}{2} S_2 + S_1 \right)^{q-1} - 1 \right] \\
= -x^{q^2-2} - x^{q^2-q-2} \left( -\frac{1}{2} S_2 + S_1 \right) \left[ \left( -\frac{1}{2} S_2 + S_1 \right)^{q-1} - 1 \right] \\
- \frac{i}{2} x^{q^2-q-2} S_2 \cdot \left[ \left( -\frac{1}{2} S_2 + S_1 \right)^{q-1} - 1 \right].
\]
In the above,

\[-x^{q^2-2} - x^{q^2-2} \left( -\frac{1}{2}S_2 + S_1 \right) \left( \left( -\frac{1}{2}S_2 + S_1 \right)^{q-1} - 1 \right) = -x^{q^2-2} - x^{q^2-2} \frac{1}{2} (x^q - x)((x - x^q)^{q-1} - 1) = -x^{q^2-2} - \frac{1}{2}x^{q^2-2}((x^q - x)^q - (x^q - x)) \equiv -x^{q^2-2} - \frac{1}{2}x^{q^2-2} \cdot 2(x - x^q) = -x^{q^2-2} - \frac{1}{2}x^{q^2-2} \cdot 2(x - x^q),\]

and, by the calculation in Case 1,

\[-\frac{i}{2}x^{q^2-2}S_2 \cdot \left[ \left( -\frac{1}{2}S_2 + S_1 \right)^{q-1} - 1 \right] \equiv -ix^{q^2-2} \sum_{j=0}^{q-2} x^{(q-1)j}.\]

So

\[g_{n,q} \equiv -x^{q^2-2} - ix^{q^2-2} \sum_{j=0}^{q-2} x^{(q-1)j}.\]

\[3.2.2 \quad \text{The Case } b = 1\]

\[\textbf{Theorem 3.2.4} \quad \text{Let } q = 2^s, \ n = q^3 - q - 1.\]

(i) For \(x \in \mathbb{F}_{q^2},\)

\[g_{n,q}(x) = \begin{cases} 0 & \text{if } x = 0, \\ x^{q^2-2} + \operatorname{Tr}_{q^2/q}(x^{-1}) & \text{if } x \neq 0. \end{cases}\]

(ii) \(g_{n,q}\) is a PP of \(\mathbb{F}_{q^2}\) if and only if \(s\) is even.
Proof. (i) It is obvious that \( g(0) = 0 \). Let \( 0 \neq x \in \mathbb{F}_{q^2} \). By (3.1.3) (with \( a_0 = 0, a_1 = 1, b_0 = 1, b_1 = 0 \)),

\[
\begin{align*}
g_{n,q}(x) &= x^{-1} + x^{q-1}S_2(x)(x^{q-1} + 1) \\
&= x^{-1} + x^{q-1}(x + x^q)(x^{q-1} + 1) \\
&= x^{-1} + x^{q-2} + x^{-q} \\
&= x^{q-2} + \text{Tr}_{q^2/q}(x^{-1}).
\end{align*}
\]

(ii) 1° We show that for every \( c \in \mathbb{F}_{q^2}^* \), the equation

\[ x^{q-2} + x^{-1} + x^{-q} = c \quad (3.2.7) \]

has at most one solution \( x \in \mathbb{F}_{q^2}^* \).

Assume that \( x \in \mathbb{F}_{q^2}^* \) is a solution of (3.2.7). Then

\[
\begin{align*}
ctx^{-q} &= x^{-2} + x^{q-1} + x^{-2q} = N_{q^2/q}(x^{-1}) + \text{Tr}_{q^2/q}(x^{-2}) \in \mathbb{F}_q.
\end{align*}
\]

Let \( t = c^{-q}x = (cx^{-q})^{-q} \in \mathbb{F}_q^* \). Then \( x = tc^q \). Making this substitution in (3.2.7), we have

\[
\frac{1}{t}(c^{q(q-2)} + c^{-q} + c^{-1}) = c.
\]

So

\[ t = c^{-2} + c^{-2q} + c^{-q-1}. \]

Hence \( x \) is unique.

2° Assume \( s \) is even. We show that

\[ x^{q-2} + \text{Tr}_{q^2/q}(x^{-1}) = 0 \quad (3.2.8) \]

has no solution in \( \mathbb{F}_{q^2}^* \). Assume to the contrary that \( x \in \mathbb{F}_{q^2}^* \) is a solution of (3.2.8). Then \( x^{q-2} \in \mathbb{F}_q \). Since \( s \) is even, we have \( \gcd(q - 2, q^2 - 1) = 1 \). So \( x \in \mathbb{F}_q \). Then \( \text{Tr}_{q^2/q}(x^{-1}) = 0 \), and \( x^{q-2} = 0 \), which is a contradiction.
Assume $s$ is odd. We show that (3.2.8) has a solution in $\mathbb{F}_{q^2}^*$. Let $x \in \mathbb{F}_{q^2} \setminus \mathbb{F}_2$.

Then $x^2 + x + 1 = 0$ and $x^3 = 1$. So

$$x^{q-2} + \text{Tr}_{q^2/q}(x^{-1}) = x^{q-2} + x^{-1} + x^{-q}$$

$$= 1 + x^2 + x \quad \text{(since } q \equiv 2 \pmod{3})$$

$$= 0.$$

\[\blacksquare\]

**Theorem 3.2.5**

(i) Assume $q > 2$. We have

$$g_{q^{2i} - q - 1, q} \equiv (i - 1)x^{q^2 - q - 1} - ix^{q-2} \pmod{x^{q^2} - x}.$$  

(ii) Assume that $q$ is odd. Then $x^{q^2 - q - 1} + x^{q-2}$ is a PP of $\mathbb{F}_{q^2}$ if and only if $q \equiv 1 \pmod{4}$.

(iii) Assume that $q$ is odd. Then $(q^{p+1} - q - 1, q)$ is desirable if and only if $q \equiv 1 \pmod{4}$.

**Proof.** In the notation of (3.1.3), we have $e = 2$, $a = 2i$, $b = 1$, $a_0 = 1$, $a_1 = i - 1$, $b_0 = 1$, $b_1 = 0$. Thus

$$g_{q^{2i} - q - 1, q} \equiv -x^{q^2 - 2} - x^{q^2 - q - 2}(i - 1)S_2 + x^q)(x^{q-1} - 1) \pmod{x^{q^2} - x}$$

$$= -x^{q^2 - 2} - x^{q^2 - q - 2}((i - 1)x + ix^q)(x^{q-1} - 1)$$

$$= -x^{q^2 - 2} - x^{q^2 - q - 2}(-x^q - (i - 1)x + ix^{2q-1})$$

$$\equiv (i - 1)x^{q^2 - q - 1} - ix^{q-2} \pmod{x^{q^2} - x}.$$  

(ii) ($\Leftarrow$) Let $f = x^{q^2 - q - 1} + x^{q-2}$. Then

$$f(x) = \begin{cases} 
0 & \text{if } x = 0, \\
x^{-q} + x^{q-2} & \text{if } x \in \mathbb{F}_{q^2}^*. 
\end{cases}$$
1° We show that for every $c \in \mathbb{F}_{q^2}^*$, the equation
\[
x^{-q} + x^{q-2} = c
\]
has at most one solution $x \in \mathbb{F}_{q^2}^*$.

Assume $x \in \mathbb{F}_{q^2}^*$ is a solution of (3.2.9). Then
\[
x^{-q}x = x^{-2q} + x^{-2} = \text{Tr}_{q^2/q}(x^{-2}) \in \mathbb{F}_q.
\]
Let $t = c^{-q}x = (cx^{-q})^{-q} \in \mathbb{F}_q^*$. Then $x = tc^q$. So (3.2.9) becomes
\[
\frac{1}{t}(c^{-1} + c^{q(q-2)}) = c.
\]
Thus $t = c^{-2} + c^{-2q}$. Hence $x$ is unique.

2° We show that $x^{-q} + x^{q-2} = 0$ has no solution $x \in \mathbb{F}_{q^2}^*$.

Assume that $x \in \mathbb{F}_{q^2}^*$ is a solution. Then $x^{2q-2} = -1$. Since $\frac{1}{2}(q + 1)$ is odd, we have $-1 = (x^{2q-2})^{\frac{1}{2}(q+1)} = x^{q^2-1} = 1$, which is a contradiction.

(⇒) Assume to the contrary that $q \equiv -1 \pmod{4}$. We show that $x^{-q} + x^{q-2} = 0$ has a solution $x \in \mathbb{F}_{q^2}^*$. Since $4(q-1) \mid q^2-1$, there exists $x \in \mathbb{F}_{q^2}^*$ with $o(x) = 4(q-1)$. Then $x^{2(q-1)} = -1$, i.e., $x^{-q} + x^{q-2} = 0$.

(iii) It follows from (i) and (ii).

We conclude this section with a conjecture that grew out of Theorem 3.2.5.

**Conjecture 3.2.6** Let $f = x^{q-2} + tx^{q^2-q-1}$, $t \in \mathbb{F}_q^*$. Then $f$ is a PP of $\mathbb{F}_{q^2}$ if and only if one of the following occurs:

(i) $t = 1$, $q \equiv 1 \pmod{4}$;

(ii) $t = -3$, $q \equiv \pm 1 \pmod{12}$;

(iii) $t = 3$, $q \equiv -1 \pmod{6}$. 

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3.2.3 The Case $a = p + i + 1$ and $b = 2i + 1$

**Theorem 3.2.7** Let $p$ be an odd prime and $q$ a power of $p$. Let $0 \leq i \leq p - 2$ and $n = q^{p+i+1} - q^{2i+1} - 1$. If

$$
(2i + 1) \equiv \begin{cases} 
1 & \text{if } i \text{ is odd,} \\
(−1)^{\frac{q−1}{2}} & \text{if } i \text{ is even,}
\end{cases} \quad (3.2.10)
$$

where $\left(\frac{a}{b}\right)$ is the Jacobi symbol, then $(q^{p+i+1} - q^{2i+1} - 1; 2; q)$ is desirable.

**Proof.** Throughout the proof, “≡” means “≡ (mod $x^q - x$)”.

Let $e = 2, a = p + i + 1, b = 2i + 1$.

**Case 1: $i$ is odd.**

In the notation of (3.1.3), $a_0 = 0, a_1 = \frac{p−i}{2}, b_0 = 1, b_1 = i$.

Write $g = g_{q^{p+i+1}-q^{2i+1}-1, q}$,

$$
g \equiv -x^{q^2−2} - x^{q^2−q−2}(\frac{p−i}{2} - S_2)((iS_2 + S_1)^q − 1) \pmod{x^q - x}
= -x^{q^2−2} + \frac{i}{2} x^{q^2−q−2}(x + x^q)\left(((i + 1)x + ix^q)^q − 1\right).
$$

Clearly, $g(0) = 0$. When $x \in \mathbb{F}_{q^2}^*$,

$$
g(x) = -x^{−1} + \frac{i}{2} x^{−q−1}(x + x^q)\left(((i + 1)x + ix^q)^q − ((i + 1)x + ix^q)\right)\frac{1}{(i + 1)x + ix^q}.
$$

Note that $(i + 1)x + ix^q \neq 0$.

$$
g(x) = -x^{−1} + \frac{i}{2} (x^{−q} + x^{−1})\frac{x^q − x}{(i + 1)x + ix^q}
= -x^{−1} + \frac{i}{2} (x^{−q} + x^{−1})\frac{x^{−1} − x^{−q}}{(i + 1)x^{−q} + ix^{−1}}
= y + \frac{i}{2} (y^q + y)\frac{y^q − y}{(i + 1)y^q + iy}\quad (y = −x^{−1})
= \frac{iy^{2q} + 2(i + 1)y^{q+1} + iy^2}{2(i + 1)y^q + 2iy}.
$$
Let \( w = 2(i+1)y^q + 2iy \). Then 
\[
y = \frac{1}{2(2i+1)}((i+1)w^q - iw).
\]
(Here \( 2(i+1)x^q + 2ix \) is a PP of \( \mathbb{F}_{q^2} \), and 
\[
\frac{1}{2(2i+1)}((i+1)x^q - ix)
\]
is its inverse PP.) So
\[
g(x) = \frac{1}{(4i+2)^2} \frac{iu^{2q} + 2(i+1)u^{q+1} + iw^2}{w},
\]
where \( u = (i+1)w^q - iw \).

The proof will be complete if we can show that for \( c \in \mathbb{F}_{q^2} \),
\[
iu^{2q} + 2(i+1)u^{q+1} + iw^2
\]
\[
\frac{w}{c}
\]
has at most one solution \( w \in \mathbb{F}_{q^2}^* \) if \( c \neq 0 \) and has no solution \( w \in \mathbb{F}_{q^2}^* \) if \( c = 0 \).

First assume \( c \neq 0 \). Let \( t = wc \). By (3.2.12), \( t \in \mathbb{F}_q \). Then (3.2.12) becomes
\[
\frac{it^2v^{2q} + 2t^2(i+1)v^{q+1} + it^2v^2}{tc^{-1}} = c
\]
where \( v = (i+1)c^{-q} - ic^{-1} \). So
\[
t = \frac{1}{iv^{2q} + 2(i+1)v^{q+1} + iv^2},
\]
which is unique. Hence \( w \) is unique.

Now assume \( c = 0 \).

Assume to the contrary that (3.2.12) has a solution \( w \in \mathbb{F}_{q^2}^* \). Then
\[
i((i+1)w^q - iw)^{2q-2} + 2(i+1)((i+1)w^q - iw)^{q-1} + i = 0.
\]
Let $z = ((i + 1)w^q - iw)^{q^{-1}} \in \mathbb{F}_{q^2}^*$. Then

$$iz^2 + 2(i + 1)z + i = 0. \quad (3.2.13)$$

Since $i$ is odd $2i + 1$ is a square in $\mathbb{F}_q$. So (3.2.13) implies that $z \in \mathbb{F}_q$. Then we have $z^2 = z^{q+1} = ((i + 1)w^q - iw)^{q^2-1} = 1$. So $z = \pm 1$, which contradicts (3.2.13).

**Case 2:** $i$ is even.

In the notation of (3.1.3), $a_0 = 1, a_1 = \frac{p - i - 1}{2}, b_0 = 1, b_1 = i$.

$$g \equiv -x^{q^2-2} - x^{q^2-q-2}left(\frac{p - i - 1}{2} \cdot S_2 + S_1^qright) \cdot ((iS_2 + x)^{q-1} - 1)$$

$$= -x^{q^2-2} + \frac{1}{2}x^{q^2-q-2}((i + 1)x + (i - 1)x^q)[((i + 1)x + ix^q)^{q-1} - 1].$$

Clearly, $g(0) = 0$. When $x \in \mathbb{F}_{q^2}^*$,

$$g(x) = -x^{-1} + \frac{1}{2}x^{-q-1} \cdot ((i + 1)x + (i - 1)x^q) \cdot \frac{((i + 1)x + ix^q)^q - ((i + 1)x + ix^q)}{(i + 1)x + ix^q}. \quad (y = -x^{-1})$$

Note that $(i + 1)x + ix^q \neq 0$.

$$g(x) = -x^{-1} + \frac{1}{2}((i + 1)x^{-q} + (i - 1)x^{-1}) \cdot \frac{x^q - x}{(i + 1)x + ix^q}$$

$$= -x^{-1} + \frac{1}{2}((i + 1)x^{-q} + (i - 1)x^{-1}) \cdot \frac{x^{-1} - x^{-q}}{(i + 1)x^{-q} + ix^{-1}}$$

$$= y + \frac{1}{2}((i + 1)y^q + (i - 1)y) \cdot \frac{y^q - y}{(i + 1)y^q + iy}$$

$$= \frac{(i + 1)y^2q + 2iy^q + 1 + (i + 1)y^2}{2(i + 1)y^q + 2iy}.$$

Let $w = 2(i + 1)y^q + 2iy$. Then $y = \frac{1}{2(2i + 1)}((i + 1)w^q - iw)$. (Here $2(i + 1)x^q + 2ix$ is a PP of $\mathbb{F}_{q^2}$, and $\frac{1}{2(2i + 1)}((i + 1)x^q - ix)$ is its inverse PP.) So

$$g(x) = \frac{1}{(4i + 2)^2} \cdot \frac{(i + 1)u^2q + 2iu^{q+1} + (i + 1)u^2}{w},$$

where $u = (i + 1)w^q - iw$. 

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The proof will be complete if we can show that for \( c \in \mathbb{F}_{q^2} \),
\[
\frac{(i + 1)u^{2q} + 2i u^{q+1} + (i + 1)u^2}{w} = c, \tag{3.2.14}
\]
i.e.,
\[
\frac{(i + 1)((i + 1)u^q - iw)^{2q} + 2i((i + 1)u^q - iw)^{q+1} + (i + 1)((i + 1)u^q - iw)^2}{w} = c
\]
has at most one solution \( w \in \mathbb{F}_{q^2}^* \) if \( c \neq 0 \) and has no solution \( w \in \mathbb{F}_{q^2}^* \) if \( c = 0 \).

Assume \( c \neq 0 \). Let \( t' = wc \). By (3.2.15), \( t' \in \mathbb{F}_q \). Then (3.2.15) becomes
\[
\frac{(i + 1)t^{2q}v^{2q} + 2it^{q+1}v^{q+1} + (i + 1)t^2v^2}{t'c^{-1}} = c,
\]
where \( v = (i + 1)c^{-q} - ic^{-1} \). So
\[
t' = \frac{1}{(i + 1)v^{2q} + 2iv^{q+1} + (i + 1)v^2},
\]
which is unique. Hence \( w \) is unique.

Now assume \( c = 0 \). Assume to the contrary that (3.2.15) has a solution \( w \in \mathbb{F}_{q^2}^* \).
Then
\[
(i + 1)((i + 1)u^q - iw)^{2q-2} + 2i((i + 1)u^q - iw)^{q-1} + (i + 1) = 0.
\]
Let \( z = ((i + 1)u^q - iw)^{q-1} \in \mathbb{F}_{q^2}^* \). Then
\[
(i + 1)z^2 + 2iz + (i + 1) = 0. \tag{3.2.16}
\]
Since \( i \) is even \( \binom{2i + 1}{q} = (-1)^{q+1} \), i.e. \( -(2i + 1) \) is a square in \( \mathbb{F}_q \). So (3.2.16) implies that \( z \in \mathbb{F}_q \). Then we have \( z^2 = z^{q+1} = ((i + 1)u^q - iw)^{q^2-1} = 1. \) So \( z = \pm 1 \), which contradicts (3.2.16).

\[\blacksquare\]
Table 3.2: Desirable triples \((q^a - q^b - 1, 2; q)\), \(q \leq 97\), \(0 < b < a < 2p\), \(b\) odd, \(b \neq p\), \((a, b) \neq (2, 1)\)

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**Notes:**
- \(q = 2, 3, 5, 7\) indicate the base values for the triples.
- The table entries represent specific values for each \((a, b)\) pair.
- The table includes all possible \((a, b)\) pairs that satisfy the conditions given in the problem statement.
Table 3.1 (Continued)

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$q = 97$

$q = 89$
This chapter is organized as follows: Section 4.1 mostly discusses the case when the base \( q \) weight of \( n \) is \( q + 1 \). We recall that for given integers \( d > 1 \) and \( a = a_0d^0 + \cdots + a_td^t \), \( 0 \leq a_i \leq d - 1 \), the base \( d \) weight of \( a \) is \( w_d(a) = a_0 + \cdots + a_t \). In Section 4.2, we further study the permutation behavior of \( g_{n,q} \) in even characteristic when the base \( q \) weight of \( n \) is arbitrary. Examples are given in each section that explain many desirable triples in Table 4.1 that can be found at the end of this chapter which contains all desirable triples \( (n,e;4) \) with \( e \leq 6 \) and \( w_q(n) > 4 \).

Even though this chapter primarily deals with the case of even characteristic we point out to the reader that in Lemmas 4.2.1, 4.2.5, 4.2.17 and Theorem 4.2.6, 4.2.11, 4.2.12, 4.2.19 the characteristic is assumed to be arbitrary.

### 4.1 Families of Desirable Triples with \( w_q(n) = q + 1 \)

**Theorem 4.1.1** Let \( q \geq 4 \) be even, and let

\[
n = 1 + q^{a_1} + q^{b_1} + \cdots + q^{a_{q/2}} + q^{b_{q/2}},
\]

where \( a_i, b_i \geq 0 \) are integers. Then

\[
g_{n,q} = \sum_i S_{a_i}S_{b_i} + \sum_{i<j}(S_{a_i} + S_{b_i})(S_{a_j} + S_{b_j}).
\]

---

\[\text{† Portions of this chapter are taken from [14] which has been published in the journal “Finite Fields and Their Applications”}\]
Proof. We write $g_n$ for $g_{n,q}$. By (1.1.10) we have

$$g_n = g_{1+2^q a_1 + q b_1 + \cdots + q^{a_q-1} b_{q-2}} + (S_{b_1} - S_{a_1}) g_{1+2^q a_1 + q b_1 + \cdots + q^{a_q-1} b_{q-2}}$$

$$= g_{1+2^q a_1 + q b_1 + \cdots + q^{a_q-1} b_{q-2}} + \sum_{i=1}^{q/2} (S_{a_i} + S_{b_i}) \left( \sum_{j=i+1}^{q/2} (S_{a_j} + S_{b_j}) \right)$$

$$= S_{a_1}^2 + \cdots + S_{a_k}^2 + \sum_{i=1}^{q/2} (S_{a_i} + S_{b_i}) \left( \sum_{j=i+1}^{q/2} (S_{a_j} + S_{b_j}) \right)$$

$$= \sum_{i} S_{a_i} S_{b_i} + \sum_{i<j} (S_{a_i} + S_{b_i})(S_{a_j} + S_{b_j}).$$

\[\blacksquare\]

**Corollary 4.1.2** Let $q \geq 4$ be even, and let

$$n = t_0 + 2t_1 q a_1 + \cdots + 2t_k q a_k,$$

where $t_0, \ldots, t_k$ and $a_1, \ldots, a_k$ are nonnegative integers with $t_0 + 2t_1 + \cdots + 2t_k = q + 1$. Then

$$g_{n,q} = (t_1 S_{a_1} + \cdots + t_k S_{a_k})^2.$$

In particular, $g_{n,q}$ is a PP of $\mathbb{F}_{q^e}$ if and only if

$$\gcd \left( \sum_{i=1}^{k} t_i (1 + x + \cdots + x^{a_i-1}), x^e - 1 \right) = 1.$$

Proof. By Theorem 4.1.1,

$$g_{n,q} = t_1 S_{a_1}^2 + \cdots + t_k S_{a_k}^2 = (t_1 S_{a_1} + \cdots + t_k S_{a_k})^2.$$
The rest is obvious.

In Theorem 4.1.1, the mapping \( g_{n,q} : \mathbb{F}_{q^e} \to \mathbb{F}_{q^e} \) is quadratic in the multivariate sense, i.e., with the identification \( \mathbb{F}_{q^e} \cong \mathbb{F}_e^q \). In general, it is difficult to tell whether a quadratic mapping is bijective. However, in some cases, such as Corollary 4.1.2, \( g_{n,q} \) can be reduced to a suitable form which allows a quick determination whether it is a PP. Here are some additional examples of Theorem 4.1.1:

**Example 4.1.3** Let \( q = 2^s, s > 1, e > 1 \) odd, \( n = q^0 + (q - 1)q^1 + q^2 \). Then

\[
g_{n,q} = S_1^2 + S_1S_2 = x^{q+1},
\]

which is a PP of \( \mathbb{F}_{q^e} \).

**Example 4.1.4** Let \( q = 4, e > 1, n = q^0 + q^1 + e + q^{e+1} + q^a, a \geq 0 \). Then

\[
g_{n,q} = S_1S_a + S_{eS_{e+1}} + (S_1 + S_a)(S_e + S_{e+1}) \\
\equiv S_1S_a + S_{eS_{e+1}} + (S_1 + S_a)S_1 \pmod{x^{q^e} - x} \\
= S_1^2 + S_{eS_{e+1}} \\
= x^2 + x\text{Tr}_{q^e/q}(x) + \text{Tr}_{q^e/q}(x)^2.
\]

We claim that when \( e \) is odd, \( g_{n,q} \) is a PP of \( \mathbb{F}_{q^e} \).

Assume to the contrary that there exist \( x, y \in \mathbb{F}_{q^e}, x \neq y \), such that \( g_{n,q}(x) = g_{n,q}(y) \). From \( \text{Tr}_{q^e/q}(g_{n,q}(x)) = \text{Tr}_{q^e/q}(g_{n,q}(y)) \), we derive that \( \text{Tr}_{q^e/q}(x) = \text{Tr}_{q^e/q}(y) = c \). Then the equation \( g_{n,q}(x) = g_{n,q}(y) \) becomes

\[
(x + y + c)(x + y) = 0.
\]

So \( x + y + c = 0 \). Thus \( c = \text{Tr}_{q^e/q}(c) = \text{Tr}_{q^e/q}(x + y) = 0 \). Hence \( (x + y)^2 = 0 \), which is a contradiction.
Example 4.1.5 Let $q = 4$, $e > 1$, $n = q^0 + 2q^1 + q^e + q^{e+1}$. Then by Theorem 4.1.1,

$$g_{q^0+2q^1+q^e+q^{e+1}} = S_1S_1 + S_eS_{e+1}$$

$$= x^2 + xTr_{q^e/q}(x) + Tr_{q^e/q}(x)^2 \pmod{x^{q^e} - x}.$$

We claim that when $e$ is odd, $g_{n,q}$ is a PP of $\mathbb{F}_{q^e}$.
Assume that there exist $x, a \in \mathbb{F}_{q^e}$ such that $g(x) = g(x + a)$. Then

$$x^2 + xTr_{q^e/q}(x) + Tr_{q^e/q}(x)^2 = (x + a)^2 + (x + a)Tr_{q^e/q}(x + a) + Tr_{q^e/q}(x + a)^2. \quad (4.1.1)$$

It leads to

$$a^2 + xTr_{q^e/q}(a) + aTr_{q^e/q}(x) + aTr_{q^e/q}(a) + Tr_{q^e/q}(a)^2 = 0. \quad (4.1.2)$$

By taking traces on both sides of (4.1.2) we get $Tr_{q^e/q}(a) = 0$.
By (4.1.2), $a(a + Tr_{q^e/q}(x)) = 0$.
If $Tr_{q^e/q}(x) = a$, taking traces on both sides gives $eTr_{q^e/q}(x) = Tr_{q^e/q}(a) = 0$. Since $e$ is odd $Tr_{q^e/q}(x) = 0$, which is a contradiction. Therefore $a = 0$.

Example 4.1.6 Let $q = 4$, $e > 1$, $n = q^0 + q^1 + 2q^{e-1} + q^e$. Then by Theorem 4.1.1,

$$g_{q^0+q^1+2q^{e-1}+q^e} = S_1S_e + S_{e-1}S_{e-1}$$

$$= xTr_{q^e/q}(x) + Tr_{q^e/q}(x)^2 + x^{2q^{e-1}}.$$

We claim that when $e$ is odd, $g_{n,q}$ is a PP of $\mathbb{F}_{q^e}$.
Assume that there exist $x, a \in \mathbb{F}_{q^e}$ such that $g(x) = g(x + a)$. Then

$$xTr_{q^e/q}(x) + Tr_{q^e/q}(x)^2 + x^{2q^{e-1}} = (x + a)Tr_{q^e/q}(x + a) + Tr_{q^e/q}(x + a)^2 + (x + a)^2q^{e-1}. \quad (4.1.3)$$

It leads to

$$xTr_{q^e/q}(a) + aTr_{q^e/q}(x) + aTr_{q^e/q}(a) + Tr_{q^e/q}(a)^2 + a^{2q^{e-1}} = 0. \quad (4.1.4)$$
By taking traces on both sides we get $\text{Tr}_{q^e/q}(a) = 0$. By (4.1.4),

$$a\text{Tr}_{q^e/q}(x) + a^{2q^{e-1}} = 0. \tag{4.1.5}$$

If $\text{Tr}_{q^e/q}(x) = 0$, $a = 0$.
If $\text{Tr}_{q^e/q}(x) = 1$, $a + a^{2q^{e-1}} = 0$. Squaring both sides gives $a(a + 1) = 0$. If $a = 1$, then since $e$ is odd, it contradicts the fact that $\text{Tr}_{q^e/q}(a) = 0$. Therefore $a = 0$.
If $\text{Tr}_{q^e/q}(x) \neq 0, 1$, squaring both sides of (4.1.5) gives $a(a\text{Tr}_{q^e/q}(x)^2 + 1) = 0$. If $a\text{Tr}_{q^e/q}(x)^2 + 1 = 0$, taking traces on both sides gives $e = 0$. It contradicts the fact that $e$ is odd. Therefore $a = 0$.

**Example 4.1.7** Let $q = 4$, $e > 2$, $n = q^0 + 2q^{e-2} + 2q^{e-1}$. Then by Theorem 4.1.1,

$$g_{q^0+2q^{e-2}+2q^{e-1}} = S_{e-2}^2 + S_{e-1}^2 = x^{2q^{e-2}}.$$ 

Since $\gcd(2q^{e-2}, q^e - 1) = 1$, $g_{n,q}$ is a PP of $\mathbb{F}_{q^e}$.

**Example 4.1.8** Let $q = 4$, $e > 2$, $n = q^0 + 2q^1 + 2q^2$. Then by Theorem 4.1.1,

$$g_{q^0+2q^1+2q^2} = S_1^2 + S_2^2 = x^{2q}.$$ 

Since $\gcd(2q, q^e - 1) = 1$, $g_{n,q}$ is a PP of $\mathbb{F}_{q^e}$.

**Example 4.1.9** Let $q = 4$, $e \geq 2$, $n = 3q^0 + 2q^1$. Then by Theorem 4.1.1,

$$g_{3q^0+2q^1} = S_0^2 + S_1^2 = x^2.$$ 

Since $\gcd(2, q^e - 1) = 1$, $g_{n,q}$ is a PP of $\mathbb{F}_{q^e}$.

**Example 4.1.10** Let $q = 2^s$, $s > 1$, $e > 1$, $n = (q - 1)q^0 + 2q^{e-1}$. Then by Theorem 4.1.1,

$$g_{(q-1)q^0+2q^{e-1}} = S_0^2 + S_{e-1}^2$$

$$= S_e^2 + x^{2q^{e-1}}.$$
We claim that when \( e \) is even, \( g_{n,q} \) is a PP of \( \mathbb{F}_{q^e} \).

Assume that there exist \( x, a \in \mathbb{F}_{q^e} \) such that \( g(x) = g(x + a) \). Then we have

\[
\text{Tr}_{q^e/q}(x)^2 + x^{2q^{e-1}} = \text{Tr}_{q^e/q}(x + a)^2 + (x + a)^{2q^{e-1}}. \tag{4.1.6}
\]

It leads to

\[
\text{Tr}_{q^e/q}(a)^2 + a^{2q^{e-1}} = 0. \tag{4.1.7}
\]

If we raise both sides to the \((q/2)\)th power, we get \( \text{Tr}(a) + a^{q^e} = 0 \). By taking traces on both sides we get \((e + 1)\text{Tr}_{q^e/q}(a) = 0 \). Since \( e \) is even \( \text{Tr}_{q^e/q}(a) = 0 \).

By (4.1.7), \( a = 0 \).

**Example 4.1.11** Let \( q = 2^s, s > 1, e > 1, n = (q-1)q^0 + 2q^{e-2} \). Then by Theorem 4.1.1,

\[
g_{(q-1)q^0+2q^{e-2}} = S_0^{q^2} + S_{e-2}^{q^2} = S_e^{q^2} + x^{2q^{e-2}} + x^{2q^{e-1}}.
\]

We claim that when \( e \) is odd, \( g_{n,q} \) is a PP of \( \mathbb{F}_{q^e} \).

Assume that there exist \( x, a \in \mathbb{F}_{q^e} \) such that \( g(x) = g(x + a) \). Then we have

\[
\text{Tr}_{q^e/q}(x)^2 + x^{2q^{e-2}} + x^{2q^{e-1}} = \text{Tr}_{q^e/q}(x + a)^2 + (x + a)^{2q^{e-2}} + (x + a)^{2q^{e-1}}. \tag{4.1.8}
\]

It leads to

\[
\text{Tr}_{q^e/q}(a)^2 + a^{2q^{e-2}} + a^{2q^{e-1}} = 0. \tag{4.1.9}
\]

By taking traces on both sides we get \( \text{Tr}_{q^e/q}(a) = 0 \).

By (4.1.9), \( a^{2q^{e-2}} + a^{2q^{e-1}} = 0 \).

Raising both sides to the \((q/2)\)th power gives \( a(a^{q^{e-2}} + 1) = 0 \).

If \( a^{q^{e-2}} + 1 = 0 \), \( \text{Tr}_{q^e/q}(a) = 1 \), since \( e \) is odd, which is a contradiction.

So \( a = 0 \).
Example 4.1.12  Let $q = 4$, $e > 2$, $n = q^0 + 2q^1 + 2q^e$. Then by Theorem 4.1.1,

$$g_{q^0+2q^1+2q^e} = S_1^2 + S_e^2$$

$$= x^2 + S_e^2.$$ 

We claim that when $e$ is even, $g_{n,q}$ is a PP of $\mathbb{F}_{q^e}$.

Assume that there exist $x, a \in \mathbb{F}_{q^e}$ such that $g(x) = g(x + a)$. Then we have

$$x^2 + \text{Tr}_{q^e/q}(x)^2 = (x + a)^2 + \text{Tr}_{q^e/q}(x + a)^2. \quad (4.1.10)$$

It leads to

$$\text{Tr}_{q^e/q}(a)^2 + a^2 = 0. \quad (4.1.11)$$

Since $e$ is even, by taking traces on both sides we get $\text{Tr}_{q^e/q}(a) = 0$.

Then by (4.1.11), $a = 0$.

4.2 More Families of Desirable Triples with Even $q$

Lemma 4.2.1  Let $n = (q - 1)q^a + (q - 1)q^b$, where $a, b \geq 0$. Then

$$g_{n,q} = -1 - (S_b - S_a)^{q - 1}. \quad (4.2.1)$$

Proof. If $a = b$, then $n = (q - 2)q^a + q^{a+1}$. By Lemma 1.1.3, $g_{n,q} = -1$.

Now assume $a < b$. We have

$$(S_b - S_a)g_{n,q} = g_{(q-1)q^a+q^{b+1},q} - g_{q^{a+1}+(q-1)q^b,q}$$

$$= -(x^{q^a} + \cdots + x^{q^b}) - (x^{q^{a+1}} + \cdots + x^{q^{b+1}})$$

$$= -(x^{q^a} + \cdots + x^{q^{b-1}}) - (x^{q^a} + \cdots + x^{q^{b-1}})^q$$

$$= -(S_b - S_a) - (S_b - S_a)^q. \quad (4.2.2)$$

Thus $g_{n,q} = -1 - (S_b - S_a)^{q - 1}$.
Theorem 4.2.2 Let \( q = 2^s, s > 1, e > 0, \) and \( n = (q-1)q^0 + (q-1)q^e + 2q^s, a \geq 0. \) Then

\[
g_{n,q} \equiv x \text{Tr}_{q^e/q}(x) + \text{Tr}_{q^e/q}(x)^2 + S_a^2 \cdot (1 + \text{Tr}_{q^e/q}(x)^{q^{-1}}) \quad (\text{mod } x^{q^e} - x),
\]

Assume that \( e \) is even and \( \gcd(a,e) = 1. \) Then \( g_{n,q} \) is a PP of \( \mathbb{F}_{q^e}. \)

Proof. Write \( g_n = g_{n,q}. \) We have

\[
g_n = g_{q+q^e+(q-1)q^e} + S_a \cdot g_{(q-1)q^0+q^e+(q-1)q^e} = g_{q+q^e+1} + (S_a - S_e)g_{q+(q-1)q^e} + S_a \cdot (g_{q+(q-1)q^e} + S_a \cdot g_{(q-1)q^0+(q-1)q^e}) \equiv S_e(S_e + S_1) + S_a(1 + S_a^{q^{-1}}) \quad (\text{mod } x^{q^e} - x) \quad \text{(Lemma 4.2.1)}
\]

\[
= x \text{Tr}_{q^e/q}(x) + \text{Tr}_{q^e/q}(x)^2 + S_a^2 \cdot (1 + \text{Tr}_{q^e/q}(x)^{q^{-1}}).
\]

To prove that \( g_n \) is a PP of \( \mathbb{F}_{q^e}, \) we assume that \( g_n(x) = g_n(y), x, y \in \mathbb{F}_{q^e}, \) and try to show that \( x = y. \) From \( \text{Tr}_{q^e/q}(g_n(x)) = \text{Tr}_{q^e/q}(g_n(y)), \) we derive that \( \text{Tr}_{q^e/q}(x) = \text{Tr}_{q^e/q}(y) = c. \) If \( c = 0, \) the equation \( g_n(x) = g_n(y) \) becomes \( S_a(x)^2 = S_a(y)^2, \) i.e., \( S_a(x+y) = 0. \) Since \( \gcd(1+x+\cdots+x^{q-1}, x^e+1) = 1, \) we have \( x = y. \) If \( c \neq 0, \) the equation \( g_n(x) = g_n(y) \) becomes \( c(x+y) = 0, \) which also gives \( x = y. \)

Example 4.2.3 Let \( q = 2^s, s > 1, e > 1, n = (q-1)q^0 + 2q^{e-1} + (q-1)q^e. \) Then

\[
g_{(q-1)q^0+2q^{e-1}+(q-1)q^e} = g_{q+q^{e-1}+(q-1)q^e} + S_{e-1}g_{(q-1)q^0+q^{e-1}+(q-1)q^e} = g_{q+q^{e+1}} + (S_{e+1} - S_e)g_{q+(q-1)q^e} + S_{e-1}g_{q+(q-1)q^e} + S_{e-1}g_{(q-1)q^0+(q-1)q^e} = S_e(S_e - x) + S_{e-1}g_{(q-1)q^0+(q-1)q^e}
\]

Note that

\[
S_e g_{(q-1)q^0+(q-1)q^e} = g_{(q-1)q^0+q^{e+1}} - g_{q+(q-1)q^e} = S_e + x^{q^e} - (S_e - x) = x^{q^e} - x = S_a^q - S_e,
\]

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\[ g_{(q-1)q^0+(q-1)q^e} = S_{q-1}^q - 1. \]  

(4.2.12)

So
\[
g_{(q-1)q^0+2q^e-1+(q-1)q^e} = S_e^2 - xS_e + S_e-1(S_e^{q-1} - 1) = xTr_{q^e/q}(x) + Tr_{q^e/q}(x)^2 + x^{2q^e-1} + Tr_{q^e/q}(x)^q x^{2q^e-1}.
\]

Thus modulo \( x^{q^e} - x \) we have
\[
g_{n,q}(x) \equiv \begin{cases} 
  x^{2q^e-1} & \text{if } Tr_{q^e/q}(x) = 0, \\
  xTr_{q^e/q}(x) + Tr_{q^e/q}(x)^2 & \text{if } Tr_{q^e/q}(x) \neq 0.
\end{cases}
\]

We claim that when \( e \) is even, \( g_{n,q} \) is a PP of \( \mathbb{F}_{q^e} \). Clearly \( x^{2q^e-1} \) and \( xTr_{q^e/q}(x) + Tr_{q^e/q}(x)^2 \) map two sets \( \{ x \in \mathbb{F}_{q^e}; Tr_{q^e/q}(x) = 0 \} \) and \( \{ x \in \mathbb{F}_{q^e}; Tr_{q^e/q}(x) \neq 0 \} \) to themselves respectively.

**Case 1.** Let \( x, y \in \{ x \in \mathbb{F}_{q^e}; Tr_{q^e/q}(x) = 0 \} \).
\[
g(x) = g(y) \Rightarrow x^{2q^e-1} = y^{2q^e-1}. \text{ Raising both sides to the } (q/2)^{th} \text{ power gives } x = y.
\]

**Case 2.** Let \( x, y \in \{ x \in \mathbb{F}_{q^e}; Tr_{q^e/q}(x) \neq 0 \} \) s.t. \( x \neq y \) and \( g(x) = g(y) \). Then
\[
xTr_{q^e/q}(x) + Tr_{q^e/q}(x)^2 = yTr_{q^e/q}(y) + Tr_{q^e/q}(y)^2. \tag{4.2.13}
\]

Since \( e \) is even, taking traces on both sides gives \( Tr_{q^e/q}(x) = Tr_{q^e/q}(y) \).

By (4.2.14), \((x - y)Tr_{q^e/q}(x) = 0\). Since \( x \neq y \), \( Tr_{q^e/q}(x) = 0 \), a contradiction.

**Example 4.2.4** Let \( q = 2^s, s > 1, e > 1, n = (q - 1)q^0 + 2q^1 + (q - 1)q^e \). Then
Lemma 4.2.5

Let $(a_1, \ldots, a_q) \geq 0$, and $n = (q - 1) + q^{a_1} + \cdots + q^{a_q}$. Then

$$g_{n,q} = -S_1 - S_{a_1} - \cdots - S_{a_q} - S_{a_1} \cdots S_{a_q}.$$
Proof. Write \( g_n = g_{n,q} \). We have

\[
g_n = g_{q+a^2+\cdots+a^q} + S_{a_1} \cdot g_{(q-1)+a^2+\cdots+a^q} = g_{q+a^2+\cdots+a^q} + S_{a_1} \cdot \left( g_{q+a^3+\cdots+a^q} + S_{a_2} \cdot g_{(q-1)+a^3+\cdots+a^q} \right)
= g_{q+a^2+\cdots+a^q} + S_{a_1} + S_{a_1} S_{a_2} \cdot g_{(q-1)+a^3+\cdots+a^q}
= \cdots
= g_{q+a^2+\cdots+a^q} - S_{a_1} + S_{a_1} \cdots S_{a_q} \cdot g_{q-1}
= -S_1 - S_{a_2} - \cdots - S_{a_q} - S_{a_1} - S_{a_1} \cdots S_{a_q}.
\]

\[\square\]

**Theorem 4.2.6** Let \( q = p^s \), \( e > 0 \), \( a > 0 \), and \( n = (q-1)q^0 + (q-1)q^e + q^a \). Then

\[
g_{n,q} = -x - S_a + \text{Tr}_{q^e/q}(x) - S_a \text{Tr}_{q^e/q}(x)^{q-1}.
\]

(4.2.15)

Assume that

(i) \(-2a - 1 + e \not\equiv 0 \pmod{p}\);

(ii) \(\gcd(x^a + x - 2, x^e - 1) = x - 1\);

(iii) \(\gcd(2x^a + x - 3, x^e - 1) = x - 1\).

Then \( g_{n,q} \) is a PP of \( \mathbb{F}_{q^e} \).

Proof. Eq. (4.2.15) follows from Lemma 4.2.5. To prove that \( g_{n,q} \) is a PP of \( \mathbb{F}_{q^e} \) under the given conditions, we assume that \( g_{n,q}(x) = g_{n,q}(y), \) \( x, y \in \mathbb{F}_{q^e} \), and try to show that \( x = y \). From \( \text{Tr}_{q^e/q}(g_{n,q}(x)) = \text{Tr}_{q^e/q}(g_{n,q}(y)) \), we derive that

\[(-2a - 1 + e)(\text{Tr}_{q^e/q}(x) - \text{Tr}_{q^e/q}(y)) = 0.\]

Since \(-2a - 1 + e \not\equiv 0 \pmod{p}\), we have \( \text{Tr}_{q^e/q}(x) = \text{Tr}_{q^e/q}(y) = c.\)
If \( c = 0 \), the equation \( g_{n,q}(x) = g_{n,q}(y) \) becomes

\[
2(x - y) + (x - y)^q + \cdots + (x - y)^{q^{a-1}} = 0.
\]

Since

\[
gcd(2 + x + \cdots + x^{a-1}, 1 + x + \cdots + x^{e-1}) = \frac{1}{x-1}gcd(x^a + x - 2, x^e - 1) = 1,
\]
we must have \( x - y = 0 \).

If \( c \neq 0 \), the equation \( g_{n,q}(x) = g_{n,q}(y) \) becomes

\[
3(x - y) + 2(x - y)^q + \cdots + 2(x - y)^{q^{a-1}} = 0.
\]

Since

\[
gcd(3 + 2x + \cdots + 2x^{a-1}, 1 + x + \cdots + x^{e-1}) = \frac{1}{x-1}gcd(2x^a + x - 3, x^e - 1) = 1,
\]
we also have \( x - y = 0 \).

**Example 4.2.7** Let \( q = 2^s \), \( s > 1 \), \( e > 1 \), \( n = (q - 1)q^0 + q^2 + (q - 1)q^e \). Then

\[
g(q-1)q^0+q^2+(q-1)q^e = g_{q+(q-1)q^e} + S_2g(q-1)q^0+(q-1)q^e
\]

\[
= x + \text{Tr}(x) + (x + x^q)g(q-1)q^0+(q-1)q^e
\]

\[
= x + \text{Tr}(x) + (x + x^q)(\text{Tr}_{q^e/q}(x)^{q^{a-1}} - 1)(4.2.12)
\]

\[
= x^q + \text{Tr}_{q^e/q}(x) + x^q\text{Tr}_{q^e/q}(x)^{q-1} + x\text{Tr}_{q^e/q}(x)^{q-1}.
\]

Thus modulo \( x^{q^e} - x \) we have

\[
g_{n,q}(x) \equiv \begin{cases} 
  x^q & \text{if } \text{Tr}_{q^e/q}(x) = 0, \\
  x + \text{Tr}_{q^e/q}(x) & \text{if } \text{Tr}_{q^e/q}(x) \neq 0.
\end{cases}
\]

We claim that when \( e \) is even, \( g_{n,q} \) is a PP of \( \mathbb{F}_{q^e} \).
Clearly \( x^q \) and \( x + \text{Tr}_{q^e/q}(x) \) map two sets \( \{ x \in \mathbb{F}_{q^e}; \text{Tr}_{q^e/q}(x) = 0 \} \) and \( \{ x \in \mathbb{F}_{q^e}; \text{Tr}_{q^e/q}(x) \neq 0 \} \) to themselves respectively.

**Case 1.** Let \( x, y \in \{ x \in \mathbb{F}_{q^e}; \text{Tr}_{q^e/q}(x) = 0 \} \). Then

\[
g(x) = g(y) \Rightarrow x^q = y^q \Rightarrow x = y.
\]

**Case 2.** Let \( x, y \in \{ x \in \mathbb{F}_{q^e}; \text{Tr}_{q^e/q}(x) \neq 0 \} \) s.t. \( g(x) = g(y) \). Then

\[
x + \text{Tr}_{q^e/q}(x) = y + \text{Tr}_{q^e/q}(y). \tag{4.2.16}
\]

Since \( e \) is even, taking traces on both sides gives \( \text{Tr}_{q^e/q}(x) = \text{Tr}_{q^e/q}(y) \). Then by (4.2.16), \( x = y \).

**Theorem 4.2.8** Let \( q = 2^s, s > 1, e > 0 \), and let \( n = (q - 1)q^0 + \frac{q}{2}q^{e-1} + \frac{q}{2}q^e \). We have

\[
g_{n,q} = x + \text{Tr}_{q^e/q}(x) + x^{\frac{1}{2}q} \text{Tr}_{q^e/q}(x)^{\frac{1}{2}q}.
\]

When \( e \) is odd, \( g_{n,q} \) is a PP of \( \mathbb{F}_{q^e} \).

**Proof.** By Lemma 4.2.5,

\[
g_{n,q} = S_1 + S_{e-1}^2 S_e^2
\]

\[
= x + (S_e^{\frac{1}{2}q} + x^{\frac{1}{2}q^e}) S_e^{\frac{1}{2}q}
\]

\[
= x + \text{Tr}_{q^e/q}(x) + x^{\frac{1}{2}q^e} \text{Tr}_{q^e/q}(x)^{\frac{1}{2}q}.
\]

Assume that \( e \) is odd. To prove that \( g_{n,q} \) is a PP of \( \mathbb{F}_{q^e} \), assume to the contrary that there exist \( x, y \in \mathbb{F}_{q^e}, x \neq y \), such that \( g_{n,q}(x) = g_{n,q}(y) \). From \( \text{Tr}_{q^e/q}(g_{n,q}(x)) = \text{Tr}_{q^e/q}(g_{n,q}(y)) \), we derive that \( \text{Tr}_{q^e/q}(x) = \text{Tr}_{q^e/q}(y) = a \). If \( a = 0 \), the equation \( g_{n,q}(x) = g_{n,q}(y) \) becomes \( x = y \), which is a contradiction. If \( a \neq 0 \), the equation \( g_{n,q}(x) = g_{n,q}(y) \) becomes

\[
(x + y)^{\frac{1}{2}q^e} a^{\frac{1}{2}q} = x + y.
\]

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i.e.,

\[(x + y)^{q^e - 2} = a^{-1}.
\]

So \(x + y = a\). Then \(a = \text{Tr}_{q^e/q}(a) = \text{Tr}_{q^e/q}(x + y) = 0\), which is a contradiction.

\[\text{Example 4.2.9}\] Let \(q = 4\), \(e > 1\) and let \(n = 3q^0 + 2q^{e-1} + 2q^e\). Then

\[g_{3q^0+2q^{e-1}+2q^e} = g_{q+2q^{e-1}+q^e} + S_e g_{3q^0+2q^{e-1}+q^e}\]

\[= x + \text{Tr}_{q^e/q}(x) + S_e g_{3q^0+2q^{e-1}+q^e}\]

\[= x + \text{Tr}_{q^e/q}(x) + S_e (1 + S_e g_{3q^0+2q^{e-1}})\]

\[= x + \text{Tr}_{q^e/q}(x) + S_e + S_e^2 g_{3q^0+2q^{e-1}}\]

\[= x + \text{Tr}_{q^e/q}(x) + S_e + S_e^2 S_{e-1}\]

\[= x + S_e^2 S_{e-1}\]

\[= x + \text{Tr}_{q^e/q}(x)^2 (\text{Tr}_{q^e/q}(x) + x^{q^{e-1}})^2\]

\[= x + \text{Tr}_{q^e/q}(x) + x^{2q^{e-1}} \text{Tr}_{q^e/q}(x)^2.\]

We claim that when \(e\) is odd, \(g_{n,q}\) is a PP of \(\mathbb{F}_{q^e}\). Assume that there exist \(x, a \in \mathbb{F}_{q^e}\) such that \(g(x) = g(x + a)\). Then we have

\[x + \text{Tr}_{q^e/q}(x) + x^{2q^{e-1}} \text{Tr}_{q^e/q}(x)^2 = (x + a) + \text{Tr}_{q^e/q}(x + a) + (x + a)^{2q^{e-1}} \text{Tr}_{q^e/q}(x + a)^2.\]

It leads to

\[a + \text{Tr}_{q^e/q}(a) + \text{Tr}_{q^e/q}(a)^2 a^{2q^{e-1}} + \text{Tr}_{q^e/q}(a)^2 x^{2q^{e-1}} + \text{Tr}_{q^e/q}(a)^2 a^{2q^{e-1}} = 0. \tag{4.2.17}\]

By taking traces on both sides we get \(\text{Tr}_{q^e/q}(a) = 0\).

By (4.2.17),

\[a + \text{Tr}_{q^e/q}(a)^2 a^{2q^{e-1}} = 0. \tag{4.2.18}\]

If \(\text{Tr}_{q^e/q}(x) = 0\), then (4.2.18) gives \(a = 0\).

If \(\text{Tr}_{q^e/q}(x) = 1\), then (4.2.18) gives \(a + a^{2q^{e-1}} = 0\). Squaring both sides of gives \(a(a + 1) = 0\). If \(a = 1\), since \(e\) is odd that contradicts the fact that \(\text{Tr}_{q^e/q}(a) = 0\).

Therefore \(a = 0\).
If $\text{Tr}_{q^e/q}(x) \neq 0, 1$, squaring both sides of (4.2.18) gives $a(a + \text{Tr}_{q^e/q}(x)) = 0$. If $\text{Tr}_{q^e/q}(x) = a$, taking traces on both sides gives $e\text{Tr}_{q^e/q}(x) = \text{Tr}_{q^e/q}(a) = 0$. Since $e$ is odd $\text{Tr}_{q^e/q}(x) = 0$, which is a contradiction. Therefore $a = 0$.

**Theorem 4.2.10** Let $q = 4$, $e > 2$, and $n = 3q^0 + 2q^{e-2} + 2q^e$. We have

$$g_{n,q} = x + \text{Tr}_{q^e/q}(x) + (x^q^{e-2} + x^{q^{e-1}})^2\text{Tr}_{q^e/q}(x)^2.$$  

Assume that $e > 2$ is even and $\gcd(1 + x^2 + x^{e-3}, x^e + 1) = 1$. Then $g_{n,q}$ is a PP of $\mathbb{F}_{q^e}$.

**Proof.** Let $q = 4$, $e > 2$, $n = 3q^0 + 2q^{e-2} + 2q^e$. Then,

$$g_{n,q} = x + \text{Tr}_{q^e/q}(x) + (x^q^{e-2} + x^{q^{e-1}})^2\text{Tr}_{q^e/q}(x)^2 \pmod{x^q^e - x}.$$  

Assume that $e$ is even and $\gcd(1 + x^2 + x^{e-3}, x^e + 1) = 1$.

To prove that $g_{n,q}$ is a PP of $\mathbb{F}_{q^e}$, we assume that $g_{n,q}(x) = g_{n,q}(y)$, $x, y \in \mathbb{F}_{q^e}$, and try to show that $x = y$. From $\text{Tr}_{q^e/q}(g_{n,q}(x)) = \text{Tr}_{q^e/q}(g_{n,q}(y))$ we derive that $\text{Tr}_{q^e/q}(x) = \text{Tr}_{q^e/q}(y)$. Let $\text{Tr}_{q^e/q}(x) = \text{Tr}_{q^e/q}(y) = a \in \mathbb{F}_q$. If $a = 0$, then $x = y$.

If $a \neq 0$, then $g_{n,q}(x) = g_{n,q}(y)$ becomes

$$z = a^2(z^{q^{e-2}} + z^{q^{e-1}}),$$  

where $z = x + y$. Substitute (4.2.19) into itself to find $z^q^3 = z + z^{q^2}$. Since $e$ is even and $\gcd(1 + x^2 + x^{e-3}, x^e + 1) = 1$, we have $z = 0$. 

\[\blacksquare\]

**Theorem 4.2.11** Let $n = 1 + q^{a_1} + \cdots + q^{a_{q+t}}$, where $-1 \leq t \leq q - 4$. Then

$$g_{n,q} = -\sum_{1 \leq i_1 < \cdots < i_{t+2} \leq q+t} S_{a_{i_1}} \cdots S_{a_{i_{t+2}}}.$$  

(4.2.20)
Proof. Use induction on $t$. When $t = -1$, $n$ is a sum of $q$ powers of $q$, in which case the conclusion is already known. Let $0 \leq t \leq q - 4$. We have

\[ g_n = g_{2+q^a + \ldots + q^a+q_t} + S_{a_1} \cdot g_{2+q^a + \ldots + q^a+q_t} \]
\[ = g_{2+q^a + \ldots + q^a+q_t} - S_{a_2} \sum_{2 \leq i_2 < \ldots < i_{t+2} \leq q+t} S_{a_{i_1}} \cdots S_{a_{i_{t+2}}} \text{ (induction hypothesis)} \]
\[ = g_{3+q^a + \ldots + q^a+q_t} + S_{a_2} \cdot g_{2+q^a + \ldots + q^a+q_t} - S_{a_1} \sum_{2 \leq i_2 < \ldots < i_{t+2} \leq q+t} S_{a_{i_2}} \cdots S_{a_{i_{t+2}}} \]
\[ = g_{3+q^a + \ldots + q^a+q_t} - S_{a_2} \sum_{3 \leq i_3 < \ldots < i_{t+2} \leq q+t} S_{a_{i_3}} \cdots S_{a_{i_{t+2}}} \text{ (induction hypothesis)} \]
\[ = \ldots \]
\[ = g_{q+t+1} - \sum_{1 \leq i_1 < \ldots < i_{t+2} \leq q+t} S_{a_{i_1}} \cdots S_{a_{i_{t+2}}}. \]

Since $w_q(q+t+1) = t+2 < q-1$, we have $g_{q+t+1} = 0$, which gives (4.2.20).

Let $q$ be even and $t = 0$ in (4.2.20). Then

\[ g_{n,q} = \sum_{1 \leq i_1 < i_2 \leq q} S_{a_{i_1}} S_{a_{i_2}} = \sum_i S_{b_i} S_{c_i} + \sum_{i < j} (S_{b_i} + S_{c_i})(S_{b_j} + S_{c_j}), \]
where $(a_1, \ldots, a_q) = (b_1, \ldots, b_{q/2}, c_1, \ldots, c_{q/2})$. This is Theorem 4.1.1. In fact, Theorem 4.2.11 is a generalized version of Theorem 4.1.1.

The next theorem is a generalization of [14, Theorem 6.12].

**Theorem 4.2.12** Let $q = p^2$, $e > 0$, and $n = (p^2 - p - 1)q^0 + (p - 1)q^e + pq^a + q^b$, $a, b \geq 0$. Then

\[ g_{n,q} = -S_{a}^p - S_{b}S_{c}^{p-1}. \quad (4.2.21) \]
Assume that $a + b \not\equiv 0 \pmod{p}$ and

$$\gcd(x(x^a - 1)^2 - \epsilon(x^b - 1)^2, (x - 1)(x^\epsilon - 1)) = (x - 1)^2,$$

for $\epsilon = 0, 1$. Then $g_{n,q}$ is a PP of $\mathbb{F}_{q^e}$.

**Proof.** Equation (4.2.21) follows from Theorem 4.2.11.

To prove that $g_{n,q}$ is a PP of $\mathbb{F}_{q^e}$ under the given conditions, we assume that $g_{n,q}(x) = g_{n,q}(y)$, $x, y \in \mathbb{F}_{q^e}$, and try to show that $x = y$. From $S_e(g_{n,q}(x)) = S_e(g_{n,q}(y))$ we derive that

$$(a + b)(S_e(x) - S_e(y))^p = 0.$$ 

Since $a + b \not\equiv 0 \pmod{p}$, we have $S_e(x) = S_e(y) = c \in \mathbb{F}_q$. Now the equation $g_{n,q}(x) = g_{n,q}(y)$ becomes

$$S_a(z)^p = -c^{q_p - 1}S_b(z),$$

where $z = x - y$. Thus

$$S_a(z) = (-c^{q_p - 1}S_b(z))^{q_p^e - 1} = -c^{q_p - 1}S_b(z^{q_p^e - 1}). \quad (4.2.22)$$

We iterate both sides of (4.2.22) to get

$$(S_a \circ S_a)(z) = -c^{q_p - p}S_b\left((-c^{q_p - p}S_b(z^{q_p^e - 1}))^{q_p^e - 1}\right) = c^{q_p - 1}(S_b \circ S_b)(z^{q_p^e - 1}),$$

i.e.,

$$[(S_a \circ S_a)(z)]^q = c^{q_p - 1}(S_b \circ S_b)(z). \quad (4.2.23)$$

Let $\epsilon = c^{q_p - 1}$, which is 0 or 1. The conventional associates of the $q$-polynomials $(S_a \circ S_a)^q$ and $S_b \circ S_b$ are $x(1 + x + \cdots + x^{a - 1})^2$ and $(1 + x + \cdots + x^{b - 1})^2$, respectively
Since
\[
gcd(x(1 + x + \cdots + x^{a-1})^2 - \epsilon(1 + x + \cdots + x^{b-1})^2, 1 + x + \cdots + x^{e-1})
\]
\[
= \frac{1}{(x-1)^2}gcd(x(x^a - 1)^2 - \epsilon(x^b - 1)^2, (x-1)(x^e - 1))
\]
\[
= 1,
\]
it follows from (4.2.23) that $z = 0$, i.e., $x = y$.

**Example 4.2.13** Let $q = 4$, $e > 3$, $n = q^0 + 2q^1 + q^2 + q^3$. Then by Theorem 4.1.1,
\[
g_{n,q} \equiv x^2 + x \text{Tr}_{q^e/q}(x) + x^q \text{Tr}_{q^e/q}(x) \pmod{x^e - x}.
\]

We claim that when $\gcd(1 + x + x^2, x^e + 1) = 1$, $g_{n,q}$ is a PP of $\mathbb{F}_{q^e}$.

To prove that $g_{n,q}$ is a PP of $\mathbb{F}_{q^e}$, we assume that $g_{n,q}(x) = g_{n,q}(y)$, $x, y \in \mathbb{F}_{q^e}$, and try to show that $x = y$. From $\text{Tr}_{q^e/q}(g_{n,q}(x)) = \text{Tr}_{q^e/q}(g_{n,q}(y))$ we derive that $\text{Tr}_{q^e/q}(x) = \text{Tr}_{q^e/q}(y)$. Let $\text{Tr}_{q^e/q}(x) = \text{Tr}_{q^e/q}(y) = a \in \mathbb{F}_q$.

If $a = 0$, then $x = y$.

If $a \neq 0$, then $g_{n,q}(x) = g_{n,q}(y)$ becomes
\[
z^2 = a(z + z^q),
\]
where $z = x + y$. Substitute (4.2.24) into itself to find $(z^2)^q = (z^2) + (z^2)^q$. Since $\gcd(1 + x + x^2, x^e + 1) = 1$, we have $z = 0$.

**Example 4.2.14** Let $q = 4$, $e > 4$, $n = q^0 + 2q^1 + q^{e-2} + q^e$. Then by Theorem 4.1.1,
\[
g_{n,q} \equiv x^2 + \text{Tr}_{q^e/q}(x)^2 + x^{q^{e-2}} \text{Tr}_{q^e/q}(x) + x^{q^{e-1}} \text{Tr}_{q^e/q}(x) \pmod{x^e - x}.
\]

We claim that when $\gcd(1 + x^2 + x^5, x^e + 1) = 1$ and $e$ is even, $g_{n,q}$ is a PP of $\mathbb{F}_{q^e}$.

To prove that $g_{n,q}$ is a PP of $\mathbb{F}_{q^e}$, we assume that $g_{n,q}(x) = g_{n,q}(y)$, $x, y \in \mathbb{F}_{q^e}$,
and try to show that $x = y$. From $\text{Tr}_{q^e/q}(g_{n,q}(x)) = \text{Tr}_{q^e/q}(g_{n,q}(y))$ we derive that $\text{Tr}_{q^e/q}(x) = \text{Tr}_{q^e/q}(y)$. Let $\text{Tr}_{q^e/q}(x) = \text{Tr}_{q^e/q}(y) = a \in \mathbb{F}_q$.

If $a = 0$ then $x = y$. If $a \neq 0$, then $g_{n,q}(x) = g_{n,q}(y)$ becomes

$$z^2 = a(z^{q^e-2} + z^{q^{e-1}}), \quad (4.2.25)$$

where $z = x + y$. Substitute $(4.2.25)$ into itself to find $(z^2)^q = (z^2) + (z^2)^q$. Since $\gcd(1 + x^2 + x^5, x^e + 1) = 1$, we have $z = 0$.

**Example 4.2.15** Let $q = 4$, $e > 3$, $n = q^0 + q^{e-2} + 2q^{e-1} + q^e$. Then by Theorem 4.1.1,

$$g_{n,q} \equiv x^{2q^{e-1}} + x^{q^{e-2}}\text{Tr}_{q^e/q}(x) + x^{q^{e-1}}\text{Tr}_{q^e/q}(x) \pmod{x^{q^e} - x}.$$ 

We claim that when $\gcd(1 + x^2 + x^3, x^e + 1) = 1$, $g_{n,q}$ is a PP of $\mathbb{F}_{q^e}$.

To prove that $g_{n,q}$ is a PP of $\mathbb{F}_{q^e}$, we assume that $g_{n,q}(x) = g_{n,q}(y)$, $x, y \in \mathbb{F}_{q^e}$, and try to show that $x = y$. From $\text{Tr}_{q^e/q}(g_{n,q}(x)) = \text{Tr}_{q^e/q}(g_{n,q}(y))$ we derive that $\text{Tr}_{q^e/q}(x) = \text{Tr}_{q^e/q}(y)$. Let $\text{Tr}_{q^e/q}(x) = \text{Tr}_{q^e/q}(y) = a \in \mathbb{F}_q$.

If $a = 0$ then $x = y$. If $a \neq 0$, then $g_{n,q}(x) = g_{n,q}(y)$ becomes

$$z = a^2(z^{2q^{e-2}} + z^{2q^{e-1}}), \quad (4.2.26)$$

where $z = x + y$. Substitute $(4.2.26)$ into itself to find $z^{q^3} = z + z^q$. Since $\gcd(1 + x^2 + x^3, x^e + 1) = 1$, we have $z = 0$.

**Example 4.2.16** Let $q = 4$, $e > 3$, $n = q^0 + q^{e-2} + q^e + 2q^{e+1}$. Then by Theorem 4.1.1,

$$g_{n,q} \equiv x^2 + x^{q^{e-2}}\text{Tr}_{q^e/q}(x) + x^{q^{e-1}}\text{Tr}_{q^e/q}(x) \pmod{x^{q^e} - x}.$$ 

We claim that when $\gcd(1 + x^2 + x^3, x^e + 1) = 1$, $g_{n,q}$ is a PP of $\mathbb{F}_{q^e}$.

To prove that $g_{n,q}$ is a PP of $\mathbb{F}_{q^e}$, we assume that $g_{n,q}(x) = g_{n,q}(y)$, $x, y \in \mathbb{F}_{q^e}$, and try to show that $x = y$. From $\text{Tr}_{q^e/q}(g_{n,q}(x)) = \text{Tr}_{q^e/q}(g_{n,q}(y))$ we derive that $\text{Tr}_{q^e/q}(x) = \text{Tr}_{q^e/q}(y)$. Let $\text{Tr}_{q^e/q}(x) = \text{Tr}_{q^e/q}(y) = a \in \mathbb{F}_q$.

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If \( a = 0 \) then \( x = y \). If \( a \neq 0 \), then \( g_{n,q}(x) = g_{n,q}(y) \) becomes

\[
z^2 = a(z^{q^e-2} + z^{q^e-1}),
\] (4.2.27)

where \( z = x + y \). Substitute (4.2.27) into itself to find \( z^{q^5} = z + z^{q^2} \). Since \( \gcd(1 + x^2 + x^5, x^e + 1) = 1 \), we have \( z = 0 \).

**Lemma 4.2.17** Let \( f : \mathbb{F}_p^n \to \mathbb{F}_p \) be a function, and assume that there exists \( y \in \mathbb{F}_p^n \) such that \( f(x + y) - f(x) \) is a nonzero constant for all \( x \in \mathbb{F}_p^n \). Then

\[
\sum_{x \in \mathbb{F}_p^n} \zeta_p^{f(x)} = 0 \quad (\zeta_p = e^{2\pi i/p}).
\]

**Proof.** Assume \( f(x + y) - f(x) = c \in \mathbb{F}_p^* \). We have

\[
\sum_{x \in \mathbb{F}_p^n} \zeta_p^{f(x)} = \sum_{x \in \mathbb{F}_p^n} \zeta_p^{f(x+y)} = \sum_{x \in \mathbb{F}_p^n} \zeta_p^{f(x)+c} = \zeta_p^c \sum_{x \in \mathbb{F}_p^n} \zeta_p^{f(x)}.
\]

Since \( \zeta_p^c \neq 1 \), we have \( \sum_{x \in \mathbb{F}_p^n} \zeta_p^{f(x)} = 0 \).

**Remark 4.2.18** If \( f : \mathbb{F}_p^n \to \mathbb{F}_p \) is quadratic, then \( \sum_{x \in \mathbb{F}_p^n} \zeta_p^{f(x)} = 0 \) if and only if there exists \( y \in \mathbb{F}_p^n \) such that \( f(x + y) - f(x) \) is a nonzero constant for all \( x \in \mathbb{F}_p^n \). See [10, Ch. VII, VIII], [19, §5.1], [30, §6.2].

Let \( f = \sum_{i=0}^{e-1} a_i x^{p^i} \in \mathbb{F}_{p^e}[x] \) be a \( p \)-linearized polynomial considered as a \( \mathbb{F}_{p^e} \)-linear map from \( \mathbb{F}_{p^e} \) to \( \mathbb{F}_{p^e} \). The adjoint of \( f \) is the \( \mathbb{F}_{p^e} \)-linear map \( f^* \) such that

\[
\text{Tr}_{p^e/p}(xf(y)) = \text{Tr}_{p^e/p}(f^*(x)y) \quad \text{for all } x, y \in \mathbb{F}_{p^e}.
\]

We have \( f^* = \sum_{i=0}^{e-1} a_{e-i} x^{p^i} \), where the subscript is taken modulo \( e \). For \( 0 \leq k \leq e \), we have

\[
(f^{p^k})^*(x) \equiv f^*(x^{p^{e-k}}) \pmod{x^{p^e} - x}.
\]

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(Here $f^p$ means product, not composition.) In fact, since $f^p = \sum_ia^p_{i-k}x^{i+k} = \sum_ia^p_{i-k}x^i$, we have

$$(f^p)^* = \sum_ia^p_{e-i-k}x^{i*} = \sum_ia^p_{e-(k+i)}x^{i*} = \sum_ia^p_{e-i}x^{i*} = f^*(x^{p-e-k}) \pmod{x^p-x}.$$

The following theorem is a generalization of [14, Theorem 6.15].

**Theorem 4.2.19** Let $p$ be a prime and $k, n$ positive integers. Let $A, B \in \mathbb{F}_{p^k}[x]$ satisfying the following conditions.

(i) $A$ is a $p$-linearized polynomial that permutes $\mathbb{F}_{p^k}$.

(ii) $B(x+y) = B(x)$ for all $x \in \mathbb{F}_{p^k}$ and $y \in \mathbb{F}_{p^k}$.

(iii) $B^p - B$ is a $p$-linearized polynomial, and all zeros of $(A^p - A)^* + (B^p - B)^*$ in $\mathbb{F}_{p^k}$ are contained in $\mathbb{F}_{p^k}$.

Then $A + B$ is a PP of $\mathbb{F}_{p^k}$.

**Proof.** By [30, Theorem 7.7], it suffices to show that for all $0 \neq a \in \mathbb{F}_{p^k}$,

$$\sum_{x \in \mathbb{F}_{p^k}} \zeta_p^{\text{Tr}(a \cdot (A+B)(x))} = 0,$$

where $\zeta_p = e^{2\pi i/p}$ and $\text{Tr} = \text{Tr}_{p^k/p}$.

**Case 1.** Assume $\text{Tr}_{p^k/p}(a) \neq 0$. By Lemma 4.2.17, It suffices to show that there exists a $y \in \mathbb{F}_{p^k}$ such that $\text{Tr}[a \cdot (A + B)(x + y) - a \cdot (A + B)(x)]$ is a nonzero constant for all $x \in \mathbb{F}_{p^k}$.

Since $\text{Tr}_{p^k/p}(a) \neq 0$ and $A$ permutes $\mathbb{F}_{p^k}$, there exists a $y \in \mathbb{F}_{p^k}$ such that
\( \text{Tr}_{p^k/p} [A(y) \text{Tr}_{p^{kn}/p^k}(a)] \neq 0 \). For all \( x \in \mathbb{F}_{p^{kn}} \) we have

\[
\begin{align*}
\text{Tr} & \left[ a \cdot (A + B)(x + y) - a \cdot (A + B)(x) \right] \\
= & \text{Tr} \left[ a(A(x + y) + B(x + y) - A(x) - B(x)) \right] \\
= & \text{Tr} \left( aA(y) \right) \quad \left( B(x + y) = B(x) \right) \\
= & \text{Tr}_{p^k/p} \left[ \text{Tr}_{p^{kn}/p^k}(aA(y)) \right] \\
= & \text{Tr}_{p^k/p} \left[ A(y) \text{Tr}_{p^{kn}/p^k}(a) \right],
\end{align*}
\]

which is a nonzero constant.

**Case 2.** Assume \( \text{Tr}_{p^{kn}/p^k}(a) = 0 \). Then \( a = b^{p^{(n-1)k}} - b \) for some \( b \in \mathbb{F}_{p^{kn}} \setminus \mathbb{F}_{p^k} \).

For \( x \in \mathbb{F}_{p^{kn}} \) we have

\[
\begin{align*}
\text{Tr} & \left[ a \cdot (A + B)(x) \right] = \text{Tr} \left[ (b^{p^{(n-1)k}} - b) \cdot (A + B)(x) \right] \\
= & \text{Tr} \left[ b((A + B)^{p^k}(x) - (A + B)(x)) \right] \\
= & \text{Tr} \left[ b((A^{p^k} - A)(x) + (B^{p^k} - B)(x)) \right] \\
= & \text{Tr} \left[ x \left((A^{p^k} - A)^*(b) + (B^{p^k} - B)^*(b)\right)\right].
\end{align*}
\]

Condition (iii) implies that for \( z \in \mathbb{F}_{p^{kn}}, \)

\[
(A^{p^k} - A)^*(z) + (B^{p^k} - B)^*(z) = 0 \iff z \in \mathbb{F}_{p^k}.
\]

Since \( b \notin \mathbb{F}_{p^k} \), we have \((A^{p^k} - A)^*(b) + (B^{p^k} - B)^*(b) \neq 0 \). Therefore

\[
\sum_{x \in \mathbb{F}_{p^{kn}}} \zeta_p \text{Tr}[a \cdot (A + B)(x)] = \sum_{x \in \mathbb{F}_{p^{kn}}} \zeta_p \text{Tr} [x((A^{p^k} - A)^*(b) + (B^{p^k} - B)^*(b))] = 0.
\]

**Corollary 4.2.20** Let \( e = 3k, \ k \geq 1, \ q = 2^s, \ s \geq 2, \) and \( n = (q-3)q^0 + 2q^1 + q^2 + q^4 \).

Then

\[
g_{n,q} \equiv x^2 + S_{2k}S_{4k} \pmod{x^e - x},
\]

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and $g_{n,q}$ is a PP of $\mathbb{F}_{q^e}$.

Proof. We write $g_n$ for $g_{n,q}$. We have

$$
\begin{align*}
g_n &= g_{(q-2)q^0+2q^1+q^2k} + S_{4k} \cdot g_{(q-3)q^0+2q^1+q^2k} \\
 &= g_{(q-1)q^0+2q^1} + S_{2k} \cdot g_{(q-2)q^0+2q^1} + S_{4k}S_{2k} \\
 &= g_{2q^1} + S_1 \cdot g_{(q-1)q^0+q^1} + S_{4k}S_{2k} \\
 &= x^2 + S_{4k}S_{2k}.
\end{align*}
$$

It follows from Theorem 4.2.19 that $g_n$ is a PP of $\mathbb{F}_{q^e}$.

Conjecture 4.2.21 Let $q = 4$, $e = 3k$, $k \geq 1$, and $n = 3q^0 + 3q^2k + q^{4k}$. It is easy to see that

$$
g_{n,q} \equiv x + S_{2k} + S_{4k} + S_{4k}^3S_{2k} \equiv x + S_{2k}^{q^2k} + S_{2k}^{q^k+3} \pmod{x^{q^e} - x}.
$$

We conjecture that $g_{n,q}$ is a PP of $\mathbb{F}_{q^e}$.

The conjecture has been verified for $e \leq 12$. 

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Table 4.1: Desirable triples \((n, e; 4), e \leq 6, w_4(n) > 4\)

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5 A PIECEWISE CONSTRUCTION OF PERMUTATION POLYNOMIALS OVER FINITE FIELDS

5.1 Introduction

Let $p$ be a prime and $q = p^n$, where $n$ is a positive integer. Let $k \mid q - 1$ and let $\omega \in \mathbb{F}_q^*$ be an element of order $k$. We shall define $\omega^\infty = 0$ and $0^0 = 0$. Let $f_\infty, f_0, \ldots, f_{k-1} \in \mathbb{F}_q[x]$ and let $\theta : \mathbb{F}_q \to \{\omega^i : i = \infty, 0, \ldots, k-1\}$. Define

$$F(x) = f_\infty(x)(1 - \theta(x)^{q-1}) + \frac{1}{k} \sum_{i=0}^{k-1} [\omega^{-0i} f_0(x) + \cdots + \omega^{-(k-1)i} f_{k-1}(x)] \theta(x)^i, \quad x \in \mathbb{F}_q. \quad (5.1.1)$$

Note that

$$F(x) = f_i(x) \quad \text{if} \quad \theta(x) = \omega^i, \quad i \in \{\infty, 0, \ldots, k-1\}. \quad (5.1.2)$$

We shall call the functions $f_i, \ i = \infty, 0, \ldots, k-1$, the case functions of $F$ and the function $\theta$ the selection function of $F$. We have

Proposition 5.1.1 The function in (5.1.1) is a permutation of $\mathbb{F}_q$ if and only if

(i) $f_i$ is 1-1 on $\theta^{-1}(\omega^i)$ for each $i \in \{\infty, 0, \ldots, k-1\}$, and

(ii) $f_i(\theta^{-1}(\omega^i)) \cap f_j(\theta^{-1}(\omega^j)) = \emptyset$ for all $i, j \in \{\infty, 0, \ldots, k-1\}, \ i \neq j$.

The idea of constructing permutation polynomials (PPs) of finite fields piecewise is not new; it has appeared in literature, at least implicitly. PPs of the form

‡This chapter consists of the paper [13] which has been published in the journal “Finite Fields and Their Applications”.
Let \( x^{m+1} + ax \), where \( m \mid q - 1 \), were considered in [4, 5, 6, 28, 33]. (In the notation of (5.1.1), one has \( k = \frac{q-1}{m}, \theta(x) = x^m, \ f_\infty(x) = 0, \ f_i(x) = (a + \omega^i)x, \ 0 \leq i \leq k - 1. \) PPs of the forms \( x^{p-1-s} + ax^{(p-1-2s)/2} \) and \( x^{p-s} + ax^{(p-s+1)/2} + bx \) were studied in [2, 3, 16, 17]. (In the notation of (5.1.1), \( q = p, \ k = 2 \) and \( \theta(x) = x^{\frac{q-1}{2}}. \))

Several recent articles on permutation polynomials suggest that the piecewise approach has more to offer. In [21], it was shown that the reversed Dickson polynomial \( D_{3^n+5}(1, x) = (1 - y - y^2)y^{3^n+1} - 1 - y + y^2, \) where \( y = 1 - x, \) is a PP over \( \mathbb{F}_{3^n} \) for even \( n. \) This particular PP was generalized by Zha and Hu in [39] in a formulation similar to (5.1.1). Also presented in [39] were several families of PPs of the form \( (x^p - x + \delta)^s + L(x), \) where \( L \) is a linearized polynomial; PPs of this form had been explored by different authors in several previous papers [18, 37, 40]. In [1, 38], new PPs were constructed through certain commutative diagrams. Such PPs can also be viewed as piecewise functions (see [1, Lemma 1.1] or [38, Lemma 2.4]) although they are not necessarily of the form (5.1.1).

Returning to Proposition 5.1.1, the challenge is to choose simple functions \( \theta \) and \( f_i \) \( (i = \infty, 0, \ldots, k - 1) \) such that conditions (i) and (ii) are satisfied. The next proposition provides a way to check condition (ii) when \( \theta \) is related to \( f_i. \)

**Proposition 5.1.2** Let \( i, j \in \{\infty, 0, \ldots, k - 1\}, i \neq j. \) Assume that there exist functions \( h_i \) and \( h_j \) from \( \mathbb{F}_q \) to \( \mathbb{F}_q \) such that the following two conditions hold.

(i) \( [(h_i \circ f_i)(x)]^{\frac{1}{2}(q-1)} = \theta(x) \) if \( \theta(x) = \omega^i; \) \( [(h_j \circ f_j)(x)]^{\frac{1}{2}(q-1)} = \theta(x) \) if \( \theta(x) = \omega^j. \)

(ii) If \( b \in f_i(\theta^{-1}(\omega^i)) \cap f_j(\theta^{-1}(\omega^j)), \) then \( (h_i(b))^{\frac{1}{2}(q-1)}, h_j(b))^{\frac{1}{2}(q-1)} \neq (\omega^i, \omega^j). \)

Then \( f_i(\theta^{-1}(\omega^i)) \cap f_j(\theta^{-1}(\omega^j)) = \emptyset. \)

**Proof.** Assume to the contrary that there exists \( b \in f_i(\theta^{-1}(\omega^i)) \cap f_j(\theta^{-1}(\omega^j)). \) Then \( b = f_i(x) = f_j(y) \) for some \( x \in \theta^{-1}(\omega^i) \) and \( y \in \theta^{-1}(\omega^j). \) By (i), \( h_i(b)^{\frac{1}{2}(q-1)} = [(h_i \circ f_i)(x)]^{\frac{1}{2}(q-1)} = \omega^i. \) In the same way, \( h_j(b)^{\frac{1}{2}(q-1)} = \omega^j. \) So we have a contradiction. 

\[ Q.E.D. \]
We will construct several families of PPs of the form (5.1.1) by choosing the selection function \( \theta \) to be \( \theta(x) = (L(x) + \delta)^{\frac{1}{k}(q-1)} \), where \( L(x) \) is a linearized polynomial, or \( \theta(x) = x^{\frac{1}{k}(q-1)} \). The PPs obtained in this paper unify and generalize several existing results, mostly from [39].

5.2 PPs with \( \theta(x) = (L(x) + \delta)^{\frac{1}{k}(q-1)} \)

**Theorem 5.2.1** Let \( k \mid q - 1 \) and let \( \omega \in F_q^* \) be an element of order \( k \). Let \( \sigma_0, \ldots, \sigma_{k-1} \in \text{Aut}(F_q/F_r) \), \( 0 \leq i \leq k - 1 \), are all distinct. Let \( L \) and \( g \) be \( r \)-linearized polynomials over \( F_r \) with \( L(1) = 0, \ g(1) = 1 \) and \( g \) a PP of \( F_q \). Let \( \delta_\infty, \delta_0, \ldots, \delta_{k-1}, \delta \in F_r \). Then

\[
F(x) = (g(x) + \delta_\infty) \left[ 1 - (L(x) + \delta)^{q-1} \right] \\
\quad + \frac{1}{k} \sum_{i=0}^{k-1} \left[ \omega^{-in}(\sigma_0(x) + \delta_0) + \cdots + \omega^{-(k-1)n}(\sigma_{k-1}(x) + \delta_{k-1}) \right] (L(x) + \delta)^{\frac{1}{k}(q-1)}. 
\]

is a PP of \( F_q \).

**Proof.** Let \( \theta(x) = (L(x) + \delta)^{\frac{1}{k}(q-1)}, \ f_\infty(x) = g(x) + \delta_\infty, \ f_i(x) = \sigma_i(x) + \delta_i, \ 0 \leq i \leq k - 1 \). We use Proposition 5.1.2 to show that \( f_i(\theta^{-1}(\omega)) \cap f_j(\theta^{-1}(\omega)) = \emptyset \) for all \( i, j \in \{\infty, 0, \ldots, k-1\}, i \neq j \). Let \( h_\infty(x) = g^{-1}(L(x)) + \delta \) and \( h_i(x) = \sigma_i^{-1}(L(x)) + \delta, \ x \in F_q \). Then

\[
(h_\infty \circ f_\infty)(x) = g^{-1}(L(g(x) + \delta_\infty)) + \delta = L(x) + \delta.
\]

(Note that \( L(\delta_\infty) = 0 \) and \( L \circ g = g \circ L \).) In the same way, \( (h_i \circ f_i)(x) = L(x) + \delta, \ 0 \leq i \leq k - 1 \). Thus

\[
[(h_i \circ f_i)(x)]^{\frac{1}{k}(q-1)} = (L(x) + \delta)^{\frac{1}{k}(q-1)} = \theta(x), \ i \in \{\infty, 0, \ldots, k - 1\}.
\]
Let $b \in \mathbb{F}_q$. For $0 \leq i \leq k - 1$ we have
\[
h_i(b)^{\frac{1}{q-1}} = \left[\sigma_i^{-1}(L(b)) + \delta\right]^{\frac{1}{q-1}} = \left[\sigma_i^{-1}(L(b) + \delta)\right]^{\frac{1}{q-1}}
\]
\[
= \sigma_i^{-1}(L(b) + \delta) = \sigma_i^{-1}(\theta(b)).
\]
(Note that $\sigma_i^{-1}(\delta) = \delta$.) Since $g^{-1}(\delta) = \delta$, we also have
\[
h_\infty(b) = g^{-1}(L(b)) + \delta = g^{-1}(L(b) + \delta).
\]
Now for $i, j \in \{0, \ldots, k - 1\}$, $i \neq j$, we have
\[
\left(h_i(b)^{\frac{1}{q-1}}, h_j(b)^{\frac{1}{q-1}}\right) = \left(\sigma_i^{-1}(\theta(b)), \sigma_j^{-1}(\theta(b))\right) \neq (\omega^i, \omega^j)
\]
since $\sigma_i(\omega^i) \neq \sigma_j(\omega^j)$. Also,
\[
\left(h_\infty(b)^{\frac{1}{q-1}}, h_i(b)^{\frac{1}{q-1}}\right) = \left((g^{-1}(L(b) + \delta))^{\frac{1}{q-1}}, \sigma_i^{-1}(\theta(b))\right) \neq (0, \omega^i)
\]
since $g^{-1}(L(b) + \delta) = 0$ implies $L(b) + \delta = 0$, which implies $\theta(b) = 0$. By Proposition 5.1.2, we have $f_i(\theta^{-1}(\omega^i)) \cap f_j(\theta^{-1}(\omega^j)) = \emptyset$ for all $i, j \in \{\infty, 0, \ldots, k - 1\}$, $i \neq j$.

\[\blacksquare\]

**Remark 5.2.2** In Theorem 5.2.1, $\delta \in \mathbb{F}_r$ can be replaced with an arbitrary function from $\mathbb{F}_q$ to $\mathbb{F}_r$. Also, each $\sigma_i$ can be replaced with $\sigma_i + \beta_i$, where $\beta_i : \mathbb{F}_q \to \ker_{\mathbb{F}_q} L$ is any function such that $\sigma_i + \beta_i$ is a PP of $\mathbb{F}_q$.

**Remark 5.2.3** In Theorem 5.2.1, if we drop the assumption that $\sigma_i(\omega^i)$, $0 \leq i \leq k - 1$, are all distinct, and maintain others, then we can describe a necessary and sufficient condition on $\sigma_0, \ldots, \sigma_{k-1}$ for $F$ to be a PP of $\mathbb{F}_q$. It is clear that $F$ is a PP of $\mathbb{F}_q$ if and only if $f_i(\theta^{-1}(\omega^i)) \cap f_j(\theta^{-1}(\omega^j)) = \emptyset$ for all $i, j \in \{\infty, 0, \ldots, k - 1\}$ with $i \neq j$. From the proof of Theorem 5.2.1, we always have $f_\infty(\theta^{-1}(0)) \cap f_j(\theta^{-1}(\omega^j)) = \emptyset$ for $j \in \{0, \ldots, k - 1\}$. For $0 \leq i < j \leq k - 1$, $f_i(\theta^{-1}(\omega^i)) \cap f_j(\theta^{-1}(\omega^j)) \neq \emptyset$ if and
only if the following system has a solution \((x, y) \in \mathbb{F}_q \times \mathbb{F}_q\):

\[
\begin{align*}
(L(x) + \delta)^{\frac{1}{k}(q-1)} &= \omega^i, \\
(L(y) + \delta)^{\frac{1}{k}(q-1)} &= \omega^j, \\
\sigma_i(x) + \delta_i &= \sigma_j(y) + \delta_j.
\end{align*}
\]

Apply \(\sigma_i\) to the first equation of (5.2.3) and \(\sigma_j\) to the second. We see that (5.2.3) is equivalent to

\[
\begin{align*}
(L(u) + \delta)^{\frac{1}{k}(q-1)} &= \sigma_i(\omega^i), \\
(L(v) + \delta)^{\frac{1}{k}(q-1)} &= \sigma_j(\omega^j), \\
u + \delta_i &= v + \delta_j,
\end{align*}
\]

where \(u = \sigma_i(x), v = \sigma_j(y)\). The third equation of (5.2.4) implies that \(L(u) = L(v)\). Therefore, (5.2.4) has a solution \((u, v) \in \mathbb{F}_q \times \mathbb{F}_q\) if and only if \(\sigma_i(\omega^i) = \sigma_j(\omega^j)\) and \((L(\mathbb{F}_q) + \delta) \cap (\sigma_i(\gamma^i) \cdot \langle \gamma^k \rangle) \neq \emptyset\), where \(\gamma\) is a primitive element of \(\mathbb{F}_q\) such that \(\omega = \gamma^{\frac{1}{k}(q-1)}\). We conclude that \(F\) is a PP of \(\mathbb{F}_q\) if and only if for each pair of distinct integers \(i, j \in \{0, \ldots, k-1\}\), either \(\sigma_i(\omega^i) \neq \sigma_j(\omega^j)\) or \((L(\mathbb{F}_q) + \delta) \cap (\sigma_i(\gamma^i) \cdot \langle \gamma^k \rangle) = \emptyset\).

The construction in Theorem 5.2.1 calls for a sequence \(\sigma_0, \ldots, \sigma_{k-1} \in \text{Aut}(\mathbb{F}_q/\mathbb{F}_r)\) such that \(\sigma_i(\omega^i), 0 \leq i \leq k-1\), are all distinct. All such sequences can be determined by the following method: Write \(q = r^m\), and let \(\sigma \in \text{Aut}(\mathbb{F}_q/\mathbb{F}_r)\) be given by \(\sigma(x) = x^r\).

1. Partition \(\{0,1,\ldots,k-1\}\) into \(r\)-cyclotomic classes modulo \(k\).

2. For each \(r\)-cyclotomic class \([i] = \{ir^0, ir^1, \ldots, ir^{s-1}\}\), choose any permutation \(\beta\) of \(\{0,1,\ldots,s-1\}\), choose \(e_j \in \mathbb{Z}_m, 0 \leq j \leq s-1\), such that \(e_j \equiv \beta(j) - j \pmod{s}\), and choose

\[
\sigma_{ir^j} = \sigma^{e_j}, \quad 0 \leq j \leq s-1.
\]

Note that \(\sigma_{ir^j}(\omega^{ir^j}) = \omega^{ir^j \cdot e_j} = \omega^{ir^j + e_j}\), where \(j + e_j, 0 \leq j \leq s-1\), is a permutation of \(0,1,\ldots,s-1\).

Theorem 5.2.1 allows several variations.
Theorem 5.2.4 Let $k \mid q - 1$ and let $\omega \in \mathbb{F}_q^*$ be an element of order $k$. Let $\mathbb{F}_r \subset \mathbb{F}_q$, $\sigma_0, \ldots, \sigma_{k-1} \in \text{Aut}(\mathbb{F}_q/\mathbb{F}_r)$, $L$ an $r$-linearized polynomial over $\mathbb{F}_r$, and

$$F(x) = \frac{1}{k} \sum_{i=0}^{k-1} \left[ \omega^{-0i} \sigma_0(x) + \cdots + \omega^{-(k-1)i} \sigma_{k-1}(x) \right] L(x)^\frac{i}{k}(q-1).$$

Then $F$ is a PP of $\mathbb{F}_q$ if and only if $L$ is a PP of $\mathbb{F}_q$ and $\sigma_i(\omega^j)$, $0 \leq i \leq k - 1$, are all distinct.

Proof.

$(\Leftarrow)$ Let $\theta(x) = L(x)^\frac{1}{k}(q-1)$, $f_\infty(x) = 0$ and $f_i(x) = \sigma_i(x)$, $0 \leq i \leq k - 1$. Note that $\theta^{-1}(0) = 0$. One only has to verify $f_i(\theta^{-1}(\omega^j)) \cap f_j(\theta^{-1}(\omega^j)) = \emptyset$ for $0 \leq i < j \leq k - 1$, which follows from the proof of Theorem 5.2.1.

$(\Rightarrow)$ Since $F$ has only one root in $\mathbb{F}_q$, $L$ must be a PP of $\mathbb{F}_q$. Assume to the contrary that $\sigma_i(\omega^j) = \sigma_j(\omega^j)$ for some $0 \leq i < j \leq k - 1$. Let $\gamma$ be a primitive element of $\mathbb{F}_q$ such that $\omega = \gamma^\frac{1}{k}(q-1)$. Then

$$\left( \frac{\sigma_i(\gamma^j)}{\sigma_j(\gamma^j)} \right)^\frac{q-1}{k} = \frac{\sigma_i(\omega^j)}{\sigma_j(\omega^j)} = 1.$$

Hence we can write $\frac{\sigma_i(\gamma^j)}{\sigma_j(\gamma^j)} = \sigma_i(\gamma^j)^l$ for some $l \in \mathbb{Z}$. Thus $\sigma_i(\gamma^{i-lk}) = \sigma_j(\gamma^j)$. Let $x = L^{-1}(\gamma^{i-lk})$ and $y = L^{-1}(\gamma^j)$. Then $\theta(x) = L(L^{-1}(\gamma^{i-lk}))^\frac{1}{k}(q-1) = \omega^i$ and $\theta(y) = L(L^{-1}(\gamma^j))^\frac{1}{k}(q-1) = \omega^j$. We have

$$F(x) = \sigma_i(x) = L^{-1}(\sigma_i(\gamma^{i-lk})) = L^{-1}(\sigma_j(\gamma^j)) = \sigma_j(y) = F(y),$$

which is a contradiction.


Theorem 5.2.5 Let $k \mid q - 1$ and let $\omega \in \mathbb{F}_q^*$ be an element of order $k$. Let $\mathbb{F}_r \subset \mathbb{F}_q$, and $\sigma_0, \ldots, \sigma_{k-1} \in \text{Aut}(\mathbb{F}_q/\mathbb{F}_r)$ such that $\sigma_i(\omega^i)$, $0 \leq i \leq k - 1$, are all distinct. Let $L$ be an $r$-linearized polynomial over $\mathbb{F}_r$ and let $\delta_0, \ldots, \delta_{k-1} \in \mathbb{F}_q$ and $\delta \in \mathbb{F}_q \setminus L(\mathbb{F}_q)$,
such that $L(\delta_i) - \sigma_i(\delta), 0 \leq i \leq k - 1$, are all equal. Then

$$F(x) = \frac{1}{k} \sum_{i=0}^{k-1} \left[ \omega^{-0i}(\sigma_0(x) + \delta_0) + \cdots + \omega^{-(k-1)i}(\sigma_{k-1}(x) + \delta_{k-1}) \right] (L(x) + \delta) \frac{1}{1}^{(q-1)}$$

is a PP of $\mathbb{F}_q$.

Proof. Let $\theta(x) = (L(x) + \delta) \frac{1}{1}^{(q-1)}$ and $f_i(x) = \sigma_i(x) + \delta_i, 0 \leq i \leq k - 1$. It suffices to show that $f_i(\theta^{-1}(\omega^i)) \cap f_j(\theta^{-1}(\omega^j)) = \emptyset$ for $0 \leq i < j \leq k - 1$.

Let $h_i(x) = \sigma_i^{-1}(L(x - \delta_i)) + \delta, 0 \leq i \leq k - 1$. Then

$$\left[ (h_i \circ f_i)(x) \right] \frac{1}{1}^{(q-1)} = \left[ (\sigma_i^{-1} \circ L \circ \sigma_i)(x) + \delta \right] \frac{1}{1}^{(q-1)} = [L(x) + \delta] \frac{1}{1}^{(q-1)} = \theta(x).$$

For each $b \in \mathbb{F}_q$ we have

$$h_i(b) = \sigma_i^{-1}(L(b - \delta_i)) + \delta = \sigma_i^{-1}(L(b) - L(\delta_i) + \sigma_i(\delta)) = \sigma_i^{-1}(c),$$

where $c = L(b) - L(\delta_i) + \sigma_i(\delta)$ is independent of $i$. So for $0 \leq i < j \leq k - 1$,

$$\left( h_i(b) \frac{1}{1}^{(q-1)}, h_j(b) \frac{1}{1}^{(q-1)} \right) = \left( \sigma_i^{-1}(c \frac{1}{1}^{(q-1)}), \sigma_j^{-1}(c \frac{1}{1}^{(q-1)}) \right) \neq (\omega^i, \omega^j)$$

since $\sigma_i(\omega^i) \neq \sigma_j(\omega^j)$. By Proposition 5.1.2, $f_i(\theta^{-1}(\omega^i)) \cap f_j(\theta^{-1}(\omega^j)) = \emptyset$.

Given $\sigma_0, \ldots, \sigma_{k-1} \in \text{Aut}(\mathbb{F}_q/\mathbb{F}_r)$ and an $r$-linearized polynomial $L$ over $\mathbb{F}_r$, the construction in Theorem 5.2.5 calls for solutions $(\delta_0, \ldots, \delta_{k-1}, \delta) \in \mathbb{F}_q^k \times (\mathbb{F}_q \setminus L(\mathbb{F}_q))$ of the system

$$L(\delta_i) - \sigma_i(\delta) = L(\delta_0) - \sigma_0(\delta), \quad 1 \leq i \leq k - 1. \quad (5.2.5)$$

Let $x_i = \delta_i - \delta_0, 1 \leq i \leq k - 1$, and $x_k = \sigma_0(\delta)$. Then (5.2.5) becomes

$$L(x_i) = \sigma_i \sigma_0^{-1}(x_k) - x_k, \quad 1 \leq i \leq k - 1, \quad (5.2.6)$$

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and we seek its solutions \((x_1, \ldots, x_{k-1}, x_k) \in \mathbb{F}_q^{k-1} \times (\mathbb{F}_q \setminus L(\mathbb{F}_q))\). Write \(q = r^m\), and let \(\sigma \in \text{Aut}(\mathbb{F}_q/\mathbb{F}_r)\) be given by \(\sigma(x) = x^r\). Write \(\sigma_i \sigma_0^{-1} = \sigma^{e_i}, 1 \leq i \leq k - 1\), and \(L = f(\sigma)\), where \(f = a_0 + \cdots + a_{m-1}x^{m-1} \in \mathbb{F}_r[x]\). Then (5.2.6) becomes

\[
f(\sigma)(x_i) = (\sigma^{e_i} - \sigma^0)(x_k), \quad 1 \leq i \leq k - 1.
\]

All solutions \((x_1, \ldots, x_{k-1}, x_k) \in \mathbb{F}_q^{k-1} \times (\mathbb{F}_q \setminus L(\mathbb{F}_q))\) of (5.2.7) can be generated by the following method: Let \(\epsilon \in \mathbb{F}_q\) such that \(\sigma^0(\epsilon), \ldots, \sigma^{m-1}(\epsilon)\) is a normal basis of \(\mathbb{F}_q\) over \(\mathbb{F}_r\).

1. Choose \(g \in \mathbb{F}_r[x]\) such that \(\text{deg } g \leq m - 1\), \(\gcd(f, x^m - 1) \nmid g\), and \(\gcd(f, x^m - 1) \mid (x^{e_i} - 1)g\) for all \(1 \leq i \leq k - 1\). (If such \(g\) does not exist, (5.2.7) has no solution \((x_1, \ldots, x_{k-1}, x_k) \in \mathbb{F}_q^{k-1} \times (\mathbb{F}_q \setminus L(\mathbb{F}_q)).\) Set

\[
x_k = g(\sigma)(\epsilon).
\]

2. Write \(\gcd(f, x^m - 1) \equiv uf \pmod{x^m - 1}, u \in \mathbb{F}_r[x]\), and let

\[
h_i = u \cdot \frac{(x^{e_i} - 1)g}{(f, x^m - 1)}, \quad 1 \leq i \leq k - 1.
\]

Set

\[
x_i = h_i(\epsilon), \quad 1 \leq i \leq k - 1.
\]

**Remark.**

(i) In Theorem 5.2.5, let \(q\) be odd, \(k = 2\), \(\sigma\) a generator of \(\text{Aut}(\mathbb{F}_q/\mathbb{F}_r)\), \(L(x) = \sigma(x) - x, \delta \in \mathbb{F}_q\) with \(\text{Tr}_{\mathbb{F}_q/\mathbb{F}_r}(\delta) \neq 0, \sigma_0 = \sigma, \delta_0 = \delta, \sigma_1 = \text{id}, \delta_1 = -\delta\). Then

\[
F(x) = \frac{1}{2}[(\sigma(x) - x + \delta)^{\frac{1}{2}(q+1)} + \sigma(x) + x],
\]

which is the PP in [39, Theorem 1].

(ii) In Theorem 5.2.1 and Remark 5.2.2, let \(q = 3^m, r = 3^l, k = 2\), \(\sigma\) a generator of \(\text{Aut}(\mathbb{F}_q/\mathbb{F}_r)\), \(L = \sigma - \text{id}, \delta \in \mathbb{F}_r, g = \text{id}, \delta_\infty = 0, \sigma_0 = \sigma, \beta_0(x) = 0, \delta_0 = \delta, \sigma_1 = \sigma^2, \beta_1(x) = -\text{Tr}_{\mathbb{F}_q/\mathbb{F}_r}(x), \delta_1 = -\delta\). (Note that \(\sigma_1 + \beta_1 = \sigma^2 - \text{Tr}_{\mathbb{F}_q/\mathbb{F}_r}\) is a
PP of \( \mathbb{F}_q \) and \( L \circ \beta_1 = 0 \) on \( \mathbb{F}_q \).) Then \( F(x) = (\sigma(x) - x + \delta)^{\frac{1}{2}(q+1)} + x \) is a PP of \( \mathbb{F}_q \).

Let \( g = \sigma, \delta_\infty = 0, \sigma_0 = \sigma^2, \beta_0(x) = -\text{Tr}_{\mathbb{F}_q/\mathbb{F}_r}(x), \delta_0 = \delta, \sigma_1 = \text{id}, \beta_1(x) = 0, \delta_1 = -\delta. \) Then \( F(x) = (\sigma(x) - x + \delta)^{\frac{1}{2}(q+1)} + \sigma(x) \) is a PP of \( \mathbb{F}_q \).

Let \( g = \sigma^2, \delta_\infty = 0, \sigma_0 = \text{id}, \beta_0(x) = \text{Tr}_{\mathbb{F}_q/\mathbb{F}_r}(x), \delta_0 = \delta, \sigma_1 = \sigma, \beta_1(x) = \text{Tr}_{\mathbb{F}_q/\mathbb{F}_r}(x), \delta_1 = -\delta. \) Then \( F(x) = (\sigma(x) - x + \delta)^{\frac{1}{2}(q+1)} + \sigma^2(x) \) is a PP of \( \mathbb{F}_q \).

These are the PPs in [39, Theorem 2].

5.3 PPs with \( \theta(x) = x^{\frac{1}{k}(q-1)} \)

**Theorem 5.3.1** Let \( k \mid q - 1 \) and let \( \omega \in \mathbb{F}_q^* \) be an element of order \( k \). Let

\[
F(x) = \frac{1}{k} \sum_{i=0}^{k-1} \left[ \omega^{-0i} x^{a_0} + \cdots + \omega^{-(k-1)i} x^{a_{k-1}} \right] x^{\frac{1}{k}(q-1)}, \tag{5.3.8}
\]

where \( a_0, \ldots, a_{k-1} \in \mathbb{Z}_{q-1}. \) Then \( F \) is a PP of \( \mathbb{F}_q \) if and only if \( \gcd(a_i, \frac{1}{k}(q-1)) = 1 \) for all \( 0 \leq i \leq k-1 \) and \( ia_i, 0 \leq i \leq k-1, \) are all distinct in \( \mathbb{Z}_k \).

**Proof.** \((\Leftarrow)\) Let \( \theta(x) = x^{\frac{1}{k}(q-1)}, f_\infty = 0, \) and \( f_i(x) = x^{a_i}, 0 \leq i \leq k-1. \)

First we show that \( f_i \) is 1-1 on \( \theta^{-1}(\omega^i) \). Let \( x_1, x_2 \in \theta^{-1}(\omega^i) \) such that \( f_i(x_1) = f_i(x_2). \) Then \((\frac{x_1}{x_2})^{a_i} = 1. \) Also,

\[
\left( \frac{x_1}{x_2} \right)^{\frac{1}{k}(q-1)} = \frac{x_1^{\frac{1}{k}(q-1)}}{x_2^{\frac{1}{k}(q-1)}} = \frac{\omega^i}{\omega^i} = 1.
\]

Since \( \gcd(a_i, \frac{1}{k}(q-1)) = 1, \) we have \( \frac{x_1}{x_2} = 1. \)

Now we show that \( f_i(\theta^{-1}(\omega^i)) \cap f_j(\theta^{-1}(\omega^j)) = \emptyset \) for \( 0 \leq i < j \leq k-1. \) Assume to the contrary that there exists \( b \in f_i(\theta^{-1}(\omega^i)) \cap f_j(\theta^{-1}(\omega^j)). \) Then \( b = f_i(x) = f_j(y) \) for some \( x \in \theta^{-1}(\omega^i) \) and \( y \in \theta^{-1}(\omega^j). \) We have

\[
b^{\frac{1}{k-1}} = (x^{a_i})^{\frac{1}{k-1}} = (x^{a_j})^{a_i} = \omega^{ia_i}.
\]

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In the same way $b^{\frac{q-1}{k}} = \omega^{ja_j}$. Thus $ia_i = ja_j$ in $\mathbb{Z}_k$, which is a contradiction.

$(\Rightarrow)$ First assume that $\gcd(a_i, \frac{1}{k}(q-1)) = l > 1$ for some $0 \leq i \leq k - 1$. Let $\epsilon \in \mathbb{F}_q^*$ such that $o(\epsilon) = l$. Then for any $x \in \theta^{-1}(\omega^i)$, we have $\epsilon x \in \theta^{-1}(\omega^i)$ and $(\epsilon x)^{a_i} = x^{a_i}$. Thus $F(x) = F(\epsilon x)$, where $x \neq \epsilon x$, which is a contradiction.

Next assume that $ia_i = ja_j$ in $\mathbb{Z}_k$ for some $0 \leq i < j \leq k - 1$. Let $\gamma$ be a primitive element of $\mathbb{F}_q$ such that $\omega = \gamma^{\frac{1}{k}(q-1)}$. Then $\gamma^i \in \theta^{-1}(\omega^i)$, $\gamma^j \in \theta^{-1}(\omega^j)$, and $(\gamma^i)^{a_i} = (\gamma^j)^{a_j}$. Hence $F(\gamma^i) = F(\gamma^j)$, which is a contradiction.

For $k \mid q - 1$, let $A_{q,k}$ denote the set of all sequences $(a_0, \ldots, a_{k-1}) \in \mathbb{Z}_{q-1}^k$ such that $\gcd(a_i, \frac{1}{k}(q-1)) = 1$ for all $0 \leq i \leq k - 1$, and $ia_i, 0 \leq i \leq k - 1$, are all distinct in $\mathbb{Z}_k$. For each $d \mid k$, let $\pi_d : \mathbb{Z}_{q-1} \to \mathbb{Z}_{k/d}$ be the canonical homomorphism. Each element of $A_{q,k}$ is generated exactly once through the following steps.

1. For each $d \mid k$, choose a permutation $\tau_d$ of $\mathbb{Z}_{k/d}^*$.

2. For each $0 \leq i \leq k - 1$, let

   $$\alpha_i = \left(\frac{i}{(i,k)}\right)^{-1} \tau_{(i,k)} \left(\frac{i}{(i,k)}\right) \in \mathbb{Z}_{k/(i,k)}^*.$$ 

   (Note that in $\mathbb{Z}_k$, $i\alpha_i = (i,k)\tau_{(i,k)}\left(\frac{i}{(i,k)}\right), 0 \leq i \leq k - 1$, which are all distinct.)

3. For each $0 \leq i \leq k - 1$, choose $a_i \in \pi_{(i,k)}^{-1}(\alpha_i)$ such that $\gcd(a_i, \frac{q-1}{k}) = 1$.

   The number of choices in Step 1 is $\prod_{d \mid k} \phi\left(\frac{k}{d}\right)! = \prod_{d \mid k} \phi(d)!$. Counting the number of choices in Step 3 requires some effort.

   For positive integers $m \mid n$ define

   $$h(m, n) = \left| \left\{ x \in \mathbb{Z}_{\frac{n}{m}} : \gcd(1 + mx, n) = 1 \right\} \right|.$$ 

   This function can be explicitly determined in terms of the prime factorizations of $m$ and $n$. 

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Lemma 5.3.2 Let $n = p_1^{e_1} \cdots p_s^{e_s}$, $m = p_1^{f_1} \cdots p_s^{f_s}$, where $p_1, \ldots, p_s$ are distinct primes and $e_i > 0$, $0 \leq f_i \leq e_i$. Without loss of generality, assume $f_1 = \cdots = f_t = 0$, $f_{t+1}, \ldots, f_s > 0$. Then

$$h(m, n) = \frac{n}{m} \left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_t}\right).$$

Proof. For $1 \leq i_1 < \cdots < i_t \leq t$, we have

$$\left|\{x \in \mathbb{Z}_{\frac{n}{m}} : 1 + mx \equiv 0 \pmod{p_{i_1} \cdots p_{i_t}}\}\right| = \frac{n}{m} \cdot \frac{1}{p_{i_1} \cdots p_{i_t}}.$$

By the inclusion-exclusion formula,

$$h(m, n) = \frac{n}{m} \sum_{t=0}^{t} (-1)^t \sum_{1 \leq i_1 < \cdots < i_t \leq t} \frac{1}{p_{i_1} \cdots p_{i_t}} = \frac{n}{m} \left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_t}\right).$$

It is quite clear that for any two positive integers $m$ and $n$ and $\alpha \in \mathbb{Z}_n^*$,

$$\left|\{x \in \mathbb{Z}_{\frac{n}{(m,n)}} : \gcd(\alpha + mx, n) = 1\}\right| = h((m, n), n).$$

Using this notation, we see that in the above Step 3, for each $0 \leq i \leq k - 1$, the number of choices for $a_i$ is

$$\left|\{x \in \mathbb{Z}_{\frac{q-1}{k(i,k)}} : \gcd\left(a_i + \frac{k}{(i,k)} x, \frac{q-1}{k}\right) = 1\}\right|$$

$$= \left|\{x \in \mathbb{Z}_{\frac{q-1}{(i,k)}} : \gcd\left(a_i + \frac{k}{(i,k)} x, \frac{q-1}{k}\right) = 1\}\right| (i,k) \left(\frac{k}{(i,k)}, \frac{q-1}{k}\right)$$

$$= (k, \frac{q-1}{k}) h\left(\left(\frac{k}{(i,k)}, \frac{q-1}{k}\right), \frac{q-1}{k}\right).$$

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Therefore the total number of choices in Step 3 is
\[
\prod_{0 \leq i \leq k-1} \left( k, \frac{q-1}{k} (i, k) \right) h\left( \left( \frac{k}{(i, k)}, \frac{q-1}{k} \right) \right)
\]
\[
= \prod_{d \mid k} \left( k, \frac{q-1}{k} d \right) h\left( \left( \frac{k}{d}, \frac{q-1}{k} \right) \right)^{\phi(d)}
\]
\[
= \prod_{d \mid k} \left( k, \frac{q-1}{d} d \right) h\left( \left( \frac{d}{k}, \frac{q-1}{k} \right) \right)^{\phi(d)}.
\]
Thus
\[
|A_{q,k}| = \prod_{d \mid k} \left( \left( k, \frac{q-1}{d} d \right) h\left( \left( \frac{d}{k}, \frac{q-1}{k} \right) \right)^{\phi(d)} \right) !.
\]

Denote the function in (5.3.8) by \( F_f \), where \( f : \mathbb{Z}_k \to \mathbb{Z}_{q-1}, \ f(i) = a_i \). Let \( \mathcal{F} = \{ f : \mathbb{Z}_k \to \mathbb{Z}_{q-1} : (f(0), \ldots, f(k-1)) \in A_{q,k} \} \). Then \( G := \{ F_f : f \in \mathcal{F} \} \) is a subgroup of the symmetric group \( \text{Sym}(\mathbb{F}_q) \). The composition in \( G \) is given by
\[
F_g \circ F_f = F_h,
\]
where
\[
h(i) = f(i)g(i \overline{f(i)}), \quad i \in \mathbb{Z}_k,
\]
and \( \overline{f(i)} \) is the image of \( f(i) \) in \( \mathbb{Z}_k \).

Now we determine the order of \( G \). Note that \( \theta^{-1}(\omega^j) = \{ x \in \mathbb{F}_q : x^{\frac{1}{k}(q-1)} = \omega^j \} = \{ \alpha^{j+l}: 0 \leq l < \frac{1}{k}(q-1) \} \), where \( \alpha \) is a primitive element of \( \mathbb{F}_q \) such that \( \omega = \alpha^{\frac{1}{k}(q-1)} \). For \( a, a' \in \mathbb{Z}_{q-1} \), we have
\[
x^a = x^{a'} \text{ for all } x \in \theta^{-1}(\omega^j)
\]
\[
\Leftrightarrow (a - a')(j + kl) \equiv 0 \pmod{q-1} \text{ for all } 0 \leq l < \frac{1}{k}(q-1)
\]
\[
\Leftrightarrow (a - a')j \equiv (a - a')k \equiv 0 \pmod{q-1}
\]
\[
\Leftrightarrow (a - a')(j, k) \equiv 0 \pmod{q-1}
\]
\[
\Leftrightarrow a - a' \equiv 0 \pmod{\frac{q-1}{(j, k)}}.
\]
It is clear that the mapping

\[ F \rightarrow G \]

\[ f \mapsto F_f \]

is \( \prod_{j=0}^{k-1} (j, k) \) to 1. Thus

\[ |G| = \frac{|F|}{\prod_{j=0}^{k-1} (j, k)} = \frac{|A_{q,k}|}{\prod_{d|k} d^{\phi(k/d)}} = \frac{|A_{q,k}|}{\prod_{d|k} (\frac{k}{d})^{\phi(d)}} \]

\[ \prod_{d|k} \left[ (d, \frac{q-1}{k}) h \left( \left( d, \frac{q-1}{k} \right), \frac{q-1}{k} \right) \right]^{\phi(d)} \phi(d)! \].

In Theorem 5.3.1, one can replace each \( x^{a_i} \) with \( c_i x^{a_i} \), where \( c_i \in \mathbb{F}_q^* \) is a \( k \)th power. The following theorem offers a more substantial extension of Theorem 5.3.1.

**Theorem 5.3.3** Let \( q, k, \omega, a_0, \ldots, a_{k-1} \) be as in Theorem 5.3.1. For each \( 0 \leq i \leq k-1 \), let \( r_i \) be a power of \( p \) such that \( k \mid r_i - 1 \) and \( b_i \in \mathbb{F}_q^* \) such that \( (-b_i)^{\frac{q-1}{k}} \neq \omega^{ia_i} \). Then

\[ F(x) = \frac{1}{k} \sum_{i=0}^{k-1} \left[ \omega^{-0i} x^{a_0} (x^{a_0} + b_0)^{r_0-1} + \cdots + \omega^{-(k-1)i} x^{a_{k-1}} (x^{a_{k-1}} + b_{k-1})^{r_{k-1}-1} \right] x^{i/q-1} \]

(5.3.9)

is a PP of \( \mathbb{F}_q \).

**Proof.** Let \( \theta(x) = x^{\frac{1}{k}(q-1)} \), \( f_i(x) = x^{a_i} (x^{a_i} + b_i)^{r_i-1} \), \( 0 \leq i \leq k-1 \). By the proof of Theorem 5.3.1, we have \( f_i(\theta^{-1}(\omega^i)) \cap f_j(\theta^{-1}(\omega^j)) = \emptyset \) for \( 0 \leq i < j \leq k-1 \). It remains to show that \( f_i \) is 1-1 on \( \theta^{-1}(\omega^i) \). Assume to the contrary that there exist \( x, y \in \theta^{-1}(\omega^i), x \neq y \), such that

\[ x^{a_i} (x^{a_i} + b_i)^{r_i-1} = y^{a_i} (y^{a_i} + b_i)^{r_i-1}. \]

Write \( r = r_i, u = \frac{x^{a_i}}{b_i}, v = \frac{y^{a_i}}{b_i} \). Then we have

\[ u(u+1)^{r-1} = v(v+1)^{r-1}. \]
Thus
\[(v + 1)u(u + 1)^r - (u + 1)v(v + 1)^r = 0.\]

The left side of the above equation equals
\[(u - v)(v + 1)[u(u - v)^{r-1} + (v + 1)^{r-1}]\).

Note that \(v + 1 \neq 0\) since otherwise, \(y^a_i \neq v = -1\), which implies that \((-b_i)^\frac{q-1}{r} = (y^a_i)^\frac{q-1}{r} = (y^\frac{q-1}{r})^a_i = \omega^a_i\), which is a contradiction. (This is perhaps an overkill. Since \(u \neq v\), we may assume that one of \(u\) and \(v\), say \(v\), is not \(-1\).) Now we have
\[u(u - v)^{r-1} + (v + 1)^{r-1} = 0,
\]
i.e.,
\[-u = \left(\frac{v + 1}{u - v}\right)^{r-1}.
\]
It follows that
\[-\left(\frac{x^a_i}{b_i}\right)^\frac{q-1}{r} = (-u)^\frac{q-1}{r} = \left(\frac{v + 1}{u - v}\right)^{(r-1)^\frac{q-1}{r}} = 1.
\]
Thus \((-b_i)^\frac{q-1}{r} = (x^\frac{q-1}{r})^a_i = \omega^ia_i\), which is a contradiction.

Remark.

(i) In Theorem 5.3.1, let \(q\) be odd, \(k = 2\), \(a_0 = t + l\), \(a_1 = l\), where \(gcd(l, q - 1) = 1\) and \(gcd(t + l, \frac{1}{2}(q - 1)) = 1\). The result is [39, Theorem 8].

(ii) In Theorem 5.3.1, let \(q\) be a power of a prime \(p\) with \(3 \mid q - 1\). Let \(k = 3\), \(a_0 = 1\), \(a_1 = 3 + \frac{2}{3}(q - 1)\), \(a_2 = p + \frac{1}{3}(q - 1)\), and assume that \(p \equiv 1 \pmod{3}\) and \(q \equiv 4 \pmod{9}\), or \(p \equiv 2 \pmod{3}\) and \(q \equiv 7 \pmod{9}\). The result is [39, Theorem 9].

(iii) In Theorem 5.3.1 let \(q\) be a power of a prime \(p\) such that \(q \equiv 1 \pmod{9}\). Let \(k = 3\), \(a_0 = 1\), \(a_1 = p^j + \frac{2}{3}(q - 1)\), \(a_2 = p + \frac{1}{3}(q - 1)\), and assume \(p^{j-1} \equiv 1 \pmod{3}\). The result is [39, Theorem 10].
(iv) In Theorem 5.3.1, let $q$ be a power of a prime $p$ with $p \equiv 1 \pmod{k}$, $q \equiv 1 \pmod{k^2}$, and let $a_i = p^i - \frac{q-1}{k}$, $0 \leq i \leq k-1$. The result is [39, Theorem 11].

(v) In Theorem 5.3.3, let $q = 3^n$, $n$ even, $k = 2$, $a_0 = 3$, $r_0 = 1$, $a_1 = 1$, $r_1 = 3$, $b_1 = 1$. The result is [21, Theorem 2.1].

(vi) In Theorem 5.3.3, let $q = 3^n$, $k = 2$, $a_0 = t$, where $\gcd\left(t, \frac{1}{2}(q - 1)\right) = 1$, $r_0 = 1$, $a_1 = 1$, $r_1 = 3$, $b_1 = -\epsilon$, where $\epsilon$ is a square of $\mathbb{F}_q^*$. The result is [39, Proposition 1].
6 Conclusion

One of the goals of this dissertation was to explore the permutation behavior of the polynomial \( g_{n,q} \) further and answer many questions about \( g_{n,q} \) that were not discussed in [22]. Many articles on permutation polynomials introduce necessary and sufficient conditions to construct permutation polynomials. In Chapters 2, 3 and 4, we explained the naturally existing families of permutation polynomials in the form of \( g_{n,q} \).

In Chapter 2, we explained the case \( e = 1 \) and several unexplained desirable triples in [22]. There are still many uncategorized cases in Table 2.1 and most of them occur when \( e = 3 \) and a few with \( e = 4 \). All desirable triples are categorized when \( e = 5, 6 \). Perhaps this is an indication that permutation property of \( g_{n,q} \) is easier to understand when \( e \) is large. However, we still do not know if the triple \((407, 3; 3)\) belongs to a family. For the time being, we believe that it is a sporadic case.

In Chapter 3, we were in new fronts and answered many questions about \( g_{n,q} \) where \( n \) is of the form \( n = q^a - q^b - 1 \). There are still many desirable triples in Table 3.2 for which no theoretic explanation has been found. Conjecture 3.1.1, and 3.1.4 are of more interest in future research in the polynomial \( g_{n,q} \). Conjecture 3.2.6 has recently been proved in [23] and its proof has led to the discovery of a hypergeometric identity.

In Chapter 4, we found many categorized cases that explained almost all desirable triples in Table 4.1. Conjecture 4.2.21 is clearly an indication that the unexplained cases in even characteristic seem to be more interesting and challenging.

One of the challenges among the remaining problems of \( g_{n,q} \) is to find a criterion for \( g_{m,q} \) and \( g_{n,q} \) to represent the same function on \( \mathbb{F}_{q^e} \), i.e., \( g_{m,q} \equiv g_{n,q} \) (mod \( x^{q^e} - x \)).
When $q = 2$, this problem has been answered in [24]. For the general case, there have only been some partial results; see [22, §4].

Computer search results have been a major tool in our effort to find new families of desirable triples of $g_{n,q}$. For example, the conjectures stated in this dissertation would not have been possible without computer search results.

Constructing permutation polynomials has been in literature for some time now and the piecewise construction had been the main focus in several recently published articles. The piecewise approach that we explained in Chapter 5 generalized several recently discovered families of permutation polynomials.
References


Appendix A - Mathematica Codes for \( g_{n,q} \)

Here we present some useful Mathematica codes used to identify the permutation behavior of the polynomial \( g_{n,q} \). Run the following command each time before you execute each code.

`Clear["Global`*"]`

**Mathematica Code 1**

The following program code, called the Fast Algorithm Code, generates the polynomial \( g_{n,q} \) for any given \( n, e, \) and \( q \) in a very short time.

```mathematica
q = ; (* input q *)
list = Flatten[FactorInteger[q]]
p = list[[1]];
e = ; (* input e*)
n = ; (* input n *)
list = {};
m = Length[IntegerDigits[n, q]];
a = IntegerDigits[n, q];
nk = a[[1]];
For[u = 0, u <= q - 1, u++,
    If[u == q - 1, g[u] = -1, g[u] = 0];
];
For[t = q, t <= 2 q, t++,
    g[t] = PolynomialMod[x* g[t - q] + g[t - q + 1], x^q^e - x, Modulus -> p];
];
For[ k = 1, k <= m - 1, k++,
For[i = 0, i <= q - 1, i++,
    g[q*nk + i*q] =
]```
Appendix A (Continued)

PolynomialMod[ g[nk + i]^q, x^q^e - x, Modulus -> p];
];
For[j = 1, j <= q - 1, j++,
  For[l = 0, l <= q - 1 - j, l++,
    g[q*nk + j + l*q] =
    PolynomialMod[ (-x* g[q*nk + j - 1 + l*q]) +
      g[q*nk + j - 1 + (l + 1)*q], x^q^e - x, Modulus -> p];
  ];
];
For[s = 2, s <= q, s++,
  For[h = 1, h <= s - 1, h++,
    g[q*nk + s*q - h] =
    PolynomialMod[
      x* g[q*nk + s*q - h - q] + g[q*nk + s*q - h - q + 1],
      x^q^e - x, Modulus -> p];
  ];
];
nk = q*nk + a[[k + 1]];
];
Print["n = ", n];
Print[g[n]];
Mathematica Code 2

The following code was executed to generate the desirable triples \((n,e;3)\) in Table 2.1 by changing the values of “e” and list “M” accordingly.

```math
\text{e = ; (* input e *)}
q = 3^\text{e};
f0 = 0;
f1 = 0;
f2 = 2;
M = \{0, 1, 2\};
n0 = 2; (* n0 = last n *)
For [n = 3, n < 3^(3\text{e}) - 1, n++,
(* Checking if n is the smallest in the cyclotomic class *)
m = \text{Min[Mod[3^M*n, 3^(3 \text{e}) - 1]]};
If [n != m, \text{Continue[]}];
For[k = n0 + 1, k <= n, k++,
  f = x*f0 + f1;
  f = \text{PolynomialMod}[f, x^q - x, \text{Modulus \rightarrow 3}] ;
  f0 = f1;
  f1 = f2;
  f2 = f;]
];
n0 = n;
(* Hermite’s criterion *)
IsPP = True;
h = 1;
For [i = 1, i < q - 1, i++,
  h = \text{PolynomialMod}[h*f, x^q - x, \text{Modulus \rightarrow 3}] ;
  If[\text{Exponent}[h, x] > q - 2, IsPP = False; \text{Goto[step3]}];
```

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Appendix A (Continued)

\[ h = \text{PolynomialMod}[h*f, x^q - x, \text{Modulus} \to 3]; \]
\[ \text{If}[\text{Exponent}[h, x] \neq q - 1, \text{IsPP} = \text{False}]; \]
\[ \text{Label}[\text{step3}]; \]
\[ \text{If}[\text{IsPP}, \text{Print}[[n, " ", \text{IntegerDigits}[n, 3]]; \text{Print}[f]];]; \]

Mathematica Code 3

The following code was executed to generate the desirable triples \((q^a - q^b - 1, 2; q)\), \(q \leq 97\), \(0 < b < a < 2p\), \(b\) odd, \(b \neq p\).

\[
\text{list1} = \{\};
\text{list2} = \{\};
\text{list3} = \{\};
\text{e} = 2;
\text{For}[k = 1, k <= 15, k++,
\text{(* Checking if } k \text{ is prime *)}
\text{If}[\text{PrimeQ}[k] \text{ || PrimePowerQ}[k], q = k, \text{Continue[]}];
\text{list1} = \text{Flatten}[\text{FactorInteger}[q]];
\text{p} = \text{list1}[[1]];
\text{Print}["q = ", q];
\text{For}[b = 1, b < p*e, b++,
\text{If}[\text{! OddQ}[b], \text{Continue[]}];
\text{If}[b == p, \text{Continue[]}]; (* avoid the case } b = p \)*
\text{For}[a = b + 1, a < p*e, a++,
\text{(* finding coefficients } a0, a1, b0 \text{ and } b1*)
\text{list2} = \text{QuotientRemainder}[b, e];
b0 = \text{list2}[[2]];
b1 = \text{list2}[[1]];
\text{list3} = \text{QuotientRemainder}[a - b, e];
\]
a0 = list3[[2]];  
a1 = list3[[1]];  

S = Sum[x^q^i, {i, 0, e - 1}];  
S1 = Sum[x^q^i, {i, 0, a0 - 1}];  
S2 = Sum[x^q^i, {i, 0, b0 - 1}];  

g = -x^(q^e - 2) -  
  x^(q^e - q^b0 - 2)*(a1*S + S1^q^b0)*((b1*S + S2)^(q - 1) - 1);  

(* Hermite's criterion *)  
IsPP = True;  
f = PolynomialMod[g, x^q^e - x, Modulus -> p];  
h = 1;  
For [i = 1, i < q^e - 1, i++,  
  h = PolynomialMod[h*f, x^q^e - x, Modulus -> p];  
  If[Exponent[h, x] > q^e - 2, IsPP = False; Goto[step3]];  
];  
h = PolynomialMod[h*f, x^q^e - x, Modulus -> p];  
If[Exponent[h, x] != q^e - 1 , IsPP = False];  
Label[step3];  
If[IsPP, Print["a = ", a , " b = ", b]];  
];
When $q > 3$ is odd,
\[
g(y)^2q^2 + 2 \equiv 8y^{-1+q^3} + 2y^{-3+q^3} + y^{-3} - 4q^2 + q^3 + 2y^{-4-4q^2 + q^3} + 4y^{-2+q-4q^2 + q^3} + 6y^{-1+2q-4q^2 + q^3} + 5y^{-1+q^2} + q^3 + 2y^{-4+3q^2 + q^3} + 4y^{-2+3q^2 + q^3} + 2y^{-3-2q^2 + q^3} + 6y^{-1+2q^2 + q^3} + 2y^{-3-2q^2 + q^3} + 2y^{-1-2q^2 + q^3} + 2y^{-3-2q^2 + q^3} + 2y^{-4+4q^2 + q^3} + 6y^{-2+2q^2 + q^3} + 6y^{-1+2q^2 + q^3} + 2y^{-3+2q^2 + q^3} + 8y^{-1+3q^2 + q^3} + 4y^{-4+2q^2 + q^3} + 4y^{-3+q^2 + q^3} + 16y^{-1+q^2 + q^3} + 12y^{-1+q^2 + q^3} + 6y^{-4+2q^2 + q^3} + 12y^{-2+2q^2 + q^3} + 6y^{-2+2q^2 + q^3} + y^{-2+2q^2 + q^3} + 2y^{-1+2q^2 + q^3} + 2y^{-1+2q^2 + q^3} + 2y^{-3+2q^2 + q^3} + 8y^{-1+3q^2 + q^3} + 4y^{-3+2q^2 + q^3} + 16y^{-1+2q^2 + q^3} + 12y^{-1+2q^2 + q^3} + 6y^{-4+3q^2 + q^3} + 12y^{-2+3q^2 + q^3} + 6y^{-2+3q^2 + q^3} + 4y^{-2+3q^2 + q^3} + 2y^{-3+2q^2 + q^3} + 8y^{-1+2q^2 + q^3} + 6y^{-4+3q^2 + q^3} + 12y^{-2+3q^2 + q^3} + 6y^{-2+3q^2 + q^3} + 2y^{-3+2q^2 + q^3} + 4y^{-3+2q^2 + q^3} + 18y^{-1+2q^2 + q^3} + 4y^{-4+3q^2 + q^3} + 8y^{-2+3q^2 + q^3} + 18y^{-1+2q^2 + q^3} + 12y^{-1+2q^2 + q^3} + 6y^{-4+3q^2 + q^3} + 12y^{-2+2q^2 + q^3} + 6y^{-2+2q^2 + q^3} + 2y^{-3+2q^2 + q^3} + 6y^{-4+3q^2 + q^3} + 4y^{-1+2q^2 + q^3} + 8y^{-2+q^2 + q^3} + 14y^{-1+2q^2 + q^3} + 12y^{-2+2q^2 + q^3} + 6y^{-2+2q^2 + q^3} + 2y^{-3+2q^2 + q^3} + 6y^{-1+2q^2 + q^3} + 4y^{-1+2q^2 + q^3} + 8y^{-2+q^2 + q^3} + 14y^{-1+2q^2 + q^3} + 12y^{-2+2q^2 + q^3} + 6y^{-2+2q^2 + q^3} + 2y^{-3+2q^2 + q^3} + 6y^{-1+3q^2 + q^3} + 4y^{-1+3q^2 + q^3} + 4y^{-4+4q^2 + q^3} + 2y^{-2+2q^2 + q^3} + 2y^{-3+2q^2 + q^3} + 6y^{-1+3q^2 + q^3} + 4y^{-1+2q^2 + q^3} + 12y^{-1+2q^2 + q^3} + 6y^{-4+3q^2 + q^3} + 12y^{-2+3q^2 + q^3} + 6y^{-2+3q^2 + q^3} + 2y^{-3+2q^2 + q^3} + 4y^{-1+2q^2 + q^3} + 18y^{-1+2q^2 + q^3} + 12y^{-1+2q^2 + q^3} + 6y^{-4+3q^2 + q^3} + 12y^{-2+2q^2 + q^3} + 6y^{-2+2q^2 + q^3} + 2y^{-3+2q^2 + q^3} + 6y^{-4+3q^2 + q^3} + 4y^{-1+2q^2 + q^3} + 8y^{-2+q^2 + q^3} + 14y^{-1+2q^2 + q^3} + 12y^{-2+2q^2 + q^3} + 18y^{-1+2q^2 + q^3} + 12y^{-1+2q^2 + q^3} + 6y^{-4+3q^2 + q^3} + 12y^{-2+2q^2 + q^3} + 6y^{-2+2q^2 + q^3} + 2y^{-3+2q^2 + q^3} + 6y^{-4+3q^2 + q^3} + 4y^{-1+2q^2 + q^3} + 14y^{-1+2q^2 + q^3} + 12y^{-2+2q^2 + q^3} + 18y^{-1+2q^2 + q^3} + 12y^{-1+2q^2 + q^3} + 6y^{-4+3q^2 + q^3} + 12y^{-2+2q^2 + q^3} + 6y^{-2+2q^2 + q^3} + 2y^{-3+2q^2 + q^3} + 6y^{-4+3q^2 + q^3} + 4y^{-1+2q^2 + q^3} + 8y^{-2+q^2 + q^3} + 14y^{-1+2q^2 + q^3} + 12y^{-2+2q^2 + q^3} + 18y^{-1+2q^2 + q^3} + 12y^{-1+2q^2 + q^3} + 6y^{-4+3q^2 + q^3} + 12y^{-2+2q^2 + q^3} + 6y^{-2+2q^2 + q^3} + 2y^{-3+2q^2 + q^3} + 6y^{-4+3q^2 + q^3} + 4y^{-1+2q^2 + q^3} + 8y^{-2+q^2 + q^3} + 14y^{-1+2q^2 + q^3} + 12y^{-2+2q^2 + q^3} + 18y^{-1+2q^2 + q^3} + 12y^{-1+2q^2 + q^3} + 6y^{-4+3q^2 + q^3} + 12y^{-2+2q^2 + q^3} + 6y^{-2+2q^2 + q^3} + 2y^{-3+2q^2 + q^3} + 6y^{-4+3q^2 + q^3} + 4y^{-1+2q^2 + q^3} + 8y^{-2+q^2 + q^3} + 14y^{-1+2q^2 + q^3} + 12y^{-2+2q^2 + q^3} + 18y^{-1+2q^2 + q^3} + 12y^{-1+2q^2 + q^3} + 6y^{-4+3q^2 + q^3} + 12y^{-2+2q^2 + q^3} + 6y^{-2+2q^2 + q^3} + 2y^{-3+2q^2 + q^3} + 6y^{-4+3q^2 + q^3} + 4y^{-1+2q^2 + q^3} + 8y^{-2+q^2 + q^3} + 14y^{-1+2q^2 + q^3} + 12y^{-2+2q^2 + q^3} + 18y^{-1+2q^2 + q^3} + 12y^{-1+2q^2 + q^3} + 6y^{-4+3q^2 + q^3} + 12y^{-2+2q^2 + q^3} + 6y^{-2+2q^2 + q^3} + 2y^{-3+2q^2 + q^3} + 6y^{-4+3q^2 + q^3} + 4y^{-1+2q^2 + q^3} + 8y^{-2+q^2 + q^3} + 14y^{-1+2q^2 + q^3} + 12y^{-2+2q^2 + q^3} + 18y^{-1+2q^2 + q^3} + 12y^{-1+2q^2 + q^3} + 6y^{-4+3q^2 + q^3} + 12y^{-2+2q^2 + q^3} + 6y^{-2+2q^2 + q^3} + 2y^{-3+2q^2 + q^3} + 6y^{-4+3q^2 + q^3} + 4y^{-1+2q^2 + q^3} + 8y^{-2+q^2 + q^3} + 14y^{-1+2q^2 + q^3} + 12y^{-2+2q^2 + q^3} + 18y^{-1+2q^2 + q^3} + 12y^{-1+2q^3} + 6y^{-1+3q} + 4y^{1+3q} + y^{-4+4q} + 2y^{-2+4q} + 6y^2 + y^4.
When $q > 3$ is even,
\[
g(y)_{2q^3 + q + 3} \equiv \\
y^{q^3 - 1} + y^{q^3 - 5} + y^{q^3 - q + 4} + y^{q^3 - q + 2} + y^{q^3 - 2q + 5} + y^{q^3 - 2q + 1} + y^{q^3 - 2q - 1} + y^{q^3 - 2q - 3} + y^{q^3 - 2q - 5} + y^{q^3 - 2q - 7} + y^{q^3 - 2q - 9} + y^{q^3 - 2q + 2q - 2} + y^{q^3 - 2q + 2q - 6} + y^{q^3 - 2q + 2q - 10} + y^{q^3 - 2q + 2q - 14} + y^{q^3 - 2q + 2q - 18} + y^{q^3 - 2q + 2q - 22} + y^{q^3 - 2q + 2q - 26} + y^{q^3 - 2q + 2q - 30} + y^{q^3 - 2q + 2q - 34} + y^{q^3 - 2q + 2q - 38} + y^{q^3 - 2q + 2q - 42} + y^{q^3 - 2q + 2q - 46} + y^{q^3 - 2q + 2q - 50} + y^{q^3 - 2q + 2q - 54} + y^{q^3 - 2q + 2q - 58} + y^{q^3 - 2q + 2q - 62} + y^{q^3 - 2q + 2q - 66} + y^{q^3 - 2q + 2q - 70} + y^{q^3 - 2q + 2q - 74} + y^{q^3 - 2q + 2q - 78} + y^{q^3 - 2q + 2q - 82} + y^{q^3 - 2q + 2q - 86} + y^{q^3 - 2q + 2q - 90} + y^{q^3 - 2q + 2q - 94} + y^{q^3 - 2q + 2q - 98}.
\]
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Appendix C (Continued)

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Neranga Fernando was born in Negombo and grew up in Ekala in Sri Lanka. He attended Maris Stella College in Negombo for his high school education. He earned his B.Sc.(special) degree majoring Mathematics from the University of Kelaniya, Sri Lanka in 2007. He then entered the Master’s program in Mathematics at the University of South Florida in 2008. After completing his master’s in 2009, he entered the Ph.D. program in Mathematics at the University of South Florida in 2010.

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