

January 2013

## Topological Degree and Variational Inequality Theories for Pseudomonotone Perturbations of Maximal Monotone Operators

Teffera Mekonnen Asfaw  
*University of South Florida*, [tefferam@yahoo.com](mailto:tefferam@yahoo.com)

Follow this and additional works at: <https://digitalcommons.usf.edu/etd>



Part of the [Mathematics Commons](#)

---

### Scholar Commons Citation

Asfaw, Teffera Mekonnen, "Topological Degree and Variational Inequality Theories for Pseudomonotone Perturbations of Maximal Monotone Operators" (2013). *USF Tampa Graduate Theses and Dissertations*. <https://digitalcommons.usf.edu/etd/4433>

This Dissertation is brought to you for free and open access by the USF Graduate Theses and Dissertations at Digital Commons @ University of South Florida. It has been accepted for inclusion in USF Tampa Graduate Theses and Dissertations by an authorized administrator of Digital Commons @ University of South Florida. For more information, please contact [digitalcommons@usf.edu](mailto:digitalcommons@usf.edu).

Topological Degree and Variational Inequality Theories for Pseudomonotone  
Perturbations of Maximal Monotone Operators

by

Teffera M. Asfaw

A dissertation submitted in partial fulfillment  
of the requirements for the degree of  
Doctor of Philosophy  
Department of Mathematics and Statistics  
College of Arts and Sciences  
University of South Florida

Major Professor: Athanassios Kartsatos, Ph.D.  
Yuncheng You, Ph.D.  
Wen-Xiu Ma, Ph.D.  
Boris Shekhtman, Ph.D.

Date of Approval:  
March 26, 2013

Keywords: Homotopy invariance, Existence of zeros, multivalued ( $S_+$ ) operators,  
Nonlinear problems, Local solvability, Quasimonotone.

Copyright © 2013, Teffera M. Asfaw

## ACKNOWLEDGMENTS

I would like to express my heartfelt gratitude to my advisor, Professor Athanassios G. Kartsatos, for supervising me with extremely valuable guidelines, advice, remarks and suggestions during the preparation of this dissertation.

I wish to express my sincere thanks to Professor Yuncheng You, Professor Wen-Xiu Ma and Professor Boris Shekhtman for their interest to be members of my Ph.D. advisory committee. My sincere thanks go to Professor Witanachchi Sarath for chairing my dissertation defence. My appreciation go to the entire faculty and staff of the USF mathematics department for their kind hospitality during my study at USF.

Many thanks go to my wife and family for their patience, interest and support throughout my student career. My thanks go to my friends for their initiatives.

## TABLE OF CONTENTS

Abstract		iii
1	Introduction and preliminaries	1
1.1	Introduction . . . . .	1
1.2	Maximal monotone and pseudomonotone operators . . . . .	6
1.2.1	Geometric properties of Banach spaces . . . . .	6
1.2.2	Maximal monotone operators . . . . .	8
1.2.3	Pseudomonotone operators . . . . .	10
1.3	Browder degree theory . . . . .	13
2	Homotopy invariance and uniqueness results of the degree mapping for multi-valued $(S_+)$ perturbations	18
2.1	New preliminary results . . . . .	19
2.2	Construction of the degree, basic properties and homotopy invariance results . . . . .	23
2.3	Uniqueness of the degree . . . . .	34
3	Degree theory for multi-valued pseudomonotone perturbations	37
3.1	Construction of the degree and basic properties . . . . .	38
3.2	Homotopy invariance theorems . . . . .	47
3.3	Uniqueness of the degree . . . . .	67
3.4	Degree theory for unbounded single multi-valued pseudomonotone operators . . . . .	68
3.5	Applications to mapping theorems of nonlinear analysis . . . . .	71

3.6	Necessary and sufficient conditions for existence of zeros . . . . .	92
3.7	Generalization for quasimonotone perturbations . . . . .	95
3.8	Applications to partial differential equations . . . . .	99
4	Variational inequality theory for perturbations of maximal monotone operators . . . . .	104
4.1	Introduction and preliminaries . . . . .	104
4.2	Variational inequalities for maximal monotone perturbations of pseudomonotone operators . . . . .	110
4.3	Variational inequalities for maximal monotone perturbations of generalized pseudomonotone operators . . . . .	127
4.3.1	Strongly quasibounded maximal monotone perturbations of generalized pseudomonotone operators . . . . .	127
4.3.2	Maximal monotone perturbations of $(pm_4)$ -generalized pseudomonotone operators . . . . .	135
4.3.3	Maximal monotone perturbations of regular generalized pseudomonotone operators . . . . .	141
4.4	Applications for parabolic partial differential equations . . . . .	146
	References . . . . .	154
	Appendix . . . . .	162
	Appendix A - Copyright and permissions . . . . .	164
	About the Author . . . . .	End Page

## ABSTRACT

Let  $X$  be a real reflexive locally uniformly convex Banach space with locally uniformly convex dual space  $X^*$ . Let  $G$  be a bounded open subset of  $X$ . Let  $T : X \supset D(T) \rightarrow 2^{X^*}$  be maximal monotone and  $S : X \rightarrow 2^{X^*}$  be bounded pseudomonotone and such that  $0 \notin \overline{(T + S)(D(T) \cap \partial G)}$ . Chapter 1 gives general introduction and mathematical prerequisites. In Chapter 2 we develop a homotopy invariance and uniqueness results for the degree theory constructed by Zhang and Chen for multivalued  $(S_+)$  perturbations of maximal monotone operators. Chapter 3 is devoted to the construction of a new topological degree theory for the sum  $T + S$  with the degree mapping  $d(T + S, G, 0)$  defined by

$$d(T + S, G, 0) = \lim_{\varepsilon \downarrow 0^+} d_{S_+}(T + S + \varepsilon J, G, 0),$$

where  $d_{S_+}$  is the degree for bounded  $(S_+)$ -perturbations of maximal monotone operators. The uniqueness and homotopy invariance result of this degree mapping are also included herein. As applications of the theory, we give associated mapping theorems as well as degree theoretic proofs of known results by Figueiredo, Kenmochi and Le. In chapter 4, we consider  $T : X \supseteq D(T) \rightarrow 2^{X^*}$  to be maximal monotone and  $S : D(S) = K \rightarrow 2^{X^*}$  at least pseudomonotone, where  $K$  is a nonempty, closed and convex subset of  $X$  with  $0 \in \overset{\circ}{K}$ . Let  $\phi : X \rightarrow (-\infty, \infty]$  be a proper, convex and lower-semicontinuous function. Let  $f^* \in X^*$  be fixed. New results are given concerning the solvability of perturbed variational inequalities for operators of the type  $T + S$  associated with the function  $\phi$ . The associated range results for nonlinear operators are also given, as well as extensions and/or improvements of known results by Kenmochi, Le, Browder, Browder and Hess, Figueiredo, Zhou, and others.

## 1.1 Introduction

This dissertation is devoted to the study of equations and variational inequalities for multi-valued pseudomonotone perturbations of maximal monotone operators in reflexive Banach spaces. Topological degree theory has been an important tool for the study of nonlinear functional equations. In this regard, the main concern of this dissertation is the development of a topological degree theory for the sum  $T + S$  where  $T : X \supseteq D(T) \rightarrow 2^{X^*}$  is maximal monotone and  $S : X \rightarrow 2^{X^*}$  is pseudomonotone. It is known that there is no degree theory with comprehensive homotopies involving the operators  $T$  and  $S$  as mentioned above. For results concerning degree theories related to the content of this dissertation, we cite Browder [12]-[18], Zhang and Chen [73], Kartsatos and Skrypnik [39]-[41], Berkovits and Mustonen [8], Ibrahimou and Kartsatos [32], Leray and Schauder [51], Lloyd [53]. For some papers on degree theories and their applications to various problems in nonlinear analysis, we cite Adhikari and Kartsatos [1]-[2], Kartsatos [35]-[36], Kartsatos and Lin [37], Kartsatos and Quarcoo [38]. For an account of degree theories during the last 50 years, the reader is referred to Mawhin's article in [55].

Our second main concern is developing a comprehensive variational inequality theory for the sum  $T + S$ , where  $S : D(S) = K \rightarrow 2^{X^*}$  is pseudomonotone or generalized pseudomonotone and  $T : X \supseteq D(T) \rightarrow 2^{X^*}$  is maximal monotone. Here  $K$  is a closed and convex subset of  $X$ . Let  $\phi : X \supseteq D(\phi) \rightarrow [-\infty, \infty]$  be a proper, convex

lower-semicontinuous function. The solvability of the variational inequality problem  $VIP(T + S, K, \phi, f^*)$  involves finding  $x_0 \in D(T) \cap D(\phi) \cap K$ ,  $v_0^* \in Tx_0$  and  $w_0^* \in Sx_0$  such that

$$\langle v_0^* + w_0^* - f^*, x - x_0 \rangle \geq \phi(x_0) - \phi(x)$$

for all  $x \in K$ . It is well known that the solvability of the problem  $VIP(T + S, K, \phi, f^*)$  is equivalent to the solvability of the inclusion  $Tx + Sx \ni f^*$  in  $D(T)$  if  $K = X$  and  $\phi = I_X$ , where  $I_X$  is indicator function on  $X$ . However, the degree theory developed in this dissertation can not be directly applied for the solvability of such inequalities if  $K \neq X$  and  $\phi \neq I_X$  because of the following basic reasons.

- (i) the solvability of the problem

$$\partial\phi(x) + T(x) + S(x) \ni f^*$$

in  $D(T) \cap D(\partial\phi) \cap K$  implies the solvability of the problem

$$VIP(T + S, K, \phi, f^*)$$

in  $D(T) \cap D(\phi) \cap K$ , and the two problems are equivalent if  $D(\phi) = D(\partial\phi) = K$ . Therefore, the solvability of the problem  $VIP(T + S, K, \phi, f^*)$  in  $D(T) \cap D(\phi) \cap K$  may be covered by range results for the sum of three monotone-type operators. However, to the author's knowledge, there are no range results involving such operators.

- (ii) If  $\phi \neq I_K$ , the solvability of the inclusion  $Tx + Sx \ni f^*$  does not necessarily imply the solvability of the problem

$$VIP(T + S, K, \phi, f^*).$$

In fact, if for some  $x_0 \in D(T) \cap D(\phi) \cap K$ ,  $v_0^* \in Tx_0$  and  $z_0^* \in Sx_0$ , the



equation  $v_0^* + z_0^* = f^*$  is satisfied, we do not necessarily have the solvability of the inequality

$$\langle v_0^* + z_0^* - f^*, x - x_0 \rangle \geq \phi(x_0) - \phi(x)$$

for all  $x \in K$  unless  $\phi(x_0) = \min_{x \in K} \phi(x)$ .

- (iii) It is known from Browder and Hess [18, Proposition 3, p. 258] that every pseudomonotone operator with effective domain all of  $X$  is generalized pseudomonotone. However, this fact is unknown if the domain is different from  $X$ . Because of this, we have treated the solvability of variational inequalities and equations separately for pseudomonotone and generalized pseudomonotone operators with domain a closed convex subset of  $X$ .

Browder and Hess [18] mentioned the difficulty of treating generalized pseudomonotone operators which are not defined everywhere on  $X$  or on a dense linear subspace. A surjectivity result for a single quasibounded coercive generalized pseudomonotone operator whose domain contains a dense linear subspace of  $X$  may be found in Browder and Hess [18, Theorem 5, p. 273]. Existence results for densely defined finitely continuous generalized pseudomonotone perturbations of maximal monotone operators may be found in Guan, Kartsatos and Skrypnik [30, Theorem 2.1, p. 335]. We should mention here that there are no range results known to the author for the sum  $T + S$ , where  $T$  is maximal monotone and  $S$  either pseudomonotone or generalized pseudomonotone with domain just  $K$ , where  $K$  is a nonempty, closed and convex subset of  $X$ . For basic results involving variational inequalities and monotone type mappings, the reader is referred to Barbu [4], Brézis [10], Browder and Hess [18], Browder [13], Browder and Brézis [19], Hartman and Stampacchia [31], Kenmochi [42]-[44], Kinderlehrer and Stampacchia [47], Kobayashi and Otani [48], Lions and Stampacchia [52], Minty [56]-[57], Moreau [58], Naniewicz and Panagiotopoulos [61], Pascali and Sburlan [63], Rockafellar [64], Stampacchia [69], Ton [70], Zeidler [72] and the references therein. A study of pseudomonotone operators and nonlinear elliptic

boundary value problems may be found in Kenmochi [43]. For a survey of maximal monotone and pseudomonotone operators and perturbation results, we cite the hand book of Kenmochi [44]. Nonlinear perturbation results of monotone type mappings, variational inequalities and their applications may be found in Guan, Kartsatos and Skrypnik [30], Guan and Kartsatos [29], Le [50], Zhou [74] and the references therein. Variational inequalities for single single-valued pseudomonotone operators in the sense of Brézis may be found in Kien, Wong, Wong and Yao [46]. Existence results for multivalued quasilinear inclusions and variational-hemivariational inequalities may be found in Carl, Le and Motreanu [22], Carl [23] and Carl and Motreanu [24] and the references therein.

In the rest of this Chapter, we give the basic definitions and mathematical ideas to be used in the next chapters. Sections 1.2.1-1.2.3 consist of main definitions and geometric properties of Banach spaces, maximal monotone, pseudomonotone and generalized pseudomonotone operators. In Section 1.2.4 through Section 1.2.6. we briefly discuss the topological degree theories of Brouwer, Leray-Schauder and Browder.

Chapter 2 is concerned with new preliminary results for proving a comprehensive homotopy invariance theorem for the degree theory developed by Zhang and Chen [73] for multivalued  $(S_+)$  perturbations of maximal monotone operators. Furthermore, the Chapter includes the uniqueness of this degree theory. For the details of this theory, the reader is referred to the book of O'Regan, Cho and Chen [62]. The contents of this Chapter will be used in developing the main degree theory for multivalued pseudomonotone perturbations of maximal monotone operators in Chapter 3.

Chapter 3 is devoted to the development of a new topological degree theory for bounded multivalued pseudomonotone perturbations of maximal monotone operators. Section 3.1 deals with the construction of the degree mapping together with its

basic properties. A new comprehensive homotopy invariance results and uniqueness of the degree mapping are included in Section 3.2 and Section 3.3 respectively. Section 3.4 is concerned with the construction of a topological degree theory for a single multivalued, possibly unbounded, pseudomonotone operator together with its basic properties. In Section 3.5 we give applications of the theory to prove new existence results as well as degree theoretic proofs of known nonlinear analysis problems. As a result, a necessary and sufficient condition for the existence of zeros is included in Section 3.6. Section 3.7 provides a possible generalization of the degree theory developed herein for multivalued quasimonotone perturbations of maximal monotone operators. We note here that the class of pseudomonotone operators is properly included in the class of quasimonotone operators. We discuss in detail that the methodology of the construction of the degree theory for pseudomonotone perturbations of maximal monotone operators can be carried out in the construction of a degree theory for quasimonotone perturbations of maximal monotone operators. Finally, an application of the theory is given for the existence of weak solution(s) for nonlinear partial differential equations.

Chapter 4 deals with the solvability of variational inequality problems involving operators of the type  $T + S$ , where  $T : X \supseteq D(T) \rightarrow 2^{X^*}$  is maximal monotone and  $S : K \rightarrow 2^{X^*}$  is bounded pseudomonotone, or finitely continuous quasibounded generalized pseudomonotone, or regular generalized pseudomonotone operator. In particular, the content of Chapter 4 addresses the problem of finding  $x_0 \in D(T) \cap K$ ,  $v_0^* \in Tx_0$  and  $w_0^* \in Sx_0$  such that

$$\langle v_0^* + w_0^* - f^*, x - x_0 \rangle \geq \phi(x_0) - \phi(x)$$

for all  $x \in K$  where  $\phi : X \rightarrow [-\infty, \infty]$  is a proper, convex and lower semicontinuous function. In Section 4.1, an introduction, motivation and basic concepts are given.

Section 4.2 and Section 4.3 deals with the solvability of variational inequalities involving pseudomonotone and generalized pseudomonotone perturbations of maximal monotone operators respectively. In Section 4.4, an application of the theory is given for a class of partial differential equations.

## 1.2 Maximal monotone and pseudomonotone operators

In this Section, we give a comprehensive overview of the geometric properties of Banach spaces, basic definitions and properties of maximal monotone as well as pseudomonotone operators, degree theories of Brouwer, Leray-Schauder and Browder.

### 1.2.1 Geometric properties of Banach spaces

**Definition 1.2.1** A normed linear space  $X$  is said to be

- (i) “*strictly convex*” if the unit sphere does not contain a line segment, i.e.  $\|(1-t)x + ty\| < 1$  for all  $x$  and  $y$  with  $\|x\| = \|y\| = 1$ ,  $x \neq y$  and all  $t \in (0, 1)$ . In other words,  $X$  is strictly convex if there are  $x, y$  with  $\|x\| = \|y\| = 1$  and  $\|(1-t)x + ty\| = 1$  for some  $t \in (0, 1)$  holds if and only if  $x = y$ .
- (ii) “*locally uniformly convex*” if for any  $\varepsilon > 0$  and  $x \in X$  with  $\|x\| = 1$ , there exists  $\delta = \delta(x, \varepsilon) > 0$  such that  $\|x - y\| \geq \varepsilon$  implies

$$\left\| \frac{x + y}{2} \right\| \leq 1 - \delta$$

for all  $y$  with  $\|y\| = 1$ .

- (iii) “*uniformly convex*” if for each  $\varepsilon > 0$ , there exists  $\delta = \delta(\varepsilon) > 0$  such that  $\|x\| = \|y\| = 1$  and  $\|x - y\| \geq \varepsilon$  imply that

$$\left\| \frac{x + y}{2} \right\| \leq 1 - \delta.$$

Based on Definition 1.2.1, it is easy to see that every uniformly convex space is locally uniformly convex and a locally uniformly convex space is strictly convex. It is well known that a Hilbert space is locally uniformly convex, the  $L^p(\Omega)$  spaces and the Sobolev spaces  $W^{m,p}(\Omega)$  are uniformly convex provided that  $1 < p < \infty$ . Important basic properties, connections and examples of these spaces can be found in the books of Deimling [27], Pascali and Sburlan [63] and Zeidler [72].

In what follows,  $X$  is real reflexive locally uniformly convex Banach space with locally uniformly convex dual space  $X^*$ . The norm of the space  $X$ , and any other normed spaces herein, will be denoted by  $\|\cdot\|$ . For  $x \in X$  and  $x^* \in X^*$ , the pairing  $\langle x^*, x \rangle$  denotes the value  $x^*(x)$ . Let  $X$  and  $Y$  be real Banach spaces. For a multi-valued mapping  $T : X \supseteq D(T) \rightarrow 2^Y$ , we define the domain  $D(T)$  of  $T$  by  $D(T) = \{x \in X : Tx \neq \emptyset\}$ , and the range  $R(T)$  of  $T$  by  $R(T) = \cup_{x \in D(T)} Tx$ . We also use the symbol  $G(T)$  for the graph of  $T : G(T) = \{(x, Tx) : x \in D(T)\}$ . A mapping  $T : X \supset D(T) \rightarrow Y$  is called

- (i) “bounded” if it maps bounded subsets of  $D(T)$  into bounded subsets of  $Y$ .
- (ii) “continuous” if it is strongly continuous, i.e. continuous from the strong topology of  $D(T)$  to the strong topology of  $Y$ .
- (iii) “demicontinuous” if it is continuous from the strong topology of  $D(T)$  to the weak topology of  $Y$ .
- (iv) “compact” if it is strongly continuous and maps bounded subsets of  $D(T)$  into relatively compact subsets of  $Y$ .

A multi-valued mapping  $T : X \supset D(T) \rightarrow 2^Y$  is called

- (i) “upper semicontinuous” denoted “usc”, if for each  $x_0 \in D(T)$  and a weak neighbourhood  $V$  of  $Tx_0$  in  $Y$ , there exists a neighbourhood  $U$  of  $x_0$  such that  $T(D(T) \cap U) \subseteq V$ .

- (ii) “compact” if  $T$  is “usc” and maps bounded subsets of  $D(T)$  into relatively compact subset of  $Y$ .
- (iii) “finitely continuous” if the restriction of  $T$  on  $D(T) \cap F$  is usc for each finite dimensional subspace of  $X$ .
- (iv) “quasibounded” if for each  $M > 0$ , there exists  $K(M) > 0$  such that , whenever  $(u, u^*) \in G(T)$  and

$$\langle u^*, u \rangle \leq M\|u\|, \quad \|u\| \leq M,$$

then  $\|u^*\| \leq K(M)$ .

- (v) “strongly quasibounded” if for each  $M > 0$ , there exists  $K(M) > 0$  such that , whenever  $(u, u^*) \in G(T)$  and

$$\langle u^*, u \rangle \leq M, \quad \|u\| \leq M,$$

then  $\|u^*\| \leq K(M)$ .

### 1.2.2 Maximal monotone operators

**Definition 1.2.2** An operator  $T : X \supset D(T) \rightarrow 2^{X^*}$  is said to be

- (i) “monotone” if for every  $x \in D(T), y \in D(T)$  and every  $u \in Tx, v \in Ty$ , we have

$$\langle u - v, x - y \rangle \geq 0.$$

- (ii) “maximal monotone” if  $T$  is monotone and the graph of  $T$  is not contained in the graph of any other monotone operator. Equivalently,  $T$  is “maximal monotone” if  $T$  is monotone and  $\langle u - u_0, x - x_0 \rangle \geq 0$  for every  $(x, u) \in G(T)$  implies  $x_0 \in D(T)$  and  $u_0 \in Tx_0$ .

Let  $J : X \rightarrow 2^{X^*}$  be the normalized duality mapping defined by

$$Jx := \{x^* \in X^* : \langle x^*, x \rangle = \|x^*\| \|x\|, \|x^*\|^2 = \|x\|^2\}.$$

Since  $X$  and  $X^*$  are locally uniformly convex, it follows that  $J$  is single-valued, bounded, bicontinuous, maximal monotone and of type  $(S_+)$ . The following characterization of maximal monotone operators is due to Rockafellar, which can be found in the book of Zeidler[72, Theorem 32. F, p.881].

**Theorem 1.2.3** (cf. Zeidler [72, Theorem 32.F, p.881]) *Let  $X$  be a reflexive Banach space with  $X$  and  $X^*$  are strictly convex. Then a monotone operator  $T : X \supseteq D(T) \rightarrow 2^{X^*}$  is maximal if and only if  $R(T + \lambda J) = X^*$  for all  $\lambda > 0$ .*

We mention here that the version of this theorem (for  $X$  a Hilbert space and  $T$  is single valued) is due to Minty, which can be found in the book of Zeidler[72, Proposition 32. 8, p. 855]. As a result of Theorem 1.2.3, if  $T : X \supseteq D(T) \rightarrow 2^{X^*}$  is maximal monotone operator, then for each  $x \in X$  and  $t > 0$ , there exist  $x_t \in D(T)$  and  $f_t \in Tx_t$  such that

$$tf_t + J(x_t - x) = 0$$

where  $J$  is the duality mapping. The *Yosida resolvent* and *Yosida approximant* of  $T$  denoted by  $J_t$  and  $T_t$  respectively are defined by

$$J_t x = x_t, \quad T_t x = f_t = \frac{1}{t} J(x - x_t) = (T^{-1} + tJ^{-1})^{-1} x,$$

for each  $x \in X$  and  $t > 0$ . In the case that  $X$  and  $X^*$  are strictly convex reflexive Banach spaces, it is well known that, for each  $t > 0$ ,  $J_t : X \rightarrow X$  is bounded and demicontinuous,  $T_t : X \rightarrow X^*$  is bounded demicontinuous maximal monotone and such that  $J_t x \in D(T)$ ,  $J_t x = x - tJ^{-1}(T_t x)$  for all  $x \in X$ ,  $J_t x \rightarrow x$  as  $t \downarrow 0^+$  for all  $x \in \overline{coD(T)}$ , where  $coD(T)$  denotes the convex hull of  $D(T)$ ,  $\|T_t x\| \leq \|T^0 x\|$  for all  $x \in D(T)$  and  $t > 0$  and  $T_t x \rightarrow T^0 x$  as  $t \rightarrow 0^+$ , where  $\|T^{(0)} x\| = \inf\{\|y^*\| : y^* \in Tx\}$ .

For further details, the reader is referred to the book of Deimling [27], Pascali and Sburlan [63] and Zeidler [72]. The following lemma can be found in the book of Zeidler [72, p. 915].

**Lemma 1.2.4** (cf. [72, p. 915]) *Let  $T : X \supset D(T) \rightarrow 2^{X^*}$  be maximal monotone. Then the following are true.*

- (i)  $\{x_n\} \subset D(T)$ ,  $x_n \rightarrow x_0$  and  $Tx_n \ni y_n \rightarrow y_0$  imply  $x_0 \in D(T)$  and  $y_0 \in Tx_0$ .
- (ii)  $\{x_n\} \subset D(T)$ ,  $x_n \rightharpoonup x_0$  and  $Tx_n \ni y_n \rightarrow y_0$  imply  $x_0 \in D(T)$  and  $y_0 \in Tx_0$ .

The following Lemma is due to Brézis, Crandall and Pazy [9, p. 136].

**Lemma 1.2.5** ([9, Brézis, Crandall and Pazy, p. 136]) *Let  $B$  be a maximal monotone set in  $X \times X^*$ . If  $(u_n, u_n^*) \in B$  such that  $u_n \rightarrow u$  in  $X$  and  $u_n^* \rightarrow u^*$  in  $X^*$  and either*

$$\limsup_{n,m \rightarrow \infty} \langle u_n^* - u_m^*, u_n - u_m \rangle \leq 0$$

or

$$\limsup_{n \rightarrow \infty} \langle u_n^* - u^*, u_n - u \rangle \leq 0,$$

then  $(u, u^*) \in B$  and  $(u_n^*, u_n) \rightarrow (u^*, u)$  as  $n \rightarrow \infty$ .

For basic properties of the operators considered in this dissertation, we refer the reader to the book of Pascali and Sburlan [63], Brézis, Crandall and Pazy [9], Browder [13], Barbu [4] and Zeidler [72].

### 1.2.3 Pseudomonotone operators

The following definition of a multi-valued pseudomonotone and a generalized pseudomonotone operator is due to Browder and Hess [18].



**Definition 1.2.6** (Browder and Hess [18]) An operator  $T : X \supset D(T) \rightarrow 2^{X^*}$  is said to be “pseudomonotone” if the following conditions are satisfied.

- (i) For each  $x \in D(T)$ ,  $Tx$  is a closed, convex and bounded subset of  $X^*$ .
- (ii)  $T$  is “weakly upper semicontinuous” on each finite-dimensional subspace  $F$  of  $X$ , i.e. for every  $x_0 \in D(T) \cap F$  and every weak neighborhood  $V$  of  $Tx_0$  in  $X^*$ , there exists a neighborhood  $U$  of  $x_0$  in  $F$  such that  $TU \subset V$ .
- (iii) For every sequence  $\{x_n\} \subset D(T)$  and every sequence  $\{y_n^*\}$  with  $y_n^* \in Tx_n$ , such that  $x_n \rightharpoonup x_0 \in D(T)$  and

$$\limsup_{n \rightarrow \infty} \langle y_n^*, x_n - x_0 \rangle \leq 0,$$

we have that for each  $x \in D(T)$  there exists  $y^*(x) \in Tx_0$  such that

$$\langle y^*(x), x_0 - x \rangle \leq \liminf_{n \rightarrow \infty} \langle y_n, x_n - x \rangle.$$

**Definition 1.2.7** (Browder and Hess [18]) An operator  $T : X \supset D(T) \rightarrow 2^{X^*}$  is said to be “generalized pseudomonotone” if for every sequence  $\{x_n\} \subset D(T)$  and every sequence  $\{y_n\}$  with  $y_n \in Tx_n$  such that  $x_n \rightharpoonup x_0 \in D(T)$ ,  $y_n \rightharpoonup y_0 \in X^*$  and

$$\limsup_{n \rightarrow \infty} \langle y_n, x_n - x_0 \rangle \leq 0,$$

we have  $y_0 \in Tx_0$  and  $\langle y_n, x_n \rangle \rightarrow \langle y_0, x_0 \rangle$  as  $n \rightarrow \infty$ .

Zhang and Chen [73] introduce the following definition of multivalued operator of type  $(S_+)$ .

**Definition 1.2.8** (Zhang and Chen [73].) An operator  $T : X \supset D(T) \rightarrow 2^{X^*}$  is said to be “of type  $(S_+)$ ” if the following conditions are satisfied.

- (i) For each  $x \in D(T)$ ,  $Tx$  is a nonempty, closed, convex and bounded subset of  $X^*$ .
- (ii)  $T$  is weakly upper semicontinuous on each finite-dimensional subspace of  $X$  (see Definition 1.2.6).
- (iii) For every sequence  $\{x_n\} \subset D(T)$  and every  $y_n \in Tx_n$ , with  $x_n \rightharpoonup x_0 \in X$  and

$$\limsup_{n \rightarrow \infty} \langle y_n, x_n - x_0 \rangle \leq 0,$$

we have  $x_n \rightarrow x_0 \in D(T)$  and  $\{y_n\}$  has a subsequence which converges weakly to  $y_0 \in Tx_0$ . A mapping  $T : X \supset D(T) \rightarrow 2^{X^*}$  is said to be “of type  $S$ ” if (i) and (ii) hold with the inequality (iii) replaced by an equality.

For basic properties of multivalued pseudomonotone and generalized pseudomonotone operators, the reader is referred to the paper by Browder and Hess [18, Proposition 2, Proposition 3, Proposition 8, Proposition 9]. For basic properties of single valued operators of type  $(S_+)$ , pseudomonotone and generalized pseudomonotone operators, the reader is referred to the book by Zeidler [72, Proposition 27.6, Proposition 27.7, pp.587-595]. The following lemma can be found in Browder and Hess [18, Proposition 7, p. 136].

**Lemma 1.2.9** (*Browder and Hess [18].*) *Let  $X$  be a reflexive Banach space and  $S : X \rightarrow 2^{X^*}$  be pseudomonotone. Suppose that  $\{u_n\}$  is a sequence in  $X$  such that  $u_n \rightharpoonup u_0 \in X$  and that  $w_n \in Su_n$  satisfies*

$$\limsup_{n \rightarrow \infty} \langle w_n, u_n - u_0 \rangle \leq 0.$$

*Then the sequence  $\{w_n\}$  is bounded in  $X^*$  and every weak limit of  $\{w_n\}$  lies in  $Su_0$ .*

### 1.3 Browder degree theory

In this Section, we briefly give an overview of the degree theory of Brouwer, Leray-Schauder and Browder. The notion of degree theory for continuous functions defined from the closure of a bounded open subset  $D$  of  $\mathbb{R}^N$  into  $\mathbb{R}^N$  was first introduced by Brouwer [11] in 1912. This degree mapping of Brouwer is normalized by the identity and invariant under homotopies of continuous functions. In 1934, Leray-Schauder [51] generalized this theory for any infinite dimensional Banach spaces  $X$  for operators of the type  $I - T$  where  $T$  is a compact operator defined from the closure of a bounded open subset  $G$  of  $X$  into  $X$  and  $I$  is the identity on  $X$ . The degree mapping of Leray-Schauder is normalized by the identity as in the case of Brouwer and invariant under homotopies of compact operators. For further details of the degree theory of Brouwer and Leray-Schauder, the reader is referred to the books of Fonesca and Gangbo [33], Kartsatos [34], Deimling [27] and Zeidler [72]. In 1983, in series of papers Browder [14, 15, 16] developed a degree theory for operators of the type  $T + f$  where  $f$  is bounded single valued demicontinuous operator of type  $(S_+)$  defined from the closure of a bounded open subset  $G$  of  $X$  into  $X^*$  and  $T : X \supseteq D(T) \rightarrow 2^{X^*}$  is maximal monotone. The theory improved the Leray-Schauder degree theory when  $X$  is a Hilbert space because the mapping of the form  $I - C$  where  $C$  is compact lies in a broader class of  $(S_+)$  operators, which includes the class of uniformly monotone operators. Browder [17, Definition 3, p. 21] introduced the following notion of homotopy of class  $(S_+)$  for a family of single valued operators of type  $(S_+)$ .

**Definition 1.3.1** (Browder [17].) Let  $G$  be a bounded open subset of  $X$  and  $\{f^t\}_{t \in [0,1]}$  be family of single valued maps from  $\overline{G}$  into  $X^*$ . Then  $\{f^t\}_{t \in [0,1]}$  is said to be a homotopy of class  $(S_+)$  if it satisfies the following condition. For any sequence  $\{t_n\}$  in  $[0, 1]$  such that  $t_n \rightarrow t_0$  and for any sequence  $\{x_n\}$  in  $\overline{G}$  such that  $x_n \rightarrow x_0$  as  $n \rightarrow \infty$

for which

$$\limsup_{n \rightarrow \infty} \langle f^{t_n}(x_n), x_n - x_0 \rangle \leq 0,$$

we have  $x_n \rightarrow x_0 \in \overline{G}$  and  $f^{t_n}(x_n) \rightarrow f^{t_0}x_0$  as  $n \rightarrow \infty$ .

Furthermore, Browder [15, Definition 2, p. 2405] introduced the concept of pseudomonotone homotopy of maximal monotone operators and proved the equivalence of the 4 conditions in the following definition.

**Definition 1.3.2** (Browder [15].) Let  $\{T^t\}_{t \in [0,1]}$  be a family of maximal monotone operators from  $X$  to  $2^{X^*}$  such that  $0 \in T^t(0)$ ,  $t \in [0, 1]$ . Then  $\{T^t\}_{t \in [0,1]}$  is called a “pseudomonotone homotopy” if it satisfies the following equivalent conditions.

- (i) Suppose that  $t_n \rightarrow t_0 \in [0, 1]$  and  $(x_n, y_n) \in G(T^{t_n})$  are such that  $x_n \rightarrow x_0$  in  $X$ ,  $y_n \rightarrow y_0$  in  $X^*$  and

$$\limsup_{n \rightarrow \infty} \langle y_n, x_n \rangle \leq \langle y_0, x_0 \rangle.$$

Then  $(x_0, y_0) \in G(T^{t_0})$  and  $\lim_{n \rightarrow \infty} \langle y_n, x_n \rangle = \langle y_0, x_0 \rangle$ .

- (ii) The mapping  $\phi : X^* \times [0, 1] \rightarrow X$  defined by  $\phi(w, t) := (T^t + J)^{-1}(w)$  is continuous.
- (iii) For each  $w \in X^*$ , the mapping  $\phi_w : [0, 1] \rightarrow X$  defined by  $\phi_w(t) := (T^t + J)^{-1}(w)$  is continuous.
- (iv) For any  $(x, y) \in G(T^{t_0})$  and any sequence  $t_n \rightarrow t_0$ , there exists a sequence  $(x_n, y_n) \in G(T^{t_n})$  such that  $x_n \rightarrow x$  and  $y_n \rightarrow y$  as  $n \rightarrow \infty$ .

The content of the following Theorem is due to Browder [15, Theorem 2, Theorem 3, p. 2406].

**Theorem 1.3.3** (Browder [15].) Let  $G$  be a bounded open subset of  $X$ . Let  $f : \overline{G} \rightarrow X^*$  be bounded demicontinuous of type  $(S_+)$  and  $T : X \supseteq D(T) \rightarrow 2^{X^*}$  be maximal

monotone with  $0 \in T(0)$ . Let the family  $\{f^t\}_{t \in [0,1]}$  be uniformly bounded homotopy of class  $(S_+)$  from  $\overline{G}$  into  $X^*$  and  $\{T^t\}_{t \in [0,1]}$  be pseudomonotone homotopy of maximal monotone operators with  $0 \in T^t(0)$  for all  $t \in [0, 1]$ . Then there exists a degree mapping for the operators of the type  $T + f$  normalized by the duality mapping  $J$  and invariant under homotopies of the type  $H(t, x) = T^t x + f^t(x)$ ,  $(t, x) \in [0, 1] \times D(T^t) \cap \overline{G}$ .

This degree of Browder extends his own degree theory for  $T = 0$ , and the degree theory for bounded demicontinuous  $(S_+)$ -mappings developed earlier by Skrypnik in separable Banach spaces in [67], again for  $T = 0$ .

Zhang and Chen [73] introduced and constructed a degree theory for multivalued  $(S_+)$  operators defined on a closure of bounded open subset  $G$  of  $X$ . In the same paper, the authors constructed a degree theory for mappings of the form  $T + S : \overline{G} \rightarrow 2^{X^*}$ , where  $T$  is maximal monotone and  $S$  is of type  $(S_+)$ . We like to mention here that this degree theory is not a generalization of Browder's theory because, here, we see that  $D(T)$  contains a bounded open subset  $G$  while it was considered arbitrary in Browder's theory. Zhang and Chen [73] introduced a homotopy of class  $(S_+)$  for multivalued operators.

**Definition 1.3.4** (Zhang and Chen [73].) For every  $t \in [0, 1]$ , consider the operator  $S^t : X \supset D(S^t) \rightarrow 2^{X^*}$ . The family  $\{S^t\}_{t \in [0,1]}$  is said to be a "homotopy of type  $(S_+)$ " if

- (i) for each  $t \in [0, 1]$ ,  $x \in D(S^t)$ ,  $S^t x$  is a nonempty, closed, convex and bounded subset of  $X^*$ ;
- (ii) for each  $t \in [0, 1]$ ,  $S^t$  is weakly upper semicontinuous on each finite-dimensional subspace of  $X$  (see Definition 1.2.6);
- (iii) Let  $\{t_n\} \subset [0, 1]$ ,  $x_n \in D(S^{t_n})$  be such that  $t_n \rightarrow t_0$  and  $x_n \rightharpoonup x_0 \in X$ . Let

$f_n \in S^{t_n} x_n$  be such that

$$\limsup_{n \rightarrow \infty} \langle f_n, x_n - x_0 \rangle \leq 0,$$

then  $x_n \rightarrow x_0 \in D(S^{t_0})$  and there exists a subsequence of  $\{f_n\}$ , denoted again by  $\{f_n\}$ , such that  $f_n \rightarrow f \in S^{t_0} x_0$  as  $n \rightarrow \infty$ .

In our future construction of the degree theory in Chapter 2 and Chapter 3, when all the operators  $S^t$  are defined on sets containing the closure of an open and bounded set  $G$ , we apply the above definition just on the set  $\overline{G}$ . We do this without further mention.

Recently, Kien et al [45] extended Browder's degree theory by omitting the conditions  $0 \in T(0)$  and  $0 \in T^t(0)$  for all  $t \in [0, 1]$ . Furthermore, Kien et al [45] obtained existence, additivity and homotopy invariance properties of the degree.

O'Regan et al [62] gave a degree theory for multi-valued bounded  $(S_+)$  perturbations of maximal monotone operators which is analogous to Browder's degree in [18] for single-valued perturbations. We would like to mention here that we believe the assumption  $0 \in T(0)$  is needed in [18]. In fact, the authors of [18] invoke Proposition 6.1.30 at the end of p. 146, which does not apply because it is assumed that  $x \in D(T)$  in that proposition. So, the required homotopy in the proof of Lemma 6.3.1. can not be obtained unless  $0 \in T(0)$ , and one uses the continuity of the mapping  $(t, x) \rightarrow T_t x$  on  $(0, \infty) \times X$ , which was first proved by Kartsatos and Skrypnik in [40]. This continuity is claimed by Zhang and Chen [73, p. 447], but their claim uses Lemma 2.4 which is given only for  $x \in D(T)$ . With such a modification, we have a degree theory for  $T + S$  with  $0 \in T(0)$  as given by authors in [62] satisfying the normalization property with normalizing map  $J$ , existence of zeros and invariant under homotopy of the type  $\{T + S^t\}_{t \in [0,1]}$  where  $T : X \supseteq D(T) \rightarrow 2^{X^*}$  is maximal monotone with  $0 \in T0$  and  $\{S^t\}_{t \in [0,1]}$  is homotopy of class  $(S_+)$ . However, this degree theory requires that

the open bounded subset  $G$  of  $X$  intersects  $D(T)$  which is not required as in Browder theory. For our goal of developing a unique degree mapping for pseudomonotone perturbations of maximal monotone operators in Chapter 3, we require a unique degree mapping for multivalued  $(S_+)$  perturbations of maximal monotone operators which is invariant under homotopy of the type  $\{T^t + S^t\}_{t \in [0,1]}$  where  $\{T^t\}_{t \in [0,1]}$  is pseudomonotone homotopy of maximal monotone operators and  $\{S^t\}_{t \in [0,1]}$  is homotopy of class  $(S_+)$ . We mention here that this degree mapping doesn't meet these important requirements. It is our aim, in Chapter 3, to address these requirements.

For degree theories related to the content of this dissertation, the reader is referred to Browder [12]-[18], Zhang and Chen [73], Kartsatos and Skrypnik [39]-[41], Berkovits and Mustonen [8], Ibrahimou and Kartsatos [32], Leray and Schauder [51], Lloyd [53]. For some papers on degree theories and their applications to various problems in non-linear analysis, we cite Adhikari and Kartsatos [1]-[2], Kartsatos [35]-[36], Kartsatos and Lin [37], Kartsatos and Quarcoo [38]. For monotone operators and associated basic results, we cite Barbu [4], Brézis, Crandall and Pazy [9], Browder [14], Cioranescu [25], Pascali and Sburlan [63] and Zeidler [72]. For an account of degree theories during the last 50 years, we cite Mawhin's article in [55].

2 HOMOTOPY INVARIANCE AND UNIQUENESS RESULTS FOR THE DEGREE  
MAPPING FOR MULTIVALUED  $(S_+)$  PERTURBATIONS <sup>1</sup>

In this Chapter we construct a degree theory for bounded multivalued  $(S_+)$  perturbations of maximal monotone operators. The contents of this Chapter are the main ingredients for the development of the degree theory for bounded multivalued pseudomonotone perturbations of maximal monotone operators possibly with  $0 \notin T(0)$ , which actually this condition should be used in the degree theory of O'Regan et al in [62]. Kien et al went to great lengths in [45] to prove that the Browder degree theory is actually valid without the assumption  $0 \in T(0)$  where the perturbation is single valued mapping of type  $(S_+)$ . This chapter contains new results on the invariance under pseudomonotone homotopies as well as the uniqueness of this degree.

The following uniform boundedness type result, which is due to Ibrahimou and Kartsatos [32] is useful.

**Lemma 2.0.5** (*Ibrahimou and Kartsatos [32].*) *Let  $T : X \subset D(T) \rightarrow 2^{X^*}$  be maximal monotone and  $G \subset X$  be bounded. Let  $0 < s_1 \leq s_2$ ,  $0 < t_1 < t_2$ . Let  $T^s := sT$ . Then there exists a constant  $K_1 > 0$ , independent of  $t, s$ , such that  $\|T_t^s u\| \leq K_1$  for all  $u \in \overline{G}$ ,  $s \in [s_1, s_2]$ ,  $t \in [t_1, t_2]$ .*

---

<sup>1</sup>The content of this chapter is part of the paper by Asfaw and Kartsatos [3]. Teffera M. Asfaw, author of this dissertation, is a Ph.D. candidate under the supervision of Professor Athanassios G.Kartsatos.



## 2.1 New preliminary results

The following three Lemmas are important for the development of our main degree theory as well as the degree theory for multi-valued  $(S_+)$ -perturbations without the basic assumption that  $0 \in T(0)$ . We begin with showing the continuity property of  $T_t x$  in  $(t, x)$ , for a maximal monotone operator  $T$  as in Kartsatos and Skrypnik [40]).

**Lemma 2.1.1** *Let  $T : X \supset D(T) \rightarrow 2^{X^*}$  be maximal monotone and, for  $t > 0$ , let  $T_t$  be the Yosida approximant of  $T$ . Then the mapping  $(t, x) \rightarrow T_t x$  is continuous on  $(0, \infty) \times X$ .*

*Proof.* Let  $\{t_n\} \subset (0, \infty)$  be such that  $t_n \rightarrow t_0 > 0$  and let  $\{x_n\}$  be a sequence in  $X$  such that  $x_n \rightarrow x_0$  as  $n \rightarrow \infty$ . Let  $G \subset X$  be open, bounded and such that  $x_n \in \overline{G}$  for all  $n$ . Furthermore, let  $t_1 > 0$ ,  $t_2 > 0$  be such that  $t_1 < t_2$  and  $t_n \in [t_1, t_2]$  for all  $n$ . By using lemma 2.0.5 for  $s = 1$ , we conclude that there exists  $K > 0$  such that  $\|T_{t_n} x_n\| \leq K$  for all  $n$ . On the other hand, by the definitions of the Yosida resolvent and the Yosida approximant of  $T$ , we know that  $J_{t_n} x_n = x_n - t_n J^{-1}(T_{t_n} x_n)$ ,  $J_{t_n} x_n \in T^{-1}(T_{t_n} x_n)$  and  $J_{t_n} x_n + t_0 J^{-1}(T_{t_n} x_n) \in (T^{-1} + t_0 J^{-1})(T_{t_n} x_n)$ . Therefore, we have

$$\begin{aligned} T_{t_0}(x_n + (t_0 - t_n)J^{-1}(T_{t_n} x_n)) &= T_{t_0}(J_{t_n} x_n + t_0 J^{-1}(T_{t_n} x_n)) \\ &= (T^{-1} + t_0 J^{-1})^{-1}(J_{t_n} x_n + t_0 J^{-1}(T_{t_n} x_n)) \\ &= T_{t_n} x_n \end{aligned}$$

for all  $n$ . Since  $\{T_{t_n} x_n\}$  is bounded, it follows that  $\{J^{-1}(T_{t_n} x_n)\}$  is bounded, and hence  $(t_0 - t_n)J^{-1}(T_{t_n} x_n) \rightarrow 0$  as  $n \rightarrow \infty$ . Therefore, by the continuity of  $T_{t_0}$ , we conclude that  $T_{t_n} x_n \rightarrow T_{t_0} x_0$  as  $n \rightarrow \infty$ . The proof is complete. ■

We next prove a useful Lemma for the extension of the definition of a pseudomonotone homotopy of maximal monotone operators introduced by Browder [15].

**Lemma 2.1.2** *Let  $\{T^t\}_{t \in [0,1]}$  be a family of maximal monotone operators with domains  $D(T^t)$ , respectively. Let  $\{x_n\} \subset X$  be bounded and let  $\{t_n\} \subset [0, 1]$ . Then, for each  $\varepsilon > 0$ , the sequence  $\{T_\varepsilon^{t_n} x_n\}$  is bounded.*

*Proof.* Let  $t_0 > 0$ ,  $x_0 \in X$ ,  $v_n = T_\varepsilon^{t_n} x_n$  and  $v_0 = T_\varepsilon^{t_0} x_0$ . Then  $v_n = ((T^{t_n})^{-1} + \varepsilon J^{-1})^{-1} x_n$  and  $v_0 = ((T^{t_0})^{-1} + \varepsilon J^{-1})^{-1} x_0$ , which implies  $x_0 = (T^{t_0})^{-1} v_0 + \varepsilon J^{-1} v_0$  and  $x_n = (T^{t_n})^{-1} v_n + \varepsilon J^{-1} v_n$  for all  $n$ . For each  $n$ , let  $w_n = (T^{t_n})^{-1} v_n$  and  $w_0 = (T^{t_0})^{-1} v_0$ . Then we have

$$\begin{aligned} \langle v_n - v_0, x_n - x_0 \rangle &= \langle v_n - v_0, w_n - w_0 \rangle \\ &\quad + \varepsilon \langle v_n - v_0, J^{-1} v_n - J^{-1} v_0 \rangle \\ &\geq \langle v_n - v_0, w_n - w_0 \rangle + \varepsilon (\|v_n\| - \|v_0\|)^2 \\ &= \langle v_n - v_0, x_n - \varepsilon J^{-1} v_n - w_0 \rangle + \varepsilon (\|v_n\| - \|v_0\|)^2. \end{aligned}$$

This says

$$\varepsilon (\|v_n\| - \|v_0\|)^2 \leq \|v_n - v_0\| \|x_n - x_0\|,$$

which implies easily the boundedness of  $\{\|v_n\|\}$  and completes the proof. ■

The following Lemma is essential for our degree theory. The equivalence of the four statements in it was given by Browder in [15] under the hypothesis that  $0 \in T^t(0)$  for all  $t \in [0, 1]$ . Lemma 2.1.2 is crucial for its proof.

**Lemma 2.1.3** *Let  $\{T^t\}_{t \in [0,1]}$  be a family of maximal monotone operators. Then the following four conditions are equivalent.*

- (i) *for any sequences  $t_n$  in  $[0, 1]$ ,  $x_n \in D(T^{t_n})$  and  $w_n \in T^{t_n} x_n$  such that  $x_n \rightharpoonup x_0$*

in  $X$ ,  $t_n \rightarrow t_0$ , and  $w_n \rightarrow w_0$  in  $X^*$  with

$$\limsup_{n \rightarrow \infty} \langle w_n, x_n - x_0 \rangle \leq 0,$$

it follows that  $x_0 \in D(T^{t_0})$ ,  $w_0 \in T^{t_0}x_0$  and  $\langle w_n, x_n \rangle \rightarrow \langle w_0, x_0 \rangle$  as  $n \rightarrow \infty$ .

- (ii) for each  $\varepsilon > 0$ , the operator defined by  $\psi(t, w) = (T^t + \varepsilon J)^{-1}w$  is continuous from  $[0, 1] \times X^*$  to  $X$ .
- (iii) for each fixed  $w \in X^*$ , the operator defined by  $\psi_w(t) = (T^t + \varepsilon J)^{-1}w$  is continuous from  $[0, 1]$  to  $X$ .
- (iv) for any given pair  $(x, u) \in G(T^{t_0})$  and any sequence  $t_n \rightarrow t_0$ , there exist sequences  $\{x_n\}$  and  $\{u_n\}$  such that  $u_n \in T^{t_n}x_n$  and  $x_n \rightarrow x$  and  $u_n \rightarrow u$  as  $n \rightarrow \infty$ .

*Proof.* Since  $0 \in T_t(0)$ ,  $t \in [0, 1]$ , is not required in the proof of the implications (ii)  $\Rightarrow$  (iii)  $\Rightarrow$  (iv)  $\Rightarrow$  (i) by Browder[15], their proof is omitted. We only need to prove (i)  $\Rightarrow$  (ii). To this end, let  $\varepsilon > 0$ ,  $t_n \rightarrow t_0 \in [0, 1]$  and  $w_n \rightarrow w_0$  as  $n \rightarrow \infty$ . Let  $u_n = (T^{t_n} + \varepsilon J)^{-1}w_n$  and  $S^{t_n} = (T^{t_n})^{-1}$ . Since  $T^{t_n}$  is maximal monotone, it follows that  $(T^{t_n})^{-1}$  is also maximal monotone. Since  $X$  is reflexive, identifying  $J^{-1}$  with the duality mapping from  $X^*$  to  $X = X^{**}$ , we observe that

$$u_n = ((S^{t_n})^{-1} + \varepsilon J)^{-1}w_n = S_\varepsilon^{t_n}w_n,$$

for all  $n$ , where  $S_\varepsilon^{t_n}$  is the Yosida approximant of  $S^{t_n}$ . Using Lemma 2.1.2, it follows that  $\{u_n\}$  is bounded. Assume, without loss of generality, that  $u_n \rightarrow u$ ,  $Ju_n \rightarrow z$  as  $n \rightarrow \infty$ . Since  $w_n = y_n + \varepsilon Ju_n$ , for some  $y_n \in T^{t_n}u_n$ , it follows that  $y_n \rightarrow y = w_0 - \varepsilon z$ . These facts and the inequality in (i) imply

$$\limsup_{n \rightarrow \infty} \langle y_n, u_n - u \rangle \leq 0$$

and, consequently,

$$\limsup_{n \rightarrow \infty} \langle y_n, u_n \rangle \leq \langle y, u \rangle.$$

Thus, using (i), we have  $y \in T^{t_0}u$  and  $\langle y_n, u_n \rangle \rightarrow \langle y, u \rangle$ , which implies

$$\limsup_{n \rightarrow \infty} \langle Ju_n, u_n - u \rangle = 0.$$

Since  $J$  is of type  $(S_+)$  and continuous, we have  $u_n \rightarrow u$  and  $Ju_n \rightarrow Ju$  as  $n \rightarrow \infty$ , which says  $y = w_0 - \varepsilon Ju$ , i.e.  $w_0 \in (T^{t_0} + \varepsilon J)u$ , implying in turn  $u = (T^{t_0} + \varepsilon J)^{-1}w_0$ . This proves the continuity of  $\psi$  on  $[0, 1] \times X^*$ . ■

We give the following definition.

**Definition 2.1.4** Let  $\{T^t\}_{t \in [0,1]}$  be a family of maximal monotone operators. Then the family  $\{T^t\}_{t \in [0,1]}$  is called a “pseudomonotone homotopy” if one of the four equivalent conditions of Lemma 2.1.3 holds true.

We observe, again, that this class of homotopies is larger than the class of pseudomonotone homotopies introduced by Browder in [15], because in our case we do not require the condition  $0 \in T^t(0)$  for all  $t \in [0, 1]$ . We also observe that the interval  $[0, 1]$  in Lemma 2.1.3 and Definition 2.1.4 may be replaced by any other finite interval  $[a, b]$  of the real line.

In this Section we construct a degree mapping for operators of the type  $T + S$ , where  $T$  is maximal monotone and  $S$  is a multi-valued bounded mapping of type  $(S_+)$  without assuming  $0 \in T(0)$ . This complements the degree theory of O’Regan et al in [62], and Kien et al in [45]. We also include results of invariance under pseudomonotone homotopies and uniqueness.

## 2.2 Construction of the degree, basic properties and homotopy invariance results

We construct the degree mapping for the sum  $T + S$ , with  $T$  maximal monotone and  $S$  of type  $(S_+)$ , based on the degree mapping  $d(T_\varepsilon + S, G, 0)$  for multi-valued  $(S_+)$ -mappings developed by Zhang and Chen in [73]. Furthermore, we give the basic properties of this degree mapping, including the important property of invariance under pseudomonotone homotopies, which is a new result in our setting.

We should mention here that the invariance of the degree mapping under pseudomonotone homotopies was considered by Kien et al in [45] for single-valued perturbations  $S$ , but was not included in [62].

**Lemma 2.2.1** *Let  $G \subset X$  be open and bounded. Let  $T : X \supset D(T) \rightarrow 2^{X^*}$  be maximal monotone, and let  $S : \overline{G} \rightarrow 2^{X^*}$  be bounded and of type  $(S_+)$  such that  $0 \notin (T + S)(D(T) \cap \partial G)$ . Then there exists  $\varepsilon_0 > 0$  such that  $d(T_\varepsilon + S, G, 0)$  is well-defined and independent of  $\varepsilon \in (0, \varepsilon_0]$ .*

*Proof.* We first claim that there exists  $\varepsilon_0 > 0$  such that  $0 \notin (T_\varepsilon + S)(\partial G)$  for all  $\varepsilon \in (0, \varepsilon_0]$ . Suppose that this is false, i.e. suppose that there exist  $\{\varepsilon_n\}$  such that  $\varepsilon_n \downarrow 0^+$ ,  $x_n \in \partial G$  and  $w_n \in Sx_n$  such that

$$v_n + w_n = 0 \tag{2.2.1}$$

for all  $n$ , where  $v_n = T_{\varepsilon_n}x_n$ . Since  $\{x_n\}$  is bounded, the boundedness of  $S$  implies that  $\{w_n\}$  is bounded, which in turn implies that  $\{v_n\}$  is bounded. Assume w.l.o.g. that  $x_n \rightharpoonup x_0$ ,  $w_n \rightharpoonup w_0$  and  $v_n \rightharpoonup v_0$  as  $n \rightarrow \infty$ . Using the  $(S_+)$ -condition on  $S$ , it is easy to see that

$$\liminf_{n \rightarrow \infty} \langle w_n, x_n - x_0 \rangle \geq 0. \tag{2.2.2}$$

Using (4.2.1) and (2.2.2), we see that

$$\limsup_{n \rightarrow \infty} \langle v_n - v_0, x_n - x_0 \rangle \leq 0.$$

Using the maximality of  $T$ , and applying Lemma 1.2.5, it follows that  $x_0 \in D(T)$ ,  $v_0 \in Tx_0$  and  $\langle v_n, x_n \rangle \rightarrow \langle v_0, x_0 \rangle$  as  $n \rightarrow \infty$ . As a result of this, we see that (4.2.1) yields

$$\lim_{n \rightarrow \infty} \langle w_n, x_n - x_0 \rangle = 0.$$

Since  $S$  is of type  $(S_+)$ , it follows that  $x_n \rightarrow x_0 \in \partial G$ , and there exists a subsequence of  $\{w_n\}$ , denoted again by  $\{w_n\}$ , such that  $w_n \rightarrow w_1 \in Sx_0$  as  $n \rightarrow \infty$ . This implies  $w_0 = w_1 \in Sx_0$  and  $v_0 + w_0 = 0$ , which contradicts our assumption.

Next, we show that  $d(T_\varepsilon + S, G, 0)$  is constant for all  $\varepsilon \in (0, \varepsilon_0]$ . Let  $\varepsilon_i \in (0, \varepsilon_0]$ ,  $i = 1, 2$ , be such that  $\varepsilon_1 < \varepsilon_2$ . To complete the proof, it suffices to show that the family  $\{H(t, \cdot)\}_{t \in [0, 1]}$  is a homotopy of type  $(S_+)$ , where

$$H(t, x) := T_{\lambda(t)}x + Sx, \quad (t, x) \in [0, 1] \times \overline{G}. \quad (2.2.3)$$

where  $\lambda(t) = t\varepsilon_1 + (1 - t)\varepsilon_2$ ,  $t \in [0, 1]$ . To this end, let  $\{t_n\} \subset [0, 1]$ ,  $\{x_n\} \subset \overline{G}$  be such that  $t_n \rightarrow t_0$ ,  $x_n \rightarrow x_0$ , as  $n \rightarrow \infty$ , and  $w_n \in Sx_n$  is such that

$$\limsup_{n \rightarrow \infty} \langle T_{\lambda_n}x_n + w_n, x_n - x_0 \rangle \leq 0, \quad (2.2.4)$$

where  $\lambda_n = \lambda(t_n)$ . It is easy to see that  $\lambda_n \rightarrow \lambda_0 > 0$  as  $n \rightarrow \infty$ . Using the monotonicity of  $T$  and (2.2.4), we obtain

$$\limsup_{n \rightarrow \infty} \langle T_{\lambda_n}x_0 + w_n, x_n - x_0 \rangle \leq 0. \quad (2.2.5)$$

Now, using Lemma 2.1.1, we see that  $T_{\lambda_n}x_0 \rightarrow T_{\lambda_0}x_0$  as  $n \rightarrow \infty$ , where  $\lambda_0 = t_0\varepsilon_1 +$

$(1 - t_0)\varepsilon_2$ , which implies

$$\lim_{n \rightarrow \infty} \langle T_{\lambda_n} x_0, x_n - x_0 \rangle = 0. \quad (2.2.6)$$

Using (2.2.5) and (2.2.6), we obtain

$$\limsup_{n \rightarrow \infty} \langle w_n, x_n - x_0 \rangle \leq 0.$$

Since  $S$  is of type  $(S_+)$ , it follows that  $x_n \rightarrow x_0$  as  $n \rightarrow \infty$ . For each  $n$ , we let

$$f_n = T_{\lambda_n} x_n + w_n.$$

Since  $S$  is of type  $(S_+)$ , we may assume w.l.o.g. that  $w_n \rightarrow w_0 \in Sx_0$  as  $n \rightarrow \infty$ . This implies  $f_n \rightarrow T_{\lambda_0} x_0 + w_0 \in H(t_0, x_0)$  as  $n \rightarrow \infty$  and proves that  $\{H(t, \cdot)\}_{t \in [0, 1]}$  is a homotopy of type  $(S_+)$  such that  $0 \notin H(t, \partial G)$  for all  $t \in [0, 1]$ . Therefore, by the invariance of the degree under homotopies of type  $(S_+)$ , we get

$$d(H(0, \cdot), G, 0) = d(H(1, \cdot), G, 0),$$

which says that  $d(T_{\varepsilon_1} + S, G, 0) = d(T_{\varepsilon_2} + S, G, 0)$ . Since  $\varepsilon_1, \varepsilon_2 \in (0, \varepsilon_0]$  are arbitrary, it follows that  $d(T_\varepsilon + S, G, 0)$  is independent of  $\varepsilon \in (0, \varepsilon_0]$ . This completes the proof.  $\blacksquare$

The definition of the degree mapping for bounded multi-valued  $(S_+)$ -perturbations of maximal monotone operators  $T$  is given below.

**Definition 2.2.2 (Degree for Multi-valued  $(S_+)$ - Perturbations)** Let  $G$  be a bounded open subset of  $X$ . Let  $T : X \supset D(T) \rightarrow 2^{X^*}$  be maximal monotone and let  $S : \overline{G} \rightarrow 2^{X^*}$  be bounded and of type  $(S_+)$  and such that  $0 \notin (T + S)(D(T) \cap \partial G)$ . We define the degree of  $T + S$ , denoted by  $d(T + S, G, 0)$ , by

$$d(T + S, G, 0) = \lim_{\varepsilon \downarrow 0^+} d(T_\varepsilon + S, G, 0)$$

where  $T_\varepsilon$  is the Yosida approximant of  $T$  and  $d(T_\varepsilon + S, G, 0)$  is the degree for multi-valued  $(S_+)$  mappings constructed by Zhang and Chen in [73]. We set

$$d(T + S, G, 0) = 0 \text{ if } G \cap D(T) = \emptyset.$$

The basic classical properties of the degree mapping are included in the following theorem.

**Theorem 2.2.3** *Let  $G$  be a bounded open subset of  $X$ . Let  $T : X \supset D(T) \rightarrow 2^{X^*}$  be maximal monotone and  $S : \bar{G} \rightarrow 2^{X^*}$  bounded and of type  $(S_+)$ . Then the following properties are true.*

- (i) *(Normalization)  $d(J, G, 0) = 1$  if  $0 \in G$ , and  $d(J, G, 0) = 0$  if  $0 \notin \bar{G}$ .*
- (ii) *(Existence) if  $0 \notin (T + S)(D(T) \cap \partial G)$  and  $d(T + S, G, 0) \neq 0$ , then the inclusion  $Tx + Sx \ni 0$  is solvable in  $D(T) \cap G$ .*
- (iii) *(Decomposition) If  $G_1$  and  $G_2$  are disjoint open subsets of  $G$  such that  $0 \notin (T + S)(D(T) \cap (\bar{G} \setminus (G_1 \cup G_2)))$ , then*

$$d(T + S, G, 0) = d(T + S, G_1, 0) + d(T + S, G_2, 0).$$

*Proof.* The proof for (i) is obvious. We give the proofs for (ii) and (iii).

(ii) Suppose that  $d(T + S, G, 0) \neq 0$ . Then there exist sequences  $\{\varepsilon_n\}, \{x_n\} \subset G$  and  $w_n \in Sx_n$  such that  $\varepsilon_n \downarrow 0^+$  and

$$T_{\varepsilon_n}x_n + w_n = 0 \tag{2.2.7}$$

for all  $n$ . Let  $v_n = T_{\varepsilon_n}x_n$ . By the boundedness of  $S$ , it follows that  $\{w_n\}$  is bounded, and hence  $\{v_n\}$  is bounded. Assume w.l.o.g. that  $x_n \rightharpoonup x_0$ ,  $v_n \rightharpoonup v_0$  and  $w_n \rightharpoonup w_0$  as



$n \rightarrow \infty$ . Since  $S$  is of type  $(S_+)$ , we have

$$\liminf_{n \rightarrow \infty} \langle w_n, x_n - x_0 \rangle \geq 0. \quad (2.2.8)$$

Since  $T_{\varepsilon_n} x_n \in T(J_{\varepsilon_n} x_n)$  for all  $n$  and  $J_{\varepsilon_n} x_n - x_n = -\varepsilon_n J^{-1}(v_n) \rightarrow 0$  as  $n \rightarrow \infty$ , using (2.2.7) and (2.2.8), we obtain

$$\limsup_{n \rightarrow \infty} \langle v_n - v_0, y_n - x_0 \rangle \leq 0.$$

Using Lemma 1.2.5, it follows that  $x_0 \in D(T)$ ,  $v_0 \in Tx_0$  and  $\langle v_n, y_n \rangle \rightarrow \langle v_0, x_0 \rangle$  as  $n \rightarrow \infty$ . Using this and (2.2.7), we see that

$$\limsup_{n \rightarrow \infty} \langle w_n, x_n - x_0 \rangle \leq 0.$$

Since  $S$  is of type  $(S_+)$ , it follows that  $x_n \rightarrow x_0 \in \overline{G}$  as  $n \rightarrow \infty$ , and  $w_0 \in Sx_0$ . Since  $0 \notin (T + S)(D(T) \cap \partial G)$ , we have  $x_0 \in D(T) \cap G$ ,  $v_0 \in Tx_0$  and  $w_0 \in Sx_0$  with  $v_0 + w_0 = 0$ .

(iii) Since  $0 \notin (T + S)(D(T) \cap (\overline{G} \setminus (G_1 \cup G_2)))$ , it is easy to show that there exists  $\varepsilon_0 > 0$  such that  $0 \notin (T_\varepsilon + S)(D(T) \cap (\overline{G} \setminus (G_1 \cup G_2)))$  for all  $\varepsilon \in (0, \varepsilon_0]$ . Since  $T_\varepsilon + S$  is of type  $(S_+)$  on  $\overline{G}$ , it follows that

$$d(T_\varepsilon + S, G, 0) = d(T_\varepsilon + S, G_1, 0) + d(T_\varepsilon + S, G_2, 0)$$

for all  $\varepsilon \in (0, \varepsilon_0]$ . As a result of this, we conclude, by the definition of the degree, that

$$d(T + S, G, 0) = d(T + S, G_1, 0) + d(T + S, G_2, 0).$$

The proof is complete. ■

In the following theorem we give a homotopy invariance result for homotopies of the type  $H(t, x) = T^t x + S^t x$ ,  $(t, x) \in [0, 1] \times D(T^t) \cap \overline{G}$ , where  $\{T^t\}_{t \in [0,1]}$  is a pseudomonotone homotopy in the sense of Definition 2.1.4 and  $\{S^t\}_{t \in [0,1]}$  is a homotopy of type  $(S_+)$ . In our proof, we follow Browder's original approach in [15] and the Kien et al approach in [45], but we have given a shorter proof for multi-valued perturbations that does not need the conclusion of Lemma 3.3 of Kien et al in [45]. This result is new and was not included in the book of O'Regan et al [62]. We first need the following lemma.

**Lemma 2.2.4** *Let  $\{S^t\}_{t \in [0,1]}$  be a homotopy of type  $(S_+)$  from  $\overline{G}$  to  $2^{X^*}$ , and let  $t_n \in [0, 1]$ ,  $x_n \in \overline{G}$  and  $w_n \in S^{t_n} x_n$  be such that  $t_n \rightarrow t_0$ ,  $x_n \rightarrow x_0 \in X$  and  $w_n \rightarrow w_0$  as  $n \rightarrow \infty$ . Then  $d \geq 0$ , where*

$$d = \liminf_{n \rightarrow \infty} \langle w_n, x_n - x_0 \rangle.$$

*Proof.* Assume that  $d < 0$ . Then w.l.o.g. we may assume that

$$\lim_{n \rightarrow \infty} \langle w_n, x_n - x_0 \rangle = d < 0.$$

Since  $\{S^t\}_{t \in [0,1]}$  is a homotopy of type  $(S_+)$ , we have  $x_n \rightarrow x_0 \in \overline{G}$  and there exists a subsequence of  $\{w_n\}$ , denoted again by  $\{w_n\}$ , such that  $w_n \rightarrow w_0 \in S^{t_0} x_0$  as  $n \rightarrow \infty$ . This says that  $d = 0$ , which is a contradiction. Consequently,

$$\liminf_{n \rightarrow \infty} \langle w_n, x_n - x_0 \rangle \geq 0.$$

and the proof is complete. ■

The following basic result establishes the invariance of the degree under pseudomonotone homotopies. It is a new result in our setting.

**Theorem 2.2.5** *Let  $G$  be a bounded open subset of  $X$ . For each  $t \in [0, 1]$ , let  $T^t : X \supset D(T^t) \rightarrow 2^{X^*}$  be a maximal monotone operator and  $S^t : \overline{G} \rightarrow 2^{X^*}$  a bounded mapping of type  $(S_+)$ . Assume that  $\{T^t\}_{t \in [0,1]}$  is a pseudomonotone homotopy of maximal monotone operators and  $\{S^t\}_{t \in [0,1]}$  is a homotopy of type  $(S_+)$ , uniformly bounded for  $t \in [0, 1]$  and such that  $0 \notin (T^t + S^t)(D(T^t) \cap \partial G)$  for all  $t \in [0, 1]$ . Then there exists  $\varepsilon_0 > 0$  such that*

- (i)  $0 \notin (T_\varepsilon^t + S^t)(\partial G)$  for all  $t \in [0, 1]$  and  $\varepsilon \in (0, \varepsilon_0]$ ;
- (ii)  $d(T_\varepsilon^t + S^t, G, 0)$  is independent of  $t \in [0, 1]$  and  $\varepsilon \in (0, \varepsilon_0]$ .

*Proof.*

(i) Suppose the assertion is false, i.e. there exist sequences  $\varepsilon_n \downarrow 0^+$ ,  $t_n \in [0, 1]$ ,  $x_n \in \partial G$  and  $w_n \in S_{t_n} x_n$  such that

$$v_n + w_n = 0, \tag{2.2.9}$$

where  $v_n = T_{\varepsilon_n}^{t_n} x_n$  for all  $n$ . By the boundedness of  $\{w_n\}$ , it follows that  $\{v_n\}$  is bounded. Assume w.l.o.g. that  $t_n \rightarrow t_0$ ,  $x_n \rightharpoonup x_0$ ,  $v_n \rightharpoonup v_0$  and  $w_n \rightharpoonup w_0$  as  $n \rightarrow \infty$ . Using Lemma 2.2.4 and (2.2.9), we have

$$\limsup_{n \rightarrow \infty} \langle v_n, x_n - x_0 \rangle \leq 0.$$

We observe that  $v_n \in T^{t_n}(y_n)$ , where  $y_n = x_n - \varepsilon_n J^{-1}(v_n)$  and  $y_n \rightharpoonup x_0$  as  $n \rightarrow \infty$ .

Thus,

$$\limsup_{n \rightarrow \infty} \langle v_n, y_n \rangle \leq \langle v_0, x_0 \rangle.$$

By the pseudomonotonicity of  $\{T^t\}_{t \in [0,1]}$ , it follows that  $(x_0, v_0) \in G(T^{t_0})$  and  $\langle v_n, y_n \rangle \rightarrow \langle v_0, x_0 \rangle$ , which implies, using (2.2.9) again,

$$\limsup_{n \rightarrow \infty} \langle w_n, x_n - x_0 \rangle \leq 0.$$

Since  $\{S^t\}_{t \in [0,1]}$  is a homotopy of type  $(S_+)$ , it follows that  $x_n \rightarrow x_0 \in \partial G$  and there exists a subsequence of  $\{w_n\}$ , denoted again by  $\{w_n\}$ , such that  $w_n \rightarrow w_0 \in S^{t_0}x_0$  as  $n \rightarrow \infty$ . Therefore, we conclude that  $x_0 \in D(T^{t_0}) \cap \partial G$ ,  $v_0 \in T^{t_0}x_0$  and  $w_0 \in S^{t_0}x_0$  with  $v_0 + w_0 = 0$ . However, this is a contradiction. Thus, (i) holds true.

(ii) Using (i), we see that, for each  $t \in [0, 1]$ ,  $d(T^t + S^t, G, 0)$  is well-defined. We show that  $d(T_\varepsilon^t + S^t, G, 0)$  is independent of  $\varepsilon \in (0, \varepsilon_0]$  and  $t \in [0, 1]$ . Suppose that there exist  $\varepsilon_n \downarrow 0^+$ ,  $\delta_n \downarrow 0^+$  and  $t_n \in [0, 1]$  such that

$$d(T_{\varepsilon_n}^{t_n} + S^{t_n}, G, 0) \neq d(T_{\delta_n}^{t_n} + S^{t_n}, G, 0) \quad (2.2.10)$$

for all  $n$ . For each  $n$ , we consider the homotopy

$$H_n(s, x) = sT_{\varepsilon_n}^{t_n}x + (1 - s)T_{\delta_n}^{t_n}x + S^{t_n}x, \quad (s, x) \in [0, 1] \times \overline{G}.$$

We observe that,  $\{H_n(t, \cdot)\}_{t \in [0,1]}$  is a homotopy of type  $(S_+)$ . We show that  $0 \notin H_n(t, \partial G)$  for all  $t \in [0, 1]$  and all large  $n$ . Suppose there exists a subsequence of  $\{n\}$ , denoted again by  $\{n\}$ , such that there exist  $x_n \in \partial G$ ,  $s_n \in [0, 1]$  and  $w_n \in S^{t_n}x_n$  such that

$$s_n T_{\varepsilon_n}^{t_n} x_n + (1 - s_n) T_{\delta_n}^{t_n} x_n + w_n = 0 \quad (2.2.11)$$

for all  $n$ . For each  $n$ , we let

$$v_n = T_{\varepsilon_n}^{t_n} x_n, \quad \tau_n = T_{\delta_n}^{t_n} x_n, \quad z_n = s_n v_n + (1 - s_n) \tau_n.$$

Since  $\{x_n\}$  is bounded, the sequence  $\{w_n\}$  is also bounded, and hence  $\{z_n\}$  is bounded.

Assume, w.l.o.g. that  $t_n \rightarrow t_0$ ,  $s_n \rightarrow s_0$ ,  $x_n \rightarrow x_0$ ,  $w_n \rightarrow w_0$  and  $z_n \rightarrow z_0$  as  $n \rightarrow \infty$ .

Using the  $(S_+)$ -condition on the family  $\{S^t\}_{t \in [0,1]}$ , in view of Lemma 2.2.4, that

$$\liminf_{n \rightarrow \infty} \langle w_n, x_n - x_0 \rangle \geq 0,$$

which implies in turn that

$$\limsup_{n \rightarrow \infty} \langle z_n, x_n - x_0 \rangle \leq 0, \text{ i.e. } \limsup_{n \rightarrow \infty} \langle z_n, x_n \rangle \leq \langle z_0, x_0 \rangle. \quad (2.2.12)$$

By the properties of the Yosida approximants and the resolvents of maximal monotone operators, we see that

$$v_n \in T^{t_n}(x_n - \varepsilon_n J^{-1}(v_n)), \quad \tau_n \in T^{t_n}(x_n - \delta_n J^{-1}(\tau_n))$$

for all  $n$ . We consider two cases.

**Case I.**  $\{s_n v_n\}$  is bounded. Since  $\{z_n\}$  is bounded,  $\{(1 - s_n)\tau_n\}$  is also bounded. Using condition (iv) of Lemma 2.1.3, for any  $(x, y) \in G(T^{t_0})$  there exists a sequence  $(u_n, u_n^*) \in G(T^{t_n})$  such that  $u_n \rightarrow x$  and  $u_n^* \rightarrow y$  as  $n \rightarrow \infty$ . Moreover, using the monotonicity of  $T^{t_n}$ , we get

$$\langle v_n - u_n^*, x_n - \varepsilon_n J^{-1}(v_n) - u_n \rangle \geq 0,$$

which implies

$$\begin{aligned} \langle v_n, x_n \rangle &\geq \langle v_n, u_n \rangle + \langle u_n^*, x_n - u_n \rangle \\ &\quad + \varepsilon_n \|v_n\|^2 - \varepsilon_n \|u_n^*\| \|v_n\|. \end{aligned} \quad (2.2.13)$$

Similarly, we have

$$\begin{aligned} \langle \tau_n, x_n \rangle &\geq \langle \tau_n, u_n \rangle + \langle u_n^*, x_n - u_n \rangle \\ &\quad + \delta_n \|v_n\|^2 - \delta_n \|u_n^*\| \|\tau_n\| \end{aligned} \quad (2.2.14)$$

for all  $n$ . Multiplying (2.2.13) and (2.2.14) by  $s_n$  and  $(1 - s_n)$ , respectively, and adding

the resulting inequalities, we obtain

$$\begin{aligned}
\langle z_n, x_n \rangle &\geq \langle z_n, u_n \rangle + \langle u_n^*, x_n - u_n \rangle \\
&\quad + s_n \varepsilon_n (\|v_n\|^2 - \|u_n^*\| \|v_n\|) \\
&\quad + (1 - s_n) \delta_n (\|\tau_n\|^2 - \|u_n^*\| \|\tau_n\|)
\end{aligned} \tag{2.2.15}$$

for all  $n$ . Consequently, using (2.2.12) and (2.2.15), we obtain

$$\begin{aligned}
\langle z_0, x_0 \rangle &\geq \liminf_{n \rightarrow \infty} \langle z_n, x_n \rangle \\
&\geq \liminf_{n \rightarrow \infty} \langle z_n, u_n \rangle + \langle u_n^*, x_n - u_n \rangle \\
&\quad - \limsup_{n \rightarrow \infty} [s_n \varepsilon_n \|v_n\| \|u_n^*\| + (1 - s_n) \delta_n \|u_n^*\| \|\tau_n\|] \\
&= \langle z_0, x \rangle + \langle y, x_0 - x \rangle
\end{aligned} \tag{2.2.16}$$

for all  $[x, y] \in G(T)$ , which yields  $\langle z_0 - y, x_0 - x \rangle \geq 0$ . Using the maximal monotonicity of  $T^{t_0}$ , we conclude that  $x_0 \in D(T^{t_0})$  and  $z_0 \in T^{t_0} x_0$ . Moreover, letting  $x = x_0$  and  $y = z_0$ , we get

$$\lim_{n \rightarrow \infty} \langle z_n, x_n \rangle = \langle z_0, x_0 \rangle.$$

Finally, from (2.2.11) we get

$$\limsup_{n \rightarrow \infty} \langle w_n, x_n - x_0 \rangle \leq 0.$$

Since  $\{S^t\}_{t \in [0,1]}$  is a homotopy of type  $(S_+)$ , it follows that  $x_n \rightarrow x_0 \in \partial G$  and there exists a subsequence of  $\{w_n\}$ , denoted again by  $\{w_n\}$ , such that  $w_n \rightarrow w_1 = w_0 \in S^{t_0} x_0$ . Thus,  $x_0 \in D(T^{t_0}) \cap \partial G$ ,  $z_0 \in T^{t_0} x_0$  and  $w_0 \in S^{t_0} x_0$  such that  $z_0 + w_0 = 0$ . This implies  $0 \in (T^{t_0} + S^{t_0})(D(T^{t_0}) \cap \partial G)$ , i.e. a contradiction.

**Case II:** Suppose  $\{s_n v_n\}$  is unbounded. Then there exists a subsequence, which we call again  $\{s_n v_n\}$ , such that  $s_n \|v_n\| \rightarrow +\infty$  as  $n \rightarrow \infty$ . Then  $\{(1 - s_n) \tau_n\}$ ,  $\{v_n\}$  and

$\{\tau_n\}$  are unbounded. Assume w.l.o.g. that  $\|v_n\| \rightarrow \infty$  and  $\|\tau_n\| \rightarrow \infty$  as  $n \rightarrow \infty$ . If either  $\{\varepsilon_n s_n \|v_n\|^2\}$  or  $\{\delta_n(1-s_n)\|\tau_n\|^2\}$  is unbounded, from (2.2.15) we obtain

$$\begin{aligned} \langle z_n, x_n \rangle &\geq \langle z_n, u_n \rangle + \langle u_n^*, x_n - u_n \rangle \\ &\quad + s_n \varepsilon_n \|v_n\|^2 \left(1 - \frac{\|u_n^*\|}{\|v_n\|}\right) \\ &\quad + (1-s_n) \delta_n (\|\tau_n\|^2) \left(1 - \frac{\|u_n^*\|}{\|\tau_n\|}\right) \end{aligned} \tag{2.2.17}$$

Assuming  $\varepsilon_n s_n \|v_n\|^2 \rightarrow \infty$  or  $\delta_n(1-s_n)\|\tau_n\|^2 \rightarrow \infty$ , and taking limits in (2.2.17), we find

$$\langle z_0 - y, x_0 - x \rangle \geq \infty,$$

which is impossible. Thus,  $\{\varepsilon_n s_n \|v_n\|^2\}$  and  $\{\delta_n(1-s_n)\|\tau_n\|^2\}$  are bounded. Consequently,

$$s_n \varepsilon_n \|v_n\| = \frac{s_n \varepsilon_n \|v_n\|^2}{\|v_n\|} \rightarrow 0$$

as  $n \rightarrow \infty$ . Similarly, we have  $(1-s_n)\delta_n\|\tau_n\| \rightarrow 0$  as  $n \rightarrow \infty$ . Using these and as in Case I, we arrive at a contradiction. Therefore, the family  $\{H_n(t, \cdot)\}_{t \in [0,1]}$  is a homotopy of type  $(S_+)$  such that  $0 \notin H_n(t, \partial G)$  for all  $t \in [0, 1]$  and all large  $n$ . Thus, for each  $n$ , we have  $d(H_n(t, \cdot), G, 0)$  is independent of  $t \in [0, 1]$ , i.e. we have

$$d(T_{\varepsilon_n}^{t_n} + S, G, 0) = d(T_{\delta_n}^{t_n} + S, G, 0),$$

which is a contradiction of (2.2.10). Therefore, we conclude that there exists  $\varepsilon_0 > 0$  such that  $d(T_\varepsilon^t + S^t, G, 0)$  is independent of  $t \in [0, 1]$  and  $\varepsilon \in (0, \varepsilon_0]$ , i.e.  $d(T^t + S^t, G, 0)$  is independent of  $t \in [0, 1]$ . This completes the proof. ■

Kien et al mentioned in [45] that Browder used the condition  $0 \in T^t(0)$ , for all  $t \in [0, 1]$ , to prove that there exists an  $M > 0$  such that  $s_n \varepsilon_n \|v_n\|^2 \leq M$  and  $(1-s_n)\delta_n\|\tau_n\|^2 \leq M$ , and subsequently conclude that  $s_n \varepsilon_n \|v_n\| \rightarrow 0$  and  $(1-s_n)\delta_n\|\tau_n\| \rightarrow$

0 as  $n \rightarrow \infty$ . On the other hand, they remarked that this scheme would collapse if the condition  $0 \in T^t(0)$  were omitted. In proof of Theorem 2.2.5 we have demonstrated the fact that Browder's approach still holds true even if the condition  $0 \in T^t(0)$  is omitted.

### 2.3 Uniqueness of the degree

The uniqueness of the degree was considered by Kien et al in [45] for densely defined operators  $T$  and single-valued operators  $S$ , but it was not included in the book of O'Regan et al in [62]. It is well known from the degree theory of Browder [17] that the degree mapping constructed for single valued demicontinuous mapping of type  $(S_+)$  is unique and invariant under affine homotopies of demicontinuous mappings of type  $(S_+)$  defined on the closure of bounded open subset  $G$  of  $X$ . The degree constructed for multi-valued mappings of type  $(S_+)$  by Zhang and Chen et al [73] can be easily shown to be unique following the construction of the uniqueness result of Browder. Furthermore, Browder [15] gave a degree theory for bounded single-valued demicontinuous  $(S_+)$  perturbation  $f : \overline{G} \rightarrow X^*$  of maximal monotone operators  $T : X \supseteq D(T) \rightarrow 2^{X^*}$ . In [17], Browder gave a uniqueness result of his degree provided that it is invariant under affine homotopies of the form

$$H(t, x) = (1 - t)(T + f_1) + tf_2, \quad t \in [0, 1],$$

where  $T$  is maximal monotone, and  $f_1$  and  $f_2$  are bounded demicontinuous and of type  $(S_+)$ . However, Kobayashi and Otani proved in [48] that the homotopy  $H(t, \cdot)$  preserves the Browder degree if and only if  $\overline{D(T)} = X$ . On the other hand, Berkovits and Miettunen [7] proved the uniqueness of the Browder degree without assuming the condition  $\overline{D(T)} = X$  provided that the degree is invariant under a homotopy of the form  $\{T^t + S^t\}_{t \in [0,1]}$  where  $\{T^t\}_{t \in [0,1]}$  is a pseudomonotone homotopy of maximal



monotone mappings, with  $0 \in T^t 0$  for all  $t \in [0, 1]$ , and  $\{S^t\}_{t \in [0, 1]}$  is a homotopy of type  $(S_+)$ . In the following theorem, we give the uniqueness result for the degree mapping. We follow the approach of Berkovits and Miettunen in [7] but we have eliminated the condition  $0 \in T(0)$ .

**Lemma 2.3.1** *Let  $T : X \supseteq D(T) \rightarrow 2^{X^*}$  be maximal monotone and let  $G$  be a bounded open subset of  $X$ . Let  $\bar{\varepsilon} > 0$  be fixed. Let*

$$H_t x = T_{t\bar{\varepsilon}} x, \quad t \in [0, 1],$$

where  $T_t x = (T^{-1} + t\bar{\varepsilon}J^{-1})^{-1}x$ ,  $D(H_t) = X$  for  $t \in (0, 1]$  and  $D(H_0) = D(T)$ . Then the family  $\{H_t\}_{t \in [0, 1]}$  is a pseudomonotone homotopy of maximal monotone mappings.

*Proof.* We observe that  $H_t$  is a single-valued continuous maximal monotone mapping with  $D(H_t) = X$  for all  $t \in (0, 1]$ , and  $H_0 = T$ . We use (iv) of Lemma 2.1.3. Let  $t_n \in [0, 1]$ ,  $t_n \rightarrow t_0 \in (0, 1]$  and  $(x, w) \in G(H_{t_0})$ , i.e.  $w = H_{t_0}x$ . Let  $x_n = x$  and  $w_n = H_{t_n}x$ . By using Lemma 2.1.1, we see that  $w_n \rightarrow H_{t_0}x = w$ , i.e.  $x_n \rightarrow x$  and  $w_n \rightarrow w$  as  $n \rightarrow \infty$ . Next we assume  $t_0 = 0$  and take  $w_n = w$  and  $x_n = x + t_n\bar{\varepsilon}J^{-1}(w)$ . Then  $w_n \in T(x_n - t_n\bar{\varepsilon}J^{-1}(w_n))$ . Thus,  $x_n \rightarrow x$ ,  $w_n \rightarrow w$  as  $n \rightarrow \infty$  and the proof is complete. ■

**Theorem 2.3.2** *Let  $G$  be a bounded open subset of  $X$ ,  $T : X \supseteq D(T) \rightarrow 2^{X^*}$  maximal monotone and  $S : \bar{G} \rightarrow 2^{X^*}$  bounded, of type  $(S_+)$  and such that  $0 \notin (T + S)(D(T) \cap \partial G)$ . Then there exists exactly one degree mapping defined on the class of mappings of the form  $T + S$ , satisfying the basic properties (i)-(iii) in Theorem 2.2.3, invariant under the homotopy in Theorem 2.2.5 and normalized by  $J$ .*

*Proof.* Let  $\tilde{d}$  be another degree mapping defined for multi-valued  $(S_+)$  perturbations of maximal monotone operators satisfying the basic properties of the degree  $d$  in Theorem 2.2.3 and Theorem 2.2.5. Suppose  $0 \notin (T + S)(D(T) \cap \partial G)$ . By the construction

of the degree, we see that  $0 \notin (T_\varepsilon + S)(\partial G)$  and

$$d(T + S, G, ) = d(T_\varepsilon + S, G, 0)$$

for all sufficiently small  $\varepsilon > 0$ . Taking  $T = 0$ , we see that both  $d$  and  $\tilde{d}$  are well-defined degree mappings on the class of bounded mappings of type  $(S_+)$  satisfying the basic properties of the unique degree. Therefore, by uniqueness of this degree on  $(S_+)$  mappings, we see that  $d = \tilde{d}$ , i.e which implies

$$d(T_\varepsilon + S, G, 0) = \tilde{d}(T_\varepsilon + S, G, 0)$$

for all sufficiently small  $\varepsilon > 0$ . Considering the pseudomonotone homotopy  $\{H_t\}_{t \in [0,1]}$  guaranteed by Lemma 2.3.1 and observing that  $0 \notin (H_t + S)(\partial G)$  for all  $t \in [0, 1]$ , we see that  $\tilde{d}$  is invariant under homotopies of the type  $\{H_t + S\}_{t \in [0,1]}$ . Consequently,  $\tilde{d}(H_1, G, 0) = \tilde{d}(H_0, G, 0)$  which implies  $\tilde{d}(T_\varepsilon + S, G, 0) = \tilde{d}(T + S, G, 0)$ . Thus,

$$\begin{aligned} d(T + S, G, 0) &= d(T_\varepsilon + S, G, 0) \\ &= \tilde{d}(T_\varepsilon + S, G, 0) \\ &= \tilde{d}(T + S, G, 0). \end{aligned}$$

This completes the proof. ■

### 3 DEGREE THEORY FOR MULTI-VALUED PSEUDOMONOTONE PERTURBATION <sup>1</sup>

In this Chapter we construct a degree theory for the sum  $T + S$ , where  $T$  is maximal monotone and  $S$  is a bounded multivalued pseudomonotone operators. Furthermore, we give results involving the construction, basic properties, homotopy invariance and uniqueness of this degree. Important relevant references for this degree are those of Berkovits [6] (single-valued pseudomonotone operators), Fitzpatrick [28] ( $A$ -properness assumptions and degree given as a sequence of numbers), and Krauss [49] (degree given as a sequence of numbers). In Section 3.1 we construct a degree theory for bounded multi-valued pseudomonotone perturbations  $S$  of maximal monotone operators  $T$ . Section 3.2 contains a rather comprehensive and new results on admissible homotopies. The homotopy invariance results generalizes the homotopy invariance results considered in Browder degree theory where the maximal monotone operator  $T$  is arbitrary instead of densely defined and the perturbation is multivalued pseudomonotone instead of single valued demicontinuous of type  $(S_+)$ . The uniqueness of this degree is also covered in Section 3.3. In Section 3.4, a degree theory is given for single multivalued possibly unbounded pseudomonotone operators. Furthermore, Section 3.5 is devoted to applications of our degree theory to new existence results as well as a degree theoretic approach to known existence results. The establishment of a necessary and sufficient condition for the existence of zeros for inclusions involving pseudomonotone perturbations of maximal monotone operators is given in Section 3.6.

---

<sup>1</sup>The content of this chapter is part of the paper by Asfaw and Kartsatos [3]. Teffera M. Asfaw, author of this dissertation, is a Ph.D. candidate under the supervision of Professor Athanassios G.Kartsatos.

In the last Section of the Chapter, we gave a generalization of the methodology of construction of the degree theory for multivalued quasimonotone perturbations of maximal monotone operators. Furthermore, examples are provided to demonstrate the applicability of the theory in solving nonlinear partial differential equations.

### 3.1 Construction of the degree and basic properties

We start with two important lemmas.

**Lemma 3.1.1** *Let  $S : X \rightarrow 2^{X^*}$  be a pseudomonotone mapping. Then, for any  $\varepsilon > 0$ ,  $S + \varepsilon J$  is a mapping of type  $(S_+)$ .*

*Proof.* By the pseudomonotonicity of  $J$  and  $S$  on  $X$ , it follows that  $S + \varepsilon J$  is pseudomonotone. Thus, for each  $x \in X$ ,  $Sx + \varepsilon Jx$  is nonempty closed convex and bounded subset of  $X^*$ . Moreover, for each finite-dimensional subspace  $F$  of  $X$ ,  $S + \varepsilon J : F \rightarrow 2^{X^*}$  is upper semicontinuous in the weak topology of  $X^*$ . Let  $\varepsilon > 0$ ,  $x_n \in X$  and  $w_n \in Sx_n$  with  $x_n \rightharpoonup x_0 \in X$  as  $n \rightarrow \infty$  be such that

$$\limsup_{n \rightarrow \infty} \langle w_n + \varepsilon Jx_n, x_n - x_0 \rangle \leq 0.$$

Since  $S$  and  $J$  are pseudomonotone with effective domain  $X$ ,  $S + \varepsilon J$  is pseudomonotone. Using Lemma 1.2.9, we see that the sequence  $\{w_n + \varepsilon Jx_n\}$  is bounded. By the boundedness of  $J$ , we have that  $\{Jx_n\}$  is bounded. Hence,  $\{w_n\}$  is bounded. Assume w.l.o.g. that  $w_n \rightharpoonup w_0$  as  $n \rightarrow \infty$ . We claim that

$$d := \liminf_{n \rightarrow \infty} \langle w_n, x_n - x_0 \rangle \geq 0.$$

Suppose that this is false, i.e. there exist subsequences  $\{x_n\}$  and  $\{w_n\}$  such that

$$\lim_{n \rightarrow \infty} \langle w_n, x_n - x_0 \rangle = d < 0.$$

Since  $S$  is pseudomonotone with domain  $X$ , it follows that  $S$  is generalized pseudomonotone. Hence  $w_0 \in Sx_0$  and  $\langle w_n, x_n \rangle \rightarrow \langle w_0, x_0 \rangle$  as  $n \rightarrow \infty$ , which implies  $d = 0$ . However, this is a contradiction. Thus, our claim is true.

We now observe that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle Jx_n, x_n - x_0 \rangle &\leq \frac{1}{\varepsilon} \limsup_{n \rightarrow \infty} \langle w_n + \varepsilon Jx_n, x_n - x_0 \rangle \\ &\quad - \liminf_{n \rightarrow \infty} \langle w_n, x_n - x_0 \rangle \\ &\leq 0. \end{aligned}$$

Since  $J$  is type  $(S_+)$ , it follows that  $x_n \rightarrow x_0 \in X$  as  $n \rightarrow \infty$ . Moreover, by the boundedness of  $\{w_n\}$ , we may assume that  $w_n \rightarrow w_0$ , which implies  $w_n + \varepsilon Jx_n \rightarrow w_0 + \varepsilon Jx_0$  as  $n \rightarrow \infty$ . Furthermore, we have

$$\limsup_{n \rightarrow \infty} \langle w_n, x_n - x_0 \rangle = 0,$$

which implies, by the pseudomonotonicity of  $S$ , that  $w_0 \in Sx_0$ . Hence  $w_0 + \varepsilon Jx_0 \in Sx_0 + \varepsilon Jx_0$ . Thus,  $\{w_n + \varepsilon Jx_n\}$  has a subsequence, which we call again  $\{w_n + \varepsilon Jx_n\}$ , such that  $w_n + \varepsilon Jx_n \rightarrow w_0 + \varepsilon Jx_0 \in Sx_0 + \varepsilon Jx_0$ . Thus,  $S + \varepsilon J$  is of type  $(S_+)$ . ■

**Lemma 3.1.2** *Let  $G$  be a nonempty bounded open subset of  $X$ . Let  $T : X \supset D(T) \rightarrow 2^{X^*}$  be maximal monotone and  $S : X \rightarrow 2^{X^*}$  bounded pseudomonotone. Assume that  $0 \notin \overline{(T + S)(D(T) \cap \partial G)}$ . Then there exists  $\varepsilon_0 > 0$  such that  $d(T + S + \varepsilon J, G, 0)$  is well-defined and constant for all  $\varepsilon \in (0, \varepsilon_0]$ .*

*Proof.* Let  $G$  be a nonempty bounded open subset of  $X$ . We first show that there exists  $\varepsilon_0 > 0$  such that  $0 \notin (T + S + \varepsilon J)(D(T) \cap \partial G)$  for all  $\varepsilon \in (0, \varepsilon_0]$ . Assume that this is false, i.e. there exist a sequence  $\{\varepsilon_n\}$  such that  $\varepsilon_n \downarrow 0^+$  and sequences

$x_n \in D(T) \cap \partial G$ ,  $w_n \in Sx_n$  and  $v_n \in Tx_n$  satisfying

$$v_n + w_n + \varepsilon_n Jx_n = 0 \quad (3.1.1)$$

for all  $n$ . By the boundedness of the sequences  $\{x_n\}$  and  $\{Jx_n\}$ , it follows that  $v_n + w_n \rightarrow 0$  as  $n \rightarrow \infty$ , which implies that  $0 \in \overline{(T+S)(D(T) \cap \partial G)}$ . However, this is a contradiction to the hypothesis of the theorem. Thus, there exists  $\varepsilon_0 > 0$  such that  $0 \notin (T+S+\varepsilon J)(D(T) \cap \partial G)$ . Hence  $d(T+S+\varepsilon J, G, 0)$  is well-defined for all  $\varepsilon \in (0, \varepsilon_0]$ .

Next we prove that  $d(T+S+\varepsilon J, G, 0)$  is constant for all  $\varepsilon \in (0, \varepsilon_0]$ . Let  $\varepsilon_i \in (0, \varepsilon_0]$ ,  $i = 1, 2$ , be such that  $\varepsilon_1 < \varepsilon_2$ . It suffices to show that

$$d(T+S+\varepsilon_1 J, G, 0) = d(T+S+\varepsilon_2 J, G, 0).$$

To this end, we consider the homotopy

$$H(t, x) = Tx + Sx + (t\varepsilon_1 + (1-t)\varepsilon_2)Jx, (t, x) \in [0, 1] \times D(T).$$

In order to use Theorem 2.2.5, it suffices to show that the homotopy

$$S(t, x) = Sx + (t\varepsilon_1 + (1-t)\varepsilon_2)Jx, (t, x) \in [0, 1] \times \overline{G}$$

is of type  $(S_+)$ . To see this, let  $\{x_n\} \subset X$  and  $\{t_n\} \subset [0, 1]$  be such that  $x_n \rightarrow x_0 \in X$  and  $t_n \rightarrow t_0$  as  $n \rightarrow \infty$  and

$$\limsup_{n \rightarrow \infty} \langle w_n + (t_n \varepsilon_1 + (1-t_n)\varepsilon_2)Jx_n, x_n - x_0 \rangle \leq 0. \quad (3.1.2)$$

Assume first that  $t_0 = 0$ . From (3.1.2), using the monotonicity of  $J$ , we obtain

$$\limsup_{n \rightarrow \infty} \langle w_n + \varepsilon_2 Jx_n, x_n - x_0 \rangle \leq 0.$$

Since  $S$  is pseudomonotone, we have that  $S + \varepsilon_2 J$  is of type  $(S_+)$  and hence  $x_n \rightarrow x_0$  as  $n \rightarrow \infty$ .

If  $t_0 = 1$ , then

$$\limsup_{n \rightarrow \infty} \langle w_n + \varepsilon_1 J x_n, x_n - x_0 \rangle \leq 0.$$

Since  $S + \varepsilon_1 J$  is of type  $(S_+)$ , we have  $x_n \rightarrow x_0$  as  $n \rightarrow \infty$ .

Assume that  $t_0 \in (0, 1)$ . Then (3.1.2) implies

$$\limsup_{n \rightarrow \infty} \langle w_n + (t_0 \varepsilon_1 + (1 - t_0) \varepsilon_2) J x_n, x_n - x_0 \rangle \leq 0.$$

We also have  $\delta_0 = t_0 \varepsilon_1 + (1 - t_0) \varepsilon_2 > 0$ . Since  $S + \delta_0 J$  is of type  $(S_+)$ ,  $x_n \rightarrow x_0$  as  $n \rightarrow \infty$ . By the boundedness of  $\{x_n\}$  and  $\{w_n\}$ , if  $f_n \in S(t_n, x_n)$  there exists a subsequence, say  $\{f_n\}$ , such that  $f_n \rightarrow f \in S(t_0, x_0)$  as  $n \rightarrow \infty$ . This proves that  $\{S(t, \cdot)\}_{t \in [0, 1]}$  is a homotopy of type  $(S_+)$ .

We now show that  $0 \notin H(t, D(T) \cap \partial G)$  for all  $t \in [0, 1]$ . Suppose that this is false, i.e. there exist  $t_1 \in [0, 1]$ ,  $x_1 \in D(T) \cap \partial G$ ,  $v_1 \in T x_1$  and  $w_1 \in S x_1$  such that

$$v_1 + w_1 + (t_1 \varepsilon_1 + (1 - t_1) \varepsilon_2) J x_1 = 0,$$

which says that  $0 \in (T + S + \tilde{\varepsilon} J)(D(T) \cap \partial G)$  where  $\tilde{\varepsilon} = t_1 \varepsilon_1 + (1 - t_1) \varepsilon_2$ . Since  $\tilde{\varepsilon} \in (0, \varepsilon_0]$ , we obtain a contradiction. Thus, we conclude that  $\{S(t, \cdot)\}_{t \in [0, 1]}$  is a homotopy of type  $(S_+)$  such that  $0 \notin (H(t, D(T) \cap \partial G))$  for all  $t \in [0, 1]$ . Consequently, by using Theorem 2.2.5, using the homotopy  $\{T^t = T\}_{t \in [0, 1]}$ , we conclude that

$$d(T + S + \varepsilon_1 J, G, 0) = d(T + S + \varepsilon_2 J, G, 0).$$

This proves that  $d(T + S + \varepsilon J, G, 0)$  is constant for all  $\varepsilon \in (0, \varepsilon_0]$  and completes the proof. ■

We are now ready to define our new degree mapping.

**Definition 3.1.3 (Degree for Pseudomonotone Perturbations)** Let  $X \supset D(T) \rightarrow 2^{X^*}$  be maximal monotone and  $S : X \rightarrow 2^{X^*}$  bounded pseudomonotone. Let  $G$  be a bounded open subset of  $X$  such that  $0 \notin \overline{(T + S)(D(T) \cap \partial G)}$ . We define

$$d(T + S, G, 0) = \lim_{\varepsilon \downarrow 0^+} d(T + S + \varepsilon J, G, 0),$$

where  $d(T + S + \varepsilon J, G, 0)$  is the degree constructed in Section 3. Furthermore, we set  $d(T + S, G, 0) = 0$  if  $D(T) \cap G = \emptyset$ .

Using the hypothesis in Definition 3.1.3, we note that

$$d(T + S, G, 0) = \lim_{\varepsilon \downarrow 0^+} \lim_{t \downarrow 0^+} d(T_t + S + \varepsilon J, G, 0).$$

This alternative definition of  $d(T + S, G, 0)$  will be frequently used in the rest of the chapters for constructing important homotopies associated with this degree.

We now give some basic properties of the new degree mapping.

**Theorem 3.1.4** *Let  $T : X \supset D(T) \rightarrow 2^{X^*}$  be maximal monotone and  $S : X \rightarrow 2^{X^*}$  bounded pseudomonotone. Let  $G$  be a bounded open subset of  $X$ . Then the following hold.*

- (i) *(Normalization)*  $d(J, G, 0) = 1$  if  $0 \in G$  and  $d(J, G, 0) = 0$  if  $0 \notin \overline{G}$ .
- (ii) *(Existence)* If  $0 \notin \overline{(T + S)(D(T) \cap \partial G)}$  with  $d(T + S, G, 0) \neq 0$ , we have  $0 \in \overline{T + S)(D(T) \cap G}$ . If, moreover,  $G$  is convex, then there exists  $x_0 \in D(T) \cap G$  such that  $0 \in Tx_0 + Sx_0$ .
- (iii) *(Decomposition)* Let  $G_1$  and  $G_2$  be disjoint open subsets of  $G$  such that  $0 \notin \overline{(T + S)(D(T) \cap (\overline{G} \setminus (G_1 \cup G_2)))}$ . Then

$$d(T + S, G, 0) = d(T + S, G_1, 0) + d(T + S, G_2, 0).$$



(iv) (*Translation Invariance*) Let  $f \notin \overline{(T + S)(D(T) \cap \partial G)}$ . Then we have

$$d(T + S - f, G, 0) = d(T + S, G, f).$$

*Proof.* (i) It suffices to show that  $d(J, G, 0)$  is well-defined. If  $0 \in G$  or  $0 \notin \overline{G}$ , then  $0 \notin J(\partial G)$ , because  $J$  is a surjective homeomorphism. Thus,  $d(J, G, 0)$  is well-defined. The conclusion follows from the result of Browder in [14].

(ii) Assume that  $0 \notin \overline{(T + S)(D(T) \cap \partial G)}$  and  $d(T + S, G, 0) \neq 0$ . Then  $0 \notin (T + S)(D(T) \cap \partial G)$  and hence, by the definition of the degree, there exists  $\varepsilon_0 > 0$  such that

$$d(T + S + \varepsilon J, G, 0) \neq 0$$

for all  $\varepsilon \in (0, \varepsilon_0]$ . Thus, for each  $\{\varepsilon_n\}$  such that  $\varepsilon_n \downarrow 0^+$ , there exists  $\{x_n\}$  in  $D(T) \cap G$ ,  $w_n \in Sx_n$  and  $v_n \in Tx_n$  such that

$$v_n + w_n + \varepsilon_n Jx_n = 0 \tag{3.1.3}$$

for all  $n$ . This implies that  $0 \in \overline{(T + S)(D(T) \cap G)}$ .

Now, assume, in addition, that  $G$  is convex. Since  $S$  is bounded and  $\{x_n\}$  is bounded, it follows that  $\{w_n\}$  is bounded and hence  $\{v_n\}$  is bounded. Assume w.l.o.g. that  $x_n \rightharpoonup x_0 \in X$ ,  $v_n \rightharpoonup v_0$  and  $w_n \rightharpoonup w_0$  as  $n \rightarrow \infty$ . Since  $\overline{G}$  is convex, it is weakly closed and hence  $x_0 \in \overline{G}$ . By the pseudomonotonicity of  $S$ , we have

$$\liminf_{n \rightarrow \infty} \langle w_n + \varepsilon_n Jx_n, x_n - x_0 \rangle \geq 0.$$

Thus, (3.1.3) implies

$$\limsup_{n \rightarrow \infty} \langle v_n - v_0, x_n - x_0 \rangle \leq 0.$$

Using Lemma 1.2.5, we conclude that  $x_0 \in D(T)$ ,  $v_0 \in Tx_0$  and  $\langle v_n, x_n \rangle \rightarrow \langle v_0, x_0 \rangle$  as  $n \rightarrow \infty$ . Thus,

$$\limsup_{n \rightarrow \infty} \langle w_n, x_n - x_0 \rangle = 0.$$

By the generalized pseudomonotonicity of  $S$ , we obtain  $w_0 \in Sx_0$ . Finally, taking limits as  $n \rightarrow \infty$  in (3.1.3), we obtain  $x_0 \in D(T) \cap \overline{G}$ ,  $v_0 \in Tx_0$ ,  $w_0 \in Sx_0$  and

$$v_0 + w_0 = 0.$$

Since  $0 \notin (T + S)(D(T) \cap \partial G)$ ,  $x_0 \in D(T) \cap G$  and  $v_0 + w_0 = 0$ . This proves that  $0 \in Tx + Sx$  is solvable in  $D(T) \cap G$ .

(iii) Since  $0 \notin \overline{(T + S)(D(T) \cap (\overline{G} \setminus (G_1 \cup G_2)))}$ , there exists  $\varepsilon_0 > 0$  such that  $0 \notin (T + S + \varepsilon J)(D(T) \cap (\overline{G} \setminus (G_1 \cup G_2)))$  for all  $\varepsilon \in (0, \varepsilon_0]$ . Using the definition of the degree and (iii) of Theorem 2.2.3, we see that

$$\begin{aligned} d(T + S, G, 0) &= d(T + S + \varepsilon J, G, 0) \\ &= d(T + S + \varepsilon J, G_1, 0) + d(T + S + \varepsilon J, G_2, 0) \end{aligned}$$

for all  $\varepsilon \in (0, \varepsilon_0]$ . By the definition of the degree, we have

$$d(T + S, G, 0) = d(T + S, G_1, 0) + d(T + S, G_2, 0).$$

This proves (iii).

(iv) Since  $f \notin \overline{(T + S)(D(T) \cap \partial G)}$ , the degree  $d(T + S - f, G, 0) = d(T + S, G, f)$  is well-defined. By the definition of our degree mapping, there exists  $\varepsilon_0 > 0$  such that

$$d(T + S - f, G, 0) = d(T + S + \varepsilon J - f, G, 0)$$

for all  $\varepsilon \in (0, \varepsilon_0]$ . By the translation invariance property of the degree in (iv) of Theorem 2.2.3, it follows that  $d(T + S + \varepsilon J - f, G, 0) = d(T + S + \varepsilon J, G, f)$  for all

$\varepsilon \in (0, \varepsilon_0]$ , which implies

$$d(T + S - f, G, 0) = d(T + S + \varepsilon J, G, f) = d(T + S, G, f).$$

for all  $\varepsilon \in (0, \varepsilon_0]$ . This completes the proof. ■

We note that in our construction of the degree, the Leray-Schauder-type boundary condition,  $0 \notin \overline{(T + S)(D(T) \cap \partial G)}$ , for the sum  $T + S$  is essential. The reader might wonder as to whether one could impose direct conditions on the operators  $T$ ,  $S$  which would guarantee the validity of this boundary condition. In fact, in the following theorem we establish sufficient conditions for this to happen.

**Theorem 3.1.5** *Let  $T : X \supset D(T) \rightarrow 2^{X^*}$  be maximal monotone and  $S : X \rightarrow 2^{X^*}$  bounded pseudomonotone. Let  $G$  be a bounded open subset of  $X$  such that  $0 \notin (T + S)(D(T) \cap \partial G)$ . Assume that one of the following is true:*

(i)  $G$  is convex and

$$((T + S)(D(T) \cap G) \cap \overline{(T + S)(D(T) \cap \partial G)}) = \emptyset;$$

(ii)  $G$  is convex and  $T$  satisfies condition (S) on  $D(T) \cap \overline{G}$ .

Then there exists  $\varepsilon_0 > 0$  such that

$$0 \notin (T + S + \varepsilon J)(D(T) \cap \partial G)$$

for all  $\varepsilon \in (0, \varepsilon_0]$ .

*Proof.* (i) Assume that the conclusion is false, i. e. there exist sequences  $\varepsilon_n \downarrow 0^+$ ,  $\{x_n\} \subset D(T) \cap \partial G$  with  $x_n \rightharpoonup x_0 \in \overline{G}$  (because  $\overline{G}$  is closed and convex and  $X$  is

reflexive),  $v_n \in Tx_n$ ,  $w_n \in Sx_n$  such that

$$v_n + w_n + \varepsilon_n Jx_n = 0. \quad (3.1.4)$$

Since  $\varepsilon_n \downarrow 0^+$  and  $\{Jx_n\}$  is bounded,  $v_n + w_n \rightarrow 0$  as  $n \rightarrow \infty$  and hence  $0 \in \overline{(T+S)(D(T) \cap \partial G)}$ . Using Lemma 2.2.4, we get

$$\liminf_{n \rightarrow \infty} \langle w_n + \varepsilon_n Jx_n, x_n - x_0 \rangle \geq 0.$$

Using this and (3.1.4), we get

$$\limsup_{n \rightarrow \infty} \langle v_n, x_n - x_0 \rangle \leq 0.$$

Since  $S$  is bounded and pseudomonotone, it follows that  $\{w_n\}$  is bounded and hence  $\{v_n\}$  is bounded. Assume w.l.o.g. that  $w_n \rightharpoonup w_0$  and  $v_n \rightharpoonup v_0$  as  $n \rightarrow \infty$ , which implies

$$\limsup_{n \rightarrow \infty} \langle v_n - v_0, x_n - x_0 \rangle \leq 0.$$

This implies in turn that  $x_0 \in D(T)$  and  $v_0 \in Tx_0$  and  $\langle v_n, x_n \rangle \rightarrow \langle v_0, x_0 \rangle$  as  $n \rightarrow \infty$ . As a result of this, we obtain

$$\limsup_{n \rightarrow \infty} \langle v_n, x_n - x_0 \rangle = 0. \quad (3.1.5)$$

Again, from (3.1.4) it follows that

$$\limsup_{n \rightarrow \infty} \langle w_n, x_n - x_0 \rangle = 0.$$

Since  $S$  is pseudomonotone with effective domain  $X$ , it is generalized pseudomonotone and hence  $w_0 \in Sx_0$ ,  $x_0 \in D(T) \cap \overline{G}$  and  $v_0 + w_0 = 0$ . Since  $0 \notin (T+S)(D(T) \cap \partial G)$ ,

we get  $x_0 \in D(T) \cap G$ , i.e.  $0 \in (T + S)(D(T) \cap G) \cap \overline{(T + S)(D(T) \cap \partial G)}$ , which is a contradiction to our hypothesis.

(ii) Suppose  $T$  satisfies condition (S) on  $D(T) \cap \overline{G}$ . Using (3.1.5), it follows that  $x_n \rightarrow x_0 \in \partial G$  as  $n \rightarrow \infty$ . Using Lemma 1.2.4 (demiclosedness of the maximal monotone mapping  $T$ ), it follows that  $x_0 \in D(T)$  and  $v_0 \in Tx_0$ , which in turn implies that  $0 \in (T + S)(D(T) \cap \partial G)$ . This is impossible by our hypothesis. Therefore, the conclusion of the theorem follows. ■

### 3.2 Homotopy invariance theorems

The theorem that follows contains comprehensive and important homotopy invariance properties of our degree. The homotopy in (iii) of Theorem 3.2.2 needs a mapping of type  $\Gamma_\phi$  which is defined below.

**Definition 3.2.1** A mapping  $T : X \supseteq D(T) \rightarrow 2^{X^*}$  is said to be “of type  $\Gamma_\phi$ ” if there exists a mapping  $\phi : [0, \infty) \rightarrow [0, \infty)$  such that  $\phi(0) = 0$  and if  $r_n > 0$ ,  $n = 1, 2, \dots$ , and

$$\lim_{n \rightarrow \infty} \phi(r_n) = 0$$

we have  $r_n \rightarrow 0^+$  as  $n \rightarrow \infty$ .  $T$  is said to be “coercive” if there exists a function  $\phi : [0, \infty) \rightarrow (-\infty, \infty)$  such that  $\phi(t) \rightarrow \infty$  as  $t \rightarrow \infty$  and  $\langle u, x \rangle \geq \phi(\|x\|)\|x\|$  for all  $x \in D(T)$ ,  $u \in Tx$ .

**Theorem 3.2.2 (*Invariance Under Pseudomonotone Homotopies*)** Let  $T : X \supset D(T) \rightarrow 2^{X^*}$  be maximal monotone and let  $S_1, S_2 : X \rightarrow 2^{X^*}$  be two operators. Let  $G$  be a bounded open subset of  $X$ . Let  $\{S^t\}_{t \in [0,1]}$  be an affine homotopy between  $S_1$  and  $S_2$ , i.e

$$S^t x = tS_1 x + (1 - t)S_2 x, \quad (t, x) \in [0, 1] \times \overline{G}.$$

Then

- (i) if both  $S_1$  and  $S_2$  are bounded pseudomonotone, then the degree is invariant under homotopies of the type

$$H_1(t, x) = Tx + S^t x, \quad (t, x) \in [0, 1] \times D(T) \cap \overline{G},$$

provided that  $0 \notin \overline{(T + S^t)(D(T) \cap \partial G)}$  for all  $t \in [0, 1]$ .

- (ii) Suppose  $\{T^t\}_{t \in [0, 1]}$  is a pseudomonotone homotopy of maximal monotone operators. If  $S_1$  is bounded pseudomonotone and  $S_2$  is bounded and of type  $(S_+)$ , then the degree is invariant under homotopies of the type

$$H_3(t, x) := T^t x + S^t x, \quad (t, x) \in [0, 1] \times D(T^t) \cap \overline{G},$$

provided that  $0 \notin \overline{(T^t + S^t)(D(T) \cap \partial G)}$  for all  $t \in [0, 1]$ .

- (iii) Suppose  $S_1$  is bounded pseudomonotone and  $S_2$  is bounded, of type  $(S_+)$  and of type  $\Gamma_\phi$ . If  $0 \in T(0)$ , then  $d(H(s, \cdot), G, 0)$  is independent of  $s \in [0, 1]$  where

$$H_4(s, x) = s(T + S_1)x + (1 - s)S_2x, \quad (s, x) \in [0, 1] \times \overline{G}$$

provided that  $0 \notin \overline{H_4(s, D(T) \cap \partial G)}$  for  $s \in [0, 1]$ .

- (iv) If  $\overline{D(T)} = X$  and  $S_1$  is bounded pseudomonotone and  $S_2$  is bounded and of type  $(S_+)$ , then  $d(H_4(s, \cdot), G, 0)$  is independent of  $s \in [0, 1]$  provided that  $0 \notin \overline{H_4(s, D(T) \cap \partial G)}$  for  $s \in [0, 1]$ .

*Proof.*

- (i) We consider the homotopy

$$H_1(t, x) := Tx + tS_1x + (1 - t)S_2x, \quad (t, x) \in [0, 1] \times D(T)$$

with  $0 \notin \overline{(H_1(t, D(T) \cap \partial G))}$  for all  $t \in [0, 1]$ . For each  $t \in [0, 1]$ , the operator  $tS_1 + (1 - t)S_2$  is bounded pseudomonotone and the degree  $d(H(t, \cdot), G, 0)$  is well-defined. We claim that there exists  $\varepsilon_0 > 0$  independent of  $t \in [0, 1]$  such that

$$0 \notin (T + tS_1 + (1 - t)S_2 + \varepsilon J)(D(T) \cap \partial G) \quad (3.2.6)$$

for all  $\varepsilon \in (0, \varepsilon_0]$  and  $t \in [0, 1]$ . Suppose this is false, i.e. there exist  $\varepsilon_n \downarrow 0^+$ ,  $t_n \in [0, 1]$  with  $t_n \rightarrow t_0$ ,  $x_n \in D(T) \cap \partial G$ ,  $v_n \in Tx_n$ ,  $w_n \in S_1x_n$  and  $z_n \in S_2x_n$  such that

$$v_n + t_n w_n + (1 - t_n)z_n + \varepsilon_n Jx_n = 0 \quad (3.2.7)$$

for all  $n$ .

**Case I:**  $t_0 = 0$ . By the boundedness of  $\{x_n\}$ ,  $S_1$  and  $S_2$ , we have  $t_n w_n \rightarrow 0$ ,  $-t_n z_n \rightarrow 0$  and  $v_n + z_n \rightarrow 0$  as  $n \rightarrow \infty$ , which implies  $0 \in \overline{(T + S_2)(D(T) \cap \partial G)}$ . However, this is a contradiction.

**Case II:**  $t_0 = 1$ . Letting  $s_n = 1 - t_n$ , we easily get the same conclusion as in Case I.

**Case III:**  $t_0 \in (0, 1)$ . From (3.2.7), we see that

$$v_n + (t_n - t_0)w_n + t_0 w_n + (t_0 - t_n)z_n + (1 - t_0)z_n + \varepsilon_n Jx_n = 0$$

for all  $n$ . By the boundedness of  $J$ ,  $S_1$ ,  $S_2$  and  $t_n \rightarrow t_0$ , we have

$$(t_n - t_0)w_n + (t_0 - t_n)z_n + \varepsilon_n Jx_n \rightarrow 0$$

as  $n \rightarrow \infty$ , which implies

$$v_n + t_0 w_n + (1 - t_0)z_n \rightarrow 0$$

as  $n \rightarrow \infty$ . This shows that  $0 \in \overline{H_1(t_0, D(T) \cap \partial G)}$ . However, this is a contradiction to the hypothesis of the theorem. Therefore our claim follows. We now show that

$$d(T + tS_1 + (1 - t)S_2 + \varepsilon J, G, 0)$$

is constant for all  $(t, \varepsilon) \in [0, 1] \times (0, \varepsilon_0]$ . To this end, let  $(t_1, \varepsilon_1), (t_2, \varepsilon_2) \in [0, 1] \times (0, \varepsilon_0]$  be such that  $(t_1, \varepsilon_1) \neq (t_2, \varepsilon_2)$ , and consider the homotopy

$$K(t, x) := \alpha_t S_1 x + \beta_t S_2 x + \gamma_t J x \quad (t, x) \in [0, 1] \times \overline{G},$$

where  $\alpha_t := tt_1 + (1 - t)t_2$ ,  $\beta_t := t(1 - t_1) + (1 - t)(1 - t_2)$  and  $\gamma_t := t\varepsilon_1 + (1 - t)\varepsilon_2$ . For each  $t \in [0, 1]$ , it is easy to see that  $\alpha_t \in [0, 1]$ ,  $\beta_t \in [0, 1]$  and  $\gamma_t \in (0, \varepsilon_0]$ .

It suffices to show that  $\{K(t, \cdot)\}_{t \in [0, 1]}$  is a homotopy of type  $(S_+)$ . We need to verify the three properties of Definition 1.3.4, which we now denote by (I)-(III) to avoid confusion.

**(I).** By the pseudomonotonicity of  $S_1$  and  $S_2$ , for each  $t \in [0, 1]$ ,  $K(t, x)$  is a nonempty closed convex and bounded subset of  $X^*$  for each  $x \in X$ .

**(II).** For each  $t \in [0, 1]$ , the upper semicontinuity of  $S_1$  and  $S_2$  on each finite dimensional subspace of  $X$  to the weak topology of  $X^*$  implies that  $K(t, \cdot)$  is upper semicontinuous from each finite dimensional subspace of  $X$  to the weak topology of  $X^*$ .

**(III).** Let  $\{x_n\} \subset X$ ,  $\{\tilde{t}_n\} \subset [0, 1]$  be such that  $x_n \rightarrow x_0$ ,  $\tilde{t}_n \rightarrow \tilde{t}_0$  as  $n \rightarrow \infty$ , and let  $w_n \in S_1 x_n$ ,  $v_n \in S_2 x_n$  be such that

$$\limsup_{n \rightarrow \infty} \langle \alpha_{\tilde{t}_n} w_n + \beta_{\tilde{t}_n} v_n + \gamma_{\tilde{t}_n} J x_n, x_n - x_0 \rangle \leq 0. \quad (3.2.8)$$



If  $\tilde{t}_0 = 0$ , then

$$\begin{aligned}
0 &\geq \limsup_{n \rightarrow \infty} \langle \alpha_{\tilde{t}_n} w_n + \beta_{\tilde{t}_n} v_n + \gamma_{\tilde{t}_n} Jx_n, x_n - x_0 \rangle \\
&= \limsup_{n \rightarrow \infty} \langle (\alpha_{\tilde{t}_n} - t_2)w_n + (\beta_{\tilde{t}_n} - (1 - t_2))v_n + (\gamma_{\tilde{t}_n} - \varepsilon_2)Jx_n, x_n - x_0 \rangle \\
&\quad + \limsup_{n \rightarrow \infty} \langle t_2 w_n + (1 - t_2)v_n + \varepsilon_2 Jx_n, x_n - x_0 \rangle \\
&= \limsup_{n \rightarrow \infty} \langle t_2 w_n + (1 - t_2)v_n + \varepsilon_2 Jx_n, x_n - x_0 \rangle.
\end{aligned} \tag{3.2.9}$$

By the monotonicity of  $J$  and (3.2.9), we see that

$$\limsup_{n \rightarrow \infty} \langle t_2 w_n + (1 - t_2)v_n, x_n - x_0 \rangle \leq 0.$$

From the boundedness of  $S_1$  and  $S_2$ , we may assume w.l.o.g. that  $t_2 w_n + (1 - t_2)v_n \rightarrow q_0$  as  $n \rightarrow \infty$ . Since  $t_2 S_1 + (1 - t_2)S_2$  is pseudomonotone with  $D(t_2 S_1 + (1 - t_2)S_2) = X$ ,  $t_2 S_1 + (1 - t_2)S_2$  is generalized pseudomonotone and hence  $q_0 \in (t_2 S_1 + (1 - t_2)S_2)x_0$  and

$$\langle t_2 w_n + (1 - t_2)v_n, x_n - x_0 \rangle \rightarrow 0$$

as  $n \rightarrow \infty$ . Thus, using (3.2.9) again, we find

$$\limsup_{n \rightarrow \infty} \langle Jx_n, x_n - x_0 \rangle = 0.$$

Since  $J$  is of type  $(S_+)$ ,  $x_n \rightarrow x_0$  as  $n \rightarrow \infty$ .

If  $\tilde{t}_0 = 1$ , letting  $s_n = 1 - \tilde{t}_n$ , we get the same conclusion as in the case  $\tilde{t}_0 = 0$ .

Assume that  $\tilde{t}_0 \in (0, 1)$ . Then from (3.2.8), we have

$$\limsup_{n \rightarrow \infty} \langle s_0 w_n + s_1 v_n + s_2 Jx_n, x_n - x_0 \rangle \leq 0,$$

where  $s_0 = \tilde{t}_0 t_1 + (1 - \tilde{t}_0)t_2$ ,  $s_1 = \tilde{t}_0(1 - t_1) + (1 - \tilde{t}_0)(1 - t_2)$  and  $s_2 = \tilde{t}_0 \varepsilon_1 + (1 - \tilde{t}_0)\varepsilon_2$ .

Since  $s_2 > 0$  and  $s_0S_1 + s_1S_2$  is pseudomonotone,  $s_0S_1 + s_1S_2 + s_2J$  is of type  $(S_+)$  and  $x_n \rightarrow x_0$  as  $n \rightarrow \infty$ .

Finally, let  $f_n = \alpha_{\tilde{t}_n} w_n + \beta_{\tilde{t}_n} v_n + \gamma_{\tilde{t}_n} Jx_n$ , where  $\tilde{t}_n \rightarrow \tilde{t}_0$  and  $x_n \rightarrow x_0$  as  $n \rightarrow \infty$ . Since  $S_1, S_2$  and  $J$  are bounded, we may assume w.l.o.g. that  $w_n \rightarrow w_0$ ,  $v_n \rightarrow v_0$  and  $Jx_n \rightarrow Jx_0$  as  $n \rightarrow \infty$ . Using the pseudomonotonicity, and hence the generalized pseudomonotonicity, of  $S_1$  and  $S_2$ , we conclude that  $w_0 \in S_1x_0$ ,  $v_0 \in S_2x_0$  and there exists a subsequence  $\{f_n\}$ , called again  $\{f_n\}$ , such that  $f_n \rightarrow f_0$ , where  $f_0 = \alpha_{\tilde{t}_0} w_0 + \beta_{\tilde{t}_0} v_0 + \gamma_{\tilde{t}_0} Jx_0 \in K(t_0, x_0)$ . Therefore,  $\{K(t, \cdot)\}_{[0,1]}$  is a homotopy of type  $(S_+)$ .

Finally, it remains to show that  $0 \notin \tilde{K}(t, D(T) \cap \partial G)$  for all  $t \in [0, 1]$ , where  $\tilde{K}(t, x) = Tx + K(t, x)$ ,  $(t, x) \in [0, 1] \times D(T)$ . Assume that there exists  $t_1 \in [0, 1]$ ,  $x_1 \in D(T) \cap \partial G$ ,  $u_1 \in Tx_1$ ,  $w_1 \in S_1x_1$  and  $v_1 \in S_2x_1$  such that

$$u_1 + \alpha_{t_1} w_1 + \beta_{t_1} v_1 + \gamma_{t_1} Jx_1 = 0.$$

Since  $\gamma_{t_1} \in (0, \varepsilon_0]$  and  $\alpha_{t_1}, \beta_{t_1} \in [0, 1]$ , we get a contradiction of (3.2.6).

Combining (I)-(III) above and using Theorem 2.2.5 and the fact that  $\{T^t : T^t = T\}_{t \in [0,1]}$  is a pseudomonotone homotopy, we see that  $d(\tilde{K}(t, \cdot), G, 0)$  is constant for all  $t \in [0, 1]$ , which implies

$$d(T + t_1S_1 + (1 - t_1)S_2 + \varepsilon_1J, G, 0) = d(T + t_2S_1 + (1 - t_2)S_2 + \varepsilon_2J, G, 0).$$

Since  $(t_1, \varepsilon_1)$  and  $(t_2, \varepsilon_2)$  are arbitrary in  $[0, 1] \times (0, \varepsilon_0]$ , we conclude that

$$d(T + tS_1 + (1 - t)S_2 + \varepsilon J, G, 0)$$

is constant for all  $(t, \varepsilon) \in [0, 1] \times (0, \varepsilon_0]$ . Thus, by the definition of our degree mapping, we conclude that  $d(H(t, \cdot), G, 0)$  is constant for all  $t \in [0, 1]$ .

(ii) The proof follows easily by using Theorem 2.2.5.

(iii) Let  $c_0 \in (0, 1]$  be arbitrarily fixed. We show that there exists  $\varepsilon_0 > 0$  such that

$$0 \notin (s(T + S_1) + (1 - s)S_2 + \varepsilon J)(D(T) \cap \partial G) \quad (3.2.10)$$

for all  $s \in [c_0, 1]$  and for all  $\varepsilon \in (0, \varepsilon_0]$ . Suppose that this is false, i.e. for each  $\varepsilon_n \downarrow 0^+$ , there exists  $s_n \in [c_0, 1]$  such that  $s_n \rightarrow s_0$ ,  $x_n \in D(T) \cap \partial G$ ,  $z_n \in Tx_n$ ,  $w_n \in S_1x_n$  and  $v_n \in S_2x_n$  such that

$$s_n(z_n + w_n) + (1 - s_n)v_n + \varepsilon_n Jx_n = 0 \quad (3.2.11)$$

for all  $n$ . If  $s_0 = 1$ , then it follows that  $z_n + w_n \rightarrow 0$  as  $n \rightarrow \infty$ , which implies  $0 \in \overline{(T + S_1)(D(T) \cap \partial G)}$ , i.e. a contradiction. So, we assume  $s_0 \in [c_0, 1)$ . Using the boundedness of the sequences  $\{x_n\}$ ,  $\{w_n\}$ ,  $\{v_n\}$  and  $\{Jx_n\}$ , we see that  $\{s_n z_n\}$  is bounded, which implies the boundedness of  $\{z_n\}$ . Assume w.l.o.g. that  $x_n \rightarrow x_0$ ,  $v_n \rightarrow v_0$ ,  $w_n \rightarrow w_0$  and  $z_n \rightarrow z_0$  as  $n \rightarrow \infty$ . Since  $s_n \rightarrow s_0 \in [c_0, 1)$ , using the pseudomonotonicity of  $S_1$  and Lemma 3.1.1, we see that

$$\liminf_{n \rightarrow \infty} s_n \langle w_n, x_n - x_0 \rangle = s_0 \liminf_{n \rightarrow \infty} \langle w_n, x_n - x_0 \rangle \geq 0.$$

Since  $x_n \in D(T)$  for all  $n$  and  $z_n \in Tx_n$ , the maximality of  $T$  implies

$$\liminf_{n \rightarrow \infty} s_n \langle z_n, x_n - x_0 \rangle \geq 0.$$

Otherwise, if for appropriate subsequences we have

$$\lim_{n \rightarrow \infty} s_n \langle z_n, x_n - x_0 \rangle < 0,$$

we see that

$$\limsup_{n \rightarrow \infty} \langle z_n, x_n - x_0 \rangle < 0.$$

Since  $x_n \in D(T)$ ,  $z_n \in Tx_n$  and  $x_n \rightarrow x_0$ ,  $z_n \rightarrow z_0$  as  $n \rightarrow \infty$  and  $T$  is maximal monotone, using Lemma 1.2.5, we get  $x_0 \in D(T)$ ,  $z_0 \in Tx_0$  and  $\langle z_n, x_n \rangle \rightarrow \langle z_0, x_0 \rangle$  as  $n \rightarrow \infty$ , which implies the contradiction  $0 < 0$ . So, the claim holds and hence, using (3.2.11), we obtain

$$\limsup_{n \rightarrow \infty} (1 - s_n) \langle v_n, x_n - x_0 \rangle \leq 0.$$

Since  $s_0 \in [c_0, 1)$ , we obtain

$$\limsup_{n \rightarrow \infty} \langle v_n, x_n - x_0 \rangle \leq 0.$$

Using the  $(S_+)$  condition on  $S_2$ , we conclude that  $x_n \rightarrow x_0 \in \partial G$  as  $n \rightarrow \infty$  and  $v_0 \in S_2 x_0$  and  $\langle v_n, x_n \rangle \rightarrow \langle v_0, x_0 \rangle$  as  $n \rightarrow \infty$ . Furthermore, we see that

$$\limsup_{n \rightarrow \infty} \langle z_n, x_n - x_0 \rangle \leq 0.$$

Since  $T$  is maximal monotone and  $(x_n, z_n) \in G(T)$  for all  $n$ , using Lemma 1.2.5 again, we get  $x_0 \in D(T)$  and  $v_0 \in Tx_0$  and  $\langle v_n, x_n \rangle \rightarrow \langle v_0, x_0 \rangle$  as  $n \rightarrow \infty$ . Similarly, by the pseudomonotonicity of  $S_1$ , it follows that  $w_0 \in S_1 x_0$ . Therefore, we obtain

$$s_0(z_0 + w_0) + (1 - s_0)v_0 = 0,$$

which is a contradiction. Thus our claim holds true. Next we show that

$$d(s(T + S_1) + (1 - s)S_2 + \varepsilon J, G, 0)$$

is independent of  $(s, \varepsilon) \in [c_0, 1] \times (0, \varepsilon_0]$ . To this end, we first show that  $\{T^s\}_{s \in [c_0, 1]}$ , where  $T^s = sT$ , is a pseudomonotone homotopy of maximal monotone mappings by verifying Property (iv) of Lemma 2.1.3. The reader is referred to the remark after Definition 2.1.4 concerning the change of the interval  $[0, 1]$  to the interval  $[c_0, 1]$ . Fix  $s_0 \in [c_0, 1]$  and let  $y \in T^{s_0}x$ , for some  $x \in D(T)$ . Then  $y = s_0v$ , for some  $v \in Tx$ . Let  $s_n \in [c_0, 1]$  be such that  $s_n \rightarrow s_0$ . Also, let  $x_n = x$  and  $y_n = s_nv$ . Then  $(x_n, y_n) \in G(T^{s_n})$ ,  $y_n \rightarrow s_0v = y$  and  $x_n \rightarrow x$ . This shows  $\{sT\}_{s \in [c_0, 1]}$  is a pseudomonotone homotopy of maximal monotone mappings.

Let now  $(s_i, \varepsilon_i) \in [c_0, 1] \times (0, \varepsilon_0]$  ( $i = 1, 2$ ) be such that  $(s_1, \varepsilon_1) \neq (s_2, \varepsilon_2)$ . We consider the homotopy

$$K(s, x) = \alpha_s S_1 x + \beta_s S_2 x + \gamma_s Jx, \quad (s, x) \in [c_0, 1] \times \overline{G},$$

where  $\alpha_s, \beta_s$  and  $\gamma_s$  are as in the proof of (i). Following the proof of (i), we see that  $\{K(s, \cdot)\}_{s \in [c_0, 1]}$  is a homotopy of type  $(S_+)$  such that

$$0 \notin (\alpha_s T + K(s, \cdot))(D(T) \cap \partial G)$$

for all  $s \in [c_0, 1]$ . We also note that the family  $\{\alpha_s T\}_{s \in [c_0, 1]}$  is a pseudomonotone homotopy of maximal monotone operators. Using Theorem 2.2.5, we conclude that

$$d(\alpha_s T + K(s, \cdot), G, 0)$$

is independent of  $s \in [c_0, 1]$ . Finally, the definition of the degree implies that

$$d(H_4(s, \cdot), G, 0)$$

is independent of  $s \in [c_0, 1]$ . Since  $c_0 \in (0, 1]$  is arbitrary, we conclude

$$d(H_4(s, \cdot), G, 0) = d(T + S_1, G, 0) \quad (3.2.12)$$

for all  $s \in (0, 1]$ . Next we show that

$$d(H_4(0, \cdot), G, 0) = d(S_2, G, 0) = d(T + S_1, G, 0).$$

For an arbitrarily but fixed  $\varepsilon \in (0, \varepsilon_0]$ , we consider the homotopy

$$H_5(s, t, \varepsilon, x) = s(T_t + S_1 + \varepsilon J)x + (1 - s)S_2x, \quad (s, t, x) \in [0, 1] \times (0, \infty) \times \overline{G}.$$

For any  $(t, \varepsilon) \in (0, \infty) \times (0, \infty)$ , the family  $\{H_5(t, s, \varepsilon, \cdot)\}_{s \in [0, 1]}$  is an admissible homotopy of type  $(S_+)$ . To show this, it is enough to show that there exists  $t_0 > 0$  such that  $0 \notin H_5(t, s, \varepsilon, \partial G)$  for all  $t \in (0, t_0]$  and all  $s \in [0, 1]$ . Suppose that this is false, i.e. there exist  $t_n \downarrow 0^+$ ,  $s_n \in [0, 1]$ ,  $x_n \in \partial G$ ,  $w_n \in S_1x_n$  and  $v_n \in S_2x_n$  such that

$$s_n(T_{t_n}x_n + w_n + \varepsilon Jx_n) + (1 - s_n)v_n = 0$$

for all  $n$ . If  $s_n = 0$  for some  $n$ , then  $v_n = 0$  with  $x_n \in \partial G$ , which is impossible by our hypothesis. Since  $s_n > 0$  for all  $n$ , we have

$$T_{t_n}x_n + w_n + \varepsilon Jx_n + \frac{1 - s_n}{s_n}v_n = 0 \quad (3.2.13)$$

for all  $n$ . For each  $n$ , we let  $z_n = T_{t_n}x_n$  and  $y_n = J_{t_n}x_n$ . We know that  $z_n \in T(y_n)$ . Since  $S_1$  and  $S_2$  are bounded, the boundedness of  $\{x_n\}$  implies the boundedness of the sequences  $\{w_n\}$  and  $\{v_n\}$ . Assume w.l.o.g. that  $s_n \rightarrow s_0$ ,  $x_n \rightarrow x_0$ ,  $w_n \rightarrow w_0$ ,  $v_n \rightarrow v_0$  as  $n \rightarrow \infty$ . If  $s_0 = 0$ , the condition  $0 \in T(0)$  gives  $T_{t_n}0 = 0$  for all  $n$ . Furthermore, by the boundedness of the sequence  $\langle w_n, x_n \rangle$ , there exists

$C > 0$  such that  $|\langle w_n, x_n \rangle| \leq C$  for all  $n$ . Using the  $\Gamma_\phi$  condition on  $S_2$  as well as the monotonicity of  $T$ , we get

$$\phi(\|x_n\|) \leq \langle v_n, x_n \rangle \leq \frac{s_n}{1-s_n}C$$

for all  $n$ , which implies  $\phi(\|x_n\|) \rightarrow 0$  as  $n \rightarrow \infty$ , i.e.  $x_n \rightarrow 0 \in \partial G$  as  $n \rightarrow \infty$ . However, this is impossible as  $0 \in G$ .

Thus, it remains to consider the case  $s_0 \in (0, 1]$ . From (3.2.13), it is easy to see that the sequence  $\{z_n\}$  is bounded. Assume w.l.o.g. that  $z_n \rightharpoonup z_0$  as  $n \rightarrow \infty$ . The pseudomonotonicity of  $S_1$  and the  $(S_+)$  condition on  $S_2$  imply

$$\liminf_{n \rightarrow \infty} \langle w_n, x_n - x_0 \rangle \geq 0 \text{ and } \liminf_{n \rightarrow \infty} \langle v_n, x_n - x_0 \rangle \geq 0,$$

and hence

$$\limsup_{n \rightarrow \infty} \langle z_n, x_n - x_0 \rangle \leq 0.$$

Since  $J_{t_n}x_n - x_n = t_n J^{-1}(v_n) \rightarrow 0$  as  $n \rightarrow \infty$ , we get

$$\limsup_{n \rightarrow \infty} \langle z_n - z_0, y_n - x_0 \rangle \leq 0.$$

Since  $T$  is maximal monotone, an application of Lemma 1.2.5 implies  $x_0 \in D(T)$ ,  $z_0 \in Tx_0$  and  $\langle z_n, y_n \rangle \rightarrow \langle z_0, x_0 \rangle$  as  $n \rightarrow \infty$ . As a result of this, we have

$$\limsup_{n \rightarrow \infty} \langle w_n, x_n - x_0 \rangle \leq 0.$$

Since  $S_1$  is pseudomonotone with  $D(S_1) = X$ , it is generalized pseudomonotone, which gives  $w_0 \in Sx_0$  and  $\langle w_n, x_n \rangle \rightarrow \langle w_0, x_0 \rangle$  as  $n \rightarrow \infty$ . From (3.2.13), we get

$$\lim_{n \rightarrow \infty} \langle Jx_n, x_n - x_0 \rangle \leq 0.$$

Since  $J$  is continuous and of type  $(S_+)$ , we see that  $x_n \rightarrow x_0 \in \partial G$  and  $Jx_n \rightarrow Jx_0$  as  $n \rightarrow \infty$ . Furthermore, by the demicontinuity of  $S_2$ , we have  $v_0 \in S_2x_0$ . Consequently, we have  $x_0 \in D(T) \cap \partial G$ ,  $z_0 \in Tx_0$ ,  $w_0 \in Sx_0$  and  $v_0 \in S_2x_0$  so that

$$s_0(z_0 + w_0) + (1 - s_0)v_0 + \tilde{\varepsilon}Jx_0 = 0, \quad (3.2.14)$$

where  $\tilde{\varepsilon} = s_0\varepsilon$ . Since  $\tilde{\varepsilon} \in (0, \varepsilon_0]$ , (3.2.14) is in contradiction of (3.2.10). Therefore, we conclude  $d(H_5(t, s, \varepsilon, \cdot), G, 0)$  is independent of all  $s \in [0, 1]$  and all sufficiently small  $t > 0$ . In conclusion, for each  $\varepsilon \in (0, \varepsilon_0]$  and all sufficiently small  $t > 0$ , we get

$$d(H_4(0, \cdot), G, 0) = d(S_2, G, \cdot) = d(T_t + S_1 + \varepsilon J, G, 0).$$

Finally, using the definition of the degree, we get

$$\begin{aligned} d(H_4(0, \cdot), G, 0) &= d(S_2, G, 0) = \lim_{\varepsilon \downarrow 0^+} \lim_{t \downarrow 0^+} d(T_t + S_1 + \varepsilon J, G, 0) \\ &= d(T + S_1, G, 0). \end{aligned} \quad (3.2.15)$$

Combining (3.2.12) and (3.2.15), we conclude that

$$d(H_4(s, \cdot), G, 0) = d(T + S_1, G, 0)$$

for all  $s \in [0, 1]$ , i.e.  $d(H_4(s, \cdot), G, 0)$  is independent of all  $s \in [0, 1]$ . This completes the proof of this part.

(iv) In this part we show that  $d(H_4(s, \cdot), G, 0)$  is independent of  $s \in [0, 1]$  using  $\overline{D(T)} = X$  without imposing any further conditions on the given operators. Using the first part of the proof of (iii), we get that  $d(H_4(s, \cdot), G, 0)$  is independent of  $s \in (0, 1]$ . In particular, we have

$$d(H_4(s, \cdot), G, 0) = d(T + S_1, G, 0)$$



for all  $s \in (0, 1]$ . Thus, it remains to show that

$$d(H_4(0, \cdot), G, 0) = d(S_2, G, 0) = d(T + S_1, G, 0).$$

To this end, we consider the homotopy  $\{H_5(s, t, \varepsilon, \cdot)\}_{s \in [0, 1]}$ . Working as in (iii), if we assume that there exist  $t_n \downarrow 0^+$ ,  $s_n \in [0, 1]$ ,  $x_n \in \partial G$ ,  $w_n \in S_1 x_n$  and  $v_n \in S_2 x_n$  such that

$$s_n(T_{t_n} x_n + w_n + \varepsilon J x_n) + (1 - s_n)v_n = 0 \quad (3.2.16)$$

for all  $n$ , we get a contradiction if  $s_0 \in (0, 1]$ .

To complete the proof, consider the case  $s_0 = 0$ . We let  $z_n = T_{t_n} x_n$  and, as before, using the monotonicity of  $T$  for each  $(x, y) \in G(T)$ , we obtain

$$\langle z_n - y, x_n - t_n J^{-1}(z_n) - x \rangle \geq 0,$$

which implies

$$s_n \langle z_n - y, x_n - x \rangle + s_n t_n \|y\| \|z_n\| \geq s_n t_n \|z_n\|^2 \geq 0$$

for all  $n$ . Since  $\{s_n z_n\}$  is bounded and  $t_n \downarrow 0^+$ , for each  $x \in D(T)$  we have

$$\liminf_{n \rightarrow \infty} \langle s_n z_n, x_n - x \rangle \geq 0.$$

Combining this with (3.2.16), we arrive at

$$\limsup_{n \rightarrow \infty} \langle v_n, x_n - x \rangle \leq 0.$$

Using the density of  $D(T)$  in  $X$ , we see that this inequality holds for each  $x \in X$ .

In particular, for  $x = x_0$  we get

$$\limsup_{n \rightarrow \infty} \langle v_n, x_n - x_0 \rangle \leq 0.$$

Since  $S_2$  is of type  $(S_+)$ , we have  $x_n \rightarrow x_0 \in \partial G$ ,  $\langle v_n, x_n \rangle \rightarrow \langle v_0, x_0 \rangle$  and  $v_0 \in S_2 x_0$ . As a consequence, for each  $x \in X$  we have  $\langle v_0, x_0 - x \rangle = 0$ . In particular, letting  $x_0 - x$  in place of  $x$ , we get  $\langle v_0, x \rangle = 0$ , i.e.  $v_0 = 0$ , which implies  $0 \in S_2(\partial G)$ . However, this is a contradiction. Therefore, for each  $\varepsilon \in (0, \varepsilon_0]$ ,  $\{H_5(s, t, \varepsilon, \cdot)\}_{s \in [0, 1]}$  is a homotopy of type  $(S_+)$  such that  $0 \notin H_5(s, t, \varepsilon, \partial G)$  for all  $s \in [0, 1]$  and all sufficiently small  $t > 0$ . Hence, we have

$$d(H_4(0, \cdot), G, 0) = d(S_2, G, 0) = d(T_t + S_1 + \varepsilon J, G, 0),$$

which implies, as in the argument in (iii), that

$$d(H_4(s, \cdot), G, 0) = d(T + S_1, G, 0)$$

for all  $s \in [0, 1]$ . The proof is complete. ■

The next Theorem contains a homotopy invariance result like the one in (iii) of Theorem 3.2.2, but we now assume that the operator  $S_2$  is bounded pseudomonotone and of type  $\Gamma_\phi$ , and the pair  $(T, S_1)$  satisfies an “inner product” boundary condition. The maximal monotone operator  $T$  is not assumed to satisfy  $0 \in T(0)$ . If  $S_2 = J$ , then this Theorem provides us with a normalization result.

**Theorem 3.2.3** *Let  $G$  be a bounded open subset of  $X$  with  $0 \in G$ , let  $T : X \supset D(T) \rightarrow 2^{X^*}$  be maximal monotone and  $S_1, S_2 : X \rightarrow 2^{X^*}$  bounded pseudomonotone*

with  $S_2$  of type  $\Gamma_\phi$ . Let

$$K_1(s, x) = s(T + S_1)x + (1 - s)S_2x, \quad (s, x) \in [0, 1] \times (D(T) \cap \overline{G}).$$

Then the following are true.

(i) Let there exist  $k_1 > 0$  such that

$$\langle v + w, x \rangle \geq k_1$$

for all  $x \in D(T) \cap \partial G$ ,  $v \in Tx$  and  $w \in S_1x$ . Then  $d(K_1(s, \cdot), G, 0)$  is well-defined and independent of  $s \in [0, 1]$ .

(ii) Suppose that the function  $\phi$  of the  $\Gamma_\phi$ -condition of  $S_2$  is  $t^2$ ,  $t \geq 0$ . Let there exist  $R > 0$  and  $k_2 > 0$  such that, for some  $v_0 \in B_{\frac{R^2}{2\beta_0}}(0)$ , we have

$$\langle v + w, x - v_0 \rangle \geq k_2$$

for all  $x \in D(T) \cap \partial B_R(0)$ ,  $v \in Tx$ ,  $w \in S_1x$ , where  $\beta_0 = \sup\{\|z\| : z \in S_2x, \|x\| = R\}$ . Then  $d(K_1(s, \cdot), G, 0)$  is well-defined and independent of  $s \in [0, 1]$ .

(iii) Under the rest of the assumptions in (ii), if  $S_2 = \alpha J$ , for some  $\alpha \in (0, 1)$ , then  $v_0$  may be taken in  $B_R(0)$ .

*Proof.* (i) Let the assumptions in (i) be satisfied. As usually, we first show that  $0 \notin \overline{K_1(s, D(T) \cap \partial G)}$  for all  $s \in [0, 1]$ , i.e. for each  $s \in [0, 1]$ , the degree  $d(K_1(s, \cdot), G, 0)$  is well-defined. Suppose that this is false, i.e. there exists  $s \in [0, 1]$ ,  $x_n \in D(T) \cap \partial G$ ,  $v_n \in Tx_n$ ,  $w_n \in S_1x_n$  and  $z_n \in S_2x_n$  such that

$$s(v_n + w_n) + (1 - s)z_n \rightarrow 0$$

as  $n \rightarrow \infty$ . If  $s = 0$ , using the condition  $\Gamma_\phi$  on  $S_2$ , we see  $\phi(\|x_n\|) \leq \langle v_n, x_n \rangle \rightarrow 0$  as  $n \rightarrow \infty$ , which in turn implies  $x_n \rightarrow 0$ , i.e. a contradiction because  $0 \in G$ . If  $s = 1$ , then

$$k_1 \leq \langle v_n + w_n, x_n \rangle \rightarrow 0,$$

which is again impossible because  $k_1 > 0$ . So, we suppose that  $s \in (0, 1)$ . Then

$$v_n + w_n + \frac{1-s}{s}z_n \rightarrow 0$$

as  $n \rightarrow \infty$ . Using the  $\Gamma_\phi$  condition on  $S_2$ , we obtain

$$k_1 \leq k_1 + \frac{1-s}{s}\phi(\|x_n\|) \leq \langle v_n + w_n + \frac{1-s}{s}z_n, x_n \rangle \rightarrow 0$$

as  $n \rightarrow \infty$ , which is impossible because  $k_1 > 0$ . Hence, for each  $s \in [0, 1]$ , the degree  $d(K_1(s, \cdot), G, 0)$  is well-defined. On the other hand, for each  $s \in [0, 1]$  and each  $\varepsilon > 0$ , we have

$$\langle s(v + w) + (1-s)z + \varepsilon Jx, x \rangle \geq (1-s)\phi(\|x\|) + \varepsilon\|x\|^2 > 0$$

for all  $v \in Tx$ ,  $w \in S_1x$ ,  $z \in S_2x$ ,  $x \in D(T) \cap \partial G$  because  $0 \in G$ , i.e.

$$0 \notin (s(T + S_1) + (1-s)S_2 + \varepsilon J)(D(T) \cap \partial G)$$

for all  $s \in [0, 1]$  and all  $\varepsilon > 0$ . Next we show that

$$d(s(T + S_1) + (1-s)S_2 + \varepsilon J, G, 0)$$

is independent of all  $(s, \varepsilon) \in (0, 1] \times (0, \infty)$ . Let  $(s_1, \varepsilon_1) \neq (s_2, \varepsilon_2)$ . Let  $\alpha_s$ ,  $\beta_s$  and  $\gamma_s$  be as in the proof of (iii) of Theorem 3.2.2. Since  $\alpha_s \in [s_1, s_2]$ , it follows, as in

that proof, that  $\{\alpha_s T\}_{s \in [0,1]}$  is a pseudomonotone homotopy of maximal monotone operators. Furthermore, as in the proof of (iii) of Theorem 3.2.2, the family  $\{\alpha_s S_1 + \beta_s S_2 + \gamma_s J\}_{s \in [0,1]}$  is a homotopy of type  $(S_+)$  such that

$$0 \notin (\alpha_s T + \alpha_s S_1 + \beta_s S_2 + \gamma_s J)(D(T) \cap \partial G).$$

Therefore, we conclude that

$$d(\alpha_s T + \alpha_s S_1 + \beta_s S_2 + \gamma_s J, G, 0)$$

is independent of  $s \in [0, 1]$ , which implies

$$d(s_1(T + S_1) + (1 - s_1)S_2 + \varepsilon_1 J, G, 0) = d(s_2(T + S_1) + (1 - s_2)S_2 + \varepsilon_2 J, G, 0).$$

Since  $(s_1, \varepsilon_1)$  and  $(s_2, \varepsilon_2)$  are arbitrary in  $(0, 1] \times (0, \infty)$ , we have

$$d(s(T + S_1) + (1 - s)S_2 + \varepsilon J, G, 0) = d(T + S_1 + \varepsilon J, G, 0).$$

Finally, using the definition of our degree, we let  $\varepsilon \downarrow 0^+$  to get

$$d(s(T + S_1) + (1 - s)S_2, G, 0) = d(T + S_1, G, 0)$$

for all  $s \in (0, 1]$ .

To show that  $d(K(0, \cdot), G, 0) = d(T + S_1, G, 0)$ , we follow exactly the same approach as in the case of (iii) of Theorem 3.2.2, which made essential use of the assumed  $\Gamma_\phi$ -condition on  $S_2$ . As a result of this, we get that  $d(K(s, \cdot), G, 0)$  is independent of  $s \in [0, 1]$ . This completes the proof of part (i).

(ii) We show first that  $0 \notin \overline{K_1(s, \cdot)(D(T) \cap \partial B_R(0))}$  for all  $s \in [0, 1]$ . Suppose that this is false, i.e. there exists  $s \in [0, 1]$ ,  $x_n \in D(T) \cap \partial B_R(0)$ ,  $v_n \in T x_n$ ,  $w_n \in S_1 x_n$

and  $z_n \in S_2 x_n$  such that

$$s(v_n + w_n) + (1 - s)z_n \rightarrow 0$$

as  $n \rightarrow \infty$ . If  $s = 0$ , using the  $\Gamma_\phi$ -condition on  $S_2$ , we see that  $x_n \rightarrow 0$  as  $n \rightarrow \infty$ , which is impossible because  $0 \in B_R(0)$ . If  $s = 1$ , using our hypothesis, we see that  $k_2 \leq \langle v_n + w_n, x_n - v_0 \rangle \rightarrow 0$  as  $n \rightarrow \infty$ , which implies  $k_2 = 0$ , i.e. a contradiction.

If  $s \in (0, 1)$ , then

$$\frac{1-s}{s}k_2 + \langle z_n, x_n - v_0 \rangle \leq \left\langle \frac{1-s}{s}(v_n + w_n) + z_n, x_n - v_0 \right\rangle,$$

which implies

$$\langle z_n, x_n - v_0 \rangle < \left\langle \frac{1-s}{s}(v_n + w_n) + z_n, x_n - v_0 \right\rangle \rightarrow 0$$

as  $n \rightarrow \infty$ . Using the  $\Gamma_\phi$  condition with  $\phi(t) \equiv t^2$ , we arrive at

$$\begin{aligned} \|x_n\|^2 &< \|z_n\| \|v_0\| + \left\langle \frac{1-s}{s}(v_n + w_n) + z_n, x_n - v_0 \right\rangle \\ &\leq \beta_0 \|v_0\| + \left\langle \frac{1-s}{s}(v_n + w_n) + z_n, x_n - v_0 \right\rangle. \end{aligned} \tag{3.2.17}$$

Combining this with the previous inequality and using  $\|x_n\| = R$  and  $v_0 \in B_{\frac{R^2}{2\beta_0}}(0)$ , we obtain the contradiction  $R^2 \leq \frac{R^2}{2}$ . Thus, we have shown that, for each  $s \in [0, 1]$ , the degree  $d(K_1(s, \cdot), B_R(0), 0)$  is well-defined. Following an argument analogous to the relevant argument above, one can verify the existence of  $\varepsilon_0 > 0$  such that

$$0 \notin (s(T + S_1) + (1 - s)S_2 + \varepsilon J)(D(T) \cap \partial B_R(0))$$

for all  $s \in [0, 1]$  and all  $\varepsilon \in (0, \varepsilon_0]$ . Again, employing the argument used in the proof of (iii) of Theorem 3.2.2 (and later in (i) above), we conclude that

$$d(K_1(s, \cdot), B_R(0), 0)$$

is independent of  $s \in [0, 1]$ .

To show (iii), all that we need to observe is that (3.2.17) will imply now the contradiction  $R^2 \leq \alpha R^2$ . The proof is complete. ■

Next we give the following normalization result.

**Corollary 3.2.4** *Let  $G$  be a bounded open subset of  $X$  with  $0 \in G$ . Let  $T : X \supset D(T) \rightarrow 2^{X^*}$  be maximal monotone and  $S : X \rightarrow 2^{X^*}$  bounded pseudomonotone. Assume that there exists  $k > 0$  such that*

$$\langle v + w, x \rangle \geq k$$

*for all  $x \in D(T) \cap \partial G$ ,  $v \in Tx$ ,  $w \in Sx$ . Then  $d(T + S, G, 0) = 1$ . Furthermore,  $k$  can be taken to be 0 provided that  $G = B_R(0)$  for some  $R > 0$ .*

*Proof.* The conclusion follows from Theorem 3.2.3 by taking  $S_1 = S$  and  $S_2 = J$ . The case  $k = 0$  can be treated using the the duality mapping  $J$  as

$$\langle v + w + \varepsilon Jx, x \rangle \geq \varepsilon \|x\|^2 = \varepsilon R^2 = k_\varepsilon > 0$$

for all  $x \in D(T) \cap \partial B_R(0)$ . This implies that

$$d(T + S + \varepsilon J, G, 0) = 1$$

for all  $\varepsilon > 0$ . By the definition of the degree, we conclude that  $d(T + S, G, 0) = 1$ . ■

More generally, the following Corollary holds.

**Corollary 3.2.5** *Let  $G$  be a bounded open subset of  $X$  with  $0 \in G$ . Let  $T : X \supset D(T) \rightarrow 2^{X^*}$  be maximal monotone with  $0 \in T(0)$  and  $S : X \rightarrow 2^{X^*}$  bounded pseudomonotone and of type  $\Gamma_\phi$ . Then*

$$d(T + S, G, 0) = d(S, G, 0).$$

*In particular, if  $S = \lambda J$ , we have  $d(T + \lambda J, G, 0) = 1$  for all  $\lambda > 0$ .*

*Proof.* Suppose  $0 \in \overline{S(\partial G)}$ , i.e. there exists  $x_n \in \partial G$  and  $w_n \in Sx_n$  such that  $w_n \rightarrow 0$  as  $n \rightarrow \infty$ . Since  $\langle w_n, x_n \rangle \geq \phi(\|x_n\|)$  for some  $\phi \in \Gamma_\phi$ , it follows that  $x_n \rightarrow 0$  as  $n \rightarrow \infty$ , i.e.  $0 \in \partial G$ , which is a contradiction. On the other hand, if  $0 \in \overline{(T + S)(D(T) \cap \partial G)}$ , then there exists  $x_n \in D(T) \cap \partial G$ ,  $v_n \in Tx_n$  and  $w_n \in Sx_n$  such that

$$v_n + w_n \rightarrow 0$$

as  $n \rightarrow \infty$ . Since  $0 \in T(0)$ , we have  $\langle v_n, x_n \rangle \geq 0$ , and hence  $x_n \rightarrow 0 \in \partial G$  as  $n \rightarrow \infty$ , i.e. a contradiction. So, the degrees  $d(S, G, 0)$  and  $d(T + S, G, 0)$  are well-defined. Similarly, we see that  $0 \notin \overline{(T_t + S + \varepsilon J)(\partial G)}$  for all  $t > 0$  and  $\varepsilon > 0$ . Thus, by the definition of our degree,

$$d(T + S, G, 0) = \lim_{\varepsilon \downarrow 0^+} \lim_{t \downarrow 0^+} d(T_t + S + \varepsilon J, G, 0). \quad (3.2.18)$$

We now consider the homotopy

$$K_2(s, t, \varepsilon, x) = sS + (1 - s)(T_t + S + \varepsilon J), \quad (s, t, \varepsilon) \in [0, 1] \times (0, \infty)^2, \quad x \in \overline{G}.$$

Since  $T_t + S + \varepsilon J$  is of type  $(S_+)$  and  $S$  is pseudomonotone, we can apply (iii) of Theorem 3.2.2 with  $T = 0$ . To this end, we observe that for each  $t > 0$  and  $\varepsilon > 0$  we get  $0 \notin \overline{K_2(s, t, \varepsilon, \partial G)}$  for any  $s \in [0, 1]$ . Therefore, applying (iii) of Theorem 3.2.2,



we conclude that

$$d(S, G, 0) = d(T_t + S + \varepsilon J, G, 0)$$

for all  $t > 0$  and  $\varepsilon > 0$ . As a result, using (3.2.18), we get  $d(T + S, G, 0) = d(S, G, 0)$ .

If  $S = \lambda J$ , then  $d(T + \lambda J, G, 0) = 1$  for all  $\lambda > 0$  because  $0 \in G$  and  $d(\lambda J, G, 0) = 1$ .

This completes the proof. ■

We observe that the conclusion of Corollary 3.2.5 does not follow, in general, from (iii) of Theorem 3.2.2 and 3.2.3 because in the former case  $S$  is not assumed of type  $(S_+)$  and in the latter case the “inner product” condition might not be satisfied.

### 3.3 Uniqueness of the degree

**Theorem 3.3.1** *There exists exactly one degree mapping defined on the class of mappings of the form  $T + S$ , where  $T : X \supset D(T) \rightarrow 2^{X^*}$  is maximal monotone,  $S : X \rightarrow 2^{X^*}$  is bounded pseudomonotone with  $0 \notin \overline{(T + S)(D(T) \cap \partial G)}$ , normalized by  $J$  and invariant under the homotopies of the type (i) and (ii) of Theorem 3.2.2.*

*Proof.* Let  $\hat{d}$  be another degree mapping satisfying the hypotheses of the theorem.

Fix  $\varepsilon > 0$  and let

$$H_t = T_{t\varepsilon} = (T^{-1} + t\varepsilon J^{-1})^{-1}, \quad 0 \leq t \leq 1.$$

By Lemma 2.3.1,  $\{H_t\}_{t \in [0,1]}$  is a pseudomonotone homotopy of maximal monotone operators. Thus, by the definition of  $d(T + S, G, 0)$ , there exists  $\varepsilon_0 > 0$  such that  $d(T + S + \varepsilon J, G, 0)$  is constant for all  $\varepsilon \in (0, \varepsilon_0]$ . This implies that, for a fixed  $\bar{\varepsilon} \in (0, \varepsilon_0]$ , we have  $d(T + S, G, 0) = d(T + S + \bar{\varepsilon} J, G, 0)$ . Similarly, there exists  $\delta_0 > 0$  such that

$$d(T + S, G, 0) = d(T_\delta + S + \bar{\varepsilon} J, G, 0)$$

for all  $\delta \in (0, \delta_0]$ . In particular, for a fixed  $\bar{\delta} \in (0, \delta_0]$ , we have

$$d(T + S, G, 0) = d(T_{\bar{\delta}} + S + \bar{\varepsilon}J, G, 0).$$

By the uniqueness of the degree mapping  $d$  on the class of bounded  $(S_+)$  mappings, we obtain

$$d(T_{\bar{\delta}} + S + \bar{\varepsilon}J, G, 0) = \hat{d}(T_{\bar{\delta}} + S + \bar{\varepsilon}J, G, 0).$$

Since  $T_{t\bar{\delta}}$  is also a pseudomonotone homotopy of maximal monotone operators, we use the assumed homotopy invariance property of  $\hat{d}$  to obtain

$$\begin{aligned} \hat{d}(T_{\bar{\delta}} + S + \bar{\varepsilon}J, G, 0) &= \hat{d}(T_{t\bar{\delta}} + t(S + \bar{\varepsilon}J) + (1-t)(S + \bar{\varepsilon}J), G, 0) \\ &= \hat{d}(T + S + \bar{\varepsilon}J, G, 0) \\ &= \hat{d}(T + tS + (1-t)(S + \bar{\varepsilon}J), G, 0) \\ &= \hat{d}(T + S, G, 0), \quad t \in [0, 1]. \end{aligned}$$

We conclude that  $d(T + S, G, 0) = \hat{d}(T + S, G, 0)$ . ■

### 3.4 Degree theory for unbounded single multi-valued pseudomonotone operators

It is our intention here to show that it is possible to develop a degree theory for a single multi-valued pseudomonotone operator  $S$  without any boundedness assumption on it.

**Theorem 3.4.1** *Let  $G$  be a bounded open subset of  $X$ . Let  $S : X \rightarrow 2^{X^*}$  be pseudomonotone and such that  $0 \notin \overline{S(\partial G)}$ . Then there exists  $\varepsilon_0 > 0$  such that  $d(S + \varepsilon J, G, 0)$  is well-defined and constant for all  $\varepsilon \in (0, \varepsilon_0]$ .*

*Proof.* Suppose that  $0 \notin \overline{S(\partial G)}$ . Then there exists  $\varepsilon_0 > 0$  such that  $0 \notin (S + \varepsilon J)(\partial G)$  for all  $\varepsilon \in (0, \varepsilon_0]$ . Indeed, if this false, then there exist  $\{\varepsilon_n\}$  with  $\varepsilon_n \downarrow 0^+$ ,  $x_n \in \partial G$  and  $w_n \in Sx_n$  such that  $w_n + \varepsilon_n Jx_n = 0$  for all  $n$ . However, this implies  $0 \in \overline{S(\partial G)}$ , which is a contradiction to our assumption. Thus, there exists  $\varepsilon_0 > 0$  such that  $d(S + \varepsilon J, G, 0)$  is well-defined for all  $\varepsilon \in (0, \varepsilon_0]$ . Furthermore, for any  $\varepsilon_1, \varepsilon_2 \in (0, \varepsilon_0]$  such that  $\varepsilon_1 < \varepsilon_2$ , we consider the homotopy

$$H(t, x) = Sx + tJx, \quad (t, x) \in [\varepsilon_1, \varepsilon_2] \times \overline{G}.$$

It is easy to see that  $\{H(t, \cdot)\}_{t \in [0, 1]}$  is a homotopy of type  $(S_+)$  such that  $0 \notin H(t, \partial G)$  for all  $t \in [\varepsilon_1, \varepsilon_2]$ . Therefore, we have

$$d(S + \varepsilon_1 J, G, 0) = d(S + \varepsilon_2 J, G, 0).$$

This implies that  $d(S + \varepsilon J, G, 0)$  is constant for all  $\varepsilon \in (0, \varepsilon_0]$  and completes the proof.

■

**Definition 3.4.2 (Degree for Multi-valued Pseudomonotone Operators)** Let  $G$  be an open bounded subset of  $X$ . Let  $S : X \rightarrow 2^{X^*}$  be a pseudomonotone operator such that  $0 \notin \overline{S(\partial G)}$ . We define

$$d(S, G, 0) := \lim_{\varepsilon \downarrow 0^+} d(S + \varepsilon J, G, 0).$$

The basic properties of this degree are contained in the following theorem.

**Theorem 3.4.3** *Let  $G$  be a bounded open subset of  $X$ . Let  $S : X \rightarrow 2^{X^*}$  be a pseudomonotone operator such that  $0 \notin \overline{S(\partial G)}$ . Then the following properties are true.*

- (i) *(Normalization)  $d(J, G, 0) = 1$  if  $0 \in G$  and  $d(J, G, 0) = 0$  if  $0 \notin \overline{G}$ .*
- (ii) *(Existence) If  $d(S, G, 0) \neq 0$ , we have  $0 \in \overline{S(G)}$ . If, moreover,  $G$  is convex, then there exists  $x_0 \in G$  such that  $0 \in Sx_0$ .*
- (iii) *(Decomposition) Let  $G_1$  and  $G_2$  be disjoint open subsets of  $G$  such that  $0 \notin \overline{S(\overline{G} \setminus (G_1 \cup G_2))}$ . Then*

$$d(S, G, 0) = d(S, G_1, 0) + d(S, G_2, 0).$$

- (iv) *(Homotopy Invariance) Let  $S_1$  and  $S_2$  be bounded pseudomonotone with effective domain  $X$  such that  $0 \notin \overline{H(t, \partial G)}$  for all  $t \in [0, 1]$  where*

$$H(t, x) = tS_1x + (1 - t)S_2x, (t, x) \in [0, 1] \times \overline{G}.$$

*Then  $d(H(t, \cdot), G, 0)$  is independent of  $t \in [0, 1]$ .*

*Proof.* The proof of (i), (iii) and (iv) follows as in the relevant proofs of Theorems 3.1.4 and 3.2.2. We give the proof of (ii). The first part of (ii) follows exactly as the first part of (ii) in Theorem 3.1.4. Suppose that  $G$  is convex and  $d(S, G, 0) \neq 0$ . Then, by the definition of the degree, there exist sequences  $\varepsilon_n \downarrow 0^+$ ,  $x_n \in G$  and  $w_n \in Sx_n$  such that

$$w_n + \varepsilon_n Jx_n = 0 \tag{3.4.19}$$

for all  $n$ . Since  $\{x_n\}$ ,  $J$  are bounded,  $\varepsilon_n Jx_n \rightarrow 0$  as  $n \rightarrow \infty$ , i.e.  $\{\varepsilon_n Jx_n\}$  is bounded and hence  $\{w_n\}$  is bounded. Assume w.l.o.g. that  $x_n \rightharpoonup x_0$  and  $w_n \rightharpoonup w_0$  as  $n \rightarrow \infty$ . Since  $\overline{G}$  is closed and convex,  $\overline{G}$  is weakly closed and  $x_0 \in \overline{G}$ . Using the monotonicity

of  $J$ , it follows that

$$\limsup_{n \rightarrow \infty} \langle w_n, x_n - x_0 \rangle \leq 0.$$

Since  $S$  is pseudomonotone with effective domain all of  $X$ , it is generalized pseudomonotone. Therefore,  $w_0 \in Sx_0$  and  $\langle w_n, x_n \rangle \rightarrow \langle w_0, x_0 \rangle$  as  $n \rightarrow \infty$ . Thus, taking limits as  $n \rightarrow \infty$  in (3.4.19), we get  $0 \in Sx_0$ . Since  $0 \notin \overline{S(\partial G)}$ , it follows that  $x_0 \in G$ , i.e.  $0 \in Sx$  is solvable in  $G$ . ■

### 3.5 Applications to mapping theorems of nonlinear analysis

In this Section we give applications of the new degree theory to the solvability of inclusions of the form  $Tx + Sx \ni f$ , where  $T : X \supset D(T) \rightarrow 2^{X^*}$  is maximal monotone and  $S : X \rightarrow 2^{X^*}$  bounded pseudomonotone. We also establish some new existence results and shorter proofs of known results. In particular, we give a degree theoretic proof of a result of Kenmochi in [44], we improve a result of Le in [50], we extend the main result of Figueiredo [26], and give a new proof of the result of Browder and Hess [18] on the maximality of the sum of maximal monotone mappings, which is originally due to Rockafellar [64].

In the following theorem we give an existence theorem extending the result of Figueiredo [26]. Figueiredo considered a single pseudomonotone operator.

**Theorem 3.5.1** *Let  $G$  be a bounded open and convex subset of  $X$  with  $0 \in G$ . Let  $T : X \supset D(T) \rightarrow 2^{X^*}$  be maximal monotone with  $0 \in D(T)$  and  $S : X \rightarrow 2^{X^*}$  bounded pseudomonotone and such that*

$$Tx + Sx + \varepsilon Jx \not\ni 0$$

*for all  $\varepsilon > 0$  and all  $x \in D(T) \cap \partial G$ . Then  $Tx + Sx \ni 0$  is solvable in  $D(T) \cap \overline{G}$ .*

*Proof.* Assume w.l.o.g. that  $0 \in T(0)$ . Indeed, otherwise, we pick  $v^* \in T(0)$  and consider instead the mappings  $\tilde{T}x = Tx - v^*$ ,  $x \in D(T)$ , and  $\tilde{S}x = Sx + v^*$ ,  $x \in X$ . We observe that  $\tilde{T}$  is maximal monotone with  $\tilde{0} \in \tilde{T}(0)$  and  $\tilde{S}$  is bounded pseudomonotone, and

$$\tilde{T}x + \tilde{S}x + \varepsilon Jx \not\equiv 0$$

for all  $x \in D(\tilde{T}) \cap \partial G$ . If  $0 \in (T + S)(D(T) \cap \partial G)$ , we are done. Assume that  $0 \notin (T + S)(D(T) \cap \partial G)$ . We consider the following cases.

**Case I.**  $0 \in \overline{(T + S)(D(T) \cap \partial G)}$

**Case II.**  $0 \notin \overline{(T + S)(D(T) \cap \partial G)}$ .

Suppose Case I holds, i.e there exist  $x_n \in D(T) \cap \partial G$ ,  $v_n \in Tx_n$  and  $w_n \in Sx_n$  such that

$$v_n + w_n \rightarrow 0 \tag{3.5.20}$$

as  $n \rightarrow \infty$ . Since  $\{x_n\}$  and  $S$  are bounded, it follows that  $\{v_n\}$  and  $\{w_n\}$  are also bounded. Assume w.l.o.g. that  $x_n \rightarrow x_0$ ,  $v_n \rightarrow v_0$  and  $w_n \rightarrow w_0$  as  $n \rightarrow \infty$ . Since the sequence  $\{\langle v_n, x_n - x_0 \rangle\}$  is bounded, we may also assume w.l.o.g. that the limit

$$\lim_{n \rightarrow \infty} \langle v_n, x_n - x_0 \rangle$$

exists. This implies

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \langle v_n + w_n, x_n - x_0 \rangle \\ &\geq \liminf_{n \rightarrow \infty} \langle v_n, x_n - x_0 \rangle + \liminf_{n \rightarrow \infty} \langle w_n, x_n - x_0 \rangle \\ &= \lim_{n \rightarrow \infty} \langle v_n, x_n - x_0 \rangle + \liminf_{n \rightarrow \infty} \langle w_n, x_n - x_0 \rangle. \end{aligned}$$

By the pseudomonotonicity of  $S$ , we know that

$$\liminf_{n \rightarrow \infty} \langle w_n, x_n - x_0 \rangle \geq 0,$$

which implies

$$\lim_{n \rightarrow \infty} \langle v_n - v_0, x_n - x_0 \rangle \leq 0.$$

By lemma 1.2.5, we conclude that  $x_0 \in D(T)$ ,  $v_0 \in Tx_0$  and  $\langle v_n, x_n \rangle \rightarrow \langle v_0, x_0 \rangle$  as  $n \rightarrow \infty$ . Using (3.5.20), we see that

$$\lim_{n \rightarrow \infty} \langle w_n, x_n - x_0 \rangle = 0.$$

Since  $S$  is pseudomonotone with effective domain all of  $X$ , we obtain that  $w_0 \in Sx_0$ , and  $v_0 + w_0 = 0$ . Since  $\overline{G}$  is closed and convex, the set  $\overline{G}$  is weakly closed and  $x_0 \in \overline{G}$ . Since  $0 \notin (T+S)(D(T) \cap \partial G)$ , we have  $x_0 \notin D(T) \cap \partial G$ , which implies  $x_0 \in D(T) \cap G$ . This shows that  $0 \in Tx + Sx$  is solvable in  $D(T) \cap G$ .

**Case II.** The degree  $d(T + S, G, 0)$  is well-defined. Fix  $\varepsilon > 0$  and consider the homotopy

$$H_\varepsilon(s, t, x) = s(T_t + S + \varepsilon J) + (1 - s)Jx, \quad (s, t, x) \in [0, 1] \times (0, \infty) \times \overline{G}.$$

For each  $t > 0$ , it is easy to see that  $\{H_\varepsilon(s, t, \cdot)\}_{s \in [0, 1]}$  is a homotopy of type  $(S_+)$ .

We claim that there exists  $t_0 > 0$  such that  $0 \notin H_\varepsilon(s, t, \partial G)$  for all  $t \in (0, t_0]$  and all  $s \in [0, 1]$ . Suppose that this is false, i.e. there exists  $t_n \downarrow 0^+$ ,  $s_n \in [0, 1]$ ,  $x_n \in \partial G$  and  $w_n \in Sx_n$  such that

$$s_n(T_{t_n}x_n + w_n + \varepsilon Jx_n) + (1 - s_n)Jx_n = 0 \tag{3.5.21}$$

for all  $n$ . We observe that  $s_n > 0$  for all  $n$ . Otherwise, if for some  $n$ ,  $s_n = 0$ , we would

have  $Jx_n = 0$ . Since  $J$  is one to one and  $J0 = 0$ , it follows that  $x_n = 0$ , which is impossible as  $0 \in G$ . Thus,

$$T_{t_n}x_n + w_n + \varepsilon Jx_n + \frac{1 - s_n}{s_n} Jx_n = 0 \quad (3.5.22)$$

for all  $n$ . Assume that  $s_n \rightarrow s_0$  as  $n \rightarrow \infty$ . Let  $v_n = T_{t_n}x_n$  and  $y_n = J_{t_n}x_n$ . If  $s_0 \in (0, 1]$ , the boundedness of  $\{w_n\}$  and  $\{x_n\}$  implies that  $\{v_n\}$  is bounded. Assume that  $v_n \rightarrow v_0$  as  $n \rightarrow \infty$ . By the pseudomonotonicity of  $S$ , the monotonicity of  $J$  and (3.5.22), we have

$$\liminf_{n \rightarrow \infty} \langle w_n, x_n - x_0 \rangle \geq 0.$$

As a result, we obtain

$$\limsup_{n \rightarrow \infty} \langle v_n, x_n - x_0 \rangle \leq 0.$$

Since  $J_{t_n}x_n - x_n = t_n J^{-1}(v_n) \rightarrow 0$  as  $n \rightarrow \infty$ , we get

$$\limsup_{n \rightarrow \infty} \langle v_n - v_0, y_n - x_0 \rangle \leq 0.$$

Now, it is well known that  $y_n = J_{t_n}x_n \in D(T)$  and  $v_n \in T(y_n)$  for all  $n$ . Therefore, using Lemma 1.2.5, we find  $x_0 \in D(T)$ ,  $v_0 \in Tx_0$  and  $\langle v_n, y_n \rangle \rightarrow \langle v_0, x_0 \rangle$  as  $n \rightarrow \infty$ .

As a result of this, we obtain

$$\limsup_{n \rightarrow \infty} \langle w_n, x_n - x_0 \rangle \leq 0.$$

Using the generalized pseudomonotonicity of  $S$ , we get  $w_0 \in Sx_0$  and  $\langle w_n, x_n \rangle \rightarrow \langle w_0, x_0 \rangle$  as  $n \rightarrow \infty$ . Thus, from (3.5.22), we obtain

$$\lim_{n \rightarrow \infty} \langle Jx_n, x_n - x_0 \rangle = 0.$$



Since  $J$  is of type  $(S_+)$ , it follows that  $x_n \rightarrow x_0 \in \partial G$  as  $n \rightarrow \infty$ . By the continuity of  $J$ , we get  $Jx_n \rightarrow Jx_0$  as  $n \rightarrow \infty$ . Finally, taking limits as  $n \rightarrow \infty$  in (3.5.22), it follows that  $x_0 \in D(T) \cap \partial G$ ,  $v_0 \in Tx_0$  and  $w_0 \in Sx_0$  such that

$$v_0 + w_0 + \tilde{\varepsilon}Jx_0 = 0,$$

where  $\tilde{\varepsilon} = \varepsilon + \frac{1-s_0}{s_0} > 0$ . However, this is a contradiction of the hypothesis of the theorem.

We now prove our claim for  $s_0 = 0$ . Since  $0 \in T(0)$ , we have  $T_{t_n}0 = 0$  for all  $n$ . Furthermore, by the boundedness of  $S$  and  $\{x_n\}$ , there exists  $K \geq 0$  such that  $\langle w_n, x_n \rangle \geq -K$  for all  $n$ . As a result of this, we have

$$\frac{1-s_n}{s_n} \|x_n\|^2 \leq K$$

for all  $n$ , which is equivalent to

$$\|x_n\|^2 \leq \frac{s_n K}{1-s_n}$$

for all  $n$ . This implies that  $x_n \rightarrow 0 \in \partial G$  as  $n \rightarrow \infty$ . Since  $0 \in G$ , this is a contradiction. Therefore, there exists  $t_0 > 0$  such that  $\{H_\varepsilon(s, t, \cdot)\}_{[0,1]}$  is a homotopy of type  $(S_+)$  such that  $0 \notin H_\varepsilon(s, t, \partial G)$  for all  $s \in [0, 1]$  and all  $t \in (0, t_0]$ . Thus, for each  $\varepsilon > 0$ , we have

$$d(T_t + S + \varepsilon J, G, 0) = d(J, G, 0) = 1$$

for all  $t \in (0, t_0]$ . Moreover, we observe that

$$\begin{aligned} d(T + S, G, 0) &= \lim_{\varepsilon \downarrow 0^+} d(T + S + \varepsilon J, G, 0) \\ &= \lim_{\varepsilon \downarrow 0^+} \lim_{t \downarrow 0^+} d(T_t + S + \varepsilon J, G, 0) = 1 \neq 0. \end{aligned}$$

Since  $0 \notin \overline{(T + S)(D(T) \cap \partial G)}$ , Theorem 3.1.4 says that  $0 \in Tx + Sx$  is solvable in  $D(T) \cap G$ . This completes the proof. ■

The following Theorem is a result of Le [50]. We give a short proof of it using the degree theory developed in this chapter.

**Theorem 3.5.2** *Let  $T : X \supset D(T) \rightarrow 2^{X^*}$  be maximal monotone and  $S : X \rightarrow 2^{X^*}$  be bounded pseudomonotone.*

(i) *Suppose there exists  $R > 0$  and  $v_0 \in B_R(0)$  such that*

$$\langle v + w, x - v_0 \rangle > 0$$

*for all  $x \in D(T) \cap \partial B_R(0)$ ,  $v \in Tx$ ,  $w \in Sx$ . Then  $0 \in R(T + S)$ .*

(ii) *If  $T + S$  is coercive, then  $R(T + S) = X^*$ .*

*Proof.* (i) For each  $\varepsilon > 0$ , we see that

$$\begin{aligned} \langle v + w + \varepsilon \tilde{J}x, x - v_0 \rangle &> \varepsilon \|x - v_0\|^2 \\ &\geq \varepsilon \left| \|x\| - \|v_0\| \right|^2 \\ &= \varepsilon |R - \|v_0\||^2 \end{aligned}$$

for all  $x \in D(T) \cap \partial B_R(0)$ , where  $\tilde{J}x = J(x - v_0)$ ,  $x \in X$ . Let  $k_\varepsilon = \varepsilon |R - \|v_0\||^2$ . Since  $\|v_0\| < R$  and  $\varepsilon > 0$ , it follows that  $k_\varepsilon > 0$  and

$$\langle v + w + \varepsilon \tilde{J}x, x - v_0 \rangle \geq k_\varepsilon$$

for all  $x \in D(T) \cap \partial B_R(0)$  and  $v \in Tx$ ,  $w \in Sx$ . Using this, we easily see that  $0 \notin \overline{(T + S + \varepsilon \tilde{J})(D(T) \cap \partial B_R(0))}$  for all  $\varepsilon > 0$ . Otherwise, we obtain  $\|v_0\| = R$  which is impossible by the hypothesis. On the other hand, since  $\tilde{J}$  is bounded, continuous

and of type  $(S_+)$  with effective domain  $X$ , it follows that  $S_1x = Sx + \varepsilon\tilde{J}$  is bounded pseudomonotone with  $D(S_1) = X$ . Therefore, letting  $S_2x = \alpha Jx$ , for some  $\alpha \in (0, 1)$ , and using (iii) of Theorem 3.2.3, for each  $\varepsilon > 0$ , we conclude that

$$d(T + S + \varepsilon\tilde{J}, B_R(0), 0) = d(\alpha J, B_R(0), 0) = 1.$$

Consequently, for each  $\varepsilon_n \downarrow 0^+$ , there exist  $x_n \in D(T) \cap B_R(0)$ ,  $v_n \in Tx_n$  and  $w_n \in Sx_n$  such that

$$v_n + w_n + \varepsilon_n\tilde{J}x_n = 0 \tag{3.5.23}$$

for all  $n$ . Since  $\{x_n\}$ ,  $\{\tilde{J}x_n\}$  and  $S$  are bounded, the sequence  $\{w_n\}$  is bounded and hence  $\{v_n\}$  is also bounded. Assume w.l.o.g. that  $x_n \rightharpoonup x_0 \in \overline{B_R(0)}$ ,  $v_n \rightharpoonup v_0$  and  $w_n \rightharpoonup w_0$  as  $n \rightarrow \infty$ . Since  $S$  is pseudomonotone, we have

$$\liminf_{n \rightarrow \infty} \langle w_n, x_n - x_0 \rangle \geq 0.$$

Using this and the monotonicity of  $\tilde{J}$ , we get

$$\limsup_{n \rightarrow \infty} \langle v_n, x_n - x_0 \rangle \leq 0.$$

Using the maximality of  $T$  and applying Lemma 1.2.5, we conclude that  $x_0 \in D(T)$ ,  $v_0 \in Tx_0$  and  $\langle v_n, x_n \rangle \rightarrow \langle v_0, x_0 \rangle$  as  $n \rightarrow \infty$ , which again implies

$$\limsup_{n \rightarrow \infty} \langle w_n, x_n - x_0 \rangle \leq 0.$$

From this, we get  $w_0 \in Sx_0$  and  $\langle w_n, x_n \rangle \rightarrow \langle w_0, x_0 \rangle$  as  $n \rightarrow \infty$ . Finally, letting  $n \rightarrow \infty$  in (3.5.23), we conclude that  $v_0 + w_0 = 0$  where  $x_0 \in D(T) \cap \overline{B_R(0)}$ ,  $v_0 \in Tx_0$  and  $w_0 \in Sx_0$ . Since, by our hypothesis,  $0 \notin (T + S)(D(T) \cap \partial B_R(0))$ , we conclude that the inclusion

$$Tx + Sx \ni 0$$

is solvable in  $D(T) \cap B_R(0)$ . This completes the proof of (i).

(ii) Let  $f \in X^*$ . Since  $T + S$  is coercive, there exists a function  $\phi : [0, \infty) \rightarrow R$  such that  $\phi(t) \rightarrow \infty$  as  $t \rightarrow \infty$  with

$$\langle v + w - f, x \rangle \geq (\phi(\|x\|) - \|f\|)\|x\|,$$

for all  $x \in D(T)$ . This in turn implies, that for every  $k > 0$  there exists  $R = R(f, k) > 0$  such that

$$\langle w + v + \varepsilon Jx - f, x \rangle \geq k$$

for all  $x \in D(T) \cap \partial B_R(0)$ . Since  $S - f$  is bounded and pseudomonotone, taking  $v_0 = 0$  in (i) and applying (i), we conclude that  $f \in R(T + S)$ . Since  $f \in X^*$  is arbitrary, the surjectivity of  $T + S$  has been proved. ■

Another existence result is contained in Theorem 3.5.3 below where the maximal monotone operator  $T$  is densely defined and omitting the hypothesis that  $0 \in D(T)$  as used in Theorem 3.5.1.

**Theorem 3.5.3** *Let  $G$  be a bounded open convex subset of  $X$  with  $0 \in G$  and let  $\varepsilon_0 \in (0, \infty)$  be fixed. Let  $T : X \supset D(T) \rightarrow 2^{X^*}$  be maximal monotone with  $\overline{D(T)} = X$  and let  $S : X \rightarrow 2^{X^*}$  be bounded pseudomonotone with*

$$Tx + Sx + \varepsilon Jx \not\ni 0$$

*for all  $x \in D(T) \cap \partial G$  and  $\varepsilon \in (0, \varepsilon_0]$ . Then the inclusion  $0 \in Tx + Sx$  is solvable in  $D(T) \cap G$ .*

*Proof.* If  $0 \in (T + S)(D(T) \cap \partial G)$ , then the proof is complete. If  $0 \notin (T + S)(D(T) \cap \partial G)$  and  $0 \in \overline{(T + S)(D(T) \cap \partial G)}$ , following the argument used in the proof of the

first part of Theorem 3.5.1, it follows that  $Tx + Sx \ni 0$  is solvable in  $D(T) \cap \overline{G}$  and hence we are done. Therefore, we assume  $0 \notin \overline{(T + S)(D(T) \cap \partial G)}$ , which implies  $d(T + S, G, 0)$  is well-defined. Let  $\varepsilon > 0$  be fixed. We show that  $d(H(s, t, \varepsilon, \cdot), G, 0)$  is well-defined for all  $s \in [0, 1]$  and for all sufficiently small  $t > 0$ , where

$$H(s, t, \varepsilon, x) := s(T_t x + Sx + \varepsilon Jx) + (1 - s)Jx, \quad (s, x) \in [0, 1] \times \partial G.$$

Suppose this assertion is false, i.e. there exist  $t_n \downarrow 0^+$ ,  $s_n \in [0, 1]$ ,  $x_n \in \partial G$  and  $w_n \in Sx_n$  such that

$$s_n(T_{t_n} x_n + w_n + \varepsilon Jx_n) + (1 - s_n)Jx_n = 0 \tag{3.5.24}$$

for all  $n$ . If  $s_n = 0$  for some  $n$ , then  $x_n = 0$ , which is a contradiction as  $0 \in G$ . Thus, we must have  $s_n \neq 0$  for all  $n$  and

$$T_{t_n} x_n + w_n + \varepsilon Jx_n + \frac{1 - s_n}{s_n} Jx_n = 0 \tag{3.5.25}$$

for all  $n$ . Assume that  $x_n \rightharpoonup x_0$  and  $s_n \rightarrow s_0$  as  $n \rightarrow \infty$ . If  $s_0 \in (0, 1]$ , following Case II of the proof of Theorem 3.5.1, we arrive at a contradiction. The case  $s_0 = 0$  is more elaborate. For each  $n$ , let  $v_n := T_{t_n} x_n$  and  $(x, y) \in G(T)$ . By the monotonicity of  $T$ , it follows that

$$\langle v_n - y, x_n - t_n J^{-1}(v_n) - x \rangle \geq 0,$$

which implies

$$s_n \langle v_n - y, x_n - x \rangle + s_n t_n \|y\| \|v_n\| \geq s_n t_n \|v_n\|^2 \geq 0$$

for all  $n$ . Since  $\{s_n v_n\}$  is bounded and  $t_n \downarrow 0^+$ , for each  $x \in D(T)$  we obtain

$$\liminf_{n \rightarrow \infty} \langle s_n v_n, x_n - x \rangle \geq 0,$$

which implies that

$$\limsup_{n \rightarrow \infty} \langle Jx_n, x_n - x \rangle \leq 0.$$

Using the monotonicity of  $J$ , it follows that  $\langle Jx, x_0 - x \rangle \leq 0$  for all  $x \in D(T)$ . Since  $\overline{D(T)} = X$ , we have  $\langle Jx, x_0 - x \rangle \leq 0$  for all  $x \in X$ . Thus, letting  $x = \alpha x_0$  with  $\alpha \in (0, 1)$ , we get

$$\alpha(1 - \alpha)\|x_0\|^2 = \alpha(1 - \alpha)\langle Jx_0, x_0 \rangle \leq 0,$$

which implies that  $x_0 = 0$ . Again using the density of  $D(T)$  in  $X$ , and the fact that  $J$  is of type  $(S_+)$ , we see that

$$\limsup_{n \rightarrow \infty} \langle Jx_n, x_n - x_0 \rangle \leq 0,$$

which implies  $x_n \rightarrow x_0 = 0 \in \partial G$ . However, this is impossible as  $0 \in G$ . As a result, for each  $\varepsilon > 0$ , there exists  $t_0 > 0$  such that  $\{H(s, t, \varepsilon, \cdot)\}_{s \in [0, 1]}$  is a homotopy of type  $(S_+)$  such that  $0 \notin H(s, t, \varepsilon, \partial G)$  for all  $s \in [0, 1]$  and all  $t \in (0, t_0]$ . By the invariance of the degree under such homotopies, we obtain

$$d(T_t + S + \varepsilon J, G, 0) = d(J, G, 0) = 1$$

for all  $t \in (0, t_0]$  and all  $\varepsilon > 0$ , i.e.  $d(T + S, G, 0) = 1$  and hence  $0 \in Tx + Sx$  is solvable in  $D(T) \cap G$ . ■

A normalization result is included in Lemma 3.5.4 below.

**Lemma 3.5.4** *Let  $T : X \supset D(T) \rightarrow 2^{X^*}$  be maximal monotone and  $S : X \rightarrow 2^{X^*}$  bounded pseudomonotone. Let  $T_t$  be the Yosida approximant of  $T$ . Suppose there exists  $R > 0$  such that*

$$\langle T_t x + w, x \rangle > 0$$

*for all  $t > 0$ ,  $x \in \partial B_R(0)$ ,  $w \in Sx$ . Then the inclusion  $Tx + Sx \ni 0$  is solvable in  $D(T) \cap \overline{B}_R(0)$ .*

We note that the inner product condition in this lemma is satisfied by any maximal monotone operator  $T$  with  $0 \in D(T)$  and any bounded pseudomonotone operator  $S$  satisfying the coercivity condition

$$\lim_{\|x\| \rightarrow \infty} \inf_{w \in Sx} \frac{\langle w, x \rangle}{\|x\|} = \infty.$$

For a verification of this, we refer the reader to the proof of Theorem 3.5.5 below.

*Proof.* [Proof of Lemma 3.5.4] If  $0 \in (T+S)(D(T) \cap \partial B_R(0))$ , then there is nothing to prove. If  $0 \notin (T+S)(D(T) \cap \partial B_R(0))$  and  $0 \in \overline{(T+S)(D(T) \cap \partial B_R(0))}$ , following the argument used in the proof of the first part of Theorem 3.5.1, we see that  $Tx + Sx \ni 0$  is solvable in  $D(T) \cap B_R(0)$ , and hence we are done. Thus, we assume that  $0 \notin \overline{(T+S)(D(T) \cap \partial B_R(0))}$ , i.e.  $d(T+S, B_R(0), 0)$  is well-defined. Let  $\varepsilon > 0$  be fixed. We show that  $d(H(s, t, \varepsilon, \cdot), B_R(0), 0)$  is well-defined for all  $s \in [0, 1]$  and all sufficiently small  $t > 0$ , where

$$H(s, t, \varepsilon, x) = s(T_t x + Sx + \varepsilon Jx) + (1-s)Jx, \quad (s, x) \in [0, 1] \times \partial B_R(0).$$

Suppose that this is false, i.e. there exist  $t_n \downarrow 0^+$ ,  $s_n \in [0, 1]$ ,  $x_n \in \partial B_R(0)$  and

$w_n \in Sx_n$  such that

$$s_n(T_{t_n}x_n + w_n + \varepsilon Jx_n) + (1 - s_n)Jx_n = 0 \quad (3.5.26)$$

for all  $n$ . If  $s_n = 0$  for some  $n$ , then  $x_n = 0$ . However, this is a contradiction. Since  $s_n \neq 0$  for all  $n$ , we get

$$T_{t_n}x_n + w_n + \varepsilon Jx_n + \frac{1 - s_n}{s_n}Jx_n = 0$$

for all  $n$ . Assume that  $s_n \rightarrow s_0$  as  $n \rightarrow \infty$ .

**Case I.** Suppose  $s_0 = 0$ . Then

$$\langle T_{t_n}x_n + w_n + \varepsilon Jx_n, x_n \rangle = \left(1 - \frac{1}{s_n}\right)R^2 \rightarrow -\infty.$$

Thus,

$$\langle T_{t_n}x_n + w_n + \varepsilon Jx_n, x_n \rangle < 0$$

for all large  $n$ . Using the monotonicity of  $J$ , we get

$$\langle T_{t_n}x_n + w_n, x_n \rangle < 0$$

for all large  $n$ , i.e. a contradiction to our assumption.

**Case II.** Let  $s_0 \in (0, 1]$ . Using the monotonicity of  $J$ , we see that

$$\langle T_{t_n}x_n + w_n, x_n \rangle \leq 0$$

for all  $n$ , which is a contradiction. Thus, our claim follows. As a result, for each  $\varepsilon > 0$ ,



there exists  $t_0 = t_0(\varepsilon) > 0$  such that

$$d(T_t + S + \varepsilon J, B_R(0), 0) = d(J, B_R(0), 0) = 1$$

for all  $t \in (0, t_0]$ . Therefore, using the definition of our degree, we let  $t \downarrow 0^+$  and  $\varepsilon \downarrow 0^+$  to get  $d(T + S, B_R(0), 0) = 1$ , which shows that  $Tx + Sx \ni 0$  is solvable in  $D(T) \cap B_R(0)$ . This completes the proof. ■

We now apply our degree theory to prove the following existence result of Kenmochi [44, Theorem 5.1, p. 236].

**Theorem 3.5.5** *Let  $T : X \supset D(T) \rightarrow 2^{X^*}$  be maximal monotone and  $S : X \rightarrow 2^{X^*}$  bounded pseudomonotone. Suppose that there exists  $v_0 \in D(T)$  such that*

$$\inf_{w \in Sx} \frac{\langle w, x - v_0 \rangle}{\|x\|} \rightarrow \infty$$

as  $\|x\| \rightarrow \infty$ . Then  $R(T + S) = X^*$ .

*Proof.* Let  $t > 0$ ,  $\tilde{J}x = J(x - v_0)$ . Since  $S - f$  is also bounded and pseudomonotone and satisfies the same coercivity condition, it suffices to assume  $f = 0$ . For each  $\varepsilon > 0$  and each  $t > 0$ , we use  $\|T_t v_0\| \leq |T v_0|$  to get

$$\begin{aligned} \inf_{w \in Sx} \frac{\langle T_t x + w + \varepsilon \tilde{J}x, x - v_0 \rangle}{\|x\|} &\geq \frac{\langle T_t v_0, x - v_0 \rangle}{\|x\|} + \inf_{w \in Sx} \frac{\langle w, x - v_0 \rangle}{\|x\|} \\ &\quad + \frac{\varepsilon \|x - v_0\|^2}{\|v_0\| + \|x - v_0\|} \rightarrow +\infty \end{aligned}$$

as  $\|x\| \rightarrow +\infty$ . Thus, for each  $k > 0$ , there exists  $R > 0$  (independent of  $\varepsilon > 0$ ,  $t > 0$ ) such that

$$\langle T_t x + w + \varepsilon \tilde{J}x, x - v_0 \rangle \geq k \tag{3.5.27}$$

for all  $x \in X \setminus B_R(0)$  and  $w \in Sx$ . One can choose w.l.o.g.  $R > \|v_0\|$ . Therefore, using (3.5.27) and the fact that  $\tilde{J}$  is continuous, monotone and of type  $(S_+)$ , the degrees

$d(T_t + S + \varepsilon\tilde{J}, B_R(0), 0)$  and  $d(J, B_R(0), 0)$  are well-defined. For each  $t > 0$  and  $\varepsilon > 0$ , we now apply (iii) of Theorem 3.2.3, where  $T = 0$ ,  $S_1 = T_t + S + \varepsilon\tilde{J}$  and  $S_2 = \alpha J$ , to conclude that

$$d(T_t + S + \varepsilon\tilde{J}, B_R(0), 0) = d(\alpha J, B_R(0), 0) = 1$$

for all  $t > 0$  and  $\varepsilon > 0$ . Therefore, for each  $\varepsilon_n \downarrow 0^+$  and  $t_n \downarrow 0^+$ , there exists  $x_n \in B_R(0)$  and  $w_n \in Sx_n$  such that

$$T_{t_n}x_n + w_n + \varepsilon_n\tilde{J}x_n = 0 \tag{3.5.28}$$

for all  $n$ . Using similar steps to those in the proof of (ii) of Theorem 3.1.4, we find that there exists  $x_0 \in D(T) \cap B_R(0)$  such that  $0 \in Tx_0 + Sx_0$ . ■

Using the homotopy invariance result of our degree theory, in particular (iii) of Theorem 3.2.2, we give a different proof of the maximality of the sum of two maximal monotone mappings from the one of Browder and Hess [18, p. 284, Theorem 9].

**Theorem 3.5.6** *Let  $T : X \supset D(T) \rightarrow 2^{X^*}$  and  $T_0 : X \supset D(T_0) \rightarrow 2^{X^*}$  be maximal monotone and such that  $0 \in D(T) \cap D(T_0)$ . If  $T_0$  is quasibounded, then  $T + T_0$  is maximal monotone.*

*Proof.* We assume w.l.o.g. that  $[0, 0] \in G(T) \cap G(T_0)$ . Otherwise, we pick  $v \in T(0)$  and  $w \in T_0(0)$  and consider instead the operators  $\tilde{T}x = Tx - v$  and  $\tilde{T}_0x = T_0x - w$ . Fix  $f \in X^*$ . We need to prove first that the operator  $T_0 + T_t + J - f$  is surjective for every  $t > 0$ . Since  $0 \in (T(0) \cap (T_0(0)))$ , it is easy to see that

$$\langle s(v_0 + T_t x + Jx - f) + (1 - s)Jx, x \rangle \geq \|x\|^2 - \|x\|\|f\|$$

for all  $x \in D(T_0)$ ,  $s \in [0, 1]$ ,  $v_0 \in T_0x$ . It follows that there exists  $R = R(f) > 0$  such that

$$\langle s(v_0 + T_t x + Jx - f) + (1 - s)Jx, x \rangle > 0 \tag{3.5.29}$$

for all  $x \in D(T_0) \cap \partial B_R(0)$ , all  $s \in [0, 1]$  and  $v_0 \in T_0x$ . For each  $t > 0$ , we consider the homotopy

$$H_t(s, x) = s(T_0 + T_t + J - f)x + (1 - s)Jx, \quad (s, x) \in [0, 1] \times (D(T_0) \cap \overline{B_R(0)}).$$

Using (3.5.29), we see that for each  $t > 0$  and all  $s \in [0, 1]$  we have  $0 \notin H_t(s, D(T_0) \cap \partial B_R(0))$ . For each  $t > 0$ , using the maximality of  $T_0 - f$ , the boundedness and property  $(S_+)$  of  $T_t + J$  and the  $\Gamma_\phi$ -property of  $J$ , and applying (iii) of Theorem 3.2.2, we get

$$d(T_0 + T_t + J - f, B_{R_f}(0), 0) = d(J, B_R(0), 0) = 1.$$

Therefore, for each  $t_n \downarrow 0^+$ , there exists  $x_n \in D(T_0) \cap B_{R_f}(0)$  and  $v_n \in T_0x_n$  such that

$$v_n + T_{t_n}x_n + Jx_n = f \tag{3.5.30}$$

for all  $n$ . The boundedness of  $\{x_n\}$  and the quasiboundedness of  $T_0$  imply that  $\{v_n\}$  is bounded. Assume w.l.o.g. that  $x_n \rightharpoonup x_0$ ,  $v_n \rightharpoonup v_0$  and  $T_{t_n}x_n \rightharpoonup w_0$  as  $n \rightarrow \infty$ . Since the sequences  $\{v_n\}$  and  $\{T_{t_n}x_n\}$  are bounded and  $T_0$  and  $T$  are maximal monotone, it is easy to show that

$$\liminf_{n \rightarrow \infty} \langle v_n, x_n - x_0 \rangle \geq 0 \text{ and } \liminf_{n \rightarrow \infty} \langle T_{t_n}x_n, x_n - x_0 \rangle \geq 0. \tag{3.5.31}$$

Using (3.5.30) and (3.5.31), we obtain

$$\limsup_{n \rightarrow \infty} \langle Jx_n, x_n - x_0 \rangle \leq - \lim_{n \rightarrow \infty} \langle f, x_n - x_0 \rangle = 0. \tag{3.5.32}$$

Since  $J$  is continuous and of type  $(S_+)$ , (3.5.32) implies  $x_n \rightarrow x_0$  and  $Jx_n \rightarrow Jx_0$  as  $n \rightarrow \infty$ . Finally, using (3.5.30) and (3.5.31), we conclude that

$$\limsup_{n \rightarrow \infty} \langle v_n, x_n - x_0 \rangle \leq 0 \text{ and } \limsup_{n \rightarrow \infty} \langle T_{t_n}x_n, J_{t_n}x_n - x_0 \rangle \leq 0,$$

where  $T_{t_n}x_n \in T(J_{t_n}x_n)$  and  $J_{t_n}x_n = x_n - t_n J^{-1}(T_{t_n}x_n) \rightarrow x_0$  as  $n \rightarrow \infty$ . Applying Lemma 1.2.4, we conclude that  $x_0 \in D(T_0) \cap D(T)$ ,  $v_0 \in T_0x_0$  and  $w_0 \in Tx_0$  with  $v_0 + w_0 + Jx_0 = f$ , i.e.  $f \in R(T_0 + T + J)$ . Since  $f \in X^*$  is arbitrary, the proof is complete. ■

As a consequence of Theorem 3.5.6, we obtain the following result of Rockaffellar [64]. We use the symbol  $\overset{\circ}{A}$  to denote the interior of the set  $A$ .

**Corollary 3.5.7** *Let  $T : X \supset D(T) \rightarrow 2^{X^*}$  and  $T_0 : X \supset D(T_0) \rightarrow 2^{X^*}$  be maximal monotone and such that  $0 \in D(T) \cap D(T_0)$ . Then  $T + T_0$  is maximal monotone.*

*Proof.* Since  $0 \in D(T)$ , it follows from Browder and Hess[18, Proposition 14, p. 284] that  $T$  is strongly quasibounded. Thus, using Theorem 3.5.6, we conclude that  $T + T_0$  is maximal monotone. ■

Browder [13, Theorem 7.8, p.92] gave the main range result on bounded pseudomonotone perturbations  $S$  (with  $D(S) = C$ ,  $C$  closed convex subset of  $X$ ) of maximal monotone operator  $T$  with  $D(T) = C$ . In Theorem 3.5.10, we use our degree theory for multi-valued (possibly unbounded) pseudomonotone perturbation of general maximal monotone operator  $T$  (with  $D(T)$  not necessarily closed and convex) to demonstrate that the same result holds true. To this end, we first prove Theorem 3.5.8 for single multi-valued pseudomonotone operator with effective domain all of  $X$ .

**Theorem 3.5.8** *Let  $G$  be a bounded open and convex subset of  $X$  with  $0 \in G$ . Let  $S : X \rightarrow 2^{X^*}$  be pseudomonotone. Let  $f \in X^*$ . Suppose that*

$$\langle w - f, x \rangle > 0$$

*for all  $x \in \partial G$ ,  $w \in Sx$ . Then the inclusion  $Sx \ni f$  is solvable in  $G$ .*

*Proof.* We observe that  $S + \lambda J$  is of type  $(S_+)$ . Since  $0 \notin (S + \lambda J - f)(\partial G)$  for all  $\lambda > 0$ , the degree  $d(S + \lambda J, G, f)$  is well-defined. We consider the following two cases.

**Case I.**  $f \in \overline{S(\partial G)}$ . Then there exist  $x_n \in \partial G$  and  $w_n \in Sx_n$  such that  $w_n \rightarrow f$  as  $n \rightarrow \infty$ . Assume w.l.o.g. that  $x_n \rightarrow x_0$  as  $n \rightarrow \infty$ . Since  $w_n \rightarrow f$  as  $n \rightarrow \infty$ , we have

$$\lim_{n \rightarrow \infty} \langle w_n, x_n - x_0 \rangle = 0.$$

Since  $\overline{G}$  is closed and convex, it is weakly closed and hence  $x_0 \in \overline{G}$ . We know that  $S$  is generalized pseudomonotone. Hence,  $f \in Sx_0$ . Thus,  $f \in Sx$  is solvable in  $\overline{G}$ .

**Case II.**  $f \notin \overline{S(\partial G)}$ . Let  $\lambda_i > 0$ ,  $i = 1, 2$ , be such that  $\lambda_1 < \lambda_2$  and consider the homotopy

$$H(\lambda, x) = Sx + \lambda Jx - f, \quad (\lambda, x) \in [\lambda_1, \lambda_2] \times \overline{G}.$$

As in the proof of Theorem 3.4.1, we see that  $\{H(\lambda, \cdot)\}_{\lambda \in [\lambda_1, \lambda_2]}$  is a homotopy of type  $(S_+)$  such that  $0 \notin H(\lambda, \partial G)$  for all  $\lambda \in [\lambda_1, \lambda_2]$ . Thus, by the invariance of the degree under homotopies of type  $(S_+)$ , we obtain

$$d(S + \lambda_1 J - f, G, 0) = d(S + \lambda_2 J - f, G, 0).$$

Therefore, we conclude that  $d(S + \lambda J - f, G, 0)$  is constant for all  $\lambda > 0$ .

Now, fix  $\lambda_0 > 0$  and let  $\Lambda$  denote the set of all finite dimensional subspaces of  $X$ . Let  $F \in \Lambda$ ,  $j_F : F \rightarrow X$  be the inclusion map and  $j_F^* : X^* \rightarrow F^*$  the adjoint of  $j_F$ . Let  $\tilde{S}x := Sx + \lambda_0 Jx$ ,  $x \in X$  and  $\tilde{S}_F x := j_F^* \circ (Sx + \lambda_0 Jx)$ ,  $x \in \overline{G} \cap F$ . Then there exists  $F_0 \in \Lambda$  such that

$$d(j_F^* \circ (S + \lambda_0 J), G \cap F, j_F^* f)$$

is well-defined and constant for all  $F \supset F_0$ . This common value equals  $d(S + \lambda_0 J, G, f)$ .

In particular, we have

$$d(S, G, f) = d(S + \lambda_0 J, G, f) = d(j_{F_0}^* \circ (S + \lambda_0 J), G \cap F_0, j_{F_0}^* f).$$

Next, we consider the homotopy

$$H_1(t, x) := j_{F_0}^* \circ (t(Sx + \lambda_0 Jx - f) + (1 - t)Jx), \quad (t, x) \in [0, 1] \times \overline{G} \cap F_0.$$

It is easy to see that  $\partial(G \cap F_0) \subset \partial G$ . We claim that  $0 \notin H(t, \partial(G \cap F_0))$  for all  $t \in [0, 1]$ . Otherwise, there exist  $t_1 \in [0, 1]$ ,  $x_1 \in \partial(G \cap F_0)$  and  $w_1 \in Sx_1$  such that  $H_1(t_1, x_1) = 0$ . It follows that

$$j_{F_0}^* (t_1(w_1 + \lambda Jx_1 - f) + (1 - t_1)Jx_1) = 0.$$

Let  $x^* = t_1(w_1 + \lambda Jx_1 - f) + (1 - t_1)Jx_1$ . Using the properties of  $j_{F_0}^*$ , we have

$$0 = \langle j_{F_0}^*(x^*), x \rangle = \langle x^*, j_{F_0} x \rangle = \langle x^*, x \rangle, \quad x \in F_0.$$

In particular, taking  $x = x_1$ , we get

$$\langle t_1(w_1 + \lambda Jx_1 - f) + (1 - t_1)Jx_1, x_1 \rangle = 0.$$

If  $t_1 = 0$ , then  $x_1 = 0$ , which is impossible as  $0 \in G \cap F_0$ . Assume that  $t_1 \in (0, 1]$ .

Using the monotonicity of  $J$ , we obtain

$$\langle w_1 - f, x_1 \rangle \leq -\left(\lambda + \frac{1 - t_1}{t_1}\right) \langle Jx_1, x_1 \rangle \leq 0.$$

But this is a contradiction with the hypothesis. Therefore, the claim holds true. Let  $\overline{G}_{F_0} = \overline{G} \cap F_0$ . We know that, for each  $t \in [0, 1]$ ,  $H_1(t, \cdot) : \overline{G}_{F_0} \rightarrow 2^{F_0^*}$  is upper semicontinuous with compact convex values in  $F_0^*$  such that  $0 \notin H_1(t, \partial(G \cap F_0))$

for all  $t \in [0, 1]$ . By the homotopy invariance of the degree for upper semicontinuous mappings with compact convex values in finite-dimensional spaces (Ma [54]) and our definition of the degree  $d(S - f, G, 0)$ , we have

$$d(S - f, G, 0) = d(S + \lambda_0 J - f, G, 0) = d(j_{F_0}^* J, G \cap F_0, 0) = 1.$$

As a result, an application of Theorem 3.4.3 implies  $f \in Sx_0$  for some  $x_0 \in G$ . ■

We remark that

- (A) the conclusion of Theorem 3.5.8 holds if the inner product condition is replaced by the condition

$$\langle w - f, x - v_0 \rangle > 0$$

for all  $x \in \partial G$ ,  $w \in Sx$  and  $v_0 \in G$ . The proof follows as in the proof of Theorem 3.5.2.

- (B) If  $S$  is locally monotone on  $G$ , it is proved in Theorem 3.6.1 that the inner product condition in the hypothesis of Theorem 3.5.8 and the Leray-Schauder condition (i.e  $Sx + \lambda Jx \not\preceq f$  for all  $\lambda > 0$ ) are equivalent.

The following surjectivity result is due to Browder and Hess [18, Theorem 3, p. 269].

**Corollary 3.5.9** *Let  $S : X \rightarrow 2^{X^*}$  be pseudomonotone and coercive. Then  $R(S) = X^*$ .*

*Proof.* Let  $\varepsilon > 0$  be arbitrary. Since  $S$  is coercive, it is easy to see that for each  $f \in X^*$ , there exists  $R = R(f) > 0$  (independent of  $\varepsilon > 0$ ) such that

$$\langle w + \varepsilon Jx - f, x \rangle > 0$$

for all  $x \in \partial B_R(0)$ ,  $w \in Sx$ . This implies  $Sx + \varepsilon Jx - f \not\equiv 0$  for all  $\varepsilon > 0$  and  $x \in \partial B_R(0)$ . Using Theorem 3.5.8, we obtain  $f \in R(S)$ . Since  $f \in X^*$  is arbitrary, we conclude that  $R(S) = X^*$ . ■

In the following Theorem we assume that either  $T$  is strongly quasibounded or  $S$  is quasibounded.

**Theorem 3.5.10** *Let  $T : X \supset D(T) \rightarrow 2^{X^*}$  be maximal monotone with  $0 \in T(0)$  and  $S : X \rightarrow 2^{X^*}$  pseudomonotone. Let  $f \in X^*$ . Suppose that either  $S$  is quasibounded or  $T$  is strongly quasibounded and there exists  $R > 0$  such that*

$$\langle w - f, x \rangle > 0$$

for all  $x \in \partial B_R(0)$  and  $w \in Sx$ . Then  $f \in R(T + S)$ .

*Proof.* Since  $0 \in T(0)$ , we have  $T_t(0) = 0$  for all  $t > 0$ . Hence, we have

$$\langle T_t x + w - f, x \rangle > 0$$

for all  $x \in \partial B_R(0)$ ,  $w \in Sx$ , i.e. for each  $t > 0$  we have  $0 \notin (T_t + S - f)(\partial B_R(0))$ . Furthermore, we observe that, for each  $t > 0$ ,  $T_t + S - f$  is pseudomonotone with effective domain all of  $X$ . Using Theorem 3.5.8, we find that for each  $t_n \downarrow 0^+$  there exist  $x_n \in B_R(0)$  and  $w_n \in Sx_n$  such that

$$T_{t_n} x_n + w_n = f \tag{3.5.33}$$

for all  $n$ . Since  $0 \in T(0)$ , we have  $T_{t_n}(0) = 0$  for all  $n$  and

$$\langle w_n, x_n \rangle \leq \|f\| \|x_n\|$$

for all  $n$ . From the boundedness of  $\{x_n\}$  and the quasiboundedness of  $S$  it follows that



$\{w_n\}$  is bounded. Furthermore, we see that  $\{T_{t_n}x_n\}$  is bounded. We assume w.l.o.g. that  $x_n \rightharpoonup x_0$ ,  $w_n \rightharpoonup w_0$  and  $T_{t_n}x_n \rightharpoonup v_0$  as  $n \rightarrow \infty$ . Since  $S$  is pseudomonotone,

$$\liminf_{n \rightarrow \infty} \langle w_n, x_n - x_0 \rangle \geq 0.$$

Thus, from (3.5.33) we get

$$\limsup_{n \rightarrow \infty} \langle T_{t_n}x_n, x_n - x_0 \rangle \leq 0.$$

We know that  $J_{t_n}x_n \in D(T)$  and  $T_{t_n}x_n \in T(J_{t_n}x_n)$  for all  $n$ . Since  $J_{t_n}x_n - x_n = t_n J^{-1}(T_{t_n}x_n) \rightarrow 0$  as  $n \rightarrow \infty$ , we have  $J_{t_n}x_n \rightharpoonup x_0$  and

$$\limsup_{n \rightarrow \infty} \langle T_{t_n}x_n - v_0, J_{t_n}x_n - x_0 \rangle \leq 0.$$

Therefore, using Lemma 1.2.5, we see that  $x_0 \in D(T)$ ,  $v_0 \in Tx_0$  and  $\langle T_{t_n}x_n, J_{t_n}x_n \rangle \rightarrow \langle v_0, x_0 \rangle$  as  $n \rightarrow \infty$ . Thus, from (3.5.33), we get

$$\lim_{n \rightarrow \infty} \langle w_n, x_n - x_0 \rangle = 0.$$

Since  $S$  is pseudomonotone, it follows that  $S$  is generalized pseudomonotone, which implies  $w_0 \in Sx_0$  and  $\langle w_n, x_n \rangle \rightarrow \langle w_0, x_0 \rangle$  as  $n \rightarrow \infty$ . Finally, taking limits as  $n \rightarrow \infty$  in (3.5.33), we obtain  $v_0 + w_0 = f$ , where  $x_0 \in D(T)$ ,  $v_0 \in Tx_0$  and  $w_0 \in Sx_0$ . This implies that  $f \in R(T + S)$ . If  $T$  is strongly quasibounded, the proof follows with a similar argument using Lemma 3.5.4. The details are omitted. ■

The following Corollary (which can be found in Naniewicz and Panagiotopoulos [61, Theorem 2.11, p. 51]), follows as an application of Theorem 3.5.10.

**Corollary 3.5.11** *Let  $T : X \supset D(T) \rightarrow 2^{X^*}$  be maximal monotone with  $0 \in T(0)$  and  $S : X \rightarrow 2^{X^*}$  pseudomonotone and coercive. Suppose that either  $S$  is quasibounded or*

$T$  is strongly quasibounded. Then  $R(T + S) = X^*$ .

*Proof.* Since  $S$  is coercive, there exists a function  $\alpha : [0, \infty) \rightarrow (-\infty, \infty)$  such that  $\alpha(t) \rightarrow \infty$  as  $t \rightarrow \infty$  and  $\langle w, x \rangle \geq \alpha(\|x\|)\|x\|$  for all  $x \in X$  and  $w \in Sx$ . This implies that for each  $f \in X^*$  there exists  $R = R(f) > 0$  such that  $\langle w - f, x \rangle > 0$  for all  $x \in \partial B_R(0)$  and  $w \in Sx$ . Thus, by Theorem 3.5.10, we have  $f \in R(T + S)$ . Since  $f$  is arbitrary, it follows that  $R(T + S) = X^*$ . ■

We remark that Theorem 3.5.10 is a new existence theorem which improves a result of Naniewicz and Panagiotopoulos [61, Theorem 2.11, p. 51].

### 3.6 Necessary and sufficient conditions for existence of zeros

In what follows,  $G$  is an open and bounded subset of  $X$ . An operator  $T : X \supset D(T) \rightarrow 2^{X^*}$  is called “locally monotone” on  $G$  if for every  $x_0 \in D(T) \cap G$  there exists a ball  $\overline{B_r(x_0)} \subset G$  such that  $T$  is monotone on  $D(T) \cap \overline{B_r(x_0)}$ .

**Theorem 3.6.1** *Let  $T : X \supset D(T) \rightarrow 2^{X^*}$  be maximal monotone with  $D(T) \neq \emptyset$  and  $S : X \rightarrow 2^{X^*}$  bounded pseudomonotone. Assume that  $G \subset X$  is open and bounded. Then, for the three statements below, we have the following implications: (ii)  $\Rightarrow$  (iii)  $\Rightarrow$  (i). If, in addition, the operator  $T + S$  is locally monotone on  $G$ , then the conditions (i) – (iii) are mutually equivalent.*

$$(i) \quad 0 \in (T + S)(D(T) \cap G).$$

$$(ii) \quad \text{There exist } r > 0 \text{ and } x_0 \in D(T) \cap G \text{ such that } \overline{B_r(x_0)} \subset D(T) \cap G \text{ and}$$

$$\langle u^* + v^*, x - x_0 \rangle \geq 0$$

for every  $(x, u^*, v^*) \in \partial B_r(x_0) \times Tx \times Sx$ .

(iii) There exist  $r > 0$  and  $x_0 \in D(T) \cap G$  such that  $\overline{B}_r(x_0) \subset D(T) \cap G$  and

$$(T + S)x \not\preceq \varepsilon J(x - x_0)$$

for every  $(\varepsilon, x) \in (-\infty, 0) \times \partial B_r(x_0)$ .

*Proof.* To show that (ii)  $\Rightarrow$  (iii), assume that (ii) holds and let  $(T + S)x \ni \varepsilon J(x - x_0)$ , for some  $(\varepsilon, x) \in (-\infty, 0) \times \partial B_r(x_0)$ . Then, for some  $u^* \in Tx$ ,  $v^* \in Sx$ ,

$$0 \leq \langle u^* + v^*, x - x_0 \rangle = \varepsilon \langle J(x - x_0), x - x_0 \rangle = \varepsilon \|x - x_0\|^2 < 0.$$

However, this is impossible.

Assume that (iii) holds. We know that

$$Tx + Sx + \varepsilon J(x - x_0) \not\preceq 0, \quad (\varepsilon, x) \in (0, \infty) \times \partial B_r(x_0). \quad (3.6.34)$$

We may also assume that  $x_0 = 0$  and  $0 \in T(0)$ . Otherwise, we consider instead the operators  $\tilde{T}$ ,  $\tilde{S}$  with  $D(\tilde{T}) = D(T) - x_0$ ,  $\tilde{T}(x) = T(x + x_0) - w_0$ , and  $\tilde{S}x = S(x + x_0) + w_0$ , where  $w_0 \in Tx_0$ . We also consider instead the set  $\tilde{G} = G - x_0$  and the ball  $B_r(0) \subset \tilde{G}$ . The operator  $\tilde{T}$  is obviously maximal monotone with effective domain  $\tilde{D}(T)$  and (3.6.34) is replaced by

$$\tilde{T}x + \tilde{S}x + \varepsilon Jx \not\preceq 0, \quad (\varepsilon, x) \in (0, \infty) \times \partial B_r(0).$$

We must also show that the operator  $\tilde{S} : X \rightarrow 2^{X^*}$  is pseudomonotone. To this end, we first observe that  $\tilde{S}$  satisfies (i) and (ii) of Definition 1.2.6. To show (iii) in that definition, assume that  $\{x_n\} \subset X$ ,  $w_n = y_n + w_0 \in \tilde{S}x_n = S(x_n - x_0) + w_0$ ,  $y_n \in S(x_n - x_0)$  are such that  $x_n \rightarrow x_1 \in X$  and

$$\limsup_{n \rightarrow \infty} \langle w_n, x_n - x_1 \rangle \leq 0.$$

This is equivalent to saying that

$$\limsup_{n \rightarrow \infty} \langle y_n, (x_n - x_0) - (x_1 - x_0) \rangle \leq 0. \quad (3.6.35)$$

Letting  $u_n = x_n - x_0$  and  $u_1 = x_1 - x_0$ , we may rewrite (3.6.35) as

$$\limsup_{n \rightarrow \infty} \langle y_n, u_n - u_1 \rangle \leq 0,$$

where  $y_n \in Su_n$ . Using property (iii) of Definition 1.2.6 on the operator  $S$ , we see that for every  $y \in X$  there exists  $y_1^*(y) \in S(u_1)$  such that

$$\liminf_{n \rightarrow \infty} \langle y_n, u_n - y \rangle \geq \langle y_1^*(y), u_1 - y \rangle.$$

Letting  $y = x - x_0$  and  $y^*(x) = y_1^*(y) = y_1^*(x - x_0)$ , we have shown that for every  $x \in X$  there exists  $y^*(x) \in S(x - x_0)$  such that

$$\liminf_{n \rightarrow \infty} \langle y_n, x_n - x \rangle \geq \langle y^*(x), x_1 - x \rangle$$

or

$$\begin{aligned} \liminf_{n \rightarrow \infty} \langle y_n + w_0, x_n - x \rangle &\geq \liminf_{n \rightarrow \infty} \langle y_n, x_n - x \rangle + \liminf_{n \rightarrow \infty} \langle w_0, x_n - x \rangle \\ &\geq \langle y(x) + w_0, x_1 - x \rangle \end{aligned}$$

or

$$\liminf_{n \rightarrow \infty} \langle w_n, x_n - x \rangle \geq \langle \tilde{y}(x), x_1 - x \rangle,$$

where  $\tilde{y}^*(x) = y^*(x) + w_0$ . Thus,  $\tilde{S}$  is pseudomonotone. In view of the above and (3.6.34), it suffices to assume the validity of the relation

$$Tx + Sx + \varepsilon Jx \not\equiv 0, \quad (\varepsilon, x) \in (0, \infty) \times \partial B_r(0).$$

This implies that  $d(T + S, B_r(0), 0)$  is well-defined and, by the definition of the degree, we have

$$d(T + S, B_r(0), 0) = \lim_{\varepsilon \downarrow 0^+} d(T + S + \varepsilon J, B_r(0), 0).$$

Consider now the homotopy

$$H(s, t, x) := s(T_t x + Sx + \varepsilon Jx) + (1 - s)Jx, \quad (s, t, x) \in [0, 1] \times (0, \infty) \times \overline{B}_r(x_0).$$

Following the steps of Case I and Case II of the proof of Theorem 3.5.1, we see that  $0 \in (T + S)(D(T) \cap G)$ . This proves (i).

Next we show that (i)  $\Rightarrow$  (ii) under the assumption that  $T + S$  is locally monotone on  $G$ . To this end, assume (i) holds and choose  $x_0 \in D(T) \cap G$  such that  $0 \in (T + S)x_0$ . Since  $T + S$  is locally monotone on  $G$ , there exists a ball  $\overline{B}_r(x_0) \subset D(T) \cap G$  such that  $T + S$  is monotone on  $\overline{B}_r(x_0)$ , which in turn implies

$$\langle u^* + v^*, x - x_0 \rangle \geq 0$$

for every  $(x, u^*, v^*) \in \partial B_r(0) \times Tx \times Sx$ . This shows that (ii) holds and completes the proof. ■

### 3.7 Generalization for quasimonotone perturbations

The reader may find several examples of pseudomonotone operators in the paper of Kenmochi [43]. The same paper contains several references of work on related

problems of variational inequalities. It is well known that the solvability of certain variational inequalities is equivalent to the solvability of certain associated inclusions involving subdifferentials. Further work and bibliography on pseudomonotone perturbations of maximal monotone operators for the existence of solutions of variational inequalities may be found in the more recent paper of Kenmochi in [44].

It should be noted that the degree theory developed herein can be used in other problems of the field of nonlinear analysis. For example, such a degree theory can be used for problems involving eigenvalues, ranges of sums, invariance of domain, multiplicity of solutions, etc.

In order to discuss the situation of possible extensions to more general operators  $S$ , we need first the following definition of a quasimonotone operator.

**Definition 3.7.1** Let  $G$  be a bounded open subset of  $X$ .

(i) An operator  $S : \overline{G} \rightarrow 2^{X^*}$  is said to be “quasimonotone“ if the following statements are true.

(i) For each  $x \in X$ ,  $Sx$  is a closed, convex and bounded subset of  $X^*$ .

(ii)  $S$  is “weakly upper semicontinuous”.

(iii) For every sequence  $\{x_n\} \subset \overline{G}$  with  $x_n \rightarrow x_0 \in X$  and every sequence  $\{w_n^*\}$  with  $w_n^* \in Sx_n$  we have

$$\limsup_{n \rightarrow \infty} \langle w_n^*, x_n - x_0 \rangle \geq 0.$$

Using this definition, if  $S : \overline{G} \rightarrow 2^{X^*}$  is quasimonotone it is easy to see that for each  $\varepsilon > 0$  the mapping  $S + \varepsilon J$  is of type  $(S_+)$ . In separable reflexive Banach spaces, Berkovits and Mustonen [8] (cf. also Berkovits [6]) developed a degree theory for  $(S_+)$ -mappings using the theory of elliptic-super regularization. Furthermore, Berkovits

gave in [6] a generalization of this degree for single single-valued demicontinuous quasimonotone operators  $S$  defined on  $\overline{G}$ .

We would like to mention here that the following two generalizations of the degree theories developed in this paper are possible.

- (I) Let  $G$  be a bounded open subset of  $X$  and let  $S : \overline{G} \rightarrow 2^{X^*}$  be quasimonotone and such that  $0 \notin \overline{S(\partial G)}$ . Using our previous approach, it is not hard to show that  $d(S + \varepsilon J, G, 0)$  is well-defined and constant for all sufficiently small  $\varepsilon > 0$ . Consequently, a degree mapping  $d(S, G, 0)$  may be defined via

$$d(S, G, 0) = \lim_{\varepsilon \downarrow 0^+} d(S + \varepsilon J, G, 0),$$

where  $d(S + \varepsilon J, G, 0)$  is the degree for the single multi-valued  $(S_+)$  mapping  $S + \varepsilon J$ . The basic properties of this degree hold exactly as before, where in the case  $d(S, G, 0) \neq 0$ , we have to conclude now that  $0 \in \overline{S(G)}$  even though the set  $G$  is convex. When  $G$  is convex, the set  $\overline{G}$  is bounded and weakly closed. If  $S$  is pseudomonotone,  $S(\overline{G})$  is closed in  $X^*$ . However, if  $S$  is just quasimonotone, we don't have, in general, the closedness of this set in  $X^*$ .

- (II) Let  $T : X \supseteq D(T) \rightarrow 2^{X^*}$  be maximal monotone and  $S : \overline{G} \rightarrow 2^{X^*}$  bounded, quasimonotone and such that  $0 \notin \overline{(T + S)(D(T) \cap \partial G)}$ . Using a similar methodology, as that of our construction of the degree for pseudomonotone perturbations, we can show that the degree mapping  $d(T + S + \varepsilon J, G, 0)$  is well-defined and constant for all sufficiently small  $\varepsilon > 0$ . This allows us to introduce a degree mapping  $d(T + S, G, 0)$ , which is defined by

$$d(T + S, G, 0) = \lim_{\varepsilon \downarrow 0^+} d(T + S + \varepsilon J, G, 0),$$

where  $d(T + S + \varepsilon J, G, 0)$  is the degree for multi-valued  $(S_+)$  perturbations of the operator  $T$ . It can be immediately seen that the basic properties are

satisfied, where in the case  $d(T + S, G, 0) \neq 0$ , we now conclude that  $0 \in \overline{(T + S)(D(T) \cap G)}$ .

For both these degree theories, relevant homotopy invariance and uniqueness results as well as the associated mapping theorems can be developed. We omit the details. This generalization extends the degree theory developed by Berkovits in [6].

The most general monotone-type mappings are the so-called operators “of type  $M$ ”. The following definition of a multi-valued operator of type  $M$  is taken from Kenmochi [42].

**Definition 3.7.2** An operator  $S : X \rightarrow 2^{X^*}$  is said to be “of type  $M$ ” if the following conditions are satisfied.

- (i) For each  $x \in X$ ,  $Sx$  is a closed, convex and bounded subset of  $X^*$ .
- (ii)  $S$  is “weakly upper semicontinuous” on each finite-dimensional subspace  $F$  of  $X$ .
- (iii) For every sequence  $\{x_n\} \subset X$  and every sequence  $\{y_n^*\}$  with  $y_n^* \in Sx_n$  such that  $x_n \rightharpoonup x_0 \in X$  and  $y_n^* \rightharpoonup y_0 \in X^*$  and

$$\limsup_{n \rightarrow \infty} \langle y_n^*, x_n - x_0 \rangle \leq 0,$$

we have  $y_0 \in Sx_0$ .

It is well-known that the class of mappings of type  $M$  contains properly mappings of type  $(S_+)$  as well as pseudomonotone mappings with effective domain all of  $X$ . Basic properties and surjectivity results for single multi-valued operators of type  $M$  and their perturbations by linear maximal monotone operators may be found in Kenmochi [42]. On the other hand, a surjectivity result for perturbations of type  $M$  of maximal monotone operators with weakly closed graphs may be found in the book of



Pascali and Sburlan [63, pp. 151-156]. Results concerning single mappings of type  $M$  can be found in Kenmochi [42] and Berkovits [6]. More properties of these mappings and range results for the sum  $T + S$  where  $T$  is maximal monotone and  $S$  is of type  $M$  can be found in Pascali and Sburlan [63]. We would also like to mention here that if  $S : X \rightarrow 2^{X^*}$  is of type  $M$ , we do not have in general that for each  $\varepsilon > 0$  the mapping  $S + \varepsilon J$  is of type  $(S_+)$ . Therefore, unlike the case of quasimonotone mappings, the generalization of the previous degree theories to these class of mappings via the established methodology seems impossible.

For a multi-valued maximal operator  $T$ , we consider a subdifferential as follows.

Let  $K$  be a proper closed convex subset of  $X$  such that  $0 \in \overset{\circ}{K}$  and  $\varphi_K : X \rightarrow \mathbb{R}_+ \cup \{\infty\}$  be the indicator function on  $K$ , defined by

$$\varphi_K(x) = \begin{cases} \{0\} & \text{if } x \in K \\ \infty & \text{if } x \in X \setminus K. \end{cases}$$

The function  $\varphi_K$  is proper, convex and lower semicontinuous on  $X$ , and  $x^* \in \partial\varphi_K(x)$ , for  $x \in K$ , if and only if  $\langle x^*, y - x \rangle \leq 0$  for all  $y \in K$ . Also, it is well known that  $D(\partial\varphi_K) = K$ ,  $0 \in \partial\varphi_K(x)$ ,  $x \in K$  and  $\partial\varphi_K(x) = \{0\}$  for all  $x \in \overset{\circ}{K}$ . The operator  $\partial\varphi_K : X \rightarrow 2^{X^*}$  is maximal monotone with  $0 \in \overset{\circ}{D}(\partial\varphi_K)$  and  $0 \in \partial\varphi_K(0)$ . It is thus strongly quasibounded. For these facts see, e.g., Kenmochi [42]. If we add to  $\partial\varphi_K$  a nontrivial maximal monotone operator  $T_0 : X = D(T) \rightarrow 2^{X^*}$ ,  $0 \in T_0(0)$ , then we end up with an operator  $\tilde{T} = \partial\varphi_K + T_0$ , which is a nontrivial example of an operator  $T$  that may be covered by our present degree theory.

### 3.8 Applications to partial differential equations

In this Section, we demonstrate the applicability of the theory to show the existence of generalized(weak) solution(s) for nonmonotone perturbation of the nonlinear differ-

ential operator  $-div(\beta(\nabla u))$ , i.e. we consider the second order differential equation given by

$$\begin{cases} -div\beta(\nabla u(x)) - \sum_{i=1}^n \frac{\partial}{\partial x_i} a_i(x, u(x), \nabla u(x)) + a_0(x, u(x), \nabla u(x)) = f(x) & \text{on } \Omega \\ \beta(\nabla u(x)) = 0 & \text{on } \partial\Omega \end{cases}$$

where  $\Omega$  is a bounded open subset of  $\mathbb{R}^n$  with smooth boundary and  $\beta : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is continuous monotone and

- (i)  $|\beta(r)| \leq C_1(1 + |r|^{p-1})$  for all  $r \in \mathbb{R}^n$ ;
- (ii)  $\beta(r)r \geq d|r|^p - C_2$  for all  $r \in \mathbb{R}^n$ ,

where  $d > 0$ ,  $p > 1$  and  $\frac{2n}{n+2} \leq p$ . Assume, further, that

(H<sub>1</sub>) the functions  $a_0, a_1, a_2, \dots, a_n$  satisfy the Carathéodory conditions, i.e.  $(\eta, \zeta) \rightarrow a_i(x, \eta, \zeta)$  is continuous for almost all  $x \in \Omega$  and  $x \rightarrow a_i(x, \eta, \zeta)$  is measurable for all  $(\eta, \zeta) \in \mathbb{R} \times \mathbb{R}^n$ ;

(H<sub>2</sub>)  $\sum_{i=1}^n [a_i(x, \eta, \zeta) - a_i(x, \eta, \zeta^*)](\zeta_i - \zeta_i^*) \geq 0$  for all  $\eta \in \mathbb{R}$ ,  $\zeta, \zeta^* \in \mathbb{R}^n$ ,  $\zeta \neq \zeta^*$  and for almost all  $x \in \Omega$ ;

(H<sub>3</sub>) Suppose either (A) or (B) or (C) of the following conditions hold.

- (A) If  $1 < p < n$  and  $1 < q < \frac{np}{n-p}$ ,  $p'$  and  $q'$  are conjugate exponents of  $p$  and  $q$  respectively, then

$$|a_i(x, \eta, \zeta)| \leq c_i(|\eta|^{\frac{q}{p'}} + |\zeta|^{\frac{p}{q'}} + k_i(x)), i = 1, 2, \dots, n$$

and

$$|a_0(x, \eta, \zeta)| \leq c_0(|\eta|^{\frac{q}{q'}} + |\zeta|^{\frac{p}{q'}} + k_0(x))$$

for almost all  $x \in \Omega$ ,  $(\eta, \zeta) \in \mathbb{R}^{n+1}$ ,  $c_i (i = 0, 1, \dots, n)$  are positive constants,  $k_i (i = 1, 2, \dots, n)$  are in  $L^{p'}(\Omega)$  and  $k_0 \in L^{q'}(\Omega)$ .

(B) If  $p > n \geq 2$ , then

$$|a_i(x, \eta, \zeta)| \leq c_i(|\eta|)(|\zeta|^{\frac{p}{p'}} + k_i(x)), i = 1, 2, \dots, n$$

and

$$|a_0(x, \eta, \zeta)| \leq c_0(|\eta|)(|\zeta|^p + k_0(x))$$

for almost all  $x \in \Omega$ , all  $(\eta, \zeta) \in \mathbb{R}^{n+1}$ ,  $c_i (i = 0, 1, \dots, n)$  are nondecreasing continuous functions from  $[0, \infty)$  to  $[0, \infty)$ ,  $k_i (i = 1, 2, \dots, n)$  are in  $L^{p'}(\Omega)$  and  $k_0 \in L^1(\Omega)$ ;

(C) If  $p = n \geq 2$ , then

$$|a_i(x, \eta, \zeta)| \leq c_i(|\eta|^{\frac{q}{q'}} + |\zeta|^{\frac{p}{p'}} + k_i(x)), i=1,2,\dots,n$$

and

$$|a_0(x, \eta, \zeta)| \leq c_0(|\eta|^{\frac{q}{q'}} + |\zeta|^{\frac{p}{p'}} + k_0(x))$$

for almost all  $x \in \Omega$ , all  $(\eta, \zeta) \in \mathbb{R}^{n+1}$ ,  $c_i (i = 0, 1, \dots, n)$  are positive constants,  $1 < q < \infty$  arbitrary,  $k_i (i = 1, 2, \dots, n)$  are in  $L^{p'}(\Omega)$  and  $k_0 \in L^{q'}(\Omega)$ ;

**Theorem 3.8.1** *Suppose that the hypothesis  $(H_1)$ - $(H_2)$  and either  $(A)$  or  $(B)$  or  $(C)$  of  $(H_3)$ , and  $\beta$  satisfies conditions  $(i)$  and  $(ii)$ . Assume, further, that there exist  $k_i \in L^1(\Omega) (i = 1, 2)$  such that*

$$\sum_{i=1}^n a_i(x, \eta, \zeta) \zeta_i \geq -k_1(x) \text{ and } a_0(x, \eta, \zeta) \eta \geq -k_2(x)$$

*for all  $\eta \in \mathbb{R}$ ,  $\zeta \in \mathbb{R}^n$  and for almost all  $x \in \Omega$ . Then for each  $f \in L^{p'}(\Omega)$ , there exists*

$u \in W_0^{1,p}(\Omega)$  such that

$$\begin{aligned} \int_{\Omega} \beta(\nabla u(x)) \cdot \nabla \phi(x) dx + \sum_{i=1}^n \int_{\Omega} a_i(x, u(x), \nabla u(x)) \frac{\partial \phi(x)}{\partial x_i} dx \\ + \int_{\Omega} (a_0(x, u(x), \nabla u(x)) - f(x)) \phi(x) dx = 0 \end{aligned} \quad (3.8.36)$$

for all  $\phi \in W_0^{1,p}(\Omega)$ .

*Proof.* Consider the second order differential operator  $A$  given by

$$(Au)(x) = - \sum_{i=1}^N \frac{\partial}{\partial x_i} a_i(x, u(x), \nabla u(x)) + a_0(x, u(x), \nabla u(x)), \quad x \in \Omega. \quad (3.8.37)$$

The operator  $A$  gives rise to an operator  $S : W_0^{1,p}(\Omega) \rightarrow W^{-1,p'}(\Omega)$  defined via

$$\langle Su, \phi \rangle = \int_{\Omega} \sum_{i=1}^N a_i(x, u(x), \nabla u(x)) \frac{\partial \phi}{\partial x_i} + \int_{\Omega} a_0(x, u(x), \nabla u(x)) \phi(x),$$

for any  $u, v \in W_0^{1,p}(\Omega)$ . From Mustonen [59], under the conditions  $(H_1)$ - $(H_3)$ , we know that the operator  $S$  is bounded, continuous and pseudomonotone. We observe that  $S$  is not necessarily monotone. For further examples, the reader is referred to Mustonen and Tienari [60]. On the other hand, we consider the operator  $Bu = -\text{div}(\beta(\nabla u))$ ,  $u \in X$ . The operator  $B$  generates a well-defined operator  $T : X \rightarrow X^*$  given by

$$\langle Tu, \phi \rangle = \int_{\Omega} \beta(\nabla u(x)) \nabla \phi(x) dx$$

for all  $u, \phi \in X$ . It is well known that  $T$  is maximal monotone. For further details, we refer, the book of Barbu [4]. Next, we show that, for each  $f \in L^{p'}(\Omega)$  and  $\langle f^*, u \rangle = \int_{\Omega} f(x)u(x)dx$ , there exists  $R = R(f^*) > 0$  such that

$$\langle Tu + Su - f^*, u \rangle \geq 0$$

for all  $u \in D(T) \cap \partial B_R(0)$ . Since  $\beta(\zeta) \cdot \zeta \geq \alpha|\zeta|^p - C$ ,  $f \in L^{p'}(\Omega)$ ,  $k_i \in L^1(\Omega)$  for  $i = 1, 2$ , using the hypothesis of the theorem and applying Hölder's inequality, we obtain that

$$\langle Tu + Su - f^*, u \rangle \geq -\|f\|_{L^{p'}(\Omega)}\|u\|_{L^p(\Omega)} - \|k_1\|_{L^1(\Omega)} - \|k_2\|_{L^1(\Omega)} + \alpha\|\nabla u\|_{L^p(\Omega)}^p$$

for all  $u \in D(T)$ , where Furthermore, using Poincaré inequality, there exists  $C_1 > 0$  such that  $\|\nabla u\|_{L^p(\Omega)}^p \geq C_1\|u\|_{W^{1,p}(\Omega)}^p$ . Thus, combining these inequalities, we obtain that

$$\begin{aligned} \langle Tu + Su - f^*, u \rangle &\geq -\|f\|_{L^{p'}(\Omega)}\|u\|_{L^p(\Omega)} - \|k_1\|_{L^1(\Omega)} - \|k_2\|_{L^1(\Omega)} + \alpha C_1\|u\|_{W^{1,p}(\Omega)}^p \\ &\geq -\|f\|_{L^{p'}(\Omega)}\|u\|_{W^{1,p}(\Omega)} - \|k_1\|_{L^1(\Omega)} - \|k_2\|_{L^1(\Omega)} + \alpha C_1\|u\|_{W^{1,p}(\Omega)}^p \\ &= \|u\|_{W^{1,p}(\Omega)}^p \left[ \alpha C_1 - \frac{\|f\|_{L^{p'}(\Omega)}}{\|u\|_{W^{1,p}(\Omega)}^{p-1}} - \frac{\|k_1\|_{L^1(\Omega)} + \|k_2\|_{L^1(\Omega)}}{\|u\|_{W^{1,p}(\Omega)}^p} \right] \end{aligned}$$

for all  $u \in D(T)$ . Since  $p > 1$ , the right side of the above inequality tends to  $\infty$  as  $\|u\|_{W^{1,p}(\Omega)} \rightarrow \infty$ , i.e. there exists  $R = R(f^*) > 0$  such that

$$\langle Tu + Su - f^*, u \rangle > 0$$

for all  $u \in D(T) \cap \partial B_R(0)$ . Applying Corollary 3.2.4, we conclude that

$$d(T + S - f^*, B_R(0), 0) = 1.$$

This proves that the equation  $Tu + Su = f^*$  is solvable, i.e. the integral equation (3.8.36) is solvable. The proof is complete. ■

4 VARIATIONAL INEQUALITY THEORY FOR PERTURBATIONS OF MAXIMAL  
MONOTONE  
OPERATORS

### 4.1 Introduction and preliminaries

This Chapter is concerned with the solvability of variational inequalities for pseudomonotone perturbations of maximal monotone operators. As mentioned in detail in the general introduction of this dissertation, we are interested to study the solvability of the variational problem denoted by  $VIP(T + S, K, \phi, f^*)$ , where  $T : X \supseteq D(T) \rightarrow 2^{X^*}$  is maximal monotone,  $S : K \rightarrow 2^{X^*}$  is either pseudomonotone or generalized pseudomonotone or regular generalized pseudomonotone,  $K$  is possibly unbounded nonempty, closed and convex subset of  $X$  and  $\phi : X \rightarrow (-\infty, \infty]$  is “proper” ( $\phi$  is not identically  $+\infty$ ), convex and “lower semicontinuous” (i.e.  $\phi(x) \leq \liminf_{y \rightarrow x} \phi(y)$ ,  $x \in X$  or equivalently, for each  $\lambda > 0$  the level set  $\{x \in X : \phi(x) \leq \lambda\}$  is closed.) We recall that the indicator function  $I_K$  on  $K$  is given by

$$I_K(x) = \begin{cases} 0 & \text{if } x \in K \\ \infty & \text{if } x \in X \setminus K. \end{cases}$$

It is known that  $I_K$  is proper, convex and lower semicontinuous on  $X$ . The subdifferential of  $I_K$  at  $x \in X$  is defined by

$$\partial I_K(x) = \{x^* \in X^* : \langle x^*, x - y \rangle \geq 0, \text{ for every } y \in K\}.$$

Here,  $D(\partial I_K) = D(I_K) = K$  and  $\partial I_K(x) = \{0\}$  for every  $x \in \overset{\circ}{K}$ . Let  $\phi : X \supseteq D(\phi) \rightarrow (-\infty, \infty]$  be a proper, lower semicontinuous and convex function on  $X$  with  $D(\phi) = \{x \in X : \phi(x) < +\infty\}$ . For each  $x \in X$ , we denote by  $\partial\phi(x)$  the set

$$\partial\phi(x) = \{x^* \in X^* : \langle x^*, x - y \rangle \geq \phi(x) - \phi(y), \text{ for every } y \in X\}.$$

It is known that  $\partial\phi : X \supseteq D(\partial\phi) \rightarrow 2^{X^*}$  is maximal monotone and such that  $D(\partial\phi)$  is dense in  $D(\phi)$  and  $\overset{\circ}{D(\phi)} \subseteq \overset{\circ}{D(\partial\phi)}$ . Furthermore, we have  $\phi(x) = \min\{\phi(y) : y \in X\}$  if and only if  $0 \in \partial\phi(x)$ . Other relevant properties may be found in Barbu [5].

Fix  $f^* \in X^*$  and  $A : X \supseteq D(A) \rightarrow 2^{X^*}$ . We denote by  $VIP(A, K, \phi, f^*)$  the variational inequality problem

$$\langle w^* - f^*, y - x \rangle \geq \phi(x) - \phi(y), \quad y \in K$$

with the unknown vector  $x \in D(A) \cap D(\phi) \cap K$  and  $w^* \in Ax$ . Since  $\overline{D(\partial\phi)} = D(\phi)$ , it is not hard to see that the solvability of the inclusion

$$\partial\phi(x) + Ax \ni f^*$$

in  $D(A) \cap D(\partial\phi) \cap K$  implies the solvability of the problem  $VIP(A, K, \phi, f^*)$  in  $D(A) \cap D(\phi) \cap K$ , and equivalence holds if  $D(\phi) = D(\partial\phi) = K$ . In particular, if  $\phi = I_K$ , we denote the problem  $VIP(A, K, I_K, f^*)$  just by  $VIP(A, K, f^*)$ , and we see that its solvability is equivalent to the solvability of the inclusion

$$\partial I_K(x) + Ax \ni f^*$$

in  $D(A) \cap K$ .

For basic results involving variational inequalities and monotone type mappings, the reader is referred to Barbu [4], Brézis [10], Browder and Hess [18], Browder [13], Browder and Brézis [19], Hartman and Stampacchia [31], Kenmochi [42], Kinderlehrer and Stampacchia [47], Kobayashi and Otani [48], Lions and Stampacchia [52], Minty [56]-[57], Moreau [58], Naniewicz and Panagiotopoulos [61], Pascali and Sburlan [63], Rockafellar [65], Stampacchia [69], Ton [70], Zeidler [72] and the references there in. A study of pseudomonotone operators and nonlinear elliptic boundary value problems may be found in Kenmochi [43]. For a survey of maximal monotone and pseudomonotone operators and perturbation results, we cite the hand book of Kenmochi [44]. Nonlinear perturbation results of monotone type mappings, variational inequalities and their applications may be found in Guan, Kartsatos and Skrypnik [30], Guan and Kartsatos [29], Le [50], Zhou [74] and the references therein. Variational inequalities for single single-valued pseudomonotone operators in the sense of Brézis may be found in Kien, Wong, Wong and Yao [46]. Existence results for multivalued quasilinear inclusions and variational-hemivariational inequalities may be found in Carl, Le and Motreanu [22], Carl [23] and Carl and Motreanu [24] and the references therein.

In this Chapter, we study the solvability of variational inequalities, where the relevant operator  $A$  could be, e.g., the sum  $T + S$  with  $T : X \supseteq D(T) \rightarrow 2^{X^*}$  maximal monotone and  $S : K \rightarrow 2^{X^*}$  at least pseudomonotone. As mentioned in the general introduction of this dissertation, the main reason for treating these perturbed problems is the fact that the solvability of the variational problem  $VIP(T + S, K, \phi, f^*)$  in  $D(T) \cap D(\phi) \cap K$  is equivalent to the solvability of the inclusion problem  $\partial\phi(x) + T(x) + S(x) \ni f^*$  in  $D(T) \cap D(\partial\phi) \cap K$  provided that  $D(\phi) = D(\partial\phi) = K$ . However, there are no known results corresponding to the solvability of inclusion problems with there monotone type operators as mentioned above. Furthermore, it is known from Browder and Hess [18, Proposition 3, p. 258] that every pseudomonotone operator with effective domain all of  $X$  is generalized pseudomonotone. However, this fact



is unknown if the domain is different from  $X$ . Browder and Hess [18] mentioned the difficulty of treating generalized pseudomonotone operators which are not defined everywhere on  $X$  or on a dense linear subspace. A surjectivity result for single quasibounded coercive generalized pseudomonotone operator whose domain contains a dense linear subspace of  $X$  may be found in Browder and Hess [18, Theorem 5, p. 273]. Existence results for densely defined finitely continuous generalized pseudomonotone perturbations of maximal monotone operators may be found in Guan, Kartsatos and Skrypnik [30, Theorem 2.1, p. 335]. We should mention that there are no range results known for the sum  $T + S$ , where  $T$  is maximal monotone and  $S$  either pseudomonotone or generalized pseudomonotone with domain  $K$ , where  $K$  is a nonempty, closed and convex subset of  $X$ .

Section 4.2 deals with the study of solvability of variational inequalities for bounded pseudomonotone perturbations of one or two maximal monotone operators. As a result, a new characterization for the maximality of the sum of two maximal monotone operators is given. In Section 4.3 we prove new existence results for the solvability of variational inequalities for finitely continuous generalized pseudomonotone,  $(pm_4)$  generalized pseudomonotone and regular generalized pseudomonotone perturbations of maximal monotone operators. In each of these sections and subsections, the corresponding range and surjectivity results are discussed. The general methodology used in this theory is to show that the problem  $VIP(T + S, G \cap K, \phi, f^*)$  solvable in  $D(T) \cap D(\phi) \cap K \cap G$ , for some bounded, open and convex subset  $G$  of  $X$  and prove that this local solution solves the global problem  $VIP(T + S, K, \phi, f^*)$  under a suitable Leray-Schauder type condition. The last Section consists of examples of single-valued as well as multivalued pseudomonotone operators which are suitable for the applicability of our theory. An example on the existence of weak(generalized) solution(s) of a nonlinear parabolic partial differential equation is provided to demonstrate the importance of the theory developed in this Chapter of the dissertation.

The following Lemma is a version of Lemma 1.2.5. Its proof in its present form may be found in Adhikari and Kartsatos [1, Lemma 1, p. 1244].

**Lemma 4.1.1** (*Adhikari and Kartsatos [1].*) *Assume that the operators  $T : X \supseteq D(T) \rightarrow 2^{X^*}$  and  $S : X \supseteq D(S) \rightarrow 2^{X^*}$  are maximal monotone with  $0 \in T(0) \cap S(0)$ . Assume, further, that  $T + S$  is maximal monotone. Assume there is a positive sequence  $\{t_n\}$  such that  $t_n \downarrow 0^+$  and a sequence  $\{x_n\}$  in  $D(S)$  such  $x_n \rightharpoonup x_0 \in X$  and  $T_{t_n}x_n + w_n^* \rightharpoonup y_0^* \in X^*$ , where  $w_n^* \in Sx_n$ . Then the following are true.*

(i) *The inequality*

$$\lim_{n \rightarrow \infty} \langle T_{t_n}x_n + w_n^*, x_n - x_0 \rangle < 0$$

*is impossible.*

(ii) *If*

$$\lim_{n \rightarrow \infty} \langle T_{t_n}x_n + w_n^*, x_n - x_0 \rangle = 0,$$

*then  $x_0 \in D(T) \cap D(S)$  and  $y_0^* \in (T + S)x_0$ .*

Browder and Hess [18] proved that a monotone mapping  $T$  with  $0 \in \overset{\circ}{D}(T)$  is strongly quasibounded. The following Lemma is due to Browder and Hess [18].

**Lemma 4.1.2** (*Browder and Hess [18].*)

*Let  $X \supseteq D(T) \rightarrow 2^{X^*}$  be strongly quasibounded maximal monotone such that  $0 \in T(0)$  and  $\{t_n\}$  be a sequence in  $(0, \infty)$  and  $\{x_n\} \subseteq X$  be such that*

$$\|x_n\| \leq S, \langle T_{t_n}x_n, x_n \rangle \leq S$$

*for all  $n$ , where  $S$  is positive constant. Then there exists  $K = K(S) > 0$  such that  $\|T_{t_n}x_n\| \leq K$  for all  $n$ .*

In what follows, we make frequent use of the following basic result of Browder and Hess [18, Proposition 15, p. 289].

**Lemma 4.1.3** (Browder and Hess [18].) *Let  $K$  be a compact convex subset of  $X$  and  $T : K \rightarrow 2^{X^*}$  an operator such that for every  $x \in K$ ,  $Tx$  is a nonempty, closed, convex and bounded subset of  $X^*$ . Assume that  $T$  is upper semicontinuous, with  $X^*$  being given its weak topology. Let  $f^* \in X^*$ . Then there exist elements  $x_0 \in K$  and  $y_0^* \in Tx_0$  such that*

$$\langle y_0^* - f^*, x - x_0 \rangle \leq 0$$

for all  $x \in K$ .

We observe that, for every  $f^* \in X^*$ ,  $-T + f^*$  is upper semicontinuous whenever  $T$  is upper semicontinuous. Under the hypothesis of the above lemma, we have the existence of  $x_0^* \in K$  and  $v_0^* \in -Tx_0 + f^*$  (i.e.  $v_0^* = -w_0^* + f^*$ , for some  $w_0^* \in Tx_0$ ) such that  $\langle -w_0^* + f^*, x - x_0 \rangle \leq 0$  for all  $x \in K$ . This implies  $\langle w_0^* - f^*, x - x_0 \rangle \geq 0$  for all  $x \in K$ , i.e. the problem  $VIP(T, K, f^*)$  is solvable in  $K$ .

The following Lemma, which is an easy application of the uniform boundedness principle, may be found in Browder [21, Lemma 1].

**Lemma 4.1.4** (Browder[21].) *Let  $X$  be a Banach space,  $\{x_n\}$  a sequence in  $X$  and  $\{\alpha_n\}$  a sequence of positive numbers such that  $\alpha_n \rightarrow 0^+$  as  $n \rightarrow \infty$ . For a fixed  $r > 0$ , assume that for every  $h^* \in X^*$  with  $\|h^*\| \leq r$ , there exists a constant  $C_{h^*}$  such that*

$$\langle h^*, x_n \rangle \leq \alpha_n \|x_n\| + C_{h^*}$$

for all  $n$ . Then the sequence  $\{x_n\}$  is bounded.

The next lemma can be found in Browder [13, Proposition 7.2, p. 81].

**Lemma 4.1.5** (Browder [13].) *Let  $X$  be a reflexive Banach space,  $A$  a bounded subset of  $X$  and  $x_0 \in \overline{A^w}$ , where  $\overline{A^w}$  is the weak closure of  $A$  in  $X$ . Then there exists a sequence  $\{x_n\}$  in  $A$  such that  $x_n \rightharpoonup x_0$  in  $X$  as  $n \rightarrow \infty$ .*

The following existence result for the solvability of a variational inequality for a single multivalued pseudomonotone operator is due to Browder and Hess [18, Theorem 15, p. 289].

**Lemma 4.1.6** (*Browder and Hess [18].*) *Let  $K$  be a nonempty, closed and convex subset of  $X$  with  $0 \in K$ . Let  $S : K \rightarrow 2^{X^*}$  be pseudomonotone and coercive. Then for each  $g^* \in X^*$  there exist  $x_0 \in K$  and  $w_0 \in Sx_0$  such that*

$$\langle w_0 - g^*, x - x_0 \rangle \geq 0$$

for all  $x \in K$ .

## 4.2 Variational inequalities for maximal monotone perturbations of pseudomonotone operators

In this Section we give some existence results for the problem  $VIP(T + S, K, \phi, f^*)$ , where  $T$  is maximal monotone and  $S$  is bounded pseudomonotone. We begin with the definition of the solvability of a variational inequality over a given set.

**Definition 4.2.1** Let  $K$  be a nonempty subset of  $X$  and  $A : X \supseteq D(A) \rightarrow 2^{X^*}$ . Let  $\phi : X \rightarrow (-\infty, \infty]$  be a proper, convex and lower semicontinuous, and fix  $f^* \in X^*$ . We say that the variational inequality problem  $VIP(A, K, \phi, f^*)$  is solvable in  $D(A) \cap D(\phi) \cap B$  if there exist  $x_0 \in D(A) \cap D(\phi) \cap B$  and  $w_0^* \in Ax_0$  such that

$$\langle w_0^* - f^*, x - x_0 \rangle \geq \phi(x_0) - \phi(x)$$

for all  $x \in K$ .

Using this definition, it follows that the problem  $VIP(A, K, \phi, f^*)$  has no solution in  $D(A) \cap D(\phi) \cap \partial K$  if and only if there exists  $u_0 \in K$  such that

$$\langle w^* - f^*, u_0 - x \rangle < \phi(x) - \phi(u_0)$$

for all  $x \in D(A) \cap D(\phi) \cap \partial K$ ,  $w^* \in Ax$ .

In what follows, we make frequent use of the following useful Lemma. A version of this Lemma is due to Lions and Stampacchia [52] when  $X$  is a Hilbert space. Another version of it is due to Hartman and Stampacchia [31] and involves monotone finitely continuous operators defined on a closed convex subset of  $X$ . For further reference, we cite book of Kinderlehrer and Stampacchia [47, Theorem 1.7, pp. 85-87, and Theorem 2.3, p. 91].

**Lemma 4.2.2** *Let  $K$  be a nonempty, closed and convex subset of  $X$  and  $A : X \supseteq D(A) \rightarrow 2^{X^*}$ . Let  $G$  be an open convex subset of  $X$ . Then the problem  $VIP(A, K, \phi, f^*)$  is solvable in  $D(A) \cap D(\phi) \cap K \cap G$  provided that the problem  $VIP(A, K \cap \overline{G}, \phi, f^*)$  is solvable in  $D(A) \cap D(\phi) \cap K \cap G$ .*

*Proof.* Suppose that  $x_0 \in D(A) \cap D(\phi) \cap K \cap G$  is a solution of the problem  $VIP(A, K \cap \overline{G}, \phi, f^*)$ , i.e. there exists  $u_0^* \in Ax_0$  such that

$$\langle u_0^* - f^*, x - x_0 \rangle \geq \phi(x_0) - \phi(x) \tag{4.2.1}$$

for all  $x \in K \cap \overline{G}$ . It suffices to show that  $x_0$  solves the inequality  $VIP(A, K, \phi, f^*)$ . We observe that, by the convexity of  $K$ , for any  $t \in (0, 1)$  and for any  $x \in K$ , we have  $tx + (1-t)x_0 \in K$ . For each  $x \in K$ , we claim that there exists  $t_0 = t_0(x) \in (0, 1)$  such that  $t_0x + (1-t_0)x_0 \in G$ . Suppose there exists  $y \in K$  such that  $ty + (1-t)x_0 \notin G$  for all  $t \in (0, 1)$ , i.e.  $ty + (1-t)x_0 \in X \setminus G$  for all  $t \in (0, 1)$ . Since  $G$  is open, letting  $t \downarrow 0^+$ , we obtain that  $x_0 \notin G$ . But this is a contradiction as  $x_0 \in G$ . Thus our claim follows, i.e.

for every  $x \in K$ , there exists  $t_0 = t_0(x) \in (0, 1)$  such that  $y = t_0x + (1 - t_0)x_0 \in K \cap G$ . Replacing  $x$  by  $y$  in (4.2.1) and using the convexity of  $\phi$ , we see that

$$\begin{aligned} t_0 \langle u_0^* - f^*, x - x_0 \rangle &= \langle u_0^* - f^*, y - x_0 \rangle \\ &\geq \phi(x_0) - \phi(y) \\ &\geq \phi(x_0) - [t_0\phi(x) + (1 - t_0)\phi(x_0)] \\ &= t_0(\phi(x_0) - \phi(x)). \end{aligned}$$

Since  $t_0 \in (0, 1)$ , we conclude that

$$\langle u_0^* - f^*, x - x_0 \rangle \geq \phi(x_0) - \phi(x)$$

for all  $x \in K$ , i.e. the problem  $VIP(A, K, \phi, f^*)$  is solvable by  $x_0 \in D(A) \cap D(\phi) \cap K \cap G$ . ■

The following Theorem will be used frequently in the sequel. For related results the reader is referred to Browder [13, Theorem 7.8, pp. 92-96] ( $D(T) = D(S) = K$ ), Kenmochi [42, Theorem 5.2, p. 236] ( $D(S) = X$ ) and Le [50] ( $D(S) = X$ ).

**Theorem 4.2.3** *Let  $K$  be a nonempty, closed and convex subset of  $X$  with  $0 \in \overset{\circ}{K}$ . Let  $T : X \supseteq D(T) \rightarrow 2^{X^*}$  be maximal monotone with  $0 \in T(0)$  and  $S : K \rightarrow 2^{X^*}$  be pseudomonotone. Fix  $f^* \in X^*$ . Assume, further, that either  $S$  is bounded or  $T$  is strongly quasibounded and there exists  $k > 0$  such that  $\langle w^*, x \rangle \geq -k$  for all  $x \in K$  and  $w^* \in Sx$ .*

- (i) *If  $K$  is bounded, then the problem  $VIP(T + S, K, f^*)$  is solvable in  $D(T) \cap K$ .*
- (ii) *If  $K$  is unbounded and there exists an open, convex and bounded subset  $G$  of  $X$  with  $0 \in G$  such that the problem  $VIP(T + S, K \cap \overline{G}, f^*)$  has no solution in  $D(T) \cap K \cap \partial G$ , then the problem  $VIP(T + S, K, f^*)$  is solvable in  $D(T) \cap K \cap G$ .*

*Proof.* We first prove (i) and (ii) assuming the boundedness of  $S$ .

(i) Suppose  $K$  is bounded. Let  $t > 0$  and  $T_t$  be the Yosida approximant of  $T$ . We notice that, for every  $t > 0$ , the operator  $T_t + S$  is bounded and pseudomonotone on  $K$ . Using the boundedness of  $K$ , instead of the coercivity of the pseudomonotone operator  $T_t + S$  in Lemma 4.1.6, we see that  $VIP(T_t + S, K, f^*)$  is solvable in  $K$ . Thus, for every  $t_n \downarrow 0^+$  there exists  $x_n \in K$  and  $w_n^* \in Sx_n$  such that

$$\langle T_{t_n}x_n + w_n^* - f^*, x - x_n \rangle \geq 0$$

for all  $n$  and all  $x \in K$ . Since the solvability of  $VIP(T_{t_n} + S, K, f)$ , with solution  $x_n \in K$ , is equivalent to the solvability of the inclusion

$$\partial I_K(x_n) + T_{t_n}x_n + w_n^* \ni f^*$$

for every  $n$ , there exists  $v_n^* \in \partial I_K(x_n)$  such that

$$v_n^* + T_{t_n}x_n + w_n^* = f^*$$

for all  $n$ . Since  $\{x_n\}$  and  $S$  are bounded, we have the boundedness of the sequence  $\{w_n^*\}$ . Since  $0 \in T(0)$ , we have  $T_{t_n}(0) = 0$  for all  $n$  and hence  $\langle v_n^*, x_n \rangle \leq \|w_n^*\| \|x_n\|$ . The boundedness of  $\{v_n^*\}$  follows from the fact that  $\partial I_K$  is strongly quasibounded. As a result, the sequence  $\{T_{t_n}x_n\}$  is also bounded. Assume, by passing to subsequences if necessary, that  $x_n \rightharpoonup x_0$ ,  $v_n^* \rightharpoonup v_0^*$ ,  $w_n^* \rightharpoonup w_0^*$  and  $T_{t_n}x_n \rightharpoonup z_0^*$  as  $n \rightarrow \infty$ . Since  $K$  is closed and convex, it is weakly closed and hence  $x_0 \in K$ . Since  $S$  is pseudomonotone and  $\partial I_K$  is maximal monotone, it is easy to see that  $\liminf_{n \rightarrow \infty} \langle v_n^*, x_n - x_0 \rangle \geq 0$  and

$$\liminf_{n \rightarrow \infty} \langle w_n^*, x_n - x_0 \rangle \geq 0.$$

Let  $J_{t_n}$  be the Yosida resolvent of  $T$ . It is well known that, for every  $n$ ,  $J_{t_n}x_n \in D(T)$ ,  $J_{t_n}x_n = x_n - t_n J^{-1}(T_{t_n}x_n)$ ,  $T_{t_n}x_n \in T(J_{t_n}x_n)$  for all  $n$  and  $J_{t_n}x_n \rightharpoonup x_0$  and  $x_n - J_{t_n}x_n \rightarrow 0$  as  $n \rightarrow \infty$ . Therefore, we have

$$\limsup_{n \rightarrow \infty} \langle T_{t_n}x_n, J_{t_n}x_n - x_0 \rangle \leq 0.$$

Using Lemma 1.2.5, we conclude that  $x_0 \in D(T)$  and  $\langle T_{t_n}x_n, J_{t_n}x_n \rangle \rightarrow \langle z_0^*, x_0 \rangle$  as  $n \rightarrow \infty$ . Similarly, using the maximality of  $\partial I_K$  and Lemma 1.2.5, we can show that  $v_0^* \in \partial I_K(x_0)$  and  $\langle v_n^*, x_n \rangle \rightarrow \langle v_0^*, x_0 \rangle$  as  $n \rightarrow \infty$ . On the other hand, we have

$$\limsup_{n \rightarrow \infty} \langle w_n^*, x_n - x_0 \rangle \leq 0.$$

Since  $S$  is pseudomonotone, for every  $x \in K$  there exists  $y^*(x) \in Sx_0$  such that

$$\langle y^*(x), x_0 - x \rangle \leq \liminf_{n \rightarrow \infty} \langle w_n^*, x_n - x \rangle = -\langle v_0^* + z_0^* - f^*, x_0 - x \rangle.$$

for all  $n$ . Since  $v_0^* \in \partial I_K(x_0)$ , we have  $\langle v_0^*, x_0 - x \rangle \geq 0$  for all  $x \in K$ . Therefore,

$$\langle y^*(x), x_0 - x \rangle \leq \liminf_{n \rightarrow \infty} \langle w_n^*, x_n - x \rangle = -\langle z_0^* - f^*, x_0 - x \rangle$$

for all  $n$ . Since  $S$  is pseudomonotone, for every  $x \in K$  there exists  $y^*(x) \in Sx_0$  such that

$$\langle y^*(x) + z_0^* - f^*, x - x_0 \rangle \geq 0.$$

By Lemma 4.1.6, using  $f^* - z_0^*$  in place of  $g^*$ , there exists  $y_0^* \in Sx_0$  such that

$$\langle y_0^* - (f^* - z_0^*), x - x_0 \rangle \geq 0$$



for all  $x \in K$ , which implies

$$\langle y_0^* + z_0^* - f^*, x - x_0 \rangle \geq 0$$

for all  $x \in K$ . This implies that  $VIP(T + S, K, f^*)$  is solvable in  $D(T) \cap K$ .

(ii) Suppose  $K$  is unbounded and the hypothesis in (ii) holds true. Since  $K \cap \overline{G}$  is a nonempty closed, convex and bounded subset of  $X$  with  $0 \in K \cap \overline{G}$ , we apply the conclusion of (i) using the closed, convex and bounded subset  $K \cap \overline{G}$  in place of  $K$ , to obtain the solvability of the problem  $VIP(T + S, K \cap \overline{G}, f^*)$  in  $D(T) \cap K \cap \overline{G}$ . Since the problem  $VIP(T + S, K \cap \overline{G}, f^*)$  has no solution in  $D(T) \cap K \cap \partial G$ , we use Lemma 4.2.2 (with  $\phi = I_K$ ) to conclude that the variational inequality  $VIP(T + S, K, f^*)$  is solvable in  $D(T) \cap K \cap G$ .

Next we assume that  $T$  is strongly quasibounded and there exists  $k > 0$  such that  $\langle w^*, x \rangle \geq -k$  for all  $x \in K$  and  $w^* \in Sx$ . We prove the result in (i). Since  $T_t + S$  is pseudomonotone on  $K$ , Lemma 4.1.6 says that for every  $t > 0$ , the problem  $VIP(T_t + S, K, f^*)$  is solvable in  $K$ . This is equivalent to the solvability of the inclusion  $\partial I_K(x) + T_t x + Sx \ni f^*$  in  $K$ . Thus, for every  $t_n \downarrow 0^+$ , there exists  $x_n \in K$ ,  $v_n^* \in \partial I_K(x_n)$  and  $w_n^* \in Sx_n$  such that

$$v_n^* + T_{t_n} x_n + w_n^* = f^* \tag{4.2.2}$$

for all  $n$ . Since  $0 \in T(0)$ , we see that  $T_{t_n}(0) = 0$  for all  $n$ . Since  $T_{t_n}$  is monotone for all  $n$ , we have  $\langle v_n^*, x_n \rangle \leq k + \|f^*\| \|x_n\| \leq Q$ , where  $Q$  is an obvious upper bound. Since  $\partial I_K$  is strongly quasibounded, it follows that  $\{v_n^*\}$  is bounded. Using a similar argument along with the strong quasiboundedness of  $T$  and Lemma 4.1.2, we obtain the boundedness of  $\{T_{t_n} x_n\}$  and, subsequently, the boundedness of  $\{w_n^*\}$  from (4.2.2). Following the argument of the proof of (i) with  $S$  bounded, we obtain the solvability of the problem  $VIP(T + S, K, f^*)$  in  $D(T) \cap K$ . The proof of (ii) under this case can

be completed as in (ii) with  $S$  bounded. The detail is omitted. ■

Le [50] gave a range result for bounded pseudomonotone perturbation  $S$  (with  $D(S) = X$ ) of maximal monotone operators satisfying an inner product condition as in the following corollary for the case  $G = B_R(0)$ . We give an analogous result below, where  $G$  is a bounded, open and convex subset of  $X$  with  $0 \in G$ , and  $D(S) = K$ , with  $K$  a nonempty, closed and convex subset of  $X$ .

**Corollary 4.2.4** *Let  $K$  be a nonempty, closed and convex subset of  $X$  with  $0 \in \overset{\circ}{K}$ . Let  $T : X \supseteq D(T) \rightarrow 2^{X^*}$  be maximal monotone with  $0 \in T(0)$  and  $S : K \rightarrow 2^{X^*}$  pseudomonotone. Assume, further, that either  $S$  is bounded or  $T$  is strongly quasibounded and there exists  $k > 0$  such that  $\langle w^*, x \rangle \geq -k$  for all  $x \in K$  and  $w^* \in Sx$ . Fix  $f^* \in X^*$ . Let  $G$  be an open, convex and bounded subset of  $X$  with  $0 \in G$  such that, for some  $u_0 \in K \cap \overline{G}$ , we have*

$$\langle v^* + w^* - f^*, x - u_0 \rangle > 0 \tag{4.2.3}$$

for all  $x \in D(T) \cap \partial(K \cap \overline{G})$ ,  $v^* \in Tx$  and  $w^* \in Sx$ . Then the inclusion  $Tx + Sx \ni f^*$  is solvable in  $D(T) \cap K \cap G$ .

*Proof.* We first observe that  $0 \in \overset{\circ}{K \cap \overline{G}}$ . By Theorem 4.2.3, the problem  $VIP(T + S, K \cap \overline{G}, f^*)$  is solvable in  $D(T) \cap K \cap \overline{G}$ . By (4.2.3), the problem  $VIP(T + S, K \cap \overline{G}, f^*)$  has no solution in  $D(T) \cap \partial(K \cap \overline{G})$ . Since the solvability of the inclusion

$$\partial I_{K \cap \overline{G}}(x) + Tx + Sx \ni f^*$$

is equivalent to the solvability of the variational inequality  $VIP(T + S, K \cap \overline{G}, f^*)$ , it follows that the inclusion  $Tx + Sx \ni f^*$  is solvable in  $D(T) \cap \overset{\circ}{K \cap \overline{G}}$ . ■

Browder [13, Theorem 7.12, pp. 100-101] showed the existence of a solution to the problem  $VIP(S, K, \phi, f)$ , where  $S$  is bounded pseudomonotone and coercive with  $D(S) = K$ ,  $0 \in K$  and  $\phi : K \rightarrow (-\infty, \infty]$  is proper, convex and lower semicontinuous having 0 as its minimum on  $K$ . Furthermore, Kenmochi [42] proved the existence of a solution to the problem  $VIP(S, K, \phi, f)$ , where  $S$  is pseudomonotone on  $K$  satisfying the  $(pm_4)$ -condition (see Definition 4.3.3 below) along with a coercivity-type condition involving  $S$  and  $\phi$ .

The following Theorem gives a new existence result for solutions of the problem  $VIP(T + S, K, \phi, f^*)$ , where  $T$  is maximal monotone and  $S$  is bounded pseudomonotone. We remark that, using the definition of  $\partial\phi$ , it is not hard to see that the solvability of the problem  $VIP(\partial\phi + T + S, K, f^*)$  in  $D(\partial\phi) \cap D(T) \cap K$  implies the solvability of the inequality  $VIP(T + S, K, \phi, f^*)$  in  $D(T) \cap D(\phi) \cap K$ . Furthermore, using Lemma 4.2.2, the solvability of  $VIP(\partial\phi + T + S, K, f^*)$  in  $D(\partial\phi) \cap D(T) \cap K$  is achieved by solving the local problem  $VIP(\partial\phi + T + S, K \cap \overline{B}_R(0), f^*)$  in  $D(T) \cap K \cap B_R(0)$ .

**Theorem 4.2.5** *Let  $K$  be a nonempty, closed and convex subset of  $X$  with  $0 \in \overset{\circ}{K}$ . Let  $T : X \supseteq D(T) \rightarrow 2^{X^*}$  be strongly quasibounded maximal monotone with  $0 \in T(0)$  and  $S : K \rightarrow 2^{X^*}$  bounded pseudomonotone. Let  $\phi : X \rightarrow (-\infty, \infty]$  be proper, convex and lower semicontinuous and such that  $0 \in D(\phi)$  and there exists  $k > 0$  such that  $\phi(x) \geq -k$  for all  $x \in X$ . Fix  $f^* \in X^*$ . Then*

- (i) *If  $K$  is bounded, then the problem  $VIP(T + S, K, \phi, f^*)$  is solvable in  $D(T) \cap K \cap D(\phi)$ .*
- (ii) *If  $K$  is unbounded and there exists a bounded open convex subset  $G$  of  $X$  with  $0 \in G$  such that the problem  $VIP(T + S, K \cap \overline{G}, \phi, f^*)$  has no solution in  $D(T) \cap D(\phi) \cap K \cap \partial G$ , then the problem  $VIP(T + S, K, \phi, f^*)$  is solvable in  $D(T) \cap K \cap D(\phi) \cap G$ .*

*Proof.* (i) Suppose that  $K$  is bounded. We first prove the solvability of the problem  $VIP(\partial\phi+T+S, K, f^*)$  in  $D(T)\cap D(\partial\phi)\cap K$ . To this end, we notice that the solvability of the problem  $VIP(\partial\phi + T + S, K, f^*)$  in  $D(\partial\phi) \cap D(T) \cap K$  is equivalent to the solvability of the inclusion

$$\partial I_K(x) + \partial\phi(x) + Tx + Sx \ni f^*$$

in  $D(\partial\phi) \cap D(T) \cap K$ . Since  $D(\partial I_K) = K$  and  $0 \in \overset{\circ}{K} \cap D(T)$ , it follows that  $\partial I_K + T$  is maximal monotone. Let  $A := \partial\phi$  and, for every  $t > 0$ , let  $A_t$  be the Yosida approximant of  $A$ . Since  $A_t + S$  is bounded pseudomonotone, using the argument in the proof of Theorem 4.2.3 with  $K$  bounded, the maximal monotone operator  $T$  and the bounded pseudomonotone operator  $A_t + S$ , we obtain that the problem  $VIP(T + A_t + S, K, f^*)$  is solvable in  $D(T) \cap K$ , which is equivalent to the solvability of the inclusion

$$\partial I_K(x) + Tx + A_t + Sx \ni f^*$$

in  $D(T) \cap K$ . Thus, for every  $t_n \downarrow 0^+$  there exist  $x_n \in D(T) \cap K$ ,  $u_n^* \in \partial I_K(x_n)$ ,  $v_n^* \in Tx_n$  and  $w_n^* \in Sx_n$  such that

$$u_n^* + v_n^* + A_{t_n}x_n + w_n^* = f^* \tag{4.2.4}$$

for all  $n$ . Next we see that

$$\begin{aligned} \langle A_{t_n}x_n, x_n \rangle &= \langle A_{t_n}x_n, x_n - J_{t_n}^A x_n \rangle + \langle A_{t_n}x_n, J_{t_n}^A x_n \rangle \\ &= t_n \langle A_{t_n}x_n, J^{-1}(A_{t_n}x_n) \rangle + \langle A_{t_n}x_n, J_{t_n}^A x_n \rangle \\ &= t_n \|A_{t_n}x_n\|^2 + \langle A_{t_n}x_n, J_{t_n}^A x_n \rangle \end{aligned} \tag{4.2.5}$$

for all  $n$ . Using the properties of the Yosida resolvent of  $A$ , we see that  $J_{t_n}^A x_n \in D(A)$  and  $A_{t_n}x_n \in A(J_{t_n}^A x_n) = \partial\phi(J_{t_n}^{\partial\phi} x_n)$  for all  $n$ . On the other hand, by the definition of

$\partial\phi$  and the assumption  $\phi(x) \geq -k$ , we have

$$\langle A_{t_n}x_n, J_{t_n}^A x_n \rangle \geq \phi(J_{t_n}^A x_n) - \phi(0) \geq -k - \phi(0) \quad (4.2.6)$$

for all  $n$ . Since  $\{x_n\}$  and  $S$  are bounded, we have the boundedness of  $\{w_n^*\}$ . From (4.2.4) and (4.2.5), we get

$$\begin{aligned} \langle u_n^*, x_n \rangle &= -\langle w_n^* - f^*, x_n \rangle - \langle A_{t_n}x_n + v_n^*, x_n \rangle \\ &\leq (\|w_n^* - f^*\|)\|x_n\| + k + \phi(0). \end{aligned}$$

Since  $0 \in D(\phi)$ , we have that  $\phi(0) < +\infty$ . The boundedness of the sequence  $\{u_n^*\}$  follows from the fact that  $\partial I_K$  is strongly quasibounded and maximal monotone with domain  $K$ . Since  $u_n^* \in \partial I_K(x_n)$ , we have  $\langle u_n^*, x_n \rangle \geq 0$  for all  $n$ . Combining (4.2.4) and (4.2.6), we have

$$\langle v_n^*, x_n \rangle \leq \|f\|\|x_n\| + \phi(0) + k$$

for all  $n$ . As a result, the boundedness of the sequence  $\{v_n^*\}$  follows because the sequence  $\{x_n\}$  is bounded and  $T$  is strongly quasibounded maximal monotone. Consequently, using the equality (4.2.4), we obtain the boundedness of  $\{A_{t_n}x_n\}$ . Assume, by passing to subsequences if necessary, that  $x_n \rightharpoonup x_0$ ,  $u_n^* \rightharpoonup u_0^*$ ,  $v_n^* \rightharpoonup v_0^*$ ,  $w_n^* \rightharpoonup w_0^*$  and  $A_{t_n}x_n \rightharpoonup z_0^*$  as  $n \rightarrow \infty$ . Since  $K$  is closed and convex, it is weakly closed and hence  $x_0 \in K$ . By using the property of pseudomonotonicity of  $S$ , it is easy to see that

$$\liminf_{n \rightarrow \infty} \langle w_n^*, x_n - x_0 \rangle \geq 0.$$

We claim that

$$d := \liminf_{n \rightarrow \infty} \langle u_n^* + v_n^*, x_n - x_0 \rangle \geq 0.$$

In fact, if this is not true, there exists a subsequence, denoted again by  $\{\langle u_n^* + v_n^*, x_n - x_0 \rangle\}$  such that

$$\lim_{n \rightarrow \infty} \langle u_n^* + v_n^*, x_n - x_0 \rangle < 0.$$

Since  $\partial I_K + T$  is maximal monotone, we use Lemma 1.2.5 to obtain  $x_0 \in D(\partial I_K + T)$ ,  $u_0^* + v_0^* \in (\partial I_K + T)(x_0)$  and  $\langle u_n^* + v_n^*, x_n \rangle \rightarrow \langle u_0^* + v_0^*, x_0 \rangle$  as  $n \rightarrow \infty$ . This implies  $d = 0$ , which is a contradiction. As a result, (4.2.4) implies

$$\limsup_{n \rightarrow \infty} \langle A_{t_n} x_n, x_n - x_0 \rangle \leq 0.$$

Let  $J_{t_n}^A$  be the Yosida resolvent of  $A$ . We know that  $J_{t_n}^A x_n \in D(A)$ ,  $J_{t_n}^A x_n = x_n - t_n J^{-1}(A_{t_n} x_n)$ ,  $A_{t_n} x_n \in A(J_{t_n}^A x_n)$  for all  $n$  and  $J_{t_n}^A x_n \rightarrow x_0$  and  $x_n - J_{t_n}^A x_n \rightarrow 0$  as  $n \rightarrow \infty$ . Therefore, we have

$$\limsup_{n \rightarrow \infty} \langle A_{t_n} x_n, J_{t_n}^A x_n - x_0 \rangle \leq 0.$$

Using Lemma 1.2.5 again, we conclude that  $x_0 \in D(A)$ ,  $z_0^* \in Ax_0$  and  $\langle A_{t_n} x_n, J_{t_n}^A x_n \rangle \rightarrow \langle z_0^*, x_0 \rangle$  as  $n \rightarrow \infty$ . Thus, (4.2.4) implies

$$\limsup_{n \rightarrow \infty} \langle u_n^* + v_n^*, x_n - x_0 \rangle \leq 0.$$

From Lemma 1.2.5, we obtain  $x_0 \in D(T) \cap K$ ,  $u_0^* + v_0^* \in (\partial I_K + T)(x_0)$  and  $\langle u_n^* + v_n^*, x_n \rangle \rightarrow \langle u_0^* + v_0^*, x_0 \rangle$  as  $n \rightarrow \infty$ . Consequently,  $x_0 \in D(A) \cap D(T) \cap K$  and

$$\limsup_{n \rightarrow \infty} \langle w_n^*, x_n - x_0 \rangle = 0.$$

Since  $S$  is pseudomonotone, for every  $x \in K$  there exists  $y^*(x) \in Sx_0$  such that

$$\langle y^*(x), x_0 - x \rangle \leq \liminf_{n \rightarrow \infty} \langle w_n^*, x_n - x \rangle = -\langle u_0^* + v_0^* + z_0^* - f^*, x_0 - x \rangle,$$

where the equality follows from (4.2.4). Thus, for every  $x \in K$  there exists  $y^*(x) \in Sx_0$  such that

$$\langle y^*(x) + u_0^* + v_0^* + z_0^* - f^*, x - x_0 \rangle \geq 0.$$

Following the proof of Theorem 4.2.3, we see that there exists a unique  $y_0^* \in Sx_0$  such that

$$\langle y_0^* + u_0^* + v_0^* + z_0^* - f^*, x - x_0 \rangle \geq 0$$

for all  $x \in K$ . Using the definition of  $\partial I_K$  and  $\partial\phi$ , since  $u_0^* \in \partial I_K(x_0)$  and  $z_0^* \in \partial\phi(x_0)$ , we see that  $\langle u_0^*, x_0 - x \rangle \geq 0$  for all  $x \in K$ , and  $\langle z_0^*, x_0 - x \rangle \geq \phi(x_0) - \phi(x)$  for all  $x \in X$ . As a consequence, we get

$$\begin{aligned} \langle v_0^* + y_0^* - f^*, x - x_0 \rangle &\geq \langle u_0^* + z_0^*, x_0 - x \rangle \\ &\geq \phi(x_0) - \phi(x) \end{aligned}$$

for all  $x \in K$ . Since  $\overline{D(\partial\phi)} = D(\phi)$ , it follows that  $x_0 \in D(T) \cap D(\phi) \cap K$ . Therefore, the problem  $VIP(T + S, K, \phi, f^*)$  is solvable in  $D(T) \cap D(\phi) \cap K$ .

(ii) Suppose that (ii) holds. Since  $K \cap \overline{G}$  is a nonempty, closed, convex and bounded subset of  $X$ , using  $K \cap \overline{G}$  in place of  $K$  in the argument of (i), we conclude that the problem  $VIP(T + S, K \cap \overline{G}, \phi, f^*)$  is solvable in  $D(T) \cap D(\phi) \cap K \cap \overline{G}$ . Since  $VIP(T + S, K \cap \overline{G}, \phi, f)$  has no solution in  $D(T) \cap D(\phi) \cap K \cap \partial G$ , we use Lemma 4.2.2 to conclude that  $VIP(T + S, K, \phi, f^*)$  is solvable in  $D(T) \cap D(\phi) \cap K \cap G$ . ■

We remark that Theorem 4.2.5 extends the result of Kenmochi [44, Theorem 4.1, p. 254] to the effect that we consider the operator  $T + S$  instead of the single pseudomonotone operator  $S$ .

In the following Corollary we use a coercivity-type condition involving the operator  $T + S$  and the function  $\phi$ .

**Corollary 4.2.6** *Let  $K$  be a nonempty, closed and convex subset of  $X$  with  $0 \in \overset{\circ}{K}$ . Let  $T : X \supseteq D(T) \rightarrow 2^{X^*}$  be strongly quasibounded maximal monotone with  $0 \in D(T)$  and  $S : K \rightarrow 2^{X^*}$  bounded pseudomonotone. Let  $\phi : X \rightarrow (-\infty, \infty]$  be proper, convex lower semicontinuous with  $0 \in D(\phi)$  and there exists a real number  $k > 0$  such that  $\phi(x) \geq -k$  for all  $x \in X$ . Assume, further, that there exists  $u_0 \in K$  with  $\phi(u_0) < \infty$  satisfying*

$$\inf_{v^* \in Tx, w^* \in Sx, x \in D(T) \cap K} \frac{\langle v^* + w^*, x - u_0 \rangle + \phi(x)}{\|x\|} \rightarrow \infty$$

as  $\|x\| \rightarrow \infty$ . Then for every  $f^* \in X^*$ , the problem  $VIP(T + S, K, \phi, f^*)$  is solvable in  $D(T) \cap D(\phi) \cap K$ .

*Proof.* Since  $\phi(u_0) < \infty$ , for every  $f^* \in X^*$  there exists  $R = R(f^*) > 0$ , which can be chosen so that  $u_0 \in \overline{B_R(0)}$ , such that

$$\langle v^* + w^* - f^*, x - u_0 \rangle + \phi(x) > \phi(u_0)$$

for all  $x \in D(T) \cap K \cap \partial B_R(0)$ . This is equivalent to saying that the problem  $VIP(T + S, K \cap \overline{B_R(0)}, \phi, f^*)$  has no solution in  $D(T) \cap D(\phi) \cap K \cap \partial B_R(0)$ . On the other hand, using the closed, convex and bounded set  $K \cap \overline{B_R(0)}$  and applying (i) of Theorem 4.2.5, we see that  $VIP(T + S, K \cap \overline{B_R(0)}, \phi, f^*)$  is solvable in  $D(T) \cap K \cap \overline{B_R(0)}$ , which implies that  $VIP(T + S, K \cap \overline{B_R(0)}, \phi, f^*)$  is solvable in  $D(T) \cap K \cap B_R(0)$ . Applying Lemma 4.2.2, we conclude that  $VIP(T + S, K, \phi, f^*)$  is solvable in  $D(T) \cap K \cap B_R(0)$ .

■

The following Theorem gives a new existence result for the solvability of the problem  $VIP(T + S + P, K, f^*)$  and the inclusion problem  $Tx + Sx + Px \ni f^*$ , where both  $T$  and  $S$  are maximal monotone and  $P$  is bounded pseudomonotone.



**Theorem 4.2.7** *Let  $K$  be nonempty, closed and convex subset of  $X$  with  $0 \in \overset{\circ}{K}$ . Let  $T : X \supseteq D(T) \rightarrow 2^{X^*}$  be maximal monotone and such that there exists  $k_1 > 0$  with  $\langle u^*, x \rangle \geq -k_1$  for all  $x \in D(T)$  and  $u^* \in Tx$ . Let  $S : X \supseteq D(S) \rightarrow 2^{X^*}$  be strongly quasibounded maximal monotone with  $0 \in S(0)$ . Suppose that  $P : K \rightarrow 2^{X^*}$  is bounded pseudomonotone. Assume, further, that there exist  $R > 0$ ,  $u_0 \in D(T) \cap D(S) \cap K \cap B_R(0)$  and  $k_2 > 2R|Tu_0|$  such that*

$$\langle w^* + z^* - f^*, x - u_0 \rangle \geq k_2$$

for all  $x \in D(T) \cap D(S) \cap K \cap \partial B_R(0)$ ,  $w^* \in Sx$  and  $z^* \in Px$ . Then the following are true.

- (i) *The problem  $VIP(T + S + P, K, f^*)$  is solvable in  $D(T) \cap D(S) \cap K \cap B_R(0)$ .*
- (ii) *If  $K = X$ , then the inclusion  $Tx + Sx + Px \ni f^*$  is solvable in  $D(T) \cap D(S) \cap B_R(0)$ .*

*Proof.* We first prove (i). Let  $\partial I_K : K \rightarrow 2^{X^*}$  be the subdifferential of the indicator function on  $K$ . It is well-known that  $D(\partial I_K) = K$  and  $\partial I_K(x) = \{0\}$  for all  $x \in \overset{\circ}{K}$ . Since  $0 \in \overset{\circ}{K}$ , we have  $0 \in \partial I_K(0)$  and  $\partial I_K$  is strongly quasibounded and maximal monotone. Let  $T_t$  be the Yosida approximant of  $T$ . Since  $u_0 \in D(T)$ , we have  $\|T_t u_0\| \leq |Tu_0|$ , where  $|Tu_0| = \inf\{\|x^*\| : x^* \in Tu_0\}$  for all  $t > 0$ . Thus, for every  $t > 0$ ,  $T_t + P$  is bounded, pseudomonotone and such that

$$\begin{aligned} \langle w^* + z^* + T_t x - f^*, x - u_0 \rangle &= \langle w^* + z^* + T_t x - T_t u_0 + T_t u_0 - f^*, x - u_0 \rangle \\ &\geq k_2 - |Tu_0| \|x - u_0\| \\ &\geq k_2 - 2R|Tu_0| > 0 \end{aligned}$$

for all  $x \in D(S) \cap K \cap \partial B_R(0)$ ,  $w^* \in Sx$  and  $z^* \in Px$ . Since  $u_0 \in K \cap \overline{B_R(0)}$ , it follows that  $VIP(P + T_t + S, K \cap \overline{B_R(0)}, f^*)$  has no solution in  $D(S) \cap K \cap \partial B_R(0)$ . Since

$T_t + P$  is bounded and pseudomonotone, we use Theorem 4.2.3 with the operators  $S$  and  $T_t + P$  to conclude that  $VIP(S + T_t + P, K, f^*)$  is solvable in  $D(S) \cap K \cap B_R(0)$ . Thus, for every  $t_n \downarrow 0^+$ , there exist  $x_n \in D(S) \cap K \cap B_R(0)$ ,  $v_n^* \in \partial I_K(x_n)$ ,  $w_n^* \in Sx_n$  and  $z_n^* \in Px_n$  such that

$$v_n^* + w_n^* + z_n^* + T_{t_n}x_n = f^* \quad (4.2.7)$$

for all  $n$ . Since  $\{x_n\}$  and  $P$  are bounded, we see that the sequence  $\{z_n^*\}$  is bounded. Next, since  $0 \in K$ , we get from the definition of  $\partial I_K$  that  $\langle v_n^*, x_n \rangle \geq 0$  for all  $n$ . Thus, using (4.2.7), we obtain

$$\begin{aligned} \langle w_n^*, x_n \rangle &\leq -\langle z_n^* - f^*, x_n \rangle - \langle T_{t_n}x_n, x_n - J_{t_n}x_n \rangle - \langle T_{t_n}x_n, J_{t_n}x_n \rangle - \langle v_n^*, x_n \rangle \\ &\leq (\|z_n^*\| + \|f^*\|)\|x_n\| - \langle T_{t_n}x_n, t_n J^{-1}(T_{t_n}x_n) \rangle + k_1 \\ &= (\|z_n^*\| + \|f^*\|)\|x_n\| - t_n \|T_{t_n}x_n\|^2 + k_1 \leq M, \end{aligned}$$

where  $M$  is an upper bound for the sequence  $\{(\|z_n^*\| + \|f^*\|)\|x_n\| + k_1\}$ . Therefore, the strong quasiboundedness of  $S$  implies the boundedness of the sequence  $\{w_n^*\}$ . Similarly, we get

$$\begin{aligned} \langle v_n^*, x_n \rangle &\leq -\langle T_{t_n}x_n, x_n - J_{t_n}x_n \rangle - \langle T_{t_n}x_n, J_{t_n}x_n \rangle \\ &\quad + (\|w_n^*\| + \|z_n^*\| + \|f^*\|)\|x_n\| \\ &\leq k_1 + (\|w_n^*\| + \|z_n^*\| + \|f^*\|)\|x_n\| \leq N, \end{aligned}$$

where  $N$  is an upper bound for the sequence  $\{k_1 + (\|w_n^*\| + \|z_n^*\| + \|f^*\|)\|x_n\|\}$ . Using the strong quasiboundedness of  $\partial I_K$ , it follows that the sequence  $\{v_n^*\}$  is bounded, which implies in turn the boundedness of the sequence  $\{T_{t_n}x_n\}$ . Assume that  $x_n \rightharpoonup x_0$ ,  $v_n^* \rightharpoonup v_0^*$ ,  $w_n^* \rightharpoonup w_0^*$ ,  $z_n^* \rightharpoonup z_0^*$  and  $T_{t_n}x_n \rightharpoonup u_0^*$  as  $n \rightarrow \infty$ . Since  $K$  is closed and convex, it is weakly closed and hence  $x_0 \in K$ . Since  $P$  is pseudomonotone, and  $S$  and  $\partial I_K$  are

monotone, we have

$$\liminf_{n \rightarrow \infty} \langle w_n^*, x_n - x_0 \rangle \geq 0, \quad \liminf_{n \rightarrow \infty} \langle v_n^*, x_n - x_0 \rangle \geq 0 \quad \text{and} \quad \liminf_{n \rightarrow \infty} \langle z_n^*, x_n - x_0 \rangle \geq 0.$$

Thus, we obtain

$$\limsup_{n \rightarrow \infty} \langle T_{t_n} x_n, x_n - x_0 \rangle \leq 0.$$

Using the maximality of  $T$  and Lemma 1.2.5, we conclude that  $x_0 \in D(T)$ ,  $u_0^* \in Tx_0$  and  $\langle T_{t_n} x_n, x_n \rangle \rightarrow \langle u_0^*, x_0 \rangle$  as  $n \rightarrow \infty$ . Similarly, we see that  $x_0 \in D(S) \cap K$ ,  $v_0^* + w_0^* \in (\partial I_K + S)x_0$  and  $\langle v_n^* + w_n^*, x_n \rangle \rightarrow \langle v_0^* + w_0^*, x_0 \rangle$  as  $n \rightarrow \infty$ . Finally, by the pseudomonotonicity of  $P$ , for every  $x \in K$  there exists  $y^*(x) \in Sx_0$  such that

$$\langle y^*(x) + w_0^* + u_0^* - f^*, x - x_0 \rangle \geq 0.$$

As in the argument of the last part of the proof of Theorem 4.2.3, there exists  $y_0^* \in Sx_0$  such that

$$\langle y_0^* + w_0^* + u_0^* - f^*, x - x_0 \rangle \geq 0$$

for all  $x \in K$ . This shows that the problem  $VIP(T + S + P, K, f^*)$  is solvable in  $D(T) \cap D(S) \cap K$ . The proof of (i) is complete.

(ii) Using (i) with  $K = X$ , we see that the inequality  $VIP(T + S + P, X, f^*)$  is solvable in  $D(T) \cap D(S) \cap B_R(0)$ . Using the definition of the solvability of a variational inequality, it is easy to see that the inclusion

$$Tx + Sx + Px \ni f^*$$

is solvable in  $D(T) \cap D(S) \cap B_R(0)$ . The proof is complete. ■

As an application of Theorem 4.2.7, the following Corollary gives a maximality criterion for the sum of two maximal monotone operators. Basic maximality criteria can be found in Browder and Hess [18] and Rockaffelar [64].

**Corollary 4.2.8** *Let  $T : X \supseteq D(T) \rightarrow 2^{X^*}$  be maximal monotone and such that there exists  $k_1 > 0$  satisfying  $\langle u^*, x \rangle \geq -k_1$  for all  $x \in D(T)$  and  $u^* \in Tx$ . Let  $S : X \supseteq D(S) \rightarrow 2^{X^*}$  be strongly quasibounded maximal monotone with  $0 \in S(0)$  such that  $D(T) \cap D(S) \neq \emptyset$ . Then  $T + S$  is maximal monotone.*

*Proof.* Choose  $u_0 \in D(T) \cap D(S)$ . Choose  $r > 0$  such that  $u_0 \in D(T) \cap D(S) \cap B_r(0)$ . Then, for  $w_0^* \in Su_0$ , using the monotonicity of  $J$  and  $S$ , we have

$$\begin{aligned} \langle w^* + Jx - f^*, x - u_0 \rangle &\geq \langle w_0^* + Jx - f^*, x - u_0 \rangle \\ &\geq \|x\|^2 - \|u_0\| \|x\| \\ &\quad - (\|w_0^*\| + \|f^*\|) \|x - u_0\| \rightarrow \infty \end{aligned}$$

as  $\|x\| \rightarrow \infty$ . Therefore, for any  $k_2 > 0$ , there exists  $R_1 > 0$  such that

$$\langle w^* + Jx - f^*, x - u_0 \rangle > k_2$$

for all  $\|x\| \geq R_1$ ,  $w^* \in Sx$ . We choose  $R = \max\{r, R_1\}$  so that  $u_0 \in D(T) \cap D(S) \cap B_R(0)$  and

$$\langle w^* + Jx - f^*, x - u_0 \rangle > k_2$$

for all  $x \in D(S) \cap \partial B_R(0)$  and  $w^* \in Sx$ . Using  $J$  in place of  $P$  in (ii) of Theorem 4.2.7, we conclude that the inclusion  $Tx + Sx + Jx \ni f^*$  is solvable in  $D(T) \cap D(S) \cap B_R(0)$ . Since  $f^* \in X^*$  is arbitrary, it follows that  $R(T + S + J) = X^*$ . This complete the maximality of  $T + S$ . ■

### 4.3 Variational inequalities for maximal monotone perturbations of generalized pseudomonotone operators

In this Section we give some results about the solvability of variational inequalities involving perturbations which are generalized pseudomonotone operators. Browder and Hess [18, Proposition 4, p. 258] showed that a bounded generalized pseudomonotone operator  $S$  is pseudomonotone if  $D(S) = X$ . However, this fact is unknown if  $D(S) \neq X$ . Because of this, we study the solvability of variational inequality problems separately for bounded pseudomonotone and bounded generalized pseudomonotone perturbations. A range result for single multivalued, densely defined, quasi-bounded, finitely continuous generalized pseudomonotone operator may be found in Browder and Hess [18, Theorem 5, p. 273]. Furthermore, range results for quasi-bounded, finitely continuous, generalized pseudomonotone perturbations  $S$  of maximal monotone operators, with  $S$  either densely defined or  $D(S) = X$ , under weaker coercivity assumptions on  $T + S$ , may be found in Guan, Kartsatos and Skrypnik [30], Guan and Kartsatos [29] respectively. Variational inequality results of the type  $VIP(T + S, K, f^*)$ , where  $T$  is maximal monotone with  $D(T) = X$ ,  $S$  is bounded, finitely continuous and generalized pseudomonotone with  $D(S) = K$  (with  $K$  closed and convex with  $0 \in K$ ) may be found in Zhou [74].

#### 4.3.1 Strongly quasibounded maximal monotone perturbations of generalized pseudomonotone operators

We now give the following existence result concerning the solvability of a variational inequality involving finitely continuous generalized pseudomonotone perturbations of a maximal monotone operator with  $D(T)$  not necessarily all of  $X$ .

**Theorem 4.3.1** *Let  $K$  be a nonempty, closed and convex subset of  $X$  with  $0 \in \overset{\circ}{K}$ . Let  $T : X \supseteq D(T) \rightarrow 2^{X^*}$  be strongly quasibounded maximal monotone with  $0 \in T(0)$*

and  $S : K \rightarrow 2^{X^*}$  finitely continuous generalized pseudomonotone such that there exists  $k > 0$  satisfying  $\langle w^*, x \rangle \geq -k$  for all  $x \in K$  and  $w^* \in Sx$ . Fix  $f^* \in X^*$ .

(i) If  $K$  is bounded, then the problem  $VIP(T + S, K, f^*)$  is solvable in  $D(T) \cap K$ .

(ii) If  $K$  is unbounded and there exists an open, bounded and convex subset  $G$  of  $X$  with  $0 \in G$  such that the variational inequality

$$VIP(T + S, K \cap \overline{G}, f^*)$$

has no solution in  $D(T) \cap K \cap \partial G$ , then the problem  $VIP(T + S, K, f^*)$  is solvable in  $D(T) \cap K \cap G$ .

Furthermore, if either (i) or (ii) holds and the inclusion  $Tx + Sx \ni f^*$  has no solution in  $D(T) \cap \partial K$ , then the inclusion  $Tx + Sx \ni f^*$  is solvable in  $D(T) \cap \overset{\circ}{K}$ .

*Proof.* (i) Assume that  $K$  is bounded. For each  $t > 0$ , let  $T_t$  be the Yosida approximant of  $T$ . It is known that  $T_t$  is bounded, continuous and maximal monotone with domain all of  $X$ . We follow in part Browder and Hess [18, Theorem 15, p. 289] who considered a single multivalued pseudomonotone operator.

Let  $\Lambda$  be the collection of all finite dimensional subspaces of  $X$ . For each  $F \in \Lambda$ , let  $j_F : F \rightarrow X$  be the inclusion mapping and  $j_F^* : X^* \rightarrow F^*$  be the adjoint of  $j_F$ . Let  $K_F := K \cap F$ . Since  $K$  is bounded,  $K_F$  is a compact subset of  $F$  for every  $F \in \Lambda$ . Since  $S$  is pseudomonotone,  $-(j_F^*(T_t + S))$  is upper semicontinuous with nonempty, closed, convex and bounded values in  $X^*$ . Thus, the operator  $j_F^*(T_t + S)j_F : K_F \rightarrow F^*$  is upper semicontinuous. Using Lemma 4.1.3, there exist  $x_F \in K_F$  and  $w_F^* \in Sx_F$  such that

$$\langle j_F^*(T_t x_F + w_F^* - f^*), x - x_F \rangle \geq 0$$

for all  $x \in K_F$ , which is equivalent to saying that

$$\langle T_t x_F + w_F^*, x - x_F \rangle \geq \langle f^*, x - x_F \rangle$$

for all  $x \in K_F$ . Since  $K$  is closed convex and bounded, the family  $\{x_F\}_{F \in \Lambda}$  is uniformly bounded and  $K$  is a weakly compact subset of  $X$ . For each  $F \in \Lambda$ , we define

$$V_F := \bigcup_{F \subset F'} \{x_{F'}\}.$$

We observe that, for every  $F$ ,  $\overline{V_F}^w$  is a weakly closed subset of the weakly compact subset  $K$ . Furthermore, the family  $\{\overline{V_F}^w\}$  satisfies the finite intersection property. Therefore, we have

$$V := \bigcap_{F \in \Lambda} \overline{V_F}^w \neq \emptyset.$$

Fix  $x \in K$  and choose  $x_0 \in V$  and a subspace  $F_0$  of  $X$  such that  $x_0, x \in F_0$ . Using Lemma 4.1.5, we choose a sequence  $\{x_n\}$  in  $V_{F_0}$  such that  $x_n \rightharpoonup x_0$  as  $n \rightarrow \infty$ . By the definition of  $V_{F_0}$ , for every  $n$  we choose  $F_n$  such that  $F_0 \subseteq F_n$  and  $x_n \in K_{F_n}$ . Since  $K$  is closed and convex, it is weakly closed and hence  $x_0 \in K$ . From the definition of  $x_n$ , it follows that

$$\langle T_t x_n + w_n^*, u - x_n \rangle \geq \langle f^*, u - x_n \rangle$$

for all  $u \in K_{F_n}$  for some  $w_n^* \in Sx_n$ , where  $K_{F_n} = K \cap F_n$ . From the definition of  $V_{F_0}$ , we have  $x \in K_{F_n}$  for all  $n$ , which implies

$$\langle T_t x_n + w_n^*, x - x_n \rangle \geq \langle f^*, x - x_n \rangle$$

for all  $n$  and for all  $x \in K$ . Thus, for every  $t_n \downarrow 0^+$ , the problem  $VIP(T_{t_n} + S, K, f^*)$  is solvable in  $K$ , i.e. there exists  $y_n \in K$ ,  $w_n^* \in Sy_n$  and  $v_n^* \in \partial I_K$  such that

$$v_n^* + T_{t_n}y_n + w_n^* = f^* \quad (4.3.8)$$

for all  $n$ . It is well known that  $D(\partial I_K) = K$ . Since  $0 \in \overset{\circ}{K}$ , the mapping  $\partial I_K(y_n)$  is strongly quasibounded maximal monotone from  $K$  in to  $X^*$ . Since  $0 \in T(0)$ , we have  $T_{t_n}(0) = 0$  and the assumption  $0 \in K$  implies  $\langle v_n^*, y_n \rangle \geq 0$  for all  $n$ . Therefore, using (4.3.8), we see that the sequence

$$\langle v_n^*, y_n \rangle \leq k + \|f^*\| \|y_n\| \leq Q$$

for all  $n$ , where  $Q$  is an upper bound for the sequence  $\{k + \|f\| \|x_n\|\}$ . The boundedness of the sequence  $\{v_n^*\}$  follows from the strong quasiboundedness of  $\partial I_K$ . In addition, using (4.3.8), we get

$$\langle T_{t_n}y_n, y_n \rangle \leq Q$$

for all  $n$ , where  $Q$  is as above. Thus, the boundedness of the sequence  $\{T_{t_n}y_n\}$  follows from Lemma 4.1.2. As a result, the sequence  $\{w_n^*\}$  is bounded. Assume w.l.o.g. that  $y_n \rightharpoonup y_0 \in K$ ,  $v_n^* \rightharpoonup v_0^*$  and  $T_{t_n}y_n \rightharpoonup z_0^*$  as  $n \rightarrow \infty$ . Using the monotonicity of  $T_{t_n}$  and  $\partial I_K$ , we see that

$$\limsup_{n \rightarrow \infty} \langle w_n^*, y_n - y_0 \rangle \leq 0.$$

Since  $S$  is generalized pseudomonotone, we have  $w_0^* \in Sy_0$  and  $\langle w_n^*, y_n \rangle \rightarrow \langle w_0^*, y_0 \rangle$  as  $n \rightarrow \infty$ . Using this and the monotonicity of  $\partial I_K$ , we get

$$\liminf_{n \rightarrow \infty} \langle v_n^* + w_n^*, y_n - y_0 \rangle \geq 0,$$



which implies

$$\limsup_{n \rightarrow \infty} \langle T_{t_n} y_n, y_n - y_0 \rangle \leq 0.$$

Let  $J_{t_n}$  be the Yosida resolvent of  $T$ . We know that  $J_{t_n} y_n = y_n - t_n J^{-1}(T_{t_n} y_n)$ ,  $J_{t_n} y_n \in D(T)$  and  $T_{t_n} y_n \in T(J_{t_n} y_n)$  for all  $n$ . Since  $\{T_{t_n} y_n\}$  is bounded,  $t_n \downarrow 0^+$  as  $n \rightarrow \infty$  and  $y_n \rightharpoonup y_0$ , it follows that  $J_{t_n} y_n \rightharpoonup y_0$  as  $n \rightarrow \infty$ . Consequently, (4.3.8) implies

$$\limsup_{n \rightarrow \infty} \langle T_{t_n} y_n, J_{t_n} y_n - y_0 \rangle \leq 0.$$

The maximality of  $T$  and Lemma 1.2.5 imply  $y_0 \in D(T)$ ,  $z_0^* \in Ty_0$  and  $\langle T_{t_n} y_n, J_{t_n} y_n \rangle \rightarrow \langle z_0^*, y_0 \rangle$  as  $n \rightarrow \infty$ . Applying a similar argument for the mapping  $\partial I_K$ , we see that  $v_0 \in \partial I_K$ . Finally, taking the limit as  $n \rightarrow \infty$  in (4.3.8), we conclude that

$$v_0^* + z_0^* + w_0^* = f^*.$$

Therefore, the problem  $VIP(T + S, K, f^*)$  is solvable in  $D(T) \cap K$ .

(ii) Suppose that the hypothesis in (ii) holds. Using the closed, convex and bounded set  $K \cap \overline{G}$  instead of  $K$  in (i), we obtain the solvability of the problem  $VIP(T + S, K \cap \overline{G}, f^*)$  in  $D(T) \cap K \cap \overline{G}$ . Since the problem  $VIP(T + S, K \cap \overline{G}, f^*)$  has no solution in  $D(T) \cap K \cap \partial G$ , we may use Lemma 4.2.2 to conclude that the problem  $VIP(T + S, K, f^*)$  is solvable in  $D(T) \cap K$ . It is known that the solvability of the problem  $VIP(T + S, K, f^*)$  is equivalent to the solvability of the inclusion  $\partial I_K(x) + Tx + Sx \ni f^*$  in  $D(T) \cap K$ . Therefore, if either (i) or (ii) holds and the inclusion  $\partial I_K(x) + Tx + Sx \ni f^*$  has no solution in  $D(T) \cap \partial K$ , then the solution lies in  $D(T) \cap \overset{\circ}{K}$ . Since  $\partial I_K(x) = \{0\}$  for all  $x \in \overset{\circ}{K}$ , we obtain the solvability of the inclusion  $Tx + Sx \ni f^*$  in  $D(T) \cap \overset{\circ}{K}$ . ■

We note that if  $K = \overline{B_R(0)}$  and  $T$  and  $S$  are as in Theorem 4.3.1, the inclusion  $Tx + Sx \ni f^*$  is solvable in  $D(T) \cap K \cap \partial B_R(0)$  provided that  $Tx + Sx + \lambda Jx \ni f^*$  has no solution in  $D(T) \cap D(S) \cap \partial B_R(0)$ ,  $v^* \in Tx$  and  $w^* \in Sx$ , for all  $\lambda \geq 0$ . This is because the subdifferential of the indicator function of  $\overline{B_R(0)}$  is given by

$$\partial I_{\overline{B_R(0)}}(x) = \begin{cases} \{0\} & \text{if } x \in B_R(0) \\ \{\lambda Jx : \lambda \geq 0\} & \text{if } x \in \partial B_R(0) \\ \emptyset & \text{if } x \in X \setminus \overline{B_R(0)}. \end{cases}$$

Let  $\Gamma_\beta$  denote the set of all functions  $\beta : R_+ \rightarrow R_+$  such that  $\beta(t) \rightarrow 0$  as  $t \rightarrow \infty$ . A range result for densely defined quasibounded, finitely continuous generalized pseudomonotone perturbation of maximal monotone operator may be found in Guan, Kartsatos and Skrypnik [30]. A new variational inequality result in the spirit of [30], is given below.

**Theorem 4.3.2** *Let  $K$  be a nonempty, closed and convex subset of  $X$  with  $0 \in \overset{\circ}{K}$ . Let  $T : X \supseteq D(T) \rightarrow 2^{X^*}$  be strongly quasibounded maximal monotone with  $0 \in T(0)$  and  $S : K \rightarrow 2^{X^*}$  finitely continuous generalized pseudomonotone. Fix  $f^* \in X^*$ . Assume, further, the following conditions hold.*

(i) *There exists a strictly increasing continuous function  $\psi : [0, \infty) \rightarrow [0, \infty)$  with  $\psi(0) = 0$  and  $\psi(t) \rightarrow \infty$  as  $t \rightarrow \infty$  satisfying  $\langle w^*, x \rangle \geq -\psi(\|x\|)$ ,  $x \in K$  and  $w^* \in Sx$ ;*

(ii) *There exist  $R > 0$ ,  $u_0 \in K$  and  $\beta \in \Gamma_\beta$  such that*

$$\langle v^* + w^* - (f^* + g^*), x - u_0 \rangle \geq -\beta(\|x\|)\|x\|$$

*for all  $g^* \in X^*$  with  $\|g^*\| \leq R$ ,  $x \in D(T) \cap K$ ,  $v^* \in Tx$  and  $w^* \in Sx$ .*

*Then the problem  $VIP(T + S, K, f^*)$  is solvable in  $D(T) \cap K$ .*

*Proof.* Fix  $f^* \in X^*$  and suppose that  $K$  is bounded, i.e. for some  $r > 0$ ,  $K \subseteq B_r(0)$ . Since  $\psi$  is strictly increasing, it follows that  $\psi(\|x\|) \leq \psi(r)$  for all  $x \in K$ . As a result, we see that  $\langle w^*, x \rangle \geq -\psi(r) = -k_r$  for all  $x \in K$  and  $w^* \in Sx$ . Applying (i) of Theorem 4.3.1, we obtain the solvability of the problem  $VIP(T + S, K, f^*)$  in  $D(T) \cap K$ . Assume  $K$  is unbounded. Let  $J_\psi$  be the duality mapping corresponding to the function  $\psi$ . For every  $\varepsilon > 0$  and  $x \neq 0$  we have

$$\langle v^* + w^* + \varepsilon J_\psi x - f^*, x \rangle \geq \psi(\|x\|)\|x\| \left( \varepsilon - \frac{1}{\|x\|} - \frac{\|f^*\|}{\psi(\|x\|)} \right) \rightarrow \infty$$

as  $\|x\| \rightarrow \infty$ , for all  $v^* \in Tx$ ,  $w^* \in Sx$ . Consequently, there exists  $R_\varepsilon = R(\varepsilon) > 0$  such that

$$\langle z^* + w^* + \varepsilon J_\psi x - f^*, x \rangle > 0 \tag{4.3.9}$$

for all  $x \in D(T) \cap K \cap \partial B_{R_\varepsilon}(0)$ ,  $z^* \in Tx$ . Since  $K \cap \overline{B_{R_\varepsilon}(0)}$  is bounded and  $S + J_\psi$  is finitely continuous generalized pseudomonotone, we may apply (i) of Theorem 4.3.1 to conclude that the problem  $VIP(T + S + \varepsilon J_\psi, K \cap \overline{B_{R_\varepsilon}(0)}, f^*)$  is solvable in  $D(T) \cap K \cap \overline{B_{R_\varepsilon}(0)}$ . Since  $0 \in K \cap \overline{B_{R_\varepsilon}(0)}$ , (4.3.9) implies that the problem  $VIP(T + S + \varepsilon J_\psi, K \cap \overline{B_{R_\varepsilon}(0)}, f^*)$  has no solution in  $D(T) \cap K \cap \partial B_{R_\varepsilon}(0)$ , i.e. the problem  $VIP(T + S + \varepsilon J_\psi, K \cap \overline{B_{R_\varepsilon}(0)}, f^*)$  is solvable in  $D(T) \cap K \cap B_{R_\varepsilon}(0)$ . Thus, using Lemma 4.2.2, we get the solvability of the problem  $VIP(T + S, K, f^*)$  in  $D(T) \cap K \cap B_{R_\varepsilon}(0)$ , i.e. for  $\varepsilon_n \downarrow 0^+$  there exist  $x_n \in D(T) \cap K \cap B_{R_{\varepsilon_n}}(0)$ ,  $w_n^* \in Sx_n$ , and  $z_n^* \in Tx_n$  such that

$$\langle z_n^* + w_n^* + \varepsilon_n J_\psi x_n - f^*, x - x_n \rangle \geq 0 \tag{4.3.10}$$

for all  $x \in K$  and all  $n$ . Equivalently, there exists  $v_n^* \in \partial I_K(x_n)$  such that

$$v_n^* + z_n^* + w_n^* + \varepsilon_n J_\psi x_n = f^* \tag{4.3.11}$$

for all  $n$ . Since  $u_0 \in K$ , we obtain from (4.3.10)

$$\begin{aligned} -\beta(\|x_n\|)\|x_n\| &\leq \langle z_n^* + w_n^* - f^* - g^*, x_n - u_0 \rangle \\ &\leq -\varepsilon_n \psi(\|x_n\|)(\|x_n\| - \|u_0\|) - \langle g^*, x_n - u_0 \rangle. \end{aligned}$$

If the sequence  $\{x_n\}$  is unbounded, then  $\|x_n\| \geq \|u_0\|$  for all large  $n$  and

$$\langle g^*, x_n \rangle \leq \langle g^*, u_0 \rangle + \beta(\|x_n\|)\|x_n\|$$

for all large  $n$ . Therefore, by Lemma 4.1.4, the sequence  $\{x_n\}$  is bounded. Since  $0 \in T(0)$ , we have  $\langle w_n^*, x_n \rangle \geq -\psi(\|x_n\|)$  and the boundedness of  $\{v_n^*\}$  follows from (4.3.11). Using a similar argument, the boundedness of the sequence  $\{z_n^*\}$  follows from the fact that  $T$  is strongly quasibounded. Consequently, we have the boundedness of the sequence  $\{w_n^*\}$ . Assume w.l.o.g that  $x_n \rightharpoonup x_0 \in K$ ,  $v_n^* \rightharpoonup v_0^*$ ,  $w_n^* \rightharpoonup w_0^*$  and  $z_n^* \rightharpoonup z_0^*$  as  $n \rightarrow \infty$ . Since  $S$  is generalized pseudomonotone, it is easy to see that

$$\liminf_{n \rightarrow \infty} \langle w_n^*, x_n - x_0 \rangle \geq 0,$$

which implies

$$\limsup_{n \rightarrow \infty} \langle v_n^* + z_n^*, x_n - x_0 \rangle \leq 0.$$

Since  $0 \in \overset{\circ}{K} \cap D(T)$ , we see that  $\partial I_K + T$  is maximal monotone. Thus, by Lemma 1.2.5, we have  $x_0 \in D(T) \cap K$ ,  $v_0^* + z_0^* \in (\partial I_K + T)(x_0)$  and  $\langle v_n^* + z_n^*, x_n \rangle \rightarrow \langle v_0^* + z_0^*, x_0 \rangle$  as  $n \rightarrow \infty$ . Consequently, (4.3.11) implies

$$\lim_{n \rightarrow \infty} \langle w_n^*, x_n - x_0 \rangle = 0.$$

The generalized pseudomonotonicity of  $S$  implies that  $w_0^* \in Sx_0$  and  $\langle w_n^*, x_n \rangle \rightarrow \langle w_0^*, x_0 \rangle$  as  $n \rightarrow \infty$ . Finally, taking the limit as  $n \rightarrow \infty$  in (4.3.11), we conclude that the problem  $VIP(T + S, K, f^*)$  is solvable in  $D(T) \cap K$ . ■

Zhou [74] proved a version of Theorem 4.3.2 with  $D(T) = X$  using the fact that  $T + S + \varepsilon J$  is of type  $(S_+)$  with  $S$  bounded. We remark that Theorem 4.3.2 improves the result of Zhou [74] in that the maximal monotone operator  $T$  may now be just strongly quasibounded with  $D(T) \neq X$ .

### 4.3.2 Maximal monotone perturbations of $(pm_4)$ – generalized pseudomonotone operators

Kenmochi [42] introduced the definition of multivalued operators of type  $(pm_4)$  as follows.

**Definition 4.3.3** An operator  $S : X \rightarrow 2^{X^*}$  is said to satisfy “Condition  $(pm_4)$ ” if for every  $x \in X$  and every bounded subset  $B$  of  $X$  there exists a number  $N(B, x)$  such that

$$\langle y^*, y - x \rangle \geq N(B, x)$$

for all  $(y, y^*) \in G(S)$  with  $y \in B$ .

Kenmochi [42] showed that an operator  $S$  with  $D(S) = X$  which satisfies (i) and (iii) of Definition 1.2.6 and Condition  $(pm_4)$  satisfies also (ii) of Definition 1.2.6, which implies that  $S$  is pseudomonotone. Furthermore, he gave various surjectivity results for perturbations of nonlinear maximal monotone operators.

In this Section we give an existence result for the problem  $VIP(T + S, K, f^*)$ , where  $T$  is maximal monotone and  $S$  is finitely continuous generalized pseudomonotone, possibly unbounded, with  $D(S) = X$  satisfying condition  $(pm_4)$ . The following uniform

boundedness result is important for our consideration.

**Lemma 4.3.4** *Assume that  $S : X \rightarrow 2^{X^*}$  satisfies Condition  $(pm_4)$ . Let  $\{x_n\} \subset X$  be bounded and  $w_n^* \in Sx_n$  be such that, for some  $y_0 \in X$ , the condition*

$$\limsup_{n \rightarrow \infty} \langle w_n^*, x_n - y_0 \rangle < +\infty$$

*is satisfied. Then the sequence  $\{w_n^*\}$  is bounded in  $X^*$ .*

*Proof.* Assume that there exists a real number  $M$  such that

$$\limsup_{n \rightarrow \infty} \langle w_n^*, x_n - y_0 \rangle \leq M.$$

Since  $\{x_n\}$  is bounded, there exists  $R > 0$  such that  $x_n \in B_R(0) := B$  for all  $n$ . Using condition  $(pm_4)$ , we see that for every  $x \in X$  there exists  $N(B, x)$  such that

$$\langle w_n^*, x_n - x \rangle \geq N(B, x)$$

for all  $n$ . Next, for every  $x \in X$  we have

$$\langle w_n^*, y_0 - x \rangle = \langle w_n^*, x_n - x \rangle - \langle w_n^*, x_n - y_0 \rangle$$

for all  $n$ , and hence

$$\liminf_{n \rightarrow \infty} \langle w_n^*, y_0 - x \rangle \geq N(B, x) - M.$$

Given  $x \in X$  and letting  $y_0 - x$  in place of  $x$  above, we know that there exists a number  $N(B, y_0 - x)$  such that

$$\liminf_{n \rightarrow \infty} \langle w_n^*, x \rangle \geq N(B, y_0 - x) - M.$$

Letting  $-x$  in place of  $x$ , there exists a number  $N(B, y_0 + x)$  such that

$$\limsup_{n \rightarrow \infty} \langle w_n^*, x \rangle \leq -N(B, y_0 + x) + M.$$

Therefore, for every  $x \in X$  the sequence  $\{\langle w_n^*, x \rangle\}$  is bounded. By the uniform boundedness principle, it follows that  $\{w_n^*\}$  is bounded. ■

We give the following result for possibly unbounded generalized pseudomonotone perturbations.

**Theorem 4.3.5** *Let  $K$  be a nonempty, closed and convex subset of  $X$  with  $0 \in \overset{\circ}{K}$ . Let  $T : X \supseteq D(T) \rightarrow 2^{X^*}$  be maximal monotone with  $0 \in T(0)$ . Assume that  $S : X \rightarrow 2^{X^*}$  is finitely continuous generalized pseudomonotone which satisfies Condition  $(pm_4)$ . Fix  $f^* \in X^*$ .*

- (i) *If  $K$  is bounded, then the problem  $VIP(T + S, K, f^*)$  is solvable in  $D(T) \cap K$ .*
- (ii) *If  $K$  is unbounded and there exists a bounded open and convex subset  $G$  of  $X$  with  $0 \in G$  such that the problem  $VIP(T + S, K \cap \overline{G}, f^*)$  has no solution in  $D(T) \cap K \cap \partial G$ , then the problem  $VIP(T + S, K, f^*)$  is solvable in  $D(T) \cap K \cap G$ .*
- (iii) *Suppose that  $G$  is a bounded, open and convex subset of  $X$  with  $0 \in G$  and there exists  $u_0 \in \overline{G}$  such that*

$$\langle v^* + w^* - f^*, x - u_0 \rangle > 0$$

*for all  $x \in D(T) \cap \partial G$ ,  $v^* \in Tx$  and  $w^* \in Sx$ . Then the inclusion  $Tx + Sx \ni f^*$  is solvable in  $D(T) \cap G$ .*

- (iv) *Suppose that either  $K$  is bounded or the hypothesis in (ii) holds. If the inclusion*

$$\partial I_K(x) + Tx + Sx \ni f^*$$

has no solution in  $D(T) \cap \partial K$ , then the inclusion  $Tx + Sx \ni f^*$  is solvable in  $D(T) \cap \overset{\circ}{K}$ .

*Proof.* (i) Let  $K$  be bounded. Since  $T_t + S$  is finitely continuous, we follow the finite dimensional argument used in the proof of (i) of Theorem 4.3.1 to conclude that there exist  $x_n \in K$ ,  $w_n^* \in Sx_n$  and  $v_n^* \in \partial I_K(x_n)$  such that

$$v_n^* + T_{t_n}x_n + w_n^* = f^* \quad (4.3.12)$$

for all  $n$ . Note that the above conclusion requires only the finite continuity of  $T_t + S$  for each  $t > 0$ . Since  $0 \in T(0)$ , it follows that  $T_{t_n}(0) = 0$  for all  $n$ . Since  $0 \in K$ , we have  $\langle v_n^*, x_n \rangle \geq 0$  for all  $n$  and

$$\limsup_{n \rightarrow \infty} \langle w_n^*, x_n \rangle \leq N,$$

where  $N$  is an upper bound for the sequence  $\{\|f^*\|\|x_n\|\}$ . Applying Lemma 4.3.4 with  $y_0 = 0$ , we conclude that the sequence  $\{w_n^*\}$  is bounded. Furthermore, we see that  $\langle v_n^*, x_n \rangle \leq M$  where  $M$  is upper bound for the sequence  $\{(\|w_n^*\| + \|f^*\|)\|x_n\|\}$ . Since  $\partial I_K$  is strongly quasibounded, the sequence  $\{v_n^*\}$  is bounded, and hence the sequence  $\{T_{t_n}x_n\}$  is bounded. Assume there exist subsequences, denoted again by  $\{x_n\}$ ,  $\{w_n^*\}$  and  $\{T_{t_n}x_n\}$ , respectively, such that  $x_n \rightharpoonup x_0 \in K$ ,  $w_n^* \rightharpoonup w_0^*$ ,  $v_n^* \rightharpoonup v_0^*$  and  $T_{t_n}x_n \rightharpoonup z_0^*$  as  $n \rightarrow \infty$ . Since  $S$  is generalized pseudomonotone and  $\partial I_K$  is monotone, we have

$$\liminf_{n \rightarrow \infty} \langle w_n^*, x_n - x_0 \rangle \geq 0$$

and

$$\liminf_{n \rightarrow \infty} \langle v_n^*, x_n - x_0 \rangle \geq 0.$$



Let  $J_{t_n}$  be the Yosida resolvent of  $T$ . We know that  $J_{t_n}x_n \in D(T)$ ,  $J_{t_n}x_n = x_n - t_n J^{-1}(T_{t_n}x_n)$  and  $x_n - J_{t_n}x_n \rightarrow 0$  and  $J_{t_n}x_n \rightarrow x_0$  as  $n \rightarrow \infty$ . From this we obtain

$$\limsup_{n \rightarrow \infty} \langle T_{t_n}x_n, J_{t_n}x_n - x_0 \rangle \leq 0.$$

Using Lemma 1.2.5, we get  $x_0 \in D(T)$ ,  $v_0 \in Tx_0$  and  $\langle T_{t_n}x_n, x_n \rangle \rightarrow \langle z_0^*, x_0 \rangle$  as  $n \rightarrow \infty$ .

On the other hand, we have

$$\limsup_{n \rightarrow \infty} \langle w_n^*, x_n - x_0 \rangle \leq 0.$$

Since  $S$  is generalized pseudomonotone,  $w_0^* \in Sx_0$  and  $\langle w_n^*, x_n \rangle \rightarrow \langle w_0^*, x_0 \rangle$  as  $n \rightarrow \infty$ .

Following a similar argument and Lemma 1.2.5, we see that the maximality of  $\partial I_K$  implies  $v_0^* \in \partial I_K(x_0)$ . Finally, taking the limit as  $n \rightarrow \infty$  in (4.3.12), we obtain

$$v_0^* + z_0^* + w_0^* = f^*.$$

This shows that the problem  $VIP(T + S, K, f^*)$  is solvable in  $D(T) \cap K$ .

(ii) Suppose (ii) holds. The conclusion follows via Lemma 4.2.2.

(iii) Using the hypothesis in (iii), we see that the problem  $VIP(T + S, \bar{G}, f^*)$  has no solution in  $D(T) \cap \partial G$ . Then, by using (ii),  $X$  instead of  $K$ , we obtain that  $VIP(T + S, X, f^*)$  is solvable in  $D(T) \cap G$ , i.e. there exists  $x_0 \in D(T) \cap G$ ,  $v_0^* \in Tx_0$  and  $w_0^* \in Sx_0$  such that  $\langle u_0^* + w_0^* - f^*, x - x_0 \rangle \geq 0$  for all  $x \in X$ . Setting  $x + x_0$  in place of  $x$ , we get  $\langle u_0^* + w_0^* - f^*, x \rangle \geq 0$ . Similarly, letting  $-x + x_0$  in place of  $x$ , we obtain  $\langle u_0^* + w_0^* - f^*, x \rangle \leq 0$ . Combining these, we conclude that  $u_0^* + w_0^* = f^*$ .

(iv) Suppose the hypothesis in (iv) holds. Using either (i) or (ii), we see that  $VIP(T + S, K, f^*)$  is solvable in  $D(T) \cap K$ , which is equivalent to the solvability of the inclusion

$$\partial I_K(x) + Tx + Sx \ni f^*$$

in  $D(T) \cap K$ . Since  $\partial I_K(x) + Tx + Sx \ni f^*$  has no solution in  $D(T) \cap \partial K$  and  $\partial I_K(x) = \{0\}$  for all  $x \in \overset{\circ}{K}$ , we conclude that the inclusion  $Tx + Sx \ni f^*$  is solvable in  $D(T) \cap \overset{\circ}{K}$ . ■

We also note that Le [50] proved (iii) of Theorem 4.3.5 for a bounded pseudomonotone operator  $S$  and  $B_R(0)$  instead of a bounded, open and convex subset  $G$ . Since every bounded pseudomonotone operator trivially satisfies the Condition  $(pm_4)$ , (iii) of Theorem 4.3.5 improves the result of Le [50]. Furthermore, Figueiredo [26] proved (iv) of Theorem 4.3.5 with  $T = 0$ ,  $K = \overline{B_R(0)}$ , for some  $R > 0$ ,  $S$  is pseudomonotone with  $D(S) = X$  and  $\lambda J$ , for all  $\lambda > 0$ , instead of  $\partial I_K$ . Kenmochi [42] improved the result of Figueiredo [26], for a pseudomonotone mapping  $S$  with  $D(S) = X$ , by assuming a Leray-Schauder-type condition with  $\partial I_K$  in place of  $\lambda J$  for all  $\lambda > 0$ . Asfaw and Kartsatos [3] proved (iv) of Theorem 4.3.5 with  $S$  bounded and using  $K = \overline{B_R(0)}$ ,  $\lambda J$ ,  $\lambda > 0$ , instead of  $\partial I_K$ . For related results, the reader is also referred to Kartsatos and Quarcoo [38, Theorem 4] and Kartsatos and Skrypnik [39, Theorem 5.8].

We now give the following surjectivity result.

**Corollary 4.3.6** *Let  $T : X \supseteq D(T) \rightarrow 2^{X^*}$  be maximal monotone with  $0 \in D(T)$ . Let  $S : X \rightarrow 2^{X^*}$  be finitely continuous generalized pseudomonotone. Assume that  $S$  satisfies Condition  $(pm_4)$  and*

$$\inf_{w^* \in Sx, z^* \in Tx} \frac{\langle z^* + w^*, x \rangle}{\|x\|} \rightarrow \infty$$

as  $\|x\| \rightarrow \infty$ . Then  $R(T + S) = X^*$ .

*Proof.* By the coercivity condition on  $T + S$ , there exists  $R = R(f^*) > 0$  such that  $\langle v^* + z^* + w^* - f^*, x \rangle > 0$  for all  $x \in D(T) \cap \partial B_R(0)$ ,  $v^* \in \partial I_{\overline{B_R(0)}}(x)$ ,  $w^* \in Sx$  and

$z^* \in Tx$ . This says that the inclusion

$$\partial I_{\overline{B_R(0)}}(x) + Tx + Sx \ni f^*$$

has no solution in  $D(T) \cap \partial B_R(0)$ . Using Theorem 4.3.5, we conclude that  $Tx + Sx \ni f^*$  is solvable in  $D(T) \cap B_R(0)$ . Since  $f^*$  is arbitrary,  $T + S$  is surjective. ■

We remark that Corollary 4.3.6 extends some results of Kenmochi [42] to unbounded generalized pseudomonotone perturbations of maximal monotone operators  $T$  with  $0 \in D(T)$ .

### 4.3.3 Maximal monotone perturbations of regular generalized pseudomonotone operators

In this Subsection we give a result concerning the existence of a solution for a variational problem involving possibly unbounded regular generalized pseudomonotone perturbations of maximal monotone operators. We cite Browder and Hess [18] for properties and range results for single regular generalized pseudomonotone operators as well as their perturbations by maximal monotone operators. It is proved in [18, Theorem 4, p. 272] that a pseudomonotone operator  $S$  with  $D(S) = X$  is regular if there exists  $k > 0$  satisfying the condition  $\langle w^*, x \rangle \geq -k\|x\|$  for all  $x \in X$  and  $w^* \in Sx$ . Browder and Hess [18, Theorem 8, p. 283] proved that the sum  $T + S$  is regular generalized pseudomonotone provided that  $T$  is strongly quasibounded maximal monotone with  $0 \in D(T)$  and  $S$  is regular generalized pseudomonotone with  $D(S) = X$  satisfying  $\langle w^*, x \rangle \geq -k\|x\|$  for all  $x \in X$ ,  $w^* \in Sx$  and some  $k > 0$ . A variational inequality result for single coercive regular generalized pseudomonotone operator may be found in Browder and Hess [18, Theorem 14, p. 288]. Kenmochi [42, Theorem 4.1, p. 254] studied the solvability of variational inequality problems of the type  $VIP(S, K, \phi, f^*)$ , where  $S$  is a multivalued pseudomonotone operator

satisfying Condition  $(pm_4)$  and  $\phi$  is proper, convex and lower semicontinuous, using coercivity-type assumptions involving  $S$  and  $\phi$ .

**Theorem 4.3.7** *Let  $K$  be a nonempty, closed and convex subset of  $X$  with  $0 \in \overset{\circ}{K}$ . Let  $T : X \supseteq D(T) \rightarrow 2^{X^*}$  be maximal monotone with  $0 \in D(T)$  and  $S : X \rightarrow 2^{X^*}$  regular generalized pseudomonotone satisfying Condition  $(pm_4)$ . Let  $\phi : X \rightarrow (\infty, \infty]$  be proper convex lower semicontinuous with  $D(\phi) = K$ . Assume, further, that there exists  $u_0 \in D(T) \cap \overset{\circ}{K}$  such that*

$$\inf_{w^* \in Sx} \frac{\langle w^*, x - u_0 \rangle}{\|x\|} \rightarrow \infty$$

as  $\|x\| \rightarrow \infty$ . Then for every  $f^* \in X^*$ , the problem  $VIP(T + S, K, \phi, f^*)$  is solvable in  $D(T) \cap K$ . Furthermore,  $Tx + Sx \ni f^*$  is solvable in  $D(T)$  provided that  $K = X$  and  $\phi = 0$  on  $X$ .

*Proof.* Let  $A = \partial\phi$ . Using Barbu [5, Proposition 1.6, p. 9], we know that  $\overline{D(A)} = K$  and  $\overset{\circ}{K} \subseteq D(A)$ . We first show that  $VIP(A + T + S, K, f^*)$  is solvable in  $D(A) \cap D(T)$ , i.e. the inclusion  $Ax + Tx + Sx \ni f^*$  is solvable in  $D(A) \cap D(T)$ . Since  $0 \in \overset{\circ}{K} \subseteq D(A)$  and  $0 \in D(T)$ , we see that  $0 \in \overset{\circ}{D(A)} \cap D(T)$ . Hence,  $B = A + T$  is maximal monotone operator. Let  $B_t$  be the Yosida approximant of  $B$  for  $t > 0$ , and  $\tilde{J}x = J(x - u_0)$ ,  $x \in X$ . Since the operator  $B_t + \tilde{J}$  is smooth for all  $t > 0$  and  $\varepsilon > 0$  and  $S$  is regular, it follows that the operator  $B_t + S + \varepsilon\tilde{J}$  is surjective for all  $t > 0$  and  $\varepsilon > 0$ . Thus, for any  $f^* \in X^*$  and every sequence  $t_n \downarrow 0^+$  and  $\varepsilon_n \downarrow 0^+$ , there exist  $x_n \in X$  and  $w_n^* \in Sx_n$  such that

$$B_{t_n}x_n + w_n^* + \varepsilon_n\tilde{J}x_n = f^* \tag{4.3.13}$$

for all  $n$ . Since  $u_0 \in \overset{\circ}{K} \cap D(T) \subseteq D(B)$ , we use the monotonicity of  $B$  and the fact that  $\|B_{t_n} u_0\| \leq |Bu_0|$  for all  $n$  to arrive at

$$\langle w_n^*, x_n - u_0 \rangle \leq \|f\| \|x_n\| + |Bu_0| \|x_n\| + (\|f\| + |Bu_0|) \|u_0\|$$

for all  $n$ , where  $|Bu_0| = \inf\{\|x^*\| : x^* \in Bu_0\}$ . The sequence  $\{x_n\}$  is bounded. Otherwise, we get the contradiction

$$\lim_{\|x_n\| \rightarrow \infty} \frac{\langle w_n^*, x_n - u_0 \rangle}{\|x_n\|} = \infty \leq \|f^*\| + |Bu_0|.$$

As a result, we have

$$\limsup_{n \rightarrow \infty} \langle w_n^*, x_n - u_0 \rangle < +\infty.$$

Since  $\tilde{J}$  is bounded, the sequence  $\{\tilde{J}x_n\}$  is bounded. Since  $S$  satisfies Condition  $(pm_4)$ , from Lemma 4.3.4, we conclude the boundedness of the sequence  $\{w_n^*\}$ . The boundedness of  $\{B_{t_n} x_n\}$  follows from (4.3.13). Let  $v_n^* = B_{t_n} x_n$  and assume that  $x_n \rightharpoonup x_0$ ,  $w_n^* \rightharpoonup w_0^*$  and  $v_n^* \rightharpoonup v_0^*$  as  $n \rightarrow \infty$ . Using the operators  $S = 0$  on  $X$  and  $B$  in place of  $T$  in Lemma 4.1.1, we conclude that  $\lim_{n \rightarrow \infty} \langle B_{t_n} x_n, x_n - x_0 \rangle \geq 0$ . Using this, the monotonicity of  $\tilde{J}$  and (4.3.13), we get

$$\limsup_{n \rightarrow \infty} \langle w_n^*, x_n - x_0 \rangle \leq 0.$$

Since  $S$  is generalized pseudomonotone,  $w_0^* \in Sx_0$  and  $\langle w_n^*, x_n \rangle \rightarrow \langle w_0^*, x_0 \rangle$  as  $n \rightarrow \infty$ .

Thus, we get

$$\limsup_{n \rightarrow \infty} \langle B_{t_n} x_n, x_n - x_0 \rangle = 0.$$

Applying Lemma 1.2.5, it follows that  $x_0 \in D(B) = D(T) \cap D(A) \subseteq D(T) \cap K$  and  $v_0^* \in Bx_0$ . Finally, taking the limit as  $n \rightarrow \infty$  in (4.3.13), we get  $v_0^* + w_0^* = f^*$ . Thus, the problem  $VIP(T + S, K, \phi, f^*)$  is solvable in  $D(T) \cap K$ . Furthermore, if  $K = X$  and  $\phi = 0$  on  $X$ , it is not hard to see that the inclusion  $Tx + Sx \ni f^*$  is solvable in  $D(T)$ . ■

We mention here that Theorem 4.3.7 is a new variational inequality as well as range result for regular generalized pseudomonotone perturbations of maximal monotone operators.

For the sake of completeness, we give the proof of the following range result for the sum  $T+S$  instead of single regular generalized pseudomonotone operator  $S$  considered in Browder and Hess [18, Theorem 11, p. 285].

**Theorem 4.3.8** *Let  $K$  be a nonempty, closed, convex and bounded subset of  $X$  with  $0 \in \overset{\circ}{K}$ . Let  $T : X \supseteq D(T) \rightarrow 2^{X^*}$  be strongly quasibounded maximal monotone with  $0 \in T(0)$  and  $S : X \rightarrow 2^{X^*}$  be regular generalized pseudomonotone such that there exists a real number  $k > 0$  satisfying  $\langle w^*, x \rangle \geq -k\|x\|$  for all  $x \in X$  and  $w^* \in Sx$ . Let  $f^* \in X^*$  be fixed. Assume, further, that*

$$\partial I_K(x) + Tx + Sx \not\ni f^*$$

*for all  $x \in D(T) \cap \partial K$ . Then the inclusion  $Tx + Sx \ni f^*$  is solvable in  $D(T) \cap K$ .*

*Proof.* To complete the proof, it is sufficient to prove that the inclusion

$$\partial I_K(x) + Tx + Sx \ni f^*$$

is solvable in  $D(T) \cap K$ . To this end, we note that  $D(\partial I_K) = K$  and  $0 \in \overset{\circ}{K}$ , and hence  $\partial I_K$  is strongly quasibounded maximal monotone operator. Furthermore, for

each  $t > 0$ , it is not hard to show that  $\partial I_K + T_t$  is strongly quasibounded maximal monotone. Using Browder and Hess [18, Theorem 8, p. 283], we conclude that  $\partial I_K + T_t + S$  is regular generalized pseudomonotone with domain  $K$ , i.e. for each  $t > 0$  and  $\varepsilon > 0$ , the operator  $\partial I_K + T_t + S + \varepsilon J$  is surjective. As a result, for each  $t_n \downarrow 0^+$  and  $\varepsilon_n \downarrow 0^+$ , there are  $x_n \in K$ ,  $v_n^* \in \partial I_K(x_n)$  and  $w_n^* \in Sx_n$  such that

$$v_n^* + T_{t_n}x_n + w_n^* + \varepsilon_n Jx_n = f^* \quad (4.3.14)$$

for all  $n$ . Since  $K$  is bounded, the sequences  $\{x_n\}$  and  $\{\varepsilon_n Jx_n\}$  are bounded. Using (4.3.14), we see that

$$\langle T_{t_n}x_n, x_n \rangle \leq (k + \|f^*\|)\|x_n\| \leq Q$$

for all  $n$ , where  $Q$  is an upper bound for the sequence  $\{(k + \|f^*\|)\|x_n\|\}$ . Since  $T$  is strongly quasibounded, by Lemma 4.1.2, we get the boundedness of the sequence  $\{T_{t_n}x_n\}$ . Using similar argument, it follows that the sequence  $\{v_n^*\}$  is bounded because  $\partial I_K$  is strongly quasibounded maximal monotone with  $0 \in \partial I_K(0)$ . Finally, from (4.3.14), we get the boundedness of the sequence  $\{w_n^*\}$ . Assume that  $x_n \rightharpoonup x_0 \in K$ ,  $w_n^* \rightharpoonup w_0^*$  and  $v_n^* + T_{t_n}x_n \rightharpoonup v_0^*$  as  $n \rightarrow \infty$ . Applying Lemma 4.1.1, we obtain that

$$\liminf_{n \rightarrow \infty} \langle v_n^* + T_{t_n}x_n, x_n - x_0 \rangle \geq 0. \quad (4.3.15)$$

As a consequence, using (4.3.14), we conclude that

$$\limsup_{n \rightarrow \infty} \langle w_n^*, x_n - x_0 \rangle \leq 0. \quad (4.3.16)$$

The generalized pseudomonotonicity of  $S$  gives that  $w_0^* \in Sx_0$  and  $\langle w_n^*, x_n \rangle \rightarrow \langle w_0^*, x_0 \rangle$  as  $n \rightarrow \infty$ . Finally, combining (4.3.14) and (4.3.15), we conclude that

$$\lim_{n \rightarrow \infty} \langle v_n^* + T_{t_n}x_n, x_n - x_0 \rangle = 0.$$

Consequently, using Lemma 1.2.5, we obtain  $x_0 \in D(T) \cap K$  and  $v_0^* \in (\partial I_K + T)(x_0)$ . In conclusion, letting  $n \rightarrow \infty$  in (4.3.14), we get that  $v_0^* + w_0^* = f^*$ , which implies the solvability of the inclusion

$$Tx + Sx + \partial I_K(x) \ni f^*$$

in  $D(T) \cap K$ . Since this inclusion has no solution in  $D(T) \cap \partial K$  and  $\partial I_K(x) = \{0\}$  for all  $x \in \overset{\circ}{K}$ , we conclude that  $x_0 \in D(T) \cap \overset{\circ}{K}$  solves the inclusion  $Tx + Sx \ni f^*$ . ■

We note that Theorem 4.3.8 is an extension of Browder and Hess [18, Theorem 11, p. 285] for the sum  $T + S$  in place of  $S$ , and the fact that we have used Leray-Schauder condition involving  $\partial I_K$  for any nonempty, closed, convex and bounded subset  $K$  of  $X$  instead of  $\lambda J$  for all  $\lambda > 0$ .

#### 4.4 Applications for parabolic partial differential equations

In this Section we give examples of multivalued pseudomonotone and maximal monotone operators. To demonstrate the applicability of the theory, we prove existence of weak solution(s) of a parabolic differential equation. Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$  with smooth boundary,  $p, p'$  such that  $1 < p < \infty$  and  $\frac{1}{p} + \frac{1}{p'} = 1$ , and  $X = W_0^{1,p}(\Omega)$ . For every  $i = 1, 2, \dots, N$ , the function  $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$  satisfies the following conditions.

- (A1)  $a_i(x, s, \xi)$  satisfies the Carathéodory conditions, i.e. it is measurable in  $x \in \Omega$  for all  $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ , and continuous in  $(s, \xi)$  a.e. w.r.t.  $x \in \Omega$ . Furthermore, there exist constants  $c_0 > 0$  and  $k_0 \in L^q(\Omega)$  such that

$$|a_i(x, s, \xi)| \leq k_0(x) + c_0(|s|^{p-1} + |\xi|^{p-1})$$

a.e. for  $x \in \Omega$ , and for all  $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ , where  $|\xi|$  denotes the norm of  $\xi$  in



$\mathbb{R}^N$ .

(A2) The functions  $a_i$  satisfy a monotonicity condition with respect to  $\xi$  in the form

$$\sum_{i=1}^N (a_i(x, s, \xi) - a_i(x, s, \xi'))(\xi - \xi') > 0$$

for a.e.  $x \in \Omega$ , and all  $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ .

(A3) There exists  $c_1 > 0$  and a function  $k_1 \in L^1(\Omega)$  such that

$$\sum_{i=1}^N a_i(x, s, \xi)\xi_i \geq -k_1(x)$$

for a.e.  $x \in \Omega$  and all  $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ .

We consider a second-order elliptic differential operator of the form

$$Au(x) = - \sum_{i=1}^n \frac{\partial}{\partial x_i} a_i(x, u, \nabla u(x)) dx, \quad x \in \Omega, \quad u \in X, \quad \nabla u = \left( \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N} \right).$$

The operator  $A$  generates an operator  $\tilde{A} : X \rightarrow X^*$  given by

$$\langle \tilde{A}u, \varphi \rangle = \int_{\Omega} \sum_{i=1}^N a_i(x, u, \nabla u) \frac{\partial \varphi}{\partial x_i} dx, \quad u \in X, \varphi \in X,$$

where  $\langle \cdot, \cdot \rangle$  denotes the duality pairing between  $X$  and  $X^*$ . It is well known that under the conditions (A1) -(A3) the operator  $\tilde{A}$  is bounded, continuous and pseudomonotone.

For the function  $j : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ , we assume the following conditions.

(J1) the function  $x \rightarrow j(x, s)$  is measurable in  $\Omega$  for all  $s \in \mathbb{R}$ , and  $s \rightarrow j(x, s)$  is locally Lipschitz continuous a.e.  $x \in \Omega$

(J2) Let  $\partial j(x, s)$  denote Clarke's generalized gradient of the function  $s \rightarrow j(x, s)$

given by

$$\partial j(x, s) = \{\xi \in \mathbb{R} : j^0(x, s; r) \geq \xi r\}$$

for all  $r \in \mathbb{R}$ , for a.e.  $x \in \Omega$ , where  $j^0(x, s; r)$  is the generalized directional derivative of the function  $s \rightarrow j(x, s)$  at  $s$  in the direction  $r$ , given by

$$j^0(x, s; r) = \limsup_{y \rightarrow s, t \downarrow 0} \frac{j(x, y + tr) - j(x, y)}{t}.$$

Assume, further, that there exist  $c > 0$ ,  $q \in [p, p^*]$  and  $k \in L^q(\Omega)$  such that

$$\eta \in \partial j(x, s) : |\eta| \leq k(x) + c|s|^{q-1}$$

for a.e.  $x \in \Omega$  and all  $s \in \mathbb{R}$ , where  $p^*$  denotes the critical Sobolev exponent with  $p^* = \frac{Np}{N-p}$  if  $p < N$  and  $p^* = \infty$  if  $p \geq N$ .

Let  $\tilde{J} : L^q(\Omega) \rightarrow \mathbb{R}$  be defined by

$$\tilde{J}(u) = \int_{\Omega} j(x, u(x)) dx.$$

By (J1) and (J2),  $\tilde{J}$  is well defined and Lipschitz continuous on bounded subsets of  $L^q(\Omega)$ . Moreover, Clarke's generalized gradient of  $\tilde{J}$ ,  $\partial \tilde{J} : L^q(\Omega) \rightarrow 2^{L^q(\Omega)}$ , is well defined and characterized by, for each  $u \in L^q(\Omega)$ ,

$$\eta \in \partial \tilde{J}(u) \Rightarrow \eta \in L^q(\Omega), \quad \eta(x) \in \partial j(x, u(x))$$

for a.e.  $x \in \Omega$ . Let  $i : X \hookrightarrow L^q(\Omega)$  be the natural embedding and  $i^* : L^q(\Omega) \hookrightarrow X^*$  the adjoint of  $i$ . Let  $S : X \rightarrow 2^{X^*}$  be defined by

$$Su = (i^* \circ \partial \tilde{J} \circ i)(u), \quad u \in X.$$

Carl and Motreanu [24, Lemma 3.1, p. 1109] showed that the operator  $S$  is bounded and pseudomonotone. By a result of Browder and Hess [18, Proposition 9, p. 267], the operator  $\tilde{A} + S : X \rightarrow 2^{X^*}$  is also bounded and pseudomonotone. The theory developed in this paper may be applied in the solvability of variational inequalities as well as inclusion problems for operators of the type  $T + \tilde{A} + S$ , where  $T : X \supseteq D(T) \rightarrow 2^{X^*}$  is an arbitrary maximal monotone operator, by using either inner product or Leray-Schauder conditions.

**Example 1:** We demonstrate the applicability of the theory for the parabolic differential equation given by

$$\left\{ \begin{array}{ll} \frac{\partial u}{\partial t} - \Delta_p u = f & \text{in } (0, T) \times \Omega \\ f = f_1 + f_2 \quad -f_1(t, x) \in \partial j(t, x) \text{ a.e.} & (0, T) \times \Omega \\ u(0, x) = u_0(x) & x \in \Omega \\ u(t, x) = 0 & \text{in } (0, T) \times \partial\Omega \end{array} \right. \quad (4.4.17)$$

where  $\Omega$  is bounded open subset of  $\mathbb{R}^N$  with smooth boundary,  $j : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is locally Lipschitz function satisfying  $(J_1)$  and  $(J_2)$  and  $\partial j(x, u)$  is the Clarke's generalized gradient of  $j$  and  $f_2$  is given function. We mention here that (4.4.17) is a model for nonmonotone semipermeability problem which arises in electrostatics, heat conduction and flows in porous media. For detailed applications of the model to mechanics, engineering and economics, the reader is referred to the book of Naniewicz and Panagiotopoulos [61, pp,126-192].

**Formulation of the problem:** We consider  $2 \leq p < \infty$ ,  $V = W^{1,p}(\Omega)$ ,  $X = L^p(0, T; V)$ ,  $H = L^2(\Omega)$  and  $X^* = L^q(0, T; V^*)$  where  $\frac{1}{p} + \frac{1}{q} = 1$ . Then we see that

$V \subseteq H \subseteq V^*$ . For  $u \in X$  and  $v^* \in X^*$ , the norm of  $u$  and  $v^*$  is given by

$$\|u\|_X^p = \int_0^T \|u(t)\|_V^p dt \text{ and } \|v^*\|_{X^*}^q = \int_0^T \|v^*(t)\|_{V^*}^q dt.$$

Let  $L : X \supseteq D(L) \rightarrow X^*$  be defined by

$$Lu = u', \tag{4.4.18}$$

where  $u'$  is understood in the sense of distributions and  $D(L) = \{u \in X : u' \in X^*, u(0) = u_0\}$ . Let  $A : X \rightarrow X^*$  be given by

$$\langle Au, v \rangle = \int_0^T \int_{\Omega} |\nabla u(t, x)|^{p-2} \cdot \nabla u(t, x) \cdot \nabla v(t, x) dx dt, \quad u \in X, v \in X. \tag{4.4.19}$$

Furthermore, we let  $J : X \rightarrow \mathbb{R}$  defined by

$$J(u) = \int_0^T \int_{\Omega} j(x, u(x, t)) dx dt, \quad u \in X.$$

Since  $j$  is locally Lipschitz, it follows that  $J : X \rightarrow \mathbb{R}$  is locally Lipschitz mapping, i.e. the Clarke generalized gradient of  $J$ , denoted by  $\partial J : X \rightarrow 2^{X^*}$  is well defined and given by

$$\partial J(u) = \{u^* \in X^* : \overset{\circ}{J}(u; v) \geq \langle u^*, v \rangle, \text{ for all } v \in X\},$$

where  $\overset{\circ}{J}(u; v)$  is the generalized directional derivative of  $J$  at  $u$  in the direction of  $v$ .

The weak formulation of the problem (4.4.17) is given as follows.

**Weak formulation:** Find  $u \in D(L)$  such that

$$\langle Lu + Au, v - u \rangle + \int_0^T \int_{\Omega} \overset{\circ}{j}(x, u; v - u) dx dt \geq \langle f_2, v - u \rangle \tag{4.4.20}$$

for all  $v \in X$ . Furthermore, using the definition of  $\partial J$ , it is easy to see that problem (4.4.20) is equivalent to finding  $u \in X$  such that

$$Lu + Au + \partial J(u) \ni f^*$$

holds, where  $f^* \in X^*$  defined by

$$\langle f^*, v \rangle = \int_0^T \int_{\Omega} f_2(x, t)v(x, t)dxdt, v \in X. \quad (4.4.21)$$

Next, we prove the following theorem.

**Theorem 4.4.1** *Let  $L : X \supseteq D(L) \rightarrow X^*$  and  $A : X \rightarrow X^*$  be as defined in (4.4.18) and (4.4.19) respectively. Let  $\partial J : X \rightarrow 2^{X^*}$  be the Clarke subgradient of  $J$ . Let  $f_2 \in L^q((0, T) \times \Omega)$ . Let  $f^* \in X^*$  given by (4.4.21). Assume, further, that the following conditions hold.*

(A) *There exists a nondecreasing function  $c : [0, \infty) \rightarrow [0, \infty)$  such that*

$$\overset{\circ}{J}(v; -v) \leq -c(\|v\|_X)\|v\|_X, \text{ for all } v \in X.$$

(B) *For any sequence  $\{u_n\}$  in  $X$  such that  $u_n \rightarrow u$  as  $n \rightarrow \infty$  and  $u_n^* \in \partial J(u_n)$ , we have*

$$\limsup_{n \rightarrow \infty} \langle u_n^*, u_n - u \rangle \leq 0 \text{ implies } J(u_n) \rightarrow J(u) \text{ as } n \rightarrow \infty.$$

*Then the problem  $Lu + Au + \partial J(u) \ni f^*$  is solvable in  $D(L)$ .*

*Proof.* Suppose the hypothesis of the theorem hold. We notice here that condition (B) holds if  $J$  is proper, convex, lower-semicontinuous function from  $X$  into  $\mathbb{R}$ . For basic properties of Clarke generalized subgradient of locally Lipschitz functions and sufficient condition(s) for  $\partial J$  to be pseudomonotone, we refer the reader to the book of Naniewicz and Panagiotopoulos [61]. According to [61, Proposition 2.19, p.59],

hypothesis (B) implies that  $\partial J : X \rightarrow 2^{X^*}$  is pseudomonotone. Furthermore, it is well known that  $A : X \rightarrow X^*$  is bounded, continuous and maximal monotone and  $L : X \supseteq D(L) \rightarrow X^*$  is densely defined linear maximal monotone. For details about the operators  $A$  and  $L$ , the reader is referred to the book of Zeidler [72, pp.354-918]. Using the definition of the subdifferential of  $J$  and applying condition (A), we see that

$$\langle u^*, u \rangle \geq c(\|u\|_X \|u\|_X)$$

for all  $u \in D(L)$  and  $u^* \in \partial J(u)$ . Since  $L$  is linear, i.e.  $L(0) = 0$  and using the definition of  $A$  and  $f^* \in X^*$  as given by (4.4.21), using the Hölder inequality, we see that

$$\begin{aligned} \langle Lu + Au + u^* - f^*, u \rangle &\geq \int_0^T \|\nabla u(t)\|_{L^p(\Omega)}^p dt - \left( \int_0^T \|u(t)\|_{L^p(\Omega)}^p dt \right)^{\frac{1}{p}} \left( \int_0^T \|f_2(t)\|_{L^q(\Omega)}^q dt \right)^{\frac{1}{q}} \\ &\quad + c(\|u\|_X) \|u\|_X \\ &\geq c(\|u\|_X) \|u\|_X - k_0 \|u\|_X \rightarrow +\infty \end{aligned}$$

as  $\|u\|_X \rightarrow \infty$ . Thus, there exists  $R = R(f^*) > 0$  such that

$$\langle Lu + Au + u^* - f^*, u \rangle > 0$$

for all  $u \in D(L) \cap \partial B_R(0)$  and  $u^* \in \partial J(u)$ . We notice here that  $\partial J : X \rightarrow 2^{X^*}$  is bounded pseudomonotone. Furthermore, we observe that  $L+A$  is maximal monotone. Therefore, by using Theorem 4.2.3, we conclude that  $VIP(L + A + \partial J, X, f^*)$  is solvable, i.e. there exists  $u \in D(L)$  such that  $Lu + Au + \partial J(u) \ni f^*$ . Therefore,  $u \in D(L)$  satisfies

$$\langle Lu + Au, v - u \rangle + \int_0^T \int_{\Omega} \overset{\circ}{j}(x, u; v - u) dx dt \geq \langle f_2, v - u \rangle$$

for all  $v \in X$ . The proof is complete. ■

We mention here that, according to Theorem 4.2.5, the solvability of a more general inclusion problem of the type  $\partial\phi(u) + Lu + Au + \partial J(u) \ni f^*$  can be treated, where  $\phi : X \rightarrow \mathbb{R}$  is proper, convex and lower-semicontinuous function.

## REFERENCES

- [1] D.R. Adhikari and A.G. Kartsatos, Topological degree theories and nonlinear operator equations in Banach spaces, *Nonl. Anal.* **69** (2008), 1235–1255.
- [2] D.R. Adhikari and A.G. Kartsatos, Strongly quasibounded maximal monotone perturbations for the Berkovits-Mustonen topological degree theory, *J. Math. Anal. Appl.* **348** (2008), 122–136.
- [3] T.M. Asfaw, A.G. Kartsatos, A Browder topological degree theory for multivalued pseudomonotone perturbations of maximal monotone operators in reflexive Banach spaces, *Adv. Math. Sci. Appl.* **22** (2012), 91-148.
- [4] V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Noordhoff Int. Publ., Leyden (The Netherlands), 1975.
- [5] V. Barbu, *Nonlinear Differential Equations of Monotone Type in Banach Spaces*, Springer Monographs in Math., 2010.
- [6] J. Berkovits, On the degree theory for nonlinear mappings of monotone type, *Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes* **58** (1986), 58 pp.
- [7] J. Berkovits and M. Miettunen, On the uniqueness of the Browder degree, *Proc. AMS* **136** (2008), 3467–3476.
- [8] J. Berkovits and V. Mustonen, On the topological degree for mappings of monotone type, *Nonl. Anal.* **10** (1986), 1373–1383.
- [9] H. Brézis, M.G. Crandall and A. Pazy, Perturbations of nonlinear maximal monotone sets in Banach spaces, *Comm. Pure Appl. Math.* **23** (1970) 123–144.



- [10] H. Brézis, Équations et inéquations non-linéaires dans les espaces vectoriels en dualité, *Ann. Inst. Fourier (Grenoble)* **18** (1968), 115–176.
- [11] L. E. J. Brouwer, Über Abbildungen von Mannichfatigkeiten, *Math. Ann.*, **71**(1912), 97-115.
- [12] F.E. Browder, Nonlinear elliptic boundary value problems and the genralized topological degree, *Bull. Amer. Math. Soc.* **76** (1970), 1771–1773.
- [13] F.E. Browder, Nonlinear operators and nonlinear equations of evolution in Banach spaces, *Proc. Sympos. Pure Math.* **18** (1976), 1–308.
- [14] F.E. Browder, Degree of mapping for nonlinear mappings of monotone type, *Proc. Nat. Acad. Sci.* **80** (1983), 1771–1773.
- [15] F.E. Browder, Degree of mapping for nonlinear mappings of monotone type: densely defined mapping, *Proc. Nat. Acad. Sci.* **80** (1983), 2405–2407.
- [16] F.E. Browder, Degree of mappings for nonlinear mappings of monotone type: strongly nonlinear mapping, *Proc. Nat. Acad. Sci.* **80** (1983), 2408–2409.
- [17] F.E. Browder, Fixed point theory and nonlinear problems, *Bull. Amer. Math. Soc.* **9** (1983), 1–39.
- [18] F.E. Browder and P. Hess, Nonlinear mappings of monotone type in Banach spaces, *J. Funct. Anal.* **11** (1972), 251–294.
- [19] F.E. Browder, H. Brézis, Strongly nonlinear parabolic variational inequalities, *Proc. Nat. Acad. Sci.* **77** (1980), 713–715.
- [20] F.E. Browder, Existence and approximation of solutions of nonlinear variational inequalities, *Proc. Nat. Acad. Sci.* **56** (1966), 1080–1086.
- [21] F. E. Browder, On the principle of H.Brézis and its applications, *J. Funct. Anal.* **25** (1977), 356-365.

- [22] S. Carl, V.Y. Le, D. Motreanu, Existence, comparison and compactness results for quasilinear variational-hemivariational inequalities, *Int. J. Math. Math. Sci.* **3** (2005), 401–417.
- [23] S. Carl, Existence and comparison results for nonconvex and nonmonotone multivalued elliptic operators, *Nonl. Anal.* **65** (2006), 1532–1546.
- [24] S. Carl, D. Motreanu, General comparison principle for quasilinear elliptic inclusions, **70** (2009), 1105–1112.
- [25] I. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, Kluwer Acad. Publ., Dordrecht, 1990.
- [26] D.G. De Figueiredo, An existence theorem for pseudo-monotone operator equations, *J. Math. Anal. Appl.* **34** (1971), 151–156.
- [27] K. Deimling, *Nonlinear functional analysis*, Springer-Verlag, New York, 1985.
- [28] P.M. Fitzpatrick,  $A$ -proper mappings and their uniform limits, *Bull. Amer. Math. Soc.* **78** (1972), 806–809.
- [29] Z. Guan and A.G. Kartsatos, Ranges of generalized pseudo-monotone perturbations of maximal monotone operators in reflexive Banach spaces, *Contemp. Math.* **204** (1997), 107–123.
- [30] Z. Guan, A.G. Kartsatos and I.V. Skrypnik, Ranges for densely defined generalized pseudomonotone perturbations of maximal monotone operators, *J. Differential Equations* **188** (2003), 332–351.
- [31] P. Hartman, G. Stampacchia, On some nonlinear elliptic differential functional equations, *Acta Math.* **115** (1966), 153–188.

- [32] B. Ibrahimou and A.G. Kartsatos, The Leray-Schauder approach to the degree theory for  $(S_+)$ -perturbations of maximal monotone operators in separable reflexive Banach spaces, *Nonl. Anal.* **70** (2009), 4350–4368.
- [33] I. Fonesca, W. Gangbo, *Degree theory in analysis and applications*, Oxford University press Inc., New York, 1995.
- [34] A.G. Kartsatos, *Advanced ordinary differential equations*, MANCORP Publishing. Tampa, 1993.
- [35] A.G. Kartsatos, New results in the perturbation theory of maximal monotone and  $m$ -accretive operators in Banach spaces, *Trans. Amer. Math. Soc.* **348** (1996), 1663–1707.
- [36] A.G. Kartsatos, On the connection between the existence of zeros and the asymptotic behavior of resolvents of maximal monotone operators in reflexive Banach spaces, *Trans. Amer. Math. Soc.* **350** (1998), 3967–3987.
- [37] A.G. Kartsatos and J. Lin, Homotopy invariance of parameter-dependent domains and perturbation theory for maximal monotone and  $m$ -accretive operators in Banach spaces, *Adv. Differential Equations* **8** (2003), 129–160.
- [38] A.G. Kartsatos and J. Quarcoo, A new topological degree theory for densely defined  $(S_+)_L$ -perturbations of multivalued maximal monotone operators in reflexive separable Banach spaces, *Nonlinear Anal.* **69** (2008), 2339–2354.
- [39] A.G. Kartsatos and I.V. Skrypnik, Topological degree theories for densely defined mappings involving operators of type  $(S_+)$ , *Adv. Differential Equations* **4** (1999), 413–456.
- [40] A.G. Kartsatos and I.V. Skrypnik, A new topological degree theory for densely defined quasibounded  $(\tilde{S}_+)$ -perturbations of multivalued maximal monotone operators in reflexive Banach spaces, *Abstr. Appl. Anal.* **2** (2005), 121–158.

- [41] A.G. Kartsatos and I.V. Skrypnik, On the eigenvalue problem for perturbed nonlinear maximal monotone operators in reflexive Banach spaces, *Trans. Amer. Math. Soc.* **358** (2005), 3851–3881.
- [42] N. Kenmochi, Nonlinear operators of monotone type in reflexive Banach spaces, and nonlinear perturbations, *Hiroshima Math. J.* **4** (1974), 229–263.
- [43] N. Kenmochi, Pseudomonotone operators and nonlinear elliptic boundary value problems, *J. Math. Soc. Japan* **27** (1975), 121–149.
- [44] N. Kenmochi, Monotonicity and compactness methods for nonlinear variational inequalities, Hand-book of differential equations, **IV**, Elsevier/North-Holland, Amsterdam, (2007), 203–298.
- [45] B.T. Kien, M.-M. Wong and N.-C. Wong, On the degree theory for general mappings of monotone type, *J. Math. Anal Appl.* **340** (2008), 707–720.
- [46] B.T. Kien, M.M. Wong, N.C. Wong, J.C. Yao, Solution existence of variational inequalities with pseudomonotone operators in the sense of Brézis, *J. Optim. Theory Appl.* **140** (2009), 249–263.
- [47] D. Kinderlehrer, G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, *Pure and Applied Mathematics*, **88**, Academic Press, New York, 1980.
- [48] J. Kobayashi and M. Otani, Topological degree for  $(S)_+$ -mappings with maximal monotone perturbations and its applications to variational inequalities, *Nonl. Anal.* **59** (2004), 147–172.
- [49] E. Krauss, A degree for operators of monotone type, *Math. Nachr.* **114** (1983), 53–62.
- [50] V. K. Le, A range and existence theorem for pseudomonotone perturbations of maximal monotone operators, *Proc. Amer. Math. Soc.* **139** (2011), 1645–1658.

- [51] J. Leray and J. Schauder, Topologie et équations fonctionnelles, *Ann. Sci. École Norm. Sup.* **51** (1934), 4–78.
- [52] J. Lions, G. Stampacchia, Variational inequalities, *Comm. Pure Appl. Math.* **20** (1967), 493–519.
- [53] N.G. Lloyd, *Degree Theory*, Cambridge Tracts in Mathematics, **73**, Cambridge Univ. Press, Cambridge-New York-Melbourne, 1978.
- [54] T.W. Ma, Topological degrees of set-valued compact fields in locally convex spaces, *Dissertationes Math. Rozprawy Mat.* **92** (1972), 43 pp.
- [55] J. Mawhin, Leray-Schauder degree: a half a century of extensions and applications, *Topol. Meth. Nonl. Anal.* **14** (1999), 195–228.
- [56] G.J. Minty, Nonlinear monotone operators in Hilbert Space, *Duke Math. J.* **29** (1962), 341–346.
- [57] G.J. Minty, On a “monotonicity” method for the solution of nonlinear equations in Banach spaces, *Proc. Nat. Acad. Sci.* **50** (1963), 1038–1041.
- [58] J.J. Moreau, Principes extrémaux pour le problème de la naissance de la cavitation, *J. Mécanique* **5** (1966), 439–470.
- [59] V. Mustonen, On elliptic operators in divergence form: old and new with applications, Proc. Conf. Function Spaces, Differential Operators and Nonlinear Analysis, FSDONA 2004, 188–200.
- [60] V. Mustonen and M. Tienari, A remark on the Leray-Lions condition, *Nonl. Anal.* **49** (2002), 1015–1022.
- [61] Z. Naniewicz and P.D. Panagiotopoulos, *Mathematical Theory of Hemivariational Inequalities and Applications*, Monographs and Textbooks in Pure and Applied Mathematics, **188**, Marcel Dekker, Inc., New York, 1995.

- [62] D. O'Regan, Y.J. Cho and Y.Q. Chen, *Topological Degree Theory and Applications*, Series in Mathematical Analysis and Applications, **10**, Chapman and Hall/CRC, Boca Raton, FL, 2006.
- [63] D. Pascali and S. Sburlan, *Nonlinear Mappings of Monotone Type*, Sijthoff and Noordhoff, Bucharest, 1978.
- [64] R.T. Rockafellar, On the maximality of sums of nonlinear monotone operators, *Trans. Amer. Math. Soc.* **149** (1970), 75–88.
- [65] R.T. Rockafellar, *Convex Analysis*, Princeton Univ. Press, Princeton, New Jersey, 1970.
- [66] A. Sard, The measure of critical values of differentiable maps, *Bull. Amer. Math. Soc.*, **48** (1942), 883–890.
- [67] I.V. Skrypnik, *Nonlinear Elliptic Equations of Higher Order*, Naukova Dumka, Kiev, 1973.
- [68] I.V. Skrypnik, *Methods for Analysis of Nonlinear Elliptic Boundary Value Problems*, Amer. Math. Soc. Transl., Ser. II, **133**, AMS, Rhode Island, 1994.
- [69] G. Stampacchia, Variational inequalities, *Actes du Congrès International des Mathématiciens (Nice)* (1970), 877–883.
- [70] B.A. Ton, Nonlinear operators on convex subsets of Banach spaces, *Math. Ann.* **181** (1969), 35–44.
- [71] G.-H. Yang, The ranges of nonlinear mappings of monotone type, *J. Math. Anal. Appl.* **173** (1993), 165–172.
- [72] E. Zeidler, *Nonlinear Functional Analysis and Its Applications*, **II/B**, Springer-Verlag, New York, 1990.

- [73] S.-S. Zhang and Y.-Q. Chen, Degree theory for multivalued ( $S$ )-type mappings and fixed point theorems, *Appl. Math. Mech.* **11** (1990), 441–454.
- [74] Y.Y. Zhou, Some results about perturbed maximal monotone mappings, *Comput. Math. Appl.* **51** (2006), 487–496.

## APPENDIX



## Appendix A - Copyright and permissions

Burke, Marilyn <msburke@usf.edu> Jan 23

to Teffera

FYI – you can use it!

*Marilyn S. Burke*

*Reference Librarian*

*Tampa Library*

*University of South Florida*

*4202 E. Fowler Ave.*

*Tampa, FL 33620, [813-974-4561](tel:813-974-4561)*

**From:** Nobuyuki Kenmochi [mailto:[amsa@vmail.plala.or.jp](mailto:amsa@vmail.plala.or.jp)]

**Sent:** Tuesday, January 22, 2013 6:33 PM

**To:** Burke, Marilyn

**Subject:** Re: query

Dear Marilyn Burke

The author can use his (or her) own materials published in AMSA and make some copies for his own university thesis and dissertation (but not for sale).

Sincerely yours

Nobuyuki Kenmochi

The editor-in-chief of AMSA

## ABOUT THE AUTHOR

I was born and raised in a large family in a country side of Ethiopia. I have 6 brothers and 2 sisters. Five of them are university graduates. Three of us studied mathematics. Others are still attending school. I love teaching and studying mathematics. I like to watch history and national geographic channels.