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Stochastic Hybrid Dynamic Systems: Modeling, Estimation and Simulation

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by

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Dedication

To my parents, my brothers and my wife

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Table of Contents

List of Figures

List of Tables

Abstract

Stochastic hybrid dynamic systems that incorporate both continuous and discrete dynamics have been an area of great interest over the recent years. In view of applications, stochastic hybrid dynamic systems have been employed to diverse fields of studies, such as communication networks, air traffic management, and insurance risk models. The aim of the present study is to investigate properties of some classes of stochastic hybrid dynamic systems.

The class of stochastic hybrid dynamic systems investigated has random jumps driven by a non-homogeneous Poisson process and deterministic jumps triggered by hitting the boundary. Its real-valued continuous dynamic between jumps is described by stochastic differential equations of the Itô-Doob type. Existing results of piecewise deterministic models are extended to obtain the infinitesimal generator of the stochastic hybrid dynamic systems through a martingale approach. Based on results of the infinitesimal generator, some stochastic stability results are derived. The infinitesimal generator and stochastic stability results can be used to compute the higher moments of the solution process and find a bound of the solution.

Next, the study focuses on a class of multidimensional stochastic hybrid dynamic systems. The continuous dynamic of the systems under investigation is described by a linear non-homogeneous systems of Itô-Doob type of stochastic differential equations with switching coefficients. The switching takes place at random jump times which are governed by a non-homogeneous Poisson process. Closed form solutions of the stochastic hybrid dynamic systems are obtained. Two important special cases for the above systems are the geometric Brownian motion process with jumps and the Ornstein-Uhlenbeck process with jumps. Based on the closed form solutions, the probability distributions of the solution processes for these two special cases are derived. The derivation em-

ploys the use of the modal matrix and transformations.

In addition, the parameter estimation problem for the one-dimensional cases of the geometric Brownian motion and Ornstein-Uhlenbeck processes with jumps are investigated. Through some existing and modified methods, the estimation procedure is presented by first estimating the parameters of the discrete dynamic and subsequently examining the continuous dynamic piecewisely.

Finally, some simulated stochastic hybrid dynamic processes are presented to illustrate the aforementioned parameter-estimation methods. One simulated insurance example is given to demonstrate the use of the estimation and simulation techniques to obtain some desired quantities.

1 Mathematical Preliminaries

1.1 Introduction

In this chapter, we shall provide a number of basic definitions and important results which are necessary for the work of the later chapters. A stochastic process is a natural model for describing the evolution of real-life dynamic processes and systems in time. In Section 1.2, we first briefly review two prominent stochastic processes, the Poisson process and the Brownian motion process. In Section 1.3, the Itô-Doob type stochastic differential equation is introduced, and the famous Itô formula is stated. Finally, in Section 1.4, we review some essential results of maximum likelihood estimation methods which are useful for estimating the parameters of the stochastic differential equations in the later chapter.

1.2 Stochastic processes

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete probability space equipped with a filtration $(\mathcal{F}_t)_{0 \leq t \leq \infty}$. A filtration is a family of σ-algebras $(\mathcal{F}_t)_{0 \leq t \leq \infty}$ that is increasing, i.e. $\mathcal{F}_s \subset \mathcal{F}_t$ if $s \leq t$. A random variable is an $\mathcal{F}$-measurable function that maps the sample space Ω to $\mathbb{R}^n$.

Definition 1.2.1 *A stochastic process is a collection of random variables with abstract time parameter $\{x(t)\}_{t \in E}$ defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ fulfilling the Kolmogorov's compatibility conditions and assuming values in $\mathbb{R}^n$.*

In most applications the parameter space E represent the time space $[0, \infty)$ or a finite time interval $[t_0, T]$. In the following work, sometimes a stochastic process is simply denoted by x. A stochastic process x is said to be $\mathcal{F}_t$-adapted if $x(t)$ is $\mathcal{F}_t$-measurable for each t. All stochastic processes discussed in this work are assumed to be defined on the complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and $\mathcal{F}_t$-adapted.

1.2.1 Poisson processes

One of the fundamental continuous-time stochastic processes is the Poisson process. Its popularity is mainly due to that it frequently appears in a wealth of physical phenomena and that it is relatively simple to analyze. Poisson processes are counting processes that count the number of events that occur between time 0 and time t, for some $t > 0$. The events of interest vary in applications. For instance, the number of claims filed for a particular insured.

Definition 1.2.2 *A counting process $\{N(t)\}_{t\geq 0}$ is said to be a (non-homogeneous) Poisson process with intensity function λ, where $\lambda(t) \geq 0$ for $t \geq 0$, if it satisfies*

(i) $N(0) = 0$,

(ii) $\{N(t)\}_{t\geq 0}$ *has independent increments,*

(iii) $P(N(t+h) - N(t) = 1) = \lambda(t)h + o(h)$, *for all* $t, h \geq 0$,

(iv) $P(N(t+h) - N(t) = 2) = o(h)$, *for all* $t, h \geq 0$.

The function Λ defined by

$$\Lambda(t) = \int_0^t \lambda(u)du, \quad t \geq 0, \tag{1.2.1}$$

is called the cumulative hazard function or the cumulative intensity function of the Poisson process. A non-homogeneous Poisson process reduces to the classical homogeneous Poisson process when its intensity function is independent of time, i.e. $\lambda(t) = \lambda$ for some $\lambda > 0$. In this case, the increments of the Poisson process are not only independent but also stationary. Hence, homogeneous Poisson process is also known as stationary Poisson process.

Theorem 1.2.1 ([61]) *Let N be a Poisson process. Then, for $t, \tau > 0$,*

$$P(N(t+\tau) - N(\tau) = n) = \frac{e^{-[\Lambda(t+\tau)-\Lambda(\tau)]}\,[\Lambda(t+\tau) - \Lambda(\tau)]^n}{n!}, \quad n \in \mathbb{N} \tag{1.2.2}$$

This implies that $N(t+\tau) - N(\tau)$ has a Poisson distribution with mean $\Lambda(t+\tau) - \Lambda(\tau) = \int_\tau^{t+\tau} \lambda(u)du$.

Some more properties of Poisson process are discussed in Chapter 5.

1.2.2 Brownian motions

We next turn our attention to the Brownian motion process, or the Weiner process. In 1828 the Scottish botanist Robert Brown observed the irregular movement of pollen suspended in water, and it is now called the Brownian movement. The motion was later defined and shown to exist in the mathematical sense by Norbert Wiener [74] in 1923. Since then the theory and application about Brownian motion process have been greatly developed.

Definition 1.2.3 *A stochastic process $w = \{w(t)\}_{0 \geq t \geq \infty}$ taking values in $\mathbb{R}^n$ is called a n-dimensional Brownian motion process, if it satisfies*

(i) for $0 \leq s < t < \infty$, $w(t) - w(s)$ is independent of $\mathcal{F}_s$,

(ii) for $0 \leq s < t < \infty$, $w(t) - w(s)$ is a normal, or Gaussian, random variable with mean zero and variance matrix $(t-s)\Sigma$, for a given, non-random positive-semidefinite matrix Σ. The Brownian motion starts at x if $P(w(0) = x) = 1$.

1.3 Stochastic differential equations

A typical Itô-Doob type stochastic differential equation is given by

$$dx(t) = \mu(x(t),t)dt + \sigma(x(t),t)dw(t), \ t > 0, \tag{1.3.3}$$

where w is a Brownian motion. We say that μ is the drift coefficient and σ is the diffusion coefficient. Indeed, we can not simply divide by dt since the Brownian motion is nowhere differentiable, almost surely. To give a meaning to the SDE (1.3.3), Itô proposed to consider it in an integral sense, that is to teat the SDE as

$$x(t+\Delta t) = x(t) + \int_t^{t+\Delta t} \mu(x(s),s)ds + \int_t^{t+\Delta t} \sigma(x(s),s)dw(s), \ t > 0 \text{ and } \Delta t > 0, \tag{1.3.4}$$

where Itô gave a meaning to the stochastic integral $\int_0^t \sigma(x(s),s)dw(s)$ as a L^2-limit of the stochastic integrals of elementary functions (see [60, 61]).

Next we will state a very important formula in stochastic calculus, the Itô formula. The result by Itô is the analog of the chain rule in ordinary differential calculus.

Theorem 1.3.1 ([60]) *Given a n-dimensional stochastic process $x(t) = (x_1(t), \ldots, x_n(t))$ satisfying the SDE (1.3.3). Let $g(t,x)$ be a C^2 map from $[0,\infty) \times \mathbb{R}^n$ into $\mathbb{R}$. Then the process $v(t) = g(t,x(t))$ satisfies the following SDE:*

$$dv(t) = \frac{\partial g}{\partial t}(t,x(t))dt + \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(t,x(t))dx_i(t) + \frac{1}{2} \sum_{1 \le i,j \le n} \frac{\partial^2 g}{\partial x_i \partial x_j}(t,x(t))dx_i(t)dx_j(t) \tag{1.3.5}$$

where $dw_i(t)dw_j(t) = 0$ for $i \neq j$, $dw_i(t)dw_i(t) = dt$, $dw_i(t)dt = dtdw_i(t) = 0$ for all i, in addition, $dw_i(t) = w_i(t+\Delta t) - w_i(t)$, $dt = \Delta t$ as $\Delta t \to 0$.

1.4 Maximum likelihood estimators

In this section, we present a method of estimating the parameters of a statistical model, the method of maximum likelihood, which is well accepted in practice and by far the most popular approach for deriving estimators. Let $X = (X_1, X_2, \ldots, X_n)$ be a random vector, and $f(x|\theta)$ be the joint density function of X with the parameter vector $\theta = (\theta_1, \theta_2, \ldots, \theta_k)$. Considering the joint density function of X as a function of θ defines the likelihood function, denoted as

$$L(\theta) \equiv L(\theta|x) = f(x|\theta) \tag{1.4.6}$$

The basic idea of the method of maximum likelihood is that given the data x, find the estimate of θ that maximizes the likelihood function.

Definition 1.4.1 *For each given sample $x = (x_1, x_2, \ldots, x_n)$, the maximum likelihood estimator of the parameter θ is defined as*

$$\hat{\theta} \equiv \hat{\theta}(x) = \arg\max_{\theta} L(\theta|x) \tag{1.4.7}$$

We sometimes refer to the maximum likelihood estimators as MLEs in the following text.

Since the log (referring to natural logarithm) function is an increasing one-to-one function, maximizing the likelihood function is the same as maximizing the log likelihood function. Considering the log likelihood is a common approach since it reduces the exponential terms to linear terms. The partial derivative with respect to θ of the log likelihood function is called the score function:

$$V = \frac{\partial}{\partial\theta} \ln L(\theta|x) \tag{1.4.8}$$

The problem of obtaining the MLE is now transferred to solving equation of setting the score function to zero. After obtaining a point estimate for the parameter, we would be interested in how "good" is the point estimate, in the sense of what is the variance of the point estimate. Let us first have the following definitions.

Definition 1.4.2 *The Fisher information matrix $I(\theta)$ of a random sample $X = (X_1, X_2, \ldots, X_n)$ is given by the $k \times k$ matrix whose $(i, j)^{th}$ element is given by*

$$I(\theta)_{i,j} = E\left[\left(\frac{\partial}{\partial\theta_i} \ln L(\theta|X)\right)\left(\frac{\partial}{\partial\theta_j} \ln L(\theta|X)\right)\right] \tag{1.4.9}$$

where the expectation is take with respect to X. If the score function is twice differentiable with respect to θ, and under certain regularity conditions, then the Fisher information has an alternative, but equivalent, form given by

$$I(\theta)_{i,j} = -E\left[\frac{\partial^2}{\partial\theta_i\partial\theta_j} \ln L(\theta|X)\right] \tag{1.4.10}$$

Strictly, the definition corresponds to the expected Fisher information. Often, the expectation is hard to compute. The other kind of information, called the observed Fisher information, is used as a sample-based version of the expected Fisher information.

Definition 1.4.3 *The observed Fisher information matrix $J(\theta)$ of a random sample $X = (X_1, X_2, \ldots, X_n)$ is given by the $k \times k$ matrix whose $(i, j)^{th}$ element is given by*

$$J(\theta)_{i,j} = -\frac{\partial^2}{\partial\theta_i\partial\theta_j} \ln L(\theta|X) \tag{1.4.11}$$

Note that the expected Fisher information $I(\theta)$ is the expected value of the observed Fisher information $J(\theta)$.

Theorem 1.4.1 (Cramèr-Rao Inequality [18, 42]) *Let $T(X)$ be a estimator for θ with $E[T(X)] = \theta$*

(known as unbiased estimator). Then,

$$Cov(T(X)) \geq I(\theta)^{-1} \tag{1.4.12}$$

where $I(\theta)^{-1}$ is called the Cramèr-Rao lower bound.

The above inequality gives a lower bound for the covariance of any unbiased estimator. In particular, if θ is a scaler parameter, we have

$$Var(T(X)) \geq \frac{1}{I(\theta)} \tag{1.4.13}$$

An unbiased estimator whose variance achieves the Cramèr-Rao lower bound is called efficient. The following theorem gives one of the main reasons of the popularity of MLEs: the MLE is asymptotically efficient.

Theorem 1.4.2 ([59, 63]) *Under some regularity conditions, as the sample size increases, the MLE is asymptotically normal, that is,*

$$I(\hat{\theta})^{1/2}(\hat{\theta} - \theta) \xrightarrow{\mathcal{D}} N(0, 1) \tag{1.4.14}$$

where the super script 1/2 is interpreted as the symmetric square root.

In other words, the above theorem states that the MLE $\hat{\theta}$ is asymptotically normal with mean θ and variance $I(\hat{\theta})^{-1}$. This implies that the MLE is asymptotically unbiased and achieving the Cramèr-Rao lower bound, and hence it is asymptotically efficient. Besides asymptotic efficiency, the MLE is also reparametrization-invariant, consistent and sufficient [42, 63].

Remark 1.4.1 *The above asymptotic theorem for MLE is also true for replacing $I(\hat{\theta})$ with $J(\hat{\theta})$. This is particularly useful when the computation of the expected Fisher information is not feasible.*

1.5 Concluding remarks

This chapter is a review of the basic concepts and results essential for the further study of stochastic hybrid systems. A thorough and detailed investigation on stochastic processes and stochastic differential equations can be found in Øksendal [60] and Protter [61]. A complete treatment of maximum likelihood estimation methods can be found in standard mathematical statistics texts, such as Casella and Berger [18], Hogg, Mckean and Craig [42], and Rohatgi and Saleh [63].

2 Stochastic Hybrid System with Non-Homogeneous Jumps

2.1 Introduction

The notion of the *hybrid system* was introduced in early 1990s and publishes special issues in engineering sciences [1, 3, 4, 5, 38]. Antsaklis and Nerode [6], Bainov and Simeonov [8], Branicy [15, 16], Brockett [17], Michel et al. [76], Varaiya [72], and several other researchers laid down the formulation and its role and scope. The idea of hybrid system was motivated by the study of sample data systems, switched systems and impulse control systems as special cases [48]. Generally speaking, hybrid systems are dynamical systems that involve the interaction of continuous and discrete dynamics.

An incorporation of the randomness into a hybrid system has been an area of great interest over the recent years. Davis [28, 29] introduced a piecewise-deterministic Markov process (PDMP), where transitions between discrete modes are triggered by random events and deterministic conditions for hitting the boundary. However, the continuous state process between jumps for the PDMP is governed by a deterministic differential equation. Hespanha [40] proposed a model where transitions between modes are triggered by stochastic events much like transitions between states of a continuous-time Markov chains.

Hu et al. [44] proposed a stochastic hybrid system where the deterministic differential equations for the evolution of the continuous state process are replaced by Itô-Doob type stochastic differential equations [46, 49]. However, in their model the transitions are only triggered by hitting the boundaries. A study of a stochastic hybrid system whose continuous time component is stochastic and altered by transitions of a finite state Markov chain can be found in Chandra and Ladde [20], Korzeniowski and Ladde [45] and Ladde [48]. Ghosh and Bagchi [35] discussed two models. In the first model the continuous state process follows a stochastic differential equation and the transitions are governed by a homogeneous Poisson measure; in the second model the continuous state process follows a switching diffusion and the transitions are triggered by hitting the boundaries.

For a dynamic system with jumps and switching triggered by hitting the boundary ∂D_ν, an intervention strategy can be considered to control the system by optimally choosing the boundary ∂D_ν for each state. The optimal boundary sets are found by minimizing specific cost functions. For some applications, when the process hits the boundary, the jump sizes and jump points can be controlled deterministically. This kind of problem is usually known as *impulse control*, a subject whose study was initiated by Bensoussan and Lions [10]. Many applied problems are of this type: for example, inventory problems in which a sequence of restocking decisions is made, resource allocation problems involving decisions to commit funds to specific projects, or management problems for vehicle dispatching, quality inspection, and capacity expansion. Interested reader may refer to Bensoussan [11], Davis [29], or Lakshmikantham [51] for further details in this subject.

In this chapter, we study a class of stochastic hybrid dynamic process where the transitions of its discrete time state are governed by either a non-homogeneous Poisson process or triggered by hitting the boundaries. The intervention of the discrete time dynamic process generates a jump in the continuous time state and switches the mode of the continuous time stochastic state dynamic. The process is assumed to follow a diffusion stochastic differential equation depending on the state of the initial point between jumps.

The organization of this chapter is as follows. In Section 2.2, the problem is formulated. A few basic definitions and auxiliary results are outlined in Section 2.3. In Section 2.4, an infinitesimal generator of a stochastic hybrid dynamic process is developed. The presented results extends the existing results in a systematic and unified way. In Section 2.5, stochastic stability results are outlined. Examples are given in Section 2.6. The presented examples illustrate the role and scope of the basic result of Section 2.4. Finally, a few conclusions are drawn to exhibit the scope of the work in this chapter.

2.2 Model formulation

The process consisting of a discrete state $\nu(t)$ and a continuous state $x(t)$ is defined as $\xi(t) = (\nu(t), x(t))$. The discrete state $\nu(t)$ takes values in a countable set, representing the modes or regimes of operation, say K. Let x takes values in $\mathbb{R}$. Then ξ is a mapping from $\mathbb{R}^+$ to $E = K \times \mathbb{R}$. Let $\mathcal{E}$ be the σ-algebra generated by the measurable subsets of E. For each $\nu \in K$, D_ν is an open subset of $\mathbb{R}$, which contains the range of $x(t)$ of the ν regime. We denote by ∂D_ν the boundary of D_ν. Let $\partial D = \bigcup_{\nu} D_\nu$. The continuous state $x(t)$ is described by

$$\begin{aligned} dx(t) =& \mu(\xi(t),t)dt + \sigma(\xi(t),t)dw(t) \\ =& \mu(\nu(t),x(t),t)dt + \sigma(\nu(t),x(t),t)dw(t) \end{aligned} \tag{2.2.1}$$

where μ, $f \in C[K \times \mathbb{R} \times \mathbb{R}^+, \mathbb{R}]$.

The jumps of the process $\xi(t)$ occur either due to the non-homogeneous Poisson process $N(t)$ or by hitting the boundary ∂D. The selection of jump times will be described later. Let T_k denote the time of the k^{th} jump, and assume $0 = T_0 < T_1 < T_2 < \ldots a.s.$ Let $\nu_k = \nu(T_k)$ and $\xi_k = \xi(T_k) = (\nu(T_k), x(T_k))$. When a jump occurs at T_k for some k, a transition of the process $\xi(t)$ takes place. The initial value after jump is determined by the value immediately before the jump and governed by a transition distribution function Q. This means that $\xi_k = (\nu(T_k), x(T_k))$ follows a distribution function $Q(\cdot, (\nu_{k-1}, x(T_k^-)))$.

We note that in the k^{th} interval, $T_{k-1} \le t < T_k$, the process $x(t)$ is governed by

$$dx(t) = \mu(\nu_{k-1}, x(t), t)dt + \sigma(\nu_{k-1}, x(t), t)dw(t),\ T_{k-1} \le t < T_k, \tag{2.2.2}$$

here in particular $\mu(\nu_{k-1}, \cdot, \cdot)$, $\sigma(\nu_{k-1}, \cdot, \cdot) \in C[D_{\nu_{k-1}} \times [T_{k-1}, T_k), \mathbb{R}]$ and are smooth enough to assure the existence and uniqueness of the solution process of initial value problem (2.2.2). The initial value of the k^{th} interval, $\xi_{k-1} = (\nu(T_{k-1}), x(T_{k-1}))$ follows the distribution function $Q(\cdot, (\nu_{k-2}, x(T_{k-1}^-)))$.

Denote the intensity function of the non-homogeneous Poisson process $N(t)$ as $\lambda(t)$. Here the Poisson process $N(t)$ and the Brownian motion $w(t)$ are assumed to be independent for $t \ge 0$. We recall that when $\lambda(t) \equiv \lambda$, the Poisson process is homogeneous.

The jump times of $\xi(t)$ are selected as follows. Starting from the origin, for $\xi(0) = \xi_0 = (\nu_0, x_0)$, we denote

$$t^*(\xi_0) = \inf\{t > 0 : \xi(0) = \xi_0 \text{ and } x(t) \in \partial D_{\nu_0}\}$$

And define that $R_z(t) = P(t^*(z) > t)$. From the uniqueness[49] of the solution process of (2.2.2), for $s > t$, R has the property $R_{\xi_0}(s) = P(t^*(\xi_0) > s) = P(t^*(\xi_0) > t \text{ and } t^*(\xi(t)) > s - t) = P(t^*(\xi_0) >$

$t)P(t^*(\xi(t)) > s-t|\xi(t)) = R_{\xi_0}(t)R_{\xi(t)}(s-t)$.

2.3 Auxiliary results

In this section, we present a few modified versions of the existing results. These results will be used, subsequently. First, we introduce, a few notations, definitions, and known results.

Lemma 2.3.1 ([2]) *Let τ_1 be the first jump time governed by the Poisson process,then its tail probability is given by*

$$P(\tau_1 > t) = P(N(t) = 0) = \exp\left\{-\int_0^t \lambda(s)ds\right\} \tag{2.3.3}$$

Definition 2.3.1 *The first jump time T_1 of the stochastic hybrid process $\xi(t)$ is defined as*

$$T_1 = \min\{\tau_1, t^*(\xi_0)\}$$

Lemma 2.3.2 *Assume that $N(t)$ and $w(t)$ are independent stochastic processes, then $t^*(\xi_0)$ and τ_1 are also independent, and moreover the tail probability of T_1 is represented by*

$$\begin{aligned} S(t) :=& P(T_1 > t) = P(t^*(\xi_0) > t, N(t) = 0) = P(t^*(\xi_0) > t)P(N(t) = 0) \\ =& R_{\xi_0}(t)\exp\left\{-\int_0^t \lambda(s)ds\right\} \end{aligned} \tag{2.3.4}$$

or

$$S(t) = \begin{cases} \exp\left\{-\int_0^t \lambda(s)ds\right\}, \ \mathit{if}\, t < t^*(\xi_0) \\ 0, \ \mathit{if}\, t \geq t^*(\xi_0) \end{cases} \tag{2.3.5}$$

where $t^*(\xi_0)$ is a random stopping time.

Then the value immediately after first jump time $\xi(T_1)$ is a random variable with the distribution function $Q(\cdot,(\nu_0, x(T_1^-)))$.

Definition 2.3.2 *Starting from $\xi(T_{k-1})$, for $k = 2,3,4,...$, the next interarrival time*

$T_k - T_{k-1}$ is defined by

$$T_k - T_{k-1} = \min\{\tau_k - T_{k-1}, t^*(\xi_{k-1})\},$$

where $\tau_k = \inf\{t > T_{k-1} : N(t) - N(t^-) \neq 0\}$, and

$t^*(\xi_{k-1}) = \inf\{t > 0 : \xi(T_{k-1}) = \xi_{k-1}$ *and* $x(t + T_{k-1}) \in \partial D_{\nu_{k-1}}\}$.

Lemma 2.3.3 *Under the assumption of Lemma 2.3.2, starting from $\xi(T_{k-1})$, for $k = 2,3,4,...$, $T_k - T_{k-1}$ has the following sequence of tail probabilities as*

$$\begin{aligned} S_k(t) :=& P(T_k - T_{k-1} > t | T_1, ..., T_{k-1}) = P(t^*(\xi_{k-1}) > t, N(T_{k-1} + t) - N(T_{k-1}) = 0) \\ =& R_{\xi_{k-1}}(t) \exp\left\{ -\int_{T_{k-1}}^{T_{k-1}+t} \lambda(s) ds \right\} \qquad (2.3.6) \end{aligned}$$

$$or \ = \begin{cases} \exp\left\{-\int_{T_{k-1}}^{T_{k-1}+t} \lambda(s)ds\right\}, \ if\ t < t^*(\xi_{k-1}) \\ 0, \ if\ t \geq t^*(\xi_{k-1}) \end{cases} \qquad (2.3.7)$$

where $t^*(\xi_{k-1})$ is a random stopping time.

Moreover, for $t < s$,

$$\begin{aligned} P(T_k > s | T_{k-1}, T_k > t) =& P(T_k - T_{k-1} > s - T_{k-1} | T_{k-1}, T_k - T_{k-1} > t - T_{k-1}) \\ =& \frac{R_{\xi_{k-1}}(s - T_{k-1})}{R_{\xi_{k-1}}(t - T_{k-1})} \exp\left\{ -\int_t^s \lambda(u) du \right\} \\ =& R_{\xi(t)}(s - t) \exp\left\{ -\int_t^s \lambda(u) du \right\} \qquad (2.3.8) \end{aligned}$$

Remark 2.3.1 *From (2.3.8), we remark that the distribution of T_k only depends on the current state $\xi(t)$, hence $\xi(t)$ is a Markov process.*

Definition 2.3.3 *Following the framework of Davis[28], we define a counting process which counts the number of jumps ending in the set $A \subset E = K \times \mathbb{R}$ as*

$$p(t, A) := \sum_{T_i \leq t} I_{(\xi(T_i) \in A)} \qquad (2.3.9)$$

and a counting process which counts the number of jumps by hitting the boundary as

$$p_t^* := \sum_{T_i \leq t} I_{(\xi(T_i) \in \partial D)} \qquad (2.3.10)$$

And, we define the following process

$$\tilde{p}(t,A) := \int_0^t Q(A,\xi(s))\lambda(s)ds + \int_0^t Q(A,\xi(s^-))dp_s^* \tag{2.3.11}$$

as a candidate for the compensator of p.

In the following, we present a modified version of the result in [28]. For the sake of completeness, we present its proof.

Lemma 2.3.4 *If $\xi(t)$ has only a single jump at time T_1, then $\left\{p(t,A) - \int_{(0,T_1\wedge t]} \frac{-dS_u^A}{S_{u^-}}\right\}$ is a $\mathcal{F}_t$- martingale, where $\mathcal{F}_t = \sigma\{\xi(s), s \le t\}$ and $S_t^A = P(T_1 > t,$ and $\xi(T_1) \in A)$.*

Proof. For $t > s$, we have

$$\begin{aligned}
E\left[p(t,A) - p(s,A)|\mathcal{F}_s\right] &= \begin{cases} E\left[p(t,A) - p(s,A)|s < T_1\right] & \text{if } s < T_1 \\ 0 & \text{if } s \ge T_1 \end{cases} \\
&= \begin{cases} \frac{S_s^A - S_t^A}{S_s} & \text{if } s < T_1 \\ 0 & \text{if } s \ge T_1 \end{cases} \\
&= I_{(s<T_1)} \frac{S_s^A - S_t^A}{S_s}
\end{aligned} \tag{2.3.12}$$

where $S_t = P(T_1 > t)$ and $S_t^A = P(T_1 > t$, and $\xi(T_1) \in A)$. For a candidate of a compensator of $p(t,A)$, consider $\int_{(0,T_1\wedge t]} -\frac{dS_u^A}{S_{u^-}}$. We then obtain that

$$\begin{aligned}
&E\left[\int_{(0,T_1\wedge t]} -\frac{dS_u^A}{S_{u^-}} - \int_{(0,T_1\wedge s]} -\frac{dS_u^A}{S_{u^-}} \middle| \mathcal{F}_s\right] \\
=&E\left[\int_{(s\wedge T_1, t\wedge T_1]} -\frac{1}{S_{u^-}} dS_u^A \middle| \mathcal{F}_s\right] \\
=&\begin{cases} E\left[\int_{(s\wedge T_1, t\wedge T_1]} -\frac{1}{S_{u^-}} dS_u^A \middle| s < T_1\right] & \text{if } s < T_1 \\ 0 & \text{if } s \ge T_1 \end{cases} \\
=&I_{(s<T_1)} \left\{\int_s^\infty \int_{(s\wedge r, t\wedge r]} -\frac{1}{S_{u^-}} dS_u^A \left(-\frac{dS_r}{S_s}\right)\right\} \\
=&I_{(s<T_1)} \left\{\int_s^t \int_{(s\wedge r, t\wedge r]} \frac{1}{S_{u^-}} dS_u^A \left(\frac{dS_r}{S_s}\right)\right. \\
&\left. + \int_t^\infty \int_{(s\wedge r, t\wedge r]} \frac{1}{S_{u^-}} dS_u^A \left(\frac{dS_r}{S_s}\right)\right\}
\end{aligned}$$

$$
\begin{aligned}
&= I_{(s<T_1)} \left\{ \int_s^t \int_s^r \frac{1}{S_{u^-}} dS_u^A \left(\frac{dS_r}{S_s} \right) \right. \\
&\qquad \left. + \int_t^\infty \int_s^t \frac{1}{S_{u^-}} dS_u^A \left(\frac{dS_r}{S_s} \right) \right\} \\
&= I_{(s<T_1)} \left\{ \frac{1}{S_s} \int_s^t \int_s^r \frac{dS_u^A}{S_{u^-}} dS_r \right. \\
&\qquad \left. + \int_s^t \left[\int_t^\infty \left(\frac{dS_r}{S_s} \right) \right] \frac{dS_u^A}{S_{u^-}} \right\} \\
&= I_{(s<T_1)} \left\{ \frac{1}{S_s} \int_s^t \int_s^r \frac{dS_u^A}{S_{u^-}} dS_r - \frac{S_t}{S_s} \int_s^t \frac{dS_u^A}{S_{u^-}} \right\} \\
&= I_{(s<T_1)} \left\{ \frac{1}{S_s} \int_s^t \frac{1}{S_{u^-}} \int_u^t dS_r dS_u^A - \frac{S_t}{S_s} \int_s^t \frac{dS_u^A}{S_{u^-}} \right\} \\
&= I_{(s<T_1)} \left\{ \frac{1}{S_s} \int_s^t \frac{1}{S_{u^-}} (S_t - S_u) dS_u^A - \frac{S_t}{S_s} \int_s^t \frac{dS_u^A}{S_{u^-}} \right\} \\
&= I_{(s<T_1)} \left\{ -\frac{1}{S_s} \int_s^t \frac{S_u}{S_{u^-}} dS_u^A \right\} \\
&= I_{(s<T_1)} \frac{S_s^A - S_t^A}{S_s}
\end{aligned} \tag{2.3.13}
$$

From equation (2.3.12) and (2.3.13), we see that

$$
E\left[\left(p(t,A) - \int_{(0,T_1 \wedge t]} \frac{-dS_u^A}{S_{u^-}} \right) - \left(p(s,A) - \int_{(0,T_1 \wedge s]} \frac{-dS_u^A}{S_{u^-}} \right) \middle| \mathcal{F}_s \right] = 0 \tag{2.3.14}
$$

This shows that when $\xi(t)$ is a single jump process, $\left\{ p(t,A) - \int_{(0,T_1 \wedge t]} \frac{-dS_u^A}{S_{u^-}} \right\}$ is a $\mathcal{F}_t$-martingale. ∎

The next result establishes that a general multi-jump process $\xi(t)$ is a $\mathcal{F}_t$-martingale.

Proposition 2.3.5 *For a general multi-jump process $\xi(t)$, $q(t,A) := p(t,A) - \tilde{p}(t,A)$ is a $\mathcal{F}_t$-martingale.*

Proof. Applying the result of Lemma 2.3.4 to a general multi-jump process $\xi(t)$, we have that $\left\{ p(t \wedge T_1, A) - \int_{(0,t\wedge T_1]} \frac{-dS_u^A}{S_{u^-}} \right\}$ is a $\mathcal{F}_t$-martingale. Further, note that

$$
\begin{aligned}
\int_{(0,t\wedge T_1]} \frac{-dS_u^A}{S_{u^-}} &= -\int_{(0,t\wedge T_1]} \frac{1}{S_{u^-}} Q(A, \xi(u-)) dS_u \\
&= \int_{(0,t\wedge T_1]} Q(A, \xi(u^-)) \lambda(u) du + \int_{(0,t\wedge T_1]} Q(A, \xi(u^-)) dp_u^*
\end{aligned}
$$

since $S_t = \begin{cases} \exp\left\{-\int_0^t \lambda(s)ds\right\}, \ t < t^*(\xi_0) \\ 0, \ t \geq t^*(\xi_0) \end{cases}$, thus $-\dfrac{dS_t}{S_{t^-}} = -\dfrac{dS_t}{S_t} = \dfrac{\lambda(t)dtS_t}{S_t}$ $= \lambda(t)dt$ for $t < t^*(\xi_0)$, and $-\dfrac{dS_{t^*(\xi_0)}}{S_{t^*(\xi_0)^-}} = 1$ for $t = t^*(\xi_0)$. From Definition 2.3.3, we have $\tilde{p}(t \wedge T_1, A) = \int_{(0,t\wedge T_1]} Q(A,\xi(u^-))\lambda(u)du + \int_{(0,t\wedge T_1]} Q(A,\xi(u^-))dp^*_u$. Then by Lemma 3.4, $q(t\wedge T_1,A) = p(t\wedge T_1,A) - \tilde{p}(t\wedge T_1,A)$ is a $\mathcal{F}_t$-martingale.

Applying the above result for single-jump process on the intervals $(T_{k-1},T_k], k = 2,3,...$, it can be showed that $q(t\wedge T_k,A) = p(t\wedge T_k,A) - \tilde{p}(t\wedge T_k,A)$ is a $\mathcal{F}_t$-martingale. Then the result follows by the principle of mathematical induction.

■

In the following, we present certain classes of functions that would be used subsequently. For this purpose, we need the following integrals. First, let the $\mathcal{F}_t$-predictable σ-field $\mathcal{P}_t$ in $\mathbb{R}^+ \times \Omega$ is the smallest σ-field of subsets with respect to which all $\mathcal{F}_t$-adapted left-continuous processes are measurable.

If $g : E \times \mathbb{R}^+ \times \Omega \to \mathbb{R}$ where $E = K \times \mathbb{R}$ is a measurable function and $t \in \mathbb{R}^+$, we can denote that

$$\int_0^t \int_E g(y,s,\omega)p(ds,dy) := \sum_{T_i \leq t} g(\xi(T_i),T_i,\omega)$$

which is well-defined if

$$E \sum_{T_i \leq t} |g(\xi(T_i),T_i,\omega)| < \infty \text{ for each } t \in \mathbb{R}^+ \tag{2.3.15}$$

Let $L_1(p)$ denote the set of functions $g : E \times \mathbb{R}^+ \times \Omega \to \mathbb{R}$ such that g is $\mathcal{E} \times P_t$-measurable and condition (2.3.15) holds.

Integrals with respect to $\{\tilde{p}(t,A)\}$ are defined by

$$\int_0^t \int_E g(y,s,\omega)\tilde{p}(ds,dy) := \int_{(0,t]} \int_E g(y,s,\omega)Q(dy,\xi(s^-))\lambda(s)ds$$

A sufficient condition for integrability is

$$E\int_0^t \int_E |g(s,y,\omega)|\tilde{p}(ds,dy) < \infty \tag{2.3.16}$$

Let $L_1(\tilde{p})$ denote the set of functions $g : E \times \mathbb{R}^+ \times \Omega \to \mathbb{R}$ such that g is $\mathcal{E} \times P_t$-measurable and condition (2.3.16) holds.

For integrals satisfying conditions (2.3.15) and (2.3.16) we define

$$\begin{aligned} M_t^g &:= \int_0^t \int_E g(y,s,\omega) q(ds,dy) \\ &:= \int_0^t \int_E g(y,s,\omega) p(ds,dy) - \int_0^t \int_E g(y,s,\omega) \tilde{p}(ds,dy) \end{aligned} \tag{2.3.17}$$

The result in Davis[28] showed that $L_1(p) = L_1(\tilde{p})$ and $\{M_t^g\}$ is a martingale if $g \in L_1(p)$. Hence, other martingales can be obtained by forming stochastic integrals with respect to the martingale $\{q(t,A)\}$.

2.4 The extended generator of the process

In this section, we develop an infinitesimal generator of the stochastic hybrid dynamic process. This work extends the existing work [28, 35, 44] in a systematic and unified way.

Prior to presenting the main result, we introduce the concept of infinitesimal generator.

Definition 2.4.1 *The infinitesimal generator $\mathcal{A}_t$ of $\xi(t)$ at t is defined by*

$$\mathcal{A}_t f(z) = \lim_{\varepsilon \downarrow 0} \frac{E_t^z[f(\xi(t+\varepsilon))] - f(z)}{\varepsilon} \quad \text{for } z \in E \tag{2.4.18}$$

where $E_t^z[f(\xi(t+\varepsilon))] = E[f(\xi(t+\varepsilon))|\xi(t) = z]$. The set of functions $f : E \to \mathbb{R}$ such that the limit exists for all z is called the domain of the generator, and it is denoted as $\mathcal{D}_{\mathcal{A}}$. It is sufficient that $f \in C^{1,2}$ and $[f(y) - f(\xi(s^-))] \in L_1(p)$ in order to belong to $\mathcal{D}_{\mathcal{A}}$, where $C^{1,2} \equiv C^{1,2}[\mathbb{R} \times \mathbb{R}, \mathbb{R}]$ is a collection of functions $f(v,x)$ such that f is continuously differentiable with respect to v and it is twice continuously differentiable with respect to x.

Proposition 2.4.1 *For $f \in C^{1,2}$, and $[f(y) - f(\xi(s^-))] \in L_1(p)$, if f satisfies the boundary condition $f(\bar{z}) = \int_E f(y) Q(dy, \xi(\bar{z}))$ for $\bar{z} \in \partial D$, then $\mathcal{A}_t f$ at $z \in E$ is given by*

$$\mathcal{A}_t f(z) = L^1 f(z) + \lambda(t) \int_E [f(y) - f(z)]\, Q(dy,z) \tag{2.4.19}$$

where $L^1 f = \left(\mu(\nu,x,t) \frac{\partial}{\partial x} + \frac{1}{2} \sigma^2(\nu,x,t) \frac{\partial^2}{\partial x^2} \right) f.$

Proof. Using q in Proposition 2.3.5 and the boundary condition, we have

$$
\begin{aligned}
&\int_t^{t+\varepsilon}\int_E \left[f(y)-f(\xi(s^-))\right] q(ds,dy)\\
=&\int_t^{t+\varepsilon}\int_E \left[f(y)-f(\xi(s^-))\right] p(ds,dy)-\int_t^{t+\varepsilon}\int_E \left[f(y)-f(\xi(s^-))\right] \tilde{p}(ds,dy)\\
=&\sum_{t\le T_i\le t+\varepsilon}\left[f(\xi(T_i))-f(\xi(T_i^-))\right]-\int_t^{t+\varepsilon}\int_E [f(y)-f(\xi(s))]\,Q(dy,\xi(s^-))\lambda(s)ds\\
&-\int_t^{t+\varepsilon}\int_E [f(y)-f(\xi(s))]\,Q(dy,\xi(s^-))dp_s^*\\
=&\sum_{t\le T_i\le t+\varepsilon}\left[f(\xi(T_i))-f(\xi(T_i^-))\right]-\int_t^{t+\varepsilon}\int_E [f(y)-f(\xi(s))]\,Q(dy,\xi(s^-))\lambda(s)ds \qquad (2.4.20)
\end{aligned}
$$

We note that the first term in (2.4.20) can be written as

$$
\begin{aligned}
&\sum_{t\le T_i\le t+\varepsilon}\left[f(\xi(T_i))-f(\xi(T_i^-))\right]=\sum_{i=1}^{l}\left[f(\xi(T_{n_i}))-f(\xi(T_{n_i}^-))\right]\\
=&\left\{\sum_{i=1}^{l}\left[f(\xi(T_{n_i}))-f(\xi(T_{n_{i-1}}))\right]+f(\xi(t+\varepsilon))-f(\xi(T_{n_l}))\right\}\\
&-\left\{\sum_{i=1}^{l}\left[f(\xi(T_{n_i}^-))-f(\xi(T_{n_{i-1}}))\right]+f(\xi(t+\varepsilon))-f(\xi(T_{n_l}))\right\} \qquad (2.4.21)
\end{aligned}
$$

where T_{n_i}, $i=1,2,...,l$, are the jump times on $[t,t+\varepsilon]$, and denote $T_{n_0}=t$. And note that the first term in (2.4.21) can be further simplified to $f(\xi(t+\varepsilon))-f(\xi(t))$.

For s in $[T_{n_{i-1}},T_{n_i})$ the state component $\nu(s)$ is equal to $\nu(T_{n_{i-1}})$ and the continuous state component $x(s)$ satisfies the stochastic differential equation

$$dx(s)=\mu(\nu(T_{n_{i-1}}),x,s)ds+\sigma(\nu(T_{n_{i-1}}),x,s)dw(s) \qquad (2.4.22)$$

Thus, by Ito's lemma[60], for s in $[t,t+\varepsilon]$ and $f\in C^{1,2}[\mathbb{R}\times\mathbb{R},\mathbb{R}]$, $f(\xi(s))\equiv f(\nu(s),x(s))$ satisfies

$$
\begin{aligned}
df(\xi(s))=&\frac{\partial f}{\partial x}(\xi(s))dx(s)+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\xi(s))(dx(s))^2\\
=&\left[\mu(\nu,x,s)\frac{\partial f}{\partial x}(\xi(s))+\frac{1}{2}\sigma^2(\nu,x,s)\frac{\partial^2 f}{\partial x^2}(\xi(s))\right]ds\\
&+\sigma(\nu,x,s)\frac{\partial f}{\partial x}(\xi(s))dw(s)\\
=&L^1f(\xi(s))ds+L^2f(\xi(s))dw(s) \qquad (2.4.23)
\end{aligned}
$$

where $L^1 f = \left(\mu(\nu,x,s)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(\nu,x,s)\frac{\partial^2}{\partial x^2}\right) f$, and $L^2 f = \left(\sigma(\nu,x,s)\frac{\partial}{\partial x}\right) f$.
Thus, as of the second term in (2.4.21), we have

$$\begin{aligned}
&\sum_{i=1}^{l}\left[f(\xi(T_{n_i}^-)) - f(\xi(T_{n_{i-1}}))\right] + f(\xi(t+\varepsilon)) - f(\xi(T_{n_l}))\\
=&\sum_{i=1}^{l}\int_{T_{n_{i-1}}}^{T_{n_i}} L^1 f(\nu(T_{n_{i-1}}),x(s))ds + \int_{T_{n_l}}^{t+\varepsilon} L^1 f(\nu(T_{n_l}),x(s))ds\\
&+\sum_{i=1}^{l}\int_{T_{n_{i-1}}}^{T_i} L^2 f(\nu(T_{n_{i-1}}),x(s))dw(s) + \int_{T_{n_l}}^{t+\varepsilon} L^2 f(\nu(T_{n_l}),x(s))dw(s)\\
=&\int_t^{t+\varepsilon} L^1 f(\nu(s),x(s))ds + \int_t^{t+\varepsilon} L^2 f(\nu(s),x(s))dw(s)\\
=&\int_t^{t+\varepsilon} L^1 f(\xi(s))ds + \int_t^{t+\varepsilon} L^2 f(\xi(s))dw(s)
\end{aligned}\tag{2.4.24}$$

Hence, from (2.4.20), (2.4.21), (2.4.23) and (2.4.24), we have

$$\begin{aligned}
f(\xi(t+\varepsilon)) - f(\xi(t)) =& \int_t^{t+\varepsilon}\int_E \left[f(y) - f(\xi(s^-))\right] q(ds,dy)\\
&+\int_t^{t+\varepsilon} L^1 f(\xi(s))ds + \int_t^{t+\varepsilon} L^2 f(\xi(s))dw(s)\\
&+\int_t^{t+\varepsilon}\int_E [f(y) - f(\xi(s))]\, Q(dy,\xi(s^-))\lambda(s)ds
\end{aligned}\tag{2.4.25}$$

From (2.4.25) and taking conditional expectation, we have

$$\begin{aligned}
&E\left[f(\xi(t+\varepsilon))\,|\, f(\xi(t)) = z\right] - f(z)\\
=&E\left[\int_t^{t+\varepsilon}\int_E \left[f(y) - f(\xi(s^-))\right] q(ds,dy)\,\middle|\, f(\xi(t)) = z\right]\\
&+E\left[\int_t^{t+\varepsilon} L^1 f(\xi(s))ds\,\middle|\, f(\xi(t)) = z\right]\\
&+E\left[\int_t^{t+\varepsilon} L^2 f(\xi(s))dw(s)\,\middle|\, f(\xi(t)) = z\right]\\
&+E\left[\int_t^{t+\varepsilon}\int_E [f(y) - f(\xi(s))]\, Q(dy,\xi(s^-))\lambda(s)ds\,\middle|\, f(\xi(t)) = z\right]
\end{aligned}\tag{2.4.26}$$

From the assumption that $[f(y) - f(\xi(s))] \in L_1(p)$, the integral in the first term in (2.4.26) is a martingale from the result in Davis[28], thus the conditional expectation becomes zero. $L^2 f(\xi(s))$ is measurable with respect to the natural filtration $\mathcal{F}_s$, then the expected value of the integral with respect to the diffusion vanishes. Now, equation (2.4.26) reduces to

$$
\begin{aligned}
&E\left[f(\xi(t+\varepsilon))|\, f(\xi(t))=z\right]-f(z)\\
=&E\left[\int_t^{t+\varepsilon} L^1 f(\xi(s))ds\,\middle|\, f(\xi(t))=z\right]\\
&+E\left[\int_t^{t+\varepsilon}\int_E [f(y)-f(\xi(s))]\,Q(dy,\xi(s^-))\lambda(s)ds\,\middle|\, f(\xi(t))=z\right]
\end{aligned}
$$

From this, we obtain the infinitesimal generator of $\xi(t)$ as

$$
\begin{aligned}
\mathcal{A}_t f(z)=&\lim_{\varepsilon\downarrow 0}\frac{E\left[f(\xi(t+\varepsilon))|\, f(\xi(t))=z\right]-f(z)}{\varepsilon}\\
=&L^1 f(z)+\lambda(t)\int_E [f(y)-f(z)]\,Q(dy,z).
\end{aligned}
\tag{2.4.27}
$$

A consequence from the above result is

$$
E\left[f(\xi(t))|\, f(\xi_0)\right]-f(\xi_0)=E\left[\int_0^t \mathcal{A}_s f(\xi(s))ds\,\middle|\, f(\xi_0)\right] \tag{2.4.28}
$$

The proof is complete.

■

If the function f depends explicitly on the time component, the infinitesimal generator can be obtain as follows.

Definition 2.4.2 *The infinitesimal generator $\bar{\mathcal{A}}_t$ of $\xi(t)$ acting on a function $f(t,\xi(t))$ in the domain of $\bar{\mathcal{A}}_t$ is defined by*

$$
\bar{\mathcal{A}}_t f(t,z)=\lim_{\varepsilon\downarrow 0}\frac{E_t[f(t+\varepsilon,\xi(t+\varepsilon))|\xi(t)=z]-f(t,z)}{\varepsilon}\quad \textit{for } z\in E \tag{2.4.29}
$$

From (2.4.27) the following result is obtained.

$$
\begin{aligned}
\bar{\mathcal{A}}_t f(t,z)=&\lim_{\varepsilon\downarrow 0}\frac{E_t[f(t+\varepsilon,\xi(t+\varepsilon))|\xi(t)=z]-f(t,z)}{\varepsilon}\\
=&\lim_{\varepsilon\downarrow 0}\frac{E_t[f(t+\varepsilon,\xi(t+\varepsilon))|\xi(t)=z]-E_t[f(t,\xi(t+\varepsilon))|\xi(t)=z]}{\varepsilon}\\
&+\lim_{\varepsilon\downarrow 0}\frac{E_t[f(t,\xi(t+\varepsilon))|\xi(t)=z]-f(t,z)}{\varepsilon}
\end{aligned}
$$

$$=\frac{\partial}{\partial t} f(t,z) + \mathcal{A}_t f(t,z) \tag{2.4.30}$$

2.5 Stability Results

In this section we consider some stochastic stability properties of the solution process of the stochastic hybrid dynamic system. First, we introduce the following definitions.

Definition 2.5.1 **([49])** *A continuously differentiable function f is positive-definite with respect to x if there exists a strictly increasing function $b(|x|)$ with $b(0) = 0$ such that $f(z) \geq b(|x|)$ for all $z = (\nu, x) \in E$.*

For the stochastic hybrid system defined in Section 2.2 and the infinitesimal generator given in (2.4.19), we define the following classes:

$\mathcal{C}_1 = \{\mu, \sigma$ in (2.2.1) and $f \in \mathcal{D}_{\mathcal{A}} : \mathcal{A}_t f(z) \leq 0$ for all $z \in E$, and f is positive-definite with respect to $x\}$.

$\mathcal{C}_2 = \{\mu, \sigma$ in (2.2.1) and $f \in \mathcal{D}_{\mathcal{A}} : \mathcal{A}_t f(z) \leq -\alpha f(z)$ for all $z \in E$, for some $\alpha > 0$, and f is positive-definite with respect to $x\}$;

$\mathcal{C}_3 = \{\mu, \sigma$ in (2.2.1) and $f \in \mathcal{D}_{\mathcal{A}} : \mathcal{A}_t f(z) \geq 0$ for all $z \in E$, and f is positive-definite with respect to $x\}$;

$\mathcal{C}_4 = \{\mu, \sigma$ in (2.2.1) and $f \in \mathcal{D}_{\mathcal{A}} : \mathcal{A}_t f(z) \geq -\alpha f(z)$ for all $z \in E$, for some $\alpha > 0$, and f is positive-definite with respect to $x\}$.

Without loss in generality, we assume that the zero is an equilibrium state of the stochastic hybrid system (2.2.2).

Definition 2.5.2 **([46])** *The zero is stochastically stable if for any $\rho > 0$ and $\varepsilon > 0$, there exists a $\delta \equiv \delta(\rho, \varepsilon) > 0$ such that, if $|x_0| < \delta$,*

$$P(\sup_{t \geq 0} |x(t)| \geq \varepsilon) \leq \rho. \tag{2.5.31}$$

Definition 2.5.3 **([46])** *The zero is stochastically exponentially stable if for any $\varepsilon > 0$ and $\bar{t} > 0$, there exists some $\alpha > 0$ and $K > 0$ such that*

$$P(\sup_{\bar{t} \leq t < \infty} |x(t)| > \varepsilon) \leq K e^{-\alpha \bar{t}} \tag{2.5.32}$$

Now, in the following, we establish the stochastic stability result.

Proposition 2.5.1 *For the stochastic hybrid system defined in Section 2.2, if $(\mu,\sigma,f)\in\mathcal{C}_1$, then, for any $t>0$,*

$$E\left[f(\xi(t))|f(\xi_0)\right]\leq f(\xi_0) \tag{2.5.33}$$

moreover, the zero is stochastically stable.

Proof. If $(\mu,\sigma,f)\in\mathcal{C}_1$, from (2.4.28), for $s<t$, we obtain

$$E\left[f(\xi(t))-f(\xi(s))|f(\xi(s))\right]=E\left[\int_s^t \mathcal{A}_u f(\xi(u))du\bigg|f(\xi(s))\right]\leq 0 \tag{2.5.34}$$

This implies that $f(\xi(t))$ is a nonnegative supermartingale.

By the positive-definiteness of f, there exists a strictly increasing function $b(|x|)$ with $b(0)=0$ such that $f(z)\geq b(|x|)$ for all $z=(\nu,x)\in E$.

For $\varepsilon>0$, let $\tau_\varepsilon=\inf\{t\geq 0:|x(t)|\geq\varepsilon\}=\inf\{t\geq 0:b(|x(t)|)\geq b(\varepsilon)\}$, where the two events are equivalent since b is a strictly increasing function. Given $\xi(0)=\xi_0=(\nu_0,x_0)$, from (2.5.34), we obtain that

$$E\left[b(|x(t\wedge\tau_\varepsilon)|)\right]\leq E\left[f(\xi(t\wedge\tau_\varepsilon))|f(\xi_0)\right]\leq f(\xi_0) \tag{2.5.35}$$

And

$$\begin{aligned}
E\left[b(|x(t\wedge\tau_\varepsilon)|)\right]=&\int_\Omega b(|x(t\wedge\tau_\varepsilon)|)dP(\omega)\\
\geq&\int_{\{\tau_\varepsilon<t\}} b(|x(t\wedge\tau_\varepsilon)|)dP(\omega)\\
\geq&\int_{\{\tau_\varepsilon<t\}} b(\varepsilon)dP(\omega)\\
=&b(\varepsilon)P(\tau_\varepsilon<t)
\end{aligned} \tag{2.5.36}$$

From (2.5.35) and (2.5.36), taking limsup as $t\to\infty$ gives

$$P(\tau_\varepsilon<\infty)=P(\sup_{t\geq 0}|x(t)|\geq\varepsilon)\leq\frac{f(\xi_0)}{b(\varepsilon)} \tag{2.5.37}$$

For given $\rho > 0$, by the continuity and positive-definiteness of f, there exists $\delta = \delta(\varepsilon, \rho) > 0$ such that $\dfrac{f(\xi_0)}{b(\varepsilon)} \leq \rho$ whenever $|x_0| < \delta$. The proof is complete.

∎

Corollary 2.5.2 *For the stochastic hybrid system defined in Section 2.2, if $(\mu, \sigma, f) \in \mathcal{C}_3$, then, for any $t > 0$,*

$$E\left[f(\xi(t)) | f(\xi_0)\right] \geq f(\xi_0) \tag{2.5.38}$$

Proposition 2.5.3 *For the stochastic hybrid system defined in Section 2.2, if $(\mu, \sigma, f) \in \mathcal{C}_2$, then, for any $t > s > 0$,*

$$E\left[f(\xi(t)) | f(\xi(s))\right] \leq e^{-\alpha(t-s)} f(\xi(s)) \tag{2.5.39}$$

moreover, the zero is stochastically exponentially stable.

Proof. By the assumption $(\mu, \sigma, f) \in \mathcal{C}_2$, from (2.4.28), for $s < t$, we have

$$\begin{aligned} E\left[f(\xi(t)) - f(\xi(s)) | f(\xi(s))\right] =& E\left[\int_s^t \mathcal{A}_u f(x(u)) du \middle| f(\xi(s))\right] \\ \leq& -\alpha \int_s^t E\left[f(\xi(u)) | f(\xi(s))\right] du \end{aligned} \tag{2.5.40}$$

By the Gronwall's lemma, it gives

$$E\left[f(\xi(t)) | f(\xi(s))\right] \leq e^{-\alpha(t-s)} f(\xi(s)) \tag{2.5.41}$$

Given $\varepsilon > 0$ and $\bar{t} > 0$, let $\tau_\varepsilon = \inf\{t > \bar{t} : |x(t)| > \varepsilon\}$.

Given $\xi(0) = \xi_0$, from (2.5.41), for $t > \bar{t}$, we obtain

$$E\left[f(\xi(t \wedge \tau) | f(\xi_0)\right] \leq e^{-\alpha t} f(\xi_0) \leq e^{-\alpha \bar{t}} f(\xi_0) \tag{2.5.42}$$

Following the same reasoning as in (2.5.36) and by the positive-definiteness of f, there exists a strictly increasing function b with $b(0) = 0$ such that

$$P(\tau_\varepsilon < \infty) = P(\sup_{\bar{t} \le t < \infty} |x(t)| \ge \varepsilon) \le \frac{f(\xi_0)}{b(\varepsilon)} e^{-\alpha \bar{t}} \tag{2.5.43}$$

The proof is complete.

■

Corollary 2.5.4 *For the stochastic hybrid system defined in Section 2.2, if $(\mu, \sigma, f) \in \mathcal{C}_4$, then, for any $t > s > 0$,*

$$E\left[f(\xi(t))|f(\xi(s))\right] \ge e^{-\alpha(t-s)} f(\xi(s)) \tag{2.5.44}$$

Remark 2.5.1 *We note that Propositions 2.5.1 and 2.5.3, and Corollaries 2.5.2 and 2.5.4 exhibit the upper and lower estimates for the solution process of (2.2.2) under given conditions. However, under different conditions on f, one can obtain various type of qualitative behavior of solution process of (2.2.2).*

Remark 2.5.2 *The results concerning other qualitative properties, namely, boundedness, convergence and stability properties [49], of the stochastic hybrid dynamic system (2.2.2), can be developed, analogously.*

2.6 Examples and applications

In this section, we present several examples to illustrate our results in previous sections.

Example 2.6.1 Insurance model I

Let U_t be a surplus process of an insurance company. The process U_t is given by

$$U_t = \int_0^t a(s)ds + \int_0^t b(s)dw(s) - V_t, \text{ and } V_t = \sum_{i=1}^{N_t} Y_i, \tag{2.6.45}$$

where $a(t)$ and $b(t)$ are deterministic functions of time, $w(t)$ is the standard Brownian motion, and N_t is a non-homogeneous Poisson process with intensity $\lambda(t)$. And Y_i's are the independent claim sizes with a common distribution function $G(y)$. Here we denote $m_1 = \int_0^\infty y dG(y)$, and $m_2 = \int_0^\infty y^2 dG(y)$.

The jump times, $T_1, T_2, \ldots$, of the surplus process are only due to the non-homogeneous Poisson process. Between jumps, say $T_k \leq t < T_{k+1}$, the surplus process is governed by the stochastic differential equation

$$dU_t = a(t)dt + b(t)dw(t), \text{ and } U_{T_k} = U_{T_k^-} - Y_{T_k}. \tag{2.6.46}$$

Applying Proposition 2.4.1 we can obtain the infinitesimal generator of U_t acting on f where f is in the domain of the generator as

$$\mathcal{A}_t f(U_t) = a(t)\frac{\partial f}{\partial u}(U_t) + \frac{1}{2}b^2(t)\frac{\partial^2 f}{\partial u^2}(U_t) + \lambda(t)\int_0^\infty [f(U_t - y) - f(U_t)]dG(y) \tag{2.6.47}$$

Setting $f(u) = u$, the generator becomes

$$\mathcal{A}_t f(U_t) = a(t) - \lambda(t)\int_0^\infty y dG(y) = a(t) - m_1\lambda(t). \tag{2.6.48}$$

From equation (2.4.28), we have

$$E[U_t|U_0] - U_0 = E\left[\int_0^t \mathcal{A}_s f(U_s)ds \middle| U_0\right], \tag{2.6.49}$$

then the first moment of the surplus process is obtained as

$$E[U_t|U_0] = U_0 + \int_0^t a(s)ds - m_1\int_0^t \lambda(s)ds. \tag{2.6.50}$$

Setting $f(u) = u^2$, the generator acting on f now is

$$\begin{aligned} \mathcal{A}_t f(U_t) =& 2a(t)U_t + b^2(t) + \lambda(t)\int_0^\infty [(U_t - y)^2 - U_t^2]dG(y) \\ =& 2a(t)U_t + b^2(t) + \lambda(t)(m_2 - 2m_1 U_t) \end{aligned} \tag{2.6.51}$$

Apply equation (2.4.28) with above formula for $E[U_t|U_0]$, we can obtain the second moment of the surplus process U_t as

$$E[U_t^2|U_0] = U_0^2 + 2\int_0^t a(s)E[U_s|U_0]ds + \int_0^t b^2(s)ds$$

$$+m_2\int_0^t \lambda(s)ds - 2m_1\int_0^t \lambda(s)E[U_s|U_0]ds \tag{2.6.52}$$

Remark 2.6.1 *Let us assume that*

$$\int_0^t a(s)ds - m_1\int_0^t \lambda(s)ds \geq 0 \tag{2.6.53}$$

From (2.6.50) and (2.6.53), we have $E[U_t|U_0] \geq U_0$*. We further note that (2.6.53) is valid provided* $a(t) - m_1\lambda(t) \geq 0$ *for* $t > 0$*. This condition is called the net profit condition in risk theory.*

Example 2.6.2 Insurance model II

Here we consider a more general risk model where the dynamic in (2.6.46) in Example 2.6.1 is replaced by

$$dU_t = [\alpha U_t + a(t)]dt + [\beta U_t + b(t)]\,dw(t), \text{ and } U_{T_k} = U_{T_k-} - Y_{T_k}. \tag{2.6.54}$$

Note that when $\alpha = \beta = 0$, the model coincides with the one in Example 2.6.1. For f in the domain of the infinitesimal generator of U_t, by the application of Proposition 2.4.1, we have

$$\begin{aligned}\mathcal{A}_t f(U_t) =& [\alpha U_t + a(t)]\frac{\partial f}{\partial u}(U_t) + \frac{1}{2}[\beta U_t + b(t)]^2\frac{\partial^2 f}{\partial u^2}(U_t)\\ &+ \lambda(t)\int_0^\infty [f(U_t - y) - f(U_t)]dG(y)\end{aligned} \tag{2.6.55}$$

For $f(u) = u^2$, equation (2.6.55) reduces to

$$\mathcal{A}_t f(U_t) = 2[\alpha U_t^2 + a(t)U_t] + [\beta U_t + b(t)]^2 + \lambda(t)(m_2 - 2m_1U_t) \tag{2.6.56}$$

Example 2.6.3 Queueing model

Here we consider a generalized version of the M/G/1 queue model in Davis[28]. Customers arrive at a single-server queue according to a non-homogeneous Poisson process with intensity $\lambda(t)$, and the service times required are i.i.d. with a common distribution function $G(y)$. Let $m_1 = \int_0^\infty ydG(y)$. Let $x(t)$ be the virtual waiting time at time t that is the time a customer would have to wait if he arrived at time t. The state process $\nu(t)$ takes the value 0 or 1, representing "empty" and "busy" server, respectively. When $\nu = 0$, the virtual waiting time $x(t) \equiv 0$. When $\nu = 1$, $x(t)$ decreases at unit rate with time plus random noise, i.e. $x(t)$ follows the SDE $dx(t) = -dt + \sigma_1 dw(t)$ for some $\sigma_1 > 0$. See Fig.2.1 for a realization of the queue process. For f in the domain of the infinitesimal

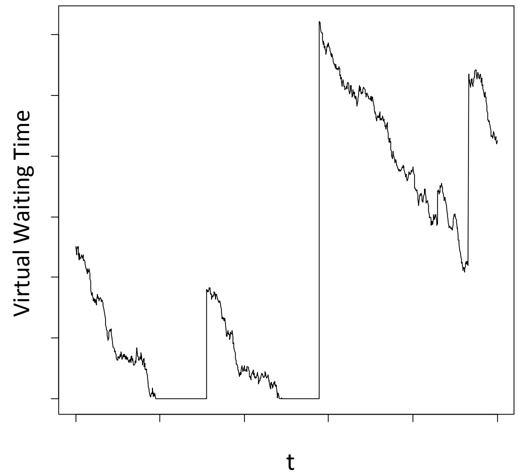

Figure 2.1: A realization of the queueing model.

generator of the process $\xi(t) = (\nu(t), x(t))$, from Proposition 2.4.1 we have

$$\begin{aligned}\mathcal{A}_t f(\xi(t)) =& \mu(\nu(t))\frac{\partial f}{\partial x}(\xi(t)) + \frac{1}{2}\sigma^2(\nu(t))\frac{\partial^2 f}{\partial x^2}(\xi(t)) \\ &+ \lambda(t)\int_0^\infty [f(1, x(t)+y) - f(\nu(t), x(t))]dG(y),\end{aligned} \tag{2.6.57}$$

where $\mu(\nu) = \begin{cases} -1, \nu = 1 \\ 0, \nu = 0 \end{cases}$, and $\sigma(\nu) = \begin{cases} \sigma_1, \nu = 1 \\ 0, \nu = 0 \end{cases}$.

For $f(\nu, x) = x$, from (2.4.28) and (2.6.57), the expected virtual waiting time is given by

$$\begin{aligned}E[x(t)] =& x_0 + E\left[\int_0^t \mathcal{A}_s f(\xi(s))ds \middle| f(\xi_0)\right] \\ =& x_0 + \int_0^t E[\mu(\nu(s))|\xi_0]ds + m_1\int_0^t \lambda(s)ds\end{aligned} \tag{2.6.58}$$

Example 2.6.4 Dam model

Let x_t be the water level of a dam where the inflow is a pure jump process V_t and the water release rate is $r(x_t)$ with a diffusion function $\sigma(x_t)$. Let $r(0) = 0$. Then x_t is governed by the differential equation

$$dx_t = -r(x_t)dt + \sigma(x_t)dw(t) \tag{2.6.59}$$

between jumps. Let $V_t = \sum_{i=1}^{N_t} Y_i$ be a compound Poisson process with intensity $\lambda(t)$ and jump sizes are i.i.d. For a jump time T_k for some k, we see that $x_{T_k} = x_{T_k^-} + Y_k$. Rewrite $x_{T_k} = \delta_{x_{T_k^-}} x_{T_k^-}$ where $\delta_{x_{T_k^-}} = \frac{x_{T_k^-} + Y_k}{x_{T_k^-}}$. With this change of variable, now the randomness of jump size is absorbed in δ_x. Denote the distribution function of δ_x as D_x with the support $\delta_x \geq 1$. Applying Proposition 2.4.1, the infinitesimal generator of x_t acting on $f(x) = x^2$ is given by

$$\begin{aligned}\mathcal{A}_t f(x) &= -2r(x)x + \sigma^2(x) + \lambda(t)\int_1^\infty \{(\delta_x x)^2 - x^2\} dD_x(\delta_x) \\ &= -2r(x)x + \sigma^2(x) + \lambda(t)x^2 \int_1^\infty (\delta_x^2 - 1) dD_x(\delta_x)\end{aligned} \tag{2.6.60}$$

Note that $f(x) = x^2$ is positive-definite. In addition, if $r(x)$ and $\sigma(x)$ satisfy

$$-\beta x^2 \leq -2r(x)x + \sigma^2(x) + \lambda(t)x^2 \int_1^\infty (\delta_x^2 - 1) dD_x(\delta_x) \leq -\alpha x^2 \tag{2.6.61}$$

for some $\beta > \alpha > 0$, then by the result of Proposition 2.5.3 and Corollary 2.5.4, we have

$$e^{-\beta t} x_0^2 \leq E\left[x^2(t))\middle| x_0\right] \leq e^{-\alpha t} x_0^2 \tag{2.6.62}$$

and the system is stochastically exponentially stable. On the other hand, if (2.6.61) is replaced by

$$-\beta x^2 \leq -2r(x)x + \sigma^2(x) + \lambda(t)x^2 \int_1^\infty (\delta_x^2 - 1) dD_x(\delta_x) \leq 0 \tag{2.6.63}$$

for some $\beta > 0$, then by the result of Proposition 2.5.1 and Corollary 2.5.4, we have

$$e^{-\beta t} x_0^2 \leq E\left[x^2(t))\middle| x_0\right] \leq x_0^2 \tag{2.6.64}$$

and the system is stochastically stable.

2.7 Concluding remarks

In this chapter, we investigated a class of stochastic hybrid dynamic systems which incorporate random jumps driven by a non-homogeneous Poisson process and deterministic jumps triggered by hitting the boundary, and the continuous state process is described by stochastic differential equations of the Itô-Doob type. We derived the extended generator of the process which can be used to compute higher moments of solution process of stochastic hybrid systems and the overall distribution of the state. The generators of insurance and queueing models are obtained. The first and second moments of the insurance models are computed through the generator. Furthermore, the stochastic stability results are also developed.

3 SOLUTIONS OF STOCHASTIC HYBRID MODEL WITH SWITCHING COEFFICIENTS AND JUMPS

3.1 Introduction

In this and the subsequent chapters, a class of multidimensional stochastic hybrid dynamic models is studied. The first system under investigation is a first-order linear non-homogeneous system of Itô-Doob type stochastic differential equations with switching coefficients. The second one is a hybrid system with continuous dynamic consisting of only drift part and additive noise, namely an Ornstein-Uhlenbeck system [62]. The switching of the system is governed by a discrete dynamic which is monitored by a non-homogeneous Poisson process. In this chapter, closed form solutions of the two systems are obtained.

The rest of the chapter is organized as follows. The formulation of the models is outlined in Section 3.2. In Section 3.3, the closed form solution processes of the multidimensional systems are obtained through utilizing the result of Ladde and Ladde [47] piecewisely on the intervals between jumps. The solution processes of two special cases, the geometric Brownian motion with jumps and the Ornstein-Uhlenbeck process with jumps are given in Section 3.4 and 3.5. The result of infinitesimal generator in Chapter 2 is generalized to a class of multivariate stochastic hybrid systems in Section 3.6. Some concluding remarks are made at the end of the chapter.

3.2 Model formulation

In this section, we develop a conceptual stochastic model for dynamic processes in chemical, biological, engineering, medical, physical, and social science [47, 64, 65] that are under the influence of discrete time events. The continuous time dynamic of the stochastic model between jumps follows a first-order linear non-homogeneous system of Itô-Doob type stochastic differential equations.

Random jump times are governed by a non-homogeneous Poisson process. The coefficients of the continuous time dynamic are switched and the process is rescaled by a random factor which results in a discontinuous random jump.

Let $x(t)$ be a real n-dimensional process. A_k and B_k^j are $n \times n$ matrices for any $k \in \mathbb{N} \cup \{0\}$ and $j = 1, 2, \ldots, q$. Let C_k^r be n-dimensional vectors for any $k \in \mathbb{N} \cup \{0\}$ and $r = 1, 2, \ldots, p$. Let the continuous dynamic of the process $x(t)$ be determined by the following system of stochastic differential equations

$$dx(t) = A_{N(t)} x(t) dt + \sum_{j=1}^{q} B_{N(t)}^{j} x(t) dw_j(t) + \sum_{r=1}^{p} C_{N(t)}^{r} d\bar{w}_r(t),\ t \geq t_0,\ x(t_0) = x_0 > 0, \tag{3.2.1}$$

where $w(t) = (w_1(t), \ldots, w_q(t))$ and $\bar{w}(t) = (\bar{w}_1(t), \ldots, \bar{w}_p(t))$ are independent q-dimensional and p-dimensional standard Wiener processes, and $N(t)$ is a non-homogeneous Poisson process with intensity $\lambda(t)$. Here we denote $x = (x^1, x^2, \ldots, x^n) > 0$ as $x^i > 0$ for all $i = 1, 2, \ldots, n$. When $C_k^r = 0$ for all k and r, system (3.2.1) reduces to a first-order linear homogeneous system of Itô-Doob type stochastic differential equations, given below

$$dx(t) = A_{N(t)} x(t) dt + \sum_{j=1}^{q} B_{N(t)}^{j} x(t) dw_j(t),\ t \geq t_0,\ x(t_0) = x_0 > 0, \tag{3.2.2}$$

Here, A_k and B_k^j are such that solution process $x(t)$ of (3.2.2) is nonnegative.

3.3 Methods of finding solution process

To obtain solution process of system (3.2.1), we first consider the solution process of the initial-value system when A, B^j, and C^r's are fixed over time, that is the solution process on a subinterval between jumps. Under the condition that the matrices $A, B^1, B^2, \ldots, B^q$ pairwise commute. The solution can be explicitly obtained, see Ladde and Ladde [47] or Movellan [56]. We state the result in the following lemma.

Lemma 3.3.1 *Let $x(t) \equiv x(t, t_0, x_0)$ be the solution of the following initial value problem (IVP):*

$$dx(t) = Ax(t) dt + \sum_{j=1}^{q} B^j x(t) dw_j(t) + \sum_{r=1}^{p} C^r d\bar{w}_r(t),\ t \geq t_0,\ x(t_0) = x_0 \tag{3.3.3}$$

Then, the $x(t)$ is expressed by

$$x(t) \equiv x(t,t_0,x_0) = \exp\left[\left(A - \frac{1}{2}\sum_{j=1}^{q}(B^j)^2\right)(t-t_0) + \sum_{j=1}^{q} B^j\left(w_j(t) - w_j(t_0)\right)\right] x_0$$
$$+ \sum_{r=1}^{p}\int_{t_0}^{t} \exp\left[\left(A - \frac{1}{2}\sum_{j=1}^{q}(B^j)^2\right)(t-s) + \sum_{j=1}^{q} B^j\left(w_j(t) - w_j(s)\right)\right] C^r d\bar{w}_r(s) \qquad (3.3.4)$$

provided that $AB^j = B^jA$ and $B^jB^{j'} = B^{j'}B^j$ for all $j, j' = 1,2,\ldots,q$.

Let $x(t,T_k,x_k)$ be the solution to system (3.3.3) with $t_0 := T_k$, $x_0 := x_k$, $A := A_k$, $B^j := B_k^j$, and $C^r := C_k^r$. Now we consider the following system of two interconnected stochastic dynamics

$$\begin{cases} dx(t) = A_{k-1}x(t)dt + \sum_{k=1}^{q} B_{k-1}^j x(t)dw_j(t) + \sum_{r=1}^{p} C_{k-1}^r d\bar{w}_r(t),\ T_{k-1} \le t < T_k,\ x(T_{k-1}) = x_{k-1} \\ x_k = z_k x(T_k^-, T_{k-1}, x_{k-1}) \end{cases} \qquad (3.3.5)$$

where z_k, $k = 1,2,3,\cdots$, are iid positive random variables with $z_0 = 1$, and $x(T_k^-,T_{k-1},x_{k-1}) = \lim_{t\to T_k^-} x(t,T_{k-1},x_{k-1})$. Here we assume that $N(t)$, $w(t)$, $\bar{w}(t)$, and z_k are mutually independent. By applying lemma 3.3.1, piecewisely, on each interval between jumps, we then obtain the solution process of system (3.3.5). The result is given in the following proposition.

Proposition 3.3.2 *If $A_kB_k^j = B_k^jA_k$ and $B_k^jB_k^{j'} = B_k^{j'}B_i^j$ for all $k \in \mathbb{N}\cup\{0\}$ and $j, j' = 1,2,\ldots,q$, then the solution to the system (3.3.5) is given by*

$$x(t) = \left(\prod_{k=1}^{N(t)} z_k\right)\exp\left[\left(A_{N(t)} - \frac{1}{2}\sum_{j=1}^{q}(B_{N(t)}^j)^2\right)(t - T_{N(t)}) + \sum_{j=1}^{q} B_{N(t)}^j\left(w_j(t) - w_j(T_{N(t)})\right)\right]$$
$$\times \prod_{k=1}^{N(t)} \exp\left[\left(A_{k-1} - \frac{1}{2}\sum_{j=1}^{q}(B_{k-1}^j)^2\right)(T_k - T_{k-1}) + \sum_{j=1}^{q} B_{k-1}^j\left(w_j(T_k) - w_j(T_{k-1})\right)\right] x_0$$
$$+ \sum_{i=1}^{N(t)}\left(\prod_{k=i}^{N(t)} z_k\right)\exp\left[\left(A_{N(t)} - \frac{1}{2}\sum_{j=1}^{q}(B_{N(t)}^j)^2\right)(t - T_{N(t)}) + \sum_{j=1}^{q} B_{N(t)}^j\left(w_j(t) - w_j(T_{N(t)})\right)\right]$$
$$\times \prod_{k=i+1}^{N(t)} \exp\left[\left(A_{k-1} - \frac{1}{2}\sum_{j=1}^{q}(B_{k-1}^j)^2\right)(T_k - T_{k-1}) + \sum_{j=1}^{q} B_{k-1}^j\left(w_j(T_k) - w_j(T_{k-1})\right)\right]$$
$$\times \left[\sum_{r=1}^{p}\int_{T_{i-1}}^{T_i} \exp\left[\left(A_{i-1} - \frac{1}{2}\sum_{j=1}^{q}(B_{i-1}^j)^2\right)(T_i - s) + \sum_{j=1}^{q} B_{i-1}^j\left(w_j(T_i) - w_j(s)\right)\right] C_{i-1}^r d\bar{w}_r(s)\right]$$

$$+\sum_{r=1}^{p}\int_{T_{N(t)}}^{t}\exp\left[\left(A_{N(t)}-\frac{1}{2}\sum_{j=1}^{q}(B^{j}_{N(t)})^{2}\right)(t-s)+\sum_{j=1}^{q}B^{j}_{N(t)}\left(w_{j}(t)-w_{j}(s)\right)\right]C^{r}_{N(t)}d\bar{w}_{r}(s) \tag{3.3.6}$$

Proof. By applying result of (3.3.4) on the subintervals $[T_{k-1},T_k)$, for $k=1,\dots,N(t)$, and $[T_{N(t)},t)$, we have the solution to system (3.3.5) as the following piecewise function

$$x(s)=\begin{cases} x(s,t_0,x_0) & x(t_0)=x_0,\ t_0\le s<T_1\\ x(s,T_1,x_1) & x(T_1)=x_1,\ T_1\le s<T_2\\ \dots & \\ x(s,T_i,x_i) & x(T_i)=x_k,\ T_k\le s<T_{k+1}\\ \dots & \\ x(s,T_{N(t)},x_{N(t)}) & x(T_{N(t)})=x_{N(t)},\ T_{N(t)}\le s<t \end{cases} \tag{3.3.7}$$

where $x(s,T_k,x_k)$ is a solution process in (3.3.4) with $t_0:=T_k$, $x_0:=x_k$, $A:=A_k$, $B^j:=B^j_k$, and $C^r:=C^r_k$. Then, we have

$$\begin{aligned}
&x(t)=x(t,T_{N(t)},x_{N(t)})\\
=&\exp\left[\left(A_{N(t)}-\frac{1}{2}\sum_{j=1}^{q}(B^{j}_{N(t)})^{2}\right)(t-T_{N(t)})+\sum_{j=1}^{q}B^{j}_{N(t)}\left(w_{j}(t)-w_{j}(T_{N(t)})\right)\right]x_{N(t)}\\
&+\sum_{r=1}^{p}\int_{T_{N(t)}}^{t}\exp\left[\left(A_{N(t)}-\frac{1}{2}\sum_{j=1}^{q}(B^{j}_{N(t)})^{2}\right)(t-s)+\sum_{j=1}^{q}B^{j}_{N(t)}\left(w_{j}(t)-w_{j}(s)\right)\right]C^{r}_{N(t)}d\bar{w}_{r}(s)\\
=&z_{N(t)}\exp\left[\left(A_{N(t)}-\frac{1}{2}\sum_{j=1}^{q}(B^{j}_{N(t)})^{2}\right)(t-T_{N(t)})+\sum_{j=1}^{q}B^{j}_{N(t)}\left(w_{j}(t)-w_{j}(T_{N(t)})\right)\right]\\
&\times\left[\exp\left[\left(A_{N(t)-1}-\frac{1}{2}\sum_{j=1}^{q}(B^{j}_{N(t)-1})^{2}\right)(T_{N(t)}-T_{N(t)-1})+\sum_{j=1}^{q}B^{j}_{N(t)-1}\left(w_{j}(T_{N(t)})-w_{j}(T_{N(t)-1})\right)\right]\right.\\
&\quad\times x_{N(t)-1}+\sum_{r=1}^{p}\int_{T_{N(t)-1}}^{T_{N(t)}}\exp\left[\left(A_{N(t)-1}-\frac{1}{2}\sum_{j=1}^{q}(B^{j}_{N(t)-1})^{2}\right)(T_{N(t)}-s)\right.\\
&\qquad\left.\left.+\sum_{j=1}^{q}B^{j}_{N(t)-1}\left(w_{j}(T_{N(t)})-w_{j}(s)\right)\right]C^{r}_{N(t)-1}d\bar{w}_{r}(s)\right]
\end{aligned}$$

$$+\sum_{r=1}^{p}\int_{T_{N(t)}}^{t}\exp\left[\left(A_{N(t)}-\frac{1}{2}\sum_{j=1}^{q}(B_{N(t)}^{j})^{2}\right)(t-s)+\sum_{j=1}^{q}B_{N(t)}^{j}\left(w_j(t)-w_j(s)\right)\right]C_{N(t)}^{r}d\bar{w}_r(s)$$

Next, substitute $x_{N(t)-1}=z_{N(t)-1}x(T_{N(t)-1}^{-},T_{N(t)-2},x_{N(t)-2})=z_{N(t)-1}x(T_{N(t)-1},T_{N(t)-2},x_{N(t)-2})$. The term $x(T_{N(t)-1}^{-},T_{N(t)-2},x_{N(t)-2})$ can be replaced by $x(T_{N(t)-1},T_{N(t)-2},x_{N(t)-2})$ because the solution process is continuous between jumps. Repeating the substitution gives the desired result.

■

3.4 Solution process of geometric Brownian motion with jumps

In the following sections, we present two important particular byproducts of Proposition 3.3.2. When $C_k^r=0$ for all k and r, system (3.3.5) reduces to the following first-order linear homogeneous system of Itô-Doob type stochastic differential equations with jumps. When $x(t)$ is a real-valued process, then the solution of the system is a geometric Brownian motion with jumps.

$$\begin{cases} dx(t)=A_{k-1}x(t)dt+\sum_{k=1}^{q}B_{k-1}^{j}x(t)dw_j(t),\ T_{k-1}\le t<T_k,\ x(T_{k-1})=x_{k-1} \\ x_k=z_kx(T_k^{-},T_{k-1},x_{k-1}) \end{cases} \tag{3.4.8}$$

The solution of the above system is given in the following corollary. The result follows from Proposition 3.3.2 by letting C_k^r be zero for all k and r.

Corollary 3.4.1 *If $A_kB_k^j=B_k^jA_k$ and $B_k^jB_k^{j'}=B_k^{j'}B_i^j$ for all $k\in\mathbb{N}\cup\{0\}$ and $j,j'=1,2,\ldots,q$, then the solution of system (3.4.8) is given by*

$$\begin{aligned} x(t)=&\left(\prod_{k=1}^{N(t)}z_k\right)\exp\left[\left(A_{N(t)}-\frac{1}{2}\sum_{j=1}^{q}(B_{N(t)}^{j})^{2}\right)(t-T_{N(t)})+\sum_{j=1}^{q}B_{N(t)}^{j}\left(w_j(t)-w_j(T_{N(t)})\right)\right] \\ &\times\prod_{k=1}^{N(t)}\exp\left[\left(A_{k-1}-\frac{1}{2}\sum_{j=1}^{q}(B_{k-1}^{j})^{2}\right)(T_k-T_{k-1})+\sum_{j=1}^{q}B_{k-1}^{j}\left(w_j(T_k)-w_j(T_{k-1})\right)\right]x_0 \end{aligned} \tag{3.4.9}$$

3.5 Solution process of Ornstein-Uhlenbeck process with jumps

In the case when B_k^j are zeros for all j and k, system (3.3.5) becomes a linear system with additive noise. The continuous dynamics between jumps are now governed by Ornstein-Uhlenbeck equations as follows

$$\begin{cases} dx(t) = A_{k-1}x(t)dt + \sum_{r=1}^{p} C_{k-1}^{r} d\bar{w}_r(t),\ T_{k-1} \le t < T_k,\ x(T_{k-1}) = x_{k-1} \\ x_k = z_k x(T_k^-, T_{k-1}, x_{k-1}) \end{cases} \tag{3.5.10}$$

Denote $C_k = (C_k^1, C_k^2, \ldots, C_k^p)$ for all k. Then the above system can be rewritten as

$$\begin{cases} dx(t) = A_{k-1}x(t)dt + C_{k-1}d\bar{w}(t),\ T_{k-1} \le t < T_k,\ x(T_{k-1}) = x_{k-1} \\ x_k = z_k x(T_k^-, T_{k-1}, x_{k-1}) \end{cases} \tag{3.5.11}$$

where C_k's are $n \times p$ matrices, and $\bar{w}(t)$ is a p-dimensional standard Wiener process.

Corollary 3.5.1 *The solution of system (3.5.10) is given by*

$$\begin{aligned} x(t) = & \left(\prod_{k=1}^{N(t)} z_k\right) \exp\left[A_{N(t)}(t - T_{N(t)})\right] \prod_{k=1}^{N(t)} \exp\left[A_{k-1}(T_k - T_{k-1})\right] x_0 \\ & + \sum_{i=1}^{N(t)} \left(\prod_{k=i}^{N(t)} z_k\right) \exp\left[A_{N(t)}(t - T_{N(t)})\right] \prod_{k=i+1}^{N(t)} \exp\left[A_{k-1}(T_k - T_{k-1})\right] \\ & \quad \times \left[\sum_{r=1}^{p} \int_{T_{i-1}}^{T_i} \exp\left[A_{i-1}(T_i - s)\right] C_{i-1}^{r} d\bar{w}_r(s)\right] \\ & + \sum_{r=1}^{p} \int_{T_{N(t)}}^{t} \exp\left[A_{N(t)}(t-s)\right] C_{N(t)}^{r} d\bar{w}_r(s) \end{aligned} \tag{3.5.12}$$

3.6 Infinitesimal generator of multivariate stochastic hybrid system

In this section, the infinitesimal generator of the multivariate stochastic hybrid system (3.4.8) is derived as an extension of the result in Chapter 2. The result and proof of the multivariate case is given in the following proposition.

Proposition 3.6.1 *The infinitesimal generator $\mathcal{A}_t$ of the process $x(t)$ at time t is given by*

$$\mathcal{A}_t f(v) = L^1 f(v) + \lambda(t) \int_{\mathbb{R}^+} [f(zv) - f(v)]\, k(z) dz$$

Proof.

$$\int_t^{t+\varepsilon} \int_{\mathbb{R}^n} \left[f(y) - f(x(s^-))\right] q(ds, dy)$$

$$
\begin{aligned}
&= \int_t^{t+\varepsilon} \int_{\mathbb{R}^n} \left[f(y) - f(x(s^-)) \right] p(ds,dy) - \int_t^{t+\varepsilon} \int_{\mathbb{R}^n} \left[f(y) - f(x(s^-)) \right] \tilde{p}(ds,dy) \\
&= \sum_{t \le T_i \le t+\varepsilon} \left[f(x(T_i)) - f(x(T_i^-)) \right] - \int_t^{t+\varepsilon} \int_{\mathbb{R}^n} \left[f(y) - f(x(s)) \right] Q(dy, x(s^-)) \lambda(s) ds \\
&= \sum_{t \le T_i \le t+\varepsilon} \left[f(x(T_i)) - f(x(T_i^-)) \right] - \int_t^{t+\varepsilon} \int_{\mathbb{R}^+} \left[f(zx(s)) - f(x(s)) \right] k(z)\, dz \lambda(s) ds \qquad (3.6.13)
\end{aligned}
$$

We note that the first term in (3.6.13) can be written as

$$
\begin{aligned}
&\sum_{t \le T_i \le t+\varepsilon} \left[f(x(T_i)) - f(x(T_i^-)) \right] = \sum_{i=1}^{l} \left[f(x(T_{n_i})) - f(x(T_{n_i}^-)) \right] \\
&= \left\{ \sum_{i=1}^{l} \left[f(x(T_{n_i})) - f(x(T_{n_{i-1}})) \right] + f(x(t+\varepsilon)) - f(x(T_{n_l})) \right\} \\
&\quad - \left\{ \sum_{i=1}^{l} \left[f(x(T_{n_i}^-)) - f(x(T_{n_{i-1}})) \right] + f(x(t+\varepsilon)) - f(x(T_{n_l})) \right\} \qquad (3.6.14)
\end{aligned}
$$

where T_{n_i}, $i = 1, 2, ..., l$, are the jump times on $[t, t+\varepsilon]$, and denote $T_{n_0} = t$. And note that the first term in (3.6.14) can be further simplified to $f(x(t+\varepsilon)) - f(x(t))$.

For s in $[T_{n_{i-1}}, T_{n_i})$ the state component $N(s)$ is equal to $N(T_{n_{i-1}})$ and the continuous state component $x(s)$ follows the stochastic differential equation

$$
dx(s) = A_{N(T_{n_{i-1}})} x(s) ds + B_{N(T_{n_{i-1}})} x(s) dw(s) \qquad (3.6.15)
$$

Thus, by Ito's formula[60], for s in $[T_{n_{i-1}}, T_{n_i})$ and $f \in C^1[\mathbb{R}^n, \mathbb{R}]$, $f(x(s))$ satisfies

$$
\begin{aligned}
df(x(s)) &= \sum_{j=1}^{n} \frac{\partial f}{\partial x^j}(x(s)) dx^j(s) + \frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial^2 f}{\partial x^j \partial x^k}(x(s)) dx^j(s) dx^k(s) \\
&= \sum_{j=1}^{n} (A_{N(T_{n_{i-1}})} x)^j(s) \frac{\partial f}{\partial x^j}(x(s)) ds + \sum_{j=1}^{n} (B_{N(T_{n_{i-1}})} x)^j(s) \frac{\partial f}{\partial x^j}(x(s)) dw(s) \\
&\quad + \frac{1}{2} \sum_{j,k=1}^{n} (B_{N(T_{n_{i-1}})} x)^j(s) (B_{N(T_{n_{i-1}})} x)^k(s) \frac{\partial^2 f}{\partial x^j \partial x^k}(x(s)) ds \\
&= L^1_{N(T_{n_{i-1}})} f(x(s)) ds + L^2_{N(T_{n_{i-1}})} f(x(s)) dw(s) \qquad (3.6.16)
\end{aligned}
$$

where

$$
L^1_{N(T_{n_{i-1}})} f = \left(\sum_{j=1}^{n} (A_{N(T_{n_{i-1}})} x)^j(s) \frac{\partial}{\partial x^j} + \frac{1}{2} \sum_{j,k=1}^{n} (B_{N(T_{n_{i-1}})} x)^j(s) (B_{N(T_{n_{i-1}})} x)^k(s) \frac{\partial^2}{\partial x^j \partial x^k} \right) f
$$

and

$$L^2_{N(T_{n_{i-1}})} f = \left(\sum_{j=1}^{n} (B_{N(T_{n_{i-1}})} x)^j (s) \frac{\partial}{\partial x^j} \right) f$$

. Then we can rewrite the second term in (3.6.14) as

$$\begin{aligned}
&\sum_{i=1}^{l} \left[f(x(T_{n_i}^-)) - f(x(T_{n_{i-1}})) \right] + f(x(t+\varepsilon)) - f(x(T_{n_l})) \\
=&\sum_{i=1}^{l} \int_{T_{n_{i-1}}}^{T_{n_i}} L^1_{N(T_{n_{i-1}})} f(x(s)) ds + \int_{T_{n_l}}^{t+\varepsilon} L^1_{N(T_{n_{i-1}})} f(x(s)) ds \\
&+ \sum_{i=1}^{l} \int_{T_{n_{i-1}}}^{T_i} L^2_{N(T_{n_{i-1}})} f(x(s)) dw(s) + \int_{T_{n_l}}^{t+\varepsilon} L^2_{N(T_{n_{i-1}})} f(x(s)) dw(s) \\
=&\int_t^{t+\varepsilon} L^1_{N(s)} f(x(s)) ds + \int_t^{t+\varepsilon} L^2_{N(s)} f(x(s)) dw(s) \\
=&\int_t^{t+\varepsilon} L^1 f(x(s)) ds + \int_t^{t+\varepsilon} L^2 f(x(s)) dw(s) \qquad (3.6.17)
\end{aligned}$$

where we denote $L^1 \equiv L^1_{N(s)}$ and $L^2 \equiv L^2_{N(s)}$. Hence, from (3.6.13), (3.6.14), (3.6.16) and (3.6.17), we have

$$\begin{aligned}
f(x(t+\varepsilon)) - f(x(t)) =& \int_t^{t+\varepsilon} \int_E \left[f(y) - f(x(s^-)) \right] q(ds, dy) \\
&+ \int_t^{t+\varepsilon} L^1 f(x(s)) ds + \int_t^{t+\varepsilon} L^2 f(x(s)) dw(s) \\
&+ \int_t^{t+\varepsilon} \int_{\mathbb{R}^+} [f(zx(s)) - f(x(s))] k(z) dz \lambda(s) ds \qquad (3.6.18)
\end{aligned}$$

After taking conditional expectation, the first integral in (3.6.18) vanishes since it is a $\mathcal{F}_s$-martingale [28]. And $L^2 f(x(s))$ is measurable with respect to the natural filtration $\mathcal{F}_s$, then the expected value of the integral with respect to the diffusion vanishes. From this, we obtain the infinitesimal generator of $x(t)$ as

$$\begin{aligned}
\mathcal{A}_t f(v) =& \lim_{\varepsilon \downarrow 0} \frac{E\left[f(x(t+\varepsilon)) | f(x(t)) = v \right] - f(v)}{\varepsilon} \\
=& L^1 f(v) + \lambda(t) \int_{\mathbb{R}^+} [f(zv) - f(v)] k(z) dz \qquad (3.6.19)
\end{aligned}$$

■

3.7 Concluding remarks

In this chapter, the closed form solutions of general linear non-homogeneous stochastic hybrid systems are obtained. Two special cases, the geometric Brownian motion with jumps and the Ornstein-Uhlenbeck process with jumps are discussed. The infinitesimal generator of a class of multivariate stochastic hybrid systems is derived.

4 DISTRIBUTIONS OF STOCHASTIC HYBRID MODEL WITH SWITCHING COEFFICIENTS AND JUMPS

4.1 Introduction

The main topic of this chapter is devoted to finding closed form probability density functions of the solution processes of linear homogeneous and Ornstein-Uhlenbeck type systems with jumps. By using the closed form solutions derived in Chapter 3, we determine the closed form probability density functions of solution processes of special cases of the general systems. The presented method provides an accessible way of obtaining the probability density functions without solving or approximating the solutions of Fokker-Planck partial differential equations [62].

The rest of the chapter is organized as follows. The problem of finding the closed form probability density functions are investigated in a systematic and coherent way. In Section 4.2 the probability density function of the solution process of one-dimensional linear homogeneous system of Itô-Doob type stochastic differential equation with jumps is derived. This result is an extension of the one-dimensional geometric Brownian motion process. Then, by using the concept of modal matrix [43], the probability density function of the solution process of n-dimensional linear homogeneous systems with jumps are obtained in Section 4.3. The probability distribution of the solution process of the system with Ornstein-Uhlenbeck type of continuous dynamic is extended to hybrid system in Section 4.4. Some concluding remarks are made in the last section.

4.2 Probability distribution of one-dimensional linear homogeneous models with jumps

In this section we will derive the probability density function of the solution process of the scalar version of system (3.4.8). Now $x(t)$ takes values in $\mathbb{R}_+$, and in this case A_k and B_k are scalars for all k in the system (3.4.8) and the solution process (3.4.9). Some auxiliary results are presented below. Suppose that on the interval $[0,t]$ we observed l jump times $t_1 < t_2 < \cdots < t_l < t$. The following lemma derives the joint density function of $N(t)$ and $T_1 < T_2 < \cdots < T_l$ where N is a non-homogeneous Poisson process.

Lemma 4.2.1 *N is a non-homogeneous Poisson process with intensity $\lambda(t)$ and given the observed jump times $t_1 < t_2 < \cdots < t_l$ on $[0,t]$. The joint density function of the $N(t)$ and T_1, T_2,..., T_l is given by*

$$f_{N(t),T_1,T_2,\ldots,T_l}(l,t_1,t_2,\ldots,t_l) = \prod_{i=1}^{l} \lambda(t_i) \exp\left\{-\int_0^t \lambda(u)du\right\} \tag{4.2.1}$$

Proof. For any $i = 1,\ldots,l+1$ and $v > s > 0$, we have

$$\begin{aligned} F_{T_i}(v|T_{i-1}=s) =& P(T_i \le v|T_{i-1}=s) \\ =& 1 - P(T_i > v|T_{i-1}=s) \\ =& 1 - P(N(v)-N(s)=0) \\ =& 1 - \exp\left\{-\int_s^v \lambda(u)du\right\} \end{aligned} \tag{4.2.2}$$

Then, differentiating (4.2.2) gives

$$f_{T_i}(v|T_{i-1}=s) = \lambda(v) \exp\left\{-\int_s^v \lambda(u)du\right\} \tag{4.2.3}$$

We then first obtain the joint density function of T_1, T_2,..., T_l, T_{l+1} as

$$\begin{aligned} & f_{T_1,T_2,\ldots,T_l,T_{l+1}}(t_1,t_2,\ldots,t_n,t_{l+1}) \\ =& f_{T_{l+1}}(t_{l+1}|T_l=t_l) f_{T_l}(t_l|T_{l-1}=t_{l-1}) \cdots f_{T_1}(t_1|T_0=0) \\ =& \prod_{i=1}^{l+1} \left[\lambda(t_i) \exp\left\{-\int_{t_{i-1}}^{t_i} \lambda(u)du\right\}\right] \end{aligned} \tag{4.2.4}$$

It follows that

$$
\begin{aligned}
& f_{N(t),T_1,T_2,\ldots,T_l}(l,t_1,t_2,\ldots,t_l) \\
=& \int_t^{\infty} f_{T_1,T_2,\ldots,T_l,T_{l+1}}(t_1,t_2,\ldots,t_l,t_{l+1})dt_{l+1} \\
=& \prod_{i=1}^{l}\left[\lambda(t_i)\exp\left\{-\int_{t_{i-1}}^{t_i}\lambda(u)du\right\}\right]\int_t^{\infty}\lambda(t_{l+1})\exp\left\{-\int_{t_l}^{t_{l+1}}\lambda(u)du\right\}dt_{l+1} \\
=& \prod_{i=1}^{l}\left[\lambda(t_i)\exp\left\{-\int_{t_{i-1}}^{t_i}\lambda(u)du\right\}\right]\left[-\exp\left\{-\int_{t_l}^{t_{l+1}}\lambda(u)du\right\}\Big|_{t_{l+1}=t}^{\infty}\right] \\
=& \prod_{i=1}^{l}\left[\lambda(t_i)\exp\left\{-\int_{t_{i-1}}^{t_i}\lambda(u)du\right\}\right]\exp\left\{-\int_{t_l}^{t}\lambda(u)du\right\} \\
=& \prod_{i=1}^{l}\lambda(t_i)\exp\left\{-\int_0^{t}\lambda(u)du\right\}
\end{aligned}
\tag{4.2.5}
$$

■

Now the joint density function of the jump times given the number of jumps due to the non-homogeneous Poisson process N is obtianed by dividing (4.2.1) by the probability of l jumps in $[0,t]$, namely, $\dfrac{\exp\left\{-\int_0^t\lambda(u)du\right\}\left(\int_0^t\lambda(u)du\right)^l}{l!}$. The result is summarized in the below lemmea.

Lemma 4.2.2 *For a non-homogeneous Poisson process N with intensity $\lambda(t)$, the joint density function of the jump times T_1, T_2,..., T_l conditioned on $N(t)=l$ is given by*

$$
f_{T_1,T_2,\ldots,T_l|N(t)=l}(t_1,t_2,\ldots,t_l)=\frac{l!\prod_{k=1}^{l}\lambda(t_k)}{\left(\int_0^t\lambda(u)du\right)^l} \tag{4.2.6}
$$

Next lemme gives the conditional probability density function of $x(t)$ given the number of jumps and the jump times.

Lemma 4.2.3 *Given that $N(t)=l$, and $T_1=t_1$,..., $T_l=t_l$, $x(t)$, defined by system (3.4.8), has a conditional probability density function as*

$$
f_{x(t)|N(t)=l,t_1,\cdots,t_l}(x)=\frac{1}{x}\int_{-\infty}^{\infty}h^{*l}(\ln x-s)\phi(s;\mu,\sigma)ds,\ x>0 \tag{4.2.7}
$$

where h^{*l} is the l^{th} convolution of the common probability density function h of $\ln z_k$, for $k=$

$1, 2, \ldots, l$*, and* $\phi(\cdot;\mu,\sigma)$ *denotes the normal density function with mean* μ *and variance* σ*, defined by*

$$\mu \equiv \mu(t, t_1, \cdots, t_l) = \ln x_0 + \left(A_l - \frac{B_l^2}{2}\right)(t - t_l) + \sum_{k=1}^{l}\left(A_{k-1} - \frac{B_{k-1}^2}{2}\right)(t_k - t_{k-1}) \tag{4.2.8}$$

and

$$\sigma \equiv \sigma(t, t_1, \cdots, t_l) = B_l^2(t - t_l) + \sum_{k=1}^{l} B_{k-1}^2(t_k - t_{k-1}) \tag{4.2.9}$$

Proof. Given that $N(t) = l$, and $T_1 = t_1, \cdots, T_l = t_l$, from (3.4.9), we have

$$\begin{aligned}\ln x(t) =& \sum_{k=1}^{l} \ln z_k + \ln x_0 + \left[\left(A_l - \frac{B_l^2}{2}\right)(t - t_l) + B_l\left(w(t) - w(t_l)\right)\right] \\ &+ \sum_{k=1}^{l}\left[\left(A_{k-1} - \frac{B_{k-1}^2}{2}\right)(t_k - t_{k-1}) + B_{k-1}\left(w(t_k) - w(t_{k-1})\right)\right] \\ =& V + S\end{aligned} \tag{4.2.10}$$

where we denote $V = \sum_{k=1}^{l} \ln z_k$ and S as the sum of last three terms in (4.2.10).

Since h is the common probability density function of $\ln z_k$, V as the sum of l iid random variables has the probability density function as the l^{th} convolution $h^{*l}(v)$. We further note that S is the sum of $l+1$ independent normal variables due to the independent increment property of Wiener process. Then S is normally distributed with mean μ and variance σ given in (4.2.8) and (4.2.9), respectively.

Since z_k and $w(t)$ are independent, then V and S are also independent. Applying transformation method [63] to $\ln x(t) = V + S$, we can obtain the conditional probability density function of $\ln x(t)$ as

$$f_{\ln x(t)|N(t)=l,t_1,\cdots,t_l}(x) = \int_{-\infty}^{\infty} h^{*l}(x - s)\phi(s;\mu,\sigma)ds \tag{4.2.11}$$

If follows that

$$\begin{aligned}f_{x(t)|N(t)=l,t_1,\cdots,t_l}(x) =& f_{\ln x(t)|N(t)=l,t_1,\cdots,t_l}(\ln x)\frac{1}{x} \\ =& \frac{1}{x}\int_{-\infty}^{\infty} h^{*l}(\ln x - s)\phi(s;\mu,\sigma)ds,\ x > 0\end{aligned} \tag{4.2.12}$$

$\blacksquare$

Having obtained the conditional probability density function of $x(t)$, we can derive the marginal probability distribution of the solution process in the one-dimensional case as follows.

Proposition 4.2.4 *The probability density function of the scalar version of the solution process $x(t)$ to the system (3.4.8) is given by*

$$f_{x(t)}(x) = \sum_{l=0}^{\infty} \left[\int_0^t \cdots \int_0^{t_3} \int_0^{t_2} \int_{-\infty}^{\infty} h^{*l}(\ln x - s)\phi(s;\mu,\sigma)ds \prod_{k=1}^{l} \lambda(t_k)dt_1 dt_2 \cdots dt_l \right.$$
$$\left. \times \frac{1}{x} \exp\left[-\int_0^t \lambda(u)du \right] \right] \tag{4.2.13}$$

Proof. From (4.2.6) and (4.2.7), the joint density function of $x(t)$, $T_1, \cdots, T_l$ given the condition $N(t) = l$ is given by

$$\begin{aligned} &f_{x(t),T_1,\cdots,T_l|N(t)=l}(x,t_1,\cdots,t_l) \\ =&f_{x(t)|N(t)=l,t_1,\cdots,t_l}(x) f_{T_1,T_2,\ldots,T_l|N(t)=l}(t_1,t_2,\ldots,t_l) \\ =&\frac{1}{x}\int_{-\infty}^{\infty} h^{*l}(\ln x - s)\phi(s;\mu,\sigma)ds \frac{l!\prod_{k=1}^{l}\lambda(t_k)}{\left(\int_0^t \lambda(u)du\right)^l} \end{aligned} \tag{4.2.14}$$

Integrating with respect to $t_1, t_2, \cdots$, and t_l yields the conditional probability density function of $x(t)$ given that $N(t) = l$ as

$$f_{x(t)|N(t)=l}(x) = \int_0^t \cdots \int_0^{t_3} \int_0^{t_2} \int_{-\infty}^{\infty} h^{*l}(\ln x - s)\phi(s;\mu,\sigma)ds \prod_{k=1}^{l} \lambda(t_k)dt_1 dt_2 \cdots dt_l$$
$$\times \frac{l!}{x\left(\int_0^t \lambda(u)du\right)^l} \tag{4.2.15}$$

By multiplying the above density by the probability of l jumps, namely

$$Pr(N(t) = l) = \exp\left[-\int_0^t \lambda(u)du \right] \frac{\left(\int_0^t \lambda(u)du\right)^l}{l!},$$

and then taking the summation over l, the marginal probability density function of $x(t)$ in (4.2.13) is established.

■

4.3 Probability distribution of multivariate geometric Brownian motion with jumps

In this section we will derive the probability density function of the solution process $x(t)$ for the n-dimensional stochastic system (3.4.8) under the following assumptions:

(i) The drift and diffusion coefficient matrices, A_k and B_k^j for all $k \in \mathbb{N} \cup \{0\}$ and $j = 1,2,\ldots,q$, are diagonalizable.

(ii) The coefficient matrices in each regime pairwise commute, i.e. $A_k B_k^j = B_k^j A_k$ and $B_k^j B_k^l = B_k^l B_k^j$ for all $k \in \mathbb{N} \cup \{0\}$ and $j,l = 1,2,\ldots,q$.

(iii) $A_k \in \mathcal{C}$ for $k \in \mathbb{N} \cup \{0\}$, where $\mathcal{C}$ denotes the set of $n \times n$ diagonalizable matrices whose eigenvector matrix, M, is such that $M^{-1}x > 0$ for any $x > 0$.

We first consider the stochastic system on the interval between jumps. Given that $N(t) = l$ and $T_1 = t_1, T_2 = t_2, \ldots, T_l = t_l$, consider the following SDE on $[t_k, t_{k+1})$, for some $k = 0,1,\ldots,l$,

$$dx(s) = A_k x(s)dt + \sum_{j=1}^{q} B_k^j x(s) dw_j(s),\ t_k \le s < t_{k+1},\ x(t_k) = x_k \tag{4.3.16}$$

In the following, we provide the necessary background material that will be used, subsequently. The following result provides a way to find a modal matrix that can diagonalize the coefficients in the above system.

Theorem 4.3.1 ([43]) *A set of diagonalizable matrices commutes if and only if the set is simultaneously diagonalizable, i.e. there exists an invertible matrix that can diagonalize all the matrices simultaneously.*

Remark 4.3.1 *In fact, the set of diagonalizable and commuting matrices shares the same set of independent eigenvectors. The eigenvector matrix is the one that simultaneously diagonalizes the all matrices in this set, and the resulting diagonal elements are the eigenvalues of each matrix.*

We recall [63] that a random vector x is said to have a n-dimensional multivariate normal distribution with mean and covariance matrix, μ_x and Σ_x, if its probability density function is given by

$$f(x) = (2\pi)^{-n/2}|\Sigma_x|^{-1/2}\exp\left[-\frac{1}{2}(x-\mu_x)^T\Sigma_x^{-1}(x-\mu_x)\right] \tag{4.3.17}$$

The following lemma gives the useful fact that the linear transformation of a multivariate normal random vector is again multivariate normally distributed.

Theorem 4.3.2 ([63]) *If x has a multivariate normal distribution with mean and covariance matrix ,μ_x and Σ_x, then $y = Px + c$ as a linear transformation of x follows also multivariate normal distribution with mean and covariance matrix, $\mu_y = P\mu_x + c$ and $\Sigma_y = P\Sigma_x P^T$, respectively.*

Proof. Since $y_i = \sum_{j=1}^{n} P_{ij}x_j + c_i$ and every linear combination of normal random variables is still normal, then y follows a multivariate normal distribution. By the linearity of expectation we have

$$\mu_y = E[y] = E[P\mu_x + c] = PE[x] + c = P\mu_x + c$$

and

$$\begin{aligned}\Sigma_y =& E[(y-\mu_y)(y-\mu_y)^T] = E[[P(x-\mu_x)]\,[P(x-\mu_x)]^T]\\ =& E[P(x-\mu_x)(x-\mu_x)^T P^T] = PE[(x-\mu_x)(x-\mu_x)^T]P^T = P\Sigma_x P^T\end{aligned}$$

■

To find the probability density function of $x(s)$ satisfying the SDE (4.3.16), we need to introduce some notations and definitions that will be used, subsequently. First note that according to Theorem 4.3.1 and Remark 4.3.1, A_k's and B_k^j's have the same eigenvector matrix, denoted by M_k. Moreover, $\tilde{A}_k \equiv M_k^{-1}A_kM_k = diag(\tilde{a}_k^1,\ldots,\tilde{a}_k^n)$ and $\tilde{B}_k^j \equiv M_k^{-1}B_k^jM_k = diag(\tilde{b}_{k,j}^1,\ldots,\tilde{b}_{k,j}^n)$ where $\{\tilde{a}_k^1,\ldots,\tilde{a}_k^n\}$ and $\{\tilde{b}_{k,j}^1,\ldots,\tilde{b}_{k,j}^n\}$ are the sets of eigenvalues of A_k and B_k^j, respectively, for all k, j. Next, we define a linear transformation $y = M_k^{-1}x$, then $x = M_ky$. Define $\ln y = (\ln y^1, \ln y^2, \ldots, \ln y^n)$ [69].

The below proposition gives the probability distribution of solution process $x(t)$ on the interval $[t_k, t_{k+1})$ over which the coefficients are constant.

Lemma 4.3.3 *Under assumptions $(i-iii)$, the process $x(s)$ satisfying the SDE (4.3.16) has a prob-*

ability density function given by

$$f_{x(s)}(x) = (2\pi)^{-n/2}|\Sigma_k(s)|^{-1/2}|M_k^{-1}|\prod_{i=1}^{n}\frac{1}{(M_k^{-1}x)^i}$$
$$\times \exp\left[-\frac{1}{2}(\ln(M_k^{-1}x)-\mu_k(s))^T(\Sigma_k(s))^{-1}(\ln(M_k^{-1}x)-\mu_k(s))\right], \textit{ for } s\in[t_k,t_{k+1}), \tag{4.3.18}$$

where

$$\mu_k(s) = \begin{pmatrix} \ln y^1(t_k) + \left(\tilde{a}_k^1 - \frac{1}{2}\sum_{j=1}^{q}(\tilde{b}_{k,j}^1)^2\right)(s-t_k) \\ \ln y^2(t_k) + \left(\tilde{a}_k^2 - \frac{1}{2}\sum_{j=1}^{q}(\tilde{b}_{k,j}^2)^2\right)(s-t_k) \\ \vdots \\ \ln y^n(t_k) + \left(\tilde{a}_k^n - \frac{1}{2}\sum_{j=1}^{q}(\tilde{b}_{k,j}^n)^2\right)(s-t_k) \end{pmatrix}, \; B_k^* = \begin{pmatrix} \tilde{b}_{k,1}^1 & \tilde{b}_{k,2}^1 & \cdots & \tilde{b}_{k,q}^1 \\ \tilde{b}_{k,1}^2 & \tilde{b}_{k,2}^2 & \cdots & \tilde{b}_{k,q}^2 \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{b}_{k,1}^n & \tilde{b}_{k,2}^n & \cdots & \tilde{b}_{k,q}^n \end{pmatrix},$$

and

$$\Sigma_k(s) = (s-t_k)B_k^*(B_k^*)^T$$

Proof. For $s\in[t_k,t_{k+1})$, we have $y(s)=M_k^{-1}x(s)$, and $x(s)=M_k y(s)$. By multiplying M_k^{-1} on both sides of the SDE (4.3.16), we obtain the transformed SDE as follows.

$$\begin{aligned} M_k^{-1}dx(s) &= M_k^{-1}A_k x(s)ds + \sum_{j=1}^{q} M_k^{-1}B_k^j x(s)dw_j(s) \\ \Rightarrow dM_k^{-1}x(s) &= M_k^{-1}A_k M_k y(s)ds + \sum_{j=1}^{q} M_k^{-1}B_k^j M_k y(s)dw_j(s) \\ \Rightarrow dy(s) &= \tilde{A}_k y(s)ds + \sum_{j=1}^{q}\tilde{B}_k^j y(s)dw_j(s) \end{aligned} \tag{4.3.19}$$

where $\tilde{A}_k$ and $\tilde{B}_k^j$ are diagonal matrices as defined before.

From the application of Lemma 3.3.1 with $C_k^r = 0$, the solution process of the transformed system (4.3.19) is

$$y(s) = \exp\left[\left(\tilde{A} - \frac{1}{2}\sum_{j=1}^{q}(\tilde{B}_k^j)^2\right)(s-t_k) + \sum_{j=1}^{q}\tilde{B}_k^j(w_j(s)-w_j(t_k))\right]y_k \tag{4.3.20}$$

for $s \in [t_k, t_{k+1})$.

It follows from assumption (iii) that $y(s) = M_k^{-1} x(s) > 0$ since $x(s) > 0$. Then, We can rewrite system (4.3.20) in the following form.

$$\ln y(s) = \mu_k(s) + B_k^*(w(s) - w(t_k)) \tag{4.3.21}$$

where $\mu_k(s)$ and B_k^* are defined above.

Since $w(t)$ is a standard Wiener process, then $w(s) - w(t_k)$ has a multivariate normal distribution with mean zero and covariance matrix $(s-t_k)I_n$, where I_n is the $n \times n$ identity matrix. Then, by Theorem 4.3.2, $\ln y(s)$ as a linear transformation of $w(s) - w(t_k)$ is also multivariate normally distributed with mean $\mu_k(s)$ and covariance matrix $\Sigma_k(s)$, where

$$\Sigma_k(s) = B_k^*[(s-t_k)I_n](B_k^*)^T = (s-t_k)B_k^*(B_k^*)^T$$

and the $(u,v)^{th}$ element of $\Sigma_k(s)$ is $(s-t_k)\sum_{j=1}^{q} \tilde{b}_{k,j}^u \tilde{b}_{k,j}^v$, for $u, v = 1, 2, \ldots, n$. The probability density function of $\ln y(s)$ is

$$f_{\ln y(s)}(\tilde{y}) = (2\pi)^{-n/2} |\Sigma_k(s)|^{-1/2} \exp\left[-\frac{1}{2}(\tilde{y} - \mu_k(s))^T (\Sigma_k(s))^{-1} (\tilde{y} - \mu_k(s))\right] \tag{4.3.22}$$

We now apply the method of transformation from $\ln y(s)$ to $y(s)$. Then,

$$f_{y(s)}(y) = f_{\ln y(s)}(\ln y) |\det(J_1)| \tag{4.3.23}$$

where $\ln y = (\ln y^1, \ln y^2, \ldots, \ln y^n)$, and J_1 is the Jacobian matrix. The Jacobian determinant can be computed as

$$\det(J_1) = \begin{vmatrix} \frac{\partial \ln y^1}{\partial y^1} & \frac{\partial \ln y^1}{\partial y^2} & \cdots & \frac{\partial \ln y^1}{\partial y^n} \\ \frac{\partial \ln y^2}{\partial y^1} & \frac{\partial \ln y^2}{\partial y^2} & \cdots & \frac{\partial \ln y^2}{\partial y^n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \ln y^n}{\partial y^1} & \frac{\partial \ln y^n}{\partial y^2} & \cdots & \frac{\partial \ln y^n}{\partial y^n} \end{vmatrix} = \begin{vmatrix} \frac{1}{y^1} & 0 & \cdots & 0 \\ 0 & \frac{1}{y^2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{y^n} \end{vmatrix} = \prod_{i=1}^{n} \frac{1}{y^i}$$

Then, from (4.3.22) and (4.3.23) we have the probability density function of $y(s)$ as

$$\begin{aligned} f_{y(s)}(y) =& (2\pi)^{-n/2}|\Sigma_k(s)|^{-1/2}\prod_{i=1}^{n}\frac{1}{y^i} \\ &\times \exp\left[-\frac{1}{2}(\ln y - \mu_k(s))^T(\Sigma_k(s))^{-1}(\ln y - \mu_k(s))\right] \end{aligned} \tag{4.3.24}$$

Next step is to find the probability density function of $x(s)$ by using method of transformation from $y(s)$ to $x(s)$. Since $x(s) = M_k y(s)$, or $y(s) = M_k^{-1}x(s)$, we have

$$f_{x(s)}(x) = f_{y(s)}(M_k^{-1}x)|\det(J_2)| \tag{4.3.25}$$

where

$$\det(J_2) = \det\left(\frac{\partial}{\partial x^v}\sum_{j=1}^{n}(M_k^{-1})^{uj}x^j\right)_{u,v} = \det\big((M_k^{-1})^{uv}\big)_{u,v} = \det(M_k^{-1})$$

The result follows from combining (4.3.24) and (4.3.25).

■

Now, we are ready to develop the conditional probability density function of solution process $x(t)$, which is parallel to the one-dimensional case in Lemma 4.2.3.

Lemma 4.3.4 *Under assumptions $(i-iii)$ and for given $N(t) = l$, $T_1 = t_1, T_2 = t_2, \ldots,$ and $T_l = t_l$, the solution process $x(t)$ to the n-dimensional system (3.4.8) has a conditional probability density function as*

$$\begin{aligned} & f_{x(t)|N(t)=l,t_1,\cdots,t_l}(x) \\ =& \int_{\mathbb{R}_+^n}\cdots\int_{\mathbb{R}_+^n} f_{x(t),x(t_l),\cdots,x(t_1)|N(t)=l,t_1,\cdots,t_l}(x,x_l,\cdots,x_1)dx_l\cdots dx_1 \\ =& \int_{\mathbb{R}_+^n}\cdots\int_{\mathbb{R}_+^n}(2\pi)^{-n/2}|\Sigma_l(t)|^{-1/2}|M_l^{-1}|\prod_{i=1}^{n}\frac{1}{(M_l^{-1}x)^i} \\ & \times\exp\left[-\frac{1}{2}(\ln(M_l^{-1}x)-\mu_l(t))^T(\Sigma_l(t))^{-1}(\ln(M_l^{-1}x)-\mu_l(t))\right] \\ & \times\prod_{k=0}^{l-1}\left[\int_0^\infty(2\pi)^{-n/2}|\Sigma_k(t_{k+1})|^{-1/2}|M_k^{-1}|\prod_{i=1}^{n}\frac{1}{(M_k^{-1}(x_{k+1}/z_{k+1}))^i}\right. \\ & \times\exp\left[-\frac{1}{2}(\ln(M_k^{-1}(x_{k+1}/z_{k+1}))-\mu_k(t_{k+1}))^T(\Sigma_k(t_{k+1}))^{-1}(\ln(M_k^{-1}(x_{k+1}/z_{k+1}))-\mu_k(t_{k+1}))\right] \end{aligned}$$

$$\times g(z_{k+1})\frac{1}{z_{k+1}^n}dz_{k+1}\Bigg]dx_l\cdots dx_1 \tag{4.3.26}$$

where g is the common probability density function of z_k, $k = 1,2,\ldots,l$.

Proof. We will apply the result in Lemma 4.3.3 piecewisely to the system (3.4.8) under the conditions $N(t) = l$ and $T_1 = t_1, T_2 = t_2, \ldots, T_l = t_l$. First we note that the joint probability density function of $(x(t), x(t_l), \ldots, x(t_1))$ can be expressed as

$$\begin{aligned}
&f_{x(t),x(t_l),\cdots,x(t_1)|N(t)=l,t_1,\cdots,t_l}(x,x_l,\cdots,x_1)\\
=&f_{x(t)|x(t_l),N(t)=l,t_1,\cdots,t_l}(x)f_{x(t_l)|x(t_{l-1}),N(t)=l,t_1,\cdots,t_l}(x_l)\cdots f_{x(t_1)|N(t)=l,t_1,\cdots,t_l}(x_1)
\end{aligned} \tag{4.3.27}$$

Then, for $k = 0,1,\ldots,l-1$, consider that $x(t_{k+1}) = x(t_{k+1}^-)z_{k+1}$ as a product of two random variables where the first one has the probability density function given in (4.3.18), and z_{k+1} is the random jump factor at time t_{k+1}. By the independence of $x(t_{k+1}^-)$ and z_{k+1}, we then have

$$\begin{aligned}
&f_{x(t_{k+1})|x(t_k),N(t)=l,t_1,\ldots,t_l}(x_{k+1})\\
=&\int_0^\infty f_{x(t_{k+1}^-)|x(t_k),N(t)=l,t_1,\ldots,t_l}(x_{k+1}/z_{k+1})g(z_{k+1})\frac{1}{z_{k+1}^n}dz_{k+1}\\
=&\int_0^\infty (2\pi)^{-n/2}|\Sigma_k(t_{k+1})|^{-1/2}|M_k^{-1}|\prod_{i=1}^n\frac{1}{(M_k^{-1}(x_{k+1}/z_{k+1}))^i}\\
&\times\exp\left[-\frac{1}{2}(\ln(M_k^{-1}(x_{k+1}/z_{k+1}))-\mu_k(t_{k+1}))^T(\Sigma_k(t_{k+1}))^{-1}(\ln(M_k^{-1}(x_{k+1}/z_{k+1}))-\mu_k(t_{k+1}))\right]\\
&\times g(z_{k+1})\frac{1}{z_{k+1}^n}dz_{k+1}
\end{aligned} \tag{4.3.28}$$

Then, (4.3.27) can be written as

$$\begin{aligned}
&f_{x(t),x(t_l),\cdots,x(t_1)|N(t)=l,t_1,\cdots,t_l}(x,x_l,\cdots,x_1)\\
=&(2\pi)^{-n/2}|\Sigma_l(t)|^{-1/2}|M_l^{-1}|\prod_{i=1}^n\frac{1}{(M_l^{-1}x)^i}\\
&\times\exp\left[-\frac{1}{2}(\ln(M_l^{-1}x)-\mu_l(t))^T(\Sigma_l(t))^{-1}(\ln(M_l^{-1}x)-\mu_l(t))\right]\\
&\times\prod_{k=0}^{l-1}\Bigg[\int_0^\infty(2\pi)^{-n/2}|\Sigma_k(t_{k+1})|^{-1/2}|M_k^{-1}|\prod_{i=1}^n\frac{1}{(M_k^{-1}(x_{k+1}/z_{k+1}))^i}
\end{aligned}$$

$$\times \exp\left[-\frac{1}{2}(\ln(M_k^{-1}(x_{k+1}/z_{k+1}))-\mu_k(t_{k+1}))^T(\Sigma_k(t_{k+1}))^{-1}(\ln(M_k^{-1}(x_{k+1}/z_{k+1}))-\mu_k(t_{k+1}))\right]$$
$$\times g(z_{k+1})\frac{1}{z_{k+1}^n}dz_{k+1}\Bigg] \tag{4.3.29}$$

The conditional probability density function of $x(t)$ given in (4.3.26) is obtained by integrating equation (4.3.29) with respect to $x_1, x_2, \ldots,$ and x_l.

∎

Now, we derive the unconditional probability distribution of the solution process of the n-dimensional system (3.4.8) in the following proposition.

Proposition 4.3.5 *Under assumptions $(i-iii)$, the probability density function of the solution process $x(t)$ to the n-dimensional system (3.4.8) is given by*

$$f_{x(t)}(x)=\sum_{l=0}^{\infty}\Bigg[\int_0^t\cdots\int_0^{t_3}\int_0^{t_2}\Bigg[\int_{\mathbb{R}_+^n}\cdots\int_{\mathbb{R}_+^n}(2\pi)^{-n/2}|\Sigma_l(t)|^{-1/2}|M_l^{-1}|\prod_{i=1}^{n}\frac{1}{(M_l^{-1}x)^i}$$
$$\times \exp\left[-\frac{1}{2}(\ln(M_l^{-1}x)-\mu_l(t))^T(\Sigma_l(t))^{-1}(\ln(M_l^{-1}x)-\mu_l(t))\right]$$
$$\times\prod_{k=0}^{l-1}\Bigg[\int_0^{\infty}(2\pi)^{-n/2}|\Sigma_k(t_{k+1})|^{-1/2}|M_k^{-1}|\prod_{i=1}^{n}\frac{1}{(M_k^{-1}(x_{k+1}/z_{k+1}))^i}$$
$$\times \exp\left[-\frac{1}{2}(\ln(M_k^{-1}(x_{k+1}/z_{k+1}))-\mu_k(t_{k+1}))^T(\Sigma_k(t_{k+1}))^{-1}(\ln(M_k^{-1}(x_{k+1}/z_{k+1}))-\mu_k(t_{k+1}))\right]$$
$$\times g(z_{k+1})\frac{1}{z_{k+1}^n}dz_{k+1}\Bigg]dx_l\cdots dx_1\Bigg]\prod_{i=1}^{l}\lambda(t_i)dt_1dt_2\cdots dt_l\exp\left[-\int_0^t\lambda(u)du\right]\Bigg] \tag{4.3.30}$$

Proof. The proof follows by the argument used in Proposition 4.2.4 and the incorporation of the random jumps.

∎

Remark 4.3.2 *It is obvious that the result (4.3.30) includes the one-dimensional result as a special case. As a result of this, the proof for Proposition 4.3.5 is considered to be an alternative proof of the one-dimensional result in Proposition 4.2.4.*

4.4 Probability distribution of multivariate Ornstein-Uhlenbeck process with jumps

In this section we derive the probability distribution of an Ornstein-Uhlenbeck model with jumps described by system (3.5.11). To obtain the desired result we need the following lemma which gives the probability distribution of an Ornstein-Uhlenbeck process which is the continuous dynamic between jumps in system (3.5.11).

Lemma 4.4.1 ([53, 62]) *The solution process $x(t)$ of the Ornstein-Uhlenbeck equation*

$$dx(t) = Ax(t)dt + Cd\bar{w}(t),\ t \geq t_0,\ x(t_0) = x_0,$$

follows a multivariate normal distribution with mean $\mu(t)$ and covariance matrix $\Sigma(t)$, where

$$\mu(t) = \exp^{A(t-t_0)} x_0, \quad \Sigma(t) = \int_{t_0}^{t} e^{A(t-u)} V e^{A^T(t-u)} du$$

and $V = CC^T$.

Supposed that the number of jumps and the jump times are given. By applying the above lemma piecewisely, the following result gives the conditional probability density function of $x(t)$.

Lemma 4.4.2 *Under the conditions $N(t) = l$, and $T_1 = t_1, T_2 = t_2, \ldots, T_l = t_l$, the solution process $x(t)$ for system (3.5.11) has a conditional probability density function*

$$\begin{aligned}
& f_{x(t)|N(t)=l,t_1,\cdots,t_l}(x) \\
= & \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} f_{x(t),x(t_l),\cdots,x(t_1)|N(t)=l,t_1,\cdots,t_l}(x, x_l, \cdots, x_1) dx_l \cdots dx_1 \\
= & \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} (2\pi)^{-n/2} |\Sigma_l(t)|^{-1/2} \exp\left[-\frac{1}{2}(x-\mu_l(t))^T (\Sigma_l(t))^{-1} (x-\mu_l(t))\right] \\
& \times \prod_{k=0}^{l-1} \left[\int_0^\infty (2\pi)^{-n/2} |\Sigma_k(t_{k+1})|^{-1/2} \right. \\
& \times \exp\left[-\frac{1}{2}\left(\frac{x_{k+1}}{z_{k+1}} - \mu_k(t_{k+1})\right)^T (\Sigma_k(t_{k+1}))^{-1} \left(\frac{x_{k+1}}{z_{k+1}} - \mu_k(t_{k+1})\right)\right] \\
& \left. \times g(z_{k+1}) \frac{1}{z_{k+1}^n} dz_{k+1}\right] dx_l \cdots dx_1
\end{aligned} \tag{4.4.31}$$

where g is the common probability density function of z_k, $k = 1, 2, \ldots, l$.

Proof. As we noted before that the joint probability density function of $(x(t), x(t_l), \ldots, x(t_1))$, under the conditions $N(t) = l$ and $T_1 = t_1, T_2 = t_2, \ldots, T_l = t_l$, can be expressed as

$$\begin{aligned} &f_{x(t),x(t_l),\cdots,x(t_1)|N(t)=l,t_1,\cdots,t_l}(x, x_l, \cdots, x_1) \\ =&f_{x(t)|x(t_l),N(t)=l,t_1,\cdots,t_l}(x) f_{x(t_l)|x(t_{l-1}),N(t)=l,t_1,\cdots,t_l}(x_l) \cdots f_{x(t_1)|N(t)=l,t_1,\cdots,t_l}(x_1) \end{aligned} \tag{4.4.32}$$

Then, by applying the result in Lemma 4.4.1 on each interval between jumps $[t_k, t_{k+1})$, we have, for $s \in [t_k, t_{k+1})$,

$$\begin{aligned} &f_{x(s)|x(t_k),N(t)=l,t_1,\ldots,t_l}(x) = \phi(x; \mu_k(s), \Sigma_k(s)) \\ =&(2\pi)^{-n/2} |\Sigma_k(s)|^{-1/2} \exp\left[-\frac{1}{2}(x - \mu_k(s))^T (\Sigma_k(s))^{-1} (x - \mu_k(s))\right] \end{aligned} \tag{4.4.33}$$

where

$$\mu_k(s) = \exp^{A_k(s-t_k)} x_k, \quad \Sigma_k(s) = \int_{t_k}^{s} e^{A_k(s-u)} V_k e^{A_k^T(s-u)} du$$

and $V_k = C_k C_k^T$. Then, for $k = 0, 1, \ldots, l-1$, consider that $x(t_{k+1}) = x(t_{k+1}^-) z_{k+1}$ as a product of two random variables where the first one has the probability density function given in (4.4.33), and z_{k+1} is the random jump factor at time t_{k+1}. By the independence of $x(t_{k+1}^-)$ and z_{k+1}, then

$$\begin{aligned} &f_{x(t_{k+1})|N(t)=l,t_1,\ldots,t_l}(x_{k+1}) \\ =&\int_0^\infty f_{x(t_{k+1}^-)|N(t)=l,t_1,\ldots,t_l}\left(\frac{x_{k+1}}{z_{k+1}}\right) g(z_{k+1}) \frac{1}{z_{k+1}^n} dz_{k+1} \\ =&\int_0^\infty (2\pi)^{-n/2} |\Sigma_k(t_{k+1})|^{-1/2} \\ &\times \exp\left[-\frac{1}{2}\left(\frac{x_{k+1}}{z_{k+1}} - \mu_k(t_{k+1})\right)^T (\Sigma_k(t_{k+1}))^{-1} \left(\frac{x_{k+1}}{z_{k+1}} - \mu_k(t_{k+1})\right)\right] \\ &\times g(z_{k+1}) \frac{1}{z_{k+1}^n} dz_{k+1} \end{aligned} \tag{4.4.34}$$

Then, the conditional joint probability density function (4.4.32) can be written as

$$\begin{aligned} &f_{x(t),x(t_l),\cdots,x(t_1)|N(t)=l,t_1,\cdots,t_l}(x, x_l, \cdots, x_1) \\ =&(2\pi)^{-n/2} |\Sigma_l(t)|^{-1/2} \exp\left[-\frac{1}{2}(x - \mu_l(t))^T (\Sigma_l(t))^{-1} (x - \mu_l(t))\right] \end{aligned}$$

$$
\times \prod_{k=0}^{l-1} \left[\int_0^\infty (2\pi)^{-n/2} |\Sigma_k(t_{k+1})|^{-1/2} \right.
$$
$$
\times \exp\left[-\frac{1}{2} \left(\frac{x_{k+1}}{z_{k+1}} - \mu_k(t_{k+1}) \right)^T (\Sigma_k(t_{k+1}))^{-1} \left(\frac{x_{k+1}}{z_{k+1}} - \mu_k(t_{k+1}) \right) \right]
$$
$$
\left. \times g(z_{k+1}) \frac{1}{z_{k+1}^n} dz_{k+1} \right] \tag{4.4.35}
$$

The conditional probability density function of $x(t)$ in (4.4.31) is obtained by integrating equation (4.4.35) with respect to $x_1, x_2, \ldots,$ and x_l.

■

Finally, following the same argument in Proposition 4.2.4, we present the unconditional probability distribution of the solution process of the Ornstein-Uhlenbeck models with jumps given in the following proposition.

Proposition 4.4.3 *The probability density function of the solution process $x(t)$ of system (3.5.11) is given by*

$$
f_{x(t)}(x) = \sum_{l=0}^{\infty} \left[\int_0^t \cdots \int_0^{t_3} \int_0^{t_2} \left[\int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} (2\pi)^{-n/2} |\Sigma_l(t)|^{-1/2} \right.\right.
$$
$$
\times \exp\left[-\frac{1}{2} (x - \mu_l(t))^T (\Sigma_l(t))^{-1} (x - \mu_l(t)) \right]
$$
$$
\times \prod_{k=0}^{l-1} \left[\int_0^\infty (2\pi)^{-n/2} |\Sigma_k(t_{k+1})|^{-1/2} \right.
$$
$$
\times \exp\left[-\frac{1}{2} \left(\frac{x_{k+1}}{z_{k+1}} - \mu_k(t_{k+1}) \right)^T (\Sigma_k(t_{k+1}))^{-1} \left(\frac{x_{k+1}}{z_{k+1}} - \mu_k(t_{k+1}) \right) \right]
$$
$$
\left. \left. \times g(z_{k+1}) \frac{1}{z_{k+1}^n} dz_{k+1} \right] dx_l \cdots dx_1 \right] \prod_{i=1}^{l} \lambda(t_i) dt_1 dt_2 \cdots dt_l
$$
$$
\left. \times \exp\left[- \int_0^t \lambda(u) du \right] \right] \tag{4.4.36}
$$

4.5 Concluding remarks

In this chapter, the methods of finding probability density functions of closed form solutions are initiated for both the linear homogeneous systems and systems with drift and additive noise (Ornstein-Uhlenbeck systems). This approach provides a procedure of finding the probability density functions without solving or approximating Fokker-Planck equations. In fact, the Fokker-Planck equation corresponding to system (3.4.8) has state dependent coefficients. For example, for $n = 1$ the Fokker-Planck equation corresponding to system (3.4.8) even in the absence of discrete time interventions is given by

$$\begin{aligned}\frac{\partial f}{\partial t} &= -\frac{\partial}{\partial x}[Axf] + \frac{\partial^2}{\partial x^2}[B^2x^2 f] \\ &= (2B^2 - A)f + (4B^2 - A)x\frac{\partial}{\partial x}f + B^2x^2\frac{\partial^2}{\partial x^2}f \end{aligned} \tag{4.5.37}$$

As a result of this, equations of this type are not easily solvable in closed form solutions. In future, we attempt to find the probability distributions of the solutions of the general linear non-homogeneous systems. In addition, by employing nonlinear transformation, we hope to develop probability distributions for nonlinear stochastic hybrid systems.

5 Parameter Estimation

5.1 Introduction

In this chapter, we discuss some estimation methods for the parameters of the stochastic hybrid systems. We will consider, for simplicity, two cases of real-valued systems discussed earlier. The first one is the one-dimensional geometric Brownian motion process with jumps described by the following system.

$$\begin{cases} dx(t) = A_{k-1}x(t)dt + B_{k-1}x(t)dw(t),\ T_{k-1} \leq t < T_k,\ x(T_{k-1}) = x_{k-1} \\ x_k = x(T_k^-, T_{k-1}, x_{k-1}) - z_k \end{cases} \tag{5.1.1}$$

The second one is the one-dimensional Ornstein-Uhlenbeck process with jumps satisfying the following system.

$$\begin{cases} dx(t) = A_{k-1}x(t)dt + C_{k-1}dw(t),\ T_{k-1} \leq t < T_k,\ x(T_{k-1}) = x_{k-1} \\ x_k = z_k x(T_k^-, T_{k-1}, x_{k-1}) \end{cases} \tag{5.1.2}$$

For both systems (5.1.1) and (5.1.2), the discrete jump times are governed by a non-homogeneous Poisson process N with intensity function $\lambda(t)$. The following two topics are the main purpose of this chapter.

1. Estimation for $\lambda(t)$.
2. Parameter estimation for the continuous dynamic.

The parameters of the continuous dynamic to be estimated are A_k and B_k for system (5.1.1), and are A_k and C_k for system (5.1.2).

Suppose that it is given a set of realizations of the process on the interval $[0,T]$, and the jump times $T_1 = t_1, T_2 = t_2, \ldots, T_l = t_l$ on the interval $[0,T]$ are observed. Denote the realizations on

$[t_k, t_{k+1})$ as $(x_k, t_k), (x_{k1}, t_{k1}), (x_{k2}, t_{k2}), \ldots, (x_{kn_k}, t_{kn_k})$, for $k = 0, 1, 2, \ldots, l$, and let $t_{l+1} \equiv T$. Assume that the realizations are equally spaced with temporal difference Δt. Given the jump times, we will first estimate the intensity function $\lambda(t)$ of the Poisson process N. Then the parameters of the continuous dynamic will be estimated piecewisely on the intervals between jumps.

In Section 5.2 and 5.3, the parameters estimation for the homogeneous Poisson process and the power-law process is discussed. Then the focus turns to estimating the parameters of the continuous dynamics. In Section 5.4, the parameters of the geometric Brownian motion process are estimated through transformation. The procedure of estimating the parameters of the Ornstein-Uhlenbeck process is given in Section 5.5. The chapter ends with some concluding remarks.

5.2 Estimation for homogeneous Poisson process

5.2.1 Point estimation

The simplest case of the Poisson process is the homogeneous Poisson process or the stationary Poisson process. In this case the intensity of the process at any instant is constant, meaning $\lambda(t) = \lambda$ for some $\lambda > 0$. For the special case of homogeneous Poisson process, we can obtain the maximum likelihood estimator (MLE) analytically, which is generally not a simple task. In order to obtain the MLE, we first discuss some properties of the interarrival times of the homogeneous Poisson process. The interarrival times are the time elapsed between two events. In our case, the jumps, hence they are also known as the interevent times. Let us denote the interarrival times as $Y_k = T_k - T_{k-1}$ for $k = 1, 2, \ldots, l$. Since the Poisson process is independent to the past, it follows that the interarrival times $Y_1, Y_2, \ldots,$ and Y_l are mutually independent to each other. The following well-known lemma gives the link between the distribution of the interarrival times and the homogeneous Poisson process. For the sake of completeness, we provide the proof of the lemma.

Lemma 5.2.1 ([54]) *If N is a homogeneous Poisson process with intensity $\lambda > 0$, then the interarrival times $Y_1, Y_2, \ldots, Y_l$ are i.i.d. exponential random variables with mean $1/\lambda$.*

Proof. First, we will show the independence. Without loss in generality, for $i < j$, consider

$$P(Y_i \leq a, Y_j \leq b) = 1 - P(Y_i > a) - P(Y_j > b) + P(Y_i > a, Y_j > b) \tag{5.2.3}$$

Note that

$$
\begin{aligned}
P(Y_i > a, Y_j > b) =& P(N(T_{i-1}+a) - N(T_{i-1}) = 0, N(T_{j-1}+b) - N(T_{j-1}) = 0) \\
=& P(N(T_{i-1}+a) - N(T_{i-1}) = 0) P(N(T_{j-1}+b) - N(T_{j-1}) = 0) \\
=& P(Y_i > a) P(Y_j > b)
\end{aligned}
\tag{5.2.4}
$$

Under the event $N(T_{i-1}+a) - N(T_{i-1}) = 0$, it follows that $T_{i-1}+a < T_i \leq T_{j-1}$ for $i < j$. This implies that the two increments $N(T_{i-1}+a) - N(T_{i-1})$ and $N(T_{j-1}+b) - N(T_{j-1})$ are disjoint. Hence the above equality is true by the independent increment property of Poisson process. From (5.2.3) and (5.2.4), we have

$$
\begin{aligned}
P(Y_i \leq a, Y_j \leq b) =& 1 - P(Y_i > a) - P(Y_j > b) + P(Y_i > a) P(Y_j > b) \\
=& (1 - P(Y_i > a))(1 - P(Y_j > b)) \\
=& P(Y_i \leq a) P(Y_j \leq b)
\end{aligned}
\tag{5.2.5}
$$

This shows the pairwise independence of $Y_1, Y_2, \ldots, Y_l$. The mutual independence of $Y_1, Y_2, \ldots, Y_l$ can be shown analogously. Next, by the homogeneity of N, we have

$$
\begin{aligned}
P(Y_i \leq y) =& P(T_i - T_{i-1} \leq y) \\
=& 1 - P(T_i - T_{i-1} > y) \\
=& 1 - P(N(T_i) - N(T_{i-1}) > y) \\
=& 1 - P(N(T_{i-1}+y) - N(T_{i-1}) = 0) \\
=& 1 - P(N(y) - N(0) = 0) \\
=& 1 - e^{-\lambda y}
\end{aligned}
\tag{5.2.6}
$$

From above we note that the cumulative distribution function of Y_i coincides with that of an exponential random variable with mean $1/\lambda$. This concludes the proof.

■

Now, we are ready to obtain the MLE of the intensity of the homogeneous Poisson process from the above lemma.

Theorem 5.2.2 ([54]) *The MLE of the intensity λ of a homogeneous Poisson process N is given by l/T.*

Proof. Given that $t_1 < t_2 < \cdots < t_l < T$, the likelihood function of λ is given by

$$\begin{aligned} L(\lambda) \equiv & L(\lambda|l, t_1, t_2, \ldots, t_l) \\ = & f(Y_1 = t_1, Y_2 = t_2 - t_1, \ldots, Y_l = t_l - t_{l-1}, Y_{l+1} > T - t_l) \\ = & f_{Y_1}(t_1) f_{Y_2}(t_2 - t_1) \cdots f_{Y_l}(t_l - t_{l-1})(1 - F_{Y_{l+1}}(T - t_l)) \\ = & \prod_{k=1}^{l} \lambda e^{-\lambda(t_k - t_{k-1})} e^{-\lambda(T - t_l)} \\ = & \lambda^l e^{\lambda T} \end{aligned} \tag{5.2.7}$$

where Y_{l+1} denote the unobserved $l + 1^{th}$ jump whose possible occurrence is outside $[0, T]$. The log likelihood of λ is

$$\ln L(\lambda) = l \ln \lambda - \lambda T \tag{5.2.8}$$

Taking the derivative with respect to λ and solving for the zero yields the MLE of λ as $\hat{\lambda} = \frac{l}{T}$. ∎

Remark 5.2.1 *In the above theorem, we obtained the likelihood function (5.2.7) by utilizing the i.i.d. exponentially distributed interarrival times. The result in Lemma 4.2.1 provides an alternative approach to the likelihood function by letting $\lambda(t) = \lambda$.*

5.2.2 Interval estimation

To obtain the standard error of the MLE, $\hat{\lambda}$, let us consider the second derivative of the score function $\ln L(\lambda)$:

$$\frac{\partial^2}{\partial \lambda^2} \ln L(\lambda) = -\frac{l}{\lambda^2} \tag{5.2.9}$$

Since $l = N(T)$ has a Poisson distribution with mean λT, the Fisher information [42, 63] can be computed as follows.

$$\begin{aligned} I(\lambda) =& E[-\frac{\partial^2}{\partial\lambda^2} \ln L(\lambda)] \\ =& \frac{E[l]}{\lambda^2} \\ =& \frac{T}{\lambda} \end{aligned} \tag{5.2.10}$$

Then, the variance of the MLE, $\hat{\lambda}$, is estimated by substituting the estimated for λ in the inverse of the Fisher information:

$$var(\hat{\lambda}) = \frac{1}{I(\lambda)}\Big|_{\lambda=\hat{\lambda}} = \frac{\hat{\lambda}}{T} = \frac{l}{T^2} \tag{5.2.11}$$

Taking the square root of the variance gives the standard error of the MLE, $\hat{\lambda}$, as

$$s.e.(\hat{\lambda}) = \frac{\sqrt{l}}{T} \tag{5.2.12}$$

As a result of Theorem 1.4.2, asymptotically, the maximum likelihood estimator has normal distribution as sample size increases.

$$\frac{\hat{\lambda} - \lambda}{s.e.(\hat{\lambda})} \xrightarrow{\mathcal{D}} N(0,1) \tag{5.2.13}$$

Consequently, an asymptotic $100(1-\gamma)\%$ confidence interval for λ is obtained, and it is given by

$$\left[\hat{\lambda} - z_{\gamma/2} s.e.(\hat{\lambda}),\ \hat{\lambda} + z_{\gamma/2} s.e.(\hat{\lambda})\right] \tag{5.2.14}$$

Another asymptotic confidence interval for λ will be discuss in later subsection.

5.3 Estimation for power-law process

5.3.1 Point estimation

In this section, we will discuss the parameters estimation for a special case of the non-homogeneous Poisson process, the power-law process. The power-law process is a well-known model used in the reliability of repairable systems. In 1964 Duane [31] was the first to discover that the cumulative number of failures of repairable systems up to time t exhibit the shape of a power-law growth function. The mathematical formulation of the power-law process as a special case of the non-homogeneous Poisson process was developed by Crow [25] in 1974.

A non-homogeneous Poisson process N is said to be a *power-law process*, if the mean value function of N takes the form of a power-law function:

$$m(t) = E[N(t)] = \int_0^t \lambda(u)du = \alpha\, t^{\beta}, \text{ for } \alpha > 0, \beta > 0, \tag{5.3.15}$$

The corresponding intensity function is

$$\lambda(t) = \frac{d}{dt}m(t) = \alpha\beta t^{\beta-1} \tag{5.3.16}$$

where α and β are the scale and shape parameters, respectively. When $\beta > 1$, the intensity function is increasing. This means that the failures, in our case, the jumps, occur more frequently as time goes on. This system is sometimes called deteriorating in the study of reliability. When $0 < \beta < 1$, the intensity function is decreasing, and the failures occur less frequently. Then the system is said to have reliability growth or be improving. For the case when $\beta = 1$, the power-law process reduces to a homogeneous Poisson process with intensity α.

One appealing reason for the popularity of the power-law process is that the maximum likelihood estimators of parameters α and β can be obtained in closed form expression [25, 34]. The following theorem gives the MLE of the parameters of the intensity of a power-law process. The proof of the result is given for the sake of completeness.

Theorem 5.3.1 ([25]) *Let N be a power-law process with intensity function given in (5.3.16). If l jumps have been observed on the interval $[0,T]$ at times $t_1, t_2, \ldots, t_l$, then the maximum likelihood estimators for the parameters α and β are given by*

$$\begin{aligned} \hat{\alpha} &= \frac{l}{T^{\hat{\beta}}}, \\ \hat{\beta} &= \frac{l}{\sum_{k=1}^{l} \ln \dfrac{T}{t_k}} \end{aligned} \tag{5.3.17}$$

Proof. By substituting the intensity function (5.3.16) in (4.2.1), we can obtain the likelihood function of α and β:

$$\begin{aligned}
L(\alpha,\beta) \equiv & L(\alpha,\beta|l,t_1,t_2,\ldots,t_l) \\
= & f_{N(T),T_1,T_2,\ldots,T_l}(l,t_1,t_2,\ldots,t_l|\alpha,\beta) \\
= & \alpha^l\beta^l\prod_{k=1}^{l} t_k^{\beta-1}e^{-\alpha T^{\beta}} \qquad (5.3.18)
\end{aligned}$$

The log likelihood (score function) is

$$\ln L(\alpha,\beta) = l\ln\alpha + l\ln\beta + (\beta-1)\sum_{k=1}^{l}\ln t_k - \alpha T^{\beta} \qquad (5.3.19)$$

The first partial derivatives with respect to α and β are

$$\begin{aligned}
\frac{\partial}{\partial\alpha}\ln L(\alpha,\beta) = & \frac{l}{\alpha} - T^{\beta}, \\
\frac{\partial}{\partial\beta}\ln L(\alpha,\beta) = & \frac{l}{\beta} + \sum_{k=1}^{l}\ln t_k - \alpha T^{\beta}\ln T
\end{aligned}$$

By setting both partial derivatives equal to zero, we obtain the MLE, $\hat{\alpha}$ and $\hat{\beta}$, given in (5.3.17). ■

5.3.2 Interval estimation

It is known (see Crow [25] and Guida [39]) that $2l\frac{\beta}{\hat{\beta}}$ possesses the chi-square distribution with $2l$ degrees of freedom, denoted by χ^2_{2l}. From this result, an exact confidence interval for the shape parameter β can be found as follows. First, we note that

$$P\left(\chi^2_{2l,1-\gamma/2} < 2l\frac{\beta}{\hat{\beta}} < \chi^2_{2l,\gamma/2}\right) = 1-\gamma$$

where $\chi^2_{\nu,\gamma}$ is the $1-\gamma$ quantile of a χ^2_{ν} random variable. Rewriting the above inequality, we obtain

$$P\left(\frac{\hat{\beta}}{2l}\chi^2_{2l,1-\gamma/2} < \beta < \frac{\hat{\beta}}{2l}\chi^2_{2l,\gamma/2}\right) = 1-\gamma$$

Hence, an exact $100(1-\gamma)\%$ confidence interval for β is

$$\left[\frac{\hat{\beta}}{2l}\chi^2_{2l,1-\gamma/2},\ \frac{\hat{\beta}}{2l}\chi^2_{2l,\gamma/2}\right] \tag{5.3.20}$$

However, the exact distribution of the scale parameter α is not known [25]. Gaudoin et al. [34] suggested various methods to find asymptotic confidence intervals for α derived from using Fisher information matrix. The estimated Fisher information matrix of the model is

$$I(\hat{\alpha},\hat{\beta}) = \left.\begin{pmatrix} -E\left[\frac{\partial^2 \ln L(\alpha,\beta)}{\partial\alpha^2}\right] & -E\left[\frac{\partial^2 \ln L(\alpha,\beta)}{\partial\alpha\partial\beta}\right] \\ -E\left[\frac{\partial^2 \ln L(\alpha,\beta)}{\partial\alpha\partial\beta}\right] & -E\left[\frac{\partial^2 \ln L(\alpha,\beta)}{\partial\beta^2}\right] \end{pmatrix}\right|_{(\alpha,\beta)=(\hat{\alpha},\hat{\beta})} \tag{5.3.21}$$

By Theorem 1.4.2, the estimated variance-covariance matrix of the MLEs is obtained by inverting the above estimated Fisher information matrix. According to Gaudoin et al. [34], the first order approximation to the estimated variance-covariance matrix is

$$I^{-1}(\hat{\alpha},\hat{\beta}) = \begin{pmatrix} \frac{\hat{\alpha}^2}{l}\left[1+\left(\ln\frac{l}{\hat{\alpha}}\right)^2\right] & -\frac{\hat{\alpha}\hat{\beta}}{l}\ln\frac{l}{\hat{\alpha}} \\ -\frac{\hat{\alpha}\hat{\beta}}{l}\ln\frac{l}{\hat{\alpha}} & \frac{\hat{\beta}^2}{l} \end{pmatrix} \tag{5.3.22}$$

Thus, the first diagonal element in (5.3.22) is the estimated variance of $\hat{\alpha}$. By the asymptotic normality of MLE, we obtain an approximate $100(1-\gamma)\%$ confidence interval for α given by

$$\left[\hat{\alpha} - z_{\gamma/2}\frac{\hat{\alpha}}{\sqrt{l}}\sqrt{1+\left(\ln\frac{l}{\hat{\alpha}}\right)^2},\ \hat{\alpha} + z_{\gamma/2}\frac{\hat{\alpha}}{\sqrt{l}}\sqrt{1+\left(\ln\frac{l}{\hat{\alpha}}\right)^2}\right] \tag{5.3.23}$$

Remark 5.3.1 *When $\beta = 1$ and $\alpha = \lambda$, the power-law process reduces to a homogeneous Poisson process discussed earlier, thus (5.3.23) is an alternative approximate confidence interval for λ of a homogeneous Poisson process.*

5.4 Estimation for geometric Brownian motion

5.4.1 Point estimation

In the preceding sections, we have discussed some methods of parameters estimation for the discrete dynamic of the stochastic hybrid system. Once the jump times are observed, the estimation of the parameters of the continuous dynamic in the one-dimensional case can be done piecewisely on each interval between jumps.

In this section, we will discuss method of estimating the parameters of the continuous dynamic in system (5.1.1). Since all increments are independent to one another, it suffices to estimate the parameters on each interval separately. According to the notation above, let $(x_k,t_k),(x_{k1},t_{k1}),(x_{k2},t_{k2}),\ldots,(x_{kn_k},t_{kn_k})$ be the realizations on the interval $[t_k,t_{k+1})$. The continuous dynamic on this interval is described by the following SDE

$$dx(t) = A_k x(t)dt + B_k x(t)dw(t),\ t_k \le t < t_{k+1},\ x(t_k) = x_k \tag{5.4.24}$$

The following theorem gives the maximum likelihood estimators for the parameters of SDE (5.4.24). We give the proof for completeness.

Theorem 5.4.1 ([70]) *Given the observations $(x_k,t_k),(x_{k1},t_{k1}),(x_{k2},t_{k2}),\ldots,(x_{kn_k},t_{kn_k})$ for the stochastic process x described by (5.4.24), the maximum likelihood estimators for the parameters A_k and B_k are given by*

$$\begin{aligned}\hat{A}_k =& \frac{m_k}{\Delta t} + \frac{1}{2}\hat{B}_k^2,\\ \hat{B}_k =& \sqrt{\frac{n_k-1}{n_k}}\frac{S_k}{\sqrt{\Delta t}}\end{aligned} \tag{5.4.25}$$

where m_k and S_k^2 be the sample mean and sample variance for $\ln x(t_{ki}) - \ln x(t_{k(i-1)})$, $i = 1,2,\ldots,n_k$.

Proof. Given the observations, A_k and B_k are the parameters to be estimated. Applying the Itô-Doob's formula on the transformation $\ln x(t)$, we have

$$\begin{aligned}d(\ln x(t)) =& \frac{\partial}{\partial x}(\ln x(t))dx(t) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(\ln x(t))(dx(t))^2\\ =& \frac{1}{x(t)}dx(t) - \frac{1}{2}\frac{1}{x^2(t)}(dx(t))^2\\ =& A_k dt + B_k dw(t) - \frac{1}{2}\frac{1}{x^2(t)}B_k^2 x^2(t)(dw(t))^2\\ =& (A_k - \frac{1}{2}B_k^2)dt + B_k dw(t)\end{aligned} \tag{5.4.26}$$

For $i = 1, 2, \ldots, n_k$, consider the difference from the above SDE,

$$\ln x(t_{ki}) - \ln x(t_{k(i-1)}) = (A_k - \frac{1}{2}B_k^2)\Delta t + B_k \left(w(t_{ki}) - w(t_{k(i-1)})\right) \tag{5.4.27}$$

Note that $w(t_{ki}) - w(t_{k(i-1)})$, as increments of a Brownian motion process, are i.i.d. normal random variables with mean zero and variance Δt. It then follows that $\ln x(t_{ki}) - \ln x(t_{k(i-1)})$ are i.i.d. normally distributed with mean $(A_k - \frac{1}{2}B_k^2)\Delta t$ and variance $B_k^2\Delta t$. Hence, the maximum likelihood estimators for the mean and variance of a normal sample [63] are given by

$$(\hat{A}_k - \frac{1}{2}\hat{B}_k^2)\Delta t = \frac{\sum_{i=1}^{n_k} \left[\ln x(t_{ki}) - \ln x(t_{k(i-1)})\right]}{n_k} = m_k \tag{5.4.28}$$

and

$$\hat{B}_k^2\Delta t = \frac{n_k - 1}{n_k} S_k^2 \tag{5.4.29}$$

where m_k and S_k^2 be the sample mean and sample variance for $\ln x(t_{ki}) - \ln x(t_{k(i-1)})$, $i = 1, 2, \ldots, n_k$. Rearranging the above equations yields the MLE for A_k and B_K in (5.4.25).

■

5.4.2 Interval estimation

In this subsection, the standard errors of the estimators will be obtained through the use of information matrix. Let $u_i = \ln x(t_{ki}) - \ln x(t_{k(i-1)})$ for $i = 1, 2, \ldots, n_k$. From (5.4.27), we know that u_i's are i.i.d. normally distributed. The log-likelihood of $u_1, u_2, \ldots, u_{n_k}$ is given by

$$\begin{aligned} \ln L \equiv& \ln L(A_k, B_k | u_1, u_2, \ldots, u_{n_k}) \\ =& -n_k \ln\sqrt{2\pi} - n_k \ln(B_k\sqrt{\Delta t}) - \frac{1}{2B_k^2\Delta t}\sum_{i=1}^{n_k}\left(u_i - (A_k - \frac{1}{2}B_k^2)\Delta t\right)^2 \end{aligned} \tag{5.4.30}$$

The second partial derivatives of $\ln L$ are as follows:

$$J_{1,1} = -\frac{\partial^2}{\partial A_k^2}\ln L = \frac{n_k\Delta t}{B_k^2}, \tag{5.4.31}$$

$$J_{1,2} = J_{2,1} = -\frac{\partial^2}{\partial A_k \partial B_k} \ln L = \frac{2\left(\sum_{i=1}^{n_k} u_i - n_k \Delta t A_k\right)}{B_k^3}, \tag{5.4.32}$$

and

$$J_{2,2} = -\frac{\partial^2}{\partial B_k^2} \ln L = n_k \Delta t + \frac{2n_k}{B_k^2}. \tag{5.4.33}$$

From the above derivatives, the observed Fisher information matrix is obtained.

$$J(A_k, B_k) = \begin{pmatrix} J_{1,1} & J_{1,2} \\ J_{1,2} & J_{2,2} \end{pmatrix} \tag{5.4.34}$$

and the inverse information matrix is

$$J^{-1}(A_k, B_k) = \frac{1}{J_{1,1}J_{2,2} - J_{1,2}^2} \begin{pmatrix} J_{2,2} & J_{1,2} \\ J_{1,2} & J_{1,1} \end{pmatrix} \tag{5.4.35}$$

Substitute the parameters by the MLE derived in (5.4.25) gives an estimate of the information matrix. From the results in Theorem 1.4.2, the variance of the estimates are then given by

$$Var(\hat{A}_k) = \left. \frac{J_{2,2}}{J_{1,1}J_{2,2} - J_{1,2}^2} \right|_{(A_k, B_k) = (\hat{A_k}, \hat{B_k})} \tag{5.4.36}$$

and

$$Var(\hat{B}_k) = \left. \frac{J_{1,1}}{J_{1,1}J_{2,2} - J_{1,2}^2} \right|_{(A_k, B_k) = (\hat{A_k}, \hat{B_k})} \tag{5.4.37}$$

The standard errors of the estimates are obtained by taking square roots of the variances above. Applying the central limit theorem, we obtain approximate $100(1-\gamma)\%$ confidence intervals for A_k and B_k as

$$\left[\hat{A}_k - z_{\gamma/2}\sqrt{Var(\hat{A_k})},\ \hat{A}_k + z_{\gamma/2}\sqrt{Var(\hat{A_k})}\right], \tag{5.4.38}$$

and

$$\left[\hat{B}_k - z_{\gamma/2}\sqrt{Var(\hat{A}_k)},\ \hat{B}_k + z_{\gamma/2}\sqrt{Var(\hat{A}_k)}\right]. \tag{5.4.39}$$

5.5 Estimation for Ornstein-Uhlenbeck process

5.5.1 Point estimation

In this section,the method of estimating the parameters of the continuous dynamic in system (5.1.2) will be investigated. Analogously to the previous section, it suffices to estimate the parameters on each interval separately. The continuous dynamic on the interval $[t_k, t_{k+1})$ follows the SDE

$$dx(t) = A_k x(t)dt + C_k dw(t),\ t_k \le t < t_{k+1},\ x(t_k) = x_k \tag{5.5.40}$$

Utilizing the ideas of discretization and least squares methods [73], we obtain the estimators for the parameters, A_k and C_k, of the above Ornstein-Uhlenbeck model. The result is given in the following theorem.

Theorem 5.5.1 *Given the observations $(x_k,t_k),(x_{k1},t_{k1}),(x_{k2},t_{k2}),\ldots,(x_{kn_k},t_{kn_k})$ for the stochastic process x described by (5.5.40), the least squares estimators for the parameters A_k and C_k are given by*

$$\begin{aligned}\hat{A}_k =& \frac{\ln\hat{b}}{\Delta t},\\ \hat{C}_k =& \widehat{s.d.(\varepsilon)}\sqrt{\frac{2\hat{A}_k}{e^{2\hat{A}_k\Delta t}-1}}\end{aligned} \tag{5.5.41}$$

where $\hat{b} = \dfrac{\sum_{i=1}^{n_k} x_{k(i-1)}x_{ki}}{\sum_{i=1}^{n_k} x_{k(i-1)}^2}$ and $\widehat{s.d.(\varepsilon)} = s.d.(\hat{\varepsilon}_i)$ is the sample standard deviation of $\hat{\varepsilon}_i'$s, defined by $\hat{\varepsilon}_i = x_{ki} - \hat{b}x_{k(i-1)}$.

Proof. In order to estimate the parameters, we consider an exact updating formula for x by itself (see Gillespie [36]), for $i = 1,2,\ldots,n_k$,

$$x_{ki} = x_{k(i-1)}e^{A_k\Delta t} + C_k\sqrt{\frac{e^{2A_k\Delta t}-1}{2A_k}}w \tag{5.5.42}$$

where w is an independent sample value of standard normal distribution.

Let $b = e^{A_k \Delta t}$ and $\varepsilon \sim N(0, C_k^2 \frac{e^{2A_k \Delta t}-1}{2A_k})$. Then, equation (5.5.42) can be viewed as a regression model with zero intercept:

$$y = bz + \varepsilon_i \tag{5.5.43}$$

By regressing x_{ki} against $x_{k(i-1)}$ for $i = 1, 2, \ldots, n_k$, we can obtain the least squares estimators given below (refer to [33, 71]).

$$\hat{b} = \frac{\sum_{i=1}^{n_k} x_{k(i-1)} x_{ki}}{\sum_{i=1}^{n_k} x_{k(i-1)}^2} \tag{5.5.44}$$

and

$$\widehat{s.d.(\varepsilon)} = s.d.(\hat{\varepsilon}_i) \tag{5.5.45}$$

where $\hat{\varepsilon}_i = y_i - \hat{b} z_i = x_{ki} - \hat{b} x_{k(i-1)}$ and $s.d.(\hat{\varepsilon}_i)$ is the sample standard deviation of $\hat{\varepsilon}_i'$s. Relating back to our original parameters gives

$$\begin{aligned}\hat{b} =& e^{\hat{A}_k \Delta t}, \\ \widehat{s.d.(\varepsilon)} =& \hat{C}_k \sqrt{\frac{e^{2\hat{A}_k \Delta t} - 1}{2\hat{A}_k}}\end{aligned} \tag{5.5.46}$$

Rearrangement of the above expressions yields the lease squares estimators for A_k and C_k in (5.5.41).

■

5.5.2 Interval estimation

Furthermore, we would like to obtain confidence intervals for estimators $\hat{A}_k$ and $\hat{C}_k$. From the standard regression text (see [21, 32]), the distributions of $\hat{b}$ and $\widehat{s.d.(\varepsilon)}$ are given by

$$\frac{\hat{b} - b}{s.d.(\hat{b})} \sim t_{n_k - 1} \tag{5.5.47}$$

and

$$\frac{(n_k-1)\left(\widehat{s.d.(\varepsilon)}\right)^2}{(s.d.(\varepsilon))^2} \sim \chi^2_{n_k-1} \tag{5.5.48}$$

where $s.d.(\hat{b}) = \frac{\widehat{s.d.(\varepsilon)}}{\sqrt{\sum_{i=1}^{n_k} x^2_{k(i-1)}}}$. From above distributions, we have

$$P\left(-t_{n_k-1,1-\gamma/2} < \frac{\hat{b}-b}{s.d.(\hat{b})} < t_{n_k-1,1-\gamma/2}\right) = 1-\gamma \tag{5.5.49}$$

and

$$P\left(\chi^2_{n_k-1,\gamma/2} < \frac{(n_k-1)\left(\widehat{s.d.(\varepsilon)}\right)^2}{(s.d.(\varepsilon))^2} < \chi^2_{n_k-1,1-\gamma/2}\right) = 1-\gamma \tag{5.5.50}$$

By substituting $b = e^{A_k \Delta t}$ and $s.d.(\varepsilon) = C_k\sqrt{\frac{e^{2\hat{A}_k\Delta t}-1}{2\hat{A}_k}}$ in the above expressions and rearranging the inequalities, we can obtain the $100(1-\gamma)\%$ confidence intervals for A_k and C_k given below, respectively.

$$\left[\frac{1}{\Delta t}\ln\left(\hat{b}-s.d.(\hat{b})t_{n_k-1,1-\gamma/2}\right),\ \frac{1}{\Delta t}\ln\left(\hat{b}+s.d.(\hat{b})t_{n_k-1,1-\gamma/2}\right)\right], \tag{5.5.51}$$

and

$$\left[\widehat{s.d.(\varepsilon)}\sqrt{\frac{2\hat{A}_k(n_k-1)}{\left(e^{2\hat{A}_k\Delta t}-1\right)\chi^2_{n_k-1,1-\gamma/2}}},\ \widehat{s.d.(\varepsilon)}\sqrt{\frac{2\hat{A}_k(n_k-1)}{\left(e^{2\hat{A}_k\Delta t}-1\right)\chi^2_{n_k-1,\gamma/2}}}\right]. \tag{5.5.52}$$

5.6 Concluding remarks

In this chapter, we provided several methods for estimating the parameters in some stochastic hybrid dynamic systems. The estimation procedure is presented by first estimating the parameters of the discrete dynamic and then examining the continuous dynamic piecewisely. The estimators for the parameters of homogeneous Poisson and power-law processes are found for the discrete evolutions. Concerning the continuous flow, the estimation is performed for the one-dimensional geometric

Brownian motion and the Ornstein-Uhlenbeck processes.

6 Simulation Study

6.1 Introduction

The attempt in this chapter is to apply two simulated stochastic processes to illustrate the estimation procedures described in Chapter 5. At the end, an example of the surplus process of an insurance company is given to exhibit the applicability of the estimation and simulation methods.

The Euler's scheme [41] for simulating solution processes for stochastic differential equations is introduced in Section 6.2. The result is then utilized to simulate a one-dimensional geometric Brownian motion process with jumps in Section 6.3. Moreover, the parameter estimation methods discussed in Chapter 5 are applied on the simulated process. Another simulation is performed for a one-dimensional Ornstein-Uhlenbeck process with jumps in Section 6.4. An illustrative example for an insurance model is given in Section 6.5. Finally, the chapter ends with some concluding remarks.

6.2 The Euler's scheme

In most applications the random sources of continuous and discrete dynamics of a stochastic hybrid dynamic systems are assumed to be independent. For the simulation examples presented in this chapter, we first generate the discrete jump times. Subsequently, the continuous trajectory is simulated on each interval between jump times.

We start this chapter with introducing the simple and commonly used simulation tool, the Euler's scheme. Let us consider a general stochastic differential equation of the form

$$dx(t) = \mu(x(t),t)dt + \sigma(x(t),t)dw(t) \tag{6.2.1}$$

We wish to simulate paths of $x(t)$ on $[0,T]$. When we simulate a stochastic differential equation, what is simulated is in fact a discretized version of the stochastic differential equations. In particular, we simulate a discretized process, $x(h), x(2h), \ldots, x(mh)$, where m is the number of points on $(0, T]$, h is the step size, and $mh = T$. The Euler's scheme is an intuitive discretization scheme that uses difference equations to approximate the differential equations. When h is small, from equation (6.2.1) we have

$$x(t+h) - x(t) \approx \mu(x(t), t)h + \sigma(x(t), t)\left(w(t+h) - w(t)\right) \tag{6.2.2}$$

The Euler's scheme stems from the above approximation. For $j = 1, 2, \ldots, m$, the simulation algorithm is given by

$$x(jh) = x((j-1)h) + \mu(x((j-1)h), (j-1)h)h + \sigma(x((j-1)h), (j-1)h)\sqrt{h}W_j \tag{6.2.3}$$

where W_j's are i.i.d. standard normal random variables.

6.3 Simulated geometric Brownian motion with jumps

The stochastic process considered in this section is a one-dimensional geometric Brownian motion process with downward jumps. The process follows the stochastic hybrid dynamics:

$$\begin{cases} dx(t) = A_{k-1}x(t)dt + B_{k-1}x(t)dw(t),\ T_{k-1} \le t < T_k,\ x(T_{k-1}) = x_{k-1} \\ x_k = x(T_k^-, T_{k-1}, x_{k-1}) - z_k \end{cases} \tag{6.3.4}$$

where the jumps are governed by a homogeneous Poisson process N with intensity λ. Figure 6.1 illustrates a simulated path of the stochastic process for $T = 2$ based on the Euler's algorithm (6.2.3).

From the simulated path in Figure 6.1, we observe that there are five large jumps on the interval $[0, T]$. To estimate the parameters of the discrete and continuous dynamics, we apply the parameter estimation methods for the intensity of the homogeneous Poisson process and the coefficients of the geometric Brownian motion process which were discussed in Section 5.2 and 5.4, respectively. Table 6.1 and 6.2 show the parameter estimates.

From the results below, we see that all the true values are within two standard errors from the point estimates. The estimates for the diffusion parameters, $B_0, B_1, \ldots, B_5$, are fairly close to the true value with small standard errors. We note that the estimate of A_3, the drift parameter of the

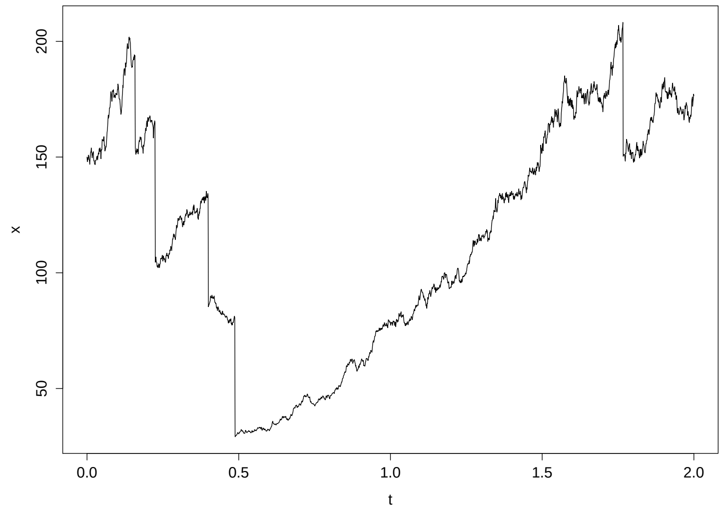

Figure 6.1: A simulation path of a geometric Brownian motion process with jumps

Table 6.1: Drift parameter estimates for GBM with jumps

	A_0	A_1	A_2	A_3	A_4	A_5
True value	1.5	1	1.5	1	1.5	1
Point estimate	1.672	1.081	1.449	-0.627	1.585	0.752
Standard error	0.786	1.225	0.688	0.912	0.270	0.611

Table 6.2: Diffusion and intensity parameter estimates for GBM with jumps

	B_0	B_1	B_2	B_3	B_4	B_5	λ
True value	0.3	0.3	0.3	0.3	0.3	0.3	0.5
Point estimate	0.312	0.312	0.287	0.269	0.305	0.294	0.400
Standard error	0.018	0.027	0.015	0.020	0.006	0.014	1.118

continuous dynamic between the third and fourth jumps, has the largest deviation from the true value among all estimates. This may be due to the fact that the interarrival time is short, hence, fewer data points are available for the parameter estimation. Then, a large error is more likely to appear. On the other hand, the estimate of A_4 has the smallest standard error among the estimates of A_j's. This shows that the precision of the estimate is greatly increased when there are more data points between jumps.

6.4 Ornstein-Uhlenbeck process with jumps

The stochastic process discussed in this section is a one-dimensional Ornstein-Uhlenbeck process with non-homogeneous jumps. The process follows the stochastic hybrid dynamics:

$$\begin{cases} dx(t) = Ax(t)dt + Cdw(t),\ T_{k-1} \le t < T_k,\ x(T_{k-1}) = x_{k-1} \\ x_k = z_k x(T_k^-, T_{k-1}, x_{k-1}) \end{cases} \tag{6.4.5}$$

where the jumps are governed by a power-law process N with intensity function $\lambda(t) = \alpha\beta t^{\beta-1}$. The model is a special case of system (5.1.2) where $A_0 = A_1 = \cdots = A_{N(T)} = A$ and $C_0 = C_1 = \cdots = C_{N(T)} = C$. Figure 6.2 illustrates a simulated path of the stochastic process for $T = 3$ based on the Euler's algorithm (6.2.3).

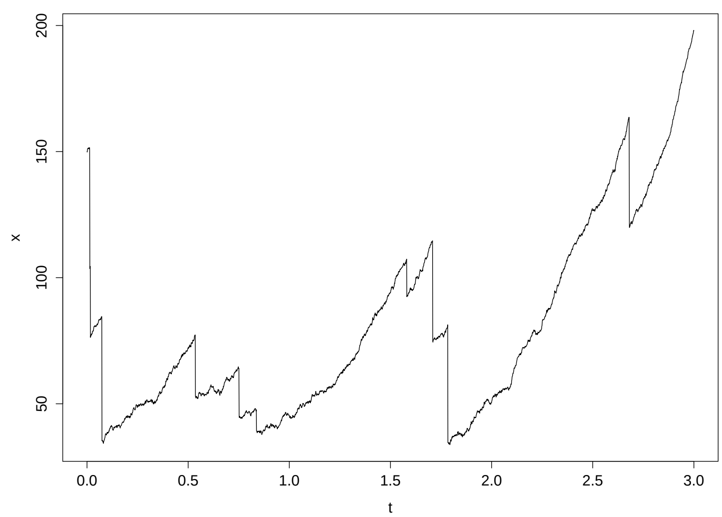

Figure 6.2: A simulation path of an Ornstein-Uhlenbeck process with jumps

From the simulated path of the Ornstein-Uhlenbeck process with jumps, ten large jumps are detected on $[0,T]$ since the differences at these locations are at least eight times larger than the rest of the differences, and the rest of the differences are all between -0.95 and 1.25. After identifying the locations of the large jumps, we are ready to estimate the discrete jump dynamic. The estimation procedure described in Section 5.3 is applied to obtain the estimates of the parameters of the power-law process, α and β.

For this simulated process, the coefficients of the continuous dynamic, the Ornstein-Uhlenbeck process, are fixed over time. Hence, we can remove the jumps and combine all the pieces together to obtain a larger sample for estimating the parameters of the continuous dynamic. The parameters of the Ornstein-Uhlenbeck process, A and C, are estimated by applying the estimation methods provided in Section 5.5. Table 6.3 gives the point estimates and 95% confidence intervals for the parameters of interest. For this simulated path, we note that all parameters are well estimated by the estimation methods discussed earlier.

Table 6.3: Parameter estimates for the power-law process and the Ornstein-Uhlenbeck process

	α	β	A	C
True value	6	0.45	1.5	10
Point estimate	5.845	0.489	1.572	9.943
95% confidence interval	(1.733, 9.957)	(0.234, 0.835)	(1.445, 1.699)	(9.697, 10.201)

6.5 Insurance Example

In this section, we will give an example to demonstrate how to apply the simulation techniques in practice. Suppose that the surplus x of an insurance company is modeled as follows.

$$\begin{cases} dx(t) = Ax(t)dt + Bx(t)dw(t),\ T_{k-1} \le t < T_k,\ x(T_{k-1}) = x_{k-1} \\ x_k = x(T_k^-, T_{k-1}, x_{k-1}) - z_k \end{cases} \tag{6.5.6}$$

where the jumps are governed by a homogeneous Poisson process N with intensity λ. Here $x(T_0) = x_0$ is the initial surplus which is subject to the initial reserve and the premium rate. Based on the model, the insurer would like to know what is the probability that the company will remain solvent in a year, and if the company remains solvent for the coming year, what is the expected surplus at the end of the year. In this situation, let $T = 1$ be the termination time. The goal is to estimate $E[x(T)|solvent]$ and the survival probability of the company. The latter has been of great interest in insurance risk theory.

Definition 6.5.1 *The ruin probability in finite time T is defined by*

$$\psi(x_0, T) = P(x(t) < 0 \textit{ for some } T_0 \leq t \leq T | x(T_0) = x_0) \tag{6.5.7}$$

Note that the survival probability is simply the ruin probability subtracted from one. An simulated path of the ruin case is given in Figure 6.3.

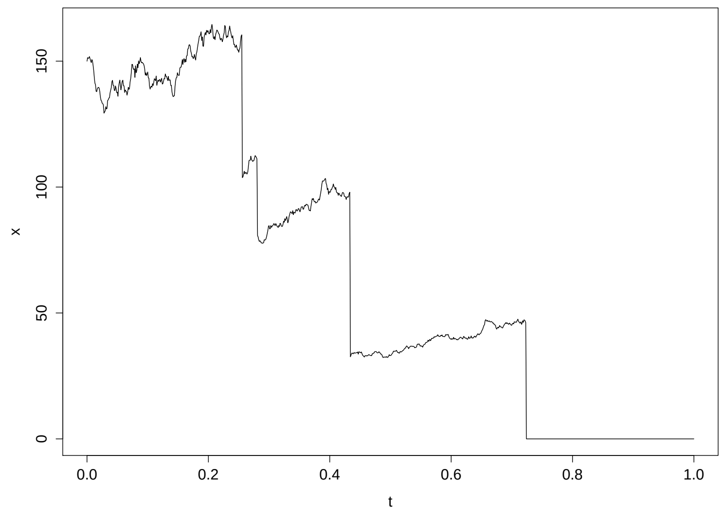

Figure 6.3: A simulated trajectory of ruin

Suppose that the past information of the insurer itself or of the competitors are available for the insurer to estimate the parameters of system (6.5.6). For illustrative purpose, let us assume that by applying the estimation techniques discussed in the preceding chapter, we have the estimates: $\hat{A} = 1$, $\hat{B} = 0.3$, and $\hat{\lambda} = 0.5$. Furthermore, the jump size distribution of z_k are estimated as lognormal with the parameters $\hat{\mu} = 3.8$ and $\hat{\sigma} = 0.2$. Let the initial surplus be $x_0 = 150$.

Having the estimates of the parameters, we are ready to answer the questions of interest based on simulation results. First, based on the Euler's scheme discussed in Section 6.2, we simulated $n = 10{,}000$ paths of the process governed by system 6.5.6). Among the n simulated paths, there are $r = 471$ ones that have dropped below zero at some time t in $(T_0, T]$. Then, the estimated ruin probability is

$$\hat{\psi}(x_0, T) = \frac{r}{n} = 0.0471 \tag{6.5.8}$$

By the strong law of large number, $\hat{\psi}(x_0, T)$ approaches the true ruin probability almost surely as n goes to infinity. The estimated probability that the company will remain solvent for one year is $1-\hat{\psi}(x_0, T) = 0.9529$.

Denote $x^1(t), x^2(t), \cdots, x^n(t)$ as the n simulated paths. Figure 6.4 gives a simulated sample path of the insurance surplus process and the 95% confidence band. Let n' be the number of non-ruin paths that never drop below zero on $(T_0, T]$. Then, the sample mean of the termination values of the non-ruin paths serves as an estimate of the expected surplus at time T given that the company is solvent from T_0 to T.

$$\text{est}\{E[x(T)|x(t) > 0 \text{ for all } T_0 < t < T]\} = \frac{1}{n'} \sum_{\{k: x^k(t) > 0 \text{ for all } T_0 < t < T\}} x^k(T) \tag{6.5.9}$$

For our simulation, we obtain est$\{E[x(T)|solvent]\} = 267.84$ with a 95% confidence interval $(57.84,\ 520.80)$. The study in this section is an illustrative example of how estimation and simulation techniques can be applied and answer questions in practice.

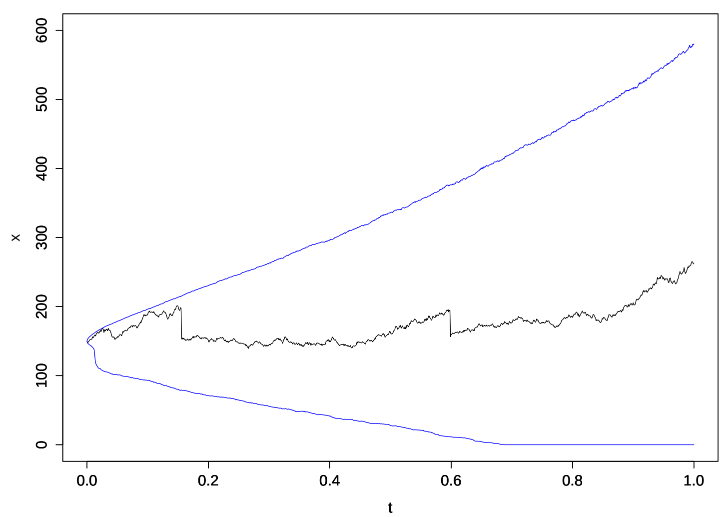

Figure 6.4: A sample path of the insurance surplus process with the 95% confidence band

6.6 Concluding remarks

Two types of hybrid stochastic processes are simulated to demonstrate the estimation procedures for both the discrete and continuous dynamics. An insurance example is discussed at the end, and the quantities of interest are obtained through estimation and simulation techniques.

7 Future Research

In this chapter, we shall pose some possible research problems resulted from the present study.

In Chapter 4, the probability distributions of the solution processes are obtained for two special classes of a linear non-homogeneous Itô-Doob type of systems with jumps, the multivariate geometric Brownian motion and Orinstein-Uhlenbeck processes with jumps. The probability distribution of the multivariate geometric Brownian motion process with jumps is obtained under some assumptions including the coefficient matrices being diagonalizable and pairwise commutable. The assumptions were made to allow our approach of using modal matrix and log transformations. Possible relaxation of the assumptions will be examined. One of the future aims is to find the probability distribution of the general linear non-homogeneous system with jumps. In addition, it is of interest to obtain the exact or approximated probability distributions for nonlinear stochastic systems with jumps. Nonlinear transformation and numerical analysis may be employed for this attempt.

In Chapter 5, we have discussed several estimation methods for some cases of the discrete and continuous dynamics. For the discrete dynamic, the parameters of homogeneous Poisson and power-law processes are estimated. A further investigation on estimation for non-homogeneous Poisson processes is proposed. Non-parametric approach may be used to estimate a general intensity function of the non-homogeneous Poisson process. Furthermore, it shall be noted that in the present study the estimation on the continuous dynamics, the geometric Brownian motion and Ornstein-Uhlenbeck processes, is restricted to the univariate cases. Extension to the multivariate cases will be of interest.

In the present study the infinitesimal generators, probability distributions and parameter estimation of some classes of stochastic hybrid dynamic systems are developed. One major aim of future research projects is to apply these analytical tools to obtain further insight and desired quantities on real-world systems.

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