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Bi-Integrable and Tri-Integrable Couplings and Their Hamiltonian Structures

by

Jinghan Meng

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
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Dedication

This dissertation is lovingly dedicated to my parents. Their support, encouragement, and constant love have sustained me throughout my life.

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Abstract

An investigation into structures of bi-integrable and tri-integrable couplings is undertaken. Our study is based on semi-direct sums of matrix Lie algebras. By introducing new classes of matrix loop Lie algebras, we form new Lax pairs and generate several new bi-integrable and tri-integrable couplings of soliton hierarchies through zero curvature equations. Moreover, we discuss properties of the resulting bi-integrable couplings, including infinitely many commuting symmetries and conserved densities. Their Hamiltonian structures are furnished by applying the variational identities associated with the presented matrix loop Lie algebras.

The goal of this dissertation is to demonstrate the efficiency of our approach and discover rich structures of bi-integrable and tri-integrable couplings by manipulating matrix Lie algebras.

Chapter 1
Introduction

We shall give a sketch of the historical origins of infinite-dimensional integrable Hamiltonian systems. Infinite-dimensional systems mean that the equations of dynamical systems are continuous in space. Soliton theory started with the empirical discovery of solitons back in 19th century. In 1834, John Scott Russell, a Scottish engineer, observed the solitary wave, a hump of water moving with constant speed and shape along the canal [1, 2]. After more than half a century, in 1894, Korteweg and de Vries (KdV) gave a convincing mathematical model for wave motion in a shallow canal. A breakthrough came in 1965, when Zabusky and Kruskal [3] proposed the concept of a soliton: a spatially localized solution of a nonlinear partial differential equation with the property that this solution always regains its initial shape and velocity after interacting with another localized disturbance. After that, further developments followed. Gardner, Greene, Kruskal and Miura [4] discovered that the KdV equation can be solved exactly by the inverse scattering transform method. We note especially the work of Zakharov and Faddeev [5], who showed that the KdV equation is a nontrivial example of an infinite-dimensional Hamiltonian system that is completely integrable. It is an integrable nonlinear evolution partial differential equation (PDE) in one spatial dimension. There do exist integrable nonlinear evolution PDEs in two spatial dimensions. For instance, a physically significant generalization of the KdV equation is the Kadomtsev-Petviashvili (KP) equation. The KdV equation is a prototype example of integrable infinite-dimensional systems (e.g., [6, 7]). Thus, the KdV equation will be an illustrative example that we will consider many times for the sake of illustration of our study presented in this dissertation.

1.1 Infinite-dimensional integrable systems

Let us now introduce our basic notation and conception in the field of infinite-dimensional integrable systems, some of which comes from Refs. [8]-[16].

Let $x \in \mathbb{R}$, $t \in \mathbb{R}$ be the independent variables, x representing position in space and t being time. Let $u_i = u_i(x,t), 1 \le i \le N$ be the dependent variables, belonging to the Schwartz space on $\mathbb{R}$ for any fixed $t \in \mathbb{R}$ and $S^N(\mathbb{R},\mathbb{R})$ be the space of all vectors $u = (u_1, u_2, \cdots, u_N)^T$ of that kind.

DEFINITION 1.1.1 *For any real-valued function $P(x,t,u)$, its Gateaux derivative with respect to u in a direction $v = (v_1, \cdots, v_N)^T \in S^N(\mathbb{R},\mathbb{R})$ is defined by*

$$P'[v] = P'(u)[v] = \frac{\partial}{\partial \varepsilon} P(u + \varepsilon v)\big|_{\varepsilon=0} = \frac{\partial}{\partial \varepsilon} P(u_1 + \varepsilon v_1, \cdots, u_N + \varepsilon v_N)\big|_{\varepsilon=0}. \tag{1.1}$$

We denote by $\mathcal{B}$ the space of all real-valued functions $P(x,t,u)$ which are C^∞-differentiable with respect to x, t and C^∞-Gateaux differentiable with respect to $u = u(x,t)$ as functions of x, and set $\mathcal{B}^N = \{(P_1, \cdots, P_N)^T | P_i \in \mathcal{B}, 1 \le i \le N\}$.

DEFINITION 1.1.2 *An evolution equation is a partial differential equation of the form*

$$u_t = K(u) \tag{1.2}$$

where $u(x,t)$ is a dependent variable, and $K(u)$ is a function of u and its derivatives with respect to x. If K is nonlinear, equation (1.2) *is called a nonlinear evolution equation.*

DEFINITION 1.1.3 *For any two vector fields $K, S \in \mathcal{B}^N$, define the product vector field to be*

$$[K, S] = K'[S] - S'[K],$$

which has been shown to be a commutator operation of $\mathcal{B}^N$.

It is easy to verify that $(\mathcal{B}^N, [\cdot,\cdot])$ constitutes a Lie algebra over the real field.

In order to study the solution and integrability of equation (1.2), we consider the infinitesimal symmetry transformations.

DEFINITION 1.1.4 *A vector field $S \in \mathcal{B}^N$ is said to be a symmetry of* (1.2) *if the infinitesimal transformation*

$$u(t) \mapsto u(t) + \epsilon S(u(t))$$

leaves (1.2) *form-invariant.*

For a solution u of (1.2), and a vector field $S \in \mathcal{B}^N$, we have

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + S'[u_t] = \frac{\partial S}{\partial t} + S'[K] = \frac{\partial S}{\partial t} + K'[S] - [K, S].$$

where $[\cdot, \cdot]$ is defined as in Definition 1.1.3.

THEOREM 1.1 *[8] For any vector field $S \in \mathcal{B}^N$ is a symmetry of* (1.2) *if and only if S satisfies*

$$\frac{\partial S}{\partial t} = [K, S]. \tag{1.3}$$

COROLLARY 1.0.1 *If a vector field S does not depend on t explicitly, i.e., $\frac{\partial S}{\partial t} = 0$, then S is a symmetry of* (1.2) *if and only if* $[K, S] = 0$.

In general, a symmetry of an equation generates a transformation that takes solutions to solutions. Knowing a symmetry of a partial differential equation allows one to find new solutions from any known solution.

Thus, if we have a symmetry, and know one solution, then we can construct a second solution by applying the symmetry, and more solutions by applying the symmetry again and again.

EXAMPLE 1 It is known that the KdV equation $u_t = u_{xxx} + 6uu_x$ is invariant under translation in x. To this invariance there corresponds the symmetry $\rho = u_x$. We verify this:

$$K'\rho = \frac{\partial}{\partial \epsilon}[u_{xxx} + \epsilon\rho_{xxx} + 6(u + \epsilon\rho)(u_x + \epsilon\rho_x)]\big|_{\epsilon=0} = \rho_{xxx} + 6u\rho_x + 6\rho u_x. \tag{1.4}$$

Therefore, we obtain $\rho_t = \rho_{xxx} + 6u\rho_x + 6\rho u_x$, which is a linear equation of ρ. Obviously, if $u(x)$ is a solution of the KdV equation (1.2), then we have

$$(u_x)_t = (u_x)_{xxx} + 6u(u_x)_x + 6u_x(u_x), \tag{1.5}$$

which implies $\rho = u_x$ is a symmetry. We can also verify that if

$$\rho = u_x, \quad \rho' = D,$$

then we obtain

$$K'[\rho] - \rho'[K] = (D^3 + 6uD + 6u_x)u_x - D(u_{xxx} + 6uu_x) = 0.$$

If two symmetries P and σ satisfy $[P, \sigma] = 0$, then we say they are commuting.

The KdV equation possesses infinitely many commuting symmetries and the first few symmetries of the KdV equation are

$$\rho^{(1)} = u_x \quad \text{(invariance under } x\text{-translation)}, \tag{1.6}$$

$$\rho^{(2)} = u_{xxx} + 6uu_x \quad \text{(invariance under } t\text{-translation)}, \tag{1.7}$$

$$\rho^{(3)} = u_{5x} + 10uu_{3x} + 20u_x u_{xx} + 30u^2 u_x \quad \text{(non-geometrical)}. \tag{1.8}$$

The equations

$$u_t = \rho^{(n)}, \quad n \in \mathbb{N}$$

define a hierarchy of exactly solvable equations associated with the KdV equation.

It is known that not only the KdV equation but also many other nonlinear evolution equations such as the Burgers equation [17], the sine-Gordon equation [18], and the Zakharov-Shabat equations [19] possess infinitely many commuting symmetries. Furthermore, for these equations, except the Burgers equation, the commuting symmetry groups are connected with the existence of infinitely many conservation laws. In addition, the symmetry groups can be generated systematically, and a recursion formula [20, 21] provides a way to find infinitely many symmetries of some general evolution equations.

DEFINITION 1.1.5 *Let $\mathcal{V}$ denote the space of linear operators from $\mathcal{B}^N$ to $\mathcal{B}^N$. A linear operator $\Phi \in \mathcal{V}$ is called a recursion operator for $u_t = K,\ K \in \mathcal{B}^N$ if for any symmetry $S \in \mathcal{B}^N$ of $u_t = K$, ΦS is again a symmetry of $u_t = K$.*

Therefore, a recursion operator $\Phi : \mathcal{B}^N \to \mathcal{B}^N$ of a system $u_t = K(u),\ K \in \mathcal{B}^N$, transforms a symmetry into another symmetry of the same system $u_t = K(u)$. It can serve as a tool of generating the a symmetry algebra of a given system and its existence is regarded as an important characterizing property for integrability of the system under study.

DEFINITION 1.1.6 *Let a linear operator $\Phi \in \mathcal{V} : \mathcal{B}^N \to \mathcal{B}^N$ and a vector field $K \in \mathcal{B}^N$. The Lie derivative $L_K\Phi \in \mathcal{V}$ of the operator Φ with respect to K is defined by*

$$(L_K\Phi)S = \Phi[K, S] - [K, \Phi S],\ S \in \mathcal{B}^N. \tag{1.9}$$

DEFINITION 1.1.7 *For a linear operator $\Phi \in \mathcal{V}$, its Gateaux derivative operator $\Phi' : \mathcal{B}^N \to \mathcal{B}^N$ is defined through*

$$\Phi'[K]S := \frac{\partial}{\partial \epsilon}\Phi(u+\epsilon K)S\big|_{\epsilon=0}, \quad K \in \mathcal{B}^N,\ S \in \mathcal{B}^N.$$

THEOREM 1.2 *[15] A linear operator $\Phi \in \mathcal{V}$ is recursion operator for $K \in \mathcal{B}^N$ if and only if*

$$L_K \Phi = \Phi'[K] - [K', \Phi] = 0, \tag{1.10}$$

i.e., Φ is invariant under K, when $\frac{\partial \Phi}{\partial t} = 0$.

DEFINITION 1.1.8 *[13] A linear operator $\Phi \in \mathcal{V}$ is called a hereditary operator, if the following equality holds:*

$$\Phi'[\Phi K]S - \Phi\Phi'[K]S - \Phi'[\Phi S]K + \Phi\Phi'[S]K = 0, \tag{1.11}$$

for all vector fields $K, S \in \mathcal{B}^N$.

THEOREM 1.3 *[15] Let $\Phi \in \mathcal{V}$. Then Φ is hereditary if and only if*

$$L_{\Phi S}\Phi = \Phi L_S \Phi, \quad \forall\ S \in \mathcal{B}^N \tag{1.12}$$

and Φ is a recursion operator of (1.2) *if and only if*

$$\frac{\partial \Phi}{\partial t} + L_K \Phi = 0. \tag{1.13}$$

THEOREM 1.4 *[15] Let $\Phi \in \mathcal{V}$ be a hereditary operator. If the Lie derivative $L_K\Phi = 0$, and let $K_n = \Phi^n K$, for $n = 0, 1, 2, \cdots$, then we have*

(i) Φ is invariant under K_n, i.e., $L_{K_n}\Phi = 0$, for $n = 0, 1, 2, \cdots$;

(ii) the vector fields K_n commute with each other, i.e., we have

$$[K_n, K_m] = 0, \quad n, m \geq 0.$$

1.2 Hamiltonian structures

Let $\mathcal{F}$ denote the space of functionals $\mathcal{H} = \int f(u)\,dx$ where the function f is in the quotient space $\mathcal{B}/\partial\mathcal{B}$, since we have that $\int \partial f(u)\,dx = 0$.

LEMMA 1.1 *For any $\mathcal{H} \in \mathcal{F}$, say $\mathcal{H} = \int f(u)\,dx,\ f \in \mathcal{B}/\partial\mathcal{B}$, we have*

$$\frac{d}{dt}\mathcal{H} = \int \frac{d}{dt} f(u)\,dx. \tag{1.14}$$

DEFINITION 1.2.1 *[12] The variational derivative $\frac{\delta \mathcal{P}}{\delta u}$ of a functional $\mathcal{P} \in \mathcal{F}$ with respect to u is determined by*

$$\int \Big(\frac{\delta \mathcal{P}}{\delta u}\Big)^T \xi\ dx = \frac{\partial}{\partial \varepsilon}\mathcal{P}(u + \varepsilon\xi)\big|_{\varepsilon=0},\ \xi \in \mathcal{B}^N. \tag{1.15}$$

PROPOSITION 1 For any $\mathcal{H} \in \mathcal{F}$, say $\mathcal{H} = \int f(u)\,dx,\ f \in \mathcal{B}/\partial\mathcal{B}$, its time derivative can be represented as

$$\frac{d}{dt}\mathcal{H} = \int \sum_{n=0}^{\infty} (-1)^n \frac{\partial^n}{\partial x^n}(\frac{\partial f}{\partial u^{(n)}})\,dx, \quad \text{where } u^{(n)} = \frac{\partial^n u}{\partial x^n}, \tag{1.16}$$

and it holds that

$$\frac{\delta \mathcal{H}}{\delta u} = \sum_{n=0}^{\infty} (-1)^n \frac{\partial^n}{\partial x^n}(\frac{\partial f}{\partial u^{(n)}}), \quad u^{(n)} = \frac{\partial^n u}{\partial x^n},$$

where f is local and does not depend on t explicitly.

REMARK 1 Generally, we have the variational derivative of a functional written $\frac{\delta \mathcal{H}}{\delta u}$ and

$$\mathcal{H}(u + \epsilon v) = \mathcal{H}(u) + \epsilon \int_{-\infty}^{\infty} \frac{\delta \mathcal{H}}{\delta u}(x) v(x)\,dx + O(\epsilon^2)$$

For example, if

$$\mathcal{H}(u) := \int_{-\infty}^{\infty} [u(x)^3 + u(x) u_{xx}(x)] dx,$$

then

$$\frac{\delta \mathcal{H}}{\delta u}(x) = 3u(x)^2 + 2u_{xx}(x).$$

The above representation helps define the Poisson bracket.

DEFINITION 1.2.2 *The adjoint operator $J^\dagger : \mathcal{B}^N \to \mathcal{B}^N$ of a linear operator $J : \mathcal{B}^N \to \mathcal{B}^N$ is determined by*

$$\int \xi^T J^\dagger \eta dx = \int \eta^T J \xi dx,\ \xi, \eta \in \mathcal{B}^N.$$

If $J^\dagger = -J$, then J is called to be skew-symmetric.

DEFINITION 1.2.3 *Let* $J : \mathcal{B}^N \to \mathcal{B}^N$ *be a linear differential operator. A bilinear product* $\{\cdot, \cdot\} : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ *defined by*

$$\{\mathcal{P}, \mathcal{Q}\}_J := \int \Big(\frac{\delta \mathcal{P}}{\delta u}\Big)^T J \frac{\delta \mathcal{Q}}{\delta u}\, dx, \quad \mathcal{P}, \mathcal{Q} \in \mathcal{F} \tag{1.17}$$

is called a Poisson bracket associated with J.

DEFINITION 1.2.4 *A linear operator* $J : \mathcal{B}^N \to \mathcal{B}^N$ *is called Hamiltonian if its Poisson bracket*

$$\{\mathcal{P}, \mathcal{Q}\} = \{\mathcal{P}, \mathcal{Q}\}_J = \int \Big(\frac{\delta \mathcal{P}}{\delta u}\Big)^T J \frac{\delta \mathcal{Q}}{\delta u}\, dx, \;\; \mathcal{P}, \mathcal{Q} \in \mathcal{F}, \tag{1.18}$$

satisfies the skew-symmetry condition

$$\{\mathcal{P}, \mathcal{Q}\} = -\{\mathcal{Q}, \mathcal{P}\}, \tag{1.19}$$

and the Jacobi identity:

$$\{\{\mathcal{P}, \mathcal{Q}\}, \mathcal{R}\} + \{\{\mathcal{Q}, \mathcal{R}\}, \mathcal{P}\} + \{\{\mathcal{R}, \mathcal{P}\}, \mathcal{Q}\} = 0. \tag{1.20}$$

DEFINITION 1.2.5 *[20] A pair of Hamiltonian operators* $J, M : \mathcal{B}^N \to \mathcal{B}^N$ *is called a Hamiltonian pair, if* $J + M$ *is also Hamiltonian.*

DEFINITION 1.2.6 *A system of evolution equations* $u_t = K$, $K \in \mathcal{B}^N$ *is called to be a Hamiltonian system, if there is a Hamiltonian operator* $J : \mathcal{B}^N \to \mathcal{B}^N$ *and a functional* $\mathcal{H} \in \mathcal{F}$, *such that*

$$u_t = K(u) = J\frac{\delta \mathcal{H}}{\delta u}. \tag{1.21}$$

The functional $\mathcal{H}$ *is called a Hamiltonian functional of the system, and we say that the system possesses a Hamiltonian structure if it can be of the form* (1.21). *To obtain Hamiltonian structures means to transform* $u_t = K(u)$ *into the form of* (1.21).

EXAMPLE 2 The KdV equation $u_t = uu_x + u_{xxx}$ is a Hamiltonian system, since it can be written in the form

$$u_t = J\frac{\delta \mathcal{H}}{\delta u}$$

where $J := \frac{\partial}{\partial x}$, and the Hamiltonian functional $\mathcal{H}$ is

$$\mathcal{H}(u) = \int_{-\infty}^{\infty} [\frac{1}{6}u(x)^3 + \frac{1}{2}u(x)u_{xx}(x)]dx.$$

It is easy to see that variational derivative of the functional $\mathcal{H}$ is

$$\frac{\delta \mathcal{H}}{\delta u} = \frac{1}{2}u^2 + u_{xx}.$$

As mentioned earlier, it is well-known that certain types of evolution equations, for example, the KdV equation, possess features such as infinitely many symmetries and conserved quantities.

DEFINITION 1.2.7 *A functional $\mathcal{P} \in \mathcal{F}$, which does not depend on t explicitly, is called a conserved quantity or conserved functional of the Hamiltonian system* (1.21) *if the Poisson bracket $\{\mathcal{P}, \mathcal{H}\}_J = 0$. We call $\mathcal{H}$ and $\mathcal{P}$ to be in involution, when their Poisson bracket is zero.*

For any solution u of (1.21), and for a functional $\mathcal{P} = \mathcal{P}(x, t, u) \in \mathcal{F}$, we have

$$\frac{d}{dt}\mathcal{P} = \frac{\partial \mathcal{P}}{\partial t} + \mathcal{P}'[u_t] = \frac{\partial \mathcal{P}}{\partial t} + \int \left(\frac{\delta \mathcal{P}}{\delta u}\right)^T u_t \, dx = \frac{\partial \mathcal{P}}{\partial t} + \int \left(\frac{\delta \mathcal{P}}{\delta u}\right)^T J \frac{\delta \mathcal{H}}{\delta u} \, dx = \frac{\partial \mathcal{P}}{\partial t} + \{\mathcal{P}, \mathcal{H}\}_J.$$

THEOREM 1.5 *For any functional $\mathcal{P} \in \mathcal{F}$, $\mathcal{P}$ is a conserved quantity of* (1.21) *if and only if*

$$\frac{\partial \mathcal{P}}{\partial t} = \{\mathcal{H}, \mathcal{P}\}_J.$$

There are various approaches for dealing with integrability of PDEs. In the context of finite-dimensional Hamiltonian systems, a $2n$-dimensional system is integrable if it has n functionally independent conserved functions (constants of the motion) that are in involution, i.e., their Poisson brackets are zero. In this dissertation, we adopt the definition involving existence of infinitely many conserved functionals.

DEFINITION 1.2.8 *An infinite-dimensional PDE is called integrable if it possesses infinitely many functionally independent conserved functionals.*

DEFINITION 1.2.9 *[20] A system of evolution equations $u_t = K$, $K \in \mathcal{B}^N$, is called to be a bi-Hamiltonian system, if there are a Hamiltonian pair $J_0, J_1 : \mathcal{B}^N \to \mathcal{B}^N$ and functionals $\mathcal{H}, \mathcal{P} \in \mathcal{F}$, such that*

$$u_t = K(u) = J_0 \frac{\delta \mathcal{H}}{\delta u} = J_1 \frac{\delta \mathcal{P}}{\delta u}. \tag{1.22}$$

We say that the system possesses a bi-Hamiltonian structure if it can be of the form (1.22). *To obtain bi-Hamiltonian structures means to transform $u_t = K(u)$ into the form of* (1.22).

THEOREM 1.6 *[15, 20] Let*

$$u_t = K(u) = J_0 \frac{\delta \mathcal{H}_1}{\delta u} = J_1 \frac{\delta \mathcal{H}_0}{\delta u}$$

be a bi-Hamiltonian system of evolution equations. Assume that the Hamiltonian operator J_0 is non-degenerate, and the linear operator Φ is defined by

$$\Phi := J_1 J_0^{-1}.$$

Also assume that for each $n = 0, 1, 2, \cdots$, we can recursively define

$$K_n = \Phi K_{n-1}, \;\; n \geq 1,$$

which means that for each $n \geq 1$, K_{n-1} lies in the image of J_0. Then there exists a sequence of Hamiltonian functionals $\{\mathcal{H}_n\}_{n\geq 0}$ such that

(i) for each $n \geq 1$, the evolution equation

$$u_{t_n} = K_n(u) = J_0 \frac{\delta \mathcal{H}_n}{\delta u} = J_1 \frac{\delta \mathcal{H}_{n-1}}{\delta u} \tag{1.23}$$

is a bi-Hamiltonian system;

(ii) the vector fields $\{K_n\}_{n\geq 0}$ commute with each other, i.e., we have

$$[K_n, K_m] = 0, \quad n, m \geq 0; \tag{1.24}$$

(iii) the Hamiltonian functionals $\{\mathcal{H}_n\}_{n\geq 0}$ are in involution with respect to both Poisson brackets

$$\{\mathcal{H}_n, \mathcal{H}_m\}_{J_0} = \{\mathcal{H}_n, \mathcal{H}_m\}_{J_1} = 0, \quad n, m \geq 0, \tag{1.25}$$

which implies there exists a sequence of infinitely many conserved quantities for each of the bi-Hamiltonian systems (1.23).

We have seen that given a bi-Hamiltonian system, the operator $\Phi = J_1 J_0^{-1}$, when applied successively to the initial equation $K_0 = J\frac{\delta \mathcal{H}_0}{\delta u}$, produces an infinite sequence of generalized symmetries and conserved quantities of the original system. Moreover, Φ is hereditary [8].

We will pay attention to systems that are bi-Hamiltonian. The idea of Magri [20] is very important because of the fact that once a bi-Hamiltonian formulation of a dynamics is found, a sequence of conserved quantities can be generated. Moreover, the bi-Hamiltonian property is closely connected with the existence of the Lax representation (more details can be found in [21]-[26]). The Magri scheme has been so far one of the most successful and systematic methods for generating bi-Hamiltonian dynamical systems.

EXAMPLE 3 The well-known KdV equation has the bi-Hamiltonian structure [20]

$$u_t = \frac{1}{4}\,u_{xxx} + \frac{3}{2}\,uu_x = J_0\frac{\delta\mathcal{H}_1}{\delta u} = J_1\frac{\delta\mathcal{H}_0}{\delta u} \tag{1.26}$$

where the Hamiltonian pair is

$$J_0 = \partial,\ \ J_1 = \frac{1}{2}\partial^3 + u\partial + \partial u,\ \ \partial = \frac{\partial}{\partial x},$$

and the Hamiltonian functionals are

$$\begin{aligned}\mathcal{H}_0 &= \int_{-\infty}^{\infty} \frac{1}{4}\,u^2 dx,\\ \mathcal{H}_1 &= \int_{-\infty}^{\infty} \frac{1}{8}\,(2u^3 - u_x^2) dx.\end{aligned}$$

Moreover, the hereditary recursion operator is

$$\Phi = J_1 J_0^{-1} = \frac{1}{2}\,(\partial^2 + 4u + 2u_x\partial^{-1}).$$

Therefore, we can write the KdV hierarchy as

$$u_{t_n} = \Phi^n u_x = J_0\frac{\delta\mathcal{H}_n}{\delta u} = J_1\frac{\delta\mathcal{H}_{n-1}}{\delta u},\ \ n \geq 1.$$

1.3 Lax pairs and zero curvature equations

In 1968, Lax [27] proposed a revolutionary technique for finding soliton solutions to nonlinear evolution equations by relating the original nonlinear PDE to two linear operators via a compatibility condition. These linear operators are now called a Lax pair. The work done by Lax was further generalized by Zakharov and Shabat [28] and Ablowitz, Kaup, Newell and Segur (AKNS) [29]. Soliton solutions were found for more complicated nonlinear PDEs, such as the nonlinear Schrödinger equation, and Lax pairs were again required for the solution technique called the Inverse Scattering Transform.

Suppose that we have been lucky enough to have a Lax pair of matrices, or operators in the infinite dimensional case, L and A, whose entries depend on the dependent variables u_i so that

$$u_t = K(u)$$

is equivalent to the equation

$$L_t = [A, L], \tag{1.27}$$

called the isospectral Lax equation, where $[\cdot,\cdot]$ denotes the commutator. With the Lax pair, we can define two linear equations

$$L\psi = \lambda\psi \;\text{ and }\; \psi_t = A\psi \tag{1.28}$$

in some Hilbert space, where the first equation represents the spectral equation for L and the second one gives the time evolution for the eigenfunction ψ. It is an isospectral problem, i.e., the eigenvalues λ of $L(t)$ are time independent.

Differentiating the spectral equation with respect to time t, by the Leibniz rule, we have

$$\frac{d}{dt}(L\psi) = \frac{dL}{dt}\psi + L\psi_t = \frac{dL}{dt}\psi + LA\psi. \tag{1.29}$$

Furthermore, applying the second relation of (1.28), we obtain

$$\frac{d}{dt}(L\psi) = \frac{d}{dt}(\lambda\psi) = \lambda\psi_t = \lambda A\Psi = A(\lambda\psi) = AL\psi. \tag{1.30}$$

Since ψ is arbitrary,

$$\frac{dL}{dt} + LA = AL, \tag{1.31}$$

which is the desired result (1.27).

We shall give a concrete example; the KdV equation $u_t = u_{xxx} + uu_x$ is the compatibility condition of two linear problems (isospectral problems) in the following example.

EXAMPLE 4 The Lax pair, L and A, takes the form

$$L = \partial^2 + \frac{1}{6}u, \;\; A = -4\,\partial^3 - u\partial - \frac{1}{2}u_x, \tag{1.32}$$

where the differential operator ∂ acting on an arbitrary function $v(x)$ is defined as $\partial v = v_x + v\partial$. Obviously

$$(L_t + [L, A])\psi = \frac{1}{6}\,(u_t + u_{xxx} + uu_x)\psi = 0. \tag{1.33}$$

Then the Lax equation

$$L_t + [L, A] = 0$$

is equivalent to the KdV equation $u_t + uu_x + u_{xxx} = 0$, when $L_t + [L, A]$ acts on function ψ. Here $[\cdot,\cdot]$ is the operator commutator.

In 1979, researchers realized that the Lax pair could be interpreted as a zero curvature condition on an appropriate connection [30].

DEFINITION 1.3.1 *A pair (U, V) of $n\times n$ smooth complex matrix functions in variables $(x,t) \in \mathbb{R}^2$, is called a compatible pair if there exists an n-dimensional smooth complex vector function $\psi(x,t)$ satisfying simultaneously the matrix linear systems*

$$\frac{\partial \psi(x,t)}{\partial x} = U\psi(x,t), \tag{1.34}$$

$$\frac{\partial \psi(x,t)}{\partial t} = V\psi(x,t). \tag{1.35}$$

If (U, V) is a compatible pair, the property of second mixed derivatives holds

$$\frac{\partial^2 \psi(x,t)}{\partial x \partial t} = \frac{\partial^2 \psi(x,t)}{\partial t \partial x}, \tag{1.36}$$

and the systems, (1.34) *and* (1.35)*, imply*

$$\frac{\partial U}{\partial t} - \frac{\partial V}{\partial x} + [U, V] = 0. \tag{1.37}$$

Equation (1.37) *is called a zero curvature equation.*

Let us now show that the KdV equation can be written as zero-curvature conditions of two linear problems. We go further from linear operators to matrices by introducing some powers of λ. Because the eigenvalue equation is second order, we need to write it in first-order form by defining

$$\psi_1 = \psi, \quad \text{and} \quad \psi_2 = \psi_x. \tag{1.38}$$

Then the equation $L\psi = \lambda\psi$ can be rewritten as

$$\frac{\partial}{\partial x}\begin{bmatrix}\psi_1 \\ \psi_2\end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{6}(u+\lambda) & 0 \end{bmatrix}\begin{bmatrix}\psi_1 \\ \psi_2\end{bmatrix} = U\begin{bmatrix}\psi_1 \\ \psi_2\end{bmatrix}. \tag{1.39}$$

Therefore, we can write these two equations as a first-order system

$$\frac{\partial}{\partial t}\begin{bmatrix}\psi_1 \\ \psi_2\end{bmatrix} = \begin{bmatrix} \frac{1}{6}u_x & \frac{2}{3}\lambda - \frac{1}{3}u \\ -\frac{1}{9}\lambda^2 - \frac{1}{18}\lambda u + \frac{1}{18}u^2 + \frac{1}{6}u_{xx} & -\frac{1}{6}u_x \end{bmatrix}\begin{bmatrix}\psi_1 \\ \psi_2\end{bmatrix} = V\begin{bmatrix}\psi_1 \\ \psi_2\end{bmatrix}. \tag{1.40}$$

Then the compatibility condition is in the form of the zero curvature equation:

$$\frac{\partial U}{\partial t} - \frac{\partial V}{\partial x} + [U, V] = 0. \tag{1.41}$$

The coefficients of λ^3, λ^2, and λ all vanish, if we equate coefficient of each power of λ in this equation. The constant term gives the matrix equation

$$\begin{bmatrix} 0 & 0 \\ -\frac{1}{6}(u_t + uu_x + u_{xxx}) & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{1.42}$$

A remarkable approach proposed by AKNS [29] and Zakharov and Shabat [28] shows that one can generate several types of nonlinear integrable systems via the Lax pair

$$U = U(u, \lambda) = \begin{bmatrix} -i\lambda & p \\ q & i\lambda \end{bmatrix} \tag{1.43}$$

and

$$V = V(u, \lambda) = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \tag{1.44}$$

where a, b, c are chosen as appropriate analytic functions of the parameter λ. In the following we shall show how this approach works (e.g., [31] for more details). Assume that a, b, c are functions of λ, q, r, and all derivatives of q, r with respect to x. Plugging U, V into the zero curvature equation (1.41), we obtain

$$\begin{cases} a_x &= qc - rb, \\ q_t &= b_x + 2i\lambda\, b + 2\, qa, \\ r_t &= c_x - 2i\lambda\, c - 2\, ra. \end{cases} \tag{1.45}$$

To be concrete, we set a, b, c be polynomials of λ,

$$a = \sum_{j=0}^{3} a_j \lambda^j, \quad b = \sum_{j=0}^{3} b_j \lambda^j, \quad c = \sum_{j=0}^{3} c_j \lambda^j \tag{1.46}$$

We combine (1.45) (1.46), and compare the coefficients of λ,

$$\begin{aligned} & b_3 = c_3 = 0,\ a_{3,x} = 0, \\ & b_{j,x} + 2i\, b_{j-1} + 2\, qa_j = 0, \\ & c_{j,x} - 2i\, c_{j-1} - 2\, ra_j = 0, \\ & a_{j,x} = q\, c_j - rb_j, \\ & q_t = b_{0,x} + 2\, a_0 q, \quad r_t = c_{0,x} + 2\, a_0 r. \end{aligned}$$

Since $a_{3,x} = 0$, we have

$$a_3 = \alpha_0 \, (\text{const.}) \;\; b_2 = \alpha_0 iq, \;\; c_2 = \alpha_0 ir; \tag{1.47}$$

Since $a_{2,x} = 0$, we have

$$\begin{aligned}
&a_2 = \beta_0 \, (\text{const.}), \\
&b_1 = -\frac{\alpha_0}{2} q_x + \beta_0 iq, \;\; c_1 = \frac{\alpha_0}{2} r_x + \beta_0 ir, \;\; a_1 = \frac{\alpha_0}{2} qr + \gamma_0 \, (\text{const.}), \\
&b_0 = \frac{i}{4} \alpha_0 \, (-q_{xx} + 2\, q^2 r) - \frac{\beta_0}{2} q_x + \gamma_0 iq, \;\; c_0 = \frac{i}{4} \alpha_0 \, (-r_{xx} + 2\, r^2 q) + \frac{\beta_0}{2} r_x + \gamma_0 ir, \\
&a_0 = -\frac{i}{4} \alpha_0 \, (q r_x - r q_x) + \frac{\beta_0}{2} qr + \mu_0.
\end{aligned}$$

Then we arrive at

$$\begin{cases} q_t \, = -\frac{i}{4} \alpha_0 \, (q_{xxx} - 6\, qr q_x) + \frac{i}{2} \beta_0 \, (-q_{xx} + 2\, q^2 r) + i\gamma_0 \, q_x + 2\mu_0 q, \\ r_t \, = -\frac{i}{4} \alpha_0 \, (r_{xxx} - 6\, qr r_x) + \frac{i}{2} \beta_0 \, (r_{xx} - 2\, r^2 q) + i\gamma_0 \, r_x - 2\mu_0 r. \end{cases} \tag{1.48}$$

For the above nonlinear system, we can reduce it by choosing different q and r.

- Letting $\beta_0 = \gamma_0 = \mu = 0$, and $\alpha_0 = -4i, r = -q$, we get the modified Korteweg-de Vries (mKdV) equation:

$$q_t + 6q^2 q_x + q_{xxx} = 0; \tag{1.49}$$

- Letting $\alpha_0 = \gamma_0 = \mu = 0$, and $\beta_0 = -2, u_x = qr = (\frac{q_x}{q})_x$, we obtain the Burgers equation:

$$u_t = 2uu_x - u_{xx}. \tag{1.50}$$

Furthermore, we can take a, b, c as Laurent polynomials $a = \dfrac{a_1}{\lambda}$, $b = \dfrac{b_1}{\lambda}$, $c = \dfrac{c_1}{\lambda}$, where a_1, b_1, c_1 are constants, and plug in (1.45) to get

$$a_{1,x} = \frac{i}{2} (qr)_t, \;\; q_{xt} = -4iaq, \;\; r_{xt} = -4iar. \tag{1.51}$$

- Letting $a_1 = \frac{i}{4} \cos u$, $b_1 = c_1 = \frac{i}{4} \sin u$ and $q = -r = -\frac{u_x}{2}$, we get the sine-Gordon equation:

$$u_{xt} = \sin u; \tag{1.52}$$

- Letting $a_1 = \frac{i}{4} \cos u$, $b_1 = c_1 = \frac{i}{4} \sin u$, we obtain the sinh-Gordon equation:

$$u_{xt} = \operatorname{sh} u. \tag{1.53}$$

We remark that the advantage of the Lax pair formulation is that it allows one to connect nonlinear integrable PDEs with pairs of linear problems, many publications have been devoted to this area [32]-[35]. However, it turns out to be very difficult to find the pair of L and A for a given PDE, and there has no completely systematic method of determining whether or not a nonlinear PDE has a Lax representation and, if so, how to obtain the associated operators, L and A. Actually, it is much easier to postulate a Lax pair, L and A, and then determine what PDE it is associated with.

We will later search for nonlinear PDEs that can be cast into such a framework with other matrix Lax pairs, U and V, and hope they will have the features that the KdV equation possesses, including infinitely many commuting symmetries and bi-Hamiltonian structures.

Chapter 2

Integrable couplings and matrix Lie algebras

Integrable equations are a significantly important class of nonlinear equations. Integrable couplings, initiated in [12], are enlarged integrable systems of an integrable evolution equation

$$u_t = K(u) = K(x, t, u, u_x, u_{xx}, \cdots) \tag{2.1}$$

where u is a column vector of dependent variables. Over the last two decades, many research papers have been dedicated to this topic [36]-[53]. It originates from an investigation on centerless Virasoro symmetry algebras of integrable systems or soliton equations [12, 54]. For a given system of soliton equations, if we take this system and each time part of Lax pairs of its hierarchy as the first component and the second component of a new enlarged system respectively, then the enlarged system will keep the same structure of Virasoro symmetry algebras as the old one [39]. In this way we form a hierarchy of integrable couplings of the original system.

Mathematically, the problem of integrable couplings may be expressed as [12, 55]: *for a given integrable system of evolution equations $u_t = K(u)$, how can we construct a non-trivial system of evolution equations which is still integrable and includes $u_t = K(u)$ as a sub-system?*

We can construct a new bigger integrable system as follows [39]:

$$\begin{cases} u_t = K(u), \\ v_t = S(u, v), \end{cases} \tag{2.2}$$

which satisfies the non-triviality condition $\partial S/\partial[u] \neq 0$, where $[u]$ denotes a vector consisting of all derivatives of u with respect to space variable x. It should be noted that the non-triviality condition helps exclude trivial diagonal systems with $S(u, v) = cK(v)$, where c is an arbitrary constant.

The symmetry problem is a good example of integrable coupling of an integrable system $u_t = K(u)$:

$$\begin{cases} u_t = K(u), \\ v_t = K'(u)[v], \end{cases} \tag{2.3}$$

which can be generated by a perturbation around a solution of the system $u_t = K(u)$ [12]. We note that the second sub-system $v_t = K'(u)[v]$ in the above integrable coupling (2.3) is linear with respect to v. Moreover, a symmetry $S(u)$ of the system $u_t = K(u)$ leads to a solution $(u, S(u))$ to the integrable coupling (2.3). However, the second component v of a solution (u, v) to the integrable coupling (2.3) is generally not a symmetry of the system $u_t = K(u)$ [38]. This is because v satisfies the linearized system $v_t = K'(u)[v]$ only for one solution, not for all solutions of the system $u_t = K(u)$. Therefore, the simple integrable coupling (2.3) is already a generalization of the symmetry problem [38]. Another basic integrable coupling of an integrable system $u_t = K(u)$ reads as

$$\begin{cases} u_t = K(u), \\ v_t = K'(u)[v] + K(u). \end{cases} \tag{2.4}$$

The two integrable couplings can be generated by zero curvature equation associated with Lax pair U and V in the following forms respectively,

$$\bar{U} = \begin{bmatrix} U & U_a \\ 0 & U \end{bmatrix}, \ \frac{\partial U_a}{\partial \lambda} = 0, \ \bar{V} = \begin{bmatrix} V & V_a \\ 0 & V \end{bmatrix}, \tag{2.5}$$

$$\bar{U} = \begin{bmatrix} U & U_a \\ 0 & U + U_a \end{bmatrix}, \ \frac{\partial U_a}{\partial \lambda} = 0, \ \bar{V} = \begin{bmatrix} V & V_a \\ 0 & V + V_a \end{bmatrix}. \tag{2.6}$$

The method of constructing integrable coupling systems by perturbation and the use of enlarged Lax pairs was initiated in [12, 39, 56]. We shall give a brief introduction in the next section.

2.1 Constructing integrable couplings from semi-direct sums of Lie algebras

We introduce some basic theory of loop algebras and related topics.

DEFINITION 2.1.1 *A Lie algebra is a vector space* $\mathfrak{g}$*, together with a law, which associates with any two elements* X, $Y \in \mathfrak{g}$*, a bracket* $[X, Y] \in \mathfrak{g}$*, satisfying*

(a) $[X, Y] = -[Y, X]$, *(skew-symmetry)*

(b) $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$, *(Jacobi identity)*

(c) $[\alpha X + \beta Y, Z] = \alpha[X, Z] + \beta[Y, Z]$. *(linearity)*

(Here α, β *are coefficients acting by scalar multiplication in the vector space* $\mathfrak{g}$*.)*

We specify the spaces of matrix differential operators [57]:

$$\mathcal{V}_0^r = \bigcup_{n=0}^{\infty} \sum_{k=0}^{n} \mathcal{V}_{(k)}^r, \;\; \mathcal{V}_{(k)}^r = \{(P^{ij}\partial^k)_{r\times r} | P^{ij} = P^{ij}(x,t,u) \in \mathcal{B}\}, \tag{2.7}$$

$$\tilde{\mathcal{V}}_0^r = \mathcal{V}_0^r \otimes C[\lambda, \lambda^{-1}] = \bigcup_{n=0}^{\infty} \sum_{k=0}^{n} \tilde{\mathcal{V}}_{(k)}^r, \;\; \tilde{\mathcal{V}}_{(k)}^r = \mathcal{V}_{(k)}^r \otimes C[\lambda, \lambda^{-1}], \tag{2.8}$$

where $\lambda \in \mathbb{R}$ is a spectral parameter, and $C[\lambda, \lambda^{-1}]$ denotes the space of Laurent polynomials in λ.

DEFINITION 2.1.2 *A Lie algebra $\mathfrak{g}$ is called simple if it is non-abelian and has no non-trivial (i.e., not $\{0\}$ and $\mathfrak{g}$) ideals.*

DEFINITION 2.1.3 *A Lie algebra $\mathfrak{g}$ is called semisimple if it is a direct sum of simple Lie algebras.*

We denote the Lie bracket on a Lie algebra $\mathfrak{g}_0$ by $[\cdot,\cdot]$. Let $C[\lambda, \lambda^{-1}]$ denote the ring of Laurent polynomials in $\lambda \in C$, and form the loop algebra

$$\mathfrak{g} := \mathfrak{g}_0 \otimes C[\lambda, \lambda^{-1}]$$

If $B \subseteq \mathfrak{g}_0$ denotes a basis for $\mathfrak{g}_0$, then we define the element $x \otimes \lambda^n$ by $x[n]$, where $x \in B$ and $n \in \mathbb{Z}$. Then

$$\{x[n] \big| x \in \mathcal{B},\, n \in \mathbb{Z}\} = \{x \otimes \lambda^n | x \in \mathcal{B},\, n \in \mathbb{Z}\}$$

forms a basis of $\mathfrak{g}$. Obviously, $\mathfrak{g}$ is infinite-dimensional.

Noting that an arbitrary Lie algebra has a semi-direct sum structure of a solvable Lie algebra and a semi-simple Lie algebra [58, 59], researchers (see [37] [48]-[52]) use semi-direct sums of Lie algebras to enlarge the original matrix Lie algebra $\mathfrak{g}$ to construct an integrable coupling of the soliton equation (1.2). Take another matrix Lie algebra $\mathfrak{g}_c$ closed under matrix multiplication and then form a semi-direct sum $\bar{\mathfrak{g}}$ of $\mathfrak{g}$ and $\mathfrak{g}_c$:

$$\bar{\mathfrak{g}} = \mathfrak{g} \Subset \mathfrak{g}_c. \tag{2.9}$$

The notion of semi-direct sums means that $\mathfrak{g}$ and $\mathfrak{g}_c$ satisfy

$$[\mathfrak{g}, \mathfrak{g}_c] \subseteq \mathfrak{g}_c, \tag{2.10}$$

where $[\mathfrak{g}, \mathfrak{g}_c] = \{[A, B] \,|\, A \in \mathfrak{g},\ B \in \mathfrak{g}_c\}$. Obviously, $\mathfrak{g}_c$ is an ideal Lie sub-algebra of $\bar{\mathfrak{g}}$. The subscript c here indicates a contribution to the construction of couplings. We also require that the closure property between $\mathfrak{g}$ and $\mathfrak{g}_c$ under matrix multiplication:

$$\mathfrak{g}\mathfrak{g}_c, \mathfrak{g}_c\mathfrak{g} \subseteq \mathfrak{g}_c, \tag{2.11}$$

where $\mathfrak{g}_1\mathfrak{g}_2 = \{AB \,|\, A \in \mathfrak{g}_1,\ B \in \mathfrak{g}_2\}$, to guarantee that a Lax pair from the semi-direct sum $\bar{\mathfrak{g}}$ can generate a coupling system (see e.g. [54]).

We choose a Lax pair in the semi-direct sum $\bar{\mathfrak{g}}$ of matrix Lie algebras:

$$\bar{U} = U + U_c,\ \bar{V} = V + V_c,\ \ U, V \in \mathfrak{g},\ U_c, V_c \in \mathfrak{g}_c, \tag{2.12}$$

to form a pair of enlarged spatial matrix spectral problems [54]:

$$\begin{cases} \bar{\phi}_x = \bar{U}\bar{\phi} = \bar{U}(\bar{u}, \lambda)\bar{\phi}, \\ \bar{\phi}_t = \bar{V}\bar{\phi} = \bar{V}(\bar{u}, \bar{u}_x, \bar{u}_{xx} \cdots \frac{\partial^{m_0}}{\partial x^{m_0}}; \lambda)\bar{\phi}, \end{cases} \tag{2.13}$$

where m_0 is a non-negative integer, the matrix U_c in $\bar{U}$ introduces additional dependent variables and $\bar{u}$ consists of both the original dependent variables and the additional dependent variables. In addition, the matrix U_c could depend on the spectral parameter λ, and the matrix V_c in $\bar{V}$ really does almost in all cases.

Therefore, the enlarged system $\bar{u}_t = K(\bar{u})$ is equivalent to the enlarged zero curvature equation

$$\bar{U}_t = \bar{V}_x - [\bar{U}, \bar{V}]. \tag{2.14}$$

The whole construction process above shows that semi-direct sums of a given Lie algebra $\mathfrak{g}$ with new Lie algebras provide a great choice of candidates of integrable couplings for the evolution equation generated from the Lie algebra $\mathfrak{g}$.

We can take semi-direct sums of matrix loop algebras introduced in [48] as

$$\bar{\mathfrak{g}} = \mathfrak{g} \Subset \mathfrak{g}_c, \quad \mathfrak{g} = \{\mathrm{diag}(\underbrace{A, \cdots, A}_{n+1})\} \quad \mathfrak{g}_c = \begin{bmatrix} 0 & B_1 & \cdots & B_n \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & B_1 \\ 0 & \cdots & \cdots & 0 \end{bmatrix}, \tag{2.15}$$

where n is an positive integer. The new enlarged Lax matrices $\bar{U}$ and $\bar{V}$ in $\bar{\mathfrak{g}} = \mathfrak{g} + \mathfrak{g}_c$ are taken as

$$\bar{U} = \begin{bmatrix} U & U_1 & \cdots & U_n \\ 0 & U & \ddots & \vdots \\ \vdots & \ddots & \ddots & U_1 \\ 0 & \cdots & 0 & U \end{bmatrix}, \ \bar{V} = \begin{bmatrix} V & V_1 & \cdots & V_n \\ 0 & V & \ddots & \vdots \\ \vdots & \ddots & \ddots & V_1 \\ 0 & \cdots & 0 & V \end{bmatrix}.$$

It is interesting to construct other possible realizations, especially those which could carry essential integrable properties of the original equations.

2.2 Variational identities and Hamiltonian structures

Variational identities are an elegant method of finding Hamiltonian structures of integrable systems generated by enlarged zero curvature equations.

DEFINITION 2.2.1 *A bilinear form $\langle \cdot, \cdot \rangle$ on a vector space is said to be non-degenerate when if $\langle A, B \rangle = 0$ for all vectors A, then $B = 0$, and if $\langle A, B \rangle = 0$ for all vectors B, then $A = 0$.*

DEFINITION 2.2.2 *Let $\mathfrak{g}$ be a Lie algebra over a field F, and $ad_X : \mathfrak{g} \to \mathfrak{g}$ be the adjoint action $ad_X Y = [X, Y]$, then the killing form on $\mathfrak{g}$ is a bilinear form*

$$B_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to F$$

defined by

$$B_{\mathfrak{g}}(X, Y) = tr(ad_X \circ ad_Y)$$

where $\circ$ means composition.

The Killing form is invariant under the adjoint action and symmetric, since the trace is symmetric. It is well-known that a Lie algebra is semisimple if and only if its killing form is non-degenerate, which implies that on non-semisimple Lie algebra, the Killing form is always degenerate. We will work on semi-direct sums of Lie algebras that are non-semisimple [60], and thus, it is impossible to obtain Hamiltonian equations by the trace identity [61, 62]. It is, therefore, natural to ask whether we can replace the Killing form with general bilinear forms to establish Hamiltonian structures for soliton equations associated with semi-direct sums of Lie algebras. Ma and his collaborators

[37, 54] give the answer by introducing the variational identity which plays an essential role in constructing Hamiltonian structures and thereby conserved quantities for integrable couplings.

The first step of applying the variational identity is to construct a general bilinear form $\langle\cdot,\cdot\rangle$ on a given algebra $\bar{\mathfrak{g}}$. Although the variational identity does not require the invariance property

$$\langle\rho(A),\rho(B)\rangle = \langle A,B\rangle \tag{2.16}$$

under an isomorphism ρ of the algebra $\bar{\mathfrak{g}}$, it keeps the symmetric property

$$\langle A,B\rangle = \langle B,A\rangle \tag{2.17}$$

and the invariance property under the multiplication

$$\langle A,BC\rangle = \langle AB,C\rangle, \tag{2.18}$$

where AB denotes the product of A and B in $\bar{\mathfrak{g}}$. Furthermore, if $\bar{\mathfrak{g}}$ is associative, then $\bar{\mathfrak{g}}$ forms a Lie algebra under

$$[A,B] = AB - BA,$$

and the invariance property under the Lie bracket holds:

$$\langle A,[B,C]\rangle = \langle [A,B],C\rangle. \tag{2.19}$$

It should be noted that the invariance property under the Lie bracket, (2.19), does not imply the invariance property under the multiplication, (2.18). In the following chapters, examples will be shown that there are many non-degenerate bilinear forms with the properties (2.17) and (2.18) on a given semi-direct sum of Lie algebras.

For a given spectral matrix $U = U(u,\lambda) \in \mathfrak{g}$, where $\mathfrak{g}$ is a matrix loop algebra, let us fix the proper ranks $\text{rank}(\lambda)$ and $\text{rank}(u)$ so that U is homogeneous in rank, i.e., we can define $\text{rank}(U)$. The rank function satisfies

$$\text{rank}(AB) = \text{rank}(A) + \text{rank}(B),$$

whenever an expression AB makes sense.

This condition on spectral problems is not only required in deducing the trace identity [61], but also in the quadratic-form identity [63] and the continuous and discrete variational identity [37, 49].

We assume that if two solutions V_1 and V_2 of the stationary zero curvature equation

$$V_x = [U, V] \tag{2.20}$$

posses the same rank, then they are linearly dependent of each other:

$$V_1 = \gamma V_2, \quad \gamma = \text{const.} \tag{2.21}$$

The condition has also been required in deducing the standard trace variational identity.

Associated with a non-degenerate bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ with the symmetric property and the invariance property, we introduce a functional [37]:

$$\mathcal{W} = \int \left(\langle V, U_\lambda \rangle + \langle \Lambda, V_x - [U, V] \rangle \right) dx, \tag{2.22}$$

while $U_\lambda = \frac{\partial U}{\partial \lambda}$, and $V, \Lambda \in \mathfrak{g}$ are two matrices to be determined.

DEFINITION 2.2.3 *[37] Under a non-degenerate bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$, the variational derivative $\nabla_A \mathcal{R} \in \mathfrak{g}$ of a functional $\mathcal{R} \in \mathcal{F}$ with respect to $A \in \mathfrak{g}$ is defined by*

$$\int \langle \nabla_A \mathcal{R}, B \rangle dx = \frac{\partial}{\partial \epsilon} \mathcal{R}(A + \epsilon B) \big|_{\epsilon=0}, \;\; B \in \mathfrak{g}. \tag{2.23}$$

Obviously, based on the non-degenerate property of the bilinear form $\langle \cdot, \cdot \rangle$ we can have

$$\nabla_B \int \langle A, B \rangle dx = A, \;\; \nabla_B \int \langle A, B_x \rangle dx = -A_x, \tag{2.24}$$

therefore, it follows from and that

$$\nabla_V \mathcal{W} = U_\lambda - \Lambda_x + [U, \Lambda], \;\; \nabla_\Lambda \mathcal{W} = V_x - [U, V]. \tag{2.25}$$

For the variational calculation of $\mathcal{W}$ with respect to u, we require the constrained conditions:

$$\nabla_V \mathcal{W} = U_\lambda \Lambda_x + [U, \Lambda] = 0, \tag{2.26}$$

$$\nabla_\Lambda \mathcal{W} = V_x - [U, V] = 0, \tag{2.27}$$

which also imply that V and Λ are related to U and thus to the potential u.

Now we have

$$\frac{\delta}{\delta u} \int \langle V, U_\lambda \rangle dx = \frac{\delta \mathcal{W}}{\delta u}, \tag{2.28}$$

where $\frac{\delta}{\delta u}$ is the variational derivative with respect to u. In this formula, we need to take the dependence of u in V and U_λ into consideration when calculating the left-hand side; however only the dependence of u in U needs to be considered in computing the right-hand side, because of the constrained conditions and the property that if $\nabla_A \mathcal{R}(A) = 0$, then $\frac{\delta}{\delta u}\mathcal{R}(A(u)) = 0$ (see e.g. [37]).

THEOREM 2.1 *[37]* **(A variational identity under general bilinear forms)** *Let $\mathfrak{g}$ be a matrix loop algebra, $U = U(u, \lambda) \in \mathfrak{g}$ be homogenous in rank, and $\langle\cdot,\cdot\rangle$ denote a non-degenerate symmetric bilinear form invariant under the matrix Lie product. Assume that the stationary zero curvature equation has a unique solution $V \in \mathfrak{g}$ of a fixed rank up to a constant multiplier. Then for any solution $V \in \mathfrak{g}$ of $V_x = [U, V]$, being homogenous in rank, we have the following variational identity*

$$\frac{\delta}{\delta u}\int \langle V, \frac{\partial U}{\partial \lambda}\rangle dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\langle V, \frac{\partial U}{\partial u}\rangle, \tag{2.29}$$

where γ is some constant.

THEOREM 2.2 *[37] Let V be a solution to the stationary zero curvature equation. Then*

(a) we have

$$\frac{d}{dx}V^m = [U, V^m],\ m \geq 1; \tag{2.30}$$

(b) for any bilinear form $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ with the properties and ,we have

$$\frac{d}{dx}\langle V^m, V^m\rangle = 0,\ m \geq 1. \tag{2.31}$$

THEOREM 2.3 *[37] Let V be a solution of the stationary zero curvature equation. If $\langle V, V\rangle \neq 0$, then the constant γ in the variational identity is given by*

$$\gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\langle V, V\rangle| \tag{2.32}$$

2.3 Integrable couplings of the AKNS hierarchy

2.3.1 The AKNS hierarchy

We consider the AKNS soliton hierarchy [28, 29, 64]. Its spectral problem is given by

$$\phi_x = U\phi,\ U = U(u, \lambda) = \begin{bmatrix} -\lambda & p \\ q & \lambda \end{bmatrix},\ u = \begin{bmatrix} p \\ q \end{bmatrix},\ \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}, \tag{2.33}$$

where matrix U is belong to a semi-simple matrix Lie algebra $\mathfrak{g}$. If we consider the stationary zero curvature equation

$$W_x = [U, W], \tag{2.34}$$

and assume that a solution W solution to (2.34) is of the form

$$W = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} = \sum_{i\geq 0} W_{0,i}\lambda^{-i} = \sum_{i\geq 0} \begin{bmatrix} a_i & b_i \\ c_i & -a_i \end{bmatrix} \lambda^{-i}. \tag{2.35}$$

By plugging (2.35) in (2.34), we obtain

$$\begin{cases} a_x = pc - qb, \\ b_x = -2\,\lambda\, b - 2\, pa, \\ c_x = 2\, qa + 2\,\lambda\, c. \end{cases}$$

Comparing the coefficient of each $\lambda^{-i}, i \geq 0$, we get

$$\begin{cases} a_{i,x} = pc_i - qb_i, \\ b_{i,x} = -2b_{i+1} - 2pa_i, \quad \text{for } i \geq 0, \\ c_{i,x} = 2qa_i + 2c_{i+1}, \end{cases} \tag{2.36}$$

i.e.,

$$\begin{cases} a_{i+1,x} = pc_{i+1} - qb_{i+1}, \\ b_{i+1} = -\frac{1}{2}b_{i,x} - pa_i, \qquad \text{for } i \geq 0. \\ c_{i+1} = \frac{1}{2}c_{i,x} - qa_i, \end{cases} \tag{2.37}$$

Assuming

$$a_0 = -1,\ b_0 = c_0 = 0, \tag{2.38}$$

and taking constants of integration as zeros, we list the first few results:

$$\begin{cases} b_1 = p,\ c_1 = q,\ a_1 = 0; \\ b_2 = -\frac{1}{2}p_x,\ c_2 = \frac{1}{2}q_x,\ a_2 = \frac{1}{2}pq; \\ b_3 = \frac{1}{4}p_{xx} - \frac{1}{2}p^2q,\ c_3 = \frac{1}{4}q_{xx} - \frac{1}{2}pq^2,\ a_3 = \frac{1}{4}(pq_x - p_xq). \end{cases}$$

We form the zero curvature equations

$$U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0,\ \ V^{[m]} = (\lambda^m W)_+,\ \ m \geq 0, \tag{2.39}$$

where P_+ denotes the polynomial part of P in λ, to generate the AKNS hierarchy of soliton equations:

$$u_{t_m} = K_m = \begin{bmatrix} -2b_{m+1} \\ \\ 2c_{m+1} \end{bmatrix} = \Phi^m \begin{bmatrix} -2p \\ \\ 2q \end{bmatrix} = J\frac{\delta \mathcal{H}_m}{\delta u}, \ \ m \geq 0, \tag{2.40}$$

with the Hamiltonian operator J, the hereditary recursion operator Φ and the Hamiltonian functionals being defined by

$$J = \begin{bmatrix} 0 & -2 \\ \\ 2 & 0 \end{bmatrix}, \ \ \Phi = \begin{bmatrix} -\frac{1}{2}\partial + p\partial^{-1}q & p\partial^{-1}p \\ \\ -q\partial^{-1}q & \frac{1}{2}\partial - q\partial^{-1}p \end{bmatrix}, \ \partial = \frac{\partial}{\partial x}, \tag{2.41}$$

$$\mathcal{H}_m = \int \frac{2a_{m+2}}{m+1}\, dx, \ m \geq 0. \tag{2.42}$$

We will enlarge the zero curvature equations to construct integrable couplings and give an example, from [39], showing the details of constructing integrable couplings of AKNS hierarchy.

2.3.2 Application to the AKNS equations

Assume that the spatial spectral matrix U depends linearly on the spectral parameter λ (see [64]-[67] for details):

$$U = U(u, \lambda) = \lambda U_0 + U_1, \ \frac{\partial U_0}{\partial \lambda} = \frac{\partial U_1}{\partial \lambda} = 0. \tag{2.43}$$

In order to construct integrable couplings of evolution equation (2.1), enlarged Lie algebra $\bar{\mathfrak{g}}$ is defined by using semi-direct sums of matrix Lie algebra $\mathfrak{g}$ with another matrix Lie algebra $\mathfrak{g}_c$:

$$\bar{\mathfrak{g}} = \mathfrak{g} \Subset \mathfrak{g}_c \tag{2.44}$$

with

$$\mathfrak{g} = \left\{ \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix} \middle| U \in \mathbb{R}[\lambda] \otimes \mathrm{sl}(2) \right\}, \quad \mathfrak{g}_c = \left\{ \begin{bmatrix} 0 & U_a \\ 0 & 0 \end{bmatrix} \middle| U_a \in \mathbb{R}[\lambda] \otimes \mathrm{sl}(2) \right\}, \tag{2.45}$$

where $sl(2)$ is the algebra of all real $2{\times}2$ matrices with trace zero and the loop algebra $\mathbb{R}[\lambda] \otimes \mathrm{sl}(2)$ is defined by $\mathrm{span}\{\lambda^n A \,|\, n \in \mathbb{Z}, \ A \in \mathrm{sl}(2)\}$.

Consider the enlarged spatial spectral matrices introduced in the last sub-section:

$$\bar{U} = M(U, U_a) = \begin{bmatrix} U & U_a \\ 0 & U \end{bmatrix}, \ \frac{\partial U_a}{\partial \lambda} = 0. \tag{2.46}$$

Note that the sub-matrices U_a in the above two enlarged spatial spectral matrices could be of different sizes. As in [36, 48], we suppose that $\bar{W} \in \bar{\mathfrak{g}}$ satisfies the enlarged stationary zero curvature equation

$$\bar{W}_x = [\bar{U}, \bar{W}], \tag{2.47}$$

with

$$W = \sum_{i\geq 0} W_{0,i}\lambda^{-i},\ W_a = \sum_{i\geq -n_0} W_{a,i}\lambda^{-i},\ \frac{\partial W_i}{\partial \lambda} = 0,\ \frac{\partial W_{a,i}}{\partial \lambda} = 0, \tag{2.48}$$

then we can directly show that the enlarged zero curvature equations

$$\bar{U}_{t_m} = \bar{V}_x^{[m]} - [\bar{U}, \bar{V}^{[m]}] \tag{2.49}$$

present

$$\begin{cases} U_{t_m} = V_x^{[m]} - [U, V^{[m]}], \\ U_{a,t_m} = V_{a,x}^{[m]} - [U, V_a^{[m]}] - [U_a, V^{[m]}], \end{cases} \tag{2.50}$$

with $V_a^{[m]} = (\lambda^m W_0)_+ + \Delta_{m,0},\ V_a^{[m]} = (\lambda^m W_a)_+ + \Delta_{m,a}$, where P_+ denotes the polynomial part of P in λ.

For the AKNS hierarchy's spectral problem (2.33), under the matrix Lie algebra (2.44), we have chosen in the last section, we define the corresponding enlarged spectral matrix by

$$\bar{U} = \bar{U}(\bar{u}, \lambda) = M(U, U_1) \in \bar{\mathfrak{g}} = \mathfrak{g} \Subset \mathfrak{g}_c, \tag{2.51}$$

$$U_1 = U_1(u_1) = \begin{bmatrix} 0 & r \\ s & 0 \end{bmatrix},$$

where $\bar{u} = (u^T, u_1^T)^T$, $u_1 = (r, s)^T$ and r, s are new dependent variables.

To solve the corresponding enlarged stationary zero curvature equation

$$\bar{W}_x = [\bar{U}, \bar{W}], \tag{2.52}$$

we set a solution of the following form

$$\bar{W} = M(W, W_1) \in \bar{\mathfrak{g}} = \mathfrak{g} \Subset \mathfrak{g}_c, \tag{2.53}$$

and assume that W as defined in (2.35), W_1 is defined in the form of

$$W_1 = W_1(u, u_1, \lambda) = \begin{bmatrix} e & f \\ g & -e \end{bmatrix} = \sum_{i\geq 0} \begin{bmatrix} e_i & f_i \\ g_i & -e_i \end{bmatrix} \lambda^{-i}. \tag{2.54}$$

It now follows from the enlarged stationary zero curvature equation (2.52) that

$$\begin{cases} W_x &= [U, W], \\ W_{1,x} &= [U, W_1] + [U_1, W]. \end{cases} \tag{2.55}$$

The above equation system equivalently leads to

$$\begin{cases} a_x &= -2\, c\lambda + 2\, qb, \\ b_x &= 2\, qa - 2\, cr, \\ c_x &= 2\, a\lambda - 2\, br, \end{cases}$$

$$\begin{cases} e_x &= pg + rc - qf - sb, \\ f_x &= -2\,\lambda\, f - 2\, pe - 2\, ra, \\ g_x &= 2\, qe + 2\,\lambda\, g + 2\, sa. \end{cases}$$

Comparing the coefficient of each $\lambda^{-i}, i \geq 0$, we obtain

$$\begin{cases} f_{i+1} &= -\frac{1}{2}\, f_{ix} - p\, e_i - r\, a_i, \\ g_{i+1} &= \frac{1}{2}\, g_{ix} - q\, e_i - s\, a_i, \\ e_{i+1,x} &= p\, g_{i+1} + r\, c_{i+1} - q\, f_{i+1} - s\, b_{i+1}, \end{cases} \quad \text{for } i \geq 0. \tag{2.56}$$

If we choose constants of integration as zero, then the recursion relation (2.56) generates the sequences of $\{e_i\}_{i\geq 1}$, $\{f_i\}_{i\geq 1}$, $\{g_i\}_{i\geq 1}$ uniquely.

Upon introducing

$$e_0 = e_0' = -1,\ f_0 = g_0 = f_0' = g_0' = 0, \tag{2.57}$$

we can compute the first few sets as follows:

$$\begin{cases} e_1 &= 0, \\ f_1 &= p + r, \\ g_1 &= q + s; \end{cases} \tag{2.58}$$

$$\begin{cases} e_2 &= \frac{1}{2}\, pq + \frac{1}{2}\, sp + \frac{1}{2}\, rq, \\ f_2 &= -\frac{1}{2}\, p_x - \frac{1}{2}\, r_x, \\ g_2 &= \frac{1}{2}\, q_x + \frac{1}{2}\, s_x; \end{cases} \tag{2.59}$$

$$\begin{cases} e_3 = \frac{1}{4}\, pq_x + \frac{1}{4}\, s_x p - \frac{1}{4}\, qp_x - \frac{1}{4}\, sp_x - \frac{1}{4}\, r_x q + \frac{1}{4}\, rq_x, \\ f_3 = \frac{1}{4}\, p_{xx} + \frac{1}{4}\, r_{xx} - \frac{1}{2}\, p^2 q - \frac{1}{2}\, sp^2 - rpq, \\ g_3 = \frac{1}{4}\, q_{xx} + \frac{1}{4}\, s_{xx} - \frac{1}{2}\, q^2 p - \frac{1}{2}\, rq^2 - spq. \end{cases} \tag{2.60}$$

Let us now define

$$\bar{V}^{[m]} = M(V^{[m]}, V_1^{[m]}) \in \bar{\mathfrak{g}} = \mathfrak{g} \Subset \mathfrak{g}_c, \tag{2.61}$$

and

$$V_1^{[m]} = (\lambda^m W_1)_+ + \Delta_{m,1} \quad m \geq 0\,, \tag{2.62}$$

where $V^{[m]}$ is defined as in (2.39), and $\Delta_{m,i}$ is chosen as the zero matrix. Then, the m-th enlarged zero curvature equation

$$\bar{U}_{t_m} = \bar{V}_x^{[m]} - [\bar{U}, \bar{V}^{[m]}] \tag{2.63}$$

gives rise to

$$\begin{cases} U_{t_m} = V_x^{[m]} - [U, V^{[m]}], \\ U_{1,t_m} = V_{1,x}^{[m]} - [U, V_1^{[m]}] - [U_1, V^{[m]}]. \end{cases} \tag{2.64}$$

Thus, a hierarchy of coupling systems are generated for the AKNS hierarchy (2.40):

$$\bar{u}_{t_m} = \begin{bmatrix} p_{t_m} \\ q_{t_m} \\ r_{t_m} \\ s_{t_m} \end{bmatrix} = \bar{K}_m(\bar{u}) = \begin{bmatrix} K_m(u) \\ S_{1,m}(u, u_1) \end{bmatrix} = \begin{bmatrix} -2b_{m+1} \\ 2c_{m+1} \\ -2f_{m+1} \\ 2g_{m+1} \end{bmatrix}, \quad m \geq 0. \tag{2.65}$$

The first integrable coupling system reads

$$\begin{cases} p_{t_2} = -\frac{1}{2} p_{xx} + p^2 q, \\ q_{t_2} = \frac{1}{2} q_{xx} - pq^2, \\ r_{t_2} = -\frac{1}{2}\, p_{xx} - \frac{1}{4}\, r_{xx} + p^2 q + sp^2 + 2\, rpq, \\ s_{t_2} = \frac{1}{2}\, q_{xx} + \frac{1}{2}\, s_{xx} - q^2 p - rq^2 - 2\, spq. \end{cases} \tag{2.66}$$

We shall present our study of bi-integrable couplings in the next chapter.

Chapter 3

Bi-integrable couplings and Hamiltonian structures

In this chapter, we are going to show a practicable way to generate bi-integrable couplings through semi-direct sums of matrix Lie algebras.

3.1 Matrix Lie algebras for bi-integrable couplings

We seek for non-semisimple matrix Lie algebras, under which we can generate bi-integrable couplings of an integrable system (2.1) by using the zero curvature equation. First, we look for matrix algebras consisting of 3×3 block matrices of the form

$$M(A_1, A_2, A_3) = \begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & \sum_{i=1}^{3} \alpha_{1,i} A_i & \sum_{i=1}^{3} \alpha_{2,i} A_i \\ 0 & 0 & \sum_{i=1}^{3} \alpha_{3,i} A_i \end{bmatrix},$$

where $\alpha_{i,j},\ 1 \le i, j \le 3$ are constants to be determined. The reason why we choose these triangular type block matrices is that Lax pair U and V of triangular types will help generate bi-integrable couplings. Thus in the next step, we want to classify classes of such matrices which form matrix Lie algebras under matrix commutator

$$[U, V] := UV - VU. \tag{3.1}$$

As a result, we require that the Lie bracket

$$[M(A_1, A_2, A_3), M(B_1, B_2, B_3)]$$

of two matrices $M(A_1, A_2, A_3)$ and $M(B_1, B_2, B_3)$ must be of the same form $M(C_1, C_2, C_3)$ for certain square submatrices C_1, C_2, C_3 of the same order as A_i and B_i, $1 \le i \le 3$, i.e.,

$$M(A_1, A_2, A_3)M(B_1, B_2, B_3) = M(C_1, C_2, C_3). \tag{3.2}$$

It thus follows that such square submatrices C_1, C_2 and C_3 must read

$$\begin{cases} C_1 = A_1B_1, \\ C_2 = A_1B_2 + \sum_{i=1}^{3} \alpha_{1,i}A_2B_i, \\ C_3 = A_1B_3 + \sum_{2\le i\le 3,\, 1\le j\le 3} \alpha_{i,j}A_iB_j. \end{cases} \tag{3.3}$$

and more relations are required by the closure property (3.2),

$$\begin{cases} \sum_{i=1}^{3} \alpha_{1,i}C_i = \sum_{1\le i,\, j\le 3} \alpha_{1,i}\alpha_{1,j}A_iB_j, \\ \sum_{i=1}^{3} \alpha_{2,i}C_i = \sum_{1\le i,\, j\le 3} (\alpha_{1,i}\alpha_{2,j} + \alpha_{2,i}\alpha_{3,j})A_iB_j, \\ \sum_{i=1}^{3} \alpha_{3,i}C_i = \sum_{1\le i,\, j\le 3} \alpha_{3,i}\alpha_{3,j}A_iB_j. \end{cases} \tag{3.4}$$

Solving (3.3) and (3.4), we find out there exist many classes of non-semisimple Lie algebras of such matrices that can form the expected subalgebras under matrix commutator. Here is a list of them:

$$\text{Class}_{\mathbf{1}} = \begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & A_1 + \alpha A_2 + \beta A_3 & 0 \\ 0 & 0 & A_1 + \alpha A_2 + \beta A_3 \end{bmatrix},$$

$$\text{Class}_{\mathbf{2}} = \begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & A_1 + \frac{\beta}{\alpha} A_2 & \alpha A_1 + \beta A_2 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\text{Class}_{\mathbf{3}} = \begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & A_1 + \alpha A_2 & \beta A_1 + \alpha A_3 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\text{Class}_{\mathbf{4}} = \begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & A_1 + \alpha A_2 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\text{Class}_{\mathbf{5}} = \begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & A_1 + \alpha A_2 & \beta A_2 + \mu A_3 \\ 0 & 0 & A_1 + \mu A_2 - \frac{\mu(\alpha-\mu)}{\beta} A_3 \end{bmatrix},$$

$$\text{Class}_{\mathbf{6}} = \begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & A_1 + \alpha A_2 & 0 \\ 0 & 0 & A_1 + \beta A_3 \end{bmatrix},$$

$$\text{Class}_{\mathbf{7}} = \begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & A_1 + \alpha A_2 & \alpha A_3 \\ 0 & 0 & A_1 \end{bmatrix},$$

$$\text{Class}_{\mathbf{8}} = \begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & A_1 + \alpha A_2 & \alpha A_3 \\ 0 & 0 & A_1 + \alpha A_2 + \beta A_3 \end{bmatrix},$$

$$\text{Class}_{\mathbf{9}} = \begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & 0 & \alpha A_1 + \alpha^2 \beta A_2 + \alpha\beta A_3 \\ 0 & 0 & A_1 + \alpha\beta A_2 + \beta A_3 \end{bmatrix},$$

$$\text{Class}_{\mathbf{10}} = \begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where α, β, μ are three arbitrarily fixed constants.

We shall focus on one class of the presented non-semisimple matrix Lie algebras and construct bi-integrable couplings by using the enlarged zero curvature equation. Moreover, the resulting bi-integrable couplings have infinitely many symmetries and conserved functionals, which further indicates that they often possess bi-Hamiltonian structures.

In what follows, we consider a class of triangular block matrices

$$M(A_1, A_2, A_3) = \begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & A_1 & \alpha\, A_2 + \alpha\, \beta\, A_3 \\ 0 & 0 & A_1 + \alpha\, \beta\, A_2 + \alpha\, \beta^2 A_3 \end{bmatrix}, \tag{3.5}$$

where A_1, A_2, A_3 are square matrices of the same order and α, β are two arbitrarily fixed constants. We see that this class of triangular block matrices is a special case of $\text{Class}_{\mathbf{5}}$, if we set α in $\text{Class}_{\mathbf{5}}$ to be zero. Obviously, under the matrix Lie bracket $[\cdot, \cdot]$ as defined in (3.1), all block matrices $M(A_1, A_2, A_3)$ as defined in (3.5) form a matrix Lie algebra, since for any square matrices A_1, A_2, A_3 and B_1, B_2, B_3 of the same order, we have

$$[M(A_1, A_2, A_3), M(B_1, B_2, B_3)] = M(C_1, C_2, C_3), \tag{3.6}$$

with

$$\begin{cases} C_1 = [A_1, B_1], \\ C_2 = [A_1, B_2] + [A_2, B_1], \\ C_3 = [A_1, B_3] + \alpha[A_2, B_2] + \alpha\beta[A_2, B_3] \\ \qquad +[A_3, B_1] + \alpha\beta[A_3, B_2] + \alpha\beta^2[A_3, B_3]. \end{cases} \tag{3.7}$$

Up to this point, we have not specified what the square matrices A_1, A_2, A_3 will be taken. In the next step, we will concentrate on this matrix Lie algebra and take its decomposition as a semi-direct sum of two subalgebras.

We define two matrix loop Lie algebras

$$\mathfrak{g}_1 = \{\, M(A_1, 0, 0) |\, \text{entries of } A_1 \text{ - Laurent series in } \lambda \}\,, \tag{3.8}$$

and

$$\mathfrak{g}_2 = \{\, M(0, A_2, A_3) |\, \text{entries of } A_2, A_3 \text{ - Laurent series in } \lambda \}\,. \tag{3.9}$$

Next, we take a semi-direct sum

$$\bar{\mathfrak{g}} = \mathfrak{g}_1 \Subset \mathfrak{g}_2. \tag{3.10}$$

of these two Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ as introduced in (3.8) and (3.9) to get

$$\bar{\mathfrak{g}} = \{\, M(A_1, A_2, A_3) |\, \text{entries of } A_1, A_2, A_3 \text{ - Laurent series in } \lambda \}\,. \tag{3.11}$$

It follows that $\bar{\mathfrak{g}}$ is an infinite-dimensional Lie algebra.

Now we have constructed the non-semisimple Lie algebra, associated with which we will formulate a scheme for constructing bi-integrable couplings.

3.2 A general scheme for constructing Hamiltonian bi-integrable couplings

In order to take advantage of zero curvature equations associated with the semi-direct sum of Lie algebras, we assume that the original integrable system

$$u_t = K(u)$$

is determined by a zero curvature equation

$$U_t - V_x + [U, V] = 0, \tag{3.12}$$

where the Lax pair $U = U(u, \lambda)$ and $V = V(u, \lambda)$, with λ being the spectral parameter, are square matrices belonging to some semisimple matrix Lie algebra [68].

Our goal is to construct bi-integrable couplings

$$\begin{cases} u_t = K(u), \\ u_{1,t} = S_1(u, u_1), \\ u_{2,t} = S_2(u, u_1, u_2), \end{cases} \tag{3.13}$$

of the system (2.1) and therefore we enlarge the original spectral matrix U and define the corresponding enlarged spectral matrix $\bar{U}$ as follows:

$$\bar{U} = \bar{U}(\bar{u}, \lambda) = M(U(u, \lambda), U_1(u_1, \lambda), U_2(u_2, \lambda)) \in \bar{\mathfrak{g}} = \mathfrak{g}_1 \in\!\!\!\!+ \mathfrak{g}_2, \tag{3.14}$$

where $\bar{u} = (u^T, u_1^T, u_2^T)^T$. We also assume that its enlarged Lax matrix $\bar{V}$ is in the form of

$$\bar{V} = \bar{V}(\bar{u}, \lambda) = M(V(u, \lambda), V_1(u, u_1, \lambda), V_2(u, u_1, u_2, \lambda)) \in \bar{\mathfrak{g}} = \mathfrak{g}_1 \in\!\!\!\!+ \mathfrak{g}_2. \tag{3.15}$$

Apparently, the Lie bracket $[\bar{U}, \bar{V}]$ of $\bar{U}$ and $\bar{V}$ is in $\bar{\mathfrak{g}} = \mathfrak{g}_1 \in\!\!\!\!+ \mathfrak{g}_2$.

Consequently, the corresponding enlarged zero curvature equation

$$\bar{U}_t - \bar{V}_x + [\bar{U}, \bar{V}] = 0 \tag{3.16}$$

is equivalent to the following triangle system

$$\begin{cases} U_t - V_x + [U,V] = 0, \\ U_{1,t} - V_{1,x} + [U,V_1] + [U_1,V] = 0, \\ U_{2,t} - V_{2,x} + [U,V_2] + \alpha[U_1,V_1] \\ \quad +\alpha\beta[U_1,V_2] + [U_2,V] + \alpha\beta[U_2,V_1] + \alpha\beta^2[U_2,V_2] = 0. \end{cases} \tag{3.17}$$

The first equation above precisely gives the system (2.1), and the second and third equations give the sub-systems $u_{1,t} = S_1(u,u_1)$ and $u_{2,t} = S_2(u,u_1,u_2)$, respectively. Thus, the triangle system gives a bi-integrable coupling system (3.13). This shows a basic idea of constructing bi-integrable couplings by using the semi-direct sum of Lie algebras in (3.8) and (3.9).

We assume that we knew U, U_1, and U_2, and then we are going to seek for a polynomial solution $\bar{V}$ of the enlarged stationary zero curvature equation of degree m (hence we denote this $\bar{V}$ by $\bar{V}^{[m]}$ and its corresponding time variable by t_m).

The constructing scheme is stated as follows.

The first step of the formulation of a hierarchy is to construct a generating function $\bar{W}$ by solving the corresponding enlarged stationary zero curvature equation

$$\bar{W}_x = [\bar{U},\bar{W}], \quad \bar{W} = \bar{W}(\bar{u},\lambda), \tag{3.18}$$

with the following form

$$\bar{W} = M(W(u,\lambda), W_1(u,u_1,\lambda), W_2(u,u_1,u_2,\lambda)) \in \bar{\mathfrak{g}} = \mathfrak{g}_1 \Subset \mathfrak{g}_2. \tag{3.19}$$

Plugging (3.19) into (3.18), we get the triangle system

$$\begin{cases} W_x = [U,W], \\ W_{1,x} = [U,W_1] + [U_1,W], \\ W_{2,x} = [U,W_2] + \alpha[U_1,W_1] + \alpha\beta[U_1,W_2] \\ \qquad\quad +[U_2,W] + \alpha\beta[U_2,W_1] + \alpha\beta^2[U_2,W_2]. \end{cases} \tag{3.20}$$

We assume that W, W_1, W_2 are in the form of

$$W = \sum_{i\geq 0} W_{0,i}\lambda^{-i},\ W_1 = \sum_{i\geq 0} W_{1,i}\lambda^{-i},\ W_2 = \sum_{i\geq 0} W_{2,i}\lambda^{-i}, \tag{3.21}$$

Then we define $\bar{V}^{[m]}$ by

$$\bar{V}^{[m]} = M(V^{[m]}, V_1^{[m]}, V_2^{[m]}) \in \bar{\mathfrak{g}} = \mathfrak{g}_1 \Subset \mathfrak{g}_2, \tag{3.22}$$

and

$$V^{[m]} = (\lambda^m W)_+ + \Delta_m,\ V_i^{[m]} = (\lambda^m W_i)_+ + \Delta_{m,i},\ i = 1, 2,\ m \geq 0, \tag{3.23}$$

where P_+ denotes the polynomial part of P in λ as before, and choose $\Delta_{m,i}$ to make sure that (3.16) with $\bar{V}^{[m]},\ m \geq 0$, i.e.,

$$\bar{U}_{t_m} - \bar{V}_x^{[m]} + [\bar{U}, \bar{V}^{[m]}] = 0,\ m \geq 0, \tag{3.24}$$

generate a soliton hierarchy of bi-integrable coupling systems

$$\bar{u}_{t_m} = \bar{K}_m(\bar{u}), \quad m \geq 0, \tag{3.25}$$

where

$$\bar{u} = \begin{bmatrix} u \\ u_1 \\ u_2 \end{bmatrix}, \quad \bar{K}_m(\bar{u}) = \begin{bmatrix} K_m(u) \\ S_{1,m}(u, u_1) \\ S_{2,m}(u, u_1, u_2) \end{bmatrix}, \quad m \geq 0. \tag{3.26}$$

We shall apply this scheme to the AKNS soliton hierarchy to construct its bi-integrable couplings in the next section.

3.3 Application to the AKNS hierarchy

3.3.1 Bi-integrable couplings of the AKNS hierarchy

For the AKNS hierarchy spectral problem (2.33), using the matrix Lie algebra (3.11) we have chosen in section 3.1, we define the corresponding enlarged spectral matrix by

$$\bar{U} = \bar{U}(\bar{u}, \lambda) = M(U, U_1, U_2) \in \bar{\mathfrak{g}} = \mathfrak{g}_1 \Subset \mathfrak{g}_2, \tag{3.27}$$

$$U_1 = U_1(u_1) = \begin{bmatrix} 0 & r \\ s & 0 \end{bmatrix},\ U_2 = U_2(u_2) = \begin{bmatrix} 0 & v \\ w & 0 \end{bmatrix}, \tag{3.28}$$

where $\bar{u} = (u^T, u_1^T, u_2^T)^T$, $u_1 = (r, s)^T$, $u_2 = (v, w)^T$, and r, s, v, w are new dependent variables.

To solve the corresponding enlarged stationary zero curvature equation

$$\bar{W}_x = [\bar{U}, \bar{W}], \tag{3.29}$$

we set a solution of the following form

$$\bar{W} = M(W, W_1, W_2) \in \bar{\mathfrak{g}} = \mathfrak{g}_1 \Subset \mathfrak{g}_2, \tag{3.30}$$

and assume that W as defined in (2.35),

$$W_1, W_2 \in \widetilde{\mathrm{sl}}(2,\mathbb{R}) = \{A \in \mathrm{sl}(2,\mathbb{R}) | \text{ entries of } A \text{ - Laurent series in } \lambda\}$$

are in the form of

$$\begin{cases} W_1 = W_1(u, u_1, \lambda) = \begin{bmatrix} e & f \\ g & -e \end{bmatrix} = \sum_{i\geq 0} \begin{bmatrix} e_i & f_i \\ g_i & -e_i \end{bmatrix} \lambda^{-i}, \\ W_2 = W_2(u, u_1, u_2, \lambda) = \begin{bmatrix} e' & f' \\ g' & -e' \end{bmatrix} = \sum_{i\geq 0} \begin{bmatrix} e'_i & f'_i \\ g'_i & -e'_i \end{bmatrix} \lambda^{-i}. \end{cases} \tag{3.31}$$

It now follows from the enlarged stationary zero curvature equation (3.29) that

$$\begin{cases} W_x &= [U, W], \\ W_{1,x} &= [U, W_1] + [U_1, W], \\ W_{2,x} &= [U, W_2] + [U_2, W] + \alpha[U_1, W_1] \\ & \quad +\alpha\beta[U_1, W_2] + \alpha\beta[U_2, W_1] + \alpha\beta^2[U_2, W_2]. \end{cases} \tag{3.32}$$

The above equation system equivalently leads to

$$\begin{cases} a_x = -2\,c\lambda + 2\,qb, \\ b_x = 2\,qa - 2\,cr, \\ c_x = 2\,a\lambda - 2\,br, \end{cases} \tag{3.33}$$

$$\begin{cases} e_x = pg + rc - qf - sb, \\ f_x = -2\,\lambda\, f - 2\,pe - 2\,ra, \\ g_x = 2\,qe + 2\,\lambda\, g + 2\,sa, \end{cases} \tag{3.34}$$

$$
\begin{cases}
e'_x = -w\,b + v\,c - \alpha\,(s+\beta\,w)\,f + \alpha\,(r+\beta\,v)\,g \\
\qquad -(q+\alpha\,\beta\,s+\alpha\,\beta^2 w)\,f' + (p+\alpha\,\beta\,r+\alpha\,\beta^2 v)\,g', \\
f'_x = -2\,a\,v - 2\,\alpha\,(r+\beta\,v)\,e \\
\qquad -2\,(p+\alpha\,\beta\,r+\alpha\,\beta^2 v)\,e' - 2\,\lambda\,f', \\
g'_x = 2\,a\,w + 2\,\alpha\,(s+\beta\,w)\,e \\
\qquad +2\,(q+\alpha\,\beta\,s+\alpha\,\beta^2 w)\,e' + 2\,\lambda\,g'.
\end{cases}
\tag{3.35}
$$

Comparing the coefficient of each $\lambda^{-i}, i \geq 0$, we obtain

$$
\begin{cases}
f_{i+1} = -\frac{1}{2}\,f_{ix} - p\,e_i - r\,a_i, \\
g_{i+1} = \frac{1}{2}\,g_{ix} - q\,e_i - s\,a_i, \\
e_{i+1,x} = p\,g_{i+1} + r\,c_{i+1} - q\,f_{i+1} - s\,b_{i+1},
\end{cases}
\quad \text{for } i \geq 0,
\tag{3.36}
$$

$$
\begin{cases}
f'_{i+1} = -\frac{1}{2}\,f'_{ix} - (p+\alpha\beta\,r+\alpha\,\beta^2\,v)\,e'_i \\
\qquad -\alpha\,(r+\beta)\,v\,e_i - v\,a_i, \\
g'_{i+1} = \frac{1}{2}\,g'_{ix} - (q+\alpha\,\beta\,s+\alpha\,\beta^2\,w)\,e'_i \\
\qquad -\alpha\,(s+\beta)\,w\,e_i - w\,a_i, \\
e'_{i+1,x} = -w\,b_{i+1} + v\,c_{i+1} - \alpha\,(s+\beta\,w)\,f_{i+1} \\
\qquad +\alpha\,(r+\beta\,v)\,g_{i+1} - (q+\alpha\,\beta\,s+\alpha\,\beta^2 w)\,f'_{i+1} \\
\qquad +(p+\alpha\,\beta\,r+\alpha\,\beta^2 v)\,g'_{i+1},
\end{cases}
\quad \text{for } i \geq 0.
\tag{3.37}
$$

Then the recursion relations (3.36) and (3.37) generate the sequences of $\{e_i\}_{i\geq 1},\ \{f_i\}_{i\geq 1},\ \{g_i\}_{i\geq 1}$ and $\{e'_i\}_{i\geq 1},\ \{f'_i\}_{i\geq 1},\ \{g'_i\}_{i\geq 1}$, upon introducing

$$
e_0 = e'_0 = -1,\ f_0 = g_0 = f'_0 = g'_0 = 0.
\tag{3.38}
$$

We can compute the first few sets as follows:

$$
\begin{cases}
e_1 = 0, \\
f_1 = p + r, \\
g_1 = q + s;
\end{cases}
\tag{3.39}
$$

$$
\begin{cases}
e_2 = \frac{1}{2}\,pq + \frac{1}{2}\,sp + \frac{1}{2}\,rq, \\
f_2 = -\frac{1}{2}\,p_x - \frac{1}{2}\,r_x, \\
g_2 = \frac{1}{2}\,q_x + \frac{1}{2}\,s_x;
\end{cases}
\tag{3.40}
$$

$$
\begin{cases}
e_3 = \frac{1}{4}\,pq_x + \frac{1}{4}\,s_xp - \frac{1}{4}\,qp_x - \frac{1}{4}\,sp_x - \frac{1}{4}\,r_xq + \frac{1}{4}\,rq_x, \\
f_3 = \frac{1}{4}\,p_{xx} + \frac{1}{4}\,r_{xx} - \frac{1}{2}\,p^2q - \frac{1}{2}\,sp^2 - rpq, \\
g_3 = \frac{1}{4}\,q_{xx} + \frac{1}{4}\,s_{xx} - \frac{1}{2}\,q^2p - \frac{1}{2}\,rq^2 - spq;
\end{cases}
\tag{3.41}
$$

and

$$
\begin{cases}
e'_1 = 0, \\
f'_1 = p + \alpha(1+\beta)r + (1+\alpha\beta+\alpha\beta^2)v, \\
g'_1 = q + \alpha(1+\beta)s + (1+\alpha\beta+\alpha\beta^2)w;
\end{cases}
\tag{3.42}
$$

$$
\begin{cases}
e'_2 = \frac{1}{2}\,pq + \frac{1}{2}\,\alpha\,(1+\beta)\,ps \\
\qquad +\frac{1}{2}\,(1+\alpha\,\beta+\alpha\,\beta^2)\,\big(pw + rq + vq + \alpha\,rs \\
\qquad +\alpha\,\beta\,rw + \alpha\,\beta\,vs + \alpha\,\beta^2\,vw\big), \\
f'_2 = -\frac{1}{2}\,p_x - \frac{1}{2}\,\alpha\,(1+\beta)\,r_x - \frac{1}{2}\,(1+\alpha\,\beta+\alpha\,\beta^2)\,v_x, \\
g'_2 = \frac{1}{2}\,q_x + \frac{1}{2}\,\alpha\,(1+\beta)\,s_x + \frac{1}{2}\,(1+\alpha\,\beta+\alpha\,\beta^2)\,w_x;
\end{cases}
\tag{3.43}
$$

$$
\begin{cases}
e'_3 = \frac{1}{4}\,(pq_x - qp_x) + \frac{1}{4}\,\alpha\,(1+\beta)\,(ps_x - p_xs + rq_x - r_xq) \\
\qquad +\frac{1}{4}\,(1+\alpha\,\beta+\alpha\,\beta^2)\,\big[(pw_x - p_xw) + \alpha\,(rs_x - r_xs + rw_x \\
\qquad -r_xw - v_xq + vq_x + vs_x - v_xs + vw_x - v_xw)\big], \\
f'_3 = \frac{1}{4}\,p_{xx} - \frac{1}{2}\,p^2q + \alpha\,(1+\beta)\,(\frac{1}{4}\,r_{xx} - \frac{1}{2}\,p^2s - prq) \\
\qquad + (1+\alpha\,\beta+\alpha\,\beta^2)\,\big[\frac{1}{4}\,v_{xx} - \,pvq - \frac{1}{2}\,p^2w - \alpha(\,prs + \frac{1}{2}\,r^2q) \\
\qquad -\alpha\,\beta\,(prw + pvs + rvq) - \alpha\,\beta^2\,(pvw + \frac{1}{2}\,v^2q) - \frac{1}{2}\,\alpha^2\beta\,r^2s \\
\qquad -\alpha^2\beta^2\,(rvs + \frac{1}{2}\,r^2w) - \alpha^2\beta^3\,(rvw + \frac{1}{2}\,v^2s) - \frac{1}{2}\,\alpha^2\beta^4\,v^2w\big], \\
g'_3 = \frac{1}{4}\,q_{xx} - \frac{1}{2}\,q^2p + \alpha(1+\beta)\,(\frac{1}{4}\,s_{xx} - \frac{1}{2}\,rq^2 - \,qps) \\
\qquad +(1+\alpha\,\beta+\alpha\,\beta^2)\,\big[\frac{1}{4}\,w_{xx} - pqw - \frac{1}{2}\,vq^2 - \alpha\,(qrs + \frac{1}{2}\,s^2p) \\
\qquad -\alpha\,\beta\,(qrw + qvs + spw) - \alpha\,\beta^2\,(\frac{1}{2}\,w^2p + qvw) - \frac{1}{2}\,\alpha^2\beta\,s^2r \\
\qquad -\alpha^2\beta^2\,(\frac{1}{2}\,s^2v + srw) - \alpha^2\beta^3\,(svw + \frac{1}{2}\,w^2r) - \frac{1}{2}\,\alpha^2\beta^4\,w^2v.
\end{cases}
\tag{3.44}
$$

Let us now define

$$\bar{V}^{[m]} = M(V^{[m]}, V_1^{[m]}, V_2^{[m]}) \in \bar{\mathfrak{g}} = \mathfrak{g}_1 \Subset \mathfrak{g}_2, \tag{3.45}$$

and

$$\begin{cases} V_1^{[m]} = (\lambda^m W_1)_+ + \Delta_{m,1}, \\ V_2^{[m]} = (\lambda^m W_2)_+ + \Delta_{m,2}, \end{cases} \quad m \geq 0\ , \tag{3.46}$$

where $V^{[m]}$ is defined as in (2.39), and $\Delta_{m,i}$ are chosen as the zero matrix. Then, the m-th enlarged zero curvature equation

$$\bar{U}_{t_m} = \bar{V}_x^{[m]} - [\bar{U}, \bar{V}^{[m]}] \tag{3.47}$$

gives rise to

$$\begin{cases} U_{t_m} = V_x^{[m]} - [U, V^{[m]}], \\ U_{1,t_m} = V_{1,x}^{[m]} - [U, V_1^{[m]}] - [U_1, V^{[m]}], \\ U_{2,t_m} = V_{2,x}^{[m]} - [U, V_2^{[m]}] - [U_2, V^{[m]}] - \alpha[U_1, V_1^{[m]}] \\ \qquad -\alpha\beta[U_1, V_2^{[m]}] - \alpha\beta[U_2, V_1^{[m]}] - \alpha\beta^2[U_2, V_2^{[m]}]. \end{cases} \tag{3.48}$$

Thus, a hierarchy of coupling systems are generated for the AKNS hierarchy (2.40):

$$\bar{u}_{t_m} = \begin{bmatrix} p_{t_m} \\ q_{t_m} \\ r_{t_m} \\ s_{t_m} \\ v_{t_m} \\ w_{t_m} \end{bmatrix} = \bar{K}_m(\bar{u}) = \begin{bmatrix} K_m(u) \\ S_{1,m}(u, u_1) \\ S_{2,m}(u, u_1, u_2) \end{bmatrix} = \begin{bmatrix} -2b_{m+1} \\ 2c_{m+1} \\ -2f_{m+1} \\ 2g_{m+1} \\ -2f'_{m+1} \\ 2g'_{m+1} \end{bmatrix}, \quad m \geq 0. \tag{3.49}$$

This suggests that (3.49) provides a hierarchy of nonlinear bi-integrable couplings for the AKNS hierarchy of soliton equations. The first nonlinear bi-integrable coupling system reads

$$\begin{cases} p_{t_2} &= -\frac{1}{2}p_{xx} + p^2q, \\ q_{t_2} &= \frac{1}{2}q_{xx} - pq^2, \\ r_{t_2} &= -\frac{1}{2}\, p_{xx} - \frac{1}{4}\, r_{xx} + \, p^2q + \, sp^2 + 2\, rpq, \\ s_{t_2} &= \frac{1}{2}\, q_{xx} + \frac{1}{2}\, s_{xx} - q^2p - rq^2 - 2\, spq, \\ v_{t_2} &= -\frac{1}{2}\, p_{xx} + \, p^2q - 2\, \alpha\, (1+\beta)\, (\frac{1}{4}\, r_{xx} - \frac{1}{2}\, p^2s - prq) \\ & \quad -2\, (1+\alpha\, \beta + \alpha\, \beta^2) \left[\frac{1}{4}\, v_{xx} - \, pvq - \frac{1}{2}\, p^2w - \alpha(\, prs + \frac{1}{2}\, r^2q) \right. \\ & \quad -\alpha\, \beta\, (prw + pvs + rvq) - \alpha\, \beta^2\, (pvw + \frac{1}{2}\, v^2q) - \frac{1}{2}\, \alpha^2\beta\, r^2s \\ & \quad \left. -\alpha^2\beta^2\, (rvs + \frac{1}{2}\, r^2w) - \alpha^2\beta^3\, (rvw + \frac{1}{2}\, v^2s) - \frac{1}{2}\, \alpha^2\beta^4\, v^2w \right], \\ w_{t_2} &= \frac{1}{2}\, q_{xx} - \, q^2p + 2\, \alpha(1+\beta)\, (\frac{1}{4}\, s_{xx} - \frac{1}{2}\, rq^2 - \, qps) \\ & \quad +2\, (1+\alpha\, \beta + \alpha\, \beta^2) \left[\frac{1}{4}\, w_{xx} - pqw - \frac{1}{2}\, vq^2 - \alpha\, (qrs + \frac{1}{2}\, s^2p) \right. \\ & \quad -\alpha\, \beta\, (qrw + qvs + spw) - \alpha\, \beta^2\, (\frac{1}{2}\, w^2p + qvw) - \frac{1}{2}\, \alpha^2\beta\, s^2r \\ & \quad \left. -\alpha^2\beta^2\, (\frac{1}{2}\, s^2v + srw) - \alpha^2\beta^3\, (svw + \frac{1}{2}\, w^2r) - \frac{1}{2}\, \alpha^2\beta^4\, w^2v. \right. \end{cases} \tag{3.50}$$

The references [39, 40] formulated integrable couplings for given integrable systems by perturbations, in which the second component of the enlarged system was just the linearized system of the original system $u_t = K(u)$, while the bi-integrable couplings constructed above are nonlinear, because the third sub-systems are nonlinear.

3.3.2 Hamiltonian structures of the bi-integrable couplings of the AKNS hierarchy

It is known that when acting on non-semisimple Lie algebras, the Killing form is always degenerate, and, the trace identity (see [69, 62] for details) will not apply in this case. To solve this problem, the variational identity was introduced in [37, 55] under more general bilinear forms, which do not require the invariance property under an isomorphism of the Lie algebra. In this section, in order to generate Hamiltonian structures of the resulting bi-integrable couplings on the presented non-semisimple Lie algebra, we use the corresponding variational identity [55]:

$$\frac{\delta}{\delta \bar{u}} \int \langle \bar{U}_\lambda, \bar{W} \rangle dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} \langle \bar{U}_{\bar{u}}, \bar{W} \rangle, \tag{3.51}$$

where $\langle \cdot, \cdot \rangle$ is a required bilinear form, which is symmetric, non-degenerate, and invariant under the Lie bracket.

One of the most important steps is to construct general bilinear forms with the symmetric, invariant, and non-degenerate properties $\langle \cdot, \cdot \rangle$ on $\bar{\mathfrak{g}}$.

First, we transform the semi-direct sum $\bar{\mathfrak{g}}$ into a vector form via defining:

$$\sigma : \bar{\mathfrak{g}} \to \mathbb{R}^9,\ A \mapsto (a_1, \cdots, a_9)^T, \tag{3.52}$$

where

$$A = A(a_1, \cdots, a_9) = M(A_1, A_2, A_3),\ A_i = \begin{bmatrix} a_{3i-2} & a_{3i-1} \\ a_{3i} & -a_{3i-2} \end{bmatrix},\ 1 \leq i \leq 3. \tag{3.53}$$

This mapping σ induces a Lie algebraic structure on $\mathbb{R}^9$, which is isomorphic to the matrix loop algebra $\bar{\mathfrak{g}}$. Next we define the corresponding Lie bracket $[\cdot, \cdot]$ on $\mathbb{R}^9$ by

$$[a, b]^T = a^T R(b), \tag{3.54}$$

for any $a = (a_1, \cdots, a_9)^T,\ b = (b_1, \cdots, b_9)^T \in \mathbb{R}^9$, and

$$R(b) = M(R_1, R_2, R_3), \tag{3.55}$$

where $R_1,\ R_2$, and R_3 are the matrices defined by

$$R_i = \begin{bmatrix} 0 & 2\,b_{3i-1} & -2\,b_{3i} \\ b_{3i} & -2\,b_{3i-2} & 0 \\ -b_{3i-1} & 0 & 2\,b_{3i-2} \end{bmatrix},\ \text{for } i = 1, 2, 3. \tag{3.56}$$

We then define a bilinear form on $\mathbb{R}^9$ by

$$\langle a, b \rangle = a^T F b, \tag{3.57}$$

where F is a constant matrix. The symmetric property $\langle a, b \rangle = \langle b, a \rangle$ requires that

$$F^T = F. \tag{3.58}$$

Under this symmetric condition, the invariance property under the Lie bracket

$$\langle a, [b, c] \rangle = \langle [a, b], c \rangle$$

equivalently requires that

$$F(R(b))^T = -R(b)F,\ b \in \mathbb{R}^9. \tag{3.59}$$

This matrix equation leads to a linear system of equations on the elements of F. Solving the resulting system yields

$$F = \begin{bmatrix} \eta_1 & \eta_2 & \eta_3 \\ \eta_2 & \alpha\,\eta_3 & \alpha\,\beta\,\eta_3 \\ \eta_3 & \alpha\,\beta\,\eta_3 & \alpha\,\beta^2\eta_3 \end{bmatrix} \otimes \begin{bmatrix} 2\;0\;0 \\ 0\;0\;1 \\ 0\;1\;0 \end{bmatrix}, \tag{3.60}$$

where η_i, $1 \le i \le 3$, are arbitrary constants, and $\otimes$ is the Kronecker product.

Now, the corresponding bilinear form on the semi-direct sum $\bar{\mathfrak{g}}$ of Lie algebras is given by

$$\begin{aligned}
\langle A, B\rangle = \langle A, B\rangle_{\bar{\mathfrak{g}}} = \langle \sigma(A), \sigma(B)\rangle_{\mathbb{R}^9} &= (a_1, \cdots, a_9)F(b_1, \cdots, b_9)^T \\
&= (2\,a_1b_1 + a_2b_3 + a_3b_2)\eta_1 \\
&\quad +(2\,a_1b_4 + a_2b_6 + a_3b_5 + 2\,a_4b_1 + a_5b_3 + a_6b_2)\eta_2 \\
&\quad +(2\,a_1b_7 + a_2b_9 + a_3b_8 + 2\,\alpha\,a_4b_4 \\
&\quad +2\,\alpha\,\beta\,a_4b_7 + \alpha\,a_5b_6 + \alpha\,\beta\,a_5b_9 + \alpha\,a_6b_5 + \alpha\,\beta\,a_6b_8 \\
&\quad +2\,a_7b_1 + 2\,\alpha\,\beta\,a_7b_4 + 2\,\alpha\,\beta^2a_7b_7 + a_8b_3 + \alpha\,\beta\,a_8b_6 \\
&\quad +\alpha\,\beta^2a_8b_9 + a_9b_2 + \alpha\,\beta^2a_9b_8 + \alpha\,\beta\,a_9b_5)\eta_3,
\end{aligned} \tag{3.61}$$

where $A = A(a_1, \cdots, a_9), B = B(b_1, \cdots, b_9) \in \bar{\mathfrak{g}}$ are as defined in (3.11).

The bilinear form (3.61) is symmetric and invariant under the Lie bracket of the matrix Lie algebra:

$$\langle A, B\rangle = \langle B, A\rangle,\ \langle A, [B, C]\rangle = \langle [A, B], C\rangle,$$

where $A = A(a_1, \cdots, a_9), B = B(b_1, \cdots, b_9), C = C(c_1, \cdots, c_9) \in \bar{\mathfrak{g}}$ are as defined in (3.11). We point out that the invariance property under the multiplication

$$\langle A, BC\rangle = \langle AB, C\rangle,$$

does not hold here. Obviously, this kind of bilinear forms is not of Killing type and is non-degenerate if and only if the determinant of the matrix F is non-zero:

$$\det(F) = 8\,\alpha^3\,(\eta_2\beta - \eta_3)^6\,{\eta_3}^3 \neq 0.$$

Therefore we can choose η_2 and η_3 such that $\det(F)$ is non-zero. Note that the two parameters α and β are arbitrary constants associated with the new class of matrix Lie algebras in (3.5), and they also should make $\det(F)$ non-zero to apply the variational identity.

Now we can compute that

$$\langle \bar{W}, \bar{U}_\lambda \rangle = -2\,\eta_1 a - 2\,\eta_2 e - 2\,\eta_3 e', \tag{3.62}$$

and

$$\langle \bar{W}, \bar{U}_{\bar{u}} \rangle = \begin{bmatrix} c\eta_1 + g\eta_2 + g'\,\eta_3 \\ b\eta_1 + f\eta_2 + f'\,\eta_3 \\ c\eta_2 + (\alpha\, g + \alpha\,\beta\, g')\,\eta_3 \\ b\eta_2 + (\alpha\, f + \alpha\,\beta\, f')\,\eta_3 \\ \left(c + \alpha\,\beta\, g + \alpha\,\beta^2 g'\right)\eta_3 \\ \left(b + \alpha\,\beta\, f + \alpha\,\beta^2 f'\right)\eta_3 \end{bmatrix}, \tag{3.63}$$

and furthermore, we have

$$\gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ \ln\,|\langle \bar{W}, \bar{W} \rangle| = 0. \tag{3.64}$$

Thus, by the previous variational identity (3.51), we obtain Hamiltonian structures for the hierarchy of bi-integrable couplings (3.49):

$$\bar{u}_{t_m} = \bar{J}\frac{\delta \bar{\mathcal{H}}_m}{\delta \bar{u}}, \tag{3.65}$$

where the Hamiltonian functionals are

$$\bar{\mathcal{H}}_m = \int \frac{2\,\eta_1 a_{m+2} + 2\,\eta_2 e_{m+2} + 2\,\eta_3\, e'_{m+2}}{m+1}\,dx,\ m \geq 0, \tag{3.66}$$

and the Hamiltonian operator is

$$\bar{J} = \begin{bmatrix} \eta_1 & \eta_2 & \eta_3 \\ \eta_2 & \alpha\,\eta_3 & \alpha\,\beta\,\eta_3 \\ \eta_3 & \alpha\,\beta\,\eta_3 & \alpha\,\beta^2\eta_3 \end{bmatrix}^{-1} \otimes J, \tag{3.67}$$

with matrix J being defined in (2.41).

It is easy to check that

$$\bar{K}_m = \bar{\Phi}\bar{K}_{m-1}, \quad m \geq 1, \tag{3.68}$$

where the hereditary recursion operator $\bar{\Phi}$ (see [70] for details) is defined by

$$\bar{\Phi} = \begin{bmatrix} \Phi & 0 & 0 \\ \Phi_1 & \Phi & 0 \\ \Phi_2 & \alpha\Phi_1 + \alpha\beta\Phi_2 & \Phi + \alpha\beta\Phi_1 + \alpha\beta^2\Phi_2 \end{bmatrix} = M^T(\Phi, \Phi_1, \Phi_2), \tag{3.69}$$

with M^T being the transpose of matrix M in (3.5), Φ being given by (2.41), and

$$\Phi_1 = \begin{bmatrix} r\partial^{-1}q + p\partial^{-1}s & r\partial^{-1}p + p\partial^{-1}r \\ -s\partial^{-1}q - q\partial^{-1}s & -s\partial^{-1}p - q\partial^{-1}r \end{bmatrix}, \tag{3.70}$$

$$\Phi_2 = \begin{bmatrix} v\partial^{-1}q + \theta_1\partial^{-1}s + \theta_2\partial^{-1}w & v\partial^{-1}p + \theta_1\partial^{-1}r + \theta_2\partial^{-1}v \\ -w\partial^{-1}q - \theta_3\partial^{-1}s - \theta_4\partial^{-1}w & -w\partial^{-1}p - \theta_3\partial^{-1}r - \theta_4\partial^{-1}v \end{bmatrix}, \tag{3.71}$$

in which

$$\begin{cases} \theta_1 := \alpha r + \alpha\beta v, \quad \theta_2 := p + \alpha\beta r + \alpha\beta^2 v, \\ \theta_3 := \alpha s + \alpha\beta w, \quad \theta_4 := q + \alpha\beta s + \alpha\beta^2 w. \end{cases} \tag{3.72}$$

It is obvious that $\bar{J}$ is skew-symmetric and

$$\bar{J}\bar{\Phi}^* = \bar{\Phi}\bar{J}, \tag{3.73}$$

where $\bar{\Phi}^*$ denote the adjoint operator of $\bar{\Phi}$. Then $\bar{J}\bar{\Phi}^*$ is also skew-symmetric. Furthermore, it can be shown that $\bar{J}$ and $\bar{M} = \bar{\Phi}\bar{J}$ form a Hamiltonian pair [20], and then it follows that $\bar{\Phi} = \bar{M}\bar{J}^{-1}$ is hereditary operator (see [8, 13]).

Consequently, there exist infinitely many commuting symmetries and conserved functionals:

$$[\bar{K}_m, \bar{K}_n] = \bar{K}_m'(\bar{u})[\bar{K}_n] - \bar{K}_n'(\bar{u})[\bar{K}_m] = 0, \quad m, n \geq 0,$$
$$\{\bar{\mathcal{H}}_m, \bar{\mathcal{H}}_n\}_{\bar{J}} = \int \left(\frac{\delta\bar{\mathcal{H}}_m}{\delta\bar{u}}\right)^T \bar{J}\frac{\delta\bar{\mathcal{H}}_n}{\delta\bar{u}}dx = 0, \quad m, n \geq 0.$$

It is easy to compute that for the m-th bi-integrable coupling system $\bar{u}_{t_m} = \bar{J}\frac{\delta\bar{\mathcal{H}}_m}{\delta\bar{u}}$,

$$\frac{d}{dt_m}\bar{\mathcal{H}}_n = \int \left(\frac{\delta\bar{\mathcal{H}}_n}{\delta\bar{u}}\right)^T \bar{u}_{t_m}dx = \int \left(\frac{\delta\bar{\mathcal{H}}_n}{\delta\bar{u}}\right)^T \bar{J}\frac{\delta\bar{\mathcal{H}}_m}{\delta\bar{u}}dx = 0, \ n \geq 0,$$

which implies that $\{\bar{\mathcal{H}}_n\}_{n\geq 0}$, are conserved, and each Hamiltonian coupling system has infinitely many commuting conserved functionals $\{\bar{\mathcal{H}}_n\}_{n\geq 0}$. Moreover, the resulting bi-integrable couplings possess the bi-Hamiltonian structure

$$\bar{u}_{t_m} = \bar{J}\frac{\delta\bar{\mathcal{H}}_m}{\delta\bar{u}} = \bar{M}\frac{\delta\bar{\mathcal{H}}_{m-1}}{\delta\bar{u}}, \ m \geq 1.$$

3.4 Application to the Dirac hierarchy

3.4.1 The Dirac hierarchy

The Dirac hierarchy is associated with the compatible condition

$$U_t - V_x + [U, V] = 0 \tag{3.74}$$

of two linear problems

$$\phi_x = U\phi,\ U = U(u,\lambda) = \begin{bmatrix} p & \lambda + q \\ -\lambda + q & -p \end{bmatrix},\ u = \begin{bmatrix} p \\ q \end{bmatrix},\ \ \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}, \tag{3.75}$$

and

$$\phi_t = V\phi,\ V = V(u,\lambda), \tag{3.76}$$

which has been studied in the area of integrable coupling by [43, 71, 72]. To derive the Dirac systems associated with (3.75), we solve the stationary zero curvature equation

$$W_x = [U, W], \tag{3.77}$$

and assume that a solution W solution to (3.77) is in the form

$$W = \begin{bmatrix} c & a+b \\ a-b & -c \end{bmatrix}. \tag{3.78}$$

By substituting (3.78) into the equation (3.77), we obtain

$$\begin{cases} a_x = 2\,pb - 2\,\lambda c, \\ b_x = 2\,pa - 2\,qc, \\ c_x = 2\,qa + 2\,\lambda\,c. \end{cases} \tag{3.79}$$

We then assume that

$$a = \sum_{i\geq 0} a_i\,\lambda^{-i},\ b = \sum_{i\geq 0} b_i\,\lambda^{-i},\ c = \sum_{i\geq 0} c_i\,\lambda^{-i}, \tag{3.80}$$

and equate the coefficient of each $\lambda^{-i}, i \geq 0$, we get

$$\begin{cases} a_{i+1} = \frac{1}{2}\,c_{i,x} + qb_i, \\ c_{i+1} = -\frac{1}{2}\,a_{i,x} + pb_i, \qquad \text{for } i \geq 0. \\ b_{i+1,x} = 2\,pa_{i+1} - 2\,qc_{i+1}, \end{cases} \tag{3.81}$$

Assuming

$$b_0 = -1,\; a_0 = c_0 = 0, \tag{3.82}$$

and $a_i|_{[u]=0} = b_i|_{[u]=0} = c_i|_{[u]=0} = 0,\; i \geq 1$, where $[u] = (u, u_x, u_{xx}, \cdots)$, which means to select zero constants for integration, we can uniquely determine a, b, c by (3.81). The first few sets are listed below:

$$\begin{cases} a_1 = -q,\; c_1 = -p,\; b_1 = 0; \\ a_2 = \frac{1}{2}p_x,\; c_2 = \frac{1}{2}q_x,\; b_2 = -\frac{1}{2}\,p^2 - \frac{1}{2}\,q^2; \\ a_3 = \frac{1}{4}\,q_{xx} - \frac{1}{2}\,qp^2 - \frac{1}{2}\,q^3,\; c_3 = \frac{1}{4}\,p_{xx} - \frac{1}{2}\,p^3 - \frac{1}{2}\,pq^2,\; b_3 = \frac{1}{2}\,q_x p - \frac{1}{2}\,qp_x. \end{cases} \tag{3.83}$$

We form the zero curvature equations

$$U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0,\;\; V^{[m]} = (\lambda^m W)_+,\;\; m \geq 0, \tag{3.84}$$

to generate the Dirac hierarchy of soliton equations:

$$u_{t_m} = K_m = \begin{bmatrix} 2a_{m+1} \\ -2c_{m+1} \end{bmatrix} = \Phi^m \begin{bmatrix} -2q \\ 2p \end{bmatrix} = J\frac{\delta \mathcal{H}_m}{\delta u},\;\; m \geq 0, \tag{3.85}$$

with the Hamiltonian operator J, the hereditary recursion operator Φ and the Hamiltonian functionals being defined by

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\;\; \Phi = \begin{bmatrix} -2p\partial^{-1}q & -\frac{1}{2}\partial + 2p\partial^{-1}p \\ \frac{1}{2}\partial - 2q\partial^{-1}q & 2q\partial^{-1}p \end{bmatrix},\;\; \mathcal{H}_m = \int \frac{2b_{m+2}}{m+1}\,dx, \tag{3.86}$$

where $m \geq 0$.

We will enlarge the zero curvature equations to construct bi-integrable couplings.

3.4.2 Bi-integrable couplings of the Dirac hierarchy

A new class of triangular block matrices will be introduced in this section for the purpose of using zero curvature equations associated with it. In order to get bi-integrable couplings, we choose triangular 3×3 block matrices of the following form [77] (a special case of $Class_{\mathbf{5}}$):

$$M(A_1, A_2, A_3) = \begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & A_1 + \alpha A_2 & \beta A_2 + \alpha A_3 \\ 0 & 0 & A_1 + \alpha A_2 \end{bmatrix}, \tag{3.87}$$

where A_1, A_2, A_3 can be any square matrices of the same order. It is easy to verify that under the matrix commutator $[\cdot,\cdot]$:

$$[M_1, M_2] = M_1 M_2 - M_2 M_1$$

for any M_1, M_2 of the same class of triangular block matrices, the collection of all block matrices of the form (3.87) is a subalgebra of the algebra of triangular 3×3 block matrices. A simple computation shows that the matrix commutator relation is

$$[M(A_1, A_2, A_3), M(B_1, B_2, B_3)] = M(C_1, C_2, C_3), \tag{3.88}$$

with

$$\begin{cases} C_1 = [A_1, B_1], \\ C_2 = [A_1, B_2] + [A_2, B_1] + \alpha[A_2, B_2], \\ C_3 = [A_1, B_3] + \beta\,[A_2, B_2] + \alpha\,[A_2, B_3] + [A_3, B_1] + \alpha\,[A_3, B_2]. \end{cases} \tag{3.89}$$

Note that closure property is satisfied by our choice of real number parameter α, β, and μ, that all block matrices defined by (3.87) form a matrix Lie algebra, which create a basis for us to generate nonlinear Hamiltonian bi-integrable couplings. Later in applications to concrete soliton hierarchies, we will show the details of how the block A_1 corresponds to the original integrable equation, and the other two blocks A_2 and A_3 contribute to the supplementary vector fields S_1 and S_2. The commutator $[A_2, B_2]$ yields nonlinear terms in the resulting bi-integrable couplings.

We remark that a reduction of (3.87) when $\alpha = 0$ is rather interesting, based on it we can construct linear bi-integrable couplings. When $\alpha \neq 0$, the block matrix (3.87) is transformed into a direct sum of two smaller block matrices under similarity transformations which means the enlarged system may have two independent subsystems.

We define two matrix loop Lie subalgebras

$$\mathfrak{g}_1 = \{\, M(A_1, 0, 0) |\, \text{entries of } A_1 \text{ - Laurent series in } \lambda \}\,, \tag{3.90}$$

and

$$\mathfrak{g}_2 = \{\, M(0, A_2, A_3) |\, \text{entries of } A_2, A_3 \text{ - Laurent series in } \lambda \}\,. \tag{3.91}$$

After defining the Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$, we form a semi-direct sum

$$\bar{\mathfrak{g}} = \mathfrak{g}_1 \Subset \mathfrak{g}_2. \tag{3.92}$$

It follows that

$$\bar{\mathfrak{g}} = \{\, M(A_1, A_2, A_3) |\, \text{entries of } A_1, A_2, A_3 \text{ - Laurent series in } \lambda \} \,, \tag{3.93}$$

and $\bar{\mathfrak{g}}$ is an infinite-dimensional Lie algebra.

For the Dirac hierarchy's spectral problem (3.75), under the matrix Lie algebra (3.93), we define the corresponding enlarged spectral matrix by

$$\bar{U} = \bar{U}(\bar{u}, \lambda) = M(U, U_1, U_2) \in \bar{\mathfrak{g}} = \mathfrak{g}_1 \Subset \mathfrak{g}_2, \tag{3.94}$$

$$U = U(u_1, \lambda) = \begin{bmatrix} p & -\lambda + q \\ \lambda + q & -p \end{bmatrix},\ U_1 = U_1(u_1) = \begin{bmatrix} r & s \\ s & -r \end{bmatrix},\ U_2 = U_2(u_2) = \begin{bmatrix} v & w \\ w & -v \end{bmatrix}, \tag{3.95}$$

where $\bar{u} = (u^T, u_1^T, u_2^T)^T$, $u_1 = (r, s)^T$, $u_2 = (v, w)^T$, and r, s, v, w are new dependent variables.

To solve the corresponding enlarged stationary zero curvature equation

$$\bar{W}_x = [\bar{U}, \bar{W}], \tag{3.96}$$

we set a solution of the following form

$$\bar{W} = M(W, W_1, W_2) \in \bar{\mathfrak{g}} = \mathfrak{g}_1 \Subset \mathfrak{g}_2, \tag{3.97}$$

and assume that W as defined in (3.78),

$$W_1, W_2 \in \widetilde{\mathrm{sl}}(2, \mathbb{R}) = \{A \in \mathrm{sl}(2, \mathbb{R}) |\, \text{entries of } A \text{ - Laurent series in } \lambda\}$$

are in the form of

$$\begin{cases} W_1 = W_1(u, u_1, \lambda) = \begin{bmatrix} g & e + f \\ e - f & -g \end{bmatrix}, \\ W_2 = W_2(u, u_1, u_2, \lambda) = \begin{bmatrix} g' & e' + f' \\ e' - f' & -g' \end{bmatrix}. \end{cases} \tag{3.98}$$

It now follows from the enlarged stationary zero curvature equation (3.96) that

$$\begin{cases} W_x = [U, W], \\ W_{1,x} = [U, W_1] + [U_1, W] + \alpha [U_1, W_1], \\ W_{2,x} = [U, W_2] + \beta\, [U_1, W_1] + \alpha\, [U_1, W_2] + [U_2, W] + \alpha\, [U_2, W_1]. \end{cases} \tag{3.99}$$

The above equation system equivalently leads to

$$\left\{\begin{aligned} a_x &= 2\,pb - 2\,\lambda\,c, \\ b_x &= 2\,pa - 2\,qc, \\ c_x &= 2\,\lambda\,a - 2\,qb, \end{aligned}\right. \tag{3.100}$$

$$\left\{\begin{aligned} e_x &= 2\,rb + 2\,(p + \alpha\,r)\,f - 2\,\lambda\,g, \\ f_x &= 2\,ra - 2\,sc + 2\,(p + \alpha\,r)\,e - 2\,(q + \alpha\,s)\,g, \\ g_x &= -2\,sb + 2\,\lambda\,e - 2\,(q + \alpha\,s)\,f, \end{aligned}\right. \tag{3.101}$$

$$\left\{\begin{aligned} e'_x &= 2\,vb + 2\,(\alpha\,v + \beta\,r)\,f + 2\,(p + \alpha\,r)\,f' - 2\,\lambda\,g', \\ f'_x &= 2\,va - 2\,wc + 2\,(\alpha\,v + \beta\,r)\,e - 2\,(\alpha\,w + \beta\,s)\,g \\ &\quad +2\,(p + \alpha\,r)\,e' - 2\,(q + \alpha\,s)\,g', \\ g'_x &= -2\,wb - 2\,(\alpha\,w + \beta\,s)\,f + 2\,\lambda\,e' - 2\,(q + \alpha\,s)\,f'. \end{aligned}\right. \tag{3.102}$$

By assuming

$$e = \sum_{i \geq 0} e_i\,\lambda^{-i},\ f = \sum_{i \geq 0} f_i\,\lambda^{-i},\ g = \sum_{i \geq 0} g_i\,\lambda^{-i}, \tag{3.103}$$

and

$$e' = \sum_{i \geq 0} e'_i\,\lambda^{-i},\ f' = \sum_{i \geq 0} f'_i\,\lambda^{-i},\ g' = \sum_{i \geq 0} g'_i\,\lambda^{-i}, \tag{3.104}$$

and comparing the coefficient of each $\lambda^{-i}, i \geq 0$, we obtain

$$\left\{\begin{aligned} e_{i+1} &= \tfrac{1}{2}\,g_{i,x} + sb_i + (q + \alpha\,s)\,f_i, \\ f_{i+1,x} &= 2\,ra_{i+1} - 2\,sc_{i+1} + 2\,(p + \alpha\,r)\,e_{i+1} \\ &\quad -2\,(q + \alpha\,s)\,g_{i+1}, \\ g_{i+1} &= -\tfrac{1}{2}\,e_{i,x} + rb_i + (p + \alpha\,r)\,f_i \end{aligned}\right. \quad \text{for } i \geq 0, \tag{3.105}$$

$$\left\{\begin{aligned} e'_{i+1} &= \tfrac{1}{2}\,g'_{i,x} + wb_i + (\beta\,s + \alpha\,w) f_i + (q + \alpha\,s)\,f'_i, \\ f'_{i+1,x} &= 2\,va_{i+1} - 2\,wc_{i+1} + 2\,(\alpha\,v + \beta\,r)\,e_{i+1} \\ &\quad -2\,(\alpha\,w + \beta\,s)\,g_{i+1} + 2\,(p + \alpha\,r)\,e'_{i+1} \\ &\quad -2\,(q + \alpha\,s)\,g'_{i+1}, \\ g'_{i+1} &= -\tfrac{1}{2}\,e'_{i,x} + vb_i + (\alpha\,v + \beta\,r)\,f_i + (p + \alpha\,r) f'_i, \end{aligned}\right. \quad \text{for } i \geq 0. \tag{3.106}$$

We are ready to calculate the sequences of $\{e_i\}_{i\geq 1},\ \{f_i\}_{i\geq 1},\ \{g_i\}_{i\geq 1}$ and $\{e'_i\}_{i\geq 1},\ \{f'_i\}_{i\geq 1},\ \{g'_i\}_{i\geq 1}$ according to the recursion relations (3.105) and (3.106).

Upon setting

$$f_0 = f'_0 = -1,\ e_0 = g_0 = e'_0 = g'_0 = 0, \tag{3.107}$$

we can uniquely compute the two sequences of $\{e_i, f_i, g_i\}_{i\geq 1}$ and $\{e'_i, f'_i, g'_i\}_{i\geq 1}$. The first few sets are listed as follows:

$$\begin{cases} e_1 = -q - (\alpha+1)s, \\ f_1 = 0, \\ g_1 = -p - (\alpha+1)r; \end{cases} \tag{3.108}$$

$$\begin{cases} e_2 = -\frac{1}{2}\,p_x - \frac{1}{2}\,(\alpha+1)r_x, \\ f_2 = -\frac{1}{2}\,p^2 - \frac{1}{2}\,q^2 - (\alpha+1)(rp+sq) - \frac{1}{2}\,\alpha\,(\alpha+1)(r^2+s^2), \\ g_2 = \frac{1}{2}\,q_x + \frac{1}{2}\,(\alpha+1)s_x; \end{cases} \tag{3.109}$$

$$\begin{cases} e_3 = -\frac{1}{2}\,[q+(\alpha+1)s]p^2 - (\alpha+1)\,(q+\alpha\,s)rp \\ \qquad -\frac{1}{2}\,q^3 - \frac{3}{2}\,(\alpha+1)sq^2 - \alpha\,(\alpha+1)\,(\frac{1}{2}\,r^2 + \frac{3}{2}\,s^2)q \\ \qquad -\frac{1}{2}\,\alpha^2(\alpha+1)\,(sr^2+s^3) + \frac{1}{4}\,q_{xx} + \frac{1}{4}\,(\alpha+1)\,s_{xx}, \\ f_3 = \frac{1}{2}\,[q_x + (\alpha+1)\,s_x]\,p - \frac{1}{2}\,[p_x + (\alpha+1)\,r_x]\,q \\ \qquad +\frac{1}{2}\,(\alpha+1)(q_x+\alpha\,s_x)\,r - \frac{1}{2}\,(\alpha+1)(p_x+\alpha\,r_x)\,s, \\ g_3 = -\frac{1}{2}\,[p+(\alpha+1)r]q^2 - (\alpha+1)\,(p+\alpha\,r)sq \\ \qquad -\frac{1}{2}\,p^3 - \frac{3}{2}\,(\alpha+1)rp^2 - \alpha\,(\alpha+1)\,(\frac{1}{2}\,s^2 + \frac{3}{2}\,r^2)p \\ \qquad -\frac{1}{2}\,\alpha^2(\alpha+1)\,(rs^2+r^3) + \frac{1}{4}\,p_{xx} + \frac{1}{4}\,(\alpha+1)\,r_{xx}; \end{cases} \tag{3.110}$$

and

$$\begin{cases} e'_1 = -q - (\alpha+\beta)\,s - (\alpha+1)\,w, \\ f'_1 = 0, \\ g'_1 = -p - (\alpha+\beta)\,r - (\alpha+1)\,v; \end{cases} \tag{3.111}$$

$$
\begin{cases}
e_2' = -\frac{1}{2}\,p_x - \frac{1}{2}\,(\alpha+\beta)\,r_x - \frac{1}{2}\,(\alpha+1)\,v_x, \\
f_2' = -\frac{1}{2}\,p^2 - [(\alpha+\beta)\,r + (\alpha+1)\,v]\,p \\
\qquad -\frac{1}{2}\,q^2 - [(\alpha+\beta)\,s + (\alpha+1)\,w]\,q \\
\qquad -\frac{1}{2}\,(\alpha^2 + 2\,\alpha\,\beta + \beta)r^2 - \alpha\,(\alpha+1)\,vr \\
\qquad -\frac{1}{2}\,(\alpha^2 + 2\,\alpha\,\beta + \beta)s^2 - \alpha\,(\alpha+1)\,ws, \\
g_2' = \frac{1}{2}\,q_x + \frac{1}{2}\,(\alpha+\beta)\,s_x + \frac{1}{2}\,(\alpha+1)\,w_x;
\end{cases}
\tag{3.112}
$$

$$
\begin{cases}
e_3' = -\frac{1}{2}\,[q + (\alpha+\beta)\,s + (\alpha+1)\,w)]\,p^2 \\
\qquad -\{[(\alpha+\beta)\,r + (\alpha+1)\,v]\,q + (\alpha^2 + 2\,\alpha\,\beta + \beta)\,sr \\
\qquad +\alpha\,(\alpha+1)(wr + vs)\}\,p - \frac{1}{2}\,q^3 - \frac{3}{2}\,[(\alpha+\beta)\,s + (\alpha+1)\,w]\,q^2 \\
\qquad -[\frac{1}{2}\,(\alpha^2 + 2\,\alpha\,\beta + \beta)\,r^2 + \alpha(\alpha+1)\,vr + \frac{3}{2}\,(\alpha^2 + 2\,\alpha\,\beta + \beta)\,s^2 \\
\qquad +3\,\alpha(\alpha+1)\,ws]\,q - \frac{1}{2}\,[(\alpha^2 + 3\,\alpha\,\beta + 2\,\beta)\,s + \alpha^2\,(\alpha+1)w]\,r^2 \\
\qquad -\alpha^2(\alpha+1)\,vsr - \frac{1}{2}\,\alpha(\alpha^2 + 3\,\alpha\,\beta + 2\,\beta)\,s^3 - \frac{3}{2}\,\alpha^2(\alpha+1)\,ws^2 \\
\qquad +\frac{1}{4}\,q_{xx} + \frac{1}{4}\,(\alpha+\beta)\,s_{xx} + \frac{1}{4}\,(\alpha+1)\,w_{xx}, \\
f_3' = \frac{1}{2}\,\{[q_x + (\alpha+\beta)\,s_x + w_x]\,p - [p_x + (\alpha+\beta)\,r_x + v_x]\,q \\
\qquad +[(\alpha+\beta)\,q_x + (\alpha^2 + \alpha\,\beta + \beta)\,s_x + \alpha\,(\alpha+1)\,w_x]\,r \\
\qquad -[(\alpha+\beta)\,p_x + (\alpha^2 + \alpha\,\beta + \beta)\,r_x + \alpha\,(\alpha+1)\,v_x]\,s \\
\qquad +[(\alpha+1)\,q_x + \alpha\,(\alpha+1)\,s_x]\,v - [(\alpha+1)\,p_x + \alpha\,(\alpha+1)\,r_x]\,w\}, \\
g_3' = -\frac{1}{2}\,[p + (\alpha+\beta)\,r + (\alpha+1)\,v)]\,q^2 \\
\qquad -\{[(\alpha+\beta)\,s + (\alpha+1)\,w]\,p + (\alpha^2 + 2\,\alpha\,\beta + \beta)\,sr \\
\qquad +\alpha\,(\alpha+1)(wr + vs)\}\,p - \frac{1}{2}\,p^3 - \frac{3}{2}\,[(\alpha+\beta)\,s + (\alpha+1)\,w]\,p^2 \\
\qquad -[\frac{1}{2}\,(\alpha^2 + 2\,\alpha\,\beta + \beta)\,s^2 + \alpha(\alpha+1)\,ws + \frac{3}{2}\,(\alpha^2 + 2\,\alpha\,\beta + \beta)\,r^2 \\
\qquad +3\,\alpha(\alpha+1)\,vr]\,p - \frac{1}{2}\,[(\alpha^2 + 3\,\alpha\,\beta + 2\,\beta)\,r + \alpha^2\,(\alpha+1)v]\,s^2 \\
\qquad -\alpha^2(\alpha+1)\,wsr - \frac{1}{2}\,\alpha(\alpha^2 + 3\,\alpha\,\beta + 2\,\beta)\,r^3 - \frac{3}{2}\,\alpha^2(\alpha+1)\,vr^2 \\
\qquad +\frac{1}{4}\,p_{xx} + \frac{1}{4}\,(\alpha+\beta)\,r_{xx} + \frac{1}{4}\,(\alpha+1)\,v_{xx}.
\end{cases}
\tag{3.113}
$$

Let us now define

$$
\bar{V}^{[m]} = M(V^{[m]}, V_1^{[m]}, V_2^{[m]}) \in \bar{\mathfrak{g}} = \mathfrak{g}_1 \Subset \mathfrak{g}_2, \tag{3.114}
$$

and

$$
\begin{cases}
V_1^{[m]} = (\lambda^m W_1)_+ + \Delta_{m,1}, \\
V_2^{[m]} = (\lambda^m W_2)_+ + \Delta_{m,2},
\end{cases}
\quad m \geq 0\,, \tag{3.115}
$$

where $V^{[m]}$ is defined as in (3.84), and the modification terms $\Delta_{m,i}$ are chosen as the zero matrix. Then, the m-th enlarged zero curvature equation

$$\bar{U}_{t_m} = \bar{V}^{[m]}_x - [\bar{U}, \bar{V}^{[m]}] \tag{3.116}$$

is equivalent to the triangular system

$$\begin{cases} U_{t_m} &= V^{[m]}_x + [U, V^{[m]}], \\ U_{1,t_m} &= V^{[m]}_{1,x} + [U, V^{[m]}_1] + [U_1, V^{[m]}] + \alpha\, [U_1, V^{[m]}_1], \\ U_{2,t_m} &= V^{[m]}_{2,x} + [U, V^{[m]}_2] + \beta\, [U_1, V^{[m]}_1] + \alpha\, [U_1, V^{[m]}_2] + [U_2, V^{[m]}] + \alpha\, [U_2, V^{[m]}_1]. \end{cases} \tag{3.117}$$

By comparing both sides of each equation in the above triangular system, a hierarchy of coupling systems is obtained for the Dirac hierarchy (3.85):

$$\bar{u}_{t_m} = \begin{bmatrix} p_{t_m} \\ q_{t_m} \\ r_{t_m} \\ s_{t_m} \\ v_{t_m} \\ w_{t_m} \end{bmatrix} = \bar{K}_m(\bar{u}) = \begin{bmatrix} K_m(u) \\ S_{1,m}(u, u_1) \\ S_{2,m}(u, u_1, u_2) \end{bmatrix} = \begin{bmatrix} -2a_{m+1} \\ 2c_{m+1} \\ -2e_{m+1} \\ 2g_{m+1} \\ -2e'_{m+1} \\ 2g'_{m+1} \end{bmatrix}, \quad m \geq 0. \tag{3.118}$$

The system (3.118) shows a hierarchy of bi-integrable couplings for the Dirac hierarchy of soliton equations. We give the first nonlinear bi-integrable coupling system as follows:

$$p_{t_2} = -2\, a_3, \;\; q_{t_2} = 2\, c_3, \;\; r_{t_2} = -2\, e_3, \;\; s_{t_2} = 2\, g_3, \;\; v_{t_2} = -2\, e'_3, \;\; w_{t_2} = 2\, g'_3. \tag{3.119}$$

3.4.3 Hamiltonian structures of the bi-integrable couplings of the Dirac hierarchy

Since the Lie algebras in this section are non-semisimple, which will result in degenerate Killing forms in this case, we are not able to apply the trace identity [69, 61]. To obtain Hamiltonian structures of the hierarchy of bi-integrable couplings, we will use the variational identity which was introduced in [37, 55] under more general bilinear forms.

The corresponding variational identity [55] is

$$\frac{\delta}{\delta \bar{u}} \int \langle \bar{U}_\lambda, \bar{W} \rangle dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} \langle \bar{U}_{\bar{u}}, \bar{W} \rangle, \tag{3.120}$$

where $\langle \cdot, \cdot \rangle$ is a required bilinear form, which is symmetric, non-degenerate, and invariant under the Lie bracket.

Let us now to construct general bilinear forms with the symmetric, invariant, and non-degenerate properties $\langle \cdot, \cdot \rangle$ on $\bar{\mathfrak{g}}$. First, we transform the semi-direct sum $\bar{\mathfrak{g}}$ into a vector form via defining:

$$\sigma : \bar{\mathfrak{g}} \to \mathbb{R}^9,\ A \mapsto (a_1, \cdots, a_9)^T, \tag{3.121}$$

where

$$A = M(A_1, A_2, A_3) \in \bar{\mathfrak{g}},\ A_i = \begin{bmatrix} a_{3i-2} & a_{3i-1} \\ a_{3i} & -a_{3i-2} \end{bmatrix},\ 1 \le i \le 3. \tag{3.122}$$

This mapping σ generates a Lie algebraic structure on $\mathbb{R}^9$, which is Lie algebra isomorphic to the matrix loop algebra $\bar{\mathfrak{g}}$. Next we define the corresponding Lie bracket $[\cdot, \cdot]$ on $\mathbb{R}^9$ by

$$[a, b]^T = a^T R(b), \tag{3.123}$$

for any $a = (a_1, \cdots, a_9)^T,\ b = (b_1, \cdots, b_9)^T \in \mathbb{R}^9$, and

$$R(b) = M(R_1, R_2, R_3), \tag{3.124}$$

where $R_1,\ R_2$, and R_3 are the matrices defined by

$$R_i = \begin{bmatrix} 0 & 2\,b_{3i-1} & -2\,b_{3i} \\ b_{3i} & -2\,b_{3i-2} & 0 \\ -b_{3i-1} & 0 & 2\,b_{3i-2} \end{bmatrix},\ \text{for } i = 1, 2, 3. \tag{3.125}$$

This Lie algebra $(\mathbb{R}^9, [\cdot, \cdot])$ is isomorphic to the matrix Lie algebra $\bar{\mathfrak{g}}$, and the mapping σ, defined by (3.121), is a Lie algebra isomorphism between the two Lie algebras defined above.

We then search for a bilinear form on $\mathbb{R}^9$ by assuming

$$\langle a, b \rangle = a^T F b, \tag{3.126}$$

where F is a constant matrix. The symmetric property $\langle a, b \rangle = \langle b, a \rangle$ yields that

$$F^T = F. \tag{3.127}$$

Furthermore, by taking the invariance property under the Lie bracket

$$\langle a, [b, c] \rangle = \langle [a, b], c \rangle$$

into consideration, we obtain that

$$F(R(b))^T = -R(b)F,\ b \in \mathbb{R}^9.$$

This matrix equation leads to a linear system of equations on the entries of F. Solving the resulting linear system yields the matrix F:

$$F = \begin{bmatrix} \eta_1 & \eta_2 & \eta_3 \\ \eta_2 & \alpha\,\eta_2 + \beta\,\eta_3 & \alpha\,\eta_3 \\ \eta_3 & \alpha\,\eta_3 & 0 \end{bmatrix} \otimes \begin{bmatrix} 2\ 0\ 0 \\ 0\ 0\ 1 \\ 0\ 1\ 0 \end{bmatrix}, \tag{3.128}$$

where η_i, $1 \le i \le 3$, are arbitrary constants and $\otimes$ is Kronecker product.

Now, the corresponding bilinear form on the semi-direct sum $\bar{\mathfrak{g}}$ of Lie algebras is determined by

$$\begin{aligned} \langle A, B\rangle_{\bar{\mathfrak{g}}} &= \langle \sigma(A), \sigma(B)\rangle_{\mathbb{R}^9} = (a_1, \cdots, a_9)F(b_1, \cdots, b_9)^T \\ &= (2\,a_1b_1 + a_2b_3 + a_3b_2)\eta_1 \\ &\quad +(2\,a_1b_4 + a_2b_6 + a_3b_5 + 2\,a_4b_1 + 2\,\alpha\,a_4b_4 \\ &\quad +a_5b_3 + \alpha\,a_5b_6 + a_6b_2 + \alpha\,a_6b_5)\eta_2 \\ &\quad +(2\,a_1b_7 + a_2b_9 + a_3b_8 + 2\,\beta\,a_4b_4 + 2\,\alpha\,a_4b_7 \\ &\quad +\alpha\,a_5b_9 + \beta\,a_5b_6 + \beta\,a_6b_5 + \alpha\,a_6b_8 + 2\,a_7b_1 \\ &\quad +2\,\alpha\,a_7b_4 + \alpha\,a_8b_6 + a_8b_3 + a_9b_2 + \alpha\,a_9b_5)\eta_3, \end{aligned} \tag{3.129}$$

where $A = A(a_1, \cdots, a_9), B = B(b_1, \cdots, b_9) \in \bar{\mathfrak{g}}$ are as defined in (3.93).

The bilinear form (3.129) is symmetric and invariant under the Lie bracket of the matrix Lie algebra:

$$\langle A, B\rangle = \langle B, A\rangle,\ \langle A, [B, C]\rangle = \langle [A, B], C\rangle,$$

where $A = A(a_1, \cdots, a_9), B = B(b_1, \cdots, b_9), C = C(c_1, \cdots, c_9) \in \bar{\mathfrak{g}}$ in (3.93). Obviously, this kind of bilinear forms is not of Killing type and is non-degenerate if and only if the determinant of the matrix F:

$$\det(F) = 8\,\left(\alpha^2\eta_1 - \alpha\,\eta_2 + \beta\,\eta_3\right)^3 {\eta_3}^6 \neq 0,$$

where the two parameters α and β are arbitrarily fixed constants chosen when the new class of matrix Lie algebras in (3.87) is introduced. Hence we should choose η_1, η_2, and η_3 such that $\det(F)$ is non-zero so as to apply the variational identity.

Similarly, plugging $\bar{W}, \bar{U}_\lambda$ and $\bar{U}_{\bar{u}}$ into the bilinear form (3.129), we obtain that

$$\langle \bar{W}, \bar{U}_\lambda \rangle = -2\, a\eta_1 - 2\, e\eta_2 - 2\, e'\, \eta_3, \tag{3.130}$$

and

$$\langle \bar{W}, \bar{U}_{\bar{u}} \rangle = \begin{bmatrix} c\eta_1 + g\eta_2 + g'\, \eta_3 \\ b\eta_1 + f\eta_2 + f'\, \eta_3 \\ (c + \alpha\, g)\, \eta_2 + (\beta\, g + \alpha\, g')\, \eta_3 \\ (b + \alpha\, f)\, \eta_2 + (\beta\, f + \alpha f')\, \eta_3 \\ (c + \alpha\, g)\, \eta_3 \\ (b + \alpha\, f)\, \eta_3 \end{bmatrix}, \tag{3.131}$$

and furthermore, we have

$$\gamma = -\frac{\lambda}{2} \frac{d}{d\lambda} \ln |\langle \bar{W}, \bar{W} \rangle| = 0.$$

Applying the variational identity (3.120), we obtain Hamiltonian structures for the hierarchy of bi-integrable couplings (3.118):

$$\bar{u}_{t_m} = \bar{J} \frac{\delta \bar{\mathcal{H}}_m}{\delta \bar{u}}, \tag{3.132}$$

where the Hamiltonian functionals are

$$\bar{\mathcal{H}}_m = \int \frac{2\, \eta_1 a_{m+2} + 2\, \eta_2 e_{m+2} + 2\, \eta_3 e'_{m+2}}{m+1}\, dx,\ m \geq 0, \tag{3.133}$$

and the Hamiltonian operator is

$$\bar{J} = \begin{bmatrix} \eta_1 & \eta_2 & \eta_3 \\ \eta_2 & \alpha\, \eta_2 + \beta\, \eta_3 & \alpha\, \eta_3 \\ \eta_3 & \alpha\, \eta_3 & 0 \end{bmatrix}^{-1} \otimes J, \tag{3.134}$$

with matrix J being defined in (3.86).

The original Dirac equation possesses infinitely many commuting symmetries $\{K_m\}_{m=1}^{\infty}$, and it is therefore natural to ask whether the enlarged system is still integrable in the same sense. Fortunately, the hereditary recursion operator $\bar{\Phi}$ (see [70]) guarantees the existence of infinitely many commuting symmetries. An algorithmic way of constructing hereditary recursion operators was published in [73, 74]. We find that the hereditary recursion operator $\bar{\Phi}$ is defined by

$$\bar{\Phi} = M^T(\Phi, \Phi_1, \Phi_2) \tag{3.135}$$

where M is defined in (3.87), the hereditary recursion operator Φ is given by (3.86), and

$$\Phi_1 = \begin{bmatrix} 2(r+\alpha s)\partial^{-1}p + 2s\partial^{-1}q & 2(r+\alpha s)\partial^{-1}s + 2s\partial^{-1}r \\ -2(q+\alpha p)\partial^{-1}p - 2p\partial^{-1}q & -2(q+\alpha p)\partial^{-1}s - 2p\partial^{-1}r \end{bmatrix}, \tag{3.136}$$

$$\Phi_2 = \begin{bmatrix} 2(r+\alpha s)\partial^{-1}(q+\alpha p) & -\frac{1}{2}\partial + 2(r+\alpha s)\partial^{-1}(r+\alpha s) \\ \frac{1}{2}\partial - 2(q+\alpha p)\partial^{-1}(q+\alpha p) & -2(q+\alpha p)\partial^{-1}(r+\alpha s) \end{bmatrix}. \tag{3.137}$$

Consequently, we can rewrite the enlarged integrable system (3.118) as

$$u_{t_m} = \bar{\Phi} K_{m-1} = \bar{\Phi}^m K_0, \quad m \geq 1.$$

We find that each enlarged system (3.118) has both infinitely many commuting symmetries and conserved functionals:

$$[\bar{K}_m, \bar{K}_n] = 0, \quad m, n \geq 0, \tag{3.138}$$

$$\{\bar{\mathcal{H}}_m, \bar{\mathcal{H}}_n\}_{\bar{J}} = 0, \quad m, n \geq 0. \tag{3.139}$$

It can be also proved that the resulting bi-integrable couplings constructed above possess a bi-Hamiltonian structure

$$\bar{u}_{t_m} = \bar{J}\frac{\delta \bar{\mathcal{H}}_m}{\delta \bar{u}} = \bar{M}\frac{\delta \bar{\mathcal{H}}_{m-1}}{\delta \bar{u}}, \ \ m \geq 2, \tag{3.140}$$

with $\bar{M} = \bar{\Phi}\bar{J}$.

The presented Lie algebras of 3×3 block matrices in section 3.1 give a number of potential algebraic structures for finding bi-integrable couplings. Based on new classes of matrix Lie algebras, and following similar schemes, we can generate bi-integrable couplings of other soliton hierarchies such as the Kaup-Newell hierarchy and the KdV hierarchy.

Chapter 4

Tri-integrable couplings and Hamiltonian structures

We will construct tri-integrable couplings. A tri-integrable coupling of a given integrable system (2.1) is the following enlarged triangular integrable system:

$$\begin{cases} u_t = K(u), \\ u_{1,t} = S_1(u, u_1), \\ u_{2,t} = S_2(u, u_1, u_2), \\ u_{3,t} = S_3(u, u_1, u_2, u_3). \end{cases} \tag{4.1}$$

The above system is called a nonlinear coupling if at least one of $S_1(u, u_1)$, $S_2(u, u_1, u_2)$ and $S_3(u, u_1, u_2, u_3)$ are nonlinear with respect to any sub-vectors u_1, u_2, u_3 of new dependent variables. We presented a new method of generating tri-integrable couplings by taking advantage of semi-direct sums of Lie algebras [75]. This method is motivated by [12, 39, 48, 76, 77], in which semi-direct sums of Lie algebras are the basis of exploring new types of tri-integrable couplings.

4.1 Matrix Lie algebras for tri-integrable couplings

We are looking for new matrix Lax pairs through block matrix Lie algebras. Undecomposable triangular Jordan blocks are ideal candidates since they correspond to undecomposable sub-systems in a given integrable system. In order to generate tri-integrable couplings, we choose triangular 4×4 in the form of

$$M(A_1, A_2, A_3, A_4) = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & A_1 & \sum_{i=1}^{4} \alpha_{1,i} A_i & \sum_{i=1}^{4} \alpha_{2,i} A_i \\ 0 & 0 & A_1 & \sum_{i=1}^{4} \alpha_{3,i} A_i \\ 0 & 0 & 0 & A_1 \end{bmatrix} \tag{4.2}$$

where A_1, A_2, A_3, A_4 are square submatrices of the same order. In order to have a class of block matrices forming a subalgebra under the matrix commutator

$$[M_1, M_2] = M_1M_2 - M_2M_1, \tag{4.3}$$

we need to have the closure property

$$M(A_1, A_2, A_3, A_4)M(B_1, B_2, B_3, B_4) = M(C_1, C_2, C_3, C_4). \tag{4.4}$$

If we multiply the elements of the first row of the first matrix $M(A_1, A_2, A_3, A_4)$ by the elements of each column in the second matrix $M(B_1, B_2, B_3, B_4)$, add the products, then we have

$$\begin{cases} C_1 = A_1B_1, \\ C_2 = A_1B_2 + A_2B_1, \\ C_3 = A_1B_3 + A_3B_1 + \sum_{i=1}^{4} \alpha_{1,i}A_2B_i, \\ C_4 = A_1B_4 + A_4B_1 + \sum_{2\le i\le 3,\, 1\le j\le 4} \alpha_{i,j}A_iB_j. \end{cases} \tag{4.5}$$

Furthermore, in order for the closure property to hold, we need

$$\begin{cases} \sum_{i=1}^{4} \alpha_{1,i}C_i = \sum_{i=1}^{4} \alpha_{1,i}(A_1B_i + A_iB_1), \\ \sum_{i=1}^{4} \alpha_{2,i}C_i = \sum_{i=1}^{4} \alpha_{2,i}(A_1B_i + A_iB_1) + \sum_{1\le i,j\le 4} \alpha_{1,i}\alpha_{3,j}A_iB_j, \\ \sum_{i=1}^{4} \alpha_{3,i}C_i = \sum_{i=1}^{4} \alpha_{3,i}(A_1B_i + A_iB_1). \end{cases} \tag{4.6}$$

Combining (4.5) and (4.6), a direct computation shows that there are the following classes of matrix Lie algebras consisting of 4×4 selected block matrices:

$$Class_{\mathbf{1}}^{*} = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & A_1 & \frac{\alpha\beta}{\mu}A_2 - \mu A_3 + \alpha A_4 & \beta A_2 - \frac{\mu^2}{\alpha}A_3 + \mu A_4 \\ 0 & 0 & A_1 & 0 \\ 0 & 0 & 0 & A_1 \end{bmatrix}, \tag{4.7}$$

$$Class^*_{\mathbf{2}} = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & A_1 & \alpha A_2 + \beta A_4 & 0 \\ 0 & 0 & A_1 & 0 \\ 0 & 0 & 0 & A_1 \end{bmatrix}, \tag{4.8}$$

$$Class^*_{\mathbf{3}} = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & A_1 & 0 & \alpha A_2 + \beta A_3 \\ 0 & 0 & A_1 & \zeta A_2 + \mu A_3 \\ 0 & 0 & 0 & A_1 \end{bmatrix}, \tag{4.9}$$

$$Class^*_{\mathbf{4}} = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & A_1 & \alpha A_2 & \beta A_2 + \mu A_3 \\ 0 & 0 & A_1 & \mu A_2 \\ 0 & 0 & 0 & A_1 \end{bmatrix}, \tag{4.10}$$

where $\alpha, \beta, \mu, \zeta$ are four arbitrarily given constants.

This closure property tells that each of the 4 classes of block matrices defined above form a block matrix Lie subalgebra. The resulting Lie algebra has a semi-direct sum decomposition

$$\bar{\mathfrak{g}} = \mathfrak{g} \Subset \mathfrak{g}_c, \tag{4.11}$$

with

$$\begin{cases} \mathfrak{g} = \{M(A_1, 0, 0, 0) |\, A_1 \text{ - arbitrary}\}, \\ \mathfrak{g}_c = \{M(0, A_2, A_3, A_4) |\, A_i \text{ - arbitrary},\ 2 \leq i \leq 4\}, \end{cases} \tag{4.12}$$

and obviously it is non-semisimple because one of nontrivial ideals of $\bar{\mathfrak{g}}$ is $\mathfrak{g}_c$.

We shall show how to construct tri-integrable couplings based on those matrix Lie algebras.

4.2 A general scheme for constructing Hamiltonian tri-integrable couplings

We follow a usual procedure to generate soliton hierarchies [61, 62, 69]. We assume that a soliton hierarchy is associated with a pair of isospectral problems:

$$\begin{cases} \phi_x = U\phi, \ U = U(u, \lambda) \in \mathfrak{g}, \\ \phi_t = V\phi, \ V = V(u, \lambda) \in \mathfrak{g}, , \end{cases} \tag{4.13}$$

where u is a dependent variable and $\mathfrak{g}$ is usually a semi-simple matrix Lie algebra. Then the compatible condition gives rise to the zero curvature equation

$$U_t - V_x + [U, V] = 0. \tag{4.14}$$

In order to obtain tri-integrable couplings, we enlarge U as the corresponding enlarged spectral matrix

$$\bar{U} = \bar{U}(\bar{u}, \lambda) = M(U, U_1, U_2, U_3) \in \bar{\mathfrak{g}}, \tag{4.15}$$

and V as the corresponding enlarged Lax matrix

$$\bar{V} = \bar{V}(\bar{u}, \lambda) = M(V, V_1, V_2, V_3) \in \bar{\mathfrak{g}}, \tag{4.16}$$

where $\bar{\mathfrak{g}}$ is defined in (4.11), $\bar{u} = (u^T, u_1^T, u_2^T, u_3^T)^T$, λ is the spectral parameter, and

$$U_i = U_i(u_i, \lambda), \ V_i = V_i(u, u_1, \cdots, u_i, \lambda), \ 1 \leq i \leq 3. \tag{4.17}$$

After enlarging the Lax pair, we get the enlarged zero curvature equation

$$\bar{U}_t - \bar{V}_x + [\bar{U}, \bar{V}] = 0. \tag{4.18}$$

Suppose that the stationary zero curvature equation

$$\bar{W}_x = [U, \bar{W}] \tag{4.19}$$

has a solution of the form

$$\bar{W} = \bar{W}(\bar{u}, \lambda) = \sum_{i \geq 0} W_i \lambda^{-i}, \tag{4.20}$$

where $W_i \in \bar{\mathfrak{g}}, \ i \geq 0$. Next, we introduce the temporal spectral problems

$$\phi_{t_m} = \bar{V}^{[m]}\phi, \ m \geq 0, \tag{4.21}$$

with the enlarged Lax matrices $\bar{V}^{[m]}$ being defined as

$$\bar{V}^{[m]} = M(V^{[m]}, V_1^{[m]}, V_2^{[m]}, V_3^{[m]}) \in \bar{\mathfrak{g}}, \tag{4.22}$$

with the submatrices $V^{[m]}$ being defined by

$$V^{[m]} = (\lambda^m W)_+ + \Delta_{m,0},\ m \geq 0, \tag{4.23}$$

and

$$V_i^{[m]} = (\lambda^m W_i)_+ + \Delta_{m,i},\ 1 \leq i \leq 3,\ m \geq 0, \tag{4.24}$$

where P_+ denotes the polynomial part of P in λ. It is important to note that the modification terms $\Delta_{m,i}$ are added so that the enlarged zero curvature equations

$$\bar{U}_{t_m} - \bar{V}_x^{[m]} + [\bar{U}, \bar{V}^{[m]}] = 0,\ m \geq 0, \tag{4.25}$$

yield a hierarchy of enlarged soliton equations with Hamiltonian structures

$$\bar{u}_{t_m} = \bar{K}_m(\bar{u}) = \bar{J}\frac{\delta \bar{\mathcal{H}}_m}{\delta u},\ m \geq 0. \tag{4.26}$$

The above Hamiltonian structures can be constructed through using the associated variational identities [37, 54], which contains the trace identities as particular examples [61, 62]:

$$\frac{\delta}{\delta \bar{u}} \int \langle \bar{U}_\lambda, \bar{W} \rangle dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma \langle \bar{U}_{\bar{u}}, \bar{W} \rangle, \tag{4.27}$$

where γ is a constant, $\bar{W}$ is a solution of (4.19), and $\langle \cdot, \cdot \rangle$ is a bilinear form on the algebra $\bar{\mathfrak{g}}$ which is non-degenerate, symmetric and ad-invariant [54]. The hierarchy (4.26) provides tri-integrable couplings for the given soliton hierarchy:

$$\bar{u}_{t_m} = \begin{bmatrix} u_{t_m} \\ u_{1,t_m} \\ u_{2,t_m} \\ u_{3,t_m} \end{bmatrix} = \bar{K}_m(\bar{u}) = \begin{bmatrix} K_m(u) \\ S_{1,m}(u, u_1) \\ S_{2,m}(u, u_1, u_2) \\ S_{3,m}(u, u_1, u_2, u_3) \end{bmatrix},\ m \geq 0. \tag{4.28}$$

4.3 Application to the AKNS hierarchy

4.3.1 Tri-integrable couplings of the AKNS hierarchy

We choose $Class_4^*$ as an example, in which triangular 4×4 block matrices take the form of

$$M(A_1, A_2, A_3, A_4) = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & A_1 & \alpha A_2 & \beta A_2 + \mu A_3 \\ 0 & 0 & A_1 & \mu A_2 \\ 0 & 0 & 0 & A_1 \end{bmatrix} \tag{4.29}$$

where A_1, A_2, A_3, A_4 are square submatrices of the same order. In the following, we define the corresponding non-semisimple Lie algebra $\bar{\mathfrak{g}}$ as semi-direct sum

$$\bar{\mathfrak{g}} = \mathfrak{g} \Subset \mathfrak{g}_c \tag{4.30}$$

with

$$\mathfrak{g} = \{ M(A_1, 0, 0, 0) | \text{ entries of } A_1 \text{ - Laurent series in } \lambda \} , \tag{4.31}$$

and

$$\mathfrak{g}_c = \{ M(0, A_2, A_3, A_4) | \text{ entries of } A_2, A_3, A_4 \text{ - Laurent series in } \lambda \} , \tag{4.32}$$

where $\mathfrak{g}$ is a semisimple subalgebra and $\mathfrak{g}_c$ is a solvable subalgebra.

Based on this special non-semisimple Lie algebra $\bar{\mathfrak{g}}$, we choose the corresponding enlarged spectral matrix as

$$\bar{U} = \bar{U}(\bar{u}, \lambda) = M(U, U_1, U_2, U_3) \in \bar{\mathfrak{g}}, \tag{4.33}$$

with $U = U(u, \lambda)$ be defined as in (2.33) and

$$U_i = U_i(u_i) = \begin{bmatrix} 0 & r_i \\ s_i & 0 \end{bmatrix}, \ u_i = \begin{bmatrix} r_i \\ s_i \end{bmatrix}, \ 1 \le i \le 3, \tag{4.34}$$

where $\bar{u} = (u^T, u_1^T, u_2^T, u_3^T)^T$ and $r_i, s_i,\ 1 \le i \le 3$, are new dependent variables.

By the compatible condition of the isospectral problems, we obtain the corresponding zero curvature equation

$$\bar{U}_t - \bar{V}_x + [\bar{U}, \bar{V}] = 0, \tag{4.35}$$

which yields the triangle system

$$
\begin{cases}
U_t \ - V_x + [U, V] = 0, \\
U_{1,t} \ - V_{1,x} + [U, V_1] + [U_1, V] = 0, \\
U_{2,t} \ - V_{2,x} + [U, V_2] + \alpha\,[U_1, V_1] + [U_2, V] = 0, \\
U_{3,t} \ - V_{3,x} + [U, V_3] + \beta\,[U_1, V_2] \\
\qquad\qquad +\mu\,([U_1, V_2] + [U_2, V_1]) + [U_3, V] = 0.
\end{cases}
\tag{4.36}
$$

We note that the first equation in (4.36) yields the originally given integrable system (2.1), and the second, third and fourth equations give rise to subsystems S_1, S_2 and S_3, respectively.

We adopt the traditional scheme for constructing soliton hierarchies [61, 62, 69], we solve the corresponding enlarged stationary zero curvature equation

$$
\bar{W}_x = [\bar{U}, \bar{W}]. \tag{4.37}
$$

We search for solutions of the following form

$$
\bar{W} = \bar{W}(\bar{u}, \lambda) = M(W, W_1, W_2, W_3) \in \bar{\mathfrak{g}}, \tag{4.38}
$$

where W is given by (2.35). Then, the enlarged stationary zero curvature equation gives

$$
\begin{cases}
W_{1,x} = [U, W_1] + [U_1, W], \\
W_{2,x} = [U, W_2] + \alpha\,[U_1, W_1] + [U_2, W], \\
W_{3,x} = [U, W_3] + \beta\,[U_1, W_2] + \mu\,([U_1, W_2] + [U_2, W_1]) + [U_3, W].
\end{cases}
\tag{4.39}
$$

Assume that

$$
W_1, W_2, W_3 \in \widetilde{\mathrm{sl}}(2, \mathbb{R}) = \{A \in \mathrm{sl}(2, \mathbb{R}) |\, \text{entries of } A \text{ - Laurent series in } \lambda\}
$$

are of the form

$$
\begin{cases}
W_1 = W_1(u, u_1, \lambda) = \begin{bmatrix} e & f \\ g & -e \end{bmatrix} = \sum_{i\geq 0} W_{1,i}\lambda^{-i}, \\
W_2 = W_2(u, u_1, u_2, \lambda) = \begin{bmatrix} e' & f' \\ g' & -e' \end{bmatrix} = \sum_{i\geq 0} W_{2,i}\lambda^{-i}, \\
W_3 = W_3(u, u_1, u_2, u_3\lambda) = \begin{bmatrix} e'' & f'' \\ g'' & -e'' \end{bmatrix} = \sum_{i\geq 0} W_{3,i}\lambda^{-i}.
\end{cases}
\tag{4.40}
$$

The above system (4.39) equivalently leads to

$$\begin{cases} e_x = -s_1 b + r_1 c - qf + pg, \\ f_x = -2\, r_1 a - 2\, pe - 2\,\lambda\, f, \\ g_x = 2\, s_1 a + 2\, qe + 2\,\lambda\, g; \end{cases} \tag{4.41}$$

$$\begin{cases} e'_x = pg' + r_1\alpha\, g + r_2 c - qf' - s_1\alpha\, f - s_2 b, \\ f'_x = -2\,\lambda\, f' - 2\, pe' - 2\, r_1\alpha\, e - 2\, r_2 a, \\ g'_x = 2\, qe' + 2\,\lambda\, g' + 2\, s_1\alpha\, e + 2\, s_2 a; \end{cases} \tag{4.42}$$

and

$$\begin{cases} e''_x \;=\; -s_3 b + r_3 c - (\beta\, s_1 + \mu\, s_2) f + g(\beta\, r_1 + \mu\, r_2) \\ \qquad\quad +pg'' - qf'' + g'\,\mu\, r_1 - f'\,\mu\, s_1, \\ f''_x \;=\; -2\, r_3 a - 2\,(\beta\, r_1 + \mu\, r_2) e - 2\,\lambda\, f'' - 2\, pe'' - 2\, e'\,\mu\, r_1, \\ g''_x \;=\; 2\, s_3 a + 2\,(\beta\, s_1 + \mu\, s_2) e + 2\, qe'' + 2\,\lambda\, g'' + 2\, e'\,\mu\, s_1. \end{cases} \tag{4.43}$$

Trying a solution $\bar{W}$ with

$$\begin{cases} e = \sum_{i\geq 0} e_i \lambda^{-i},\; f = \sum_{i\geq 0} f_i \lambda^{-i},\; g = \sum_{i\geq 0} g_i \lambda^{-i}, \\ e' = \sum_{i\geq 0} e'_i \lambda^{-i},\; f' = \sum_{i\geq 0} f'_i \lambda^{-i},\; g' = \sum_{i\geq 0} g'_i \lambda^{-i}, \\ e'' = \sum_{i\geq 0} e''_i \lambda^{-i},\; f'' = \sum_{i\geq 0} f''_i \lambda^{-i},\; g'' = \sum_{i\geq 0} g''_i \lambda^{-i}, \end{cases} \tag{4.44}$$

and taking

$$e_0 = e'_0 = e''_0 = -1,\; f_0 = g_0 = f'_0 = g'_0 = f''_0 = g''_0 = 0, \tag{4.45}$$

we obtain that

$$\begin{cases} f_{i+1} \;=\; -\frac{1}{2}\, f_{i,x} - pe_i - r_1 a_i, \\ g_{i+1} \;=\; \frac{1}{2}\, g_{i,x} - qe_i - s_1 a_i, \\ e_{i+1,x} \;=\; pg_{i+1} + r_1 c_{i+1} - qf_{i+1} - s_1 b_{i+1}, \end{cases} \tag{4.46}$$

$$\begin{cases} f'_{i+1} \;=\; -\frac{1}{2}\, f'_{i,x} - pe'_i - \alpha\, r_1 e_i - r_2 a_i, \\ g'_{i+1} \;=\; \frac{1}{2}\, g'_{i,x} - qe'_i - \alpha\, s_1 e_i - s_2 a_i, \\ e'_{i+1,x} \;=\; pg'_{i+1} + \alpha\, r g_{i+1} + r_1 c_{i+1} - qf'_{i+1} - \alpha\, s f_{i+1} - s_1 b_{i+1}, \end{cases} \tag{4.47}$$

and

$$\begin{cases} f''_{i+1} &= -\frac{1}{2}\, f''_{i,x} - r_3 a_i - (\beta\, r + \mu\, r_2) e_i - p e''_i - \mu\, r_1\, e'_i, \\ g''_{i+1} &= \frac{1}{2}\, g''_{i,x} - s_3 a_i - (\beta\, s_1 + \mu\, s_2) e_i - q e''_i - \mu\, s_1\, e'_i, \\ e''_{i+1,x} &= -s_3 b_{i+1} + r_3 c_{i+1} - (\beta\, s_1 + \mu\, s_2) f_{i+1} + (\beta\, r_1 + \mu\, r_2) g_{i+1} \\ & \quad + p g''_{i+1} - q f''_{i+1} + g'_{i+1} \mu\, r_1 - \mu\, s_1 f'_{i+1}. \end{cases} \tag{4.48}$$

The recursion relations (4.46), (4.47) and (4.48) uniquely determine three sequences of $\{e_i, f_i, g_i\}_{i\geq 1}$, $\{e'_i, f'_i, g'_i\}_{i\geq 1}$ and $\{e''_i, f''_i, g''_i\}_{i\geq 1}$, respectively, when we take constants of integration as zero. The first three sets of functions in the first sequence are as follows:

$$\begin{cases} e_1 = 0, \\ f_1 = p + r_1, \\ g_1 = q + s_1; \end{cases} \tag{4.49}$$

$$\begin{cases} e_2 = \frac{1}{2}\, [(q + s_1) p + r_1 q], \\ f_2 = -\frac{1}{2}\, (p_x + r_{1,x}), \\ g_2 = \frac{1}{2}\, (q_x + s_{1,x}); \end{cases} \tag{4.50}$$

$$\begin{cases} e_3 \,=\, \frac{1}{4}\, [(q_x + s_{1,x}) p - (p_x + r_{1,x}) q + q_x r_1 - p_x s_1], \\ f_3 \,=\, -\frac{1}{2}\, (q + s_1) p^2 - r_1 q p + \frac{1}{4}\, p_{xx} + \frac{1}{4}\, r_{1,xx}, \\ g_3 \,=\, -\frac{1}{2}\, (p + r_1) q^2 - s_1 p q + \frac{1}{4}\, q_{xx} + \frac{1}{4}\, s_{1,xx}. \end{cases} \tag{4.51}$$

The first three sets of functions in the second sequence are as follows:

$$\begin{cases} e'_1 \,=\, 0, \\ f'_1 \,=\, p + \alpha\, r_1 + r_2, \\ g'_1 \,=\, q + \alpha\, s_1 + s_2; \end{cases} \tag{4.52}$$

$$\begin{cases} e'_2 \,=\, \frac{1}{2}\, (q + \alpha\, s_1 + s_2)\, p + \frac{1}{2}\, (\alpha\, r_1 + r_2)\, q + \frac{1}{2}\, \alpha r_1 s_1, \\ f'_2 \,=\, -\frac{1}{2}\, p_x - \frac{1}{2}\, \alpha\, r_{1,x} - \frac{1}{2}\, r_{2,x}, \\ g'_2 \,=\, \frac{1}{2}\, q_x + \frac{1}{2}\, \alpha\, s_{1,x} + \frac{1}{2}\, s_{2,x}; \end{cases} \tag{4.53}$$

$$\begin{cases} e_3' = \frac{1}{4}(q_x + \alpha s_{1,x} + s_{2,x})p - \frac{1}{4}(p_x + \alpha r_{1,x} + r_{2,x})q \\ \qquad +\frac{1}{4}\alpha(q_x + s_{1,x})r - \frac{1}{4}\alpha(p_x + r_{1,x})s - \frac{1}{4}s_{2,x}p_x + \frac{1}{4}r_{2,x}q_x, \\ f_3' = -\frac{1}{2}(q + \alpha s_1 + s_2)p^2 - [(\alpha r_1 + r_2)q - \alpha r_1 s_1]p \\ \qquad +\frac{1}{4}p_{xx} + \frac{1}{4}\alpha r_{1,xx} + \frac{1}{4}r_{2,xx} - \frac{1}{2}\alpha q r_1^2, \\ g_3' = -\frac{1}{2}(p + \alpha r_1 + r_2)q^2 - [(\alpha s_1 + s_2)p - \alpha r_1 s_1]q \\ \qquad +\frac{1}{4}q_{xx} + \frac{1}{4}\alpha s_{1,xx} + \frac{1}{4}s_{2,xx} - \frac{1}{2}\alpha p s_1^2. \end{cases} \tag{4.54}$$

The first three sets of functions in the third sequence are as follows:

$$\begin{cases} e_1'' = 0, \\ f_1'' = p + (\beta + \mu)r_1 + \mu r_2 + r_3, \\ g_1'' = q + (\beta + \mu)s_1 + \mu s_2 + s_3; \end{cases} \tag{4.55}$$

$$\begin{cases} e_2'' = \frac{1}{2}[q + (\beta + \mu)s_1 + \mu s_2 + s_3]p + \frac{1}{2}[(\beta + \mu)r_1 + \mu r_2 + r_3]q \\ \qquad +\frac{1}{2}[(\alpha\mu + \beta)s_1 + \mu s_2]r_1 + \frac{1}{2}\mu s_1 r_2, \\ f_2'' = -\frac{1}{2}p_x - \frac{1}{2}(\beta + \mu)r_{1,x} - \frac{1}{2}\mu r_{2,x} - \frac{1}{2}r_{3,x}, \\ g_2'' = \frac{1}{2}q_x + \frac{1}{2}(\beta + \mu)s_{1,x} + \frac{1}{2}\mu s_{2,x} + \frac{1}{2}s_{3,x}; \end{cases} \tag{4.56}$$

$$\begin{cases} e_3'' = \frac{1}{4}[q_x + (\beta + \mu)s_{1,x} + \mu s_{2,x} + s_{3,x}]p \\ \qquad -\frac{1}{4}[p_x + (\beta + \mu)r_{1,x} + \mu r_{2,x} + r_{3,x}]q \\ \qquad +\frac{1}{4}[(\beta + \mu)q_x + (\alpha\mu + \beta)s_{1,x} + \mu s_{2,x}]r_1 \\ \qquad -\frac{1}{4}[(\beta + \mu)p_x + (\alpha\mu + \beta)r_{1,x} + \mu r_{2,x}]s_1 \\ \qquad +\frac{1}{4}(\mu q_x + \mu s_{1,x})r_2 - \frac{1}{4}(\mu p_x + \mu r_{1,x})s_2 \\ \qquad +\frac{1}{4}q_x r_3 - \frac{1}{4}p_x s_3, \\ f_3'' = -\frac{1}{2}[q + (\beta + \mu)s_1 + \mu s_2 + s_3]p^2 - \{[(\beta + \mu)r_1 \\ \qquad +\mu r_2 + r_3]q + [(\alpha\mu + \beta)s_1 + \mu s_2]r_1 + \mu s_1 r_2\}p \\ \qquad -[\frac{1}{2}(\alpha\mu + \beta)r_1^2 + \mu r_1 r_2]q - \frac{1}{2}\alpha\mu r_1^2 s_1 \\ \qquad +\frac{1}{4}p_{xx} + \frac{1}{4}\mu r_{2,xx} + \frac{1}{4}r_{3,xx}, \\ g_3'' = -\frac{1}{2}[p + (\beta + \mu)r_1 + \mu r_2 + r_3]q^2 - \{[(\beta + \mu)s_1 \\ \qquad +\mu s_2 + s_3]p + [(\alpha\mu + \beta)r_1 + \mu r_2]s_1 + \mu r_1 s_2\}q \\ \qquad -[\frac{1}{2}(\alpha\mu + \beta)s_1^2 + \mu s_1 s_2]p - \frac{1}{2}\alpha\mu s_1^2 r_1 \\ \qquad +\frac{1}{4}q_{xx} + \frac{1}{4}\mu s_{2,xx} + \frac{1}{4}s_{3,xx}. \end{cases} \tag{4.57}$$

Let us now introduce the enlarged Lax matrices

$$\bar{V}^{[m]} = M(V^{m]}, V_1^{[m]}, V_2^{[m]}, V_3^{[m]}) \in \bar{\mathfrak{g}},\ m \geq 0, \tag{4.58}$$

where $V^{[m]}$ is given in (2.39) and

$$V_i^{[m]} = (\lambda^m W_i)_+,\ m \geq 0, \tag{4.59}$$

which means that modification terms $\Delta_{m,i}$ are chosen as zero.

Finally, the m-th enlarged zero curvature equation

$$\bar{U}_{t_m} = \bar{V}_x^{[m]} + [\bar{U}, \bar{V}^{[m]}] \tag{4.60}$$

determines a hierarchy of coupling systems for the AKNS equations:

$$\bar{u}_{t_m} = \begin{bmatrix} p_{t_m} \\ q_{t_m} \\ r_{1,t_m} \\ s_{1,t_m} \\ r_{2,t_m} \\ s_{2,t_m} \\ r_{3,t_m} \\ s_{3,t_m} \end{bmatrix} = \bar{K}_m(\bar{u}) = \begin{bmatrix} K_m(u) \\ S_{1,m}(u, u_1) \\ S_{2,m}(u, u_1, u_2) \\ S_{3,m}(u, u_1, u_2, u_3) \end{bmatrix} = \begin{bmatrix} -2b_{m+1} \\ 2c_{m+1} \\ -2f_{m+1} \\ 2g_{m+1} \\ -2f'_{m+1} \\ 2g'_{m+1} \\ -2f''_{m+1} \\ 2g''_{m+1} \end{bmatrix},\ m \geq 0. \tag{4.61}$$

Obviously that all members in (4.61) give tri-integrable couplings for the AKNS equations.

4.3.2 Hamiltonian structures of the tri-integrable couplings of the AKNS hierarchy

The trace identity [61, 62], which is based on the killing form on a semisimple Lie algebra, provides an effective way to search for Hamiltonian structures of soliton equations. However, when it comes to non-semisimple Lie algebras, it is no longer applicable, because the Killing form is degenerate under non-semisimple Lie algebras. Variational identities were introduced in [54, 55] to solve this problem.

Now we shall generate Hamiltonian structures for the tri-integrable couplings (4.61) by applying the variational identity [54]:

$$\frac{\delta}{\delta \bar{u}} \int \langle \bar{U}_\lambda, \bar{W} \rangle dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma \langle \bar{U}_{\bar{u}}, \bar{W} \rangle. \tag{4.62}$$

The first step to construct symmetric and ad-invariant bilinear forms on $\bar{\mathfrak{g}}$ is to transform the semi-direct sum $\bar{\mathfrak{g}}$ into a vector form. To do this, we define a mapping

$$\sigma : \bar{\mathfrak{g}} \to \mathbb{R}^{12},\ A \mapsto (a_1, \cdots, a_{12})^T, \tag{4.63}$$

where

$$A = M(A_1, A_2, A_3, A_4) \in \bar{\mathfrak{g}},\ A_i = \begin{bmatrix} a_{3i-2} & a_{3i-1} \\ a_{3i} & -a_{3i-2} \end{bmatrix},\ 1 \le i \le 4. \tag{4.64}$$

It is easy to see that this mapping σ induces a Lie algebraic structure on $\mathbb{R}^{12}$, isomorphic to the matrix Lie algebra $\bar{\mathfrak{g}}$. The corresponding Lie bracket $[\cdot,\cdot]$ on $\mathbb{R}^{12}$ can be computed as follows

$$[a, b]^T = a^T R(b),\ a = (a_1, \cdots, a_{12})^T,\ b = (b_1, \cdots, b_{12})^T \in \mathbb{R}^{12}, \tag{4.65}$$

where

$$R(b) = M(R_1, R_2, R_3, R_4), \tag{4.66}$$

and

$$R_i = \begin{bmatrix} 0 & 2\,b_{3i-1} & -2\,b_{3i} \\ b_{3i} & -2\,b_{3i-2} & 0 \\ -b_{3i-1} & 0 & 2\,b_{3i-2} \end{bmatrix},\ 1 \le i \le 4. \tag{4.67}$$

This Lie algebra $(\mathbb{R}^{12}, [\cdot,\cdot])$ is Lie algebra isomorphic to the matrix Lie algebra $\bar{\mathfrak{g}}$, and the mapping σ, defined by (4.63), is a Lie algebra isomorphism between the two Lie algebras introduced above.

A bilinear form on $\mathbb{R}^{12}$ can be defined by

$$\langle a, b \rangle = a^T F b, \tag{4.68}$$

where F is a constant matrix. First the symmetric property $\langle a, b \rangle = \langle b, a \rangle$ needs that

$$F^T = F. \tag{4.69}$$

Then together with the ad-invariance property

$$\langle a, [b, c] \rangle = \langle [a, b], c \rangle$$

we find out that the matrix F should satisfy the equation

$$F(R(b))^T = -R(b)F,\ \text{for any}\ b \in \mathbb{R}^{12}. \tag{4.70}$$

Since b is arbitrary, this matrix equation is equivalent to a linear system of equations of the entries of the matrix F. We then solve the resulting linear system of equations and obtain that

$$F = \begin{bmatrix} \eta_1 & \eta_2 & \eta_3 & \eta_4 \\ \eta_2 & \alpha\eta_3 + \beta\eta_4 & \mu\eta_4 & 0 \\ \eta_3 & \mu\eta_4 & 0 & 0 \\ \eta_4 & 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 2\,0\,0 \\ 0\,0\,1 \\ 0\,1\,0 \end{bmatrix}, \tag{4.71}$$

where η_i, $1 \leq i \leq 4$, are arbitrary constants. Therefore, the corresponding bilinear form $\langle \cdot, \cdot \rangle$ on the semi-direct sum $\bar{\mathfrak{g}}$ of the two Lie subalgebras $\mathfrak{g}$ and $\mathfrak{g}_c$ is given by

$$\begin{aligned} \langle A, B \rangle_{\bar{\mathfrak{g}}} = \langle \sigma(A), \sigma(B) \rangle_{\mathbb{R}^{12}} &= (a_1, \cdots, a_{12}) F (b_1, \cdots, b_{12})^T \\ &= (2\,a_1b_1 + a_2b_3 + a_3b_2)\eta_1 \\ &\quad +(2\,a_1b_4 + a_2b_6 + a_3b_5 + 2\,a_4b_1 + a_5b_3 + a_6b_2)\eta_2 \\ &\quad +(2\,a_1b_7 + a_2b_9 + a_3b_8 + 2\,\alpha\,a_4b_4 + \alpha\,a_5b_6 + \alpha\,a_6b_5 + 2\,a_7b_1 + a_8b_3 + a_9b_2)\eta_3 \\ &\quad +(2\,a_1b_{10} + a_2b_{12} + a_3b_{11} + 2\,\beta\,a_4b_4 + 2\,\mu\,a_4b_7 + \beta\,a_5b_6 + \mu\,a_5b_9 + \beta\,a_6b_5 \\ &\quad +\mu\,a_6b_8 + 2\,\mu\,a_7b_4 + \mu\,a_8b_6 + \mu\,a_9b_5 + 2\,a_{10}b_1 + a_{11}b_3 + a_{12}b_2)\eta_4, \end{aligned} \tag{4.72}$$

where

$$A = \sigma^{-1}((a_1, \cdots, a_{12})^T) \in \bar{\mathfrak{g}},\ B = \sigma^{-1}((b_1, \cdots, b_{12})^T) \in \bar{\mathfrak{g}}.$$

Because σ is an isomorphism, the bilinear form (4.72) is also symmetric and ad-invariant:

$$\langle A, B \rangle_{\bar{\mathfrak{g}}} = \langle B, A \rangle_{\bar{\mathfrak{g}}}, \quad A, B, C \in \bar{\mathfrak{g}}, \tag{4.73}$$

$$\langle A, [B, C] \rangle_{\bar{\mathfrak{g}}} = \langle [A, B], C \rangle_{\bar{\mathfrak{g}}},\ A, B, C \in \bar{\mathfrak{g}}. \tag{4.74}$$

At this point, the bilinear form (4.72) is symmetric and ad-invariant. We also want it to be non-degenerate. Deducing from its definition (4.68), the bilinear form is non-degenerate if and only if determinant of F is not zero, i.e.,

$$\det(F) = 16\,{\eta_4}^{12}\mu^6 \neq 0. \tag{4.75}$$

We therefore require that η_4 and μ should be non-zero constants, otherwise the bilinear form (4.72) is degenerate.

Since the bilinear form required in the variational identity is ready, we can compute

$$\langle \bar{W}, \bar{U}_\lambda \rangle_{\bar{\mathfrak{g}}} = -2\, a\eta_1 - 2\, e\eta_2 - 2\, e'\eta_3 - 2\, e''\eta_4, \tag{4.76}$$

and

$$\langle \bar{W}, \bar{U}_{\bar{u}} \rangle_{\bar{\mathfrak{g}}} = \begin{bmatrix} c\eta_1 + g\eta_2 + g'\eta_3 + g''\eta_4 \\ b\eta_1 + f\eta_2 + f'\eta_3 + f''\eta_4 \\ c\eta_2 + \alpha\, g\eta_3 + (\beta\, g + \mu\, g')\,\eta_4 \\ b\eta_2 + \alpha\, f\eta_3 + (\beta\, f + \mu\, f')\,\eta_4 \\ c\eta_3 + \mu\, g\eta_4 \\ b\eta_3 + \mu\, f\eta_4 \\ c\eta_4 \\ b\eta_4 \end{bmatrix}. \tag{4.77}$$

In the article [37], a formula

$$\gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\langle \bar{W}, \bar{W}\rangle|,$$

is given to help calculate the parameter γ in the variational identity (4.62), where $\bar{W}$ is a solution of the stationary zero curvature equation. We therefore directly get the parameter $\gamma = 0$. Thus, by applying the corresponding variational identity (4.62) and equating coefficients of λ^m, $m \geq 0$, we obtain a Hamiltonian structure for the hierarchy (4.61) of tri-integrable couplings:

$$\bar{u}_{t_m} = \bar{J}\frac{\delta \bar{\mathcal{H}}_m}{\delta \bar{u}},\ m \geq 0, \tag{4.78}$$

with

$$\bar{\mathcal{H}}_m = \int \frac{2\,\eta_1 a_{m+2} + 2\,\eta_2 e_{m+2} + 2\,\eta_3\, e'_{m+2} + 2\,\eta_4\, e''_{m+2}}{m+1}\, dx,\ m \geq 0, \tag{4.79}$$

and

$$\bar{J} = \begin{bmatrix} \eta_1 & \eta_2 & \eta_3 & \eta_4 \\ \eta_2 & \alpha\,\eta_3 + \beta\,\eta_4 & \mu\,\eta_4 & 0 \\ \eta_3 & \mu\,\eta_4 & 0 & 0 \\ \eta_4 & 0 & 0 & 0 \end{bmatrix}^{-1} \otimes J, \tag{4.80}$$

where J is given in (2.41) and $\otimes$ is the Kronecker product.

The hierarchy (4.61) has a recursion relation

$$\bar{K}_m = \bar{\Phi}\bar{K}_{m-1},\ m \geq 1, \tag{4.81}$$

in which the recursion operator $\bar{\Phi}$ can be computed by

$$\bar{\Phi} = M^T(\Phi, \Phi_1, \Phi_2, \Phi_3) \tag{4.82}$$

where M^T means the transpose of the matrix M defined in (4.29) and Φ is defined in (2.41), Φ_1, Φ_2, Φ_3 are shown below:

$$\Phi_1 = \begin{bmatrix} r_1\partial^{-1}q + p\partial^{-1}s_1 & r_1\partial^{-1}p + p\partial^{-1}r_1 \\ -s_1\partial^{-1}q - q\partial^{-1}s_1 & -s_1\partial^{-1}p - q\partial^{-1}r_1 \end{bmatrix}, \tag{4.83}$$

$$\Phi_2 = \begin{bmatrix} p\partial^{-1}s_2 + r_1\partial^{-1}s_1 + r_2\partial^{-1}p & p\partial^{-1}r_2 + r_1\partial^{-1}r_1 + r_2\partial^{-1}q \\ -q\partial^{-1}s_2 - s_1\partial^{-1}s_1 - s_2\partial^{-1}p & -q\partial^{-1}r_2 - s_1\partial^{-1}r_1 - s_2\partial^{-1}q \end{bmatrix}, \tag{4.84}$$

$$\Phi_3 = \begin{bmatrix} r_3\partial^{-1}q + (\beta r_1 + \mu r_2)\partial^{-1}s_1 & r_3\partial^{-1}p + (\beta r_1 + \mu r_2)\partial^{-1}r_1 \\ +\mu r_1\partial^{-1}s_2 + p\partial^{-1}s_3, & +\mu r_1\partial^{-1}r_2 + p\partial^{-1}r_3 \\ -s_3\partial^{-1}q - (\beta s_1 + \mu s_2)\partial^{-1}s_1 & -s_3\partial^{-1}p - (\beta s_1 + \mu s_2)\partial^{-1}r_1 \\ -\mu s_1\partial^{-1}s_2 - q\partial^{-1}s_3, & -\mu s_1\partial^{-1}r_2 - q\partial^{-1}r_3 \end{bmatrix}. \tag{4.85}$$

It can be verified by definition that $\bar{\Phi}$ is hereditary [13], and $\bar{J}$ and $\bar{M} = \bar{\Phi}\bar{J}$ constitute a Hamiltonian pair [20], and thus we can construct a bi-Hamiltonian structure for the hierarchy (4.61) as follows:

$$\bar{u}_{t_m} = \bar{J}\frac{\delta\bar{\mathcal{H}}_m}{\delta\bar{u}} = \bar{M}\frac{\delta\bar{\mathcal{H}}_{m-1}}{\delta\bar{u}},\ \ m \geq 1. \tag{4.86}$$

Once a bi-Hamiltonian formulation of a dynamics system is found, a hierarchy of conserved functionals can be computed [20]. We state that the resulting tri-integrable couplings of AKNS hierarchy have infinitely many commuting symmetries $\{K_n\}_{n\geq 0}$ and functionally independent conserved quantities $\{\bar{\mathcal{H}}_n\}_{n\geq 0}$ since we have

$$[\bar{K}_m, \bar{K}_n] = 0, \quad m, n \geq 0, \tag{4.87}$$

and

$$\{\bar{\mathcal{H}}_m, \bar{\mathcal{H}}_n\}_{\bar{J}} = \{\bar{\mathcal{H}}_m, \bar{\mathcal{H}}_n\}_{\bar{M}} = 0,\ m, n \geq 0. \tag{4.88}$$

Since the hierarchy (4.61) of tri-integrable couplings is bi-Hamiltonian, it is Liouville integrable.

In the next section, we will choose some special reduction of the $Class_{\mathbf{3}}$ of non-semisimple matrix Lie algebras consisting of 4×4 block matrices, and generate tri-integrable couplings from the resulting Lie algebras. We shall present an application to the KdV soliton hierarchy as a concrete example, and show that the resulting tri-integrable couplings of the KdV soliton hierarchy have bi-Hamiltonian structures.

4.4 Application to the KdV hierarchy

The aim of this section is to show that tri-integrable couplings may provide an answer to the question that how two integrable couplings

$$\bar{u}_{1,t} = \bar{K}_1(\bar{u}_1) = \begin{bmatrix} K(u) \\ S_1(u, u_1) \end{bmatrix}, \quad \bar{u}_1 = \begin{bmatrix} u \\ u_1 \end{bmatrix}, \tag{4.89}$$

and

$$\bar{u}_{2,t} = \bar{K}_2(\bar{u}_2) = \begin{bmatrix} K(u) \\ S_2(u, u_1) \end{bmatrix}, \quad \bar{u}_2 = \begin{bmatrix} u \\ u_2 \end{bmatrix}, \tag{4.90}$$

can be cast into a new bigger system which is still integrable and has a bi-Hamiltonian structure.

It is obvious that the enlarged system in the form of

$$\bar{u}_t = \bar{K}(\bar{u}) = \begin{bmatrix} K(u) \\ S(u, u_1) \\ T(u, u_2) \end{bmatrix}, \quad \bar{u} = \begin{bmatrix} u \\ u_1 \\ u_2 \end{bmatrix}, \tag{4.91}$$

is degenerate in the sense that the two dependent variables u_1 and u_2 are separated. The above coupled system couples two integrable couplings together but it does not have any Hamiltonian structures found so far [53].

4.4.1 The KdV hierarchy

Let us recall the KdV soliton hierarchy [54, 78]. The typical spectral problem for the KdV hierarchy is given by

$$\phi_x = U\phi, \; U = U(u, \lambda) = \begin{bmatrix} 0 & 1 \\ \lambda - u & 0 \end{bmatrix}, \; \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}. \tag{4.92}$$

The stationary zero curvature equation

$$W_x = [U, W] \tag{4.93}$$

gives rise to

$$\begin{cases} a_x = (-\lambda + u)b + c, \\ b_x = -2\,a, \\ c_x = -2\,(-\lambda + u)a, \end{cases} \tag{4.94}$$

if we assume that W is of the form

$$W = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} = \sum_{i\geq 0} W_{0,i}\lambda^{-i} = \sum_{i\geq 0} \begin{bmatrix} a_i & b_i \\ c_i & -a_i \end{bmatrix} \lambda^{-i}. \tag{4.95}$$

The systems (4.94) equivalently yields

$$\begin{cases} b_{i+1} = \frac{1}{4}\,b_{i,xx} + ub_i - \frac{1}{2}\,\partial^{-1}u_x b_i, \\ c_i = -\frac{1}{2}\,b_{i,xx} + b_{i+1} - ub_i, \qquad\qquad i \geq 0. \\ a_i = -\frac{1}{2}\,b_{i,x}, \end{cases} \tag{4.96}$$

Upon taking the initial values

$$b_0 = 0,\; b_1 = 1, \tag{4.97}$$

we can uniquely compute the sequence $\{b_i\}_{i=1}^{\infty}$ and selecting constants of integration to be zero, the first few sets are as follows:

$$b_2 = \frac{1}{2}\,u,\; b_3 = \frac{1}{8}\,u_{xx} + \frac{3}{8}\,u^2, b_4 = \frac{1}{32}\,u_{xxxx} + \frac{5}{32}\,u_x^2 + \frac{5}{16}\,uu_{xx} + \frac{5}{16}\,u^3. \tag{4.98}$$

Now, the zero curvature equations

$$U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0 \text{ with } V^{[m]} = (\lambda^{m+1}W)_+ + \Delta_m,\; \Delta_m = \begin{bmatrix} 0 & 0 \\ -b_{m+2} & 0 \end{bmatrix}, m \geq 0, \tag{4.99}$$

generate the KdV hierarchy of soliton equations:

$$u_{t_m} = K_m = 2\,b_{m+2,x} \tag{4.100a}$$

$$K_m = \Phi K_{m-1},\;\; \Phi = \frac{1}{4}\partial^2 + u + \frac{1}{2}u_x\partial^{-1},\;\; m \geq 1. \tag{4.100b}$$

The KdV hierarchy has the bi-Hamiltonian structure [20]:

$$u_{t_m} = K_m = J\frac{\delta \mathcal{H}_m}{\delta u} = M\frac{\delta \mathcal{H}_{m-1}}{\delta u}, \;\; m \geq 1, \tag{4.101}$$

with the Hamiltonian pair:

$$J = \partial, \;\; M = \Phi J = \frac{1}{4}\,\partial^3 + \partial u + \frac{1}{2}\,u_x, \tag{4.102}$$

and Hamiltonian functionals [54]

$$\mathcal{H}_m = \int \frac{4b_{m+3}}{2m+3}\,dx,\; m \geq 0. \tag{4.103}$$

4.4.2 Tri-integrable couplings of the KdV hierarchy

In the following, we choose a class of block matrices as a special case of $Class^*_{\mathbf{3}}$ when $\zeta = \beta$:

$$M(A_1, A_2, A_3, A_4) = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & A_1 & 0 & \alpha\, A_2 + \beta\, A_3 \\ 0 & 0 & A_1 & \beta\, A_2 + \mu\, A_3 \\ 0 & 0 & 0 & A_1 \end{bmatrix}. \tag{4.104}$$

In particular, we take a special non-semisimple Lie algebra $\bar{\mathfrak{g}}$ as a semi-direct sum decomposition of a semisimple subalgebra $\mathfrak{g}$ and a solvable subalgebra $\mathfrak{g}_c$:

$$\bar{\mathfrak{g}} = \mathfrak{g} \Subset \mathfrak{g}_c \tag{4.105}$$

with

$$\mathfrak{g} = \{\, M(A_1, 0, 0, 0) |\, \text{entries of } A_1 \text{ - Laurent series in } \lambda \}\,, \tag{4.106}$$

and

$$\mathfrak{g}_c = \{\, M(0, A_2, A_3, A_4) |\, \text{entries of } A_2, A_3, A_4 \text{ - Laurent series in } \lambda \}\,. \tag{4.107}$$

To construct tri-integrable couplings for the KdV equations, we introduce the corresponding enlarged spectral matrix

$$\bar{U} = \bar{U}(\bar{u}, \lambda) = M(U, U_1, U_2, U_3) \in \bar{\mathfrak{g}}, \tag{4.108}$$

with $U = U(u, \lambda)$ be defined by (4.92) and

$$U_i = U_i(u_i) = \begin{bmatrix} 0 & 0 \\ -u_i & 0 \end{bmatrix}, \quad 1 \le i \le 3, \tag{4.109}$$

where $\bar{u} = (u, u_1, u_2, u_3)^T$, and u_1, u_2, u_3 are new dependent variables.

To solve the corresponding enlarged stationary zero curvature equation

$$\bar{W}_x = [\bar{U}, \bar{W}], \tag{4.110}$$

we search for solutions of the following form

$$\bar{W} = \bar{W}(\bar{u}, \lambda) = M(W, W_1, W_2, W_3) \in \bar{\mathfrak{g}}, \tag{4.111}$$

where W is given by (4.95).

Assume that

$$W_1, W_2, W_3 \in \widetilde{\mathrm{sl}}(2, \mathbb{R}) = \{A \in \mathrm{sl}(2, \mathbb{R}) |\, \text{entries of } A \text{ - Laurent series in } \lambda\}$$

are of the form

$$W_1 = \begin{bmatrix} e & f \\ g & -e \end{bmatrix}, \quad W_2 = \begin{bmatrix} e' & f' \\ g' & -e' \end{bmatrix}, \quad W_3 = \begin{bmatrix} e'' & f'' \\ g'' & -e'' \end{bmatrix}. \tag{4.112}$$

Plugging (4.112) in the enlarged stationary zero curvature equation (4.110), we get

$$\begin{cases} e_x = u_1\, b + (u - \lambda)\, f + g, \\ f_x = -2\, e, \\ g_x = -2\, u_1\, a + 2\, (\lambda - u)\, e; \end{cases} \tag{4.113}$$

$$\begin{cases} e'_x = u_2\, b + (u - \lambda)\, f' + g', \\ f'_x = -2\, e', \\ g'_x = -2\, u_2 a + 2\, (\lambda - u)\, e'; \end{cases} \tag{4.114}$$

and

$$\begin{cases} e''_x = u_3 b + (\alpha\, u_1 + \beta\, u_2) f + (\beta\, u_1 + \mu\, u_2) f' + (u - \lambda) f'' + g'', \\ f''_x = -2\, e'', \\ g''_x = -2\, u_3 a - 2\, (\alpha\, u_1 + \beta\, u_2) e - 2\, (\beta\, u_1 + \mu\, u_2) e' + 2\, (\lambda - u) e''. \end{cases} \tag{4.115}$$

Trying a solution $\bar{W}$ with

$$\begin{cases} e = \sum_{i\geq 0} e_i\lambda^{-i},\ f = \sum_{i\geq 0} f_i\lambda^{-i},\ g = \sum_{i\geq 0} g_i\lambda^{-i}, \\ e' = \sum_{i\geq 0} e'_i\lambda^{-i},\ f' = \sum_{i\geq 0} f'_i\lambda^{-i},\ g' = \sum_{i\geq 0} g'_i\lambda^{-i}, \\ e'' = \sum_{i\geq 0} e''_i\lambda^{-i},\ f'' = \sum_{i\geq 0} f''_i\lambda^{-i},\ g'' = \sum_{i\geq 0} g''_i\lambda^{-i}, \end{cases} \tag{4.116}$$

and taking

$$f_0 = f_1 = f'_0 = f'_1 = f''_0 = f''_1 = 0, \tag{4.117}$$

then we can have

$$\begin{cases} f_{i+1} = u_1\, b_i - \frac{1}{2}\,\partial^{-1}u_{1,x}b_i + \frac{1}{4}\, f_{i,xx} + u\, f_i - \frac{1}{2}\,\partial^{-1}u_x f_i, \\ e_i = -\frac{1}{2}\, f_{i,x}, \\ g_i = -\frac{1}{2}\, f_{i,xx} - u_1\, b_i + f_{i+1} - u\, f_i, \end{cases} \tag{4.118}$$

$$\begin{cases} f'_{i+1} = u_2\, b_i - \frac{1}{2}\,\partial^{-1}u_{2,x}\, b_i + \frac{1}{4}\, f'_{i,xx} + u\, f'_i - \frac{1}{2}\,\partial^{-1}u_x\, f'_i, \\ e'_i = -\frac{1}{2}\, f'_{i,x}, \\ g'_i = -\frac{1}{2}\, f'_{i,xx} - u_2\, b_i + f'_{i+1} - u\, f'_i, \end{cases} \tag{4.119}$$

and

$$\begin{cases} f''_{i+1} = u_3\, b_i - \frac{1}{2}\,\partial^{-1}u_{3,x}\, b_i \\ \qquad +(\alpha\, u_1 + \beta\, u_2)\, f_i - \frac{1}{2}\,\partial^{-1}(\alpha\, u_{1,x} + \beta\, u_{2,x})\, f_i \\ \qquad +(\beta\, u_1 + \mu\, u_2)\, f'_i - \frac{1}{2}\,\partial^{-1}(\beta\, u_{1,x} + \mu\, u_{2,x})\, f'_i \\ \qquad +\frac{1}{4}\, f''_{i,xx} + u\, f''_i - \frac{1}{2}\,\partial^{-1}u_x\, f''_i, \\ e''_i = -\frac{1}{2}\, f''_{i,x}, \\ g''_i = -\frac{1}{2}\, f''_{i,xx} - u_3\, b_i - (\alpha\, u_1 + \beta\, u_2)\, f_i \\ \qquad -(\beta\, u_1 + \mu\, u_2)\, f'_i + f''_{i+1} - u\, f''_i. \end{cases} \tag{4.120}$$

The recursion relations (4.118), (4.119) and (4.120) uniquely generate three sequences of $\{e_i, f_i, g_i\}_{i\geq 1}$, $\{e'_i, f'_i, g'_i\}_{i\geq 1}$ and $\{e''_i, f''_i, g''_i\}_{i\geq 1}$, respectively if we take constants of integration as zero.

The first few sets of functions are as follows:

$$\begin{cases} f_2 \; = \frac{1}{2}\, u_1, \\ f_2' \; = \frac{1}{2}\, u_2, \\ f_2'' \; = \frac{1}{2}\, u_3; \end{cases} \tag{4.121}$$

$$\begin{cases} f_3 \; = \frac{1}{8}\, u_{1,xx} + \frac{3}{4}\, uu_1, \\ f_3' \; = \frac{1}{8}\, u_{2,xx} + \frac{3}{4}\, uu_2, \\ f_3'' = \frac{1}{8}\, u_{3,xx} + \frac{3}{4}\, uu_3 + \frac{3}{4}\beta\, u_1u_2 + \frac{3}{8}\,\alpha\, u_1^2 + \frac{3}{8}\,\mu\, u_2^2; \end{cases} \tag{4.122}$$

$$\begin{cases} f_4 \; = \frac{5}{16}\, u_{xx}u_1 + \frac{15}{16}\, u^2u_1 + \frac{5}{16}\, u_{1,x}u_x + \frac{5}{16}\, u_{1,xx}u + \frac{1}{32}\, u_{1,xxxx}, \\ f_4' \; = \frac{5}{16}\, u_{xx}u_2 + \frac{15}{16}\, u^2u_2 + \frac{5}{16}\, u_{2,x}u_x + \frac{5}{16}\, u_{2,xx}u + \frac{1}{32}\, u_{2,xxxx}, \\ f_4'' = \frac{15}{16}\, u^2u_3 + \frac{1}{32}\, u_{3,xxxx} + \frac{15}{16}\,(\alpha\, u_1^2 + \mu\, u_2^2 + 2\,\beta\, u_1u_2 + \frac{1}{3}\, u_{3,xx})u \\ \qquad +\frac{5}{16}\,(\alpha\, u_{1,xx} + \beta\, u_{2,xx})u_1 + \frac{5}{16}\,(\beta\, u_{1,xx} + \mu\, u_{2,xx})u_2 \\ \qquad +\frac{5}{16}\,(\beta\, u_{1,x}u_{2,x} + u_3u_{xx} + u_{3,x}u_x) + \frac{5}{32}\,(\alpha\, u_{1,x}^2 + \mu\, u_{2,x}^2); \end{cases} \tag{4.123}$$

Let us now introduce the enlarged Lax matrices

$$\bar{V}^{[m]} = M(V^{m}], V_1^{[m]}, V_2^{[m]}, V_3^{[m]}) \in \bar{\mathfrak{g}},\ m \geq 0, \tag{4.124}$$

where $V^{[m]}$ is defined as in (4.99) and

$$V_i^{[m]} = (\lambda^{m+1}W_i)_+ + \Delta_{m,i},\ m \geq 0, \tag{4.125}$$

and we choose

$$\Delta_{m,1} = \begin{bmatrix} 0 & 0 \\ -f_{m+2} & 0 \end{bmatrix}, \quad \Delta_{m,2} = \begin{bmatrix} 0 & 0 \\ -f_{m+2}' & 0 \end{bmatrix}, \quad \Delta_{m,3} = \begin{bmatrix} 0 & 0 \\ -f_{m+2}'' & 0 \end{bmatrix}. \tag{4.126}$$

Then the enlarged zero curvature equations

$$\bar{U}_{t_m} = \bar{V}_x^{[m]} + [\bar{U}, \bar{V}^{[m]}],\ \ m \geq 0, \tag{4.127}$$

determine a hierarchy of coupling systems for the KdV equations:

$$\bar{u}_{t_m} = \begin{bmatrix} u_{t_m} \\ u_{1,t_m} \\ u_{2,t_m} \\ u_{3,t_m} \end{bmatrix} = \bar{K}_m(\bar{u}) = \begin{bmatrix} K_m(u) \\ K_m'[u_1] \\ K_m'[u_2] \\ S_{3,m}(u, u_1, u_2, u_3) \end{bmatrix} = \begin{bmatrix} 2\, b_{m+2,x} \\ 2\, f_{m+2,x} \\ 2\, f_{m+2,x}' \\ 2\, f_{m+2,x}'' \end{bmatrix},\ \ m \geq 0. \tag{4.128}$$

It is direct to check that all members in (4.128), with $m \geq 2$, provide tri-integrable couplings for the KdV equations. The first two tri-integrable couplings of the KdV equation are listed below:

$$\begin{cases} u_{t_1} &= \frac{3}{2}\,uu_x + \frac{1}{4}\,u_{xxx}, \\ u_{1,t_1} &= \frac{3}{2}\,uu_{1,x} + \frac{3}{2}\,u_xu_1 + \frac{1}{4}\,u_{1,xxx}, \\ u_{2,t_1} &= \frac{3}{2}\,uu_{2,x} + \frac{3}{2}\,u_xu_2 + \frac{1}{4}\,u_{2,xxx}, \\ u_{3,t_1} &= \frac{3}{2}\,[u_xu_3 + uu_{3,x} + (\alpha\, u_{1,x} + \beta\, u_{2,x})u_1 + (\beta\, u_{1,x} + \mu\, u_{2,x})u_2] + \frac{1}{4}\,u_{3,xxx}; \end{cases} \tag{4.129}$$

$$\begin{cases} u_{t_2} &= \frac{15}{8}\,u^2u_x + \frac{5}{8}\,uu_{xxx} + \frac{1}{16}\,u_{xxxxx} + \frac{5}{4}\,u_xu_{xx}, \\ u_{1,t_2} &= \frac{15}{8}\,u_{1,x}u^2 + \frac{1}{16}\,u_{1,xxxxx} + (\frac{15}{4}\,u_xu_1 + \frac{5}{8}\,u_{1,xxx})\,u \\ &\quad +\frac{5}{8}\,u_{xxx}u_1 + \frac{5}{4}\,(u_{1,xx}u_x + u_{xx}u_{1,x}), \\ u_{2,t_2} &= \frac{15}{8}\,u_{2,x}u^2 + \frac{1}{16}\,u_{2,xxxxx} + (\frac{15}{4}\,u_xu_2 + \frac{5}{8}\,u_{2,xxx})\,u \\ &\quad +\frac{5}{8}\,u_{xxx}u_2 + \frac{5}{4}\,(u_{2,xx}u_x + u_{xx}u_{2,x}), \\ u_{3,t_2} &= \frac{15}{8}\,u_{3,x}u^2 + [\frac{15}{4}\,(\alpha\, u_{1,x} + \beta\, u_{2,x})u_1 + \frac{15}{4}\,(\beta\, u_{1,x} + \mu\, u_{2,x})u_2 + \frac{15}{4}\,u_xu_3 + \frac{5}{8}\,u_{3,xxx}]u \\ &\quad +\frac{15}{8}\,\alpha\, u_xu_1^2 + (\frac{15}{4}\,\beta\, u_xu_2 + \frac{5}{8}\,\alpha\, u_{1,xxx} + \frac{5}{8}\,\beta\, u_{2,xxx})u_1 + \frac{15}{8}\,\mu\, u_xu_2^2 \\ &\quad +\frac{5}{8}\,(\beta\, u_{1,xxx} + \mu\, u_{2,xxx})u_2 + \frac{5}{8}\,u_{xxx}u_3 + \frac{5}{8}\,\mu\, u_{2,x}u_{2,xx} + \frac{5}{8}\,\alpha\, u_{1,x}u_{1,xx} \\ &\quad +\frac{5}{4}\,u_xu_{3,xx} + \frac{5}{4}\,u_{3,x}u_{xx} + \frac{5}{8}\,(\alpha\, u_{1,xx} + \beta\, u_{2,xx})u_{1,x} \\ &\quad +\frac{5}{8}\,(\beta\, u_{1,xx} + \mu\, u_{2,xx})u_{2,x} + \frac{1}{16}\,u_{3,xxxxx} + \frac{5}{8}\,\beta\,(u_{2,x}u_{1,xx} + u_{1,x}u_{2,xx}). \end{cases} \tag{4.130}$$

4.4.3 Hamiltonian structures of the tri-integrable couplings of the KdV hierarchy

For the presented tri-integrable couplings in (4.128), its Hamiltonian structures will be generated by applying the variational identity [55, 54]:

$$\frac{\delta}{\delta \bar{u}} \int \langle \bar{U}_\lambda, \bar{W} \rangle dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} \langle \bar{U}_{\bar{u}}, \bar{W} \rangle. \tag{4.131}$$

In the above formula, $\langle \cdot, \cdot \rangle$ is a symmetric, ad-invariant and non-degenerate bilinear forms on $\bar{\mathfrak{g}}$.

We need to construct a specific non-degenerate bilinear form on $\bar{\mathfrak{g}}$ with the symmetric and ad-invariance properties.

To do so, we first transform the semi-direct sum $\bar{\mathfrak{g}}$ into a vector form by introducing a mapping

$$\sigma : \bar{g} \to \mathbb{R}^{12},\ A \mapsto (a_1, \cdots, a_{12})^T, \tag{4.132}$$

where

$$A = M(A_1, A_2, A_3, A_4) \in \bar{\mathfrak{g}},\ A_i = \begin{bmatrix} a_{3i-2} & a_{3i-1} \\ a_{3i} & -a_{3i-2} \end{bmatrix},\ 1 \le i \le 4. \tag{4.133}$$

This mapping σ engenders a Lie algebraic structure on $\mathbb{R}^{12}$, isomorphic to the matrix Lie algebra $\bar{\mathfrak{g}}$. The corresponding Lie bracket $[\cdot,\cdot]$ on $\mathbb{R}^{12}$ can be computed as follows

$$[a,b]^T = a^T R(b),\ a = (a_1, \cdots, a_{12})^T,\ b = (b_1, \cdots, b_{12})^T \in \mathbb{R}^{12}, \tag{4.134}$$

where

$$R(b) = M(R_1, R_2, R_2, R_3),\ R_i = \begin{bmatrix} 0 & 2\,b_{3i-1} & -2\,b_{3i} \\ b_{3i} & -2\,b_{3i-2} & 0 \\ -b_{3i-1} & 0 & 2\,b_{3i-2} \end{bmatrix},\ 1 \le i \le 4. \tag{4.135}$$

This Lie algebra $(\mathbb{R}^{12}, [\cdot,\cdot])$ is isomorphic to the matrix Lie algebra $\bar{\mathfrak{g}}$, and the mapping σ, defined by (4.132), is a Lie algebra isomorphism between the two Lie algebras defined above.

We now define a bilinear form on $\mathbb{R}^{12}$ is determined by

$$\langle a, b\rangle = a^T F b, \tag{4.136}$$

where F is a constant matrix.

The symmetric property $\langle a, b\rangle = \langle b, a\rangle$ demands that

$$F^T = F. \tag{4.137}$$

Together with this symmetric condition, the ad-invariance property

$$\langle a, [b, c]\rangle = \langle [a, b], c\rangle$$

equivalently requires that

$$F(R(b))^T = -R(b)F,\ b \in \mathbb{R}^{12}. \tag{4.138}$$

This matrix equation with an arbitrary b leads to a linear system of equations on the entries of the matrix F. Solving the resulting system tells

$$F = \begin{bmatrix} \eta_1 & \eta_2 & \eta_3 & \eta_4 \\ \eta_2 & \alpha\,\eta_4 & \beta\,\eta_4 & 0 \\ \eta_3 & \beta\,\eta_4 & \mu\,\eta_4 & 0 \\ \eta_4 & 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 2\,0\,0 \\ 0\,0\,1 \\ 0\,1\,0 \end{bmatrix}, \tag{4.139}$$

where η_i, $1 \leq i \leq 4$, are arbitrary constants, and $\otimes$ is the Kronecker product. Now, the corresponding bilinear form on the semi-direct sum $\bar{\mathfrak{g}}$ of the two Lie subalgebras $\mathfrak{g}$ and $\mathfrak{g}_c$ is given as follows:

$$\begin{aligned}\langle A, B\rangle_{\bar{\mathfrak{g}}} = \langle \sigma(A), \sigma(B)\rangle_{\mathbb{R}^{12}} = (a_1, \cdots, a_{12}) F (b_1, \cdots, b_{12})^T \\ = (2\, a_1 b_1 + a_2 b_3 + a_3 b_2)\eta_1 + (2\, a_1 b_4 + a_2 b_6 + a_3 b_5 + 2\, a_4 b_1 \\ + a_5 b_3 + a_6 b_2)\eta_2 + (2\, a_1 b_7 + a_2 b_9 + a_3 b_8 + 2\, a_7 b_1 + a_8 b_3 + a_9 b_2)\eta_3 \\ + (2\, a_1 b_{10} + a_2 b_{12} + a_3 b_{11} + 2\,\alpha\, a_4 b_4 + 2\,\beta\, a_4 b_7 + \alpha\, a_5 b_6 + \beta\, a_5 b_9 \\ + \alpha\, a_6 b_5 + \beta\, a_6 b_8 + 2\,\beta\, a_7 b_4 + 2\,\mu\, a_7 b_7 + \beta\, a_8 b_6 + \mu\, a_8 b_9 \\ + \beta\, a_9 b_5 + \mu\, a_9 b_8 + 2\, a_{10} b_1 + a_{11} b_3 + a_{12} b_2)\eta_4,\end{aligned} \tag{4.140}$$

where

$$A = \sigma^{-1}((a_1, \cdots, a_{12})^T) \in \bar{\mathfrak{g}},\ B = \sigma^{-1}((b_1, \cdots, b_{12})^T) \in \bar{\mathfrak{g}}.$$

Because the mapping σ is an isomorphism, the bilinear form (4.140) is also symmetric and as-invariant:

$$\langle A, B\rangle_{\bar{\mathfrak{g}}} = \langle B, A\rangle_{\bar{\mathfrak{g}}},\ \langle A, [B, C]\rangle_{\bar{\mathfrak{g}}} = \langle [A, B], C\rangle_{\bar{\mathfrak{g}}},\ A, B, C \in \bar{\mathfrak{g}}. \tag{4.141}$$

It is obvious that the bilinear form (4.140) is non-degenerate if and only if the determinant of matrix F is non-zero, i.e.,

$$\det(F) = -16\, {\eta_4}^{12} (\alpha\,\mu - \beta^2)^3 \neq 0. \tag{4.142}$$

Therefore, in order to get non-degenerate bilinear forms over $\bar{\mathfrak{g}}$, we should have $\eta_4 \neq 0$ and $\alpha\mu \neq \beta^2$, so that $\det(F)$ is non-zero.

It is now direct to compute that

$$\langle \bar{W}, \bar{U}_\lambda\rangle_{\bar{\mathfrak{g}}} = \eta_1 b + \eta_2 f + \eta_3 f' + \eta_4 f'',$$

and

$$\langle \bar{W}, \bar{U}_{\bar{u}}\rangle_{\bar{\mathfrak{g}}} = \begin{bmatrix} -\eta_1 b - \eta_2 f - \eta_3 f' - \eta_4 f'' \\ -\eta_2 b - \eta_4 \alpha\, f - \eta_4 \beta\, f' \\ -\eta_3 b - \eta_4 \beta\, f - \eta_4 \mu\, f' \\ -\eta_4 b \end{bmatrix}. \tag{4.143}$$

To calculate the parameter γ in the variational identity (4.131), we use the formula [37]:

$$\gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\langle \bar{W}, \bar{W}\rangle|,$$

and find out that $\gamma = \frac{1}{2}$. Consequently, applying the corresponding variational identity, we obtain a Hamiltonian structure for the hierarchy (4.128) of tri-integrable couplings:

$$\bar{u}_{t_m} = \bar{J}\frac{\delta \bar{\mathcal{H}}_m}{\delta \bar{u}},\ m \geq 0, \tag{4.144}$$

with

$$\bar{\mathcal{H}}_m = \int \frac{4(\eta_1 b_{m+3} + \eta_2 f_{m+3} + \eta_3 f'_{m+3} + \eta_4 f''_{m+3})}{2m+3} dx,\ m \geq 0, \tag{4.145}$$

and

$$\bar{J} = \begin{bmatrix} \eta_1 & \eta_2 & \eta_3 & \eta_4 \\ \eta_2 & \alpha\,\eta_4 & \beta\,\eta_4 & 0 \\ \eta_3 & \beta\,\eta_4 & \mu\,\eta_4 & 0 \\ \eta_4 & 0 & 0 & 0 \end{bmatrix}^{-1} \otimes \partial. \tag{4.146}$$

The recursion relation

$$\bar{K}_m = \bar{\Phi}\bar{K}_{m-1},\ m \geq 1, \tag{4.147}$$

tells that the recursion operator $\bar{\Phi}$ reads

$$\bar{\Phi} = \bar{\Phi}(\bar{u}) = M^T(\Phi, \Phi_1, \Phi_2, \Phi_3), \tag{4.148}$$

where M is defined in (4.104), Φ is given by (4.100b) and

$$\Phi_i = u_i + \frac{1}{2}u_{i,x}\partial^{-1},\ \text{ for } i = 1, 2, 3. \tag{4.149}$$

It can be verified directly that $\bar{\Phi}$ is hereditary [13], and $\bar{J}$ and $\bar{M} = \bar{\Phi}\bar{J}$ constitute a Hamiltonian pair [20].

Therefore, the hierarchy (4.128) of tri-integrable couplings is bi-Hamiltonian,

$$u_{t_m} = K_m = \bar{J}\frac{\delta \bar{\mathcal{H}}_m}{\delta \bar{u}} = \bar{M}\frac{\delta \bar{\mathcal{H}}_{m-1}}{\delta \bar{u}},\ \ m \geq 1, \tag{4.150}$$

and so it is Liouville integrable, i.e., we have

$$[\bar{K}_m, \bar{K}_n] = 0,\ m, n \geq 0, \tag{4.151}$$

and

$$\{\bar{\mathcal{H}}_m, \bar{\mathcal{H}}_n\}_{\bar{J}} = \{\bar{\mathcal{H}}_m, \bar{\mathcal{H}}_n\}_{\bar{M}} = 0,\ m, n \geq 0. \tag{4.152}$$

It is worth pointing out that although we cannot decide whether the degenerate system (4.91) is integrable or not, we successfully constructed a bi-Hamiltonian tri-integrable coupling (4.128) which contains the dynamical system (4.91) as a sub-system. This tri-integrable coupling has many significant integrable properties, including infinitely many commuting symmetries and conserved functionals, which may help us understand more about mathematical properties that degenerate systems possess.

Chapter 5

Conclusions and remarks

In this dissertation, we discuss the problem of bi-integrable and tri-integrable couplings and present two methods for constructing bi-integrable and tri-integrable couplings by use of semi-direct sums of matrix Lie algebras.

There are a few methods of generating integrable couplings developed, such as the perturbation method [12, 56, 39], enlarging spectral problems [36, 38], and constructing new matrix loop Lie algebras [45, 35]. It is remarkable that multi-integrable couplings, and their infinitely many symmetries and conserved quantities, can be derived in a straightforward way just by performing rather natural manipulations on the non-semisimple matrix Lie algebras and applying the variational identities.

Among all 10 classes of non-semisimple Lie algebras we presented for generating bi-integrable couplings, $\mathrm{Class}_\mathbf{5}$ and $\mathrm{Class}_\mathbf{8}$ are two of the most interesting ones. They have more than one parameter and, taking special reductions of the parameters, we can obtain interesting classes of Lie algebras of block matrices to construct bi-integrable couplings:

$$\begin{cases} u_t = K(u), \\ u_{1,t} = S_1(u, u_1), \\ u_{2,t} = S_2(u, u_1, u_2). \end{cases}$$

Note that we don't keep all the parameters, and otherwise, the subsystems $u_t = K(u)$, $u_{1,t} = S_1(u, u_1)$ and $u_{2,t} = S_2(u, u_1, u_2)$ might be independent of each other and the enlarged integrable systems will be trivial bi-integrable couplings.

Some other classes, for example, $\mathrm{Class}_\mathbf{1}$, $\mathrm{Class}_\mathbf{6}$ and $\mathrm{Class}_\mathbf{7}$, might not produce Hamiltonian structures. One example of $\mathrm{Class}_\mathbf{6}$ has already been studied in [53], and the Lax pair is in the form of

$$\bar{U} = \begin{bmatrix} U & U_1 & U_2 \\ 0 & U & 0 \\ 0 & 0 & U \end{bmatrix}, \qquad \bar{V} = \begin{bmatrix} V & V_1 & V_2 \\ 0 & V & 0 \\ 0 & 0 & V \end{bmatrix}.$$

However, it is difficult to determine whether the integrable couplings generated by the above type of non-semisimple Lie algebras possess Hamiltonian structures since any bilinear form satisfying the three conditions required in the variational identity is degenerate. This is the case also for Class$_{\mathbf{7}}$. Our question is: for those non-semisimple matrix Lie algebras of 3×3 block matrices, can we release some restrictions on bilinear forms in the variational identity to find Hamiltonian structures?

Among all classes of block matrix Lie algebras for constructing tri-integrable couplings, we notice that $Class^*_{\mathbf{1}}$ has no non-degenerate bilinear form required in the variational identity [37, 54, 55]. It is the same case for $Class^*_{\mathbf{2}}$ when $\beta \neq 0$, and $Class^*_{\mathbf{3}}$ with $\beta \neq \zeta$. The bilinear form on the algebra $\bar{\mathfrak{g}}$ has to be non-degenerate, symmetric and ad-invariant in [37, 54, 55]; however, it might be possible for us to loosen any of these restrictions in the variational identity, and find Hamiltonian structures of the tri-integrable couplings produced via $Class^*_{\mathbf{1}}, Class^*_{\mathbf{2}}, Class^*_{\mathbf{3}}$ block matrix Lie algebras. We remark that based on one class of matrix Lie algebras, another way to obtain different bi-integrable and tri-integrable couplings is to choose different types of submatrices U_i in the enlarged spectral matrix $\bar{U}$.

In conclusion, due to the rich structure of block matrices, non-semisimple matrix loop algebras become a feasible way to construct various integrable couplings [12, 37, 48, 56, 76], bi-integrable couplings [77], and tri-integrable couplings. Of course, the construction scheme can be applied to many other existing soliton hierarchies.

However, we are only at the beginning of finding integrable couplings and multi-integrable couplings by using zero curvature equations on non-semisimple matrix loop Lie algebras, there are still many questions. Can we find nonlinear integrable couplings by using enlarged matrix loop algebras? Is it possible to guarantee the existence of Hamiltonian structures of the resulting integrable couplings by setting preconditions on matrix Lie algebras? Note that in this dissertation, we enlarge Lax pair matrices by adding the first order perturbations which have also been successfully used in [56, 79, 40, 38] to generate Hamiltonian integrable systems. Our guess is that the second order perturbations may be good choices for enlarging Lax pair matrices, too. Moreover using the Kronecker product to combine zero curvature equations of two multi-integrable couplings to generate a new one is also option to find multi-integrable couplings (for details see, e.g., [47, 80]).

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