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Stochastic Modeling of Network-Centric Epidemiological Processes

Divine Wanduku
University of South Florida, wandukudivine@yahoo.com

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by

Divine T Wanduku

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University of South Florida

Major professor: Gangaram S Ladde, Ph.D.
Chris P. Tsokos, Ph.D.
Marcus S. McWaters, Ph.D.
Kandethody Ramachandran, Ph.D.

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Dedication

This doctoral dissertation is dedicated to my Dad, my late mum, my baby sister Esther, to my other sisters and brother.

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Abstract

The technological changes and educational expansion have created the heterogeneity in the human species. Clearly, this heterogeneity generates a structure in the population dynamics, namely: citizen, permanent resident, visitor, and etc. Furthermore, as the heterogeneity in the population increases, the human mobility between meta-populations patches also increases. Depending on spatial scales, a meta-population patch can be decomposed into sub-patches, for examples: homes, neighborhoods, towns, etc. The dynamics of human mobility in a heterogeneous and scaled structured population is still its infancy level. We develop and investigate (**1**) an algorithmic two scale human mobility dynamic model for a meta-population. Moreover,the two scale human mobility dynamic model can be extended to multi-scales by applying the algorithm. The subregions and regions are interlinked via intra-and inter regional transport network systems. Under various types of growth order assumptions on the intra and interregional residence times of the residents of a sub region, different patterns of static behavior of the mobility process are studied. Furthermore, the human mobility dynamic model is applied to a two-scale population dynamic exhibiting a special real life human transportation network pattern. The static evolution of all categories of residents of a given site (homesite, visiting sites within the region, and visiting sites in other regions) over continuous changes in the intra and inter-regional visiting times is also analyzed.

The development of the two scale human mobility dynamic model provides a suitable approach to undertake the study of the non-uniform global spread of emergent infectious diseases of humans in a systematic and unified way. In view of this, we derive (**2**) a SIRS stochastic epidemic dynamic process in a two scale structured population. By defining a positively self invariant set for the dynamic model the stochastic asymptotic stability

results of the disease free equilibrium are developed(**2**). Furthermore, the significance of the stability results are illustrated in a simple real life scenario that is under controlled quarantine disease strategy. In addition, the epidemic dynamic model (**2**) is applied to a SIR influenza epidemic in a two scale population that is under the influence of a special real life human mobility pattern. The simulated trajectories for the different states (susceptible, Infective, Removal) with respect to current location in the two-scale population structure are presented. The simulated findings reveal comparative evolution patterns for the different states and current locations over time.

The SIRS stochastic epidemic dynamic model (**2**) is extended to a SIR delayed stochastic epidemic dynamic model(**3**). The delay effects in the dynamic model (**3**) is temporary and account for natural or infection acquired immunity conferred by the disease after disease recovery. Again, we justify the model validation as a prerequisite for the dynamic modeling. Moreover, we also exhibit the real life scenario under controlled quarantine disease strategy.In addition, the developed delayed SIR dynamic model is also applied to SIR influenza epidemic with temporary immunity to an influenza disease strain. The simulated results reveal an oscillatory effect in the trajectory of the naturally immune population. Moreover, the oscillations are more significant at the homesite.

We further extended the stochastic temporary delayed epidemic dynamic model (**3**) into a stochastic delayed epidemic dynamic model with varying immunity period(**4**). The varying immunity period accounts for the varying time lengths of natural immunity against the infectious agent exhibited within the naturally immune population. Obviously, the stochastic dynamic model with varying immunity period generalizes the SIR temporary delayed dynamic.

1 A TWO-SCALE NETWORK DYNAMIC MODEL FOR HUMAN MOBILITY PROCESS

1.1 Introduction

Over the centuries human societies across the globe have progressively established closer bilateral relationships and contacts. With the recent advent of high technology in the area of communication, transportation and basic services, multilateral interactions have been facilitated. As a result of this, the world has become like a neighborhood. Furthermore, the national and binational problems have become the multinational problems. This has generated a sense of cooperation and understanding about the basic needs of human species in the global community. In short, the idea of globalization is spreading in almost all aspects of the human species on the surface of earth.

The human mobility plays a very significant role in the globalization process[7]. Cultural changes and understandings, flow of ideas about the current events, the occurrence and endemicity of new infectious diseases of humans in new areas, the world events of disease pandemic, outsourcing jobs and resources, economics and environmental conditions are a few byproducts of human mobility. In fact the 1918-19 influenza pandemic [1, 2, 3, 4] and the sociocultural changes in societies [6] are a few illustrations exhibiting the movement of people.

Many studies regarding the mobility of the human species consider its impact in spreading infectious diseases between communities as a result of the movement of people, goods, vectors and animals across the globe. Different mathematical modeling techniques have been proposed to study the mobility. The discrete time difference equations in continuous space [8, 9] is used to study the global spread of influenza and the geographic spread of infectious diseases. The dynamics of diseases between two patches and a finite number of patches resulting from human dispersal among the patches are modeled by ordinary differential equations [42, 11, 12].

The effects of human movements among a finite number of patches on the persistence of vector-born diseases are described with ordinary differential equations[13].

Human mobility models are increasingly being used to evaluate and increase the efficiency, effectiveness and feasibility of network systems for mobile wireless devices. Finding a realistic human mobility model is a very important component of the study of network systems for mobile wireless devices. Using real network data captured from a campus situation[15] and the movements of pedestrians in downtown Osaka[14], models for mobile networks were designed. Also simulated mobile network models have been studied in [16, 17, 18, 19].

A large population exhibits structure at many scales. The movement of people within and between these scales affects the population dynamics and demography. We define scale here as a single level of interaction of people within the large population. For example, the population of human species in/at the home level, in/at a town level, city level and so on. In fact, a population is considered an n scales or n levels if there are $n-1$ levels nested in the n^{th} level or scale of the hierarchy. Also, in these n scales population, movement can occur between spatially separated patches of the same size or scale, beginning from patches of the $(n-1)^{th}$ scale down to patches of the lowest level or lowest scale in the hierarchy. In addition, an n scales movement can be reduced to a single scale if there is only one level movement between spatially separated patches of the same scale. For example, a country level in a population can be considered to be five scales with four nested levels namely: homes, neighborhoods, towns, counties and states in increasing order of nested scales. If movement occurs in this five scales structure between patches of the lowest-level group (homes), then we have a five scales movement. If the lowest scale is the neighborhoods, then we have a four scales movement. If only one level is considered in the country structure made up of patches of the same scale: homes only or neighborhoods only or towns only, then there is a single scale movement.

Many attempts have been made to describe human mobility in a metapopulation [8, 9, 42, 20].Several of these investigations characterize human movements on a single scale framework[8, 9, 42, 11, 12, 13, 16, 20, 22].A human mobility inter-geographical location model was designed to study mobile network devices [16]. Some studies of the spread of diseases in a structured population have considered the human mobility on multiple hierarchy of scales [21, 22, 23, 24]. Generally, we can categorize all models describing human mobility into two types of mobility approaches, namely the Lagrangian approach or the Euler approach in [13].

The Lagrangian approach labels individuals by home site and current location. The Euler approach only labels individuals by their current location.

In this work, we consider human mobility of a two scale population with a formulation that allows the possibility of considering the two forms of movement of people: permanent displacement (migration) and temporal displacement (visits) to patches within and between scales. The presented model allows the possibility of simultaneous study of the intra and inter scale temporal displacement of people in the structure. Hence, the model extends and generalizes the multiscale mobility models [21, 22, 23, 24] in a systematic and unified way. This two scaled structure, formulation of the mobility process provides an algorithmic framework to expand and extend the multiscales mobility process of the human species. The byproduct of this multiscale human mobility model would play a significant role in the study of mobile network wireless devices [16].

Of particular interest to our formulation is in the spirit of the single scale model by Sattenspiel and Dietz [25] that incorporates the Lagrangian and Eulerian approaches. The same model was used again by the authors to study the spread of the 1918-1919 influenza epidemic and later to investigate the effects of quarantine on the spread of the epidemic among the Cree and Metis people in the central Canadian Subarctic [26, 27]. By following the framework of the single scale model of Sattenspiel and Dietz [25], we extend and expand their model into two scales, a local and a global scale. The local scale is the sub-regional level consisting of a finite number of patches or subregions. At this local level, there is a transport network of residents between the patches. The global scale is the regional level consisting of a finite number of bigger patches or regions. Also at the global level, there is a transport network of residents between the regions.

This chapter is organized as follows; in Section 1.2 we describe the general mobility process, define our notations and state the assumptions of our model. We then present an explicit probabilistic formulation of the travel rates of our mobility dynamic model in section 1.3. Using a compartmental framework[28, 29], we derive a deterministic dynamical model for the mobility process described by ordinary differential equations. In Section 1.4 we give a detailed analysis of the general static mobility dynamic model structure. In addition, a description and the analysis of specific scenarios of the general dynamic mobility process at its steady state are outlined. Moreover, we compare the inter and intra regional visiting times on the distribution of the residents of a give region and draw a few conclusions.

1.2 Large-Scale Two Level Hierarchic Mobility Formulation Process

In this section, we introduce the idea of the mobility of human species in two level interconnected hierarchic population. This study can be applied to any two level interconnected hierarchic system. We define the following notations.

Definition 1.2.1

i. *Let M be a positive integer, $\mathbb{R}$ be the set of real numbers, $\bar{x} = (x_1, x_2) \in \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ be arbitrary and $y_r = (y_{1_r}, y_{2_r}) \in \mathbb{R}^2$ be fixed for all $r \in \{1, 2, \ldots M\} = I(1, M)$, $c_r > 0$. Also, let $||\bar{x}||_{\mathbb{R}^2} = \sqrt{x_1^2 + x_2^2}$. The open ball in $\mathbb{R}^2$*

$$B(y_r, c_r) = C_r = B_{\mathbb{R}^2}(y_r; c_r) = \{\bar{x} \in \mathbb{R}^2 : ||\bar{x} - y_r||_{\mathbb{R}^2} < c_r\}, \tag{1.2.1}$$

where for each $r, q \in I(1, M), r \neq q$, $C_r \cap C_q = \{\}$. Also, define $C = \bigcup_{r=1}^{M} C_r$.

ii. *Let $r \in I(1, M)$, and let n_r be a positive integer and let $s_i^r \in C_r, i \in \{1, 2, \ldots, n_r\} = I(1, n_r)$. For every $i, j \in I(1, n_r), i \neq j$, let $||s_i^r - s_j^r||_{\mathbb{R}^2} > 0$. Also, let $C_{rr}(s^r) = \{s_i^r \in C_r : i \in I(1, n_r)\}$ be a finite collection of the n_r distinct points in C_r. And $C(s) = \{s_i^r \in C_r : i \in I(1, n_r), r \in I(1, M)\} = \bigcup_{r=1}^{M} C_{rr}(s^r)$ be the finite collection of all distinct points in C. The cardinality of $C(s)$ is $n = \sum_{r=1}^{M} n_r$.*

Definition 1.2.2 ***Decomposition of Hierarchic Process:*** *Let us consider a population that is distributed into M distinct spatial regions $C_1, C_2, \ldots, C_M$. Each region $C_r, r \in I(1, M)$ consists of n_r distinct sites $s_1^r, s_2^r, \ldots, s_{n_r}^r$ spatially distributed within the region. Residents of sites in a region can either visit other sites within the region or visit sites in other regions.*

Definition 1.2.3 ***Population Decomposition and Aggregation Process:*** *Let N_{ii}^{rr} be the number of residents of site $s_i^r, i \in I(1, n_r)$ in region $C_r, r \in I(1, M)$ who are actually present in their home site at time t. Let N_{ij}^{rr} be the number of residents of $s_i^r, i, j \in I(1, n_r)$ in region $C_r, r \in I(1, M)$ visiting site s_j^r within the region. Let N_{il}^{rq} be the number of residents of $s_i^r, i \in I(1, n_r)$ in region $C_r, r, q \in I(1, M)$ visiting site $s_l^q, l \in I(1, n_q)$ in region $C_q, q \neq r$. Let N_{i0}^{rr} be the total number of residents of site s_i^r within C_r and visiting other regions, then*

$$N_{i0}^{rr} = \sum_{j=1}^{n_r} N_{ij}^{rr} + \sum_{q \neq r}^{M} \sum_{l=1}^{n_q} N_{il}^{rq}, i \in I(1, n_r). \tag{1.2.2}$$

Definition 1.2.4 ***Intra and Interregional Probabilistic visiting Rates:*** *For each $r \in I(1,M)$, residents of site s_i^r in region C_r leave on trips to other sites within the region at a per capita rate σ_i^r. The visitors then distribute themselves among the $n_r - 1$ sites s_j^r, $j \neq i$ with the probabilistic rate ν_{ij}^{rr}. Also, residents of site s_i^r in region C_r leave on trips to other regions at a per capita rate γ_i^r. The residents of site s_j^r that leave on trips to other regions distribute themselves among $M-1$ destinations with probability γ_{i0}^{rq} to region $C_q, q \neq r$. Collectively, the residents of sites in region C_r leave their region to visit other sites in region C_q with a grand total rate $\gamma^{rq} = \sum_{i=1}^{n_r} \gamma_{i0}^{rq}$. The residents that leave region C_r to visit sites in region C_q distribute themselves among n_q destinations with probability γ_{0l}^{rq} to site s_l^q, $l \in I(1,n_q)$.*

Definition 1.2.5 ***Inter regional Probabilistic Return Rates:*** *For each $r \in I(1,M)$, persons traveling from site s_i^r to s_j^r in region C_r, have a per capita probabilistic return rate ρ_{ij}^{rr}. Also, for each $q \in I(1,M), q \neq r$ residents from all other regions that came to site $s_l^q, l \in I(1,n_q)$ in region $C_q, q \neq r$, leave the site s_l^q to return to their home region with rate ρ_l^q. This rate ρ_l^q further distributes among the $M-1$ regions $C_r, r \neq q$ regions with probabilities ρ_{0l}^{rq}. Hence, the grand total per capita return rate of the residents of region C_r that came to the n_q sites in region $C^q, q \neq r$ is $\rho^{rq} = \sum_{l=1}^{n_q} \rho_{0l}^{rq}$. This return rate ρ^{rq} of residents of region C_r from region C_q then distribute among the n_r sites $s_i^r, i \in I(1,n_r)$ in region $C_r, r \neq q$ with the probability rate ρ_{i0}^{rq}.*

1.3 Probabilistic Mobility Dynamic Model Formulation Process

Here we define and derive our probabilistic rates at which residents leave and return to their home sites and regions. The probabilistic formulation of the return rates is similar to the visiting rates by virtue of the symmetry in these travel patterns. Therefore, below we give a detailed derivation of the visiting rates and refer this frame work for the derivation of the return rates.

In the following, we define the accessible domain of residents of site s_i^r in region C_r, which is composed of sites within the region C_r and also in other regions C_q, that are accessible to residents of site s_i^r. For this purpose, we introduce a few notions and definitions. For $r \in I(1,M)$, $i \in I(1,n_r)$, we define

$$I^r(1,M) = \{q \in I(1,M) : r \neq q, \gamma^{rq} > 0\}, I^r(1,M) \subseteq I(1,M). \tag{1.3.3}$$

$$I_i^r(1,n_r) = \{j \in I(1,n_r) : j \neq i, \nu_{ij}^{rr} > 0\}, I_i^r(1,n_r) \subseteq I(1,n_r). \tag{1.3.4}$$

Let $q \in I(1, M_r)$,

$$I_i^r(1, n_q) = \{l \in I(1, n_q) : \gamma^{rq} > 0, \quad and \quad \gamma_{0l}^{rq} > 0\}, I_i^r(1, n_q) \subseteq I(1, n_q). \tag{1.3.5}$$

Definition 1.3.1 ***Inter and Intra Regional Accessible Domain:*** *For each $i \in I(1, n_r)$ and $r \in I(1, M)$, $C_{rr}(s_i^r) = \{s_j^r \in C_{rr}(s^r) : j \in I_i^r(1, n_r)\}$ is the intra regional accessible domain of residents of site s_i^r in region C_r.*

For each $r \in I(1, M)$, $i \in I(1, n_r)$ and $q \in I^r(1, M)$, $C_{rq}(s_i^r) = \{s_l^q \in C_{qq}(s^q) : l \in I_i^r(1, n_q)\}$ is the inter regional accessible domain of residents of site s_i^r in region C_r.

Given $i \in I(1, n_r)$, $C(s_i^r) = \{s_j^q \in C(s) : j \in I_i^r(1, n_r), q \in I^r(1, M)\}$ $= \bigcup_{q \in I^r(1,M)} \bigcup_{j \in I_i^r(1,n_r)} C_{rq}(s_i^r) = [C_{rr}(s_i^r)] \cup [\bigcup_{q \in I^r(1,M)} C_{rq}(s_i^r)]$, is the aggregate inter and intra regional accessible domain of residents of site s_i^r in region C_r.

Definition 1.3.2 ***Intra Regional Visiting Rates:*** *Residents of site s_i^r leave on trips to other sites s_j^r within the region at a per capita rate $\sigma_{ij}^{rr} = \sigma_i^r \nu_{ij}^{rr}, i, j \in I(1, n_r), j \neq i$, where σ_i^r and ν_{ij}^{rr} are defined in the previous section.*

Indeed, let $N_{i0}^{rr}(C_{rr}(s_i^r))$ be the total number of residents of site s_i^r that leave the site to visit other sites in $C_{rr}(s_i^r)$. Furthermore, let $T_{i,total}^{rr}$ be the total time during which the visiting to sites in $C_{rr}(s_i^r)$ takes place.

$$\sigma_i^r = \frac{N_{i0}^{rr}(C_{rr}(s_i^r))}{N_{i0}^{rr} * T_{i,total}^{rr}} \tag{1.3.6}$$

Also, let E_i^r be the event that residents leave their site s_i^r to visit other sites in $C_{rr}(s_i^r)$, and let E_{ij}^{rr} be the event that residents leave their site s_i^r and visit site s_j^r. Then the intra regional probability visiting rates are given by

$$\begin{aligned} P(E_i^r) &= \sigma_i^r, \quad \nu_{ij}^{rr} = P(E_{ij}^{rr}|E_i^r) = \frac{N_{ij}^{rr}}{N_{i0}^{rr}(C_{rr}(s_i^r))}, \\ \sigma_{ij}^{rr} &= P(E_{ij}^{rr}) = P(E_{ij}^{rr}|E_i^r)P(E_i^r) = \sigma_i^r \nu_{ij}^{rr}. \end{aligned} \tag{1.3.7}$$

Definition 1.3.3 ***Inter Regional Visiting Rates:*** *Residents of site s_i^r in region $C_r, r \in I(1, M)$ leave on trips to other sites s_l^q in other regions $C_q, q \neq r$ at a per capita rate $\gamma_{il}^{rq} = \gamma_i^r \gamma_{i0}^{rq} \gamma^{rq} \gamma_{0l}^{rq}, i \in I(1, n_r), l \in I(1, n_q), r \neq q$.*

In fact, it can be justified as follows: for all $q \in I^r(1,M)$, let $N_{i0}^{rq}(C_{rq}(s_i^r))$ be the total number of residents of site s_i^r that leave the site to visit other sites in $C_{rq}(s_i^r)$ and let F_i^r be the event representing this movement. In addition, we let $T_{i,total}^{rq}$ be the total time during which the visiting to sites in $C_{rq}(s_i^r)$ takes place. Then

$$P(F_i^r) = \gamma_i^r = \frac{\sum_{t \in I^r(1,M)} N_{i0}^{rt}(C_{rt}(s_i^r))}{N_{i0}^{rr} * \sum_{t \in I^r(1,M)} T_{i,total}^{rt}}. \tag{1.3.8}$$

Now for each $q \in I(1,M)$, and $l \in I_i^r(1,n_q)$, let F_{i0}^{rq} be the event that residents leave the site s_i^r, and go to region C_q (the specific destination in C_q is not taken into account at this point). Furthermore, let F^{rq} be the event that residents coming from region C_r, go to region Cq, and F_{0l}^{rq} be the event that the residents coming from C_r to region C_q, go to site s_l^q in C_q (the site of origin in C_r is not taken into account at this point). Then we can formulate the conditional probability rates as follows

$$P(F_{i0}^{rq}|F_i^r) = \gamma_{i0}^{rq} = \frac{N_{i0}^{rq}(C_{rq}(s_{i0}^r))}{\sum_{t \in I^r(1,M)} N_{i0}^{rt}(C_{rt}(s_i^r))}, \tag{1.3.9}$$

$$P(F^{rq}) = \gamma^{rq} = \sum_{i=1}^{n_r} P(F_{i0}^{rq}|F_i^r) = \sum_{i=1}^{n_r} \gamma_{i0}^{rq}. \tag{1.3.10}$$

$$P(F_{0l}^{rq}|F^{rq}) = \gamma_{0l}^{rq} = \frac{N_{il}^{rq}}{N_{i0}^{rq}(C_{rq}(s_i^r))}. \tag{1.3.11}$$

Therefore given that F_{il}^{rq} is the event that residents of site s_i^r in region C_r travel to site s_l^q in region C_q, then $F_{il}^{rq} = F_{i0}^{rq} \cap F_{0l}^{rq}$, where F_{i0}^{rq} and F_{0l}^{rq} are independent events. This is because sites of origin in region C_r of residents from region C_r that travel to region C_q is not taken into account when they arrive at site s_i^q; that is, F_{0l}^{rq} is independent of s_i^r. Also, the destination in other regions C_q, of residents of site s_i^r in region C_r is not taken into account when defining F_{i0}^{rq}.

$$\begin{aligned} \gamma_{il}^{rq} &= P(F_{il}^{rq}) = P(F_{i0}^{rq} \cap F_{0l}^{rq}) = P(F_{i0}^{rq}) * P(F_{0l}^{rq}) = P(F_{i0}^{rq}|F_i^r)P(F_i^r) * P(F_{0l}^{rq}|F^{rq})P(F^{rq}) \\ &= \gamma_i^r \gamma_{i0}^{rq} \gamma^{rq} \gamma_{0l}^{rq}. \end{aligned} \tag{1.3.12}$$

Definition 1.3.4 ***Intra Regional Return Rates:*** *Persons traveling from site s_i^r to s_j^r within a region $C_r, r \in I(1,M)$ have a per capita return rate $\rho_{ij}^{rr}, i, j \in I(1,n_r), i \neq j$.*

Definition 1.3.5 ***Inter Regional Return Rates:*** *Persons traveling from site s_i^r in region C_r to site s_l^q in region $C_q, q \neq r$ have a per capita return rate $\rho_{il}^{rq} = \rho_{0l}^{rq}\rho_l^q\rho_{i0}^{rq}\rho^{rq}, i \in I(1,n_r), l \in I(1,n_q), r \neq q$. The large two-scale hierarchic mobility structure is illustrated in Figure 1.1 and Figure 1.2.*

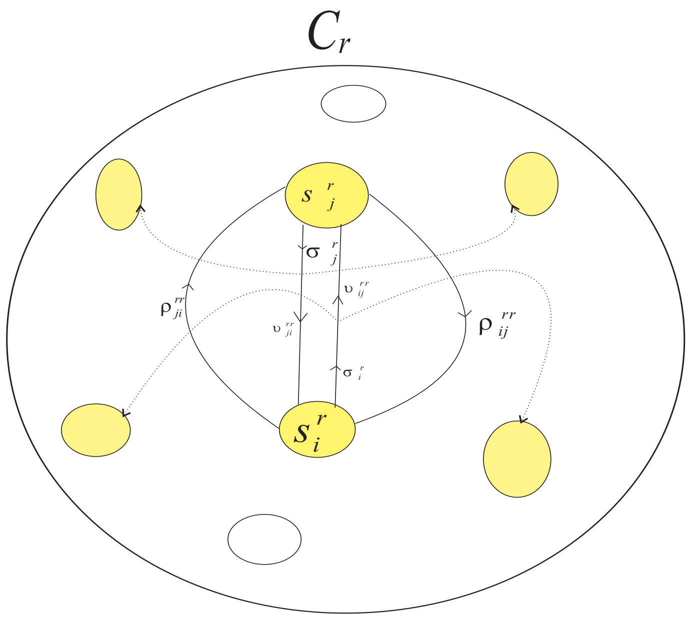

Figure 1.1: shows the intra-regional mobility network between n_r sites in $C_r, r \in I(1,M)$. Dotted lines and curves represent connections with other sites in region C_r. Furthermore, the parameters in the diagram are defined in Section 1.3.

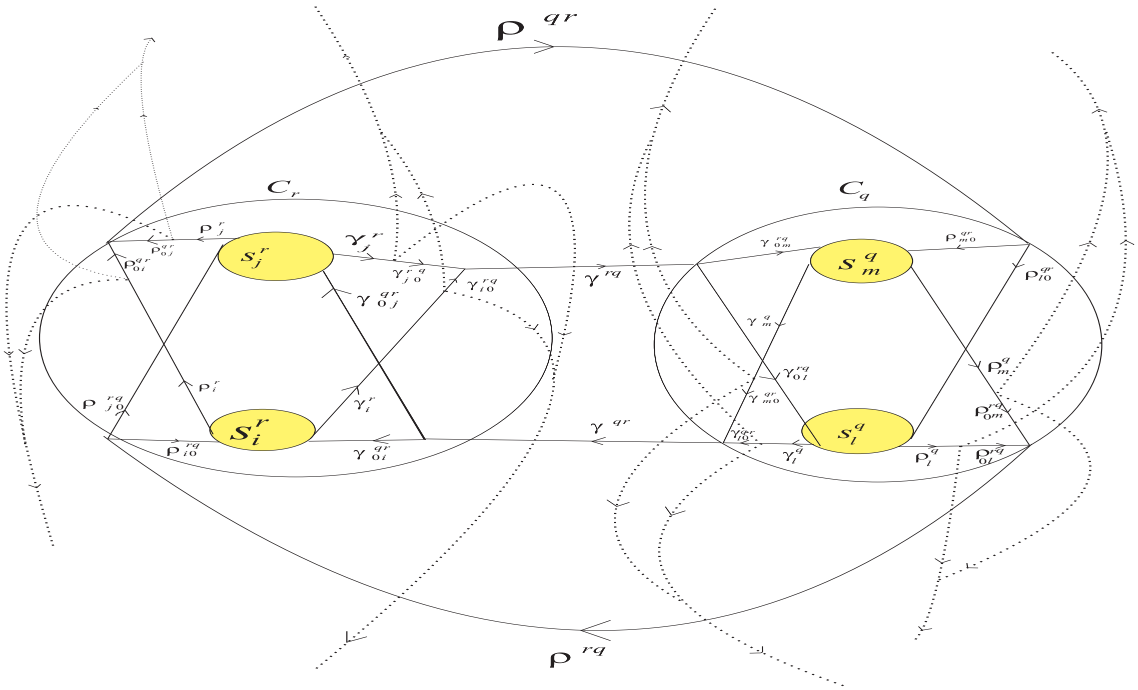

Figure 1.2: shows the interregional mobility network between M regions $C_r, r \in I(1,M)$ and n_r sites that are present in each region C_r. Dotted lines and curves represent connections with other sites in other regions. The parameters are defined in Section 1.3.

Using the above defined mobility rates, the travel pattern of individuals among all sites and all regions leads to the following large-scale interconnected linear system of differential equations

$$\begin{aligned}
\frac{dN_{ii}^{rr}}{dt} &= \sum_{k=1}^{n_r} \rho_{ik}^{rr} N_{ik}^{rr} + \sum_{q \neq r}^{M} \sum_{l=1}^{n_q} \rho_{il}^{rq} N_{il}^{rq} - (\gamma_i^r + \sigma_i^r) N_{ii}^{rr}, && (1.3.13)\\
\frac{dN_{ij}^{rr}}{dt} &= \sigma_{ij}^{rr} N_{ii}^{rr} - \rho_{ij}^{rr} N_{ij}^{rr}, i \neq j, && (1.3.14)\\
\frac{dN_{il}^{rq}}{dt} &= \gamma_{il}^{rq} N_{ii}^{rr} - \rho_{il}^{rq} N_{il}^{rq}, r \neq q, && (1.3.15)\\
& \quad i \in I(1,n_r), l \in I_i^r(1,n_q); r,q \in I^r(1,M),
\end{aligned}$$

where all the parameters in (1.3.13)-(1.3.15), are nonnegative and at time $t = 0$, $N_{ii}^{rr}(0) = N_{i0}^{rr}$, $N_{ij}^{rr}(0) = 0$ and $N_{il}^{rq}(0) = 0$. And N_{ii}^{rr}, N_{ij}^{rr} and N_{il}^{rq} $i, j \in I_i^r(1,n_r), l \in I_i^r(1,n_q), r,q \in I(1,M)$ are as defined before.

Remark 1.3.1 *It is important to note that residents of every site $s_i^r, i \in I(1,n_r)$ can only reach out to other sites in their accessible domain $C(s_i^r)$. Thus the summations in (1.3.13) reduce to summation over all $q \in I^r(1,M)$, $k \in I_i^r(1,n_r)$ and $l \in I_i^r(1,n_q)$. Keeping this in mind, for easy presentation we keep the current expressions.*

In the following we analyze the steady states of the mobility process determined by the system of differential equations. The analysis of this section also gives the equilibrium states of a general mobility system whose sites and regions are connected. In real life many mobility patterns that occur frequently, are specific scenarios of this general mobility process. In the following we shall consider a few of these cases.

We denote the equilibrium states of N_{ii}^{rr}, N_{ij}^{rr} and N_{il}^{rq} by N_{ii}^{*rr}, N_{ij}^{*rr} and N_{il}^{*rq}, respectively. Hence at the equilibrium, we have $\frac{dN_{ii}^{rr}}{dt} = 0$, $\frac{dN_{ij}^{rr}}{dt} = 0$ and $\frac{dN_{il}^{rq}}{dt} = 0$. Therefore, setting $(1.3.13)$, $(1.3.14)$ and $(1.3.15)$ to zero, one can see that

$$N_{ij}^{*rr} = \frac{\sigma_{ij}^{rr} N_{ii}^{*rr}}{\rho_{ij}^{rr}}, i \neq j, \tag{1.3.16}$$

$$N_{il}^{*rq} = \frac{\gamma_{il}^{rq} N_{ii}^{*rr}}{\rho_{il}^{rq}}, r \neq q, \tag{1.3.17}$$

$$i, j \in I(1,n_r), l \in I(1,n_q); r,q \in I(1,M).$$

We rewrite (1.2.2) in terms of steady states, then we have

$$N_{i0}^{*rr} = N_{ii}^{*rr} + \sum_{k \neq i}^{n_r} N_{ik}^{*rr} + \sum_{q \neq r}^{M} \sum_{l=1}^{n_q} N_{il}^{*rq}, i \in I(1,n_r). \tag{1.3.18}$$

Now substituting $(1.3.16)$ and $(1.3.17)$ into $(1.3.18)$ and factorizing N_{ii}^{*rr}, we have

$$N_{i0}^{*rr} = N_{ii}^{*rr}\big(1 + \sum_{k \neq i}^{n_r} \frac{\sigma_{ik}^{rr}}{\rho_{ik}^{rr}} + \sum_{q \neq r}^{M} \sum_{l=1}^{n_q} \frac{\gamma_{il}^{rq}}{\rho_{il}^{rq}}\big), i \in I(1,n_r). \tag{1.3.19}$$

From (1.3.19), we have

$$N_{ii}^{*rr} = N_{i0}^{*rr}\big(1 + \sum_{k \neq i}^{n_r} \frac{\sigma_{ik}^{rr}}{\rho_{ik}^{rr}} + \sum_{q \neq r}^{M} \sum_{l=1}^{n_q} \frac{\gamma_{il}^{rq}}{\rho_{il}^{rq}}\big)^{-1}, i \in I(1,n_r). \tag{1.3.20}$$

Now substituting (1.3.20) into (1.3.16) and (1.3.17), N_{ij}^{*rr} and N_{il}^{*rq} are represented by

$$N_{ij}^{*rr} = N_{i0}^{*rr} \frac{\frac{\sigma_{ij}^{rr}}{\rho_{ij}^{rr}}}{\left(1+\sum_{k\neq i}^{n_r} \frac{\sigma_{ik}^{rr}}{\rho_{ik}^{rr}} + \sum_{q\neq r}^{M} \sum_{l=1}^{n_q} \frac{\gamma_{il}^{rq}}{\rho_{il}^{rq}}\right)}, j \in I_i^r(1,n_r), \quad and \quad N_{ij}^{*rr} = 0 \quad otherwise, \quad (1.3.21)$$

and

$$N_{il}^{*rq} = N_{i0}^{*rr} \frac{\frac{\gamma_{il}^{rq}}{\rho_{il}^{rq}}}{\left(1+\sum_{k\neq i}^{n_r} \frac{\sigma_{ik}^{rr}}{\rho_{ik}^{rr}} + \sum_{q\neq r}^{M} \sum_{l=1}^{n_q} \frac{\gamma_{il}^{rq}}{\rho_{il}^{rq}}\right)}, l \in I_i^r(1,n_q) \quad and \quad N_{il}^{*rq} = 0, \quad otherwise. \quad (1.3.22)$$

Let us denote

$$S_{ii}^{*rr} = \left(1+\sum_{k\neq i}^{n_r} \frac{\sigma_{ik}^{rr}}{\rho_{ik}^{rr}} + \sum_{q\neq r}^{M} \sum_{l=1}^{n_q} \frac{\gamma_{il}^{rq}}{\rho_{il}^{rq}}\right)^{-1}, i \in I(1,n_r), \quad (1.3.23)$$

$$U_{ij}^{*rr} = \frac{\frac{\sigma_{ij}^{rr}}{\rho_{ij}^{rr}}}{\left(1+\sum_{k\neq i}^{n_r} \frac{\sigma_{ik}^{rr}}{\rho_{ik}^{rr}} + \sum_{q\neq r}^{M} \sum_{l=1}^{n_q} \frac{\gamma_{il}^{rq}}{\rho_{il}^{rq}}\right)}, j \in I_i^r(1,n_r), \quad and \quad U_{ij}^{*rr} = 0, \quad otherwise, \quad (1.3.24)$$

$$V_{il}^{*rq} = \frac{\frac{\gamma_{ij}^{rq}}{\rho_{il}^{rq}}}{\left(1+\sum_{k\neq i}^{n_r} \frac{\sigma_{ik}^{rr}}{\rho_{ik}^{rr}} + \sum_{q\neq r}^{M} \sum_{l=1}^{n_q} \frac{\gamma_{il}^{rq}}{\rho_{il}^{rq}}\right)}, l \in I_i^r(1,n_q), \quad and \quad V_{il}^{*rq} = 0, \quad otherwise. \quad (1.3.25)$$

The quantities in (1.3.20), (1.3.21) and (1.3.22) represent the equilibrium sizes of residents of site s_i^r, present at their home site, visiting the j^{th} site $s_j^r, j \neq i$ in their intra regional accessible domain $C_{rr}(s_i^r)$, and also visiting the l^{th} site s_l^q in their inter regional accessible domain $C_{rq}(s_i^r)$, respectively. Thus it follows that the quantities in (1.3.23), (1.3.24) and (1.3.25) represent the fraction of the equilibrium size of residents in the different categories present at the corresponding locations.

We further observe that for each $r,q \in I(1,M)$ and $r \neq q$, persons traveling from site s_i^r to s_j^r have a per capital return rate ρ_{ij}^{rr}, and persons traveling from site s_i^r in region C_r to site s_l^q in region $C_q, q \neq r$ have a per capita return rate ρ_{il}^{rq}. The average length of time spent visiting site s_j^r and site s_l^q is denoted by τ_{ij}^{rr} and τ_{il}^{rq}, respectively, where

$$\tau_{ij}^{rr} = \frac{1}{\rho_{ij}^{rr}}, j \in I_i^r(1,n_r), r \in I(1,M) \quad and \quad \tau_{ij}^{rr} = 0 \quad otherwise. \quad (1.3.26)$$

$$\tau_{il}^{rq} = \frac{1}{\rho_{il}^{rq}}, l \in I_i^r(1,n_q), q \in I^r(1,M), \quad and \quad \tau_{il}^{rq} = 0 \quad otherwise, \quad (1.3.27)$$

In addition, given $r \in I(1,M)$ and $i \in I(1,n_r)$, for every $s,t \in I^r(1,M)$, we let

$$K_{imn}^{rst} \quad \equiv \quad K_{imn}^{rst}(\tau_{im}^{rs},\tau_{in}^{rt}) = \frac{\tau_{im}^{rs}}{\tau_{in}^{rt}}, m \in I_i^r(1,n_s), n \in I_i^r(1,n_t) \tag{1.3.28}$$

be the ratio of visiting times of residents of the i^{th} site s_i^r in region C_r visiting the m^{th} and n^{th} sites in region C_s and C_t respectively, where $m \neq n$, and $m,n \neq i$ whenever $s = t = r$. Now, substituting (1.3.26), (1.3.27) and (1.3.28) into (1.3.23), (1.3.24) and (1.3.25) and further simplifying we have the following

$$S_{ii}^{*rr} = \frac{1}{\left(1 + \sum_{k\neq i}^{n_r} \sigma_{ik}^{rr}\tau_{ik}^{rr} + \sum_{q\neq r}^{M}\sum_{a=1}^{n_q}\gamma_{ia}^{rq}\tau_{ia}^{rq}\right)}, i \in I(1,n_r), \tag{1.3.29}$$

$$U_{ij}^{*rr} = \frac{\sigma_{ij}^{rr}}{\left(\frac{1}{\tau_{ij}^{rr}} + \sum_{k\neq i}^{n_r} \sigma_{ik}^{rr}K_{ikj}^{rrr} + \sum_{q\neq r}^{M}\sum_{a=1}^{n_q}\gamma_{ia}^{rq}K_{iaj}^{rqr}\right)}, j \in I_i^r(1,n_r), \tag{1.3.30}$$

and

$$V_{il}^{*rq} = \frac{\gamma_{il}^{rq}}{\left(\frac{1}{\tau_{il}^{rq}} + \sum_{k\neq i}^{n_r} \sigma_{ik}^{rr}K_{ikl}^{rrq} + \sum_{q\neq r}^{M}\sum_{a=1}^{n_q}\gamma_{ia}^{rq}K_{ial}^{rqq}\right)}, l \in I_i^r(1,n_q). \tag{1.3.31}$$

We define for each $i \in I(1,n_r), r \in I(1,M)$,

$$\tau_{i,min}^{rr} \quad = \quad \min_{1\leq j\leq n_r} \tau_{ij}^{rr}, \tag{1.3.32}$$

$$\tau_{i,min}^{rq} \quad = \quad \min_{1\leq l\leq n_q} \tau_{il}^{rq}, q \in I^r(1,M), \tag{1.3.33}$$

$$\tau_{i,min}^{r,min} \quad = \quad \min_{q\neq r} \tau_{i,min}^{rq}, \tag{1.3.34}$$

$$\tau_{i,max}^{rr} \quad = \quad \max_{1\leq j\leq n_r} \tau_{ij}^{rr} \tag{1.3.35}$$

$$\tau_{i,max}^{rq} \quad = \quad \max_{1\leq l\leq n_q} \tau_{il}^{rq}, q \in I^r(1,M) \tag{1.3.36}$$

$$\tau_{i,max}^{r,max} \quad = \quad \max_{q\neq r} \tau_{i,max}^{rq}, \tag{1.3.37}$$

1.4 Special Mobility Patterns

The special mobility patterns are characterized by the qualitative behavior of the mobility rates of the large-scale hierarchic regional mobility dynamics process. In order to understand the mobility patterns we need to classify the qualitative behavior of the mobility rates. We define the following.

Definition 1.4.1 *Given two real valued functions f and g,*

1. *if $\exists k > 0$, and n_0, such that $\forall n > n_0, |f(n)| \leq k|g(n)|$, we say that f is big-o of g, and is denoted by $f(n) = 0(g(n))$ or $f = 0(g)$. If $f(n) \to 0$, as $n \to \infty$, that is, f turns in the limit to a zero function for sufficiently large n, we write $f = 0(\varepsilon)$ or $f(n) = 0(\frac{1}{n})$, for $\varepsilon > 0$. If $f(n)$ is a constant function as $n \to \infty$, we write $f(n) = 0(1)$.*

2. *if $\exists k_1, k_2 > 0$, and n_0, such that $\forall n > n_0, k_1|g(n)| \leq |f(n)| \leq k_2|g(n)|$, we say that f is big-theta of g, and is denoted by $f(n) = \theta(g(n))$. If $f(n) \to \infty$ as $n \to \infty$, we write $f(n) = \theta(n)$ or $f = \theta(\frac{1}{\varepsilon})$, for $\varepsilon > 0$.*

To classify the qualitative behavior of the mobility rates we make the following assumptions about the mobility rate functions. Assume that for fixed $r \in I(1,M)$ and $i \in I(1,n_r)$, and for any $q \in I^r(1,M)$, $j \in I_i^r(1,n_r)$ and $l \in I_i^r(1,n_q)$, the inter and intra regional visiting times of residents of the i^{th} site visiting the j^{th} site within the r^{th} region, and the i^{th} site visiting l^{th} site in the q^{th} region satisfy

Hypothesis 1.4.1 *Using Definition 1.4.1, we assume that*

H_1: $\tau_{ij}^{rr} \to 0$ and $\tau_{il}^{rq} < \infty \iff \tau_{ij}^{rr} = 0(\varepsilon)$ and $\tau_{il}^{rq} = 0(1)$;

H_2: $\tau_{il}^{rq} \to 0$ and $\tau_{ij}^{rr} < \infty \iff \tau_{il}^{rq} = 0(\varepsilon)$ and $\tau_{ij}^{rr} = 0(1)$;

H_3: $\tau_{ij}^{rr} < \infty$ and $\tau_{il}^{rq} < \infty \iff \tau_{il}^{rq} = 0(1)$ and $\tau_{ij}^{rr} = 0(1)$;

H_4: $\tau_{ij}^{rr} \to 0$ and $\tau_{il}^{rq} \to 0 \iff \tau_{il}^{rq} = 0(\varepsilon)$ and $\tau_{ij}^{rr} = 0(\varepsilon)$;

H_5: $\tau_{ij}^{rr} \to \infty$ and $\tau_{il}^{rq} \to \infty \iff \tau_{il}^{rq} = \theta(\frac{1}{\varepsilon})$ and $\tau_{ij}^{rr} = \theta(\frac{1}{\varepsilon})$, for $\varepsilon > 0$.

The interpretation of H_1 is that residents of site s_i^r that visit sites in their intra regional accessible domain $C_{rr}(s_i^r)$, tend to spend infinitesimally small time, while residents of the same site that visit sites in their inter regional accessible domain $C_{rq}(s_i^r)$, tend to spend a finite amount of time. The interpretation of H_2 and H_3 is similar to H_1. On the other hand, H_4 means that residents of site s_i^r that visit sites in their inter and intra regional accessible domain $C(s_i^r)$, tend to spend infinitesimally small time. Finally, H_5 means that residents of site s_i^r that visit sites in their inter and intra regional accessible domain $C(s_i^r)$, tend to stay permanently. The following result is a characterization of the steady states as intra regional visiting time approaches zero and the inter regional visiting time is finite.

Theorem 1.4.1 *If H_1 holds, then the steady state of the mobility system satisfies*

$$\begin{aligned} N_{ii}^{*rr} &\rightarrow N_i^{rr}\frac{1}{\left(1+\sum_{q\neq r}^{M}\sum_{a=1}^{n_q}\gamma_{ia}^{rq}\tau_{ia}^{rq}\right)}, \quad N_{ij}^{*rr}\rightarrow 0, \\ N_{il}^{*rq} &\rightarrow N_i^{rr}\frac{\gamma_{il}^{rq}}{\left(\frac{1}{\tau_{il}^{rq}}+\sum_{q\neq r}^{M}\sum_{a=1}^{n_q}\gamma_{ia}^{rq}K_{ial}^{rqq}\right)}. \end{aligned} \tag{1.4.38}$$

for $j\in I_i^r(1,n_r), q\in I^r(1,M), l\in I_i^r(1,n_q)$.

Proof: Letting $\tau_{ij}^{rr}=0(\varepsilon)$ and $\tau_{il}^{rq}=0(1)$ in (1.3.29), (1.3.30) and (1.3.31), we have

$$\begin{aligned} S_{ii}^{*rr} &\rightarrow \frac{1}{\left(1+\sum_{q\neq r}^{M}\sum_{a=1}^{n_q}\gamma_{ia}^{rq}\tau_{ia}^{rq}\right)}, \quad U_{ij}^{*rr}\rightarrow 0 \\ V_{il}^{*rq} &\rightarrow \frac{\gamma_{il}^{rq}}{\left(\frac{1}{\tau_{il}^{rq}}+\sum_{q\neq r}^{M}\sum_{a=1}^{n_q}\gamma_{ia}^{rq}K_{ial}^{rqq}\right)}. \end{aligned} \tag{1.4.39}$$

for $l\in I_i^r(1,n_q)$. Then substituting (1.4.39) into (1.3.20), (1.3.21) and (1.3.22) gives us (1.4.38).

Remark 1.4.1 *Theorem 1.4.1 suggests that a fraction of residents tend to be life long residents of their home site and the remaining fraction of residents tend to reside a finite time to the sites in the inter regional accessible domain $C_{rq}(s_i^r)$. This situation is illustrated in Figure 1.3.*

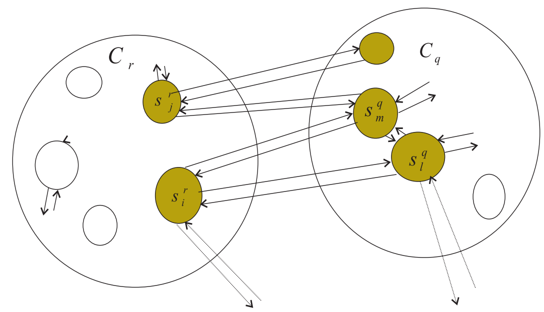

Figure 1.3: Shows that residents of site s_i^r are only present at their home site and also at those sites s_l^q that are in their interregional accessible domain $C_{rq}(s_i^r)$. The arrows represent a transport network between any two sites and regions. Furthermore, the dotted lines and arrows indicate a connection with other accessible sites in other regions.

In the next theorem we characterize the steady states when there is finite intra regional and short inter regional visiting time.

Theorem 1.4.2 *If H_2 holds, then the steady state of the mobility system satisfies*

$$N_{ii}^{*rr} \to N_i^{rr} \frac{1}{\left(1+\sum_{k\neq i}^{n_r} \sigma_{ik}^{rr}\tau_{ik}^{rr}\right)}, N_{il}^{*rq} \to 0 \quad and \quad N_{ij}^{*rr} \to N_i^{rr} \frac{\sigma_{ij}^{rr}}{\left(\frac{1}{\tau_{ij}^{rr}} + \sum_{k\neq i}^{n_r} \sigma_{ik}^{rr} K_{ikj}^{rrr}\right)}, \tag{1.4.40}$$

for $j \in I_i^r(1, n_r(s_i^r)), l \in I_i^r(1, n_q(s_i^r))$.

Proof: Similar to Theorem 1.4.2.

Remark 1.4.2 *Theorem 1.4.2 suggest that a fraction of residents tend to be life long residents of the home sites and the remaining fraction of the residents tend to stay for a finite time to sites in their intra regional accessible domain $C_{rr}(s_i^r)$. This special pattern is illustrated in Figure 1.4.*

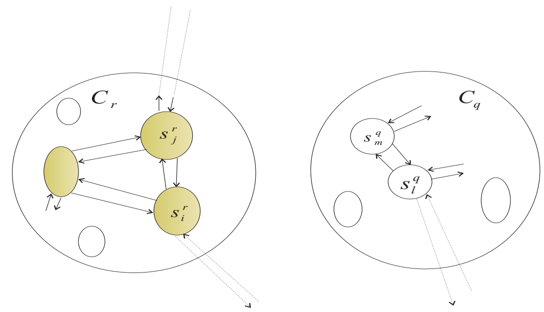

Figure 1.4: shows that residents of site s_i^r are present at their home site, at sitesin their intra and inter-regional accessible domain $C(s_i^r)$ whenever the average interregional visiting times are sufficiently small. The arrows represent a transport network between any two sites and regions. Furthermore, as average interregional visiting times tends to zero, (a) the solid lines represent the intraregional mobility of the equilibrium states of the residents which approaches to finite fractions, and (b) the dotted lines indicate interregional mobility of the equilibrium states of residents that approaches to arbitrarily small fractions.

We now characterize the steady states of the system when the inter and intra regional visiting times are constant and the same.

Theorem 1.4.3 *If H_3 holds and $\tau_{ij}^{rr} = \tau_{il}^{rq} = \tau_i^r, \exists \tau > 0$, then the steady states of the mobility system satisfy*

$$\begin{aligned} N_{ii}^{*rr} &\to N_i^{rr} \frac{1}{\left(1+\tau\sigma_i^r+\tau\gamma_i^r\gamma_{i0}^{rq}\right)}, \quad N_{ij}^{*rr} \to N_i^{rr} \frac{\sigma_{ij}^{rr}\tau}{\left(1+\tau\sigma_i^r+\tau\gamma_i^r\gamma_{i0}^{rq}\right)}, \\ N_{il}^{*rq} &\to N_i^{rr} \frac{\gamma_{il}^{rq}}{\left(1+\tau\sigma_i^r+\tau\gamma_i^r\gamma_{i0}^{rq}\right)} \end{aligned} \tag{1.4.41}$$

for $j \in I_i^r(1,n_r), q \in I^r(1,M), l \in I_i^r(1,n_q)$.

Proof: Letting $\tau_{ij}^{rr} = \tau_{il}^{rq} = \tau_i^r$ in (1.3.29), (1.3.30) and (1.3.31) and remembering that $\sigma_{ij}^{rr} = \sigma_i^r \nu_{ij}^{rr}$, $\gamma_{il}^{rq} = \gamma_i^r \gamma_{i0}^{rq} \gamma^{rq} \gamma_{0l}^{rq}, j \in I_i^r(1,n_r), l \in I_i^r(1,n_q)$, are probabilities with sum equal one, that is $\sum_{k\neq i}^{n_r} \nu_{ik}^{rr} = 1$ and $\sum_{q\neq r}^{M} \sum_{l=1}^{n_q} \gamma^{rq}\gamma_{0l}^{rq} = 1$. Thus we have the following reduced formula

$$\begin{aligned} S_{ii}^{*rr} &= \frac{1}{\left(1+\tau_i^r\sigma_i^r+\tau_i^r\gamma_i^r\gamma_{i0}^{rq}\right)}, \quad U_{ij}^{*rr} = \frac{\sigma_{ij}^{rr}\tau_i^r}{\left(1+\tau_i^r\sigma_i^r+\tau_i^r\gamma_i^r\gamma_{i0}^{rq}\right)}, \\ V_{il}^{*rq} &= \frac{\gamma_{il}^{rq}}{\left(1+\tau_i^r\sigma_i^r+\tau_i^r\gamma_i^r\gamma_{i0}^{rq}\right)}. \end{aligned} \tag{1.4.42}$$

for $j \in I_i^r(1,n_r), l \in I_i^r(1,n_q), q \in I_i^r(1,M)$. Hence substituting (1.4.42) into (1.3.20), (1.3.21) and (1.3.22) gives (1.4.41).

Remark 1.4.3 *Under the assumption of Theorem 1.4.3, almost all the residents of any site s_i^r, tend to be permanent residents at their home site s_i^r, and at all site in their intra and inter regional accessible domain $C(s_i^r)$, for finite time. Also, the fraction of the residents that reside a finite time at a given site in $C(s_i^r)$, is primarily determined by the probabilistic rate at which the residents leave their original home site s_i^r, to visit other sites in this domain $C(s_i^r)$. This special mobility pattern is illustrated in Figure 1.5.*

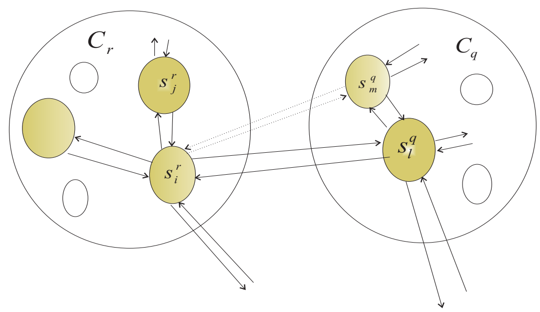

Figure 1.5: Shows that residents of site s_i^r are present at all sites in their inter and intra regional accessible domain $C(s_i^r)$. The arrows represent a transport network between any two sites and regions. The dotted lines and arrows indicate connection with other sites in other regions.

We now consider the case when intra and inter regional visiting times are decreasing at the same rate.

Theorem 1.4.4 *If H_4 holds, then the steady states of the mobility system satisfies*

$$N_{ii}^{*rr} \to N_i^{rr}, \quad N_{ij}^{*rr} \to 0 \quad and \quad N_{il}^{*rq} \to 0. \tag{1.4.43}$$

for $j \in I_i^r(1,n_r), q \in I^r(1,M), l \in I_i^r(1,n_q)$.

Proof: Letting $\tau_{ij}^{rr} \to 0$ and $\tau_{il}^{rq} \to 0$, in (1.3.29), (1.3.30) and (1.3.31). This leads us to

$$S_{ii}^{*rr} \to 1, \quad U_{ij}^{*rr} \to 0 \quad and \quad V_{il}^{*rq} \to 0, l \in I_i^r(1,n_q), j \in I_i^r(1,n_r). \tag{1.4.44}$$

Substituting (1.4.44) into (1.3.20), (1.3.21) and (1.3.22) gives us (1.4.43).

Remark 1.4.4 *Theorem 1.4.4 suggests the residents of site s_i^r are isolated from all other sites in their intra and inter regional accessible domain $C(s_i^r)$. That is, all the residents of site s_i^r tend to be life long stationary residents at their home site. However this does not mean site s_i^r is isolated from visitors from other sites in $C(s_i^r)$. This special case is illustrated in Figure 1.6.*

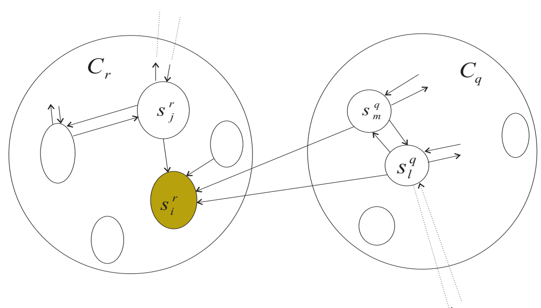

Figure 1.6: Shows that residents of site s_i^r are present only at their home site s_i^r. Hence they isolate every site from their inter and intra reginal accessible domain $C(s_i^r)$. Site s_i^r is a 'sink' in the context of the compartmental system. The arrows represent a transport network between any two sites and regions. Furthermore, the dotted lines and arrows indicate connection with other sites in regions.

We now characterize the steady states when the intra and inter regional visiting times grow unboundedly at the same rate.

Theorem 1.4.5 *Let H_5 holds. Let $\tau_{il}^{rq} = \theta(f_0), \tau_{ij}^{rr} = \theta(g_0)$ where f_0, and g_0 are positive real valued functions satisfy $f_0, g_0 = \theta(\frac{1}{\varepsilon})$. For $j \in I_i^r(1,n_r), q \in I^r(1,M), l \in I_i^r(1,n_q)$.*

1. *If $\theta(f_0) = \theta(g_0)$, then the steady state of the mobility system satisfies*

$$N_{ii}^{*rr} \to 0, \quad N_{ij}^{*rr} \to N_i^{*rr} \frac{\sigma_{ij}^{rr}}{\left(\sum_{k\neq i}^{n_r} \sigma_{ik}^{rr} K_{ikj}^{rrr} + \sum_{q\neq r}^{M} \sum_{a=1}^{n_q} \gamma_{ia}^{rq} K_{iaj}^{rqr}\right)},$$
$$\textit{and} \quad N_{il}^{*rq} \to N_i^{*rr} \frac{\gamma_{il}^{rq}}{\left(\sum_{k\neq i}^{n_r} \sigma_{ik}^{rr} K_{ikl}^{rrq} + \sum_{q\neq r}^{M} \sum_{a=1}^{n_q} \gamma_{ia}^{rq} K_{ial}^{rqq}\right)}. \tag{1.4.45}$$

2. *If $\theta(f_0) > \theta(g_0)$, then*

$$N_{ii}^{*rr} \to 0, \quad N_{ij}^{*rr} \to 0, \quad N_{il}^{*rq} \to N_i^{*rr} \frac{\gamma_{il}^{rq}}{\left(\sum_{q\neq r}^{M} \sum_{a=1}^{n_q} \gamma_{ia}^{rq} K_{ial}^{rqq}\right)}. \tag{1.4.46}$$

3. *If $\theta(f_0) < \theta(g_0)$, then*

$$N_{ii}^{*rr} \to 0, \quad N_{ij}^{*rr} \to N_i^{*rr} \frac{\sigma_{ij}^{rr}}{\left(\sum_{k\neq i}^{n_r} \sigma_{ik}^{rr} K_{ikj}^{rrr}\right)}, \quad N_{il}^{*rq} \to 0. \tag{1.4.47}$$

Proof: For $j \in I_i^r(1,n_r), q \in I^r(1,M), l \in I_i^r(1,n_q)$, $\theta(K_{ilj}^{rqr}) = \frac{\theta(f_0)}{\theta(g_0)}$. If $\theta(f_0) = \theta(g_0)$ then $K_{ilj}^{rqr} = 0(1)$ and $K_{ijl}^{rrq} = 0(1)$ as $\tau_{ij}^{rr} \to \infty$ and $\tau_{il}^{rq} \to \infty$. If $\theta(f_0) > \theta(g_0)$ then $K_{iaj}^{rqr} \to \infty$ and $K_{ijl}^{rrq} \to 0$ as $\tau_{ij}^{rr} \to \infty$ and $\tau_{il}^{rq} \to \infty$. And if If $\theta(f_0) < \theta(g_0)$ then $K_{ilj}^{rqr} \to 0$ and $K_{ijl}^{rrq} \to \infty$ as $\tau_{ij}^{rr} \to \infty$ and $\tau_{il}^{rq} \to \infty$. By substituting each of these conditions into (1.3.29), (1.3.30) and (1.3.31), and then substituting the results into (1.3.20), (1.3.21) and (1.3.22), we obtain (1.4.45), (1.4.46) and (1.4.47) respectively.

Remark 1.4.5 *Under the assumption of Theorem 1.4.5, the condition $\theta(f_0) = \theta(g_0)$ states that τ_{il}^{rq} and τ_{ij}^{rr} are increasing at the same order. The significance of this relationship is that for sufficiently large values of the intra and interregional visiting times, all the residents of site s_i^r leave their homes and emigrate to sites in their intra and interregional accessible domain $C(s_i^r)$. Therefore at the steady state, the original home site of the residents is occupied by visitors from other sites in C_r and C_q. This special pattern is illustrated in Figure 1.7.*

Also, the results under the condition $\theta(f_0) > \theta(g_0)$ signifies that for sufficiently large values of intra and interregional visiting time, the residents of s_i^r emigrate from their home region C_r, and become permanent residents at sites in their interregional accessible domain $C_{rq}(s_i^r)$.

Hence their original home site s_i^r is occupied by residents from other sites in C_r and C_q. This special pattern is illustrated in Figure 1.8.

The results under the condition $\theta(f_0) < \theta(g_0)$ now signifies that for sufficiently large values of intra and interregional visiting time, the residents of site s_i^r now leave their home site s_i^r and become permanent residents only at sites in their intra regional accessible domain $C_{rr}(s_i^r)$. This special pattern is illustrated in Figure 1.9.

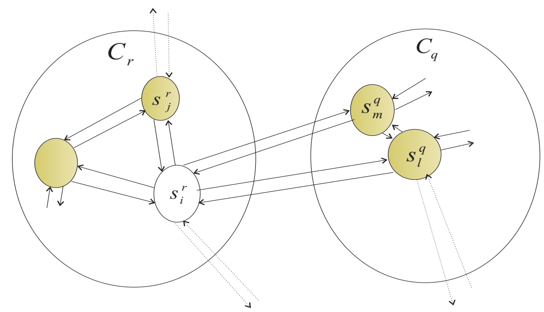

Figure 1.7: Shows that residents of site s_i^r are present only at sites s_j^r in their intra regional accessible domain and at sites s_l^r in their interregional accessible domain $C_{rq}(s_i^r)$. The arrows represent a transport network between any two sites and regions. Moreover dotted arrows indicate connections with other sites in other regions.

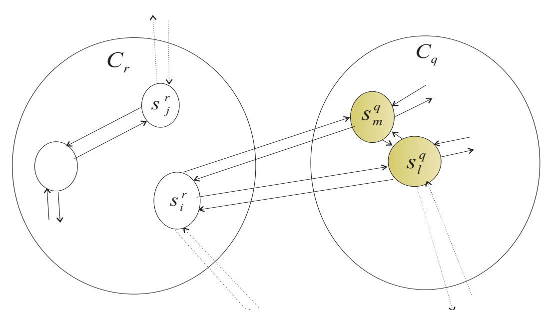

Figure 1.8: Shows that residents of site s_i^r are present only at sites s_l^r in their interregional accessible domain $C_{rq}(s_i^r)$. The arrows represent a transport network between any two sites and regions. In addition, the dotted lines and arrows indicate a connection with other sites in other regions.

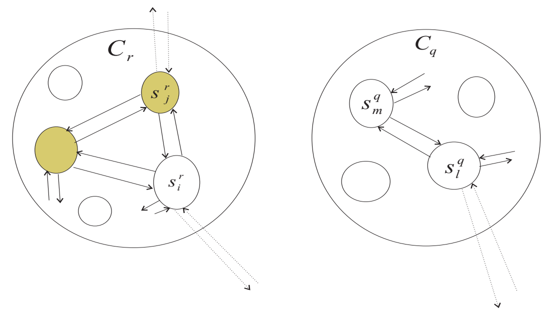

Figure 1.9: Shows that residents of site s_i^r are present only at sites s_j^r in their intra-regional accessible domain $C_{rq}(s_i^r)$. The arrows represent a transport network between any two sites and regions. Furthermore, the dotted lines and arrows indicate a connection with other sites in other regions.

Assuming the mobility rate functions change at different rates, we now characterize the steady states of the system. Consider the following assumption.

Assumption 1.4.1 *Given $\varepsilon > 0$, and for all $j \in I_i^r(1,n_r), l \in I_i^r(1,n_q)$, let τ_{ij}^{rr} and τ_{il}^{rq} be related as follows,*

$$\begin{aligned}
&\tau_{ij}^{rr} = 0(f(\tau_{i,min}^{rr})), \quad \tau_{i,min}^{rq} = 0(h(\tau_{i,max}^{rr})), \quad \tau_{il}^{rq} = 0(g(\tau_{i,min}^{rq}),\\
&\qquad \texttt{whenever } \tau_{i,max}^{rr} = 0(1) \texttt{ and } \tau_{i,max}^{rq} = 0(1)\texttt{, and}\\
&\tau_{ij}^{rr} = \theta(f(\tau_{i,min}^{rr})), \quad \tau_{i,min}^{rq} = \theta(h(\tau_{i,max}^{rr})), \quad \tau_{il}^{rq} = \theta(g(\tau_{i,min}^{rq}),\\
&\qquad \texttt{whenever } \tau_{i,min}^{rr} = \theta(\frac{1}{\varepsilon}) \texttt{ and } \tau_{i,min}^{rq} = \theta(\frac{1}{\varepsilon})\texttt{,}
\end{aligned} \tag{1.4.48}$$

where h has the explicit form

$$h :]0,\infty[\mapsto [0,\infty[, x \mapsto h(x) = x^c, c > 0, \quad \text{and} \quad f,g :]0,\infty[\mapsto [0,\infty[\tag{1.4.49}$$

are arbitrary positive real valued functions.

For the sake of easy reference and simplicity, we state the following hypotheses.

Hypothesis 1.4.2 *Assume that τ_{ij}^{rr} and τ_{il}^{rq} satisfy Assumption 1.4.1. Further assume that*

H_6: $0 < \tau_{i,max}^{rr} < 1$ and $0 < c < 1$;

H_7: $0 < \tau_{i,max}^{rr} < 1$, and $c \geq 1$;

H_8: $\tau^{rr}_{i,max} \geq 1$, *and* $0 < c < 1$;

H_9: $\tau^{rr}_{i,max} \geq 1$, *and* $c \geq 1$.

In the following theorems we describe the steady states of the mobility system under these hypotheses.

Theorem 1.4.6 *Suppose H_6 holds and for any given $\varepsilon > 0$.*

1. *If $\tau^{rr}_{i,max} \to 0^+$ and $c \to 0^+$, then for all $j \in I^r_i(1,n_r), l \in I^r_i(1,n_q)$, $\tau^{rr}_{ij} = 0(\varepsilon)$ and $\tau^{rq}_{il} = 0(1)$ and $\tau^{rr}_{ij} \leq \tau^{rq}_{il}$.*

   Moreover, the steady state of the mobility system satisfies

$$N^{*rr}_{ii} \to N^{*rr}_i \frac{1}{\left(1 + \sum_{q \neq r}^{M} \sum_{a=1}^{n_q} \gamma^{rq}_{ia} \tau^{rq}_{ia}\right)}, \quad N^{*rr}_{ij} \to 0$$
$$\text{and} \quad N^{*rq}_{il} \to N^{*rr}_i \frac{\gamma^{rq}_{il}}{\left(\frac{1}{\tau^{rq}_{il}} + \sum_{q \neq r}^{M} \sum_{a=1}^{n_q} \gamma^{rq}_{ia} K^{rqq}_{ial}\right)}. \tag{1.4.50}$$

2. *If $\tau^{rr}_{i,max} \to 0^+$, and $c \to 1^-$, then $\tau^{rr}_{ij} = 0(\varepsilon)$ for all $j \in I^r_i(1,n_r)$, and there exists $j_0 \in I^r_i(1,n_r)$ and $l_0 \in I^r_i(1,n_q)$ such that $\tau^{rq}_{il_0} \to 0^+$ slower than $\tau^{rr}_{ij_0} \to 0^+$. Moreover,*

   i. *the steady state of the mobility system satisfies*

$$N^{*rr}_{ii} \to N^{*rr}_i \frac{1}{\left(1 + \sum_{q \neq r}^{M} \sum_{a=1}^{n_q} \gamma^{rq}_{ia} \tau^{rq}_{ia}\right)}, \quad N^{*rr}_{ij} \to 0 \quad \text{and}$$
$$N^{*rq}_{il_0} \to 0, \quad N^{*rq}_{il} \to N^{*rr}_i \frac{\gamma^{rq}_{il}}{\left(\frac{1}{\tau^{rq}_{il}} + \sum_{q \neq r}^{M} \sum_{a=1}^{n_q} \gamma^{rq}_{ia} K^{rqq}_{ial}\right)}, l \neq l_0, \tag{1.4.51}$$

   whenever $\tau^{rq}_{il} = 0(1)$

   ii. *the steady state is given by Theorem1.4.4, whenever $\tau^{rq}_{il} = 0(\varepsilon)$, for all $l \in I^r_i(1,n_q)$.*

3. *If $\tau^{rr}_{i,max} \to 1^-$ and $c \to 0^+$, then for all $j \in I^r_i(1,n_r), l \in I^r_i(1,n_q)$, $\tau^{rq}_{il} = 0(1)$. Also, there exists $j_1 \in I^r_i(1,n_r)$ such that $\tau^{rr}_{ij_1} = 0(1)$. Furthermore,*

   i. *if $\tau^{rr}_{ij} = 0(1)$ for all $j \in I^r_i(1,n_r)$ then the steady state of the mobility system is given by (1.3.20), (1.3.21) and (1.3.22),* *ii.* *if* $\tau_{ij}^{rr} = 0(\varepsilon)$*, for all* $j \neq j_1,\ j, j_1 \in I_i^r(1,n_r)$*, then the steady state of the mobility system satisfies*

$$\begin{aligned} N_{ii}^{*rr} &\to N_i^{rr} \frac{1}{\left(1+\sigma_{ij_1}^{rr}\tau_{ij_1}^{rr} + \sum_{q\neq r}^{M}\sum_{a=1}^{n_q}\gamma_{ia}^{rq}\tau_{ia}^{rq}\right)}, N_{ij}^{*rr} \quad \to \quad 0, j \neq j_1, \\ N_{ij_1}^{*rr} &\to N_i^{*rr} \frac{\sigma_{ij_1}^{rr}}{\left(\frac{1}{\tau_{ij_1}^{rr}} + \sigma_{ij_1}^{rr} + \sum_{q\neq r}^{M}\sum_{a=1}^{n_q}\gamma_{ia}^{rq}K_{iaj_1}^{rqr}\right)}, \quad \textit{and} \\ N_{il}^{*rq} &\to N_i^{*rr} \frac{\gamma_{il}^{rq}}{\left(\frac{1}{\tau_{il}^{rq}} + \sigma_{ij_1}^{rr}K_{ij_1l}^{rrq} + \sum_{q\neq r}^{M}\sum_{a=1}^{n_q}\gamma_{ia}^{rq}K_{ial}^{rqq}\right)}. \end{aligned} \tag{1.4.52}$$

4. *If* $\tau_{i,max}^{rr} \to 1^-$ *and* $c \to 1^-$*, then for all* $j \in I_i^r(1,n_r), l \in I_i^r(1,n_q)$*,* $\tau_{il}^{rq} = 0(1)$*. Also, there exists* $j_2 \in I_i^r(1,n_r)$ *such that* $\tau_{ij_2}^{rr} = 0(1)$*. In addition,*

   i. *if* $\tau_{ij}^{rr} = 0(1)$ *for all* $j \in I_i^r(1,n_r)$ *then the steady states of the mobility system is given by (1.3.20), (1.3.21) and (1.3.22).*

   ii. *if* $\tau_{ij}^{rr} = 0(\varepsilon)$*, for all* $j \neq j_2,\ j, j_2 \in I_i^r(1,n_r)$*, then the steady state of the mobility system is given by (1.4.52).*

Proof: Under H_6, we have

$$\tau_{i,min}^{rq} = 0((\tau_{i,max}^{rr})^c), r \neq q. \tag{1.4.53}$$

1. If $\tau_{i,max}^{rr} \to 0^+$ and $c \to 0^+$, then it follows that $(\tau_{i,max}^{rr})^c \to 1^-$. Thus from (1.4.48) and (1.4.53), for all $j \in I_i^r(1,n_r), l \in I_i^r(1,n_q)$, $0 \leq \tau_{ij}^{rr} \leq \tau_{i,max}^{rr} \to 0^+$, we obtain

   $$\tau_{i,min}^{rq} = 0(1), \tau_{il}^{rq} = 0(g(\tau_{i,min}^{rq})) = 0(1) \quad \textit{and} \quad \tau_{ij}^{rr} = 0(\varepsilon). \tag{1.4.54}$$

   Also, $\tau_{i,min}^{rq} = 0(1) \Rightarrow \tau_{il}^{rq} > 0$, for all $l \in I_i^r(1,n_q)$. Hence $\tau_{il}^{rq} \geq \tau_{ij}^{rr}$.

   Finally, from (1.4.54), (1.3.29), (1.3.30) and (1.3.31), (1.3.20), (1.3.21) and (1.3.22), we get (1.4.50).

2. If $\tau_{i,max}^{rr} \to 0^+$ and $c \to 1^-$, then from (1.4.53), $(\tau_{i,max}^{rr})^c \to 0^+$ at a slower rate. Hence, take $\tau_{i,min}^{rq} = \tau_{il_0}^{rq}$ and $\tau_{i,max}^{rr} = \tau_{ij_0}^{rr}$, and the existence of l_0 and j_0 is verified. Now, from $\tau_{ij}^{rr} = 0(\varepsilon)$ and $\tau_{il}^{rq} = 0(1), l \neq l_0$, the proof of the steady state follows from Theorem 1.4.6(1). From $\tau_{ij}^{rr} = 0(\varepsilon)$ and $\tau_{il}^{rq} = 0(\varepsilon)$, for all $l \in I_i^r(1,n_q)$, the proof of the steady state follows from Theorem 1.4.4.

3. If $\tau^{rr}_{i,max} \to 1^-$ and $c \to 1^+$, then it suffices to note $\tau^{rr}_{ij_1} = \tau^{rr}_{i,max}$, for some $j_1 \in I^r_i(1,n_r)$, and this proves the existence of j_1. The rest of the proof follows from part (1) and (2) above.

4. The proof of Theorem 1.4.6(4) follow from part (1), (2) and (3) above.

Remark 1.4.6 *The interpretation of Theorem 1.4.6(1) is similar to Theorem 1.4.1. Theorem 1.4.6(2) signifies that, whenever all the residents of a given site s^r_i that travel to sites within their intra regional accessible domain $C_{rr}(s^r_i)$ spend infinitesimally small amount of time visiting the sites, and there is also one site $s^q_{l_0}$ in the inter regional accessible domain $C_{rq}(s^r_i)$, where all the residents of site s^r_i that travel to this site also spend infinitesimally small amount of time visiting that site, that is, a fraction of the residents of s^r_i would remain permanent residents of their home site s^r_i, and a fraction would relocate to all sites $s^q_l \neq s^q_{l_0}$ in their interregional accessible domain $C_{rq}(s^r_i)$, and spend a finite amount of time visiting these sites.*

Theorem 1.4.6(3) also states that all the residents of site s^r_i that travel to sites in their interregional accessible domain $C_{rq}(s^r_i)$, spend a finite amount of time visiting the sites, and there is a site $s^r_{j_1}$ in $C_{rr}(s^r_i)$, where the residents of site s^r_i can spend a finite amount of time visiting. From Assumption in Theorem 1.4.6(3)(ii) the residents of site s^r_i that travel in the intra regional accessible domain $C_{rr}(s^r_i)$ spend infinitesimally small amount of time visiting, this implies that the distribution of the residents of site s^r_i, (i) a fraction would remain permanent residents at their home site s^r_i, (ii) a fraction would migrate to site $s^r_{j_1}$ in $C_{rr}(s^r_i)$, and (iii) the remaining fraction would migrate to all sites in $C_{rq}(s^r_i)$ and become residents of those sites for a finite amount of time. Under Theorem 1.4.6(3)(i) a fraction the residents of site s^r_i would remain permanent residents at home site s^r_i and the remaining fraction would distribute among the visiting sites for a finite visiting time. Theorem 1.4.6(4) has similar interpretations to Theorem 1.4.6(2).

The conditions on c in Theorem 1.4.6(1) and Theorem 1.4.6(2) indicate the existence of a critical value for $c \in]0,1[$ denoted by c_0. For $0 < c < c_0$, the steady state is given by (1.4.50), and for $c_0 < c < 1$, the steady state is given by(1.4.51). Similar conclusion with regards to Theorem 1.4.6(3)&(4) can be drawn about a critical value for $c \in]0,1[$ denoted by c_1. Moreover, in the case of Theorem 1.4.6(3) we have $0 < c < c_1$, and the steady state is given by (1.4.52). In the case of Theorem 1.4.6(4) we have $c_1 < c < 1$, and the steady state is given by Theorem 1.4.6(4).

The conditions on $\tau^{rr}_{i,max}$ in Theorem 1.4.6 also indicate the existence of a critical value for $\tau^{rr}_{i,max} \in]0,1[$ denoted by τ^{rr}_{i,max_0}, such that for $0 < \tau^{rr}_{i,max} < \tau^{rr}_{i,max_0}$, whenever $0 < c < c_0$, the steady *state is given by (1.4.50), and whenever $c_0 < c < 1$, the steady state is given by(1.4.51). Also, for $\tau^{rr}_{i,max_0} < \tau^{rr}_{i,max} < 1$, when $0 < c < c_1$, the steady state is given by (1.4.52), and when $c_1 < c < 1$, the steady state is given by Theorem 1.4.6(4).*

Remark 1.4.7 *The condition on the intra regional visiting times in (1.4.48) signifies that these times are of the same order. A similar conclusion can be drawn with respect to the interregional visiting times. We further note that the functions f, g and h, can take arbitrary forms such as quadratic, cubic, exponential, or logarithmic functions depending on the kind of mobility process that is being modeled. Under this consideration, we incorporate more details about the mobility process. For instance, if one site $s^r_{j_0}$ in the intra regional accessible domain receives more visitors than all other sites in the domain, then we could have a hub in the domain. And one possible representation of this detail about the mobility process, can be $\tau^{rr}_{ij} = 0(g(\tau^{rr}_{i,min})), \forall j \neq i, j \neq j_0, j \in I^r_i(1,n_r)$, $\tau^{rr}_{ij_0} = 0(g_1(\tau^{rr}_{i,min})), j_0 \in I^r_i(1,n_r)$, where $g \neq g_1$. Where g_1 is an arbitrary positive real valued function describing the return rate of visitors to site $s^r_{j_0}$.*

We now describe the steady states of the system under H_7. Observe that H_7 is composed of two conditions ($\tau^{rr}_{i,max} \to 0^+$ and $c \geq 1$) and ($\tau^{rr}_{i,max} \to 1^-$ and $c \geq 1$). Under $\tau^{rr}_{i,max} \to 0^+$ and $c \geq 1$, the construction of the steady state is similar to Theorem 1.4.6(2). Similarly, the steady states is similar to Theorem 1.4.6(4) whenever $\tau^{rr}_{i,max} \to 1^-$ and $c \geq 1$. We further remark that $\tau^{rr}_{i,max}$ has a critical value $\tau^{rr}_{i,max_1} \in]0,1[$, such that for $c \geq 1$ the steady states are given by Theorem 1.4.6(2) whenever $0 < \tau^{rr}_{i,max} < \tau^{rr}_{i,max_1}$, and the steady states are given by Theorem 1.4.6(4) whenever $\tau^{rr}_{i,max_1} < \tau^{rr}_{i,max} < 1$.

Finally we describe the steady states of the system under H_8 and H_9.

Theorem 1.4.7 *Suppose H_8 and H_9 hold. Given for any $\varepsilon > 0$,*

1. *If $\tau^{rr}_{i,max} = 0(1)$ and $0 < c < 1$, for all $j \in I^r_i(1,n_r)$, and $l \in I^r_i(1,n_q)$, then (a) $\tau^{rq}_{il} = 0(1)$, (b) there exists $j_3 \in I^r_i(1,n_q)$ and $l_3 \in I^r_i(1,n_r)$ such that $\tau^{rr}_{ij_3} = 0(1)$ and $\tau^{rq}_{il_3} \leq \tau^{rr}_{ij_3}$, and (c) furthermore,*

    i. *if $\tau^{rr}_{ij} = 0(1)$ for all $j \in I^r_i(1,n_r)$, then the steady state of the mobility system satisfies (1.3.20), (1.3.21) and (1.3.22).*

    ii. *if $\tau^{rr}_{ij} = 0(\varepsilon)$ for all $j \neq j_3$, $j, j_3 \in I^r_i(1,n_r)$, then the steady state of the mobility system* *satisfies*

$$\begin{aligned}
N_{ii}^{*rr} &\to N_i^{rr}\frac{1}{\left(1+\sigma_{ij_3}^{rr}\tau_{ij_3}^{rr}+\sum_{q\neq r}^{M}\sum_{a=1}^{n_q}\gamma_{ia}^{rq}\tau_{ia}^{rq}\right)}, \quad N_{ij}^{*rr}\to 0, j\neq j_3,\\
N_{ij_3}^{*rr} &\to N_i^{rr}\frac{\sigma_{ij_3}^{rr}}{\left(\frac{1}{\tau_{ij_3}^{rr}}+\sigma_{ij_3}^{rr}+\sum_{q\neq r}^{M}\sum_{a=1}^{n_q}\gamma_{ia}^{rq}K_{iaj_3}^{rqr}\right)},\\
N_{il}^{*rq} &\to N_i^{rr}\frac{\gamma_{il}^{rq}}{\left(\frac{1}{\tau_{il}^{rq}}+\sigma_{ij_3}^{rr}K_{ij_3l}^{rrq}+\sum_{q\neq r}^{M}\sum_{a=1}^{n_q}\gamma_{ia}^{rq}K_{ial}^{rqq}\right)}.
\end{aligned} \tag{1.4.55}$$

2. *If $\tau_{i,max}^{rr}=0(1)$ and $c\geq 1$, then (a) for all $l\in I_i^r(1,n_r)$, $\tau_{il}^{rq}=0(1)$, (b) there exists $j_4\in I_i^r(1,n_r)$ such that $\tau_{ij_4}^{rr}=0(1)$ and $\tau_{ij}^{rr}\leq\tau_{il}^{rq}$ for all $j\in I_i^r(1,n_r), l\in I_i^r(1,n_q)$, and (c) furthermore,*

   i. *if $\tau_{ij}^{rr}=0(1)$ for all $j\in I_i^r(1,n_r)$ then the steady state of the mobility system satisfies (1.3.20), (1.3.21) and (1.3.22).*

   ii. *if $\tau_{ij}^{rr}=0(\varepsilon)$, for all $j\neq j_4$, $j, j_4\in I_i^r(1,n_r)$, then the steady state of the mobility system satisfies (1.4.55) (where, we replace j_3 with j_4).*

3. *If $\tau_{i,max}^{rr}=\theta(\frac{1}{\varepsilon})$ and $0<c<1$, then (a) for all $j\in I_i^r(1,n_r), l\in I_i^r(1,n_q)$, $\tau_{il}^{rq}=\theta(\frac{1}{\varepsilon})$, (b) there exists $j_5\in I_i^r(1,n_r)$ such that $\tau_{ij_5}^{rr}=\theta(\frac{1}{\varepsilon})$, and (c) furthermore,*

   i. *for $\tau_{ij}^{rr}=0(1)$ or $\tau_{ij}^{rr}=0(\varepsilon)$, for all $j\neq j_5$, if $\theta(f)>\theta(g)$, then*

$$N_{ii}^{*rr}\to 0, \quad N_{ij_5}^{*rr}\to N_i^{*rr}, \quad N_{ij}^{*rr}\to 0, j\neq j_5, \quad N_{il}^{*rq}\to 0, \tag{1.4.56}$$

   and if $\theta(f)<\theta(g)$, then

$$N_{ii}^{*rr}\to 0, \quad N_{ij}^{*rr}\to 0, \quad N_{il}^{*rq}\to N_i^{rr}\frac{\gamma_{il}^{rq}}{\left(\sum_{q\neq r}^{M}\sum_{a=1}^{n_q}\gamma_{ia}^{rq}K_{ial}^{rqq}\right)}. \tag{1.4.57}$$

   Finally, if $\theta(f)=\theta(g)$, then

$$\begin{aligned}
N_{ii}^{*rr} &\to 0, \quad N_{ij_5}^{*rr}\to N_i^{rr}\frac{\sigma_{ij_5}^{rr}}{\left(\sigma_{ij_5}^{rr}+\sum_{q\neq r}^{M}\sum_{a=1}^{n_q}\gamma_{ia}^{rq}K_{iaj_5}^{rqr}\right)},\\
N_{ij}^{*rr} &\to 0, j\neq j_5, \quad N_{il}^{*rq}\to N_i^{rr}\frac{\gamma_{il}^{rq}}{\left(\sigma_{ij_5}^{rr}K_{ij_5l}^{rrq}+\sum_{q\neq r}^{M}\sum_{a=1}^{n_q}\gamma_{ia}^{rq}K_{ial}^{rqq}\right)}.
\end{aligned} \tag{1.4.58}$$

   ii. *For $\tau_{ij}^{rr}=0(\frac{1}{\varepsilon})$ for all $j\in I_i^r(1,n_r)$, the steady states are similar to Theorem 1.4.5.*

4. *If $\tau^{rr}_{i,max} = \theta(\frac{1}{\varepsilon})$ and $c \geq 1$, then for all $j \in I^r_i(1,n_r), l \in I^r_i(1,n_q)$, $\tau^{rq}_{il} = \theta(\frac{1}{\varepsilon})$ and $\tau^{rr}_{ij} \leq \tau^{rq}_{il}$. Also, there exists $j_6 \in I^r_i(1,n_r)$ such that $\tau^{rr}_{ij_6} = \theta(\frac{1}{\varepsilon})$. Furthermore, for $\tau^{rr}_{ij} = 0(1)$ or $\tau^{rr}_{ij} = 0(\varepsilon)$, for all $j \neq j_6$, if $\theta(f) > \theta(g)$, then steady state of the mobility system satisfies (1.4.56); if $\theta(f) < \theta(g)$, then the steady state satisfies (1.4.57); and if $\theta(f) = \theta(g)$, then the steady state satisfies (1.4.58).*

Proof: The proofs of parts (1)& (2) follow from Theorem 1.4.6. Parts (3)& (4) follow from Theorem 1.4.5.

Remark 1.4.8 *A remark similar to Remark 1.4.5 and Remark 1.4.6 can be formulated with regards to Theorem 1.4.7.*

We also examine the situation where the growth rate of the minimum intra regional visiting time is compared with a power function of the maximum inter regional visiting time. We shall consider the cases where the power function is a fractional power and a polynomial function.

Assumption 1.4.2 *Given $\varepsilon > 0$, and for all $j \in I^r_i(1,n_r), l \in I^r_i(1,n_q)$, let τ^{rr}_{ij} and τ^{rq}_{il} be related as follows,*

$$\begin{gathered}
\tau^{rr}_{ij} = 0(g(\tau^{rr}_{i,min})), \tau^{rr}_{i,min} = 0(h(\tau^{r,max}_{i,max})), \tau^{rq}_{il} = 0(f(\tau^{rq}_{i,min})), \\
\texttt{whenever } \tau^{r,max}_{i,max} = 0(1) \texttt{ and } \tau^{rr}_{i,max} = 0(1)\texttt{, and} \\
\tau^{rr}_{ij} = \theta(g(\tau^{rr}_{i,min})), \tau^{rr}_{i,min} = \theta(h(\tau^{r,max}_{i,max})), \tau^{rq}_{il} = \theta(f(\tau^{rq}_{i,min})), \\
\texttt{whenever } \tau^{rr}_{i,min} = \theta(\frac{1}{\varepsilon}) \texttt{ and } \tau^{rq}_{i,min} = \theta(\frac{1}{\varepsilon})\texttt{,}
\end{gathered} \tag{1.4.59}$$

where f, g, and h are defined in Assumption 1.4.1.

We state the following hypotheses:

Hypothesis 1.4.3 *Resident time τ^{rr}_{ij} and τ^{rq}_{il} satisfy Assumption 1.4.2 and moreover,*

H_{10}: $0 < \tau^{r,max}_{i,max} < 1$, *and* $0 < c < 1$;

H_{11}: $0 < \tau^{r,max}_{i,max} < 1$, *and* $c \geq 1$;

H_{12}: $\tau^{r,max}_{i,max} \geq 1$, *and* $0 < c < 1$;

H_{13}: $\tau^{r,max}_{i,max} \geq 1$, *and* $c \geq 1$.

We shall characterize the steady states of the mobility process under H_{13}. The static behavior of the system under the other hypotheses can be derived in a similar manner.

Theorem 1.4.8 *Suppose H_{13} holds. Given $\varepsilon > 0$,*

1. *If $\tau_{i,max}^{r,max} = 0(1)$ and $c \geq 1$, then for all $q \in I^r(1,M)$, $j \in I_i^r(1,n_r)$ and $l \in I_i^r(1,n_q)$, $\tau_{ij}^{rr} = 0(1)$ and $\tau_{il}^{rq} \leq \tau_{ij}^{rr}$. Furthermore,*

   i. *if $\tau_{il}^{rq} = 0(\varepsilon)$ for all $q \in I^r(1,M)$ and $l \in I_i^r(1,n_r)$, then the steady states of the mobility system satisfy*

$$\begin{aligned} N_{ii}^{*rr} &\to N_i^{*rr}\frac{1}{\left(1+\sum_{j\neq i}^{n_r}\sigma_{ij}^{rr}\tau_{ij}^{rr}\right)}, \quad N_{ij}^{*rr} \to N_i^{*rr}\frac{\sigma_{ij}^{rr}}{\left(\frac{1}{\tau_{ij}^{rr}}+\sum_{k\neq i}^{n_r}\sigma_{ik}^{rr}K_{ikj}^{rrr}\right)}, j \in I_i^r(1,n_r), \\ N_{il}^{*rq} &\to 0. \end{aligned} \tag{1.4.60}$$

   ii. *if for some $q_1 \in I^r(1,M)$, and $l \in I_i^r(1,n_{q_1})$, $\tau_{il}^{rq_1} = 0(\varepsilon)$ and for all $q \neq q_1$, $q \in I^r(1,M)$ and $l \in I_i^r(1,n_q)$, $\tau_{il}^{rq} = 0(1)$, then the steady state of the mobility system satisfies*

$$\begin{aligned} N_{ii}^{*rr} &\to N_i^{*rr}\frac{1}{\left(1+\sum_{j\neq i}^{n_r}\sigma_{ij}^{rr}\tau_{ij}^{rr}+\sum_{q\neq r,q_1}^{M}\sum_{l=1}^{n_q}\gamma_{il}^{rq}\tau_{il}^{rq}\right)}, \\ N_{ij}^{*rr} &\to N_i^{*rr}\frac{\sigma_{ij}^{rr}}{\left(\frac{1}{\tau_{ij}^{rr}}+\sum_{k\neq i}^{n_r}\sigma_{ik}^{rr}K_{ikj}^{rrr}+\sum_{q\neq r,q_1}^{M}\sum_{a=1}^{n_q}\gamma_{ia}^{rq}K_{ilj}^{rqr}\right)}, \\ N_{il}^{*rq_1} &\to 0, l \in I_i^r(1,n_{q_1}), \quad \texttt{and for } q \neq q_1, l \in I_i^r(1,n_q), \\ N_{il}^{*rq} &\to N_i^{*rr}\frac{\gamma_{il}^{rq}}{\left(\frac{1}{\tau_{il}^{rq}}+\sum_{k\neq i}^{n_r}\sigma_{ik}^{rr}K_{ikj}^{rrr}+\sum_{q\neq r,q_1}^{M}\sum_{a=1}^{n_q}\gamma_{ia}^{rq}K_{ilj}^{rqr}\right)}. \end{aligned} \tag{1.4.61}$$

   iii. *if for all $q \in I^r(1,M)$ and $l \in I_i^r(1,n_r)$, $\tau_{il}^{rq} = 0(1)$ then the steady state of the mobility system satisfies (1.3.20), (1.3.21) and (1.3.22).*

2. *If $\tau_{i,max}^{r,max} = \theta(\frac{1}{\varepsilon})$ and $c \geq 1$, then (a) for all $j \in I_i^r(1,n_r)$, $\tau_{ij}^{rr} = \theta(\frac{1}{\varepsilon})$ and (b) for all $l \in I_i^r(1,n_q)$,*

   i. *if $\tau_{il}^{rq} = 0(\varepsilon)$ or $\tau_{il}^{rq} = 0(1)$ for all $q \in I(1,M)$, and $l \in I_i^r(1,n_q)$, then for $j \in I_i^r(1,n_r)$, the steady state of the mobility system satisfies*

$$N_{ii}^{*rr} \to 0, \quad N_{ij}^{*rr} \to N_i^{rr}\frac{\sigma_{ij}^{rr}}{\left(\sum_{k\neq i}^{n_r}\sigma_{ik}^{rr}K_{ikj}^{rrr}\right)}, \quad N_{il}^{*rq} \to 0. \tag{1.4.62}$$

ii. *if for some* $q_2 \in I^r(1,M)$, *and* $l \in I_i^r(1,n_{q_2})$, $\tau_{il}^{rq_2} = 0(\varepsilon)$ *and for all* $q \neq q_2$, $q \in I^r(1,M)$, *and* $l \in I_i^r(1,n_q)$, $\tau_{il}^{rq} = 0(1)$ *then the steady state of the mobility system satisfies*

$$\begin{aligned} N_{ii}^{*rr} &\to 0, N_{ij}^{*rr} \to N_i^{rr} \frac{\sigma_{ij}^{rr}}{\left(\sum_{k\neq i}^{n_r} \sigma_{ik}^{rr} K_{ikj}^{rrr}\right)}, \\ N_{il}^{*rq} &\to 0, \quad \texttt{for all } q \in I^r(1,M), l \in I_i^r(1,n_q(s_i^r). \end{aligned} \tag{1.4.63}$$

iii. *if* $\tau_{il}^{rq} = \theta(\frac{1}{\varepsilon})$ *for all* $q \in I^r(1,M)$, *and* $l \in I_i^r(1,n_q)$, *then for* $j \in I_i^r(1,n_r)$, *if* $\theta(g) < \theta(f)$, *the steady state of the mobility system satisfies*

$$N_{ii}^{*rr} \to 0, \quad N_{ij}^{*rr} \to N_i^{rr} \frac{\sigma_{ij}^{rr}}{\left(\sum_{k\neq i}^{n_r} \sigma_{ik}^{rr} K_{ikj}^{rrr}\right)}, \quad N_{il}^{*rq} \to 0; \tag{1.4.64}$$

if $\theta(g) > \theta(f)$, *the steady state of the mobility system satisfies*

$$\begin{aligned} N_{ii}^{*rr} &\to 0, \quad N_{ij}^{*rr} \to 0, \\ N_{il}^{*rq} &\to N_i^{rr} \frac{\gamma_{il}^{rq}}{\left(\sum_{q\neq r}^{M} \sum_{a=1}^{n_q} \gamma_{ia}^{rq} K_{ilj}^{rqr}\right)}, \end{aligned} \tag{1.4.65}$$

and if $\theta(g) = \theta(f)$, *the steady state of the mobility system satisfies*

$$\begin{aligned} N_{ii}^{*rr} &\to 0, \quad N_{ij}^{*rr} \to N_i^{rr} \frac{\sigma_{ij}^{rr}}{\left(\sum_{k\neq i}^{n_r} \sigma_{ik}^{rr} K_{ikj}^{rrr} + \sum_{q\neq r}^{M} \sum_{a=1}^{n_q} \gamma_{ia}^{rq} K_{ilj}^{rqr}\right)}, \\ N_{il}^{*rq} &\to N_i^{rr} \frac{\gamma_{il}^{rq}}{\left(\sum_{k\neq i}^{n_r} \sigma_{ik}^{rr} K_{ikj}^{rrr} + \sum_{q\neq r}^{M} \sum_{a=1}^{n_q} \gamma_{ia}^{rq} K_{ilj}^{rqr}\right)}, \end{aligned} \tag{1.4.66}$$

iv. *Assume that there exists* $q_3 \in I^r(1,M)$ *such that, for all* $q \neq q_3$, $\tau_{il}^{rq} = \theta(\frac{1}{\varepsilon})$ *and* $\tau_{il}^{rq_3} = 0(\varepsilon)$ *or* $\tau_{il}^{rq_3} = 0(1)$, $l \in I_i^r(1,n_{q_3})$). *(a) If* $\theta(g) < \theta(f)$, *then the steady state of the mobility system satisfies (1.4.64);*

(b) If $\theta(g) > \theta(f)$, *then the steady state of the mobility system satisfies*

$$\begin{aligned} N_{ii}^{*rr} &\to 0, \quad N_{ij}^{*rr} \to 0, \\ N_{il}^{*rq_3} &\to 0, \quad N_{il}^{*rq} \to N_i^{rr} \frac{\gamma_{il}^{rq}}{\left(\sum_{q\neq r}^{M} \sum_{a=1}^{n_q} \gamma_{ia}^{rq} K_{ilj}^{rqr}\right)}, q \neq q_3, \end{aligned} \tag{1.4.67}$$

and (c) if $\theta(g) = \theta(f)$*, the steady state of the mobility system satisfies*

$$\begin{aligned}
N_{ii}^{*rr} &\to 0, N_{ij}^{*rr} \to N_i^{rr} \frac{\sigma_{ij}^{rr}}{\left(\sum_{k\neq i}^{n_r} \sigma_{ik}^{rr} K_{ikj}^{rrr} + \sum_{q\neq r}^{M} \sum_{a=1}^{n_q} \gamma_{ia}^{rq} K_{ilj}^{rqr}\right)}, \\
N_{il}^{*rq_3} &\to 0, N_{il}^{*rq} \to N_i^{rr} \frac{\gamma_{il}^{rq}}{\left(\sum_{k\neq i}^{n_r} \sigma_{ik}^{rr} K_{ikj}^{rrr} + \sum_{q\neq r}^{M} \sum_{a=1}^{n_q} \gamma_{ia}^{rq} K_{ilj}^{rqr}\right)}, \\
for \quad q &\neq q_3.
\end{aligned} \tag{1.4.68}$$

v. *Finally, assume that there exists* $q_4, q_5 \in I(1,M)$ *such that, for all* $q \neq q_4, q_5$, $\tau_{il}^{rq} = \theta(\frac{1}{\varepsilon})$ *for* $l \in I_i^r(1,n_q)$*, and* $\tau_{il}^{rq_4} = 0(\varepsilon)$*, for* $l \in I_i^r(1,n_{q_4})$ *and* $\tau_{il}^{rq_5} = 0(1)$*, for* $l \in I_i^r(1,n_{q_5})$*, then (a) if* $\theta(g) < \theta(f)$ *the steady states of the mobility system are given by (1.4.65), (b) if* $\theta(g) > \theta(f)$ *the steady states of the mobility system are given by (1.4.67), and (c) if* $\theta(g) = \theta(f)$ *the steady states of the mobility system are given by (1.4.68). Where* N^{*rq_4} *and* N^{*rq_5} *take the same value as* N^{*rq_3} *in each case.*

Proof: The proof follows immediately from the definition of τ_{ij}^{rr} and τ_{il}^{rq} in Assumption 1.4.2, and the proof of Theorem 1.4.5 and Theorem 1.4.7.

Remark 1.4.9 *The interpretations of the results of Theorem 1.4.8 are formulated in a similar manner to the results of Theorem 1.4.7(3).*

1.5 Conclusion

The rapid technological changes, scientific developments and educational expansion have created the heterogeneity in the human species. This heterogeneity generates a structure in the human population dynamics. The two-scale network dynamic model formulation for human mobility process makes a transition from its current infancy level to a teen-age level. Moreover, the dynamic model provides a bench mark to quantify the interactions between various scale levels generated by the increase in heterogeneity in the human mobility process in systematic and unified way. In fact, this work provides probabilistic and mathematical algorithmic tools to develop different level nested type interaction rates as well as network-centric dynamic equations. Naturally, the derived network-centric dynamic equations lead to network-centric steady-states of various types of steady-state level population structures. Of course, the steady-state population structure varies according to the: (a) various degrees of variations in the magnitude of intra-inter-regional visiting times, (b) various changes in mobility rate functions at different rates, and (c) various growth rates of the minimum intra regional visiting time compared with a power function of the maximum inter regional visiting time. Several results are successfully developed and analyzed.

2 SIMULATION RESULTS AND PROTOTYPE TWO-SCALE NETWORK HUMAN MOBILITY DYNAMIC PROCESS

In this chapter we characterize the steady state behavior of a two scale network human dynamic population that is under the influence of a special case real life human mobility process. The chapter is organized as follows: In Section 2.1, we characterize the two scale population structure and the human mobility process represented in this example. In Section 2.2, we describe the mathematical algorithm for generating the steady state population, and also give graphical representations of the steady state population.

2.1 The Two-Scale Hierarchic Population of the Special Real Life Mobility Process

By using three community single-scale model, the mobility dynamic structure determined by the simulated data set for the three communities in the district of Central Manitoba, Canada, and the data set for the interdistrict movement of the people in the West Indian Island of Dominica [27], we develop a two-scale mobility model. We note that the estimates of the underlying parameters under both simulated and real data sets are recorded in [27].

The development of this example is based on the following assumptions: In the absence of data set and without loss of generality, we assume that the structure determined by the simulated data set of the three communities in the district of Central Manitoba, Canada, is the structure of our two scale model at the interregional level. The three districts in the West Indian Island of Dominica represent 'sites' for each of the these three regions. Conceptually this assumption is not representative in the sense of geography/size but in the sense of sample drawn from small community or vice versa, that is representative of the big or small region. In short, we identify the three communities in the district of Central Manitoba, Canada as 'regions', and the districts of the West Indian Island of Dominica as 'sites'. Therefore every region has three sites, and we denote regions by $C_r, r = 1, 2, 3$, and the sites by $s_i^r, i = 1,2,3$. The intra-regional visiting/travel rates are displayed in Table 2.1. This table was rerecorded from the data set under the column of 'Dominica mobility' in ([27] page 14). We used the following information about the structure of the one scale(interregional) mobility simulated data set (NH mobility [27] page 14) for the above defined three regions and generate the interregional visiting/travel rates as follows: (1) communication patterns between two of the three regions is completely symmetric, (2) the third region is partially symmetric with one of the two regions. This is because of the fact that there is a zero flow rate into itself from one of the two regions. That is, one of the two complete symmetric regions is a 'sink' for the third region. From the this description, we conclude that the travel pattern in [27] includes the human mobility structure of our presented model as a special case. Using the structural and probabilistic understanding we constructed the interregional visiting/travel rates. We display the interregional travel rates in Table 2.2. Furthermore, the large scale two level population structure and the underlying human mobility pattern are exhibited in Figure 2.1

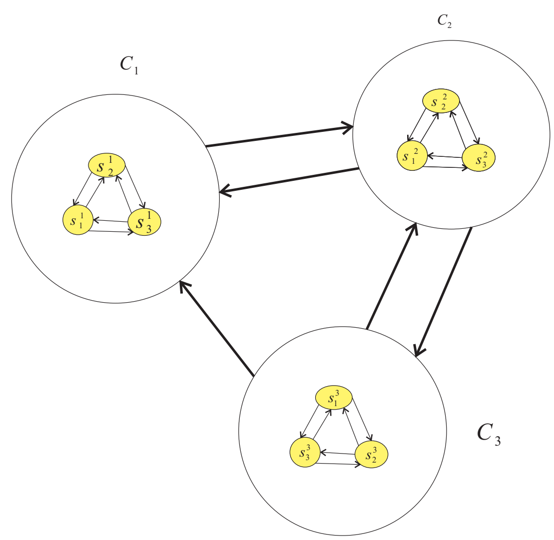

Figure 2.1: A two scale network of three spatial regions $C_r, r = 1,2,3$ of human habitation and three interconnected sites $s_i^r, i = 1,2,3$ in each region. The arrows represent direction of human mobility and summarize the heterogeneities in the epidemic process at each site and region. C_1 & C_2, and C_2 & C_3 are symmetric in the human mobility process. C_1 is a sink for C_3 in human mobility. All sites in each region are completely symmetric in the human mobility process.

We further assume that for every site $s_i^r, i = 1,2,3$ in any region C_r,$r = 1,2,3$, for all $j \neq i$, intra regional visiting times τ_{ij}^{rr}, of residents are the same, moreover it is assumed that the corresponding interregional visiting times τ_{il}^{rq}, of residents of site s_i^r in region C_r, are also the same for all $l = 1,2,3$ and $q \neq r$.

Table 2.1: Intra regional visiting/travel rates for sites in region $C_r, r = 1,2,3$. The intra regional travel rates within each of the three regions $C_r, r = 1,2,3$ are assumed to be the same. The estimates of the parameters are derived from data[27] that was collected from the Island of Dominica in 1991. The parameter estimates reflect the rates of travel that can be obtained in regions that have a low technological development.

Parameter	Intra-regional mobility
σ_1^{rr}	0.00147
σ_2^{rr}	0.03695
σ_3^{rr}	0.03754
ν_{12}^{rr}	0.7432
ν_{13}^{rr}	0.2568
ν_{21}^{rr}	0.9860
ν_{23}^{rr}	0.014.
ν_{31}^{rr}	0.8852
ν_{32}^{rr}	0.1147

Table 2.2: The interregional visiting/travel rates between the three regions. The derivation of these rates is based on the the structural understanding of the data set under the column 'NH mobility' in [27], page 14 and probabilistic understanding of our presented model.

Parameter	Interregional mobility
$(\gamma_1^1, \gamma_2^1, \gamma_3^1)$	(0.5,0.3.24,0.176)
$(\gamma_{10}^{12}, \gamma_{20}^{12}, \gamma_{30}^{12})$	(1,1,1)
$(\gamma_{10}^{13}, \gamma_{20}^{13}, \gamma_{30}^{13})$	(0,0,0)
$(\gamma_1^2, \gamma_2^2, \gamma_3^2)$	(0.16,0.06,0.03)
$(\gamma_{10}^{21}, \gamma_{20}^{21}, \gamma_{30}^{21})$	(0.222,0.12,0.658)
$(\gamma_{10}^{23}, \gamma_{20}^{23}, \gamma_{30}^{23})$	(0.778,0.88,0.342)
$(\gamma_1^3, \gamma_2^3, \gamma_3^3)$	(0.12,0.011,0.001)
$(\gamma_{10}^{31}, \gamma_{20}^{31}, \gamma_{30}^{31})$	(0.200,0.080,0.006)
$(\gamma_{10}^{32}, \gamma_{20}^{32}, \gamma_{30}^{32})$	(0.800,0.920,0.994)
$(\gamma^{12}, \gamma^{13})$	(3,0)
$(\gamma^{21}, \gamma^{23})$	(0.6,2)
$(\gamma^{31}, \gamma^{32})$	(0.286,2.714)
$(\gamma_{01}^{13}, \gamma_{02}^{13}, \gamma_{03}^{13})$	(0, 0, 0)
$(\gamma_{01}^{rq}, \gamma_{02}^{rq}, \gamma_{03}^{rq})_{r,q=2,3.r\neq q}$	(0.333, 0.27, 0.397)

2.2 Mathematical Algorithm and Simulation Results

In this section, we perform simulations for three general scenarios. Each scenario is based on restrictions on the intra- and inter- regional visiting times between zero and hundred days. We recall the fraction of the steady state population of residents of site s_i^r in region C_r at the home site, intra-regional and interregional accessible domains are given in (1.3.29), (1.3.30) and (1.3.31) respectively. Therefore all categories of the steady state population of the residents of site s_i^r in region C_r subject to continuous changes in the intra- and inter-regional visiting times can be characterized by the following functions:

$$S_{ii}^{*rr}, U_{ij}^{*rr}, V_{il}^{*rq} : [0,\infty] \times [0,\infty] \to [0,\infty], \forall j \in I_i^r(1,n_r), l \in I_i^r(1,n_q)$$

where,

$$S_{ii}^{*rr}(\tau_{ij}^{rr}, \tau_{il}^{rq}) = \frac{1}{\left(1 + \sum_{k\neq i}^{n_r} \sigma_{ik}^{rr}\tau_{ik}^{rr} + \sum_{q\neq r}^{M}\sum_{a=1}^{n_q}\gamma_{ia}^{rq}\tau_{ia}^{rq}\right)}, i \in I(1,n_r), \tag{2.2.1}$$

$$U_{ij}^{*rr}(\tau_{ij}^{rr}, \tau_{il}^{rq}) = \frac{\sigma_{ij}^{rr}}{\left(\frac{1}{\tau_{ij}^{rr}} + \sum_{k\neq i}^{n_r} \sigma_{ik}^{rr}K_{ikj}^{rrr} + \sum_{q\neq r}^{M}\sum_{a=1}^{n_q}\gamma_{ia}^{rq}K_{iaj}^{rqr}\right)}, j \in I_i^r(1,n_r), \tag{2.2.2}$$

and

$$V_{il}^{*rq}(\tau_{ij}^{rr}, \tau_{il}^{rq}) = \frac{\gamma_{il}^{rq}}{\left(\frac{1}{\tau_{il}^{rq}} + \sum_{k\neq i}^{n_r} \sigma_{ik}^{rr}K_{ikl}^{rrq} + \sum_{q\neq r}^{M}\sum_{a=1}^{n_q}\gamma_{ia}^{rq}K_{ial}^{rqq}\right)}, l \in I_i^r(1,n_q). \tag{2.2.3}$$

Furthermore, the notation $W_{ia}^{*ru}(\tau_{ij}^{rr}, \tau_{il}^{rq}), W \in \{S,U,V\}, \forall u \in I(1,M), a \in I(1,n_r)$ denotes the two hierarchic population interaction levels, and should not be understood as representing a function of two variables. In the following, we present the three scenarios. We fix $r = 1$, and $i = 1$.

Case 1: Constant Interregional Visiting Time and Varying Intra-regional Visiting Time: Suppose that the assumptions of the example presented in the previous section are satisfied, we further assume that interregional visiting time τ_{1l}^{rq} of residents of a given site s_1^1 in community C_1 is 10 days, and the intra-regional visiting time of the residents $\tau_{1j}^{11}, j = 2,3$, equally vary between zero and 100days. By utilizing (2.2.1), (2.2.2), (2.2.3) and the methods of graph sketching[85], we obtain values and graphs for the steady state population for the different categories of the residents of site s_1^1 over all $0 \leq \tau_{1j}^{11} \leq 100, j = 2,3$. The results are displayed in Figure 2.2

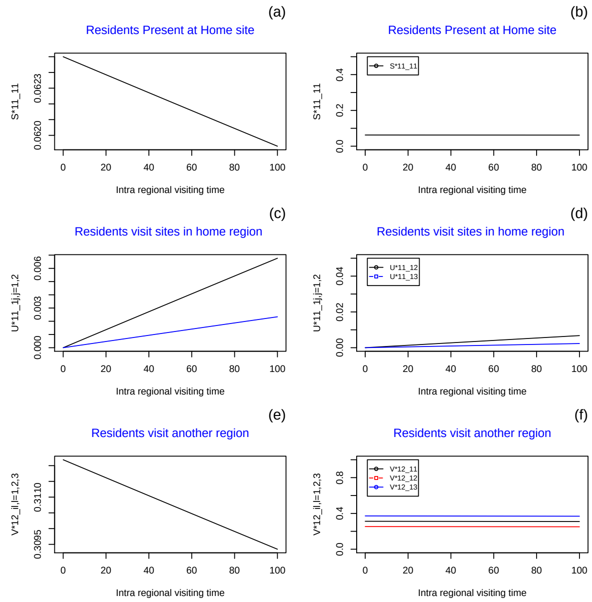

Figure 2.2: Exhibits diagrams (b), (d) and (f) corresponding to (a), (c) and (e) respectively, under magnified scales. Furthermore, we note that the axes labels for (a)&(b), (c)&(d), and (e)& (f) respectively, are the same, and the axes represent quantities on different scale units. The figure displays the behavior of the proportions S_{11}^{*11}, $U_{1j}^{*11}, j = 2,3$ and $V_{1l}^{*12}, l = 1,2,3$ of the different classes of residents of site s_1^1, subject to continuous changes in intraregional visiting time over the interval from zero to 100 days, given that interregional visiting time is fixed at 10 days. More comments about this figure are given in Remark 2.2.1.

Remark 2.2.1 We observe from Figure 2.2 that as the intraregional visiting time continuously changes value from zero to 100, diagrams (a) & (b) for S_{11}^{*11} and (e) & (f) for $V_{1l}^{*12}, l = 1,2,3$, show a smooth decrease, indicating that for larger values of intraregional visiting time, S_{11}^{*11}, and $V_{1l}^{*12}, l = 1,2,3$ turn to be smaller values. This qualitative behavior of S_{11}^{*11}, and $V_{1l}^{*12}, l = 1,2,3$ is exhibited in Diagrams (a) and (e), respectively. Diagrams (d) for $U_{1j}^{*11}, j = 2,3$ on the other hand has a smooth rise with intraregional visiting time, also illustrating the growth of $U_{1j}^{*11}, j = 2,3$ with larger values of intraregional visiting time. Furthermore, for low intraregional visiting time,(i) S_{11}^{*11}, and V_{1l}^{*12} are maximum, for $l = 1,2,3$ and (ii) $U_{1j}^{*11}, j = 2,3$ is minimum. That is, residents of site s_1^1 distribute them selves between home sites and sites in other regions.

Case 2: Varying Interregional Visiting Time and Constant Intra-regional Visiting Time: When the intra-regional visiting time of residents of a given site s_1^1 in region C_1 is 20 days and the corresponding interregional visiting time of the residents, $\tau_{1l}^{12}, l = 1,2,3$, equally vary between zero and 100 days. We utilize (2.2.1), (2.2.2), (2.2.3) and the basic methods of graph sketching[85], to obtain values and graphs for the steady state population for the different categories of the residents of site s_1^1 over all $0 \le \tau_{1l}^{12} \le 100, l = 1,2,3$. The results are illustrated in Figure 2.3.

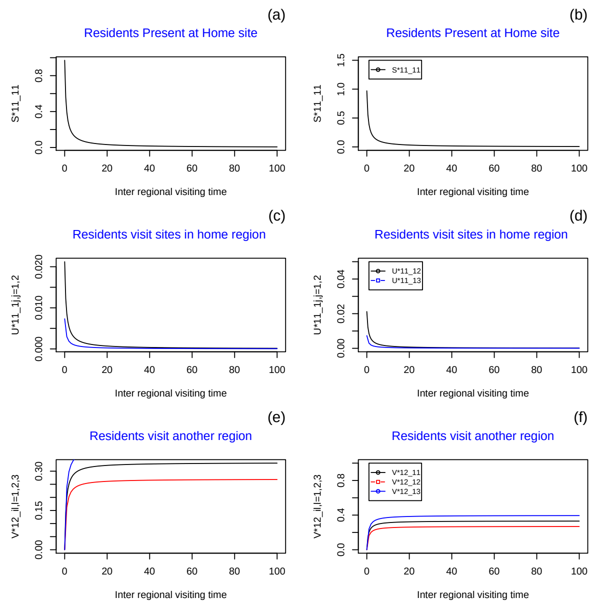

Figure 2.3: Exhibits diagrams (b), (d) and (f) corresponding to (a), (c) and (e) respectively, under magnified scales. Furthermore, we note that the axes labels for (a)&(b), (c)&(d), and (e)& (f) respectively, are the same, and the axes represent quantities on different scale units. The figure shows the behavior of the proportions S_{11}^{*11}, $U_{1j}^{*11}, j = 2,3$ and $V_{1l}^{*12}, l = 1,2,3$ of the different classes of residents of site s_1^1, when the proportions are subject to continuous changes in interregional visiting time over the interval from zero to 100 days. And the intraregional visiting time is 20 days. More comments about this figure are given in Remark 2.2.2.

Remark 2.2.2 We observe from Figure 2.3 that as the interregional visiting time continuously changes value from zero to 100, diagrams (a)&(b) for S_{11}^{*11}, and (b) & (c) for $U_{1j}^{*11}, j = 2,3$, show a continuous decrease, and thus indicating that for larger values of interregional visiting time, S_{11}^{*11}, and $U_{1j}^{*11}, j = 2,3$ tend to be very small values. This qualitative behavior of S_{11}^{*11}, and $U^{*11}_{1j}, j = 2,3$ is exhibited in Diagrams (a) and (c), respectively. Diagrams (e) for $l = 1,2,3$, V^{12}_{1l}, has a continuous rise with an intraregional visiting time. Thus there is a tendency for V^{*12}_{1l} to increase with larger values of interregional visiting time. Also, for this example, the rising of $V^{*12}_{1l}, l = 1,2,3$ approaches one, for large values of interregional visiting time. These observations signify that for sufficiently larger values of interregional visiting time, the residents of site s^1_1 tend to distribute them selves only among sites in other regions. For this specific scenario, it is clear that for sufficiently large values of interregional visiting time, the residents of site s^1_1 totally isolate their home region. Therefore the fixed intraregional visiting time does not change the residents' decision to emigrate to another region. That is, after spending more that 100 days in another region, all residents of site s^1_1 would become permanent residents of other sites in those regions C_r.

Case 3: Varying Inter- and Intra-regional Visiting Times: When the intra-regional $\tau^{11}_{1j}, j = 2,3$, and interregional $\tau^{12}_{1l}, l = 1,2,3$ visiting times of residents of the site s^1_1 in region C_1 equally vary between zero and 100 days. From (2.2.1), (2.2.2), (2.2.3) and the basic methods of graph sketching[85], we obtain values and graphs for the steady state population for the different categories of the residents of site s^1_1 over all $0 \le \tau^{11}_{1j}, \tau^{12}_{1l} \le 100, j = 2,3, l = 1,2,3$. The results are shown in Figure 2.4. We note that the results under these scenarios, exhibit the behavior of the steady state population as visitors are allowed to spend up to 100 days at their destinations.

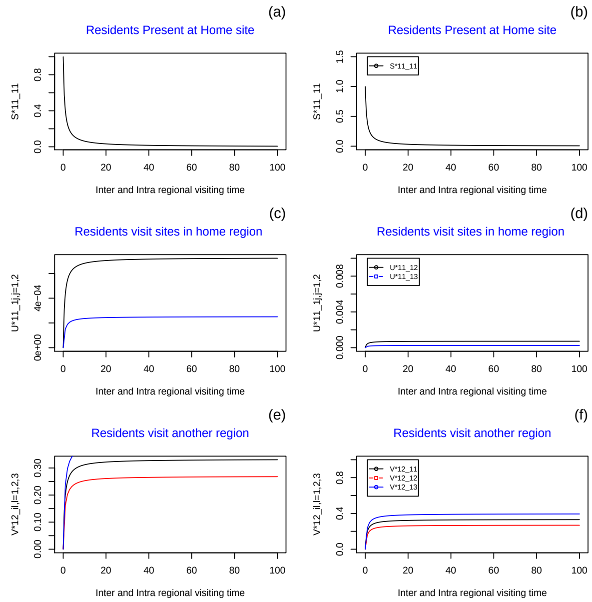

Figure 2.4: Exhibits diagrams (b), (d) and (f) corresponding to (a), (c) and (e) respectively, under magnified scales. Furthermore, we note that the axes labels for (a)&(b), (c)&(d), and (e)& (f) respectively, are the same, and the axes represent quantities on different scale units. The figure further exhibits the behavior of the proportions S^{*11}_{11}, $U^{*11}_{1j}, j = 2,3$ and $V^{*12}_{1l}, l = 1,2,3$ of the different classes of residents of site s^1_1, when the proportions are subject to continuous change in interregional and intra-regional visiting time simultaneously over the interval from zero to 100 days.The interregional and intra-regional visiting time are are assumed to be equal. More comments about this figure are given in Remark 2.2.3.

Remark 2.2.3 Figure 2.4 exhibit that as both visiting times continuously change values from 0 to 100, for S^{11}_{11}, graphs (b) show a smooth decrease. Diagrams (d) for $U^{*11}_{1j}, j = 2,3,$ and (f) for $V^{*12}_{1l}, l = 1,2,3$ show a continuous rise with intra-regional and interregional visiting time and vice versa. This qualitative behavior of $U^{*11}_{1j}, j = 2,3,$ and $V^{*12}_{1l}, l = 1,2,3$ is exhibited in Diagrams (d) and (f), respectively. This suggests that more residents are found visiting other sites that are within their regions or in other regions when intra-regional and interregional visiting time are large. This also signifies that for smaller values of intra-regional and interregional visiting times, more residents are found in their home site s^1_1, which could lead to an isolation of home site.

For this particular example, with the increase visiting times the rise of both U_{1j}^{*11}, $j = 2,3$, and $V_{1l}^{*12}, l = 1,2,3$ saturate at different values less than one. This signifies that there is always a fraction of residents of site s_1^1 at other sites within their region and in other regions.

2.3 Conclusion

A special two-scale human mobility dynamic model with underlying real life human mobility pattern and specified travel and return rates is implicitely defined. Comparative graphical representations of the steady state population behavior of residents of a given site with respect to their current locations are exhibited for different human mobility pattern strutures. The different human mobility pattern structures are influnced by continuous changes in the intra and inter-regional visiting times. The simulated findings reveal different steady state population displacement trends over the contiuous changes in the intra and inter-regional visiting times.

3 FUNDAMENTAL PROPERTIES OF A TWO-SCALE NETWORK STOCHASTIC HUMAN EPIDEMIC DYNAMIC MODEL

3.1 Introduction

The recent advent of high technology in the areas such as communication and transportation has increased the rate and effects of globalization in many aspects of the human species. Of particular importance is the rate of globalization of human infectious diseases[7]. For instance, the 2009 H1N1 flu pandemic[39] is a result of the many inter-patch connections facilitated human transportation. Several mathematical models describing the dynamics of infectious diseases of humans have been studied. Models describing the dynamics of insect vector born diseases[13, 52], influenza[8], HIV[48, 49, 51] and AIDS [50] are studied.

There has also been many studies[8, 9, 11, 12, 13, 26, 27, 20, 32, 42, 53, 54, 25] describing the dynamics of human mobility and disease in meta-populations. Generally, these models can be called multi-group models as they describe the dynamics of diseases in a network of the patches of a meta-population. These models can be further categorized into two general classes based on the modeling approach, namely: Langrangian[53, 54, 25, 13, 26, 27] and Eulerian [20, 32, 42, 11, 12, 8, 9] models. In addition, individuals in the population are labeled based on their residence or their current location. In Langrangian models, individuals do not change their residence, but are allowed to visit other patches in the meta-population. The Eulerian models on the other hand label individuals in the population based only on the current location. Moreover, this model can be considered to be migration models because only the present location of individuals is important.

Many authors have investigated the dynamics of diseases described with SIRS models. A significant portion of SIRS models study the dynamics of the disease under variant incident rates[40, 41, 42, 43, 44, 45, 46]. Using Lyapunov functions, the local nonlinear and global stability of the equilibria is established[40]. By constructing a Lyapunov function based on the structure of the biological system [43, 28, 29], the existence, uniqueness and global stability of the endemic equilibrium are investigated. Furthermore, the bifurcation and stability analysis of the disease free and endemic equilibria, are investigated in [42, 45, 46]. SIRS epidemic models have also been described and studied using complex network of human contacts[47]. In [58], a special SIRS epidemic model is formulated with a proportional direct transfer from the infectious state to the susceptible state immediately after the infectious period.

Stochastic models offer a better representation of the reality. Several stochastic models describing single and multi-group disease dynamics have been investigated[55, 56, 50, 51]. Assuming random perturbation about the endemic equilibrium of a two-group SIR model, the stochastic asymptotic stability of the endemic equilibrium via constructing a Lyapunov function according to the structure the system is established in [55]. Also, the stability of the competitive equilibrium [61], disease free equilibrium for SIRS[57] and SIR[56] single-group epidemic models are studied. Furthermore, by showing the existence of nonnegative solution for a stochastic model, the stochastic asymptotic stability behavior of the equilibria is proved in[50, 51, 61, 62].

In more complex meta-population structures, the understanding of the dynamics of infectious diseases is still in the infancy level. This is due to the high degree of heterogeneities and complexity of spatial human population structures. In Chapter 1, we characterized various patterns of static behavior of multi-scale structured meta-population human mobility process described by the Langrangian type dynamic model (1.3.13)-(1.3.15).

In this paper we incorporate the multi-scale structured meta-population human mobility process (1.3.13)-(1.3.15) into an SIRS human epidemic model that is under the influence of random environmental fluctuations. The resulting two-scale network structured SIRS human epidemic stochastic dynamic model is an extension, expansion and generalization of the structured deterministic epidemic model [25] that is under the influence of mobility process.The presented stochastic two-scale network human dynamic epidemic process is described by a large-scale system of Ito-Doob stochastic differential equations. In addition to well defined underlying system parameter domains for disease eradication in the large-scale two level dynamic structure, the results are algebraically simple, computationally attractive and explicit system parameter dependent threshold values.

This chapter is organized as follows. In Section 3.2 we describe the general stochastic SIRS epidemic process that is under the influence of human mobility process[59]. In Section 3.3, the model validation is exhibited. The existence and asymptotic stability of the disease free equilibrium is shown in Section 3.4.

3.2 Large Scale Two Level SIRS Epidemic Process

In this section, we define the structure of the SIRS epidemic dynamic process in the two-scale network population dynamic structure. The human mobility dynamic structure of the intra and inter-regional levels of the SIRS epidemic dynamic model of this study are exhibited in [Fig. 1,[30]] and [Fig. 2.[30]] respectively. Furthermore, the characterization of the human mobility hierarchic process in the two-scale population dynamic structure is also exhibited in [30]. The general SIRS disease structure with dual conversions to the susceptible class from the infectious and immune populations exhibited in this study is inspired by the work [58]. We make the following definitions related to the SIRS disease process.

Definition 3.2.1 ***Endemic population decomposition and Aggregation:*** *For each $r \in I(1,M)$, let $i \in I_i^r(1,n_r)$. The total population N_{i0}^{rr} of residents of site s_i^r at time t is distributed among the sites in their intra and inter regional domain $C(s_i^r)$, and it is partitioned into three general disease compartments namely, susceptible (S), infectious (I) and removals (R) (those who were previously sick and have acquired immunity from the disease). That is, A_{il}^{rq} is the number of residents of site s_i^r whose disease status is of type $A, A \in \{S,I,R\}$, and are visiting to site $s_l^q, l \in I_i^q(1,n_q)$ in region C_q, where $q \in I^r(1,M)$. Furthermore, when $r = q$, A_{ik}^{rr} is the number of residents of site s_i^r with disease status $A \in \{S,I,R\}$, and are visiting to site $s_k^r, k \in I_i^r(1,n_r)$ in their home region C_r. Moreover, when $k = i$, A_{ii}^{rr} is the number of residents of site s_i^r who have disease status of type $A, A \in \{S,I,R\}$ and remain as permanent residents at their home site. Hence N_i^{rr} is given by*

$$N_{i0}^{rr} = S_{i0}^{rr} + I_{i0}^{rr} + R_{i0}^{rr}, \tag{3.2.1}$$

where

$$S_{i0}^{rr} = \sum_{q=1}^{M}\sum_{k=1}^{n_q} S_{ik}^{rq}, \quad I_{i0}^{rr} = \sum_{q=1}^{M}\sum_{k=1}^{n_q} I_{ik}^{rq}, \quad \text{and} \quad R_{i0}^{rr} = \sum_{q=1}^{M}\sum_{k=1}^{n_q} R_{ik}^{rq}. \tag{3.2.2}$$

Remark 3.2.1 *We note that the effective population* $eff(N_{i0}^{rr})$ *present at the site* s_i^r *at anytime is different from the census population or the total number of residents* N_{i0}^{rr} *(3.2.1) with permanent residence site* s_i^r*. At anytime* t*, the effective community size of site* s_i^r *is made up of the permanent residents of site* s_i^r *and all visitors of to site* s_i^r*. This is as given below*

$$eff(N_i^{rr}) = \sum_{q=1}^{M}\sum_{k=1}^{n_q} S_{ki}^{qr} + \sum_{q=1}^{M}\sum_{k=1}^{n_q} I_{ki}^{qr} + \sum_{q=1}^{M}\sum_{k=1}^{n_q} R_{ki}^{qr}. \tag{3.2.3}$$

$eff(N_i^{rr})$ *represents the population that is at risk for infection at site* s_i^r *and it is the population size resulted by the mobility process in the two-scale network structure.*

Definition 3.2.2 ***Disease Transmission Process:*** *The disease transmission process in any site* s_i^r *in region* C_r *in a mobile population necessitates: (1) a susceptible person to travel from site* s_k^u *in region* C_u *to site* s_i^r*, (* $u = r$ *and* $k = i$ *if there is no traveling), (2) an infectious person traveling from site* s_l^q *in region* $C_q, q \neq r$ *to site* s_i^r*, (3) the susceptible and infectious persons meeting at a contact zone* z *(which may be the home, market place or recreational facility etc) in site* s_i^r *with a probability* p *of a person being at a zone* z *at anytime* t*, and (4)* β *is the probability of the infectious agent being transmitted from the infectious person to the susceptible person knowing that the contact between the susceptible and the infectious individual took place.*

Let n_{ri} *be the number of contact zones denoted by* $z_{i_b}^r, b \in \{1,2,\ldots,n_{r_i}\} \equiv I(1,n_{r_i})$ *at each site* s_i^r*.Furthermore, let* $p_{i_b}^r$ *be the probability that a member of the effective population would be in a zone* $z_{i_b}^r$ *at a time* t*; in addition, we assume that the events of visiting contact zones are independent, and the probability* $p_{i_b}^r$ *of being in a given zone* $z_{i_b}^r$ *is independent of the permanent residence of the individual. In each zone* $z_{i_b}^r$*, there is random mixing and transmission of the infectious agent from an infectious person to a susceptible person via a direct contact between the two individuals. Moreover, let* β_{ikj}^{ruv*} *be the probability that the transmission takes place given that the contact occurs in any zone* $z_{i_b}^r, \forall b \in I(1,n_{r_i})$ *in site* s_i^r *between a susceptible* S_{ki}^{ur} *from site* s_k^u *in region* C_u *and an infectious individual* I_{mi}^{vr} *from site* s_m^v *in region* C_v*. Then the infectious rate (average number of contacts per individual per unit time required to transmit the disease),* $\beta_{i_bkm}^{ruv*}$*, in zone* $z_{i_b}^r$ *between* S_{ki}^{ur} *and* I_{mi}^{vr} *is given by*

$$\beta_{i_bkm}^{ruv*} = (p_{i_b}^r)^2 \beta_{ikm}^{ruv*}, \tag{3.2.4}$$

whenever $v, u \in I(1, M)$, and $v \neq u$. The infection process in zone $z^r_{i_b}$ is illustrated by the following transition.

$$S^{ur}_{ki} + I^{vr}_{mi} \xrightarrow{\beta^{ruv*}_{i_b km}} I^{ur}_{ki} + I^{vr}_{mi}. \tag{3.2.5}$$

Hence, the net conversion rate to the infectious class from the susceptible class during the disease transmission process at the site s^r_i in region C_r of the meta-population with M regions is given by

$$\sum_{v=1}^{M} \sum_{u=1}^{M} \sum_{m=1}^{n_v} \sum_{k=1}^{n_u} \sum_{b=1}^{n_{r_i}} \beta^{ruv*}_{i_b km} I^{vr}_{mi} S^{ur}_{ki} \tag{3.2.6}$$

We set

$$\beta^{ruv}_{ikm} = \sum_{b=1}^{n_{r_i}} \beta^{ruv*}_{i_b km} \tag{3.2.7}$$

We further assume that the disease status of an individual in the population does not affect travel rates and the mobility pattern.

A diagram illustrating the disease transmission and mobility processes in the two scale dynamic structure described in Definition 3.2.2 is exhibited in Figure 3.1.

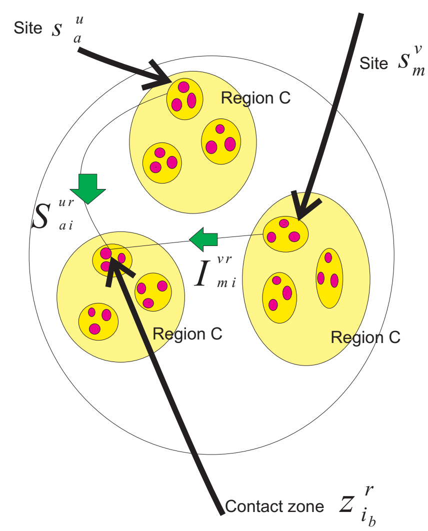

Figure 3.1: Shows the movement of susceptible (S^{ur}_{ai}) and infective (I^{vr}_{mi}) from arbitrary home site s^u_a in region C_u and from site s^v_m in region C_v, to visit an arbitrary contact zone $z^r_{i_b}$ in site s^r_i, which is in region C_r. Disease transmission takes place in zone $z^r_{i_b}$.

Definition 3.2.3 ***Acquisition and Loss of Immunity Process:*** *The changes in environmental conditions influence the immunity systems of individuals in the large scale two level population dynamic structure. This leads to dependence of the acquisition and loss of immunity rates of residents of all sites in all regions in the two-scale structured population, on the current locations of the residents in the population dynamic structure. In each site* s_i^r*, let* $\frac{1}{\rho_i^r}$ *be the average active infectious period of infected individual* (I) *who recovered from the disease and acquired immunity (R),immediately after the infectious period. Also, let* $\frac{1}{\eta_i^r}$ *be the average infectious period of infected person in site* s_i^r*, who is recovered from the disease and become susceptible (S), immediately, after the infectious period. Furthermore, let* $\frac{1}{\alpha_i^r}$ *be the average immunity period of removal person (R) in site* s_i^r*, who has lost his/her their immunity and become susceptible (S) again immediately after the immunity period. The recovery process of an infected person in site* s_i^r *as well as the loss of immunity of a removal person is illustrated in the following disease transition processes:*

$$I_{ki}^{ur} \xrightarrow{\rho_i^r} R_{ki}^{ur}, \quad I_{ki}^{ur} \xrightarrow{\eta_i^r} S_{ki}^{ur}, \quad R_{ki}^{ur} \xrightarrow{\alpha_i^r} S_{ki}^{ur}, \tag{3.2.8}$$

for $u \in I(1,M)$ *and* $k \in I(1,n_u)$.

Definition 3.2.4 ***Population Demography:*** *The current SIRS infectious disease involves time scales that are comparable with the life-time of individuals in the population. Furthermore, all births occur at home site and deaths occur at current locations of residents in the two-scale population structure. Let* B_i^r *be a constant birthrate of the human population at site* s_i^r *and at time* t. *We assume that every new born is a susceptible and becomes a resident of the site of birth. Let* δ_i^r *be the per capita natural mortality rate, and let* d_i^r *be the per capita disease related mortality rate of all members of the effective population at site* s_i^r.

A compartmental framework illustrating the different process and stages in the SIRS epidemic described above is exhibited in Figure 3.2.

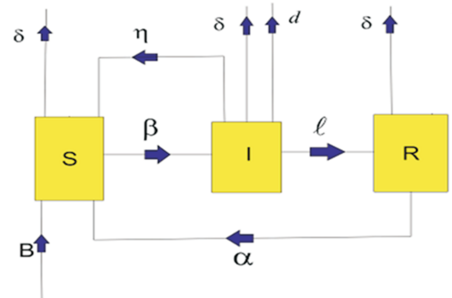

Figure 3.2: Compartmental framework summarizing the transition stages in the SIRS epidemic process. All the parameters presented in this figure are define in Section 3.2 for particular sites and regions.

From Definition 3.2.1-Definition 3.2.4, the complete SIRS epidemic model under the influence of a large scale two-level population mobility process[30] is described by:

$$\frac{dS_{il}^{rq}}{dt} = \begin{cases} [B_i^r + \sum_{k=1}^{n_r} \rho_{ik}^{rr} S_{ik}^{rr} + \sum_{q\neq r}^{M} \sum_{a=1}^{n_q} \rho_{ia}^{rq} S_{ia}^{rq} + \eta_i^r I_{ii}^{rr} + \alpha_i^r R_{ii}^{rr} \\ -(\gamma_i^r + \sigma_i^r + \delta_i^r) S_{ii}^{rr} - \sum_{u=1}^{M} \sum_{a=1}^{n_u} \beta_{iia}^{rru} S_{ii}^{rr} I_{ai}^{ur}], \quad for \quad q = r, l = i \\ [\sigma_{ij}^{rr} S_{ii}^{rr} + \eta_j^r I_{ij}^{rr} + \alpha_j^r R_{ij}^{rr} - (\rho_{ij}^{rr} + \delta_j^r) S_{ij}^{rr} \\ -\sum_{u=1}^{M} \sum_{a=1}^{n_u} \beta_{jia}^{rru} S_{ij}^{rr} I_{aj}^{ur}], \quad for \quad q = r, l = j, j \neq i, \\ [\gamma_{il}^{rq} S_{ii}^{rr} + \eta_l^q I_{il}^{rq} + \alpha_l^q R_{il}^{rq} - (\rho_{il}^{rq} + \delta_l^q) S_{il}^{rq} \\ -\sum_{u=1}^{M} \sum_{a=1}^{n_u} \beta_{lia}^{qru} S_{il}^{rq} I_{al}^{uq}], \quad for \quad q \neq r, \end{cases} \tag{3.2.9}$$

$$\frac{dI_{il}^{rq}}{dt} = \begin{cases} [\sum_{k=1}^{n_r} \rho_{ik}^{rr} I_{ik}^{rr} + \sum_{q\neq r}^{M} \sum_{a=1}^{n_q} \rho_{ia}^{rq} I_{ia}^{rq} - \eta_i^r I_{ii}^{rr} - \rho_i^r I_{ii}^{rr} \\ -(\gamma_i^r + \sigma_i^r + \delta_i^r + d_i^r) I_{ii}^{rr} + \sum_{u=1}^{M} \sum_{a=1}^{n_u} \beta_{iia}^{rru} S_{ii}^{rr} I_{ai}^{ur}], \quad for \quad q = r, l = i \\ [\sigma_{ij}^{rr} I_{ii}^{rr} - \eta_j^r I_{ij}^{rr} - \rho_j^r I_{ij}^{rr} - (\rho_{ij}^{rr} + \delta_j^r + d_j^r) I_{ij}^{rr} \\ + \quad \sum_{u=1}^{M} \sum_{a=1}^{n_u} \beta_{jia}^{rru} S_{ij}^{rr} I_{aj}^{ur}], \quad for \quad q = r, l = j, i \neq j, \\ [\gamma_{il}^{rq} I_{ii}^{rr} - \eta_l^q I_{il}^{rq} - \rho_l^q I_{il}^{rq} - (\rho_{il}^{rq} + \delta_l^q + d_l^q) I_{il}^{rq} \\ +\sum_{u=1}^{M} \sum_{a=1}^{n_u} \beta_{lia}^{qru} S_{il}^{rq} I_{al}^{uq}], \quad for \quad q \neq r, \end{cases} \tag{3.2.10}$$

$$\frac{dR_{il}^{rq}}{dt} = \begin{cases} [\sum_{k=1}^{n_r} \rho_{ik}^{rr} R_{ik}^{rr} + \sum_{q \neq r}^{M} \sum_{l=1}^{n_q} \rho_{il}^{rq} R_{il}^{rq} + \rho_i^r I_{ii}^{rr} - (\gamma_i^r + \sigma_i^r + \alpha_i^r + \delta_i^r) R_{ii}^{rr}], \\ for \quad q = r, l = i \\ [\sigma_{ij}^{rr} R_{ii}^{rr} + \rho_j^r I_{ij}^{rr} - (\rho_{ij}^{rr} + \alpha_j^r + \delta_j^r) R_{ij}^{rr}], \quad for \quad q = r, l = j, i \neq j, \\ [\gamma_{il}^{rq} R_{ii}^{rr} + \rho_l^q I_{il}^{rq} - (\rho_{il}^{rq} + \alpha_l^q + \delta_l^q) R_{il}^{rq}], \quad for \quad q \neq r, \end{cases} \tag{3.2.11}$$

where $i \in I(1,n_r), l \in I_i^r(1,n_q); r \in I(1,M), q \in I^r(1,M)$. Furthermore, the parameters B_i^r, η_a^u, α_a^u, δ_a^u and d_a^u are nonnegative, and ρ_a^u is positive for $r,u \in I(1,M)$, $i \in I(1,n_r)$, and $a \in I(1,n_u)$. Also, at time $t = t_0$, and for each $r \in I(1,M)$, and $i \in I(1,n_r)$, $(S_{ii}^{rr}(t_0), S_{ij}^{rr}(t_0), S_{il}^{rq}(t_0)) = (S_{ii0}^{rr}, S_{ij0}^{rr}, S_{il0}^{rq})$, $(I_{ii}^{rr}(t_0), I_{ij}^{rr}(t_0), I_{il}^{rq}(t_0)) = (I_{ii0}^{rr}, I_{ij0}^{rr}, I_{il0}^{rq})$, $(R_{ii}^{rr}(t_0), R_{ij}^{rr}(t_0), R_{il}^{rq}(t_0)) = (R_{ii0}^{rr}, R_{ij0}^{rr}, R_{il0}^{rq})$, whenever $j \in I_i^r(1,n_r)$ and $l \in I_i^r(1,n_q)$. Furthermore, we denote $n = \sum_{u=1}^{M} n_u$. We now incorporate the effects of the random environmental perturbations into the modeling epidemic dynamic process described in (3.2.9)-(3.2.11).

The random fluctuations lead to variabilities in the disease transmission, human mobility, birth and death processes of the system. In this chapter, we assume that the effects of the fluctuating environment manifest mainly as variations in the infectious rate β. Generally, we represent the variability in the infectious rate by a white noise process as:

$$\beta \to \beta + v\xi(t), \quad dw(t) = \xi(t)dt, \quad and \quad var(\beta(t)) = v^2, \tag{3.2.12}$$

where $\xi(t)$ is the standard white noise process, and $w(t)$ is corresponding normalized Wiener process or a homogenous Brownian motion process with the following properties: $w(0) = 0$, $E(w(t)) = 0$ and $var(w(t)) = t$.

Given $t \geq t_0$, we let $(\Omega, \mathcal{F}, P)$ be a complete probability space, and $\mathcal{F}_t$ is a filtration (that is sub $\sigma - algebra\ \mathcal{F}_t$ satisfies the following: given $t_1 \leq t_2 \Rightarrow \mathcal{F}_{t_1} \subset \mathcal{F}_{t_2}$; $E \in \mathcal{F}_t$ and $P(E) = 0 \Rightarrow E \in \mathcal{F}_0$), for each $r \in I(1,M)$, and $i \in I(1,n_r)$, the variability in the infectious process at sites s_i^r, s_j^r and s_l^q between a susceptible from site s_k^u and an infective from an arbitrary site s_m^v, can be represented as follows:

$$\begin{aligned} \beta_{ikm}^{ruv} &\to \beta_{ikm}^{ruv} + v_{ikm}^{ruv} \xi_{ikm}^{ruv}(t), \quad dw_{ikm}^{ruv}(t) = \xi_{ikm}^{ruv}(t)dt \\ \beta_{jkm}^{ruv} &\to \beta_{jkm}^{ruv} + v_{jkm}^{ruv} \xi_{jkm}^{ruv}(t), dw_{jkm}^{ruv}(t) = \xi_{jkm}^{ruv}(t)dt \\ \beta_{lkm}^{quv} &\to \beta_{lkm}^{quv} + v_{lkm}^{quv} \xi_{lkm}^{quv}(t), dw_{lkm}^{quv}(t) = \xi_{lkm}^{quv}(t)dt \end{aligned} \tag{3.2.13}$$

and

$$var(\beta_{ikm}^{ruv}(t)) = (v_{ikm}^{ruv})^2, \quad var(\beta_{jkm}^{ruv}(t)) = (v_{jkm}^{ruv})^2, \quad var(\beta_{lkm}^{quv}(t)) = (v_{lkm}^{quv})^2, \tag{3.2.14}$$

where $q,u,v \in I^r(1,M)$, $k \in I_i^u(1,n_u)$, $m \in I_i^v(1,n_v)$, and $l \in I_i^r(1,n_q)$.

We substitute (3.2.13) into (3.2.9)-(3.2.11), and obtain the following two level large scale stochastic epidemic model under the influence of human mobility process [30]

$$dS_{il}^{rq} = \begin{cases} \big[B_i^r + \sum_{k=1}^{n_r} \rho_{ik}^{rr} S_{ik}^{rr} + \sum_{q\neq r}^{M} \sum_{a=1}^{n_q} \rho_{ia}^{rq} S_{ia}^{rq} + \eta_i^r I_{ii}^{rr} + \alpha_i^r R_{ii}^{rr} \\ -(\gamma_i^r + \sigma_i^r + \delta_i^r) S_{ii}^{rr} - \sum_{u=1}^{M} \sum_{a=1}^{n_u} \beta_{iia}^{rru} S_{ii}^{rr} I_{ai}^{ur}\big] dt \\ -\big[\sum_{u=1}^{M} \sum_{a=1}^{n_u} v_{iia}^{rru} S_{ii}^{rr} I_{ai}^{ur} dw_{iia}^{rru}(t)\big], for \quad q=r, l=i, \\ \big[\sigma_{ij}^{rr} S_{ii}^{rr} + \eta_j^r I_{ij}^{rr} + \alpha_j^r R_{ij}^{rr} - (\rho_{ij}^{rr} + \delta_j^r) S_{ij}^{rr} - \sum_{u=1}^{M} \sum_{a=1}^{n_u} \beta_{jia}^{rru} S_{ij}^{rr} I_{aj}^{ur}\big] dt \\ -\big[\sum_{u=1}^{M} \sum_{a=1}^{n_u} v_{jia}^{rru} S_{ij}^{rr} I_{aj}^{ur} dw_{jia}^{rru}(t)\big], for \quad q=r, l=j, j\neq i, \\ \big[\gamma_{il}^{rq} S_{ii}^{rr} + \eta_l^q I_{il}^{rq} + \alpha_l^q R_{il}^{rq} - (\rho_{il}^{rq} + \delta_l^q) S_{il}^{rq} \\ -\sum_{u=1}^{M} \sum_{a=1}^{n_u} \beta_{lia}^{qru} S_{il}^{rq} I_{al}^{uq}\big] dt - \big[\sum_{u=1}^{M} \sum_{a=1}^{n_u} v_{lia}^{qru} S_{il}^{rq} I_{al}^{uq} dw_{lia}^{qru}(t)\big], for \quad q\neq r, \end{cases} \tag{3.2.15}$$

$$dI_{il}^{rq} = \begin{cases} \big[\sum_{k=1}^{n_r} \rho_{ik}^{rr} I_{ik}^{rr} + \sum_{q\neq r}^{M} \sum_{a=1}^{n_q} \rho_{ia}^{rq} I_{ia}^{rq} - \eta_i^r I_{ii}^{rr} - \rho_i^r I_{ii}^{rr} \\ -(\gamma_i^r + \sigma_i^r + \delta_i^r + d_i^r) I_{ii}^{rr} + \sum_{u=1}^{M} \sum_{a=1}^{n_u} \beta_{iia}^{rru} S_{ii}^{rr} I_{ai}^{ur}\big] dt \\ +\big[\sum_{u=1}^{M} \sum_{a=1}^{n_u} v_{iia}^{rru} S_{ii}^{rr} I_{ai}^{ur} dw_{iia}^{rru}(t)\big], for \quad q=r, l=i \\ \big[\sigma_{ij}^{rr} I_{ii}^{rr} - \eta_j^r I_{ij}^{rr} - \rho_j^r I_{ij}^{rr} - (\rho_{ij}^{rr} + \delta_j^r + d_j^r) I_{ij}^{rr} + \sum_{u=1}^{M} \sum_{a=1}^{n_u} \beta_{jia}^{rru} S_{ij}^{rr} I_{aj}^{ur}\big] dt \\ +\big[\sum_{u=1}^{M} \sum_{a=1}^{n_u} v_{jia}^{rru} S_{ij}^{rr} I_{aj}^{ur} dw_{jia}^{rru}(t)\big], for \quad q=r, l=j, j\neq i, \\ \big[\gamma_{il}^{rq} I_{ii}^{rr} - \eta_l^q I_{il}^{rq} - \rho_l^q I_{il}^{rq} - (\rho_{il}^{rq} + \delta_l^q + d_l^q) I_{il}^{rq} \\ +\sum_{u=1}^{M} \sum_{a=1}^{n_u} \beta_{lia}^{qru} S_{il}^{rq} I_{al}^{uq}\big] dt + \big[\sum_{u=1}^{M} \sum_{a=1}^{n_u} v_{lia}^{qru} S_{il}^{rq} I_{al}^{uq} dw_{lia}^{qru}(t)\big], for \quad q\neq r, \end{cases} \tag{3.2.16}$$

$$dR_{il}^{rq} = \begin{cases} \big[\sum_{k=1}^{n_r} \rho_{ik}^{rr} R_{ik}^{rr} + \sum_{q\neq r}^{M} \sum_{l=1}^{n_q} \rho_{il}^{rq} R_{il}^{rq} + \rho_i^r I_{ii}^{rr} - (\gamma_i^r + \sigma_i^r + \alpha_i^r + \delta_i^r) R_{ii}^{rr}\big] dt, \\ for \quad q=r, l=i \\ \big[\sigma_{ij}^{rr} R_{ii}^{rr} + \rho_j^r I_{ij}^{rr} - (\rho_{ij}^{rr} + \alpha_j^r + \delta_j^r) R_{ij}^{rr}\big] dt, for \quad q=r, l=j, j\neq i, \\ \big[\gamma_{il}^{rq} R_{ii}^{rr} + \rho_l^q I_{il}^{rq} - (\rho_{il}^{rq} + \alpha_l^q + \delta_l^q) R_{il}^{rq}\big] dt, for \quad q\neq r, \end{cases} \tag{3.2.17}$$

where $i \in I(1,n_r), l \in I_i^r(1,n_q); r \in I(1,M), q \in I^r(1,M)$; all parameters are as defined before.

At time $t=t_0$, for each $r \in I(1,M)$ and $i \in I(1,n_r)$, $(S^{rr}_{ii}(t_0), S^{rr}_{ij}(t_0), S^{rq}_{il}(t_0)) = (S^{rr}_{ii0}, S^{rr}_{ij0}, S^{rq}_{il0})$, $(I^{rr}_{ii}(t_0), I^{rr}_{ij}(t_0), I^{rq}_{il}(t_0)) = (I^{rr}_{ii0}, I^{rr}_{ij0}, I^{rq}_{il0})$, $(R^{rr}_{ii}(t_0), R^{rr}_{ij}(t_0), R^{rq}_{il}(t_0)) = (R^{rr}_{ii0}, R^{rr}_{ij0}, R^{rq}_{il0})$, whenever $j \in I^r_i(1,n_r)$ and $l \in I^r_i(1,n_q)$, where the random variables $(S^{rr}_{ii}(t_0), S^{rr}_{ij}(t_0), S^{rq}_{il}(t_0))$, $(I^{rr}_{ii}(t_0), I^{rr}_{ij}(t_0), I^{rq}_{il}(t_0))$ and $(R^{rr}_{ii}(t_0), R^{rr}_{ij}(t_0), R^{rq}_{il}(t_0))$ are $\mathcal{F}_0 - measurable$, and are independent of $w(t)$ whenever $t \geq t_0$.

We express the state of system (3.2.15)-(3.2.17) in vector form and use it, subsequently. We denote

$$
\begin{aligned}
x^{ru}_{ia} &= (S^{ru}_{ia}, I^{ru}_{ia}, R^{ru}_{ia})^T \in \mathbb{R}^3 \\
x^{ru}_{i0} &= (x^{ruT}_{i1}, x^{ruT}_{i2}, \ldots, x^{ruT}_{i,nu})^T \in \mathbb{R}^{3n_u}, \\
x^{ru}_{00} &= (x^{ruT}_{10}, x^{ruT}_{20}, \ldots, x^{ruT}_{n_r0})^T \in \mathbb{R}^{3n_r n_u}, \\
x^{r0}_{00} &= (x^{r1T}_{00}, x^{r2T}_{00}, \ldots, x^{rMT}_{00})^T \in \mathbb{R}^{3n_r \sum_{u=1}^{M} n_u}, \\
x^{00}_{00} &= (x^{10}_{00}, x^{20}_{00}, \ldots, x^{M0}_{00})^T \in \mathbb{R}^{3\left(\sum_{r=1}^{M} n_r\right)\left(\sum_{u=1}^{M} n_u\right)}, \qquad (3.2.18)
\end{aligned}
$$

where $r,u \in I(1,M)$, $i \in I(1,n_r)$, $a \in I^r_i(1,n_u)$. We set $n = \sum_{u=1}^{M} n_u$.

Definition 3.2.5

1. $\mathbf{p-norm}$ ***in*** $\mathbb{R}^{\mathbf{3n^2}}$*: Let $z^{00}_{00} \in \mathbb{R}^{3n^2}$ be an arbitrary vector defined in (3.2.18), where $z^{ru}_{ia} = (z^{ru0}_{ia1}, z^{ru0}_{ia2}, z^{ru0}_{ia3})^T$ whenever $r,u \in I(1,M)$, $i \in I(1,n_r)$, $a \in I^r_i(1,n_u)$. The $p-norm$ on $\mathbb{R}^{3n^2}$ is defined as follows*

$$||z^{00}_{00}||_p = \left(\sum_{r=1}^{M} \sum_{u=1}^{M} \sum_{i=1}^{n_r} \sum_{a=1}^{n_u} \sum_{j=1}^{3} |z^{ru0}_{iaj}|^p \right)^{\frac{1}{p}} \qquad (3.2.19)$$

whenever $1 \leq p < \infty$, and

$$\bar{z} \equiv ||z^{00}_{00}||_p = \max_{1 \leq r,u \leq M, 1 \leq i \leq n_r, 1 \leq a \leq n_u, 1 \leq j \leq 3} |z^{ru0}_{iaj}|, \qquad (3.2.20)$$

whenever $p = \infty$. Let

$$\underline{k} \equiv k^{00}_{00min} = \min_{1 \leq r,u \leq M, 1 \leq i \leq n_r, 1 \leq a \leq n_u} |k^{ru}_{ia}|. \qquad (3.2.21)$$

2. ***Closed Ball in*** $\mathbb{R}^{\mathbf{3n^2}}$*: Let $z^{*00}_{00} \in \mathbb{R}^{3n^2}$ be fixed. The closed ball in $\mathbb{R}^{3n^2}$ with center at z^{*00}_{00} and radius $r > 0$ denoted $\bar{\mathfrak{B}}_{\mathbb{R}^{3n^2}}(z^{*00}_{00}; r)$ is the set*

$$\bar{\mathfrak{B}}_{\mathbb{R}^{3n^2}}(z^{*00}_{00}; r) = \{z^{00}_{00} \in \mathbb{R}^{3n^2} : ||z^{00}_{00} - z^{*00}_{00}||_p \leq r\} \qquad (3.2.22)$$

3. ***Ito-Doob Differential:*** *Let* $x \in \mathbb{R}^{3n}$ *be a stochastic process described by the following equation*

$$dx = f(x,t)dt + g(x,t)d\omega \tag{3.2.23}$$

where ω *is a Wiener process. Furthermore, let* $V \in \mathcal{C}^{2,1}(\mathbb{R}^{3n}, \mathbb{R}^{+})$. *The Ito-Doob stochastic differential of* V *with respect to (3.2.23) is given by*

$$dV = V_t dt + V_x dx + \frac{1}{2} dx^T V_{xx} dx, \tag{3.2.24}$$

where V_t, V_x *and* V_{xx} *are the first and second order differentials of* V *respectively.*

3.3 Model Validation Results

We now show that the initial value problem associated with the system (3.2.15)-(3.2.17) has a unique solution. We observe that the rate functions of the system are nonlinear and locally Lipschitz continuous with respect to x_{00}^{00} but do not satisfy the linear growth condition. As a result of this the classical existence and uniqueness results[59] are not applicable. Therefore, we use the Lyapunov energy function method (cf.[50, 51, 59, 60]) to prove the existence and uniqueness of solution process of the system. We first state and prove two lemmas that are useful for the proof of the existence and uniqueness result. From (3.2.15)-(3.2.17), define the vector $y_{00}^{00} \in \mathbb{R}^{n^2}$ as follows: For $i \in I(1, n_r), l \in I_i^r(1, n_q), r \in I(1, M)$ and $q \in I^r(1, M)$,

$$\begin{aligned}
y_{ia}^{ru} &= S_{ia}^{ru} + I_{ia}^{ru} + R_{ia}^{ru} \in \mathbb{R}_{+} = [0, \infty) \\
y_{i0}^{ru} &= (y_{i1}^{ru}, y_{i2}^{ru}, \ldots, y_{i,nu}^{ru})^T \in \mathbb{R}_{+}^{n_u}, \\
y_{00}^{ru} &= (y_{10}^{ruT}, y_{20}^{ruT}, \ldots, y_{n_r0}^{ruT})^T \in \mathbb{R}_{+}^{n_r n_u}, \\
y_{00}^{r0} &= (y_{00}^{r1T}, y_{00}^{r2T}, \ldots, y_{00}^{rMT})^T \in \mathbb{R}_{+}^{n_r \sum_{u=1}^{M} n_u}, \\
y_{00}^{00} &= (y_{00}^{10T}, y_{00}^{20T}, \ldots, y_{00}^{M0T})^T \in \mathbb{R}_{+}^{\left(\sum_{r=1}^{M} n_r\right)\left(\sum_{u=1}^{M} n_u\right)},
\end{aligned} \tag{3.3.25}$$

and obtain

$$dy_{il}^{rq} = \begin{cases} \left[B_i^r + \sum_{k\neq i}^{n_r} \rho_{ik}^{rr} y_{ik}^{rr} + \sum_{q\neq r}^{M} \sum_{a=1}^{n_q} \rho_{ia}^{rq} y_{ia}^{rq} - (\gamma_i^r + \sigma_i^r + \delta_i^r) y_{ii}^{rr} - d_i^r I_{ii}^{rr}\right] dt, \\ for \quad q = r, l = i \\ \left[\sigma_{ij}^{rr} y_{ii}^{rr} - (\rho_{ij}^{rr} + \delta_j^r) y_{ij}^{rr} - d_j^r I_{ij}^{rr}\right] dt, forq = r, a = j \quad and \quad i \neq j, \\ \left[\gamma_{il}^{rq} y_{ii}^{rr} - (\rho_{il}^{rq} + \delta_l^q) y_{il}^{rq} - d_l^q I_{il}^{rq}\right] dt, for \quad q \neq r, y_{il}^{rq}(t_0) \geq 0, \end{cases} \tag{3.3.26}$$

In the following, we show that the solution process of the initial value problem (3.3.26) is nonnegative. That is for all $t \geq 0$, $y_{ia}^{ru}(t) \geq 0$ is nonnegative, whenever $y_{ia}^{ru}(t_0) \geq 0$.

Lemma 3.3.1 *Let $r,u \in I(1,M)$, $i \in I^r(1,n_r)$ and $a \in I_i^r(1,n_u)$. For all $t \geq t_0$, from (3.3.25), if $y_{ia}^{ru}(t_0) \geq 0$, then $y_{ia}^{ru}(t) \geq 0$.*

Proof:

It follows from (3.3.25) and (3.2.15)-(3.2.17) that the system (3.3.26) is of the form $u' = A(t,u)w(t,u), u(t_0) \geq 0$, in [[33],equation (8)] and satisfies the quasimonotonicity condition. Furthermore, from Remark 4 in [33], we assert that this system (3.3.26) has nonnegative solutions whenever $y_{il}^{rq}(0) \geq 0, \quad \forall i \in I(1,n_r), l \in I_i^r(1,n_q), r \in I(1,M)$, and $q \in I^r(1,M)$.

Remark 3.3.1 *From the decomposition described in (3.2.1), we observe that $y_{ia}^{ru}(t) = N_{ia}^{ru} = S_{ia}^{ru}(t) + I_{ia}^{ru}(t) + R_{ia}^{ru}(t)$. Furthermore, that $N_{i0}^{rr} = \sum_{u=1}^{M} \sum_{a=1}^{n_u} y_{ia}^{ru}$. Therefore, Lemma 3.3.1 established that for any nonnegative initial endemic population, the number of residents of site s_i^r present at home, y_{ii}^{rr}, or visiting any given site s_{ia}^{ru} in any other region C_u, y_{ia}^{ru}, is nonnegative. This implies that the total population of residents of site s_i^r present at home and also visiting sites in regions in their intra and intra-regional accessible domains, $N_{i0}^{rr}(t)$, is nonnegative. Moreover, Lemma 3.3.1 exibits that the effective population at any site in any region given by (3.2.3) is nonnegative at all time $t \geq t_0$. Furthermore, $R_+^{n^2} = \{y \in R^{n^2} : y \geq 0\}$ is a self-invariant set with respect to (3.3.26).*

In the following lemma, we use Lemma 3.3.1 to find an upper bound for the solution process of (3.2.15)-(3.2.17)

Lemma 3.3.2 *Let* $\mu = \min_{1\leq u\leq M, 1\leq a\leq n_u}(\delta_a^u)$. *If*

$$\sum_{r=1}^{M}\sum_{u=1}^{M}\sum_{i=1}^{n_r}\sum_{a=1}^{n_u} y_{ia}^{ru}(t_0) \leq \frac{1}{\mu}\sum_{r=1}^{M}\sum_{i=1}^{n_r} B_i^r, \tag{3.3.27}$$

then

$$\sum_{r=1}^{M}\sum_{u=1}^{M}\sum_{i=1}^{n_r}\sum_{a=1}^{n_u} y_{ia}^{ru}(t) \leq \frac{1}{\mu}\sum_{r=1}^{M}\sum_{i=1}^{n_r} B_i^r, \quad for \quad t \geq 0, a.s. \tag{3.3.28}$$

Proof:

From 3.3.25, define

$$\sum_{r=1}^{M}\sum_{u=1}^{M}\sum_{i=1}^{n_r}\sum_{a=1}^{n_u} dy_{ia}^{ru} = \sum_{r=1}^{M}\sum_{i=1}^{n_r}\left[dy_{ii}^{rr} + \sum_{a\neq i}^{n_r} dy_{ia}^{rr} + \sum_{u\neq r}^{M}\sum_{a=1}^{n_r} dy_{ia}^{ru}\right] \tag{3.3.29}$$

From (3.2.15)-(3.2.17) and (3.3.29), one can see that

$$\sum_{r=1}^{M}\sum_{u=1}^{M}\sum_{i=1}^{n_r}\sum_{a=1}^{n_u} dy_{ia}^{ru} = \left[\sum_{r=1}^{M}\sum_{i=1}^{n_r} B_i^r - \sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u=1}^{M}\sum_{a=1}^{n_u}(\delta_a^u y_{ia}^{ru} + d_a^u I_{ia}^{ru})\right] dt \tag{3.3.30}$$

From lemma 3.3.1, and (3.3.30), we have

$$d\left\{\sum_{r=1}^{M}\sum_{u=1}^{M}\sum_{i=1}^{n_r}\sum_{a=1}^{n_u} y_{ia}^{ru}\right\} \leq \left[\sum_{r=1}^{M}\sum_{i=1}^{n_r} B_i^r - \mu\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u=1}^{M}\sum_{a=1}^{n_u} y_{ia}^{ru}\right] dt \tag{3.3.31}$$

for a nonnegative differential of t. We note that(3.3.31) is a first order deterministic differential inequality[59], and its solution is given by

$$\sum_{r=1}^{M}\sum_{u=1}^{M}\sum_{i=1}^{n_r}\sum_{a=1}^{n_u} y_{ia}^{ru}(t) \leq \frac{1}{\mu}\sum_{r=1}^{M}\sum_{i=1}^{n_r} B_i^r + \left[\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u=1}^{M}\sum_{a=1}^{n_u} y_{ia}^{ru}(t_0)\right] e^{-\mu t} \tag{3.3.32}$$

Therefore, (3.3.28) is satisfied provided (3.3.27) is valid.

Remark 3.3.2 *From Lemma 3.3.2, we conclude that a closed ball in* R^{3n^2} *under the sum norm with radius* $r = \frac{1}{\mu}\sum_{r=1}^{M}\sum_{i=1}^{n_r} B_i^r$ *is self-invariant with regard to a two-scale network dynamic of human epidemic process that is under the influence of human mobility process[30].*

Prior to presenting the model validation result, we need to establish an auxiliary result. this result provides a fundamental tool in the context of the energy Lyapunov function approach.

Lemma 3.3.3 *Let us assume that the hypotheses of Lemma 3.3.2 be satisfied. Let V be a function defined by* $V : \mathbb{R}_+^{3n^2} \times \mathbb{R}_+ \to \bar{\mathbb{R}}_+$ *as follows*

$$V(x_{00}^{00}) = \sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u=1}^{M}\sum_{a=1}^{n_u} V_{ia}^{ru}(x_{ia}^{ru}), \tag{3.3.33}$$

where

$$V_{ia}^{ru}(x_{ia}^{ru}) = \left[(S_{ia}^{ru} - 1 - logS_{ia}^{ru}) + (I_{ia}^{ru} - 1 - logI_{ia}^{ru}) + (R_{ia}^{ru} - 1 - logR_{ia}^{ru})\right]. \tag{3.3.34}$$

Furthermore, let us denote

$$\begin{aligned}
M_{001}^{00} &= \max_{1\le r,q\le M, q\ne r, 1\le l\le n_q} 1 + \frac{S_{il}^{rq}}{I_{il}^{rq}},\\
M_{002}^{00} &= \max_{1\le r,q\le M, q\ne r, 1\le l\le n_q} 1 + \frac{(S_{il}^{rq})^2}{(I_{il}^{rq})^2},\\
N_{00}^{00} &= \max_{1\le r,q\le M, q\ne r, 1\le l\le n_q} 1 + S_{il}^{rq},\\
\beta_{000}^{000} &= \max_{1\le r,q,u\le M, q\ne r, 1\le l,a\le n_{q,u}} \beta_{ail}^{urq}\\
v_{000}^{000} &= \max_{1\le r,q,u\le M, q\ne r, 1\le l,a\le n_{q,u}} v_{ail}^{urq}\\
(\rho_{00}^{00}, \alpha_0^0, \delta_0^0, d_0^0, \sigma_{i0}^{00}, \rho_0^0) &= \max_{1\le r,u\le M, 1\le a\le n_u} (\rho_{ia}^{ru}, \alpha_a^u, \delta_a^u, d_a^u, \sigma_{ia}^{ru}, \rho_a^u),
\end{aligned} \tag{3.3.35}$$

Then there exists $\tilde{K} > 0$ *such that*

$$dV(x_{00}^{00}) \le \tilde{K}dt + \sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u=1}^{M}\sum_{a=1}^{n_u}\sum_{v=1}^{M}\sum_{b=1}^{n_v}\left(1 - \frac{S_{ia}^{ru}}{I_{ia}^{ru}}\right) v_{aib}^{urv} I_{ba}^{vu} dw_{aib}^{urv} \tag{3.3.36}$$

Proof:

For $r,u \in I(1,M)$, $i \in I^r(1,n_r)$ and $a \in I_i^r(1,n_u)$, under the assumptions of Lemma 3.3.2, and the definitions of S_{ia}^u, I_{ia}^u and R_{ia}^u, the function defined in (3.3.33) belongs to $V \in C^{2,1}(\mathbb{R}_+^{3n^2} \times \mathbb{R}_+, \bar{\mathbb{R}}_+)$. Moreover, we rewrite (3.3.33) as

$$\begin{aligned}
V(x_{00}^{00}) &= \sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u=1}^{M}\sum_{a=1}^{n_u} V_{ia}^{ru}(x_{00}^{00}),\\
&= \sum_{r=1}^{M}\sum_{i=1}^{n_r}\left\{V_{ii}^{rr}(x_{00}^{00}) + \sum_{a\ne i}^{n_r} V_{ia}^{rr}(x_{00}^{00}) + \sum_{u\ne r}^{M}\sum_{a=1}^{n_u} V_{ia}^{ru}(x_{00}^{00})\right\},
\end{aligned} \tag{3.3.37}$$

where

$$V_{ia}^{ru}(x_{00}^{00}) = (S_{ia}^{ru} - 1 - logS_{ia}^{ru}) + (I_{ia}^{ru} - 1 - logI_{ia}^{ru}) + (R_{ia}^{ru} - 1 - logR_{ia}^{ru}). \tag{3.3.38}$$

From (3.3.37) and (3.3.38), it follows that

$$dV(x_{00}^{00}) = \sum_{r=1}^{M}\sum_{i=1}^{n_r}\left\{dV_{ii}^{rr}(x_{00}^{00}) + \sum_{a\neq i}^{n_r} dV_{ia}^{rr}(x_{00}^{00}) + \sum_{u\neq r}^{M}\sum_{a=1}^{n_u} dV_{ia}^{ru}(x_{00}^{00})\right\}, \tag{3.3.39}$$

where

$$\begin{aligned} dV_{ia}^{ru}(x_{00}^{00}) &= \left[\left(1-\frac{1}{S_{ia}^{ru}}\right)dS_{ia}^{ru} + \frac{1}{2(S_{ia}^{ru})^2}(dS_{ia}^{ru})^2\right] + \left[\left(1-\frac{1}{I_{ia}^{ru}}\right)dI_{ia}^{ru} + \frac{1}{2(I_{ia}^{ru})^2}(dI_{ia}^{ru})^2\right] \\ &\quad + \left[\left(1-\frac{1}{R_{ia}^{ru}}\right)dR_{ia}^{ru} + \frac{1}{2(R_{ia}^{ru})^2}(dR_{ia}^{ru})^2\right]. \end{aligned} \tag{3.3.40}$$

In the following, by considering positive differential of t $(0 < \Delta t \approx dt)$, using the nature of the rate coefficients of (3.2.15)-(3.2.17) and definitions (3.3.35), we carefully estimate the three terms in the righthand side of (3.3.40). This is achieved by the usage of nested argument process.

Site level: the estimates on terms in the righthand side of (3.3.40) for the case of $u = r$, and $a = i$

$$\begin{aligned} &\left[\left(1-\frac{1}{S_{ii}^{rr}}\right)dS_{ii}^{rr} + \frac{1}{2(S_{ii}^{rr})^2}(dS_{ii}^{rr})^2\right] \\ &\leq \left\{B_i^r + \sum_{b=1}^{n_r}\rho_{ib}^{rr}S_{ib}^{rr} + \sum_{v\neq r}^{M}\sum_{b=1}^{n_v}\rho_{ib}^{rv}S_{ib}^{rv} + \eta_i^r I_{ii}^{rr} + \alpha_i^r R_{ii}^{rr} + (\gamma_i^r + \sigma_i^r + \delta_i^r + d_i^r)\right. \\ &\left. + \sum_{v=1}^{M}\sum_{b=1}^{n_v}\beta_{iib}^{rrv}I_{bi}^{vr} + \frac{1}{2}\sum_{v=1}^{M}\sum_{b=1}^{n_v}(\nu_{iib}^{rrv})^2(I_{bi}^{vr})^2\right\}dt + \left[(1-S_{ii}^{rr})\sum_{v=1}^{M}\sum_{b=1}^{n_v}\nu_{iib}^{rrv}I_{bi}^{vr}dw_{iib}^{rrv}\right], \end{aligned} \tag{3.3.41}$$

$$\begin{aligned} &\left[\left(1-\frac{1}{I_{ii}^{rr}}\right)dI_{ii}^{rr} + \frac{1}{2(I_{ii}^{rr})^2}(dI_{ii}^{rr})^2\right] \\ &\leq \left\{\sum_{b=1}^{n_r}\rho_{ib}^{rr}I_{ib}^{rr} + \sum_{v\neq r}^{M}\sum_{b=1}^{n_v}\rho_{ib}^{rv}I_{ib}^{rv} + (\rho_i^r + \eta_i^r + \gamma_i^r + \sigma_i^r + \delta_i^r + d_i^r) + \sum_{v=1}^{M}\sum_{b=1}^{n_v}\beta_{iib}^{rrv}\frac{S_{ii}^{rr}I_{bi}^{vr}}{I_{ii}^{rr}}\right. \\ &\left. + \frac{1}{2(I_{ii}^{rr})^2}(S_{ii}^{rr})^2\sum_{v=1}^{M}\sum_{b=1}^{n_v}(\nu_{iib}^{rrv})^2(I_{bi}^{vr})^2\right\}dt + \left(S_{ii}^{rr} - \frac{S_{ii}^{rr}}{I_{ii}^{rr}}\right)\left[\sum_{v=1}^{M}\sum_{b=1}^{n_v}\nu_{iib}^{rrv}I_{bi}^{vr}dw_{iib}^{rrv}\right], \end{aligned} \tag{3.3.42}$$

and

$$\left(1-\frac{1}{R_{ii}^{rr}}\right)dR_{ii}^{rr} \leq \left[\sum_{b=1}^{n_r}\rho_{ib}^{rr}R_{ib}^{rr}+\sum_{v\neq r}^{M}\sum_{b=1}^{n_v}\rho_{ib}^{rv}R_{ib}^{rv}+\rho_i^r I_{ii}^{rr}+(\gamma_i^r+\alpha_i^r+\sigma_i^r+\delta_i^r+d_i^r)\right]dt. \tag{3.3.43}$$

Regional Level: The estimated on terms in the righthand side of (3.3.40) for the case of $u = r$ and $a \neq i$:

$$\begin{aligned}\sum_{a\neq i}^{n_r}\left[\left(1-\frac{1}{S_{ia}^{rr}}\right)dS_{ia}^{rr}+\frac{1}{2(S_{ia}^{rr})^2}(dS_{ia}^{rr})^2\right] \quad \leq \quad & \sum_{a\neq i}^{n_r}\Bigg\{\Bigg[\sigma_{ia}^{rr}S_{ii}^{rr}+\eta_a^r I_{ia}^{rr}+\alpha_a^r R_{ia}^{rr}+\sum_{v=1}^{M}\sum_{b=1}^{n_v}\beta_{aib}^{rrv}I_{ba}^{vr}\\ & +(\sigma_{ia}^{rr}+\delta_a^r+d_a^r)\\ & +\frac{1}{2}\left(\sum_{v=1}^{M}\sum_{b=1}^{n_v}(v_{aib}^{rrv})^2(I_{bi}^{vr})^2\right)\Bigg]dt\\ & +(1-S_{ia}^{rr})\sum_{v=1}^{M}\sum_{b=1}^{n_v}v_{aib}^{rrv}I_{bi}^{vr}dw_{aib}^{rrv}\Bigg\},\end{aligned} \tag{3.3.44}$$

$$\begin{aligned}\sum_{a\neq i}^{n_r}\left[\left(1-\frac{1}{I_{ia}^{rr}}\right)dI_{ia}^{rr}+\frac{1}{2(I_{ia}^{rr})^2}(dI_{ia}^{rr})^2\right] \quad \leq \quad & \sum_{a\neq i}^{n_r}\Bigg\{\Bigg[\sigma_{ia}^{rr}I_{ii}^{rr}+\sum_{v=1}^{M}\sum_{b=1}^{n_v}\beta_{aib}^{rrv}S_{ia}^{rr}I_{ba}^{vr}\\ & +(\rho_a^r+\eta_a^r+\rho_{ia}^{rr}+\delta_a^r+d_a^r)\\ & +\frac{(S_{ia}^{rr})^2}{2(I_{ia}^{rr})^2}\sum_{v=1}^{M}\sum_{b=1}^{n_v}(v_{aib}^{rrv})^2(I_{ba}^{vr})^2\Bigg]dt\\ & +\left(S_{ia}^{rr}-\frac{S_{ia}^{rr}}{I_{ii}^{rr}}\right)\left[\sum_{v=1}^{M}\sum_{b=1}^{n_v}v_{aib}^{rrv}I_{ba}^{vr}dw_{aib}^{rrv}\right]\Bigg\},\end{aligned} \tag{3.3.45}$$

and

$$\sum_{a\neq i}^{n_r}\left(1-\frac{1}{R_{ia}^{rr}}\right)dR_{ia}^{rr} \quad \leq \quad \sum_{a\neq i}^{n_r}[\sigma_{ia}^{rr}R_{ii}^{rr}+\rho_a^r I_{ia}^{rr}+(\rho_{ia}^{rr}+\alpha_a^r+\delta_a^r+d_a^r)]\,dt. \tag{3.3.46}$$

Interregional Level: the estimate on terms in the righthand side of (3.3.40) for the case of $u \neq r$, $a \in I(1,n_u)$:

$$\begin{aligned}\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\left[\left(1-\frac{1}{S_{ia}^{ru}}\right)dS_{ia}^{ru}+\frac{1}{2(S_{ia}^{ru})^2}(dS_{ia}^{ru})^2\right] \quad &\leq \quad \sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\{[\gamma_{ia}^{ru}S_{ii}^{rr}+\eta_a^u I_{ia}^{rr}+\alpha_a^u R_{ia}^{ru} \\ & \qquad +(\rho_{ia}^{ru}+\delta_a^u+d_a^u) \\ &+ \quad \sum_{v=1}^{M}\sum_{b=1}^{n_v}\beta_{aib}^{urv}I_{ba}^{vr}+\frac{1}{2}\left[\sum_{v=1}^{M}\sum_{b=1}^{n_v}(v_{aib}^{urv})^2(I_{ba}^{vr})^2\right]\Bigg]\,dt \\ &+ \quad (1-S_{ia}^{ru})\sum_{v=1}^{M}\sum_{b=1}^{n_v}v_{aib}^{urv}I_{ba}^{vr}dw_{aib}^{urv}\Bigg\}, \end{aligned} \tag{3.3.47}$$

$$\begin{aligned}\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\left[\left(1-\frac{1}{I_{ia}^{ru}}\right)dI_{ia}^{ru}+\frac{1}{2(I_{ia}^{ru})^2}(dI_{ia}^{ru})^2\right] \quad &\leq \quad \sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\left\{\left[\gamma_{ia}^{ru}I_{ii}^{rr}+\sum_{v=1}^{M}\sum_{b=1}^{n_v}\beta_{aib}^{urv}S_{ia}^{ru}I_{ba}^{vu}\right.\right. \\ &+ \quad (\eta_a^r+\rho_a^u+\rho_{ia}^{ru}+\delta_a^u+d_a^u) \\ &+ \quad \left.\frac{(S_{ia}^{ru})^2}{2(I_{ia}^{ru})^2}\left(\sum_{v=1}^{M}\sum_{b=1}^{n_v}(v_{aib}^{urv})^2(I_{ba}^{vr})^2\right)\right]dt \\ &+ \quad \left.\left(S_{ia}^{ru}-\frac{S_{ia}^{ru}}{I_{ia}^{ru}}\right)\sum_{v=1}^{M}\sum_{b=1}^{n_v}v_{aib}^{urv}I_{ba}^{vr}dw_{aib}^{urv}\right\}, \end{aligned} \tag{3.3.48}$$

and

$$\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\left(1-\frac{1}{R_{ia}^{ru}}\right)dR_{ia}^{ru}\leq\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}[\gamma_{ia}^{ru}R_{ii}^{rr}+\rho_a^u I_{ia}^{ru}+(\rho_{ia}^{ru}+\alpha_a^u\delta_a^u+d_a^u)]\,dt. \tag{3.3.49}$$

From (3.3.40) and (3.3.41)-(3.3.43), the first term in the righthand side of (3.3.37) can be estimated as follows:

$$\begin{aligned}
\sum_{r=1}^{M}\sum_{i=1}^{n_r} dV_{ii}^{rr}(x_{00}^{00}) &= \sum_{r=1}^{M}\sum_{i=1}^{n_r}\left\{\left[\left(1-\frac{1}{S_{ii}^{rr}}\right)dS_{ii}^{rr}+\frac{1}{2(S_{ii}^{rr})^2}(dS_{ii}^{rr})^2\right]\right.\\
&\quad+\left[\left(1-\frac{1}{I_{ii}^{rr}}\right)dI_{ii}^{rr}+\frac{1}{2(I_{ii}^{rr})^2}(dI_{ii}^{rr})^2\right]\\
&\quad\left.+\left[\left(1-\frac{1}{R_{ii}^{rr}}\right)dR_{ii}^{rr}+\frac{1}{2(R_{ii}^{rr})^2}(dR_{ii}^{rr})^2\right]\right\}\\
&\le \sum_{r=1}^{M}\sum_{i=1}^{n_r}\left\{\left[B_i^r+\sum_{b=1}^{n_r}\rho_{ib}^{rr}(S_{ib}^{rr}+I_{ib}^{rr}+R_{ib}^{rr})+\sum_{b\neq r}^{M}\sum_{b=1}^{n_v}\rho_{ib}^{rv}(S_{ib}^{rv}+I_{ib}^{rv}+R_{ib}^{rv})\right.\right.\\
&\quad+(\rho_i^r+\eta_i^r+\alpha_i^r)(S_{ii}^{rr}+I_{ii}^{rr}+R_{ii}^{rr})+3(\gamma_i^r+\sigma_i^r+\alpha_i^r+\delta_i^r+d_i^r)\\
&\quad+\left(1+\frac{S_{ii}^{rr}}{I_{ii}^{rr}}\right)\sum_{v=1}^{M}\sum_{b=1}^{n_v}\beta_{iib}^{rrv}(S_{bi}^{vr}+I_{bi}^{vr}+R_{bi}^{vr})\\
&\quad\left.+\frac{1}{2}\left(1+\frac{(S_{ii}^{rr})^2}{(I_{ii}^{rr})^2}\right)\sum_{v=1}^{M}\sum_{b=1}^{n_v}(v_{iib}^{rrv})^2(S_{bi}^{vr}+I_{bi}^{vr}+R_{bi}^{vr})^2\right]dt\\
&\quad\left.+\left(1-\frac{S_{ii}^{rr}}{I_{ii}^{rr}}\right)\left[\sum_{v=1}^{M}\sum_{b=1}^{n_v}v_{iib}^{rrv}I_{bi}^{vr}dw_{iib}^{rrv}\right]\right\},
\end{aligned}$$

$$\begin{aligned}
\le\ &\left\{\left[\sum_{r=1}^{M}\sum_{i=1}^{n_r}B_i^r+\rho_{00}^{00}\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{v=1}^{M}\sum_{b=1}^{n_v}(S_{ib}^{rv}+I_{ib}^{rv}+R_{ib}^{rv})\right.\right.\\
&+(\rho_0^0+\eta_0^0+\alpha_0^0)\sum_{r=1}^{M}\sum_{i=1}^{n_r}(S_{ii}^{rr}+I_{ii}^{rr}+R_{ii}^{rr})+3\sum_{r=1}^{M}\sum_{i=1}^{n_r}(\gamma_0^0+\sigma_0^0+\alpha_0^0+\delta_0^0+d_0^0)\\
&+M_{001}^{00}\beta_{000}^{000}\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{v=1}^{M}\sum_{b=1}^{n_v}(S_{bi}^{vr}+I_{bi}^{vr}+R_{bi}^{vr})\\
&\left.+M_{002}^{00}(v_{000}^{000})^2\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{v=1}^{M}\sum_{b=1}^{n_v}(S_{bi}^{vr}+I_{bi}^{vr}+R_{bi}^{vr})^2\right]dt\\
&\left.+\sum_{r=1}^{M}\sum_{i=1}^{n_r}\left(1-\frac{S_{ii}^{rr}}{I_{ii}^{rr}}\right)\left[\sum_{v=1}^{M}\sum_{b=1}^{n_v}v_{iib}^{rrv}I_{bi}^{vr}dw_{iib}^{rrv}\right]\right\}
\end{aligned}\tag{3.3.50}$$

From Lemma 3.3.2, (3.3.50) becomes

$$\sum_{r=1}^{M}\sum_{i=1}^{n_r}dV_{ii}^{rr}\le\tilde{K}_1dt+\sum_{r=1}^{M}\sum_{i=1}^{n_r}\left(1-\frac{S_{ii}^{rr}}{I_{ii}^{rr}}\right)\left[\sum_{v=1}^{M}\sum_{b=1}^{n_v}v_{iib}^{rrv}I_{bi}^{vr}dw_{iib}^{rrv}\right],\tag{3.3.51}$$

where

$$\tilde{K}_1 = \Bigg\{\Bigg[\sum_{r=1}^{M}\sum_{i=1}^{n_r} B_i^r + \rho_{00}^{00}\frac{1}{\mu}\sum_{r=1}^{M}\sum_{i=1}^{n_r} B_i^r(\rho_0^0+\eta_0^0+\alpha_0^0)\frac{1}{\mu}\sum_{r=1}^{M}\sum_{i=1}^{n_r} B_i^r$$
$$+3\sum_{r=1}^{M}\sum_{i=1}^{n_r}(\gamma_0^0+\sigma_0^0+\alpha_0^0+\delta_0^0+d_0^0)+M_{001}^{00}\beta_{000}^{000}\frac{1}{\mu}\sum_{r=1}^{M}\sum_{i=1}^{n_r} B_i^r$$
$$+\ M_{002}^{00}(v_{000}^{000})^2\frac{1}{\mu^2}\left(\sum_{r=1}^{M}\sum_{i=1}^{n_r} B_i^r\right)^2\Bigg]\Bigg\} > 0. \qquad (3.3.52)$$

Similarly from (3.3.40) and (3.3.44)-(3.3.46) the second term in the righthand side of (3.3.37) is estimated as

$$\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{a\neq i}^{n_r} dV_{ia}^{rr} =$$
$$\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{a\neq i}^{n_r}\left[\left(1-\frac{1}{S_{ia}^{rr}}\right)dS_{ia}^{rr}+\frac{1}{2(S_{ia}^{rr})^2}(dS_{ia}^{rr})^2\right]+\sum_{a\neq i}^{n_r}\left[\left(1-\frac{1}{I_{ia}^{rr}}\right)dI_{ia}^{rr}+\frac{1}{2(I_{ia}^{rr})^2}(dI_{ia}^{rr})^2\right]$$
$$+\sum_{a\neq i}^{n_r}\left[\left(1-\frac{1}{R_{ia}^{rr}}\right)dR_{ia}^{rr}+\frac{1}{2(R_{ia}^{rr})^2}(dR_{ia}^{rr})^2\right]$$
$$\leq \sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{a\neq i}^{n_r}\Big\{\sigma_{00}^{00}(S_{ii}^{rr}+I_{ii}^{rr}+R_{ii}^{rr})+(\eta_0^0+\alpha_0^0+\rho_0^0)(S_{ia}^{rr}+I_{ia}^{rr}+R_{ia}^{rr})$$
$$+\beta_{000}^{000}N_{00}^{00}\sum_{v=1}^{M}\sum_{b=1}^{n_v}(S_{ba}^{vr}+I_{ba}^{vr}+R_{ba}^{vr})$$
$$+3(\rho_{00}^{00}+\rho_0^0+\eta_0^0+\alpha_0^0+\delta_0^0+d_0^0)+\frac{M_{002}^{00}(v_{000}^{000})^2}{2}\left[\sum_{v=1}^{M}\sum_{b=1}^{n_v}(S_{ba}^{vr}+I_{ba}^{vr}+R_{ba}^{vr})^2\right]\Bigg\}dt$$
$$+\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{a\neq i}^{n_r}\left(1-\frac{S_{ia}^{rr}}{I_{ia}^{rr}}\right)\left[\sum_{v=1}^{M}\sum_{b=1}^{n_v} v_{aib}^{rrv}I_{ba}^{vr}dw_{aib}^{rrv}\right]$$
$$= \Bigg\{\sigma_{00}^{00}\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{a\neq i}^{n_r}(S_{ii}^{rr}+I_{ii}^{rr}+R_{ii}^{rr})+(\eta_0^0+\alpha_0^0+\rho_0^0)\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{a\neq i}^{n_r}(S_{ia}^{rr}+I_{ia}^{rr}+R_{ia}^{rr})$$
$$+\beta_{000}^{000}N_{00}^{00}\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{a\neq i}^{n_r}\sum_{v=1}^{M}\sum_{b=1}^{n_v}(S_{ba}^{vr}+I_{ba}^{vr}+R_{ba}^{vr})+\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{a\neq i}^{n_r}3(\rho_{00}^{00}+\rho_0^0+\eta_0^0+\alpha_0^0+\delta_0^0+d_0^0)$$
$$+\ \frac{M_{002}^{00}(v_{000}^{000})^2}{2}\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{a\neq i}^{n_r}\left[\sum_{v=1}^{M}\sum_{b=1}^{n_v}(S_{ba}^{vr}+I_{ba}^{vr}+R_{ba}^{vr})^2\right]\Bigg\}dt$$
$$+\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{a\neq i}^{n_r}\left(1-\frac{S_{ia}^{rr}}{I_{ia}^{rr}}\right)\left[\sum_{v=1}^{M}\sum_{b=1}^{n_v} v_{aib}^{rrv}I_{ba}^{vr}dw_{aib}^{rrv}\right]. \qquad (3.3.53)$$

Again from and Lemma 3.3.2, the above random differential inequality reduces to

$$\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{a\neq i}^{n_r} dV_{ia}^{rr} \leq \tilde{K}_2 dt + \sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{a\neq i}^{n_r}\left(1-\frac{S_{ia}^{rr}}{I_{ia}^{rr}}\right)\left[\sum_{v=1}^{M}\sum_{b=1}^{n_v} v_{aib}^{rrv} I_{ba}^{vr} dw_{aib}^{rrv}\right]. \qquad (3.3.54)$$

where

$$\begin{aligned}\tilde{K}_2 = & \left\{\sigma_{00}^{00}\frac{1}{\mu}\sum_{r=1}^{M}\sum_{i=1}^{n_r} B_i^r + (\eta_0^0+\alpha_0^0+\rho_0^0)\frac{1}{\mu}\sum_{r=1}^{M}\sum_{i=1}^{n_r} B_i^r + \beta_{000}^{000}N_{00}^{00}\frac{1}{\mu}\sum_{r=1}^{M}\sum_{i=1}^{n_r} B_i^r\right. \\ & + \left.\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{a\neq i}^{n_r} 3(\rho_{00}^{00}+\rho_0^0+\eta_0^0+\alpha_0^0+\delta_0^0+d_0^0) + \frac{M_{002}^{00}(v_{000}^{000})^2}{2}\frac{1}{\mu^2}\left(\sum_{r=1}^{M}\sum_{i=1}^{n_r} B_i^r\right)^2\right\}\end{aligned} \qquad (3.3.55)$$

Finally from (3.3.37), (3.3.40) and (3.3.47)-(3.3.49), the third term in (3.3.37) is estimated as below we get

$$\begin{aligned}&\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u\neq r}^{M}\sum_{a=1}^{n_u} dV_{ia}^{ru} = \\ &\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\left[\left(1-\frac{1}{S_{ia}^{ru}}\right)dS_{ia}^{ru} + \frac{1}{2(S_{ia}^{ru})^2}(dS_{ia}^{ru})^2\right] \\ &+\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\left[\left(1-\frac{1}{I_{ia}^{ru}}\right)dI_{ia}^{ru} + \frac{1}{2(I_{ia}^{ru})^2}(dI_{ia}^{ru})^2\right] \\ &+\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\left[\left(1-\frac{1}{R_{ia}^{ru}}\right)dR_{ia}^{ru} + \frac{1}{2(R_{ia}^{ru})^2}(dR_{ia}^{ru})^2\right] \\ &\leq \sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\{\gamma_{ia}^{ru}(S_{ii}^{rr}+I_{ii}^{rr}+R_{ii}^{rr}) + (\eta_a^u+\alpha_a^u+\rho_a^u)(S_{ia}^{ru}+I_{ia}^{ru}+R_{ia}^{ru}) \\ &+ 3(\eta_a^u+\rho_a^u+\alpha_a^u+\rho_{ia}^{ru}+\delta_a^u+d_a^u) \\ &+ (1+S_{ia}^{ru})\sum_{v=1}^{M}\sum_{b=1}^{n_v}\beta_{aib}^{urv}(S_{ba}^{vu}+I_{ba}^{vu}+R_{ba}^{vu}) \\ &+\frac{1}{2}\left(1+\frac{(S_{ia}^{ru})^2}{(I_{ia}^{ru})^2}\right)\sum_{v=1}^{M}\sum_{b=1}^{n_v}(v_{aib}^{urv})^2(S_{ba}^{vu}+I_{ba}^{vu}+R_{ba}^{vu})^2\Big\}dt \\ &+\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\sum_{v=1}^{M}\sum_{b=1}^{n_v}\left(1-\frac{S_{ia}^{ru}}{I_{ia}^{ru}}\right)v_{aib}^{urv}I_{ba}^{vu}dw_{aib}^{urv}\end{aligned}$$

$$\leq \Bigg\{ \gamma_{00}^{00} \sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}(S_{ii}^{rr}+I_{ii}^{rr}+R_{ii}^{rr}) + (\eta_0^0+\alpha_0^0+\rho_0^0)\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}(S_{ia}^{ru}+I_{ia}^{ru}+R_{ia}^{ru})$$
$$+\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u\neq r}^{M}\sum_{a=1}^{n_u} 3(\eta_0^0+\rho_0^0+\alpha_0^0+\rho_{00}^{00}+\delta_0^0+d_0^0)$$
$$+N_{00}^{00}\beta_{000}^{000}\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\sum_{v=1}^{M}\sum_{b=1}^{n_v}(S_{ba}^{vu}+I_{ba}^{vu}+R_{ba}^{vu})$$
$$+\frac{M_{002}^{00}(v_{000}^{000})^2}{2}\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\sum_{v=1}^{M}\sum_{b=1}^{n_v}(S_{ba}^{vu}+I_{ba}^{vu}+R_{ba}^{vu})^2\Bigg\} dt$$
$$+\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\sum_{v=1}^{M}\sum_{b=1}^{n_v}\left(1-\frac{S_{ia}^{ru}}{I_{ia}^{ru}}\right)v_{aib}^{urv}I_{ba}^{vu}dw_{aib}^{urv}. \qquad (3.3.56)$$

By using Lemma 3.3.2, differential inequality (3.3.57) becomes

$$\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u\neq r}^{M}\sum_{a=1}^{n_u} dV_{ia}^{ru} \leq \tilde{K}_3 dt + \sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\sum_{v=1}^{M}\sum_{b=1}^{n_v}\left(1-\frac{S_{ia}^{ru}}{I_{ia}^{ru}}\right)v_{aib}^{urv}I_{ba}^{vu}dw_{aib}^{urv}, \qquad (3.3.57)$$

where

$$\begin{aligned}\tilde{K}_3 = & \Bigg\{\gamma_{00}^{00}\frac{1}{\mu}\sum_{r=1}^{M}\sum_{i=1}^{n_r}B_i^r + (\eta_0^0+\alpha_0^0+\rho_0^0)\frac{1}{\mu}\sum_{r=1}^{M}\sum_{i=1}^{n_r}B_i^r \\ & +\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}3(\eta_0^0+\rho_0^0+\alpha_0^0+\rho_{00}^{00}+\delta_0^0+d_0^0) \\ + \; & N_{00}^{00}\beta_{000}^{000}\frac{1}{\mu}\sum_{r=1}^{M}\sum_{i=1}^{n_r}B_i^r + \frac{M_{002}^{00}(v_{000}^{000})^2}{2}\frac{1}{\mu^2}\left(\sum_{r=1}^{M}\sum_{i=1}^{n_r}B_i^r\right)^2\Bigg\}\end{aligned} \qquad (3.3.58)$$

Hence, from (3.3.51), (3.3.54) and (3.3.57), we arrive at the following stochastic differential inequality

$$\begin{aligned}dV(x_{00}^{00}(t)) \leq \; & (\tilde{K}_1+\tilde{K}_2+\tilde{K}_3)\,dt + \sum_{r=1}^{M}\sum_{i=1}^{n_r}\left(1-\frac{S_{ii}^{rr}}{I_{ii}^{rr}}\right)\left[\sum_{v=1}^{M}\sum_{b=1}^{n_v}v_{iib}^{rrv}I_{bi}^{vr}dw_{iib}^{rrv}\right] \\ + \; & \sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{a\neq i}^{n_r}\left(1-\frac{S_{ia}^{rr}}{I_{ia}^{rr}}\right)\left[\sum_{v=1}^{M}\sum_{b=1}^{n_v}v_{aib}^{rrv}I_{ba}^{vr}dw_{aib}^{rrv}\right] \\ + \; & \sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\sum_{v=1}^{M}\sum_{b=1}^{n_v}\left(1-\frac{S_{ia}^{ru}}{I_{ia}^{ru}}\right)v_{aib}^{urv}I_{ba}^{vu}dw_{aib}^{urv}.\end{aligned} \qquad (3.3.59)$$

Therefore choosing $\tilde{K} = \tilde{K}_1 + \tilde{K}_2 + \tilde{K}_3 > 0$, and combining the last three summations, concludes the proof of the theorem. We now show the existence of a unique solution of the system (3.2.15)-(3.2.17) in the following theorem.

Theorem 3.3.4 *Given any initial condition $x_{00}^{00}(t_0) \in \mathbb{R}_+^{3n^2}$ under the assumptions of Lemma 3.4.1, there is a unique solution process of the system (3.2.15)-(3.2.17) in $\mathbb{R}_+^{3n^2}$, for $t \geq t_0$, almost surely.*

Proof:

Given that the rate functions of the system are locally Lipschitz continuous in x_{00}^{00}, it follows that for any initial value $x_{00}^{00}(t_0) \in \mathbb{R}_+^{3n^2}$, there is a unique local solution of the system (3.2.15)-(3.2.17) $x_{00}^{00}(t)$, for $t \in (t_0, t_e)$, where at $t = t_e$ is the first exit time of x_{00}^{00}. Therefore to show the solution process of the system exists for all $t \geq t_0$, it suffices to show that $t_e = \infty$.

Let $k_{00}^{00} \in \mathbb{R}_+^{n^2}$. From (3.2.20) and (3.2.21), we have

$$||k_{00}^{00}||_\infty = \max_{1\leq r,u\leq M, 1\leq i\leq n_r, 1\leq a\leq n_u} |k_{ia}^{ru}|, \quad k_{00min}^{00} = \min_{1\leq r,u\leq M, 1\leq i\leq n_r, 1\leq a\leq n_u} |k_{ia}^{ru}|. \tag{3.3.60}$$

We denote

$$k \equiv k_{00min}^{00}. \tag{3.3.61}$$

We choose $k_{00}^{*00} \in \mathbb{R}_+^{n^2}$ with each component k_{ia}^{*ru}, sufficiently large such that $S_{ia}^{ru}(t_0), I_{ia}^{ru}(t_0), R_{ia}^{ru}(t_0) \in [\frac{1}{k_{ia}^{*ru}}, k_{ia}^{*ru}] \equiv \bar{\mathfrak{B}}_{\mathbb{R}}(\frac{\frac{1}{k_{ia}^{*ru}} + k_{ia}^{*ru}}{2}; \frac{\frac{1}{k_{ia}^{*ru}} - k_{ia}^{*ru}}{2})$, for $i \in I(1,n_r), a \in I(1,n_u)$, and $r,u \in I(1,M)$. In other words, from (3.2.18), $x_{00}^{00}(t_0) \in \prod_{r=1}^{M}\prod_{u=1}^{M}\prod_{i=1}^{n_i}\prod_{a=1}^{n_u}[\frac{1}{k_{ia}^{*ru}}, k_{ia}^{*ru}] \times [\frac{1}{k_{ia}^{*ru}}, k_{ia}^{*ru}] \times [\frac{1}{k_{ia}^{*ru}}, k_{ia}^{*ru}]$. From (3.3.61) let $k_0 \equiv k_{00min}^{*00}$.

Let $k_{00}^{00} \in \mathbb{R}_+^{n^2}$ be an arbitrary vector whose components k_{ia}^{ru} satisfy $k_{ia}^{ru} \geq k_{ia}^{*ru}, \forall i \in I(1,n_i), a \in I(1,n_u)$, and $r,u \in I(1,M))$. And let the local solution $x_{00}^{00}(t) \in \prod_{r=1}^{M}\prod_{u=1}^{M}\prod_{i=1}^{n_i}\prod_{a=1}^{n_u}[\frac{1}{k_{ia}^{ru}}, k_{ia}^{ru}] \times [\frac{1}{k_{ia}^{ru}}, k_{ia}^{ru}] \times [\frac{1}{k_{ia}^{ru}}, k_{ia}^{ru}]$, for $t \in (0, t_e)$ where t_e is the first hitting time of the solution process. For $t \leq t_e$, it follows that $S_{ia}^{ru}(t), I_{ia}^{ru}(t), R_{ia}^{ru}(t) \in [\frac{1}{||k_{00}^{00}||_\infty}, k_{00min}^{00}]$, for all $i \in I(1,n_r), a \in I(1,n_u), r,u \in I(1,M)$. Using (3.3.61), define a stoping time for the process as follows

$$\begin{aligned}
\tau_k &= inf\Big\{t \in (0,t_e) : \min_{1\leq r,u\leq M, 1\leq i\leq n_r, 1\leq a\leq n_u} (S_{ia}^{ru}(t), I_{ia}^{ru}(t), R_{ia}^{ru}(t)) \leq \frac{1}{||k_{00}^{00}||_\infty}, \\
&\quad \max_{1\leq r,u\leq M, 1\leq i\leq n_r, 1\leq a\leq n_u} (S_{ia}^{ru}(t), I_{ia}^{ru}(t), R_{ia}^{ru}(t)) \geq k\Big\}, \quad and \qquad (3.3.62) \\
\tau_k(t) &= \min\{t, \tau_k\}, \quad for \quad t \geq t_0. \qquad (3.3.63)
\end{aligned}$$

where k is defined in (3.3.61). Furthermore, we set $inf\varnothing = \infty$. It follows from (3.3.63) that τ_k increases as $k \to \infty$. We let $\tau_\infty = \lim_{k\to\infty} \tau_k$. From (3.3.63) it implies that

$$\tau_\infty \leq t_e \quad a.s. \tag{3.3.64}$$

Therefore to show $t_e = \infty$, we only show that $\tau_\infty = \infty$ a.s.

On the contrary suppose $\tau_\infty < \infty$, then $\exists T > 0$, such that for a given $0 < \varepsilon < 1$, $P(\tau_\infty \leq T) > \varepsilon$. This means that $\{\tau_k\}$ is a finite sequence. Moreover, from the definition of a finite sequence there exists a vector $k_{00}^{100} \in \mathbb{R}^{n^2}$, with $k_{00min}^{100} \equiv k_1 \geq k_0$, (where $k_1 \equiv k_{00min}^{100}$ is defined by (3.3.61) and (3.3.60),)

$$P(\tau_k \leq T) \geq \varepsilon, \tag{3.3.65}$$

whenever $k \geq k_1$. From (3.3.38), (3.3.37) can be rewritten as

$$\begin{aligned} V(x_{00}^{00}) \quad = \quad & \sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u=1}^{M}\sum_{a=1}^{n_u} [(S_{ia}^{ru} - 1 - logS_{ia}^{ru}) + (I_{ia}^{ru} - 1 - logI_{ia}^{ru}) \\ & + (R_{ia}^{ru} - 1 - logR_{ia}^{ru})]. \end{aligned} \tag{3.3.66}$$

From Lemma 3.3.2& 3.3.3, the stopped solution process (3.2.15)-(3.2.17) satisfies the following stochastic inequality for some $\tilde{K} > 0$.

$$dV(x_{00}^{00}(t)) \leq \tilde{K}dt + \sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u=1}^{M}\sum_{a=1}^{n_u}\sum_{v=1}^{M}\sum_{b=1}^{n_v}\left(1 - \frac{S_{ia}^{ru}}{I_{ia}^{ru}}\right) v_{aib}^{urv} I_{ba}^{vu} dw_{aib}^{urv} \tag{3.3.67}$$

Furthermore, for $\forall t_1 \leq T$, integrating both sides of (3.3.67) on $[t_0, t_1 \wedge \tau_k]$, and taking the expected values of both sides, it implies that

$$\begin{aligned} E(V(x_{00}^{00}(t_1 \wedge \tau_k))) \quad & \leq \quad V(x_{00}^{00}(t_0)) + \tilde{K}(t_1 \wedge \tau_k) \\ & \leq \quad V(x_{00}^{00}(t_0)) + \tilde{K}T \end{aligned} \tag{3.3.68}$$

Given that $k \geq k_1$, we set $E_k = \{\tau_k \leq T\}$. Then from (3.3.65), we see that $P(E_k) \geq \varepsilon$. If $\omega \in E_k$, then ω is an event at the stopping time where at least one of $S_{ia}^{ru}(\tau_k, \omega)$, $I_{ia}^{ru}(\tau_k, \omega)$, or $R_{ia}^{ru}(\tau_k, \omega)$ whenever

$r,u \in I(1,M)$, $i \in I(1,n_r)$ and $a \in I(1,n_u)$ is $\frac{1}{||k_{00}^{00}||_\infty}$ or $k \equiv k_{min}$. This implies from (3.3.66) that

$$V(x_{00}^{00}(\tau_k,\omega)) \geq [k_{min} - 1 - logk_{min}] \wedge \left[\frac{1}{||k_{00}^{00}||_\infty} - 1 - log||k_{00}^{00}||_\infty\right], \forall\omega \in E_k. \tag{3.3.69}$$

It follows from (3.3.68) and (3.3.69) that

$$\begin{aligned} V(x_{i0}^{00}(t_0)) + \tilde{K}T &\geq E(I_{E_k(\omega)}V(x_{00}^{00}(\tau_k,\omega))) \\ &\geq \varepsilon\left\{[k_{min} - 1 - logk_{min}] \wedge \left[\frac{1}{||k_{00}^{00}||_\infty} - 1 - log||k_{00}^{00}||_\infty\right]\right\}, \end{aligned} \tag{3.3.70}$$

where $I_{E_k(\omega)}$ is the indicator function of E_k.

Hence as $k = k_{min} \to \infty$, (3.3.70) implies that $V(x_{00}^{00}(t_0)) + \tilde{K}T \to \infty$ which leads to a contradiction to the existence of a local solution. Therefore, we must have $\tau_\infty = \infty$, and the rest of the proof follows.

Remark 3.3.3 *For any $r \in I(1,M)$ and $i \in I(1,n_r)$, Lemmas 3.3.1, 3.3.2, 3.3.3 and Theorem 3.3.4 show that there exists a positive self-invariant set for system (3.2.15)-(3.2.17) given by*

$$A = \left\{(S_{ia}^{ru}, I_{ia}^{ru}, R_{ia}^{ru}) : y_{ia}^{ru}(t) \geq 0 \quad and \quad \sum_{r=1}^{M}\sum_{u=1}^{M}\sum_{i=1}^{n_r}\sum_{a=1}^{n_u} y_{ia}^{ru}(t) \leq \frac{1}{\mu}\sum_{r=1}^{M}\sum_{i=1}^{n_r} B_i^r\right\} \tag{3.3.71}$$

whenever $u \in I^r(1,M)$ and $a \in I_i^r(1,n_u)$. We shall denote

$$\bar{B} \equiv \frac{1}{\mu}\sum_{r=1}^{M}\sum_{i=1}^{n_r} B_i^r \tag{3.3.72}$$

3.4 Existence and Asymptotic Behavior of Disease Free Equilibrium

In this section, we study the existence and the asymptotic behavior of the disease free equilibrium state of the system (3.2.15)-(3.2.17). The disease free equilibrium is obtained by solving the system of algebraic equations obtained by setting the drift and the diffusion parts of the system of stochastic differential equations to zero. In addition, conditions that $I = R = 0$ in the event when there is no disease in the population. We summarize the results as follows.

For any $r,u \in I(1,M)$, $i \in I(1,n_r)$ and $a \in I(1,n_u)$, let

$$D_i^r = \gamma_i^r + \sigma_i^r + \delta_i^r - \sum_{a=1}^{n_r} \frac{\rho_{ia}^{rr}\sigma_{ia}^{rr}}{\rho_{ia}^{rr} + \delta_a^r} - \sum_{u \neq r}^{M} \sum_{a=1}^{n_u} \frac{\rho_{ia}^{rr}\gamma_{ia}^{ru}}{\rho_{ia}^{ru} + \delta_a^u} > 0. \tag{3.4.73}$$

Furthermore, let $(S_{ia}^{ru*}, I_{ia}^{ru*}, R_{ia}^{ru*})$, be the equilibrium state of the system (3.2.15)-(3.2.17). One can see that the disease free equilibrium state is given by $E_{ia}^{ru} = (S_{ia}^{ru*}, 0, 0)$, where

$$S_{ia}^{ru*} = \begin{cases} \frac{B_i^r}{D_i^r}, & for \quad u = r, a = i, \\ \frac{B_i^r}{D_i^r} \frac{\sigma_{ij}^{rr}}{\rho_{ij}^{rr} + \delta_j^r}, & for \quad u = r, a \neq i, \\ \frac{B_i^r}{D_i^r} \frac{\gamma_{ia}^{ru}}{\rho_{ia}^{ru} + \delta_a^u}, & for \quad u \neq r. \end{cases} \tag{3.4.74}$$

The asymptotic stability property of E_{ia}^{ru} will be established by verifying the conditions of the stochastic version of the Lyapunov second method given in [[34],Theorem 2.4],[59], and [[34],Theorem 4.4],[59] respectively. In order to study the qualitative properties of (3.2.15)-(3.2.17) with respect to the equilibrium state $(S_{ia}^{ru*}, 0, 0)$, first, we use the change of variable. For this purpose, we use the following transformation:

$$\begin{cases} U_{ia}^{ru} &= S_{ia}^{ru} - S_{ia}^{ru*} \\ V_{ia}^{ru} &= I_{ia}^{ru} \\ W_{ia}^{ru} &= R_{ia}^{ru}. \end{cases} \tag{3.4.75}$$

By employing this transformation, system (3.2.15)-(3.2.17) is transformed into the following forms

$$dU_{il}^{rq} = \begin{cases} \left[\sum_{q \neq r}^{M} \sum_{a=1}^{n_q} \rho_{ia}^{rq} U_{ia}^{rq} + \eta_i^r V_{ii}^{rr} + \alpha_i^r W_{ii}^{rr} \right. \\ \left. -(\gamma_i^r + \sigma_i^r + \delta_i^r) U_{ii}^{rr} - \sum_{u=1}^{M} \sum_{a=1}^{n_u} \beta_{iia}^{rru} (S_{ii}^{rr*} + U_{ii}^{rr}) V_{ai}^{ur} \right] dt \\ -\left[\sum_{u=1}^{M} \sum_{a=1}^{n_u} \nu_{iia}^{rru} (S_{ii}^{rr*} + U_{ii}^{rr}) V_{ai}^{ur} dw_{iia}^{rru}(t)\right], for \quad q = r, l = i \\ \left[\sigma_{ij}^{rr} U_{ii}^{rr} + \eta_j^r V_{ij}^{rr} + \alpha_j^r W_{ij}^{rr} - (\rho_{ij}^{rr} + \delta_j^r) U_{ij}^{rr} - \sum_{u=1}^{M} \sum_{a=1}^{n_u} \beta_{jia}^{rru} (S_{ij}^{rr*} + U_{ij}^{rr}) V_{aj}^{ur}\right] dt \\ -\left[\sum_{u=1}^{M} \sum_{a=1}^{n_u} \nu_{jia}^{rru} (S_{ij}^{rr*} + U_{ij}^{rr}) V_{aj}^{ur} dw_{jia}^{rru}(t)\right], for \quad q = r, l = j, j \neq i, \\ \left[\gamma_{il}^{rq} U_{ii}^{rr} + \eta_l^q V_{il}^{rq} + \alpha_l^q W_{il}^{rq} - (\rho_{il}^{rq} + \delta_l^q) U_{il}^{rq} \right. \\ \left. -\sum_{u=1}^{M} \sum_{a=1}^{n_u} \beta_{lia}^{qru} S_{il}^{rq} I_{al}^{uq}\right] dt - \left[\sum_{u=1}^{M} \sum_{a=1}^{n_u} \nu_{lia}^{qru} (S_{il}^{rq*} + U_{il}^{rq}) V_{al}^{uq} dw_{lia}^{qru}(t)\right], \\ for \quad q \neq r, \end{cases} \tag{3.4.76}$$

$$
dV_{il}^{rq} = \begin{cases}
\left[\sum_{q=1}^{M}\sum_{a=1}^{n_q}\rho_{ia}^{rq}V_{ia}^{rq} - (\eta_i^r+\rho_i^r+\gamma_i^r+\sigma_i^r+\delta_i^r+d_i^r)W_{ii}^{rr}\right.\\
\left.+\sum_{u=1}^{M}\sum_{a=1}^{n_u}\beta_{iia}^{rru}(S_{ii}^{rr*}+U_{ii}^{rr})V_{ai}^{ur}\right]dt\\
+\left[\sum_{u=1}^{M}\sum_{a=1}^{n_u}\nu_{iia}^{rru}(S_{ii}^{rr*}+U_{ii}^{rr})V_{ai}^{ur}dw_{iia}^{rru}(t)\right], for \quad q=r, l=i\\
\left[\sigma_{ij}^{rr}V_{ii}^{rr}-(\eta_j^r+\rho_j^r+\rho_{ij}^{rr}+\delta_j^r+d_j^r)V_{ij}^{rr}+\sum_{u=1}^{M}\sum_{a=1}^{n_u}\beta_{jia}^{rru}(S_{ij}^{rr*}+U_{ij}^{rr})V_{aj}^{ur}\right]dt\\
+\left[\sum_{u=1}^{M}\sum_{a=1}^{n_u}\nu_{jia}^{rru}(S_{ij}^{rr*}+U_{ij}^{rr})V_{aj}^{ur}dw_{jia}^{rru}(t)\right], for \quad q=r, l=j, j\neq i,\\
\left[\gamma_{il}^{rq}V_{ii}^{rr}-(\eta_l^q+\rho_l^q+\rho_{il}^{rq}+\delta_l^q+d_l^q)V_{il}^{rq}\right.\\
\left.\sum_{u=1}^{M}\sum_{a=1}^{n_u}\beta_{lia}^{qru}(S_{il}^{rq*}+U_{il}^{rq})V_{al}^{uq}\right]dt+\left[\sum_{u=1}^{M}\sum_{a=1}^{n_u}\nu_{lia}^{qru}(S_{il}^{rq*}+U_{il}^{rq})V_{al}^{uq}dw_{lia}^{qru}(t)\right],\\
for \quad q\neq r,
\end{cases}
\tag{3.4.77}
$$

and

$$
dW_{il}^{rq} = \begin{cases}
\left[\sum_{q\neq r}^{M}\sum_{l=1}^{n_q}\rho_{il}^{rq}W_{il}^{rq}+\rho_i^rV_{ii}^{rr}-(\gamma_i^r+\sigma_i^r+\alpha_i^r+\delta_i^r)W_{ii}^{rr}\right]dt, for \quad q=r, l=i\\
\left[\sigma_{ij}^{rr}W_{ii}^{rr}+\rho_j^rV_{ij}^{rr}-(\rho_{ij}^{rr}+\alpha_j^r+\delta_j^r)W_{ij}^{rr}\right]dt, for \quad q=r, l=j, j\neq i\\
\left[\gamma_{il}^{rq}W_{ii}^{rr}+\rho_l^qV_{il}^{rq}-(\rho_{il}^{rq}+\alpha_l^q+\delta_l^q)W_{il}^{rq}\right]dt, for \quad q\neq r
\end{cases}
\tag{3.4.78}
$$

We state and prove the following lemmas that would be useful in the proofs of the stability results.

Lemma 3.4.1 *Let $V : \mathbb{R}^{3n^2}\times\mathbb{R}_+ \to \mathbb{R}_+$ be a function defined by*

$$
V(\tilde{x}_{00}^{00}) = \sum_{r=1}^{M}\sum_{u=1}^{M}\sum_{i=1}^{n_r}\sum_{a=1}^{n_u} V(\tilde{x}_{ia}^{ru}), \tag{3.4.79}
$$

where,

$$
\begin{aligned}
V(\tilde{x}_{ia}^{ru}) &= (S_{ia}^{ru}-S_{ia}^{ru*}+I_{ia}^{ru})^2+c_{ia}^{ru}(I_{ia}^{ru})^2+(R_{ia}^{ru})^2 \\
\tilde{x}_{00}^{00} &= (U_{ia}^{ru}, V_{ia}^{ru}, W_{ia}^{ru})^T \quad and \quad c_{ia}^{ru}\geq 0.
\end{aligned}
\tag{3.4.80}
$$

Then $V\in\mathcal{C}^{2,1}(\mathbb{R}^{3n^2}\times\mathbb{R}_+,\mathbb{R}_+)$, and it satisfies

$$
b(||\tilde{x}_{00}^{00}||) \leq V(\tilde{x}_{00}^{00}(t)) \leq a(||\tilde{x}_{00}^{00}||) \tag{3.4.81}
$$

where

$$
\begin{aligned}
b(||\tilde{x}_{00}^{00}||) &= \min_{1\le r,u\le M, 1\le i\le n_r, 1\le a\le n_u} \left\{\frac{c_{ia}^{ru}}{2+c_{ia}^{ru}}\right\} \sum_{r=1}^{M}\sum_{u=1}^{M}\sum_{i=1}^{n_r}\sum_{a=1}^{n_u}\left[(U_{ia}^{ru})^2+(V_{ia}^{ru})^2+(W_{ia}^{ru})^2\right] \\
a(||\tilde{x}_{00}^{00}||) &= \max_{1\le r,u\le M, 1\le i\le n_r, 1\le a\le n_u} \{c_{ia}^{ru}+2\} \sum_{r=1}^{M}\sum_{u=1}^{M}\sum_{i=1}^{n_r}\sum_{a=1}^{n_u}\left[(U_{ia}^{ru})^2+(V_{ia}^{ru})^2+(W_{ia}^{ru})^2\right].
\end{aligned}
\tag{3.4.82}
$$

Proof: From (3.4.78), (3.4.79) can be written as

$$
\begin{aligned}
V(x_{ia}^{ru}) &= (U_{ia}^{ru}+V_{ia}^{ru})^2 + c_{ia}^{ru}(V_{ia}^{ru})^2 + (W_{ia}^{ru})^2 \\
&= (U_{ia}^{ru})^2 + 2U_{ia}^{ru}V_{ia}^{ru} + (c_{ia}^{ru}+1)(V_{ia}^{ru})^2 + (W_{ia}^{ru})^2 \\
&= (U_{ia}^{ru})^2 + (c_{ia}^{ru}+1)(V_{ia}^{ru})^2 + 2\left(\frac{1}{\sqrt{1+\frac{c_{ia}^{ru}}{2}}}U_{ia}^{ru}\right)\left(\sqrt{1+\frac{c_{ia}^{ru}}{2}}V_{ia}^{ru}\right) + (W_{ia}^{ru})^2 \\
&= \left(-\frac{1}{1+\frac{c_{ia}^{ru}}{2}}+1\right)(U_{ia}^{ru})^2 + \left(-\left(1+\frac{c_{ia}^{ru}}{2}\right)+c_{ia}^{ru}+1\right)(V_{ia}^{ru})^2 + (W_{ia}^{ru})^2 \\
&\quad + \left[\left(\frac{1}{\sqrt{1+\frac{c_{ia}^{ru}}{2}}}U_{ia}^{ru}\right) + \left(\sqrt{1+\frac{c_{ia}^{ru}}{2}}V_{ia}^{ru}\right)\right]^2
\end{aligned}
$$

Therefore, by nothing the fact that $\min\{1-\frac{1}{1+\frac{c_{ia}^{ru}}{2}}, \frac{c_{ia}^{ru}}{2}, 1\}$, we have

$$
V(x_{ia}^{ru}) \ge \frac{c_{ia}^{ru}}{2+c_{ia}^{ru}}\left[(U_{ia}^{ru})^2+(V_{ia}^{ru})^2+(W_{ia}^{ru})^2\right] \tag{3.4.83}
$$

Hence from (3.4.83) we have

$$
\begin{aligned}
V(\tilde{x}_{00}^{00}) &\ge \sum_{r=1}^{M}\sum_{u=1}^{M}\sum_{i=1}^{n_r}\sum_{a=1}^{n_u}\frac{c_{ia}^{ru}}{2+c_{ia}^{ru}}\left[(U_{ia}^{ru})^2+(V_{ia}^{ru})^2+(W_{ia}^{ru})^2)\right] \\
&\ge b(||\tilde{x}_{00}^{00}||).
\end{aligned}
\tag{3.4.84}
$$

On the other hand, it follows from (3.4.79) that

$$\begin{aligned} V(x_{ia}^{ru}) &= (U_{ia}^{ru})^2 + 2U_{ia}^{ru}V_{ia}^{ru} + (c_{ia}^{ru}+1)(V_{ia}^{ru})^2 + (W_{ia}^{ru})^2 \\ &\leq 2(U_{ia}^{ru})^2 + (c_{ia}^{ru}+2)(V_{ia}^{ru})^2 + (W_{ia}^{ru})^2 \\ &\leq (c_{ia}^{ru}+2)\left[(U_{ia}^{ru})^2 + (V_{ia}^{ru})^2 + (W_{ia}^{ru})^2\right] \end{aligned} \tag{3.4.85}$$

Thus, from (3.4.83) and (3.4.85) we have

$$\begin{aligned} V(x_{00}^{00}(t)) &\leq \sum_{r=1}^{M}\sum_{u=1}^{M}\sum_{i=1}^{n_r}\sum_{a=1}^{n_u}(c_{ia}^{ru}+2)\left[(U_{ia}^{ru})^2 + (V_{ia}^{ru})^2 + (W_{ia}^{ru})^2\right] \\ &\leq a(||\tilde{x}_{00}^{00}||) \end{aligned} \tag{3.4.86}$$

Therefore from (3.4.79), (3.4.84) and (3.4.86), we establish the desired inequality.

Remark 3.4.1 *Lemma 3.4.1 shows that the Lyapunov function V defined in (3.4.79) is positive definite((3.4.84)), decrescent and radially unbounded ((3.4.86)) function[34, 59].*

We now state the following lemma

Lemma 3.4.2 *Assume that the hypothesis of Lemma 3.4.1 are satisfied. For each $r,u,v \in I(1,M)$, $i \in I(1,n_r)$, $a \in I(1,n_u)$ and $b \in I(1,n_v)$, let*

$$d_{ai}^{ur} = \sum_{v=1}^{M}\sum_{b=1}^{n_r} c_{ba}^{vu}\left[\beta_{abi}^{uvr}\left(\frac{S_{ba}^{vu*}}{\mu_{ba}^{vu}} + \frac{\bar{B}^2}{\mu_{ba}^{vu}}\right) + (\nu_{abi}^{uvr})^2(S_{ba}^{vu*}+\bar{B})^2\right]. \tag{3.4.87}$$

for some positive numbers c_{ia}^{ru}. Furthermore, let

$$\mathfrak{U}_{ia}^{ru} = \begin{cases} \dfrac{\left[2\sum_{u=1}^{M}\sum_{a=1}^{n_u}\mu_{ia}^{ru} + \sum_{u\neq r}^{M}\sum_{a=1}^{n_r}\frac{(\gamma_{ia}^{ru})^2}{\mu_{ii}^{rr}} + \sum_{a\neq i}^{n_r}\frac{(\sigma_{ia}^{rr})^2}{\mu_{ii}^{rr}} + \frac{3}{2}\mu_{ii}^{rr}\right]}{(\gamma_i^r+\sigma_i^r+\delta_i^r)} for \quad u=r, i=a \\ \dfrac{\left[\frac{(\rho_{ia}^{rr})^2}{\mu_{ia}^{rr}} + \mu_{ii}^{rr} + \frac{3}{2}\mu_{ia}^{rr}\right]}{(\rho_{ia}^{rr}+\delta_a^r)}, for \quad u=r, a\neq i \\ \dfrac{\left[\frac{(\rho_{ia}^{ru})^2}{\mu_{ia}^{ru}} + \mu_{ii}^{rr} + \frac{3}{2}\mu_{ia}^{ru}\right]}{(\rho_{ia}^{ru}+\delta_a^u)}, for \quad u\neq r, \end{cases} \tag{3.4.88}$$

$$\mathfrak{V}_{ia}^{ru} = \begin{cases} \dfrac{\sum_{u=1}^{M}\sum_{a=1}^{n_u}\frac{1}{2}\mu_{ia}^{ru} + \sum_{v=1}^{M}\sum_{b=1}^{n_v}\frac{1}{2}\beta_{iib}^{rrv}(S_{ii}^{rr*}+\mu_{ii}^{rr}) + \frac{1}{2}d_{ii}^{rr}}{\eta_i^r+\rho_i^r+\gamma_i^r+\sigma_i^r+\delta_i^r+d_i^r}, for \quad a=i, u=r \\ \dfrac{\frac{1}{2}\mu_{ii}^{rr} + \sum_{v=1}^{M}\sum_{b=1}^{n_v}\frac{1}{2}\beta_{aib}^{rrv}(S_{ia}^{rr*}+\mu_{ia}^{rr}) + \frac{1}{2}d_{ai}^{rr}}{\eta_a^r+\rho_a^r+\rho_{ia}^{rr}+\delta_a^r+d_a^r}, for \quad a\neq i, u=r \\ \dfrac{\frac{1}{2}\mu_{ii}^{rr} + \sum_{v=1}^{M}\sum_{b=1}^{n_v}\frac{1}{2}\beta_{aib}^{urv}(S_{ii}^{ru*}+\mu_{ia}^{ru}) + \frac{1}{2}d_{ai}^{ur}}{\eta_a^u+\rho_a^u+\rho_{ia}^{ru}+\delta_a^u+d_a^u}, for \quad u\neq r. \end{cases} \tag{3.4.89}$$

and

$$\mathfrak{W}_{ia}^{ru} = \begin{cases} \dfrac{\left[\frac{1}{2}\sum_{u=1}^{M}\sum_{a=1}^{n_u}\mu_{ia}^{ru}+\frac{1}{2}\mu_{ii}^{rr}+\frac{1}{2}\sum_{a\neq i}^{n_r}\frac{(\sigma_{ia}^{rr})^2}{\mu_{ii}^{rr}}+\frac{1}{2}\sum_{u\neq r}^{M}\sum_{a=1}^{n_r}\frac{(\gamma_{ia}^{ru})^2}{\mu_{ii}^{rr}}+\frac{(\alpha_i^r)^2}{\mu_{ii}^{rr}}\right]}{(\gamma_i^r+\sigma_i^r+\alpha_i^r+\delta_i^r)}, for \quad u=r, a=i, \\ \dfrac{\left[\frac{1}{2}\frac{(\rho_{ia}^{rr})^2}{\mu_{ia}^{rr}}+\frac{1}{2}\mu_{ii}^{rr}+\frac{1}{2}\mu_{ia}^{rr}+\frac{(\alpha_a^r)^2}{\mu_{ia}^{rr}}\right]}{(\rho_{ia}^{rr}+\alpha_a^r+\delta_a^r)}, for \quad u=r, a\neq i, \\ \dfrac{\left[\frac{1}{2}\frac{(\rho_{ia}^{ru})^2}{\mu_{ia}^{ru}}+\frac{1}{2}\mu_{ii}^{rr}+\frac{1}{2}\mu_{ia}^{ru}+\frac{(\alpha_a^u)^2}{\mu_{ia}^{ru}}\right]}{(\rho_{ia}^{ru}+\alpha_a^u+\delta_a^u)}, for \quad u\neq r \end{cases} \tag{3.4.90}$$

for some suitably defined positive number μ_{ia}^{ru}, depending on δ_a^u, for all $r,u \in I^r(1,M)$, $i \in I(1,n)$ and $a \in I_i^r(1,n_r)$. Assume that $\mathfrak{U}_{ia}^{ru} \leq 1$, $\mathfrak{V}_{ia}^{ru} < 1$ and $\mathfrak{W}_{ia}^{ru} \leq 1$. There exist positive numbers ϕ_{ia}^{ru}, ψ_{ia}^{ru} and φ_{ia}^{ru} such that the differential operator LV associated with Ito-Doob type stochastic system (3.2.15)-(3.2.17)satisfies the following inequality

$$\begin{aligned} LV(\tilde{x}_{00}^{00}) \leq \sum_{r=1}^{M}\sum_{i=1}^{n_r}\Big[&-[\phi_{ii}^{rr}(U_{ii}^{rr})^2+\psi_{ii}^{rr}(V_{ii}^{rr})^2+\varphi_{ii}^{rr}(W_{ii}^{rr})^2] \\ &-\sum_{a\neq i}^{n_r}[\phi_{ia}^{rr}(U_{ia}^{rr})^2+\psi_{ia}^{rr}(V_{ia}^{rr})^2+\varphi_{ia}^{rr}(W_{ia}^{rr})^2] \\ &-\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}[\phi_{ia}^{ru}(U_{ia}^{rr})^2+\psi_{ia}^{ru}(V_{ia}^{ru})^2+\varphi_{ia}^{ru}(W_{ia}^{ru})^2]\Big]. \end{aligned} \tag{3.4.91}$$

Moreover,

$$LV(\tilde{x}_{00}^{00}) \leq -cV(\tilde{x}_{00}^{00}) \tag{3.4.92}$$

where a positive constant c is defined by

$$c = \frac{\min_{1\leq r,u\leq M, 1\leq i\leq n_r, 1\leq a\leq n_u}(\phi_{ia}^{ru},\psi_{ia}^{ru},\varphi_{ia}^{ru})}{\max_{1\leq r,u\leq M, 1\leq i\leq n_r, 1\leq a\leq n_u}\{C_{ia}^{ru}+2\}} \tag{3.4.93}$$

Proof:

The computation of differential operator[59, 34] applied to the Lyapunov function V in (3.4.79) with respect to the large-scale system of Ito-Doob type stochastic differential equation (3.2.15)-(3.2.17) is as follows:

$$\begin{aligned} LV(\tilde{x}_{ii}^{rr}) &= 2\sum_{u=1}^{M}\sum_{a=1}^{n_u}[(1+C_{ii}^{rr})\rho_{ia}^{ru}V_{ia}^{ru}V_{ii}^{rr}+\rho_{ia}^{ru}U_{ia}^{ru}U_{ii}^{rr}+\rho_{ia}^{ru}V_{ia}^{ru}U_{ii}^{rr}+\rho_{ia}^{ru}U_{ia}^{ru}V_{ii}^{rr} \\ &\quad +\rho_{ia}^{ru}W_{ia}^{ru}W_{ii}^{rr}]+2\alpha_i^r U_{ii}^{rr}W_{ii}^{rr}+2(\alpha_i^r+\rho_i^r)V_{ii}^{rr}W_{ii}^{rr} \\ &\quad -2[\rho_i^r+d_i^r+2(\gamma_i^r+\sigma_i^r+\delta_i^r)]V_{ii}^{rr}U_{ii}^{rr}-2(\gamma_i^r+\sigma_i^r+\delta_i^r)(U_{ii}^{rr})^2 \\ &\quad -2[c_{ii}^{rr}\eta_i^r+2(c_{ii}^{rr}+1)(\rho_i^r+\gamma_i^r+\sigma_i^r+\delta_i^r+d_i^r)](V_{ii}^{rr})^2-2(\gamma_i^r+\sigma_i^r+\alpha_i^r+\delta_i^r)(W_{ii}^{rr})^2 \\ &\quad +2\sum_{u=1}^{M}\sum_{a=1}^{n_u}c_{ii}^{rr}\beta_{iia}^{rru}(S_{ii}^{rr*}+U_{ii}^{rr})V_{ai}^{ur}V_{ii}^{rr}+c_{ii}^{rr}\sum_{u=1}^{M}\sum_{a=1}^{n_u}(v_{iia}^{rru})^2(S_{ii}^{rr*}+U_{ii}^{rr})^2(V_{ai}^{ur})^2, \\ &\quad for \quad u=r, a=i \end{aligned} \tag{3.4.94}$$

$$\begin{aligned} \sum_{a\neq i}^{n_r}LV(\tilde{x}_{ia}^{rr}) &= \sum_{a\neq r}^{n_r}\{2(1+c_{ia}^{rr})\sigma_{ia}^{rr}V_{ia}^{rr}V_{ii}^{rr}+2\sigma_{ia}^{rr}U_{ia}^{rr}U_{ii}^{rr}+2\sigma_{ia}^{rr}V_{ia}^{rr}U_{ii}^{rr}+2\sigma_{ia}^{rr}U_{ia}^{rr}V_{ii}^{rr} \\ &\quad +2\sigma_{ia}^{rr}W_{ia}^{rr}W_{ii}^{rr} \\ &\quad -2[c_{ia}^{rr}\eta_a^r+2(c_{ia}^{rr}+1)(\rho_a^r+\rho_{ia}^{rr}+\delta_a^r)](V_{ia}^{rr})^2-2(\rho_{ia}^{rr}+\delta_a^r)(U_{ia}^{rr})^2 \\ &\quad -2(\rho_{ia}^{rr}+\alpha_a^r+\delta_a^r)(W_{ia}^{rr})^2+2\alpha_a^r W_{ia}^{rr}U_{ia}^{rr}+2(\alpha_a^r+\rho_a^r)V_{ia}^{rr}W_{ia}^{rr} \\ &\quad -2[\rho_a^r+d_a^r+2(\rho_{ia}^{rr}+\delta_a^r)]V_{ia}^{rr}U_{ia}^{rr}\}+2\sum_{a\neq r}^{n_r}\sum_{v=1}^{M}\sum_{b=1}^{n_v}c_{ia}^{rr}\beta_{aib}^{rrv}(S_{ia}^{rr*}+U_{ia}^{rr})V_{ba}^{vr}V_{ia}^{rr} \\ &\quad +\sum_{a\neq r}^{n_r}c_{ia}^{rr}\sum_{v=1}^{M}\sum_{b=1}^{n_v}(v_{aib}^{rrv})^2(S_{ia}^{rr*}+U_{ia}^{rr})^2(V_{ba}^{vr})^2\ , for \quad u=r, \quad a\neq i \end{aligned} \tag{3.4.95}$$

$$\begin{aligned} \sum_{u=1}^{M}\sum_{a=1}^{n_r}LV(\tilde{x}_{ia}^{ru}) &= \sum_{u=1}^{M}\sum_{a=1}^{n_u}\{2(1+c_{ia}^{ru})\gamma_{ia}^{ru}V_{ia}^{ru}V_{ii}^{rr}+2\gamma_{ia}^{ru}U_{ia}^{ru}U_{ii}^{rr}+2\gamma_{ia}^{ru}V_{ia}^{ru}U_{ii}^{rr}+2\gamma_{ia}^{ru}U_{ia}^{ru}V_{ii}^{rr} \\ &\quad +2\gamma_{ia}^{ru}W_{ia}^{ru}W_{ii}^{rr}-2[c_{ia}^{ru}\eta_a^u+2(c_{ia}^{ru}+1)(\rho_a^u+\rho_{ia}^{ru}+\delta_a^u+d_a^u)](V_{ia}^{ru})^2 \\ &\quad -2(\rho_{ia}^{ru}+\delta_a^u)(U_{ia}^{ru})^2 \\ &\quad -2(\rho_{ia}^{ru}+\alpha_a^u+\delta_a^u)(W_{ia}^{rr})^2+2\alpha_a^u W_{ia}^{ru}U_{ia}^{ru}+2(\alpha_a^u+\rho_a^u)V_{ia}^{ru}W_{ia}^{ru} \\ &\quad -2[\rho_a^u+d_a^u+2(\rho_{ia}^{ru}+\delta_a^u)]V_{ia}^{ru}U_{ia}^{ru}\} \\ &\quad +2\sum_{u=1}^{M}\sum_{a=1}^{n_u}\sum_{v=1}^{M}\sum_{b=1}^{n_v}c_{ia}^{ru}\beta_{aib}^{urv}(S_{ia}^{ru*}+U_{ia}^{ru})V_{ba}^{vu}V_{ia}^{ru} \\ &\quad +\sum_{u=1}^{M}\sum_{a=1}^{n_r}c_{ia}^{ru}\sum_{v=1}^{M}\sum_{b=1}^{n_v}(v_{aib}^{urv})^2(S_{ia}^{ru*}+U_{ia}^{ru})^2(V_{ba}^{vu})^2, for \quad u\neq r \end{aligned} \tag{3.4.96}$$

By using (3.3.71) and the algebraic inequality

$$2ab \leq \frac{a^2}{g(c)} + b^2 g(c) \tag{3.4.97}$$

where $a, b, c \in \mathbb{R}$, and the function g is such that $g(c) \geq 0$. The sixth term in (3.4.94), (3.4.95) and (3.4.96) is estimated as follows:

$$\begin{aligned}
&2\sum_{v=1}^{M}\sum_{b=1}^{n_v} c_{ii}^{rr}\beta_{iib}^{rrv}(S_{ii}^{rr*}+U_{ii}^{rr})V_{bi}^{vr}V_{ii}^{rr} \\
&\leq \sum_{v=1}^{M}\sum_{b=1}^{n_v} c_{ii}^{rr}\beta_{iib}^{rrv}(S_{ii}^{rr*}g_i^r(\delta_i^r)+g_i^r(\delta_i^r))(V_{ii}^{rr})^2 \\
&+\sum_{v=1}^{M}\sum_{b=1}^{n_v} c_{ii}^{rr}\beta_{iib}^{rrv}\left(\frac{S_{ii}^{rr*}}{g_i^r(\delta_i^r}) + \frac{\bar{B}^2}{g_i^r(\delta_i^r)}\right)(V_{bi}^{vr})^2 \\
&\sum_{a\neq r}^{n_r} 2\sum_{v=1}^{M}\sum_{b=1}^{n_v} c_{ia}^{rr}\beta_{aib}^{rrv}(S_{ia}^{rr*}+U_{ia}^{rr})V_{ba}^{vr}V_{ia}^{rr} \\
&\leq \sum_{a\neq r}^{n_r}\sum_{v=1}^{M}\sum_{b=1}^{n_v} c_{ia}^{rr}\beta_{aib}^{rrv}(S_{ia}^{rr*}g_i^r(\delta_a^r)+g_i^r(\delta_a^r))(V_{ia}^{rr})^2 \\
&+\sum_{a\neq r}^{n_r}\sum_{v=1}^{M}\sum_{b=1}^{n_v} c_{ia}^{rr}\beta_{aib}^{rrv}\left(\frac{S_{ia}^{rr*}}{g_i^r(\delta_a^r}) + \frac{\bar{B}^2}{g_i^r(\delta_a^r)}\right)(V_{bi}^{vr})^2
\end{aligned}$$

and

$$\begin{aligned}
&2\sum_{u=1}^{M}\sum_{a=1}^{n_u}\sum_{v=1}^{M}\sum_{b=1}^{n_v} c_{ia}^{ru}\beta_{aib}^{urv}(S_{ia}^{ru*}+U_{ia}^{ru})V_{ba}^{vu}V_{ia}^{ru} \\
&\leq \sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\sum_{v=1}^{M}\sum_{b=1}^{n_v} c_{ia}^{ru}\beta_{aib}^{urv}(S_{ia}^{ru*}g_i^r(\delta_a^u)+g_i^r(\delta_a^u))(V_{ia}^{ru})^2 \\
&+\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\sum_{v=1}^{M}\sum_{b=1}^{n_v} c_{ia}^{ru}\beta_{aib}^{urv}\left(\frac{S_{ia}^{ru*}}{g_i^r(\delta_a^u}) + \frac{\bar{B}^2}{g_i^r(\delta_a^u)}\right)(V_{ba}^{vu})^2
\end{aligned} \tag{3.4.98}$$

From (3.4.94), (3.4.95) and repeated usage of inequality (3.4.97) and (3.4.98) coupled with algebraic manipulations and simplifications, we have the following inequality

$$
\begin{aligned}
LV(\tilde{x}_{00}^{00}) \leq \; & \sum_{r=1}^{M}\sum_{i=1}^{n_r}\Bigg\{\Bigg[2\sum_{u=1}^{M}\sum_{a=1}^{n_u}\mu_{ia}^{ru}+3\mu_{ii}^{rr}+2\sum_{a\neq i}^{n_r}\frac{(\sigma_{ia}^{rr})^2}{\mu_{ii}^{rr}}+2\sum_{a\neq r}^{M}\sum_{a=1}^{n_r}\frac{(\gamma_{ia}^{ru})^2}{\mu_{ii}^{rr}} \\
& -2(\gamma_i^r+\sigma_i^r+\delta_i^r)\Bigg](U_{ii}^{rr})^2 \\
& +\Bigg[\sum_{u=1}^{M}\sum_{a=1}^{n_u}[(2+c_{ii}^{rr})\mu_{ia}^{ru}]+\mu_{ii}^{rr}+\frac{(\rho_i^r)^2}{\mu_{ii}^{rr}}+\frac{(\rho_i^r+d_i^r)^2}{\mu_{ii}^{rr}}+4\frac{(\gamma_i^r+\sigma_i^r+\delta_i^r)^2}{\mu_{ii}^{rr}} \\
& -2[c_{ii}^{rr}\eta_i^r+(c_{ii}^{rr}+1)(\rho_i^r+\gamma_i^r+\sigma_i^r+\delta_i^r+d_i^r)]+\sum_{a\neq r}^{n_r}\frac{(2+c_{ia}^{rr})(\sigma_{ia}^{rr})^2}{\mu_{ii}^{rr}} \\
& +\sum_{u\neq r}^{M}\sum_{a=1}^{n_r}\frac{(2+c_{ia}^{ru})(\gamma_{ia}^{ru})^2}{\mu_{ii}^{rr}}+c_{ii}^{rr}\sum_{v=1}^{M}\sum_{b=1}^{n_r}\beta_{iib}^{rrv}(S_{ii}^{rr*}\mu_{ii}^{rr}+\mu_{ii}^{rr})\Bigg](V_{ii}^{rr})^2 \\
& +\Bigg[\sum_{u=1}^{M}\sum_{a=1}^{n_u}\mu_{ia}^{ru}+2\frac{(\alpha_i^r)^2}{\mu_{ii}^{rr}}+\mu_{ii}^{rr}+\sum_{a\neq i}^{n_r}\frac{(\sigma_{ia}^{rr})^2}{\mu_{ii}^{rr}}+\sum_{a\neq r}^{M}\sum_{a=1i}^{n_r}\frac{(\gamma_{ia}^{ru})^2}{\mu_{ii}^{rr}} \\
& -2(\gamma_i^r+\sigma_i^r+\alpha_i^r+\delta_i^r)\Bigg](W_{ii}^{rr})^2 \\
& +\sum_{a\neq i}^{n_r}\Bigg\{\Bigg[2\frac{(\rho_{ia}^{rr})^2}{\mu_{ia}^{rr}}+2\mu_{ii}^{rr}+3\mu_{ia}^{rr}-2(\rho_{ia}^{rr}+\delta_a^r)\Bigg](U_{ia}^{rr})^2+\Bigg[(2+c_{ii}^{rr})\frac{(\rho_{ia}^{rr})^2}{\mu_{ia}^{rr}} \\
& (2+c_{ia}^{rr})\mu_{ii}^{rr}-2[c_{ia}^{rr}\eta_a^r+(1+c_{ia}^{rr})(\eta_a^r+\rho_{ia}^{rr}+\delta_a^r+d_a^r)]+\frac{(\rho_a^r+d_a^r)^2}{\mu_{ia}^{rr}} \\
& +4\frac{(\rho_{ia}^{rr}+\delta_a^r)^2}{\mu_{ia}^{rr}}+\mu_{ia}^{rr}+c_{ia}^{rr}\sum_{v=1}^{M}\sum_{b=1}^{n_r}\beta_{aib}^{rrv}(S_{ia}^{rr}\mu_{ia}^{rr}+\mu_{ia}^{rr})\Bigg](V_{ia}^{rr})^2 \\
& +\Bigg[\frac{(\rho_{ia}^{rr})^2}{\mu_{ia}^{rr}}+\mu_{ii}^{rr}+\mu_{ia}^{rr}+\frac{2(\alpha_a^r)^2}{\mu_{ia}^{rr}}-2(\rho_{ia}^{rr}+\alpha_a^r+\delta_a^r)\Bigg](W_{ia}^{rr})^2\Bigg\} \\
& +\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\Bigg\{\Bigg[2\frac{(\rho_{ia}^{ru})^2}{\mu_{ia}^{ru}}+2\mu_{ii}^{rr}+3\mu_{ia}^{ru}-2(\rho_{ia}^{ru}+\delta_a^u)\Bigg](U_{ia}^{ru})^2 \\
& +\Bigg[(2+c_{ii}^{rr})\frac{(\rho_{ia}^{ru})^2}{\mu_{ia}^{ru}}+(2+c_{ia}^{ru})\mu_{ii}^{rr}-2[c_{ia}^{ru}\eta_a^u+(1+c_{ia}^{ru})(\eta_a^u+\rho_{ia}^{ru}+\delta_a^u+d_a^u)] \\
& +\frac{(\rho_a^u+d_a^u)^2}{\mu_{ia}^{ru}}+4\frac{(\rho_{ia}^{ru}+\delta_a^u)^2}{\mu_{ia}^{ru}}+\mu_{ia}^{ru}+c_{ia}^{ru}\sum_{v=1}^{M}\sum_{b=1}^{n_r}\beta_{aib}^{urv}(S_{ia}^{ru*}\mu_{ia}^{ru}+\mu_{ia}^{ru})\Bigg](V_{ia}^{ru})^2 \\
& +\Bigg[\frac{(\rho_{ia}^{ru})^2}{\mu_{ia}^{ru}}+\mu_{ii}^{rr}+\mu_{ia}^{ru}+\frac{2(\alpha_a^u)^2}{\mu_{ia}^{ru}}-2(\rho_{ia}^{ru}\alpha_a^u+\delta_a^u)\Bigg](W_{ia}^{ru})^2\Bigg\}\Bigg\}
\end{aligned}
$$

$$+\sum_{r=1}^{M}\sum_{i=1}^{n_r}c_{ii}^{rr}\sum_{v=1}^{M}\sum_{b=1}^{n_r}\left[\beta_{iib}^{rrv}(\frac{S_{ii}^{rr*}}{\mu_{ii}^{rr}}+\frac{\bar{B}^2}{\mu_{ii}^{rr}})+(v_{iib}^{rrv})^2(S_{ii}^{rr*}+\bar{B})^2\right](V_{bi}^{vr})^2$$

$$+\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{a\neq i}^{n_r}c_{ia}^{rr}\sum_{v=1}^{M}\sum_{b=1}^{n_r}\left[\beta_{aib}^{rrv}(\frac{S_{ia}^{rr*}}{\mu_{ia}^{rr}}+\frac{\bar{B}^2}{\mu_{ia}^{rr}})+(v_{aib}^{rrv})^2(S_{ia}^{rr*}+\bar{B})^2\right](V_{ba}^{vr})^2$$

$$+\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u\neq r}^{M}\sum_{a=1}^{n_r}c_{ia}^{ru}\sum_{v=1}^{M}\sum_{b=1}^{n_r}\left[\beta_{aib}^{urv}(\frac{S_{ia}^{ru*}}{\mu_{ia}^{ru}}+\frac{\bar{B}^2}{\mu_{ia}^{ru}})+(v_{aib}^{urv})^2(S_{ia}^{ru*}+\bar{B})^2\right](V_{ba}^{vu})^2, \quad (3.4.99)$$

where $\mu_{ia}^{ru}=g_i^r(\delta_a^u)$, g_i^r is appropriately defined by (3.4.97).

For each $r,u\in I(1,M)$, $i\in I(1,n_r)$, and $a\in I(1,n_u)$, using algebraic manipulations and (3.4.88), (3.4.89) and (3.4.90), the coefficients of $(U_{ia}^{ru})^2$, $(V_{ia}^{ru})^2$ and $(W_{ia}^{ru})^2$ in (3.4.99) defined by ϕ_{ia}^{ru}, ψ_{ia}^{ru} and φ_{ia}^{ru} respectively:

$$\phi_{ia}^{ru}=\begin{cases} 2(\gamma_i^r+\sigma_i^r+\delta_i^r)(1-\mathfrak{U}_{ia}^{ru}), for \quad u=r,a=i \\ 2(\rho_{ia}^{rr}+\delta_a^r+\delta_a^r)(1-\mathfrak{U}_{ia}^{ru}), for \quad u=r,a\neq i \\ 2(\rho_{ia}^{ru}+\delta_a^u+\delta_a^u)(1-\mathfrak{U}_{ia}^{ru}), for \quad u\neq r, \end{cases} \quad (3.4.100)$$

$$\psi_{ia}^{ru}=\begin{cases} [2c_{ii}^{rr}(1-\mathfrak{V}_{ii}^{rr})(\eta_i^r+\rho_i^r+\gamma_i^r+\sigma_i^r+\delta_i^r+d_i^r)-\mathfrak{E}_{ii}^{rr}] \\ +2(\rho_i^r+\gamma_i^r+\sigma_i^r+\delta_i^r+d_i^r), for \quad u=r,a=i \\ [2c_{ia}^{rr}(1-\mathfrak{V}_{ia}^{rr})(\eta_a^r+\rho_a^r+\rho_{ia}^{rr}+\delta_a^r+d_a^r)-\mathfrak{E}_{ia}^{rr}] \\ +2(\rho_a^r+\rho_{ia}^{rr}+\delta_a^r+d_a^r), for \quad u=r,a\neq i \\ [2c_{ia}^{ru}(1-\mathfrak{V}_{ia}^{ru})(\eta_a^u+\rho_a^u+\rho_{ia}^{ru}+\delta_a^u+d_a^u)-\mathfrak{E}_{ia}^{ru}] \\ +2(\rho_a^u+\rho_{ia}^{ru}+\delta_a^u+d_a^u), for \quad u\neq r \end{cases}$$

and

$$\varphi_{ia}^{ru}=\begin{cases} 2(\gamma_i^r+\sigma_i^r+\alpha_i^r+\delta_i^r)(1-\mathfrak{W}_{ia}^{ru}), for \quad u=r,a=i, \\ 2(\rho_{ia}^{rr}+\delta_a^r)(1-\mathfrak{W}_{ia}^{ru}), \quad for u=r,a\neq i, \\ 2(\rho_{ia}^{ru}+\delta_a^u)(1-\mathfrak{W}_{ia}^{ru}), for \quad u\neq r \end{cases} \quad (3.4.101)$$

where

$$\mathfrak{E}_{ia}^{ru} = \begin{cases} \Big[2\sum_{u=1}^{M}\sum_{a=1}^{n_u}\mu_{ia}^{ru} + \mu_{ii}^{rr} + \frac{(\rho_i^r)^2}{\mu_{ii}^{rr}} + \frac{(\rho_i^r+d_i^r)^2}{\mu_{ii}^{rr}} + 4\frac{(\gamma_i^r+\sigma_i^r+\delta_i^r}{\mu_{ii}^{rr}} + \sum_{a\neq r}^{n_r}\frac{(2+c_{ia}^{rr})(\sigma_{ia}^{rr})^2}{\mu_{ii}^{rr}} \\ +\sum_{u\neq r}^{M}\sum_{a=1}^{n_r}\left(\frac{(2+c_{ia}^{ru})(\gamma_{ia}^{ru})^2}{\mu_{ii}^{rr}}\right) + \sum_{b\neq i}^{n_r} c_{bi}^{rr}d_{ii}^{rr} + \sum_{v\neq r}^{M}\sum_{b=1}^{n_r} c_{bi}^{vr}d_{ii}^{rr}\Big], \\ for \quad u=r, a=i, \\ (2+c_{ii}^{rr})\frac{(\rho_{ia}^{rr})^2}{\mu_{ia}^{rr}} + \frac{(\rho_a^r)^2}{\mu_{ia}^{rr}} + 2\mu_{ii}^{rr} + \frac{(\rho_a^r+d_a^r)^2}{\mu_{ia}^{rr}} \\ +4\frac{(\rho_{ia}^{rr}+\delta_a^r)^2}{\mu_{ia}^{rr}} + \mu_{ia}^{rr} + \sum_{b\neq i}^{n_r} c_{ba}^{rr}d_{ia}^{rr} + \sum_{v\neq r}^{M}\sum_{b=1}^{n_r} c_{ba}^{vr}d_{ia}^{rr}, for \quad u=r, a\neq i, \\ (2+c_{ii}^{rr})\frac{(\rho_{ia}^{ru})^2}{\mu_{ia}^{ru}} + \frac{(\rho_a^u)^2}{\mu_{ia}^{ru}} + 2\mu_{ii}^{rr} + \frac{(\rho_a^u+d_a^u)^2}{\mu_{ia}^{ru}} \\ +4\frac{(\rho_{ia}^{ru}+\delta_a^u)^2}{\mu_{ia}^{ru}} + \mu_{ia}^{ru} + \sum_{b\neq i}^{n_r} c_{ba}^{ru}d_{ia}^{ru} + \sum_{v\neq r}^{M}\sum_{b=1}^{n_r} c_{ba}^{vu}d_{ia}^{ru}, u\neq r \end{cases}$$

Under the assumptions on $\mathfrak{U}_{ia}^{ru}$, $\mathfrak{V}_{ia}^{ru}$ and $\mathfrak{W}_{ia}^{ru}$, it is clear that $\phi_{ia}^{ru}, \psi_{ia}^{ru}$ and φ_{ia}^{ru} are positive for suitable choice of c_{ia}^{ru} defined in (3.4.80). We substitute (3.4.87), (3.4.100), (3.4.101) and (3.4.102) into (3.4.99). Thus inequality (3.4.99) can be rewritten as

$$\begin{aligned} LV(\tilde{x}_{00}^{00}) \quad \leq \quad & \sum_{r=1}^{M}\sum_{i=1}^{n_r} -\big\{[\phi_{ii}^{rr}(U_{ii}^{rr})^2 + \psi_{ii}^{rr})(V_{ii}^{rr})^2 \\ & \varphi_{ii}^{rr}(W_{ii}^{rr})^2] + \sum_{a\neq r}^{n_r}[\phi_{ia}^{rr}(U_{ia}^{rr})^2 + \psi_{ia}^{rr}(V_{ia}^{rr})^2 \\ & +\varphi_{ia}^{rr}(W_{ia}^{rr})^2] + \sum_{u\neq r}^{M}\sum_{a=1}^{n_u}[\phi_{ia}^{ru}(U_{ia}^{ru})^2 + \psi_{ia}^{ru}(V_{ia}^{ru})^2 \\ & + \varphi_{ia}^{ru}(W_{ia}^{ru})^2]\big\} \end{aligned} \tag{3.4.102}$$

This proves the inequality (3.4.91). Now, the validity of (3.4.92) follows from (3.4.91), that is,

$$LV(\tilde{x}_{00}^{00}) \leq -cV(\tilde{x}_{00}^{00}),$$

where $c = \frac{\min_{1\leq r,u\leq M, 1\leq i\leq n_r, 1\leq a\leq n_u}(\phi_{ia}^{ru}, \psi_{ia}^{ru}, \varphi_{ia}^{ru})}{\max_{1\leq r,u\leq M, 1\leq i\leq n_r, 1\leq a\leq n_u}\{C_{ia}^{ru}+2\}}$. This establishes the result.

We now formally state the stochastic stability theorems for the disease free equilibria.

Theorem 3.4.3 *Given $r,u \in I(1,M)$, $i \in I(1,n_r)$ and $a \in I(1,n_u)$. Let us assume that the hypotheses of Lemma 3.4.2 are satisfied. Then the disease free solutions E_{ia}^{ru}, are asymptotically stable in the large. Moreover, the solutions E_{ia}^{ru} are exponentially mean square stable.*

Proof:

From the application of comparison result[34, 59], the proof of stochastic asymptotic stability follows immediately Moreover, the disease free equilibrium state is exponentially mean square stable. We now consider the following corollary to Theorem 3.4.3.

Corollary 3.4.4 *Let $r \in I(1,M)$ and $i \in I(1,n_r)$. Assume that $\sigma_i^r = \gamma_i^r = 0$, for all $r \in I(1,M)$ and $i \in I(1,n_r)$.*

$$\mathfrak{U}_{ia}^{ru} = \begin{cases} \dfrac{\frac{1}{(\delta_i^r)}}{\left[\frac{1}{2\sum_{u=1}^{M}\sum_{a=1}^{n_u}\mu_{ia}^{ru}+\frac{3}{2}\mu_{ii}^{rr}}\right]} for \quad u=r, i=a \\ \dfrac{\frac{1}{(\delta_a^r)}}{\frac{1}{[\mu_{ii}^{rr}+\mu_{ia}^{rr}]}}, for \quad u=r, a\neq i \\ \dfrac{\frac{1}{(\delta_a^u)}}{\frac{1}{[\mu_{ii}^{rr}+\mu_{ia}^{ru}]}}, for \quad u\neq r, \end{cases} \tag{3.4.103}$$

$$\mathfrak{V}_{ia}^{ru} = \begin{cases} \dfrac{\sum_{u=1}^{M}\sum_{a=1}^{n_u}\frac{1}{2}\mu_{ia}^{ru}+\sum_{v=1}^{M}\sum_{b=1}^{n_v}\frac{1}{2}\beta_{iib}^{rrv}(S_{ii}^{rr*}+\mu_{ii}^{rr})+\frac{1}{2}d_{ii}^{rr}}{\eta_i^r+\rho_i^r+\delta_i^r+d_i^r}, for \quad a=i, u=r \\ \dfrac{\frac{1}{2}\mu_{ii}^{rr}+\sum_{v=1}^{M}\sum_{b=1}^{n_v}\frac{1}{2}\beta_{aib}^{rrv}(S_{ia}^{rr*}+\mu_{ia}^{rr})+\frac{1}{2}d_{ia}^{rr}}{\eta_a^r+\rho_a^r+\delta_a^r+d_a^r}, for \quad a\neq i, u=r \\ \dfrac{\frac{1}{2}\mu_{ii}^{rr}+\sum_{v=1}^{M}\sum_{b=1}^{n_v}\frac{1}{2}\beta_{aib}^{urv}(S_{ii}^{ru*}+\mu_{ia}^{ru})+\frac{1}{2}d_{ia}^{ru}}{\eta_a^u+\rho_a^u+\delta_a^u+d_a^u}, for \quad u\neq r. \end{cases} \tag{3.4.104}$$

and

$$\mathfrak{W}_{ia}^{ru} = \begin{cases} \dfrac{\left[\frac{1}{2}\sum_{u=1}^{M}\sum_{a=1}^{n_u}\mu_{ia}^{ru}+\frac{(\alpha_i^r)^2}{\mu_{ii}^{rr}}+\frac{1}{2}\mu_{ii}^{rr}\right]}{(\alpha_i^r+\delta_i^r)}, for \quad u=r, a=i, \\ \dfrac{\frac{1}{2}\mu_{ii}^{rr}+\frac{1}{2}\mu_{ia}^{rr}+\frac{(\alpha_a^r)^2}{\mu_{ia}^{rr}}}{(\alpha_a^r+\delta_a^r)}, for \quad u=r, a\neq i, \\ \dfrac{\left[\frac{1}{2}\mu_{ii}^{rr}+\frac{1}{2}\mu_{ia}^{ru}+\frac{(\alpha_a^u)^2}{\mu_{ia}^{ru}}\right]}{\alpha_a^u+\delta_a^u}, for \quad u\neq r \end{cases} \tag{3.4.105}$$

The equilibrium state E_{ii}^{rr} is stochastically asymptotically stable provided that $\mathfrak{U}_{ia}^{ru}, \mathfrak{W}_{ia}^{ru} \leq 1$ and $\mathfrak{V}_{ia}^{ru} < 1$, for all $u \in I^r(1,M)$ and $a \in I_i^r(1,n_u)$.

Proof: Follows immediately from the hypotheses of Lemma 3.4.2,(letting $\sigma_i^r = \gamma_i^r = 0$), the conclusion of Theorem 3.4.3 and some algebraic manipulations.

Remark 3.4.2 *The presented results about the two-level large scale SIRS disease dynamic model depend on the underlying system parameters. In particular, the sufficient conditions are algebraically simple, computationally attractive and explicit in terms of the rate parameters. As a result of this, several scenarios can be discussed and exhibit practical course of action to control the disease. For simplicity, we present an illustration as follows: the conditions of $\sigma_i^r = \gamma_i^r = 0, \forall r, i$ in Corollary 3.4.4 signify that the arbitrary site s_i^r is a sink[28, 29] for all other sites in the inter and intra-regional accessible domain. This scenario is displayed in Figure 7.1. The condition $\mathfrak{U}_{ia}^{ru} \leq 1$ exhibits that the average infectious period is smaller than the joint average life span of individuals* *in the intra and inter-regional accessible domain of site s_i^r. Furthermore, the condition $\mathfrak{V}_{ia}^{ru} < 1$ signifies that the magnitude of disease inhibitory processes for example, the magnitude of the recovery process is greater than the disease transmission process. A future detailed study of the disease dynamics in the two scale network dynamic structure for many real life scenarios using the presented two level large-scale SIRS disease dynamic model will appear elsewhere.*

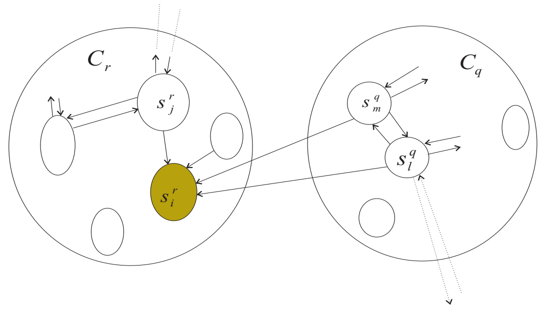

Figure 3.3: Shows that residents of site s_i^r are present only at their home site s_i^r. Hence they isolate every site from their inter and intra reginal accessible domain $C(s_i^r)$. Site s_i^r is a 'sink' in the context of the compartmental system[28, 29]. The arrows represent a transport network between any two sites and regions. Furthermore, the dotted lines and arrows indicate connection with other sites and regions.

3.5 Conclusion

The recent high technological changes and scientific developments have led to many\ variant structure types inter-patch connections interactions in the global human population. This has further afforded efficient mass flow of human beings, animals, goods and equipments between patches thereby causing the appearance of new disease strains and infectious agents at non-endemic zones. The two-scale network disease dynamic model formulation characterizes the dynamics of an SIRS epidemic in a population with various scale levels created by the heterogeneities in the population. Moreover, the disease dynamics is subject to random environmental perturbations at the disease transmission stage of the disease. Furthermore, the SIRS epidemic has a proportional transfer to the susceptible class immediately after the infectiousness.

This work provides a mathematical and probabilistic algorithmic tool to develop different levels nested type disease transmission rates as well as the variability in the transmission process in the framework of the network-centric Ito-Doob type dynamic equations.

The model validation results are developed and a positively self invariant set for the dynamic model is defined. Moreover, the globalization of the solution process existence is obtained via the construction of the two-scale dynamic structure motivated Lypunov function. The detailed stochastic asymptotic stability results of the disease free equilibrium are also exhibited in this chapter. Moreover, the system parameter dependent threshold values controlling the stochastic asymptotic stability of the disease free equilibrium are also defined. The presented analysis of Chapter 3 are illustrated in a simple real life scenario.

We note that the disease dynamics is subject to random environmental perturbations from other related processes such as the mobility, recovery, birth and death processes. The presented stochastic epidemic dynamic model will be extended to the variability in the mobility, recovery and birth and death processes in our further work. A further detailed study of the oscillation of the epidemic process about the ideal endemic equilibrium of the dynamic epidemic model will also appear else where.

4 SIMULATION RESULTS FOR A TWO-SCALE STOCHASTIC NETWORK SIR INFLUENZA EPIDEMIC DYNAMIC MODEL

4.1 The Two-Scale Hierarchic Population Structure and Special SIR Epidemic Dynamic Process

By using the two scale human mobility model and the underlying human mobility dynamic structure determined by the respective intra and interregional mobility data recorded in Tables 1& 2 in the example of Chapter 2, and also the influenza pandemic simulation model in [35], we develop a two-scale SIR influenza epidemic dynamic model. The compartmental framework for the SIR epidemic model is exhibited in Figure 3.2 in Chapter 3 with the restrictions $\eta_i^r = \alpha_i^r = 0, \forall r \in I(1,M), i \in I(1,n_r)$. Furthermore, the diagram illustrating the inter-patch connections in the example for two scale dynamic epidemic model represented in this example is shown in Figure 2.1 in Chapter 2. In the absence of intra and interregional mobility return rates, based on the mobility structure and the probabilistic formulation of the mobility process, we simulate intra and interregional mobility return rates. We display the intra and inter-regional mobility return rates in Table 4.1 and Table 4.2 respectively.

Table 4.1: The intra-regional return rates of residents of sites in the two scale network of spatial patches illustrated in Figure 2.1 are simulated based on the special human mobility pattern and the probabilistic formulation for the mobility process. (cf.Chapter 1 or [30]).

$(\rho_{12}^{11}, \rho_{13}^{11}, \rho_{21}^{11}, \rho_{23}^{11})$	(0.000092504,0.000177496,0.164327,0.0001173)
$(\rho_{31}^{11}, \rho_{32}^{11})$	(0.013230408,0.001305838)
$(\rho_{12}^{22}, \rho_{13}^{22}, \rho_{21}^{22}, \rho_{23}^{22})$	(0.000092504,0.000177496,0.164327,0.0001173)
$(\rho_{31}^{22}, \rho_{32}^{22})$	(,0.013230408,0.001305838)
$(\rho_{12}^{33}, \rho_{13}^{33}, \rho_{21}^{33}, \rho_{23}^{33})$	(0.000092504,0.000177496,0.164327,0.0001173)
$(\rho_{31}^{33}, \rho_{32}^{33})$	(0.013230408,0.001305838)

Table 4.2: The inter-regional return rates of residents of sites in the two scale network of spatial patches illustrated in Figure 2.1 are simulated based on the mobility structure and the probabilistic formulation for the mobility process. (cf. Chapter 1 or [30]).

$(\rho^{12}_{11}, \rho^{12}_{12}, \rho12_{13}, \rho12_{21}, \rho12_{22})$	(0.1995,0.035,0.0985,0.007892,0.02748)
$(\rho12_{23}, \rho12_{31}, \rho12_{32}, \rho12_{33})$	(0.075824,0.04256,0.009616,0.028628)
$(\rho^{21}_{11}, \rho^{21}_{12}, \rho21_{13}, \rho21_{21}, \rho21_{22})$	(0.002096896,0.00175424,0.003460864,
	0.00043856, 0.0001664)
$(\rho21_{23}, \rho21_{31}, \rho21_{32}, \rho21_{33})$	(0.00071504, 0.001944052,0.00119788,
	0.0001713912)
$(\rho^{23}_{11}, \rho^{23}_{12}, \rho23_{13}, \rho23_{21}, \rho23_{22})$	(0.018512, 0.03290368,0.0272192,0.04883712,
	0.00151648)
$(\rho23_{23}, \rho23_{31}, \rho23_{32}, \rho23_{33})$	(0.0219232, 0.00383316,0.0025404,
	0.000414644)
$(\rho^{31}_{11}, \rho^{31}_{12}, \rho31_{13}, \rho31_{21}, \rho31_{22})$	(0.001285712,0.00085328,0.001725008,
	0.0004380944,0.000379536)
$(\rho31_{23}, \rho31_{31}, \rho31_{32}, \rho31_{33})$	(0.0005991696 ,0.000000371428,0.00000026332,
	0.000000281252)
$(\rho^{32}_{11}, \rho^{32}_{12}, \rho32_{13}, \rho32_{21}, \rho31_{22})$	(0.0003230096,0.00036224,0.0004619664,
	0.00043146104,0.0003741576)
$(\rho32_{23}, \rho32_{31}, \rho32_{32}, \rho32_{33})$	(0.00059126136, 0.000498339428,0.00042838332,
	0.000070993252)

The following assumptions are made concerning the influenza epidemic process represented in this example:

(a_1) The population structure and influenza transmission process at every site $s^r_i, r = 1,2,3, i = 1,2,3$ in region $C_r, r = 1,2,3$ is similar to the population structure and the influenza transmission process represented in the simulation model of [35]. That is, we assume that every person in site s^r_i belongs to one age dependent stratum (ages≥ 0). In addition, each individual belongs to three mixing or contact groups $z_j, j = 1,2,3$, for example, household, marketplace, and the community. In each day, a susceptible person, A, has contacts with other individuals in his or her contact zones. The probability of acquiring infection depends on (a) the number of different persons A has contacts within the contact group, (b) the time duration, in minutes, of all contacts (c) the rate of infection transmission per-minute if the contacted person is infectious (see [35]). We assume that in a given day, a susceptible person makes three contacts in mixing group z_1, ten contacts in mixing group z_2, and three contacts in mixing group z_3. In addition, each contacted person is infectious. Furthermore, the time duration d and the per minute influenza transmission rate λ per contact in all contact zones are [zone z_1: $d \approx 92$ minutes, $\lambda = 0.00062$], [zone z_2: $d \approx 120$ minutes, $\lambda = 0.00061$] and [zone z_3: $d \approx 51$ minutes, $\lambda = 0.00061$]. Furthermore, we assume that the number and duration of contacts are the same on weekdays and weekend days. We utilize the probability model $1 - \exp(-\lambda d)$ for the influenza transmission occurring during a contact of d minutes and a transmission rate λ (see [35]) to find the infection probability β^{ury}_{aib} of the two-scale SIRS epidemic dynamic model. It is easy to see that the infection probability per day for a susceptible person at site s^r_i is $\beta^{ury}_{aib} = 1 - Pr(\text{No disease transmission in zones} \quad z_1, \quad z_2, \quad z_3) = 1 - exp(-3(92)(0.00062) - 10(120)(0.00061) - 3(51)(0.00061)) \approx 0.6277$.

(a_2) In the absence of data for the recovery and disease related death processes, we take the recovery and disease mortality rate to be $\rho^u_a = 0.05067$ and $d^u_a = 0.01838, u = 1,2,3; a,i = 1,2,3$ respectively.

(a_3) The population in this example assumed to be remote and lacking the high technological facilities found in the developed world. Furthermore, we assume that influenza is highly endemic in this population. As a result, we can assume that the time duration of the epidemic is comparable with the average life span of individuals in the population. In the absence of data concerning average birth rates, we use the yearly birth rate data from [36] for the people of the Dominican republic, $B = \frac{births}{1000} = \frac{22.39}{1000}$ as an estimate. Furthermore, we assume this birth rate is the same for all residents of sites in the population. That is, the constant birth rate is $B^u_a = \frac{births}{1000} = \frac{22.39}{1000} per \quad year, \quad for \quad u = 1,2,3; a,i = 1,2,3.$

(a_4) In addition, using the average life span of the people of Dominican Republic [37], the natural death rate of the residents at all sites and regions are the same and is calculated as the reciprocal of the average life span of individuals in the population, that is, $\delta_a^u = \frac{1}{77.15\times365}, u = 1,2,3; a,i = 1,2,3$ per day.

(a_5) The effects of the fluctuating environment on the dynamics of the influenza epidemic is assumed to be the same at all sites and regions. We take the standard deviation of the environmental fluctuations to be $\nu_{aib}^{urv} = 0.5, r,u,v = 1,2,3; a,b,i = 1,2,3$.

4.2 Mathematical Algorithm and Simulation Results

We use the standard Euler-Maruyama method stochastic approximation scheme[38] to generate the trajectories for the residents of sites s_1^1, s_1^2 and s_1^3 in regions C_1, C_2 and C_3 respectively, for the different population disease classifications (S,I,R), and current locations at some sites in the intra and inter-regional accessible domain of the sites. Given a scalar autonomous stochastic differential equation

$$dX(t) = f(X(t))dt + g(X(t))dW(t), \quad X(0) = X_0, \quad T_0 \leq t \leq T, \tag{4.2.1}$$

let $T_0 = \tau_0 \leq \tau_1 \leq \tau_2 \leq \ldots \leq \tau_n = T$, be a regular partition of $[T_0, T]$, where $\Delta\tau = \tau_j - \tau_{j-1} = \frac{T-T_0}{L}$, $\tau_j = T_0 + j\Delta\tau, \quad j = 1,\ldots,L$ and L is a positive integer. The Euler-Maruyama method takes the form

$$X(\tau_j) = X(\tau_{j-1}) + f(X(\tau_{j-1}))\Delta\tau + g(X(\tau_{j-1}))(W(\tau_j) - W(\tau_{j-1})), \quad j = 1,\ldots,L. \tag{4.2.2}$$

Using (4.2.1) as a general representation of the system (3.2.15)-(3.2.17) in the context of the scenario considered in this example (see Section 4.1), the algorithm to execute the Euler-Maruyama method to finding the solution process of (4.2.1) consists of the following steps:

Step one: Parameter Specification: The system rate parameters for the epidemic model (3.2.15)-(3.2.17) represented in this example are specified in Section 4.1. Further the following convenient initial conditions are used for the simulation process: for $r,u \in I(1,3), i,a \in I(1,3)$,

$$S_{ia}^{ru}(0) = \begin{cases} 9, for & r = u, i = a \\ 8, for & r = u, i \neq a \\ 7, for & r \neq u, \end{cases}$$

$$I_{ia}^{ru}(0) = \begin{cases} 6, for & r = u, i = a \\ 4, for & r = u, i \neq a \\ 3, for & r \neq u \end{cases}$$

and $R_{ia}^{ru}(0) = 2, \forall r, u, i, a \in I(1,3)$. Furthermore, the trajectories were generated over the time interval $t \in [0,1]$.

Step Two: **Generate Brownian Path:** Given that $W(t)$ is a standard Brownian motion or Wierner process over the time interval $[T_0, T]$, then (1) $W(0) = 0$, (2) for $T_0 \leq s < t \leq T$, the increments $W(t) - W(s) \forall s, t$ are independent and have normal distribution with mean equal to 0 and variance equal to $t - s$. In other words, $W(t) - W(s) \sim \sqrt{t-s}N(0,1)$, where $N(0,1)$ represents the normally distributed random variable with zero mean and unit variance. From conditions (1)&(2), we discretize the Brownian motion as follows: we let $T_0 = t0 \leq t_1 \leq t_2 \leq \ldots \leq t_n = T$, be a regular partition of $[T_0, T]$, where $\delta t = t_j - t_{j-1} = \frac{T-T_0}{N}$, $t_j = T_0 + j\delta t, \quad j = 1, \ldots, N$ and N is a positive integer. The Brownian path is generated as the solution to the following difference equation

$$\begin{cases} W(0) = 0, \\ W(t_j) = W(t_{j-1}) + dW(t_j), \quad j = 1, \ldots, N. \end{cases} \tag{4.2.3}$$

We simulated 1000 sample points for the Brownian motion over the interval $[0,1]$.

Step Three: **Generate Solution Path for the Susceptible, Infectious and Removal Populations:** Using (4.2.1) as a general representation of each equation in the system (3.2.15)-(3.2.17), we use the discretization (4.2.2) to find solutions path for each equation in the system. For convenience, we choose $\Delta\tau = R\delta t$,[38], where the positive integer $R \geq 1$. Moreover, from (4.2.2), it follows that

$$W(\tau_j) - W(\tau_{j-1}) = W(jR\delta t) - W((j-1)R\delta t) = \sum_{k=jR-R+1}^{jR} dW_k, \tag{4.2.4}$$

where dW_k is given by the Brownian path (4.2.3). We choose $R = 1$ for this example. Moreover, from (4.2.2), (4.2.3), and (4.2.4), we obtain trajectories for susceptible, infectious and removal populations of residents of sites s_1^1, s_1^2 and s_1^3 in regions C_1, C_2 and C_3 over the time interval $[0,1]$. The trajectories for the residents of sites s_1^1, s_1^2 and s_1^3 in regions C_1, C_2 and C_3 are exhibited in Figure 4.1, Figure 4.2 and Figure 4.3 respectively.

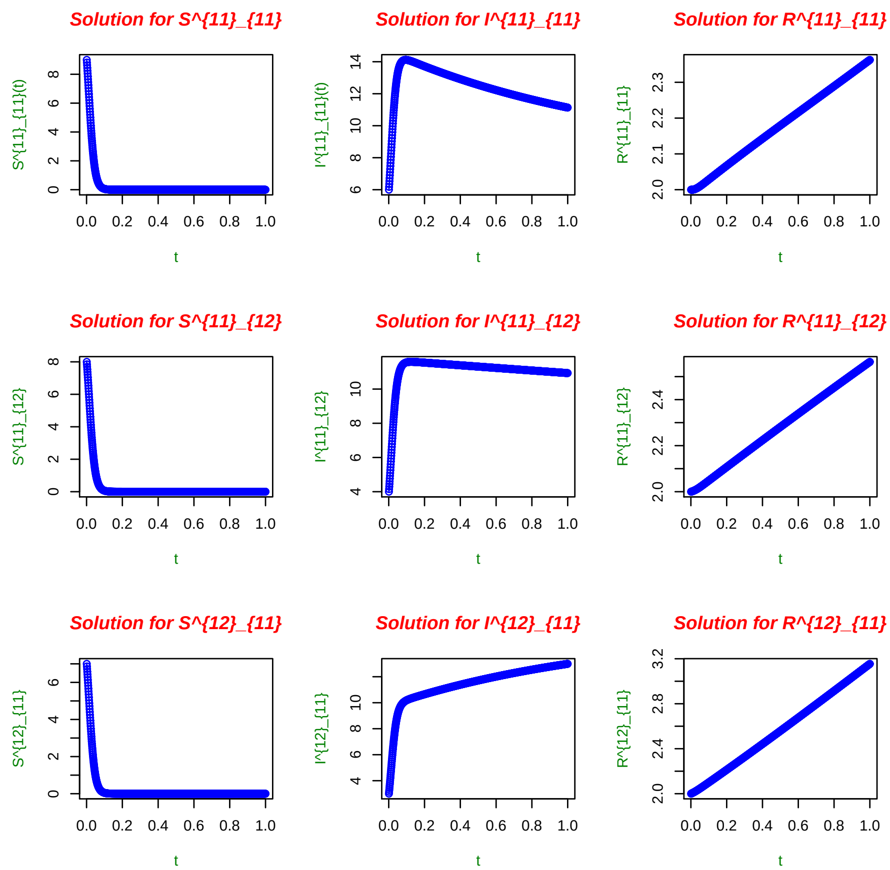

Figure 4.1: Trajectories of the disease classification (S, I, R) for residents of site s_1^1 in region C_1 at their home site and the current location in the two-scale spatial patch dynamic structure. See Remark 4.2.1 for more comments on this figure.

Remark 4.2.1 From Figure 4.1, we observe that Figures (a),(b) & (c) represent the trajectories of the different disease classes of residents of site s_1^1 at home. Figures (d),(e) & (f) represent the trajectories of the different disease classes of residents of site s_1^1 visiting site s_2^1 in home region C_1. These two groups of figures are representative of the disease dynamics of influenza affecting the residents of site s_1^1 at the intra-regional level. Figures (g),(h) & (i) represent the trajectories of the different disease classes of residents of site s_1^1 visiting site s_1^2 in region C_2. These figures reflect the behavior of the disease affecting the residents of site s_1^1 at the inter-regional level.

Furthermore, we observe that the trajectories of the susceptible (S) and infectious(I) populations saturate to their equilibrium states. We further note that the trajectory paths are random in character but because of the scale of the pictures presented in this figure, they appear to be smooth.

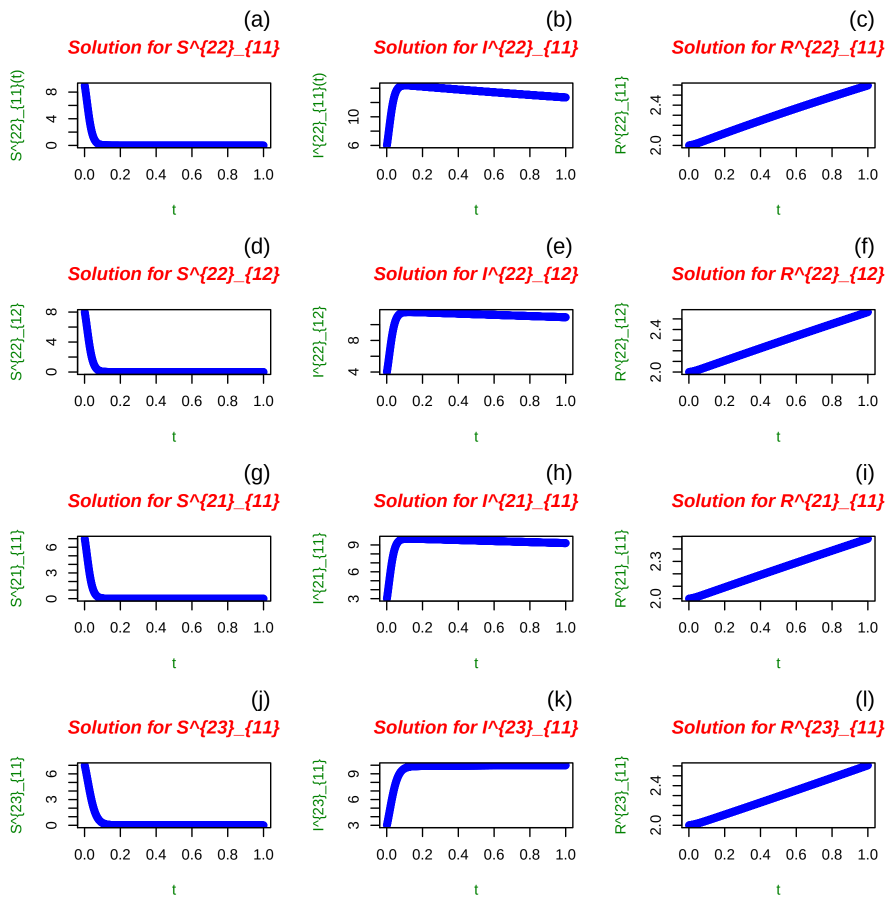

Figure 4.2: Trajectories of the disease classification (S, I, R) for residents of site s_1^2 in region C_2 at their home site and at their current locations in the two-scale spatial patch dynamic structure. See Remark 4.2.2 for more comments on this figure.

Remark 4.2.2 From Figure 4.2, we observe that Figures (a),(b) & (c) represent the trajectories of the different disease classes of residents of site s_1^2 at home. Figures (d),(e) & (f) represent the trajectories of the different disease classes of residents of site s_1^2 visiting site s_2^2 in home region C_2. These two groups of figures are representative of the disease dynamics of influenza affecting the residents of site s_1^2 at the intra-regional level. Figures (g),(h) & (i) represent the trajectories of the different disease classes of residents of site s_1^2 visiting site s_1^1 in region C_1. Figures (j),(k) & (l) represent the trajectories of the different disease classes of residents of site s_1^2 visiting site s_1^3 in region C_3. These last two groups of figures reflect the behavior of the disease affecting the residence of site s_1^2 at the inter-regional level. Furthermore, we observe that the trajectories of the susceptible (S) and infectious(I) populations saturate to their equilibrium states. We further note that the trajectory paths are random in character but because of the scale of the pictures presented in this figure, they appear to be smooth.

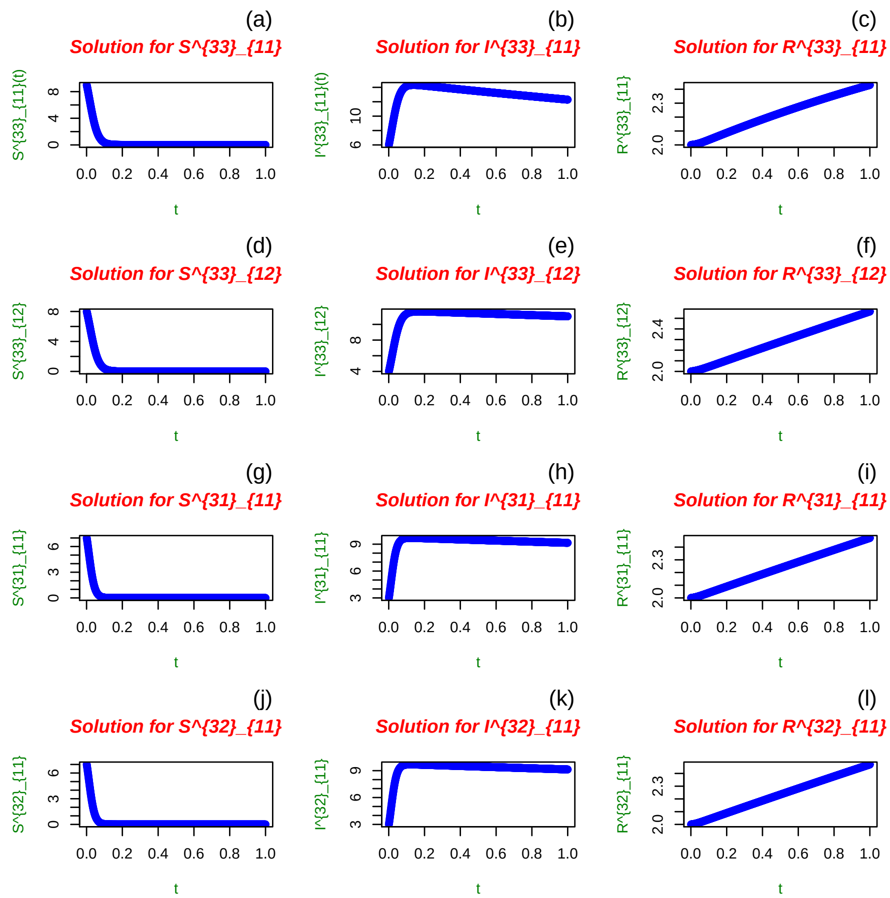

Figure 4.3: Trajectories of the disease classification (S,I,R) for residents of site s_1^3 in region C_3 at their current location in the two-scale spatial patch dynamic structure. See Remark 4.2.3 for more comments on this figure.

Remark 4.2.3 From Figure 4.3, we observe that Figures (a),(b) & (c) represent the trajectories of the different disease classes of residents of site s_1^3 at home. Figures (d),(e) & (f) represent the trajectories of the different disease classes of residents of site s_1^3 visiting site s_2^3 in home region

C_3. These two groups of figures are representative of the disease dynamics of influenza affecting the residents of site s_1^3 at the intra-regional level. Figures (g),(h) & (i) represent the trajectories of the different disease classes of residents of site s_1^3 visiting site s_1^1 in region C_1. Figures (j),(k) & (l) represent the trajectories of the different disease classes of residents of site s_1^3 visiting site s_1^2 in region C_2. The last two groups of figures reflect the behavior of the disease affecting the residence of site s_1^3 at the inter-regional level. Furthermore, we observe that the trajectories of the susceptible (S) and infectious(I) populations saturate to their equilibrium states. We further note that the trajectory paths are random in character but because of the scale of the pictures presented in this figure, they appear to be smooth.

4.3 Conclusion

An influenza stochastic epidemic dynamic model in a two-scale population structure with specific model parameters is implicitly defined in the framework of the epidemic dynamic model studied in Chapter 3. The influenza transmission process at the site level is elaborated. In addition, a suitable disease transmission rate function developed in [35] is modified and computed in the context of the influenza transmission scenario presented in this example. The Euler-Maruyama stochastic simulation scheme and application process is developed for the two-scale network centric Ito-Doob system of stochastic differential equations. Furthermore, simulated trajectories for the different state processes (susceptible, infective, removal) of residents of some sites in the three regions with respect to the current locations in the intra and interregional levels are developed and presented. The simulated findings reveal comparative evolution patterns for the different state processes and current locations over time.

5 GLOBAL PROPERTIES OF A TWO-SCALE NETWORK STOCHASTIC DELAYED HUMAN EPIDEMIC DYNAMIC MODEL

5.1 Introduction

Delay epidemic dynamic models are more realistic than ordinary epidemic dynamic models because they represent finer aspects of the disease process such as the hereditary features of the disease. There are generally two sources of the time delay in most epidemic processes namely:- disease latency and immunity. Disease latency is the time lapse between acquisition of the infectious agent and infectiousness. On the other hand, disease immunity is conferred to the endangered population in two general ways namely:- artificial immunity through vaccination of the susceptible individuals or natural immunity (infection acquired immunity) conferred by the disease infection after recovery from the disease. Most often the effectiveness of natural or artificial immunity wanes after a period of time due to low disease exposure and therefore require boosting. For diseases such as measles, vaccinated individuals are less immune than those with natural immunity[84]; for pertussis, the immunity declines 6-12 years after the last disease episode or booster dose[63]. Several studies representing the effects of disease latency or immunity of the epidemic process into the epidemic dynamic model have been done[67, 68, 71, 72, 73, 74, 77].

Some of the main issues addressed in the study of mathematical delay epidemic dynamic models include: model validation (existence and uniqueness of positive solution) and the stability of the disease free equilibrium. The global positive solution existence is establish using an extension criterion of a local solution. This approach is exhibited in [65, 68]. Moreover, the extension of the local solution is exhibited in [68] by applying a Lyapunov energy function method.

The global asymptotic stability of disease free equilibrium for delay epidemic dynamic models is established by applying the Lyapunov functional approach [76, 50, 51, 61, 62, 68]. Furthermore, the disease free equilibrium for SIRS[75, 57] and SIR[56] single-group delay epidemic models are studied.

In this chapter we extend the stochastic epidemic dynamic model studied in Chapter 3 by incorporating the temporary immunity delay period of the disease. We consider an infectious disease that confers natural immunity to all recovered individuals immediately after infectiousness. This work is organized as follows. In Section 5.2, we derive the natural or infection-acquired immunity delay part of the epidemic process. In Section 5.3, we present the model validation results. In Section 5.4, we show the stochastic asymptotic stability of the disease free equilibrium.

5.2 Derivation of the SIR Delayed Stochastic dynamic Model

We assume that the epidemic represented in this chapter is an SIR(susceptible-infective-removal) satisfying all assumptions for the population structure, human mobility process, and disease dynamics described in Chapter 3. Furthermore, all removals (R) are those who have acquired natural immunity against the disease. For an exclusive SIR epidemic process with no fractional transfer from the infectious to the susceptible states, the conditions for the recovery rates of the disease η^u_a and α^u_a, $\forall u \in I(1,M)$ and $\forall a \in I(1,n_u)$ represented in the SIRS epidemic model (3.2.9)-(3.2.11) reduce to $\eta^u_a = \alpha^u_a = 0$, $\forall u \in I(1,M)$ and $\forall a \in I(1,n_u)$. Furthermore, we assume that for each $r \in I(1,M)$, and $i \in I(1,n_r)$, an infectious (I^{ru}_{ia}) resident of site s^r_i in region C_r visiting site s^u_a in region C_u recovers from the disease and acquires temporary natural or infection-acquired immunity against the disease immediately after recovery. The recovered individual further looses immunity against the disease and becomes susceptible to the disease after a period of time T^r_i. We assume that the natural immunity period T^r_i is constant for all naturally immune residents of site s^r_i present at their home site and at all visiting sites s^u_a in region $C_u, \forall u \in I(1,M)$ and $\forall a \in I(1,n_u)$ in the large scale two level dynamic structure. We incorporate the natural or infection acquired immunity into the epidemic dynamic model (3.2.9)-(3.2.11) by introducing the term $\rho^u_a I^{ru}_{ia}(t-T^r_i)e^{-\delta^u_a T^r_i}$, where $e^{-\delta^u_a T^r_i}$ is the probability that an individual has survived from natural death during the immunity period T^r_i, before becoming susceptible again[67]. The two level large scale stochastic SIR delayed epidemic dynamic model that is under the influence of human mobility process is as follows:

$$dS_{ia}^{ru} = \begin{cases} \big[B_i^r + \sum_{k=1}^{n_r} \rho_{ik}^{rr} S_{ik}^{rr} + \sum_{q\neq r}^{M} \sum_{a=1}^{n_q} \rho_{ia}^{rq} S_{ia}^{rq} + \rho_i^r I_{ii}^{rr}(t - T_i^r) e^{-\delta_i^r T_i^r} \\ -(\gamma_i^r + \sigma_i^r + \delta_i^r) S_{ii}^{rr} - \sum_{u=1}^{M} \sum_{a=1}^{n_u} \beta_{iia}^{rru} S_{ii}^{rr} I_{ai}^{ur}\big] dt \\ -\big[\sum_{u=1}^{M} \sum_{a=1}^{n_u} v_{iia}^{rru} S_{ii}^{rr} I_{ai}^{ur} dw_{iia}^{rru}(t)\big], u = r, a = i \\ \big[\sigma_{ij}^{rr} S_{ii}^{rr} + \rho_j^r I_{ij}^{rr}(t - T_i^r) e^{-\delta_j^r T_i^r} - (\rho_{ij}^{rr} + \delta_j^r) S_{ij}^{rr} \\ -\sum_{u=1}^{M} \sum_{a=1}^{n_u} \beta_{jia}^{rru} S_{ij}^{rr} I_{aj}^{ur}\big] dt - \big[\sum_{u=1}^{M} \sum_{a=1}^{n_u} v_{jia}^{rru} S_{ij}^{rr} I_{aj}^{ur} dw_{jia}^{rru}(t)\big], u = r, a = j, j \neq i, \\ \big[\gamma_{il}^{rq} S_{ii}^{rr} + \rho_l^q I_{il}^{rq}(t - T_i^r) e^{-\delta_l^q T_i^r} - (\rho_{il}^{rq} + \delta_l^q) S_{il}^{rq} \\ -\sum_{u=1}^{M} \sum_{a=1}^{n_u} \beta_{lia}^{qru} S_{il}^{rq} I_{al}^{uq}\big] dt - \big[\sum_{u=1}^{M} \sum_{a=1}^{n_u} v_{lia}^{qru} S_{il}^{rq} I_{al}^{uq} dw_{lia}^{qru}(t)\big], u = q, a = l, q \neq r, \end{cases} \tag{5.2.1}$$

$$dI_{ia}^{ru} = \begin{cases} \big[\sum_{k=1}^{n_r} \rho_{ik}^{rr} I_{ik}^{rr} + \sum_{q\neq r}^{M} \sum_{a=1}^{n_q} \rho_{ia}^{rq} I_{ia}^{rq} - \rho_i^r I_{ii}^{rr} \\ -(\gamma_i^r + \sigma_i^r + \delta_i^r + d_i^r) I_{ii}^{rr} + \sum_{u=1}^{M} \sum_{a=1}^{n_u} \beta_{iia}^{rru} S_{ii}^{rr} I_{ai}^{ur}\big] dt \\ +\big[\sum_{u=1}^{M} \sum_{a=1}^{n_u} v_{iia}^{rru} S_{ii}^{rr} I_{ai}^{ur} dw_{iia}^{rru}(t)\big], u = r, a = i \\ \big[\sigma_{ij}^{rr} I_{ii}^{rr} - \rho_j^r I_{ij}^{rr} - (\rho_{ij}^{rr} + \delta_j^r + d_j^r) I_{ij}^{rr} + \sum_{u=1}^{M} \sum_{a=1}^{n_u} \beta_{jia}^{rru} S_{ij}^{rr} I_{aj}^{ur}\big] dt \\ +\big[\sum_{u=1}^{M} \sum_{a=1}^{n_u} v_{jia}^{rru} S_{ij}^{rr} I_{aj}^{ur} dw_{jia}^{rru}(t)\big], u = r, a = j, j \neq i, \\ \big[\gamma_{il}^{rq} I_{ii}^{rr} - \rho_l^q I_{il}^{rq} - (\rho_{il}^{rq} + \delta_l^q + d_l^q) I_{il}^{rq} \\ +\sum_{u=1}^{M} \sum_{a=1}^{n_u} \beta_{lia}^{qru} S_{il}^{rq} I_{al}^{uq}\big] dt + \big[\sum_{u=1}^{M} \sum_{a=1}^{n_u} v_{lia}^{qru} S_{il}^{rq} I_{al}^{uq} dw_{lia}^{qru}(t)\big], u = q, a = l, q \neq r, \end{cases} \tag{5.2.2}$$

$$R_{ia}^{ru} = \begin{cases} \big[\sum_{k=1}^{n_r} \rho_{ik}^{rr} R_{ik}^{rr} + \sum_{q\neq r}^{M} \sum_{l=1}^{n_q} \rho_{il}^{rq} R_{il}^{rq} + \rho_i^r I_{ii}^{rr} - \rho_i^r I_{ii}^{rr}(t - T_i^r) e^{-\delta_i^r T_i^r} \\ -(\gamma_i^r + \sigma_i^r + \delta_i^r) R_{ii}^{rr}\big] dt, u = r, a = i \\ \big[\sigma_{ij}^{rr} R_{ii}^{rr} + \rho_j^r I_{ij}^{rr} - \rho_j^r I_{ij}^{rr}(t - T_i^r) e^{-\delta_j^r T_i^r} \\ -(\rho_{ij}^{rr} + \delta_j^r) R_{ij}^{rr}\big] dt, u = r, a = j, j \neq i, \\ \big[\gamma_{il}^{rq} R_{ii}^{rr} + \rho_l^q I_{il}^{rq} - \rho_l^q I_{il}^{rq}(t - T_i^r) e^{-\delta_l^q T_i^r} \\ -(\rho_{il}^{rq} + \delta_l^q) R_{il}^{rq}\big] dt, u = q, a = l, q \neq r, \end{cases} \tag{5.2.3}$$

where all parameters are previously defined. Furthermore, for each $r \in I(1,M)$, and $i \in I(1,n_r)$, we have the following initial conditions

$$(S_{ia}^{ru}(t,w), I_{ia}^{ru}(t,w), R_{ia}^{ru}(t,w)) = (\varphi_{ia1}^{ru}(t), \varphi_{ia2}^{ru}(t), \varphi_{ia3}^{ru}(t)), t \in [-T_i^r, t_0],$$
$$\varphi_{iak}^{ru} \in \mathcal{C}([-T_i^r, t_0], \mathbb{R}_+), \forall k = 1,2,3, \forall r, q \in I(1,M), a \in I(1,n_u), i \in I(1,n_r),$$
$$\varphi_{iak}^{ru}(t_0) > 0, \forall k = 1,2,3, \tag{5.2.4}$$

where $\mathcal{C}([-T_i^r,t_0],\mathbb{R}_+)$ is the space of continuous functions with the supremum norm

$$||\varphi||_\infty = Sup_{-T_i^r\leq t\leq t_0}|\varphi(t)|, \tag{5.2.5}$$

and w is a Wierner process. Furthermore, the random continuous functions $\varphi_{iak}^{ru}, k=1,2,3$ are $\mathcal{F}_0-measurable$, or independent of $w(t)$ for all $t\geq 0$.

We utilize (3.2.18) to express the state of system (5.2.1)-(5.2.3) in vector form. Furthermore, using the expression (3.3.25), it follows from (5.2.1)-(5.2.3) that for each $i\in I(1,n_r), l\in I_i^r(1,n_q), r\in I(1,M)$ and $q\in I^r(1,M)$,

$$dy_{il}^{rq} = \begin{cases} \left[B_i^r+\sum_{k\neq i}^{n_r}\rho_{ik}^{rr}y_{ik}^{rr}+\sum_{q\neq r}^{M}\sum_{a=1}^{n_q}\rho_{ia}^{rq}y_{ia}^{rq}-(\gamma_i^r+\sigma_i^r+\delta_i^r)y_{ii}^{rr}-d_i^rI_{ii}^{rr}\right]dt, for \quad q=r, l=i \\ \left[\sigma_{ij}^{rr}y_{ii}^{rr}-(\rho_{ij}^{rr}+\delta_j^r)y_{ij}^{rr}-d_j^rI_{ij}^{rr}\right]dt, forq=r, a=j \quad and \quad i\neq j, \\ \left[\gamma_{il}^{rq}y_{ii}^{rr}-(\rho_{il}^{rq}+\delta_l^q)y_{il}^{rq}-d_l^qI_{il}^{rq}\right]dt, for \quad q\neq r, y_{il}^{rq}(t_0)\geq 0, \end{cases} \tag{5.2.6}$$

5.3 Model Validation Results

In the following we state and prove a positive solution process existence theorem for the delayed system (5.2.1)-(5.2.3). We utilize the Lyapunov energy function method[68] to establish the results of this theorem. We observe from (5.2.1)-(5.2.3) that (5.2.3) decouples from the first two equations in the system. Therefore, it suffices to prove the existence of positive solution process for $(S_{ia}^{ru}, I_{ia}^{ru})$. We utilize the notations (3.2.18) and keep in mind that $X_{ia}^{ru}=(S_{ia}^{ru},I_{ia}^{ru})^T$.

Theorem 5.3.1 *Let $r,u\in I(1,M)$, $i\in I(1,n_r)$ and $a\in I(1,n_u)$. Given any initial conditions (5.2.4) and (5.2.5), there exists a unique solution process $X_{ia}^{ru}(t,w)=(S_{ia}^{ru}(t,w),I_{ia}^{ru}(t,w))^T$ satisfying (5.2.1) and (5.2.2), for all $t\geq t_0$. Moreover, the solution process is positive for all $t\geq t_0$ a.s. That is, $S_{ia}^{ru}(t,w)>0, I_{ia(t,w)}^{ru}>0, \forall t\geq t_0$ a.s.*

Proof:

It is easy to see that the coefficients of (5.2.1) and (5.2.2) satisfy the local Lipschitz condition for the given initial data (5.2.4). Therefore there exist a unique maximal local solution $X_{ia}^{ru}(t,w)$ on $t\in[-T_i^r,\tau_e(w)]$, where $\tau_e(w)$ is the first hitting time or the explosion time[34]. We show subsequently that $S_{ia}^{ru}(t,w), I_{ia}^{ru}(t,w) > 0$ for all $t \in [-T_i^r, \tau_e(w)]$ almost surely. We define the stopping time

$$\begin{cases} \tau_+ &= sup\{t \in (t_0, \tau_e(w)) : S_{ia}^{ru}|_{[t_0,t]} > 0 \quad and \quad I_{ia}^{ru}|_{[t_0,t]} > 0\}, \\ \tau_+(t) &= \min(t, \tau_+), \quad for \quad t \geq t_0. \end{cases} \tag{5.3.7}$$

and we show that $\tau_+(t) = \tau_e(w)$ a.s. Suppose on the contrary that $P(\tau_+(t) < \tau_e(w)) > 0$. Let $w \in \{\tau_+(t) < \tau_e(w)\}$, and $t \in [t_0, \tau_+(t))$. Define

$$\begin{cases} V(X_{00}^{00}) = \sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u=1}^{M}\sum_{a=1}^{n_u} V(X_{ia}^{ru}), \\ V(X_{ia}^{ru}) = \ln(S_{ia}^{ru}) + \ln(I_{ia}^{ru}), \forall t \leq \tau_+(t). \end{cases} \tag{5.3.8}$$

We rewrite (5.3.8) as follows

$$V(X_{00}^{00}) = \sum_{r=1}^{M}\sum_{i=1}^{n_r}\left[V(X_{ii}^{rr}) + \sum_{j\neq i}^{n_r} V(X_{ij}^{rr}) + \sum_{q\neq r}^{M}\sum_{l=1}^{n_q} V(X_{il}^{rq})\right], \tag{5.3.9}$$

And (5.3.9) further implies that

$$dV(X_{00}^{00}) = \sum_{r=1}^{M}\sum_{i=1}^{n_r}\left[dV(X_{ii}^{rr}) + \sum_{j\neq i}^{n_r} dV(X_{ij}^{rr}) + \sum_{q\neq r}^{M}\sum_{l=1}^{n_q} dV(X_{il}^{rq})\right], \tag{5.3.10}$$

where dV is the Ito-Doob differential operator with respect to the system (5.2.1)-(5.2.3). We express the terms of the right-hand-side of (5.3.10) in the following:

Site Level: From (5.3.8) the terms of the right-hand-side of (5.3.10) for the case of $u = r, a = i$

$$\begin{aligned} dV(X_{ii}^{rr}) \quad = \quad & \left[\frac{B_i^r}{S_{ii}^{rr}} + \sum_{k\neq i}^{n_r} \rho_{ik}^{rr}\frac{S_{ik}^{rr}}{S_{ii}^{rr}} + \sum_{q\neq r}^{M}\sum_{l=1}^{n_q} \rho_{ia}^{rq}\frac{S_{ia}^{rq}}{S_{ii}^{rr}} + \frac{\rho_i^r I_{ii}^{rr}(t-T_i^r)e^{-\delta_i^r T_i^r}}{S_{ii}^{rr}}\right. \\ & \left. -(\gamma_i^r + \sigma_i^r + \delta_i^r) - \sum_{u=1}^{M}\sum_{a=1}^{n_u} \beta_{iia}^{rru} I_{ai}^{ur} - \frac{1}{2}\sum_{u=1}^{M}\sum_{a=1}^{n_u}(v_{iia}^{rru})^2 (I_{ai}^{ur})^2\right] dt \\ & \left[\sum_{k\neq i}^{n_r} \rho_{ik}^{rr}\frac{I_{ik}^{rr}}{S_{ii}^{rr}} + \sum_{q\neq r}^{M}\sum_{l=1}^{n_q} \rho_{ia}^{rq}\frac{I_{ia}^{rq}}{S_{ii}^{rr}} - \rho_i^r - (\gamma_i^r + \sigma_i^r + \delta_i^r + d_i^r)\right. \\ & \left. -\sum_{u=1}^{M}\sum_{a=1}^{n_u} \beta_{iia}^{rru}\frac{S_{ii}^{rr}}{I_{ii}^{rr}} I_{ai}^{ur} - \frac{1}{2}\sum_{u=1}^{M}\sum_{a=1}^{n_u}(v_{iia}^{rru})^2\frac{(S_{ii}^{rr})^2}{(I_{ii}^{rr})^2}(I_{ai}^{ur})^2\right] dt \\ & -\sum_{u=1}^{M}\sum_{a=1}^{n_u} v_{iia}^{rru} I_{ai}^{ur} dw_{iia}^{rru}(t) + \sum_{u=1}^{M}\sum_{a=1}^{n_u} v_{iia}^{rru}\frac{S_{ii}^{rr}}{I_{ii}^{rr}} I_{ai}^{ur} dw_{iia}^{rru}(t) \end{aligned} \tag{5.3.11}$$

Intra-regional Level: From (5.3.8) the terms of the right-hand-side of (5.3.10) for the case of $u = r, a = j, j \neq i$

$$\begin{aligned} dV(X_{ij}^{rr}) &= \Bigg[\sigma_{ij}^{rr}\frac{S_{ii}^{rr}}{S_{ij}^{rr}} + \frac{\rho_j^r I_{ij}^{rr}(t-T_i^r)e^{-\delta_j^r T_i^r}}{S_{ij}^{rr}} \\ &\quad -(\rho_{ij}^{rr}+\delta_j^r) - \sum_{u=1}^{M}\sum_{a=1}^{n_u}\beta_{jia}^{rru}I_{aj}^{ur} - \frac{1}{2}\sum_{u=1}^{M}\sum_{a=1}^{n_u}(v_{jia}^{rru})^2(I_{aj}^{ur})^2\Bigg]dt \\ &\quad + \Bigg[\sigma_{ij}^{rr}\frac{I_{ii}^{rr}}{I_{ij}^{rr}} - \rho_j^r - (\rho_{ij}^{rr}+\delta_j^r+d_j^r) \\ &\quad + \sum_{u=1}^{M}\sum_{a=1}^{n_u}\beta_{jia}^{rru}\frac{S_{ij}^{rr}}{I_{ij}^{rr}}I_{aj}^{ur} - \frac{1}{2}\sum_{u=1}^{M}\sum_{a=1}^{n_u}(v_{jia}^{rru})^2\frac{(S_{ij}^{rr})^2}{(I_{ij}^{rr})^2}(I_{aj}^{ur})^2\Bigg]dt \\ &\quad - \sum_{u=1}^{M}\sum_{a=1}^{n_u}v_{jia}^{rru}I_{aj}^{ur}dw_{jia}^{rru}(t) + \sum_{u=1}^{M}\sum_{a=1}^{n_u}v_{jia}^{rru}\frac{S_{ij}^{rr}}{I_{ij}^{rr}}I_{aj}^{ur}dw_{jia}^{rru}(t) \end{aligned} \tag{5.3.12}$$

Regional Level: From (5.3.8) the terms of the right-hand-side of (5.3.10) for the case of $u = q, q \neq r, a = l$,

$$\begin{aligned} dV(X_{il}^{rq}) &= \Bigg[\gamma_{il}^{rq}\frac{S_{ii}^{rr}}{S_{iq}^{rq}} + \frac{\rho_l^q I_{il}^{rq}(t-T_i^r)e^{-\delta_l^q T_i^r}}{S_{il}^{rq}} \\ &\quad -(\rho_{il}^{rq}+\delta_l^q) - \sum_{u=1}^{M}\sum_{a=1}^{n_u}\beta_{lia}^{qru}I_{al}^{uq} - \frac{1}{2}\sum_{u=1}^{M}\sum_{a=1}^{n_u}(v_{lia}^{qru})^2(I_{al}^{uq})^2\Bigg]dt \\ &\quad + \Bigg[\gamma_{il}^{rq}\frac{I_{ii}^{rr}}{I_{il}^{rq}} - \rho_l^q - (\rho_{il}^{rq}+\delta_l^q+d_l^q) \\ &\quad + \sum_{u=1}^{M}\sum_{a=1}^{n_u}\beta_{lia}^{qru}\frac{S_{il}^{rq}}{I_{il}^{rq}}I_{al}^{uq} - \frac{1}{2}\sum_{u=1}^{M}\sum_{a=1}^{n_u}(v_{lia}^{qru})^2\frac{(S_{il}^{rq})^2}{(I_{il}^{rq})^2}(I_{al}^{uq})^2\Bigg]dt \\ &\quad - \sum_{u=1}^{M}\sum_{a=1}^{n_u}v_{lia}^{qru}I_{al}^{uq}dw_{lia}^{qru}(t) + \sum_{u=1}^{M}\sum_{a=1}^{n_u}v_{lia}^{qru}\frac{S_{il}^{rq}}{I_{il}^{rq}}I_{al}^{uq}dw_{lia}^{qru}(t) \end{aligned} \tag{5.3.13}$$

It follows from (5.3.11)-(5.3.13), (5.3.10), and (5.3.7) that for $t < \tau_+(t)$,

$$
\begin{aligned}
V(X_{00}^{00}(t)) - V(X_{00}^{00}(t_0)) \quad \geq \quad & \sum_{r=1}^{M}\sum_{i=1}^{n_r}\int_{t_0}^{t}\Bigg[\frac{\rho_i^r I_{ii}^{rr}(t-T_i^r)e^{-\delta_i^r T_i^r}}{S_{ii}^{rr}} - (\gamma_i^r + \sigma_i^r + \delta_i^r) - \sum_{u=1}^{M}\sum_{a=1}^{n_u}\beta_{iia}^{rru} I_{ai}^{ur} \\
& -\frac{1}{2}\sum_{u=1}^{M}\sum_{a=1}^{n_u}(v_{iia}^{rru})^2 (I_{ai}^{ur})^2\Bigg] ds + \sum_{r=1}^{M}\sum_{i=1}^{n_r}\int_{t_0}^{t}[-\rho_i^r - (\gamma_i^r + \sigma_i^r + \delta_i^r + d_i^r) \\
& -\sum_{u=1}^{M}\sum_{a=1}^{n_u}\beta_{iia}^{rru}\frac{S_{ii}^{rr}}{I_{ii}^{rr}} I_{ai}^{ur} - \frac{1}{2}\sum_{u=1}^{M}\sum_{a=1}^{n_u}(v_{iia}^{rru})^2\frac{(S_{ii}^{rr})^2}{(I_{ii}^{rr})^2}(I_{ai}^{ur})^2\Bigg] ds \\
& -\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u=1}^{M}\sum_{a=1}^{n_u}\int_{t_0}^{t} v_{iia}^{rru} I_{ai}^{ur} dw_{iia}^{rru}(s) \\
& +\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u=1}^{M}\sum_{a=1}^{n_u}\int_{t_0}^{t} v_{iia}^{rru}\frac{S_{ii}^{rr}}{I_{ii}^{rr}} I_{ai}^{ur} dw_{iia}^{rru}(s) \\
& \sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{j\neq i}^{n_r}\int_{t_0}^{t}\Bigg[\frac{\rho_j^r I_{ij}^{rr}(t-T_i^r)e^{-\delta_j^r T_i^r}}{S_{ij}^{rr}} \\
& -(\rho_{ij}^{rr} + \delta_j^r) - \sum_{u=1}^{M}\sum_{a=1}^{n_u}\beta_{jia}^{rru} I_{aj}^{ur} - \frac{1}{2}\sum_{u=1}^{M}\sum_{a=1}^{n_u}(v_{jia}^{rru})^2 (I_{aj}^{ur})^2\Bigg] ds \\
& +\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{j\neq i}^{n_r}\int_{t_0}^{t}\Bigg[-\rho_j^r - (\rho_{ij}^{rr} + \delta_j^r + d_j^r) - \frac{1}{2}\sum_{u=1}^{M}\sum_{a=1}^{n_u}(v_{jia}^{rru})^2\frac{(S_{ij}^{rr})^2}{(I_{ij}^{rr})^2}(I_{aj}^{ur})^2\Bigg] ds \\
& -\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{j\neq i}^{n_r}\sum_{u=1}^{M}\sum_{a=1}^{n_u}\int_{t_0}^{t} v_{jia}^{rru} I_{aj}^{ur} dw_{jia}^{rru}(s) \\
& +\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{j\neq i}^{n_r}\sum_{u=1}^{M}\sum_{a=1}^{n_u}\int_{t_0}^{t} v_{jia}^{rru}\frac{S_{ij}^{rr}}{I_{ij}^{rr}} I_{aj}^{ur} dw_{jia}^{rru}(s) \\
& +\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{q\neq r}^{M}\sum_{l=1}^{n_q}\int_{t_0}^{t}\Bigg[\frac{\rho_l^q I_{il}^{rq}(t-T_i^r)e^{-\delta_l^q T_i^r}}{S_{il}^{rq}} - (\rho_{il}^{rq} + \delta_l^q) - \sum_{u=1}^{M}\sum_{a=1}^{n_u}\beta_{lia}^{qru} I_{al}^{uq} \\
& -\frac{1}{2}\sum_{u=1}^{M}\sum_{a=1}^{n_u}(v_{lia}^{qru})^2 (I_{al}^{uq})^2\Bigg] ds + \sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{q\neq r}^{M}\sum_{l=1}^{n_q}\int_{t_0}^{t}\big[-\rho_l^q - (\rho_{il}^{rq} + \delta_l^q + d_l^q) \\
& -\frac{1}{2}\sum_{u=1}^{M}\sum_{a=1}^{n_u}(v_{lia}^{qru})^2\frac{(S_{il}^{rq})^2}{(I_{il}^{rq})^2}(I_{al}^{uq})^2\Bigg] ds \\
& -\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{q\neq r}^{M}\sum_{l=1}^{n_q}\sum_{u=1}^{M}\sum_{a=1}^{n_u}\int_{t_0}^{t} v_{lia}^{qru} I_{al}^{uq} dw_{lia}^{qru}(s) \\
& +\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{q\neq r}^{M}\sum_{l=1}^{n_q}\sum_{u=1}^{M}\sum_{a=1}^{n_u}\int_{t_0}^{t} v_{lia}^{qru}\frac{S_{il}^{rq}}{I_{il}^{rq}} I_{al}^{uq} dw_{lia}^{qru}(s) \qquad (5.3.14)
\end{aligned}
$$

Taking the limit on (5.3.14) as $t \to \tau_+(t)$, it follows from (5.3.8) and (5.3.7) that $V(X_{00}^{00}(t)) - V(X_{00}^{00}(t_0)) \leq -\infty$. This contradicts the finiteness of the right-hand-side of the inequality (5.3.14).

Hence $\tau_+(t) = \tau_e(w)$ a.s. We show subsequently that $\tau_e(w) = \infty$. Let $k > 0$ be a positive integer such that $||\varphi^{00}_{00}||_1 \le k$, where the vector of initial values $\varphi^{00}_{00} = (\varphi^{ru}_{ia})_{1\le r,u\le M, 1\le i\le n_r, 1\le a\le n_u} \in \mathbb{R}^{2n^2}$ is defined in (5.2.4). Furthermore, $||.||_1$ is the p-sum norm (3.2.19) for the case of $p = 1$. We define the stopping time

$$\begin{cases} \tau_k = sup\{t \in [t_0, \tau_e) : ||X^{00}_{00}(s)||_1 \le k, s \in [0,t]\} \\ \tau_k(t) = \min(t, \tau_k). \end{cases} \tag{5.3.15}$$

where from (3.2.19),

$$||X^{00}_{00}(s)||_1 = \sum_{r=1}^{M}\sum_{u=1}^{M}\sum_{i=1}^{n_r}\sum_{a=1}^{n_u}(S^{ru}_{ia}(s) + I^{ru}_{ia}(s)). \tag{5.3.16}$$

It is easy to see that as $k \to \infty$, $\tau_k(t)$ increases. Set $\lim_{k\to\infty}\tau_k(t) = \tau_\infty$. Then $\tau_\infty \le \tau_e$ a.s. We show in the following that: (1.) $\tau_e = \tau_\infty \quad a.s. \Leftrightarrow P(\tau_e \ne \tau_\infty) = 0$, (2.) $\tau_\infty = \infty \quad a.s. \Leftrightarrow P(\tau_\infty = \infty) = 1$.

Suppose on the contrary that $P(\tau_\infty < \tau_e) > 0$. Let $w \in \{\tau_\infty < \tau_e\}$ and $t \le \tau_\infty$. In the same structure line as (5.3.8) and (5.3.10), define

$$\begin{cases} V_1(X^{00}_{00}) = \sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u=1}^{M}\sum_{a=1}^{n_u} V(X^{ru}_{ia}), \\ V_1(X^{ru}_{ia}) = e^{\delta^u_a t}(S^{ru}_{ia} + I^{ru}_{ia}), \forall t \le \tau_k(t). \end{cases} \tag{5.3.17}$$

From (5.3.17), using the expression (5.3.10), the Ito-Doob differential dV_1 with respect to the system (5.2.1)-(5.2.3) is given as follows:

Site Level: From (5.3.17), the terms of the right-hand-side of (5.3.10) for the case of $u = r, a = i$

$$\begin{aligned} dV_1(X^{rr}_{ii}) &= e^{\delta^r_i t}\Bigg[B^r_i + \sum_{k\ne i}^{n_r}\rho^{rr}_{ik}S^{rr}_{ik} + \sum_{q\ne r}^{M}\sum_{l=1}^{n_q}\rho^{rq}_{ia}S^{rq}_{ia} + \rho^r_i I^{rr}_{ii}(t - T^r_i)e^{-\delta^r_i T^r_i} \\ &\quad -(\gamma^r_i + \sigma^r_i)S^{rr}_{ii}\Bigg]dt + e^{\delta^r_i t}\Bigg[\sum_{k\ne i}^{n_r}\rho^{rr}_{ik}I^{rr}_{ik} + \sum_{q\ne r}^{M}\sum_{l=1}^{n_q}\rho^{rq}_{ia}I^{rq}_{ia} - \rho^r_i I^{rr}_{ii} \\ &\quad -(\gamma^r_i + \sigma^r_i + d^r_i)I^{rr}_{ii}\Bigg]dt \end{aligned} \tag{5.3.18}$$

Intra-regional Level: From (5.3.17), the terms of the right-hand-side of (5.3.10) for the case of $u = r, a = j, j \ne i$

$$\begin{aligned} dV_1(X^{rr}_{ij}) &= e^{\delta^r_i t}\left[\sigma^{rr}_{ij}S^{rr}_{ii} + \rho^r_j I^{rr}_{ij}(t - T^r_i)e^{-\delta^r_j T^r_i} - \rho^{rr}_{ij}S^{rr}_{ij}\right]dt \\ &\quad + e^{\delta^r_j t}\left[\sigma^{rr}_{ij}I^{rr}_{ii} + \rho^r_j I^{rr}_{ij} - (\rho^{rr}_{ij} + d^r_j)I^{rr}_{ij}\right]dt \end{aligned} \tag{5.3.19}$$

Regional Level: From (5.3.17), the terms of the right-hand-side of (5.3.10) for the case of $u = q, q \neq r, a = l$

$$\begin{aligned} dV_1(X_{il}^{rq}) &= e^{\delta_l^q t}\left[\gamma_{il}^{rq} S_{ii}^{rr} + \rho_l^q I_{il}^{rq}(t - T_i^r)e^{-\delta_l^q T_i^r} - \rho_{il}^{rq} S_{il}^{rq}\right] dt \\ &\quad + e^{\delta_l^q t}\left[\gamma_{il}^{rq} I_{ii}^{rr} + \rho_l^q I_{il}^{rq} - (\rho_{il}^{rq} + d_l^q) I_{il}^{rq}\right] dt \end{aligned} \tag{5.3.20}$$

From (5.3.18)-(5.3.20), (5.3.10), integrating (5.3.10) over $[t_0, \tau]$ leads to the following

$$\begin{aligned} & V_1(X_{00}^{00}(\tau)) \\ = \; & V_1(X_{00}^{00}(t_0)) + \sum_{r=1}^{M}\sum_{i=1}^{n_r}\int_{t_0}^{\tau} e^{\delta_i^r s}\left[B_i^r + \sum_{k\neq i}^{n_r}\rho_{ik}^{rr} S_{ik}^{rr} + \sum_{q\neq r}^{M}\sum_{l=1}^{n_q}\rho_{ia}^{rq} S_{ia}^{rq} + \rho_i^r I_{ii}^{rr}(t - T_i^r)e^{-\delta_i^r T_i^r}\right. \\ & \left. - (\gamma_i^r + \sigma_i^r) S_{ii}^{rr}\right] ds + \sum_{r=1}^{M}\sum_{i=1}^{n_r}\int_{t_0}^{\tau} e^{\delta_i^r s}\left[\sum_{k\neq i}^{n_r}\rho_{ik}^{rr} I_{ik}^{rr} + \sum_{q\neq r}^{M}\sum_{l=1}^{n_q}\rho_{ia}^{rq} I_{ia}^{rq} - \rho_i^r I_{ii}^{rr}\right. \\ & \left. - (\gamma_i^r + \sigma_i^r + d_i^r) I_{ii}^{rr}\right] ds \\ & + \sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{j\neq i}^{n_r}\int_{t_0}^{\tau} e^{\delta_i^r s}\left[\sigma_{ij}^{rr} S_{ii}^{rr} + \rho_j^r I_{ij}^{rr}(t - T_i^r)e^{-\delta_j^r T_i^r} - \rho_{ij}^{rr} S_{ij}^{rr}\right] ds \\ & + \sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{j\neq i}^{n_r}\int_{t_0}^{\tau} e^{\delta_j^r s}\left[\sigma_{ij}^{rr} I_{ii}^{rr} - \rho_j^r I_{ij}^{rr} - (\rho_{ij}^{rr} + d_j^r) I_{ij}^{rr}\right] ds \\ & + \sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{q\neq r}^{n_r}\sum_{l=1}^{n_q}\int_{t_0}^{\tau} e^{\delta_l^q s}\left[\gamma_{il}^{rq} S_{ii}^{rr} + \rho_l^q I_{il}^{rq}(t - T_i^r)e^{-\delta_l^q T_i^r} - \rho_{il}^{rq} S_{il}^{rq}\right] ds \\ & + \sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{q\neq r}^{n_r}\sum_{l=1}^{n_q}\int_{t_0}^{\tau} e^{\delta_l^q s}\left[\gamma_{il}^{rq} I_{ii}^{rr} - \rho_l^q I_{il}^{rq} - (\rho_{il}^{rq} + d_l^q) I_{il}^{rq}\right] ds \end{aligned} \tag{5.3.21}$$

From (5.3.21), we let $\tau = \tau_k(t)$, where $\tau_k(t)$ is defined in (5.3.15). It is easy to see from (5.3.21), (5.3.15), (5.3.16), and (5.3.17) that

$$k = ||X_{00}^{00}(\tau_k(t))||_1 \leq V_1(X_{00}^{00}(\tau_k(t))) \tag{5.3.22}$$

Taking the limit on (5.3.22) as $k \to \infty$ leads to a contradiction because the left-hand-side of the inequality (5.3.22) is infinite, and the right-hand-side is finite. Hence $\tau_e = \tau_\infty$ a.s. In the following, we show that $\tau_e = \tau_\infty = \infty$ a.s.

We let $w \in \{\tau_e < \infty\}$. Applying some algebraic manipulations and simplifications to (5.3.21), we have the following

$$
\begin{aligned}
& I_{\{\tau_e<\infty\}} V_1(X_{00}^{00}(\tau)) \\
= \;& I_{\{\tau_e<\infty\}} V_1(X_{00}^{00}(t_0)) + I_{\{\tau_e<\infty\}} \sum_{r=1}^{M} \sum_{i=1}^{n_r} \frac{B_i^r}{\delta_i^r}(e^{\delta_i^r \tau} - 1) \\
& + I_{\{\tau_e<\infty\}} \sum_{r=1}^{M} \sum_{i=1}^{n_r} \sum_{q=1}^{M} \sum_{l=1}^{n_q} \left[\rho_l^q \int_{-T_i^r}^{t_0} I_{il}^{rq}(s) e^{\delta_l^q s} ds - \rho_l^q \int_{\tau - T_i^r}^{\tau} I_{il}^{rq}(s) e^{\delta_l^q s} ds \right] \\
& - I_{\{\tau_e<\infty\}} \sum_{r=1}^{M} \sum_{i=1}^{n_r} \int_{t_0}^{\tau} \left[\sigma_i^r e^{\delta_i^r s} - \sum_{j \neq i}^{n_r} \sigma_{ij}^{rr} e^{\delta_j^r s} \right] (S_{ii}^{rr} + I_{ii}^{rr}) ds \\
& - I_{\{\tau_e<\infty\}} \sum_{r=1}^{M} \sum_{i=1}^{n_r} \int_{t_0}^{\tau} \left[\gamma_i^r e^{\delta_i^r s} - \sum_{q=1}^{M} \sum_{l=1}^{n_q} \gamma_{il}^{rq} e^{\delta_l^q s} \right] (S_{ii}^{rr} + I_{ii}^{rr}) ds \\
& - I_{\{\tau_e<\infty\}} \sum_{r=1}^{M} \sum_{i=1}^{n_r} d_i^r \int_{t_0}^{\tau} I_{ii}^{rr} e^{\delta_i^r s} ds \\
& - I_{\{\tau_e<\infty\}} \sum_{r=1}^{M} \sum_{i=1}^{n_r} \sum_{j \neq i}^{n_r} d_j^r \int_{t_0}^{\tau} I_{ij}^{rr} e^{\delta_j^r s} ds \\
& - I_{\{\tau_e<\infty\}} \sum_{r=1}^{M} \sum_{i=1}^{n_r} \sum_{q \neq r}^{M} \sum_{l=1}^{n_q} d_l^q \int_{t_0}^{\tau} I_{il}^{rq} e^{\delta_l^q s} ds,
\end{aligned}
\tag{5.3.23}
$$

where I_A is the indicator function of the set A.

We recall [30], $\sigma_i^r = \sum_{j \neq i}^{n_r} \sigma_{ij}^{rr}$ and $\gamma_i^r = \sum_{q \neq r}^{M} \sum_{l=1}^{n_q} \gamma_{il}^{rq}$. Hence the fourth and fifth terms on the right-hand-side of (5.3.23) are such that $\left[\sigma_i^r e^{\delta_i^r s} - \sum_{j \neq i}^{n_r} \sigma_{ij}^{rr} e^{\delta_j^r s}\right] \geq 0, \forall \delta_i^r \geq \delta_j^r, j \neq i$ and $\left[\gamma_i^r e^{\delta_i^r s} - \sum_{q=1}^{M} \sum_{l=1}^{n_q} \gamma_{il}^{rq} e^{\delta_l^q s}\right] \geq 0, \forall \delta_i^r \geq \delta_l^q, q \neq r, l \in I(1, n_q)$. We now let $\tau = \tau_k(t) \wedge T$ in (5.3.23), $\exists T > 0$, where $\tau_k(t)$ is defined in (5.3.15). The expected value of (5.3.23) is estimated as follows

$$
\begin{aligned}
E\left[I_{\{\tau_e<\infty\}} V_1(X_{00}^{00}(\tau_k(t) \wedge T))\right] \;\leq\; & V_1(X_{00}^{00}(t_0)) + \sum_{i=1}^{n_r} \frac{B_i^r}{\delta_i^r} e^{\delta_i^r \tau_k(t) \wedge T} \\
& + \sum_{r=1}^{M} \sum_{i=1}^{n_r} \sum_{q=1}^{M} \sum_{l=1}^{n_q} \left[\rho_l^q \int_{-T_i^r}^{t_0} \varphi_{il2}^{rq}(s) e^{\delta_l^q s} ds \right]
\end{aligned}
\tag{5.3.24}
$$

Furthermore, from (5.3.16), (5.3.17) and the definition of the indicator function I_A it follows that

$$I_{\{\tau_e<\infty,\tau_k(t)\leq T\}}||X_{00}^{00}(\tau_k(t))||_1 \leq I_{\{\tau_e<\infty\}}V_1(X_{00}^{00}(\tau_k(t)\wedge T)) \tag{5.3.25}$$

It follows from (5.3.24), (5.3.25) and (5.3.15) that

$$\begin{aligned} P(\{\tau_e<\infty,\tau_k(t)\leq T\})k &= E\left[I_{\{\tau_e<\infty,\tau_k(t)\leq T\}}||X_{00}^{00}(\tau_k(t))||_1\right] \\ &\leq E\left[I_{\{\tau_e<\infty\}}V(X_{00}^{00}(\tau_k(t)\wedge T))\right] \\ &\leq V_1(X_{00}^{00}(t_0))+\sum_{i=1}^{n_r}\frac{B_i^r}{\delta_i^r}e^{\delta_i^r T} \\ &\quad +\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{q=1}^{M}\sum_{l=1}^{n_q}\left[\rho_l^q\int_{-T_i^r}^{t_0}\varphi_{il2}^{rq}(s)e^{\delta_l^q s}ds\right] \end{aligned} \tag{5.3.26}$$

It follows immediately from (5.3.26) that $P(\{\tau_e<\infty,\tau_\infty\leq T\})\to 0$ as $k\to\infty$. Furthermore, since $T<\infty$ is arbitrary, we conclude that $P(\{\tau_e<\infty,\tau_\infty<\infty\})=0$.

Finally, by the total probability principle,

$$\begin{aligned} P(\{\tau_e<\infty\}) &= P(\{\tau_e<\infty,\tau_\infty=\infty\})+P(\{\tau_e<\infty,\tau_\infty<\infty\}) \\ &\leq P(\{\tau_e\neq\tau_\infty\})+P(\{\tau_e<\infty,\tau_\infty<\infty\}) \\ &= 0. \end{aligned} \tag{5.3.27}$$

Thus from (5.3.27), $\tau_e=\tau_\infty=\infty$ a.s. as was required to show.

Remark 5.3.1 *For any $r\in I(1,M)$ and $i\in I(1,n_r)$, Theorem 5.3.1 signifies that the number of residents of site s_i^r of all categories present at home site s_i^r, or visiting intra and inter-regional sites s_j^r and s_l^q respectively, are nonnegative. This implies that the total population of residents of site s_i^r present at home and also visiting sites in regions in their intra and inter-regional accessible domains[66], given by the sum $N_{i0}^{rr}(t)=\sum_{u=1}^{M}\sum_{a=1}^{n_u}y_{ia}^{ru}$, is nonnegative. Moreover, the total effective population[66], defined by $eff(N_{i0}^{rr})(t)=\sum_{u=1}^{M}\sum_{a=1}^{n_u}y_{ai}^{ur}$, at any site s_i^r in region C_r is also nonnegative at all time $t\geq t_0$.*

The following result defines an upper bound for the solution process of the system (5.2.1)-(5.2.3). We use of Theorem 5.3.1 to establish this result.

Theorem 5.3.2 *Suppose the hypotheses of Theorem 5.3.1 is satisfied. Let* $\mu = \min_{1\leq u\leq M, 1\leq a\leq n_u}(\delta_a^u)$. *If*

$$\sum_{r=1}^{M}\sum_{u=1}^{M}\sum_{i=1}^{n_r}\sum_{a=1}^{n_u} y_{ia}^{ru}(t_0) \leq \frac{1}{\mu}\sum_{r=1}^{M}\sum_{i=1}^{n_r} B_i^r, \tag{5.3.28}$$

then

$$\sum_{r=1}^{M}\sum_{u=1}^{M}\sum_{i=1}^{n_r}\sum_{a=1}^{n_u} y_{ia}^{ru}(t) \leq \frac{1}{\mu}\sum_{r=1}^{M}\sum_{i=1}^{n_r} B_i^r, \quad for \quad t \geq t_0 a.s. \tag{5.3.29}$$

Proof:

See Lemma 3.3.2.

Remark 5.3.2 *From Theorem 5.3.1 and Theorem 5.3.2, we conclude that a closed ball* $\bar{\mathfrak{B}}_{R^{3n^2}}(\vec{0}; r)$ *in* R^{3n^2} *under the sum norm* $||.||_1$ *centered at the origin* $\vec{0} \in R^{3n^2}$, *with radius* $r = \frac{1}{\mu}\sum_{r=1}^{M}\sum_{i=1}^{n_r} B_i^r$ *is self-invariant with regard to a two-scale network dynamics of human epidemic process (5.2.1)-(5.2.3) that is under the influence of human mobility process[30]. That is,*

$$\bar{\mathfrak{B}}_{R^{3n^2}}(\vec{0}; r) = \left\{ (S_{ia}^{ru}, I_{ia}^{ru}, R_{ia}^{ru}) : y_{ia}^{ru}(t) \geq 0 \quad and \quad ||x_{00}^{00}||_1 = \sum_{r=1}^{M}\sum_{u=1}^{M}\sum_{i=1}^{n_r}\sum_{a=1}^{n_u} y_{ia}^{ru}(t) \leq \frac{1}{\mu}\sum_{r=1}^{M}\sum_{i=1}^{n_r} B_i^r \right\} \tag{5.3.30}$$

is a positive self-invariant set for system (5.2.1)-(5.2.3). We shall denote

$$\bar{B} \equiv \frac{1}{\mu}\sum_{r=1}^{M}\sum_{i=1}^{n_r} B_i^r \tag{5.3.31}$$

5.4 Existence and Asymptotic Behavior of Disease Free Equilibrium

In this section, we study the existence and the asymptotic behavior of the disease free equilibrium state of the system (5.2.1)-(5.2.3). The disease free equilibrium is obtained by solving the system of algebraic equations obtained by setting the drift and the diffusion parts of the system of stochastic differential equations to zero. In addition, we utilize the conditions that $I = R = 0$ in the event when there is no disease in the population. We summarize the results in the following. For any $r, u \in I(1, M)$, $i \in I(1, n_r)$ and $a \in I(1, n_u)$, let

$$D_i^r = \gamma_i^r + \sigma_i^r + \delta_i^r - \sum_{a=1}^{n_r} \frac{\rho_{ia}^{rr}\sigma_{ia}^{rr}}{\rho_{ia}^{rr} + \delta_a^r} - \sum_{u\neq r}^{M}\sum_{a=1}^{n_u} \frac{\rho_{ia}^{rr}\gamma_{ia}^{ru}}{\rho_{ia}^{ru} + \delta_a^u} > 0. \tag{5.4.32}$$

Furthermore, let $(S_{ia}^{ru*}, I_{ia}^{ru*}, R_{ia}^{ru*})$, be the equilibrium state of the delayed system (5.2.1)-(5.2.3). One can see that the disease free equilibrium state is given by $E_{ia}^{ru} = (S_{ia}^{ru*}, 0, 0)$, where

$$S_{ia}^{ru*} = \begin{cases} \frac{B_i^r}{D_i^r}, \quad for \quad u = r, a = i, \\ \frac{B_i^r}{D_i^r}\frac{\sigma_{ij}^{rr}}{\rho_{ij}^{rr}+\delta_j^r}, \quad for \quad u = r, a \neq i, \\ \frac{B_i^r}{D_i^r}\frac{\gamma_{ia}^{ru}}{\rho_{ia}^{ru}+\delta_a^u}, \quad for \quad u \neq r. \end{cases} \tag{5.4.33}$$

The asymptotic stability property of E_{ia}^{ru} will be established by verifying the conditions of the stochastic version of the Lyapunov second method given in [[34],Theorem 2.4], and [[34],Theorem 4.4],[59] respectively. In order to study the qualitative properties of (5.2.1)-(5.2.3) with respect to the equilibrium state $(S_{ia}^{ru*}, 0, 0)$, first, we use the change of variable that shifts the equilibrium to the origin. For this purpose, we use the following transformation:

$$\begin{cases} U_{ia}^{ru} = S_{ia}^{ru} - S_{ia}^{ru*} \\ V_{ia}^{ru} = I_{ia}^{ru} \\ W_{ia}^{ru} = R_{ia}^{ru}. \end{cases} \tag{5.4.34}$$

By employing this transformation, system (5.2.1)-(5.2.3) is transformed into the following forms

$$dU_{il}^{rq} = \begin{cases} \left[\sum_{q=1}^{M}\sum_{a=1}^{n_q}\rho_{ia}^{rq}U_{ia}^{rq} + \rho_i^r V_{ii}^{rr}(t-T_i^r)e^{-\delta_i^r T_i^r} \right. \\ \left. -(\gamma_i^r + \sigma_i^r + \delta_i^r)U_{ii}^{rr} - \sum_{u=1}^{M}\sum_{a=1}^{n_u}\beta_{iia}^{rru}(S_{ii}^{rr*} + U_{ii}^{rr})V_{ai}^{ur}\right]dt \\ -\left[\sum_{u=1}^{M}\sum_{a=1}^{n_u}\nu_{iia}^{rru}(S_{ii}^{rr*} + U_{ii}^{rr})V_{ai}^{ur}dw_{iia}^{rru}(t)\right], for \quad q = r, l = i \\ \left[\sigma_{ij}^{rr}U_{ii}^{rr} + \rho_j^r V_{ij}^{rr}(t-T_i^r)e^{-\delta_j^r T_i^r} - (\rho_{ij}^{rr} + \delta_j^r)U_{ij}^{rr} - \sum_{u=1}^{M}\sum_{a=1}^{n_u}\beta_{jia}^{rru}(S_{ij}^{rr*} + U_{ij}^{rr})V_{aj}^{ur}\right]dt \\ -\left[\sum_{u=1}^{M}\sum_{a=1}^{n_u}\nu_{jia}^{rru}(S_{ij}^{rr*} + U_{ij}^{rr})V_{aj}^{ur}dw_{jia}^{rru}(t)\right], for \quad q = r, l = j, j \neq i, \\ \left[\gamma_{il}^{rq}U_{ii}^{rr} + \rho_l^q V_{il}^{rq}(t-T_i^r)e^{-\delta_l^q T_i^r} - (\rho_{il}^{rq} + \delta_l^q)U_{il}^{rq} \right. \\ \left. -\sum_{u=1}^{M}\sum_{a=1}^{n_u}\beta_{lia}^{qru}S_{il}^{rq}I_{al}^{uq}\right]dt - \left[\sum_{u=1}^{M}\sum_{a=1}^{n_u}\nu_{lia}^{qru}(S_{il}^{rq*} + U_{il}^{rq})V_{al}^{uq}dw_{lia}^{qru}(t)\right], for \quad q \neq r, \end{cases} \tag{5.4.35}$$

$$dV_{il}^{rq} = \begin{cases} \left[\sum_{q=1}^{M}\sum_{a=1}^{n_q}\rho_{ia}^{rq}V_{ia}^{rq} - (\rho_i^r+\gamma_i^r+\sigma_i^r+\delta_i^r+d_i^r)W_{ii}^{rr} \right. \\ \left. +\sum_{u=1}^{M}\sum_{a=1}^{n_u}\beta_{iia}^{rru}(S_{ii}^{rr*}+U_{ii}^{rr})V_{ai}^{ur}\right]dt \\ +\left[\sum_{u=1}^{M}\sum_{a=1}^{n_u}\nu_{iia}^{rru}(S_{ii}^{rr*}+U_{ii}^{rr})V_{ai}^{ur}dw_{iia}^{rru}(t)\right], for \quad q=r, l=i \\ \left[\sigma_{ij}^{rr}V_{ii}^{rr} - (\rho_j^r+\rho_{ij}^{rr}+\delta_j^r+d_j^r)V_{ij}^{rr} + \sum_{u=1}^{M}\sum_{a=1}^{n_u}\beta_{jia}^{rru}(S_{ij}^{rr*}+U_{ij}^{rr})V_{aj}^{ur}\right]dt \\ +\left[\sum_{u=1}^{M}\sum_{a=1}^{n_u}\nu_{jia}^{rru}(S_{ij}^{rr*}+U_{ij}^{rr})V_{aj}^{ur}dw_{jia}^{rru}(t)\right], for \quad q=r, l=j, j\neq i, \\ \left[\gamma_{il}^{rq}V_{ii}^{rr} - (\rho_l^q+\rho_{il}^{rq}+\delta_l^q+d_l^q)V_{il}^{rq}\right. \\ \left.\sum_{u=1}^{M}\sum_{a=1}^{n_u}\beta_{lia}^{qru}(S_{il}^{rq*}+U_{il}^{rq})V_{al}^{uq}\right]dt + \left[\sum_{u=1}^{M}\sum_{a=1}^{n_u}\nu_{lia}^{qru}(S_{il}^{rq*}+U_{il}^{rq})V_{al}^{uq}dw_{lia}^{qru}(t)\right], \\ for \quad q\neq r, \end{cases} \tag{5.4.36}$$

and

$$dW_{il}^{rq} = \begin{cases} \left[\sum_{q=1}^{M}\sum_{l=1}^{n_q}\rho_{il}^{rq}W_{il}^{rq} + \rho_i^rV_{ii}^{rr} - \rho_i^rV_{ii}^{rr}(t-T_i^r)e^{-\delta_i^rT_i^r} - (\gamma_i^r+\sigma_i^r+\delta_i^r)W_{ii}^{rr}\right]dt, \\ for \quad q=r, l=i \\ \left[\sigma_{ij}^{rr}W_{ii}^{rr} + \rho_j^rV_{ij}^{rr} - \rho_j^rV_{ij}^{rr}(t-T_i^r)e^{-\delta_j^rT_i^r} - (\rho_{ij}^{rr}+\delta_j^r)W_{ij}^{rr}\right]dt, for \quad q=r, l=j, j\neq i \\ \left[\gamma_{il}^{rq}W_{ii}^{rr} + \rho_l^qV_{il}^{rq} - \rho_l^qV_{il}^{rq}(t-T_i^r)e^{-\delta_l^qT_i^r} - (\rho_{il}^{rq}+\delta_l^q)W_{il}^{rq}\right]dt, for \quad q\neq r \end{cases} \tag{5.4.37}$$

We state and prove the following lemmas that would be useful in the proofs of the stability results.

Lemma 5.4.1 *Let $V_1 : \mathbb{R}^{3n^2}\times\mathbb{R}_+ \to \mathbb{R}_+$ be a function defined by*

$$\begin{cases} V_1(\tilde{x}_{00}^{00}) &= \sum_{r=1}^{M}\sum_{u=1}^{M}\sum_{i=1}^{n_r}\sum_{a=1}^{n_u} V(\tilde{x}_{ia}^{ru}), \\ V_1(\tilde{x}_{ia}^{ru}) &= (S_{ia}^{ru}-S_{ia}^{ru*}+I_{ia}^{ru})^2 + c_{ia}^{ru}(I_{ia}^{ru})^2 + (R_{ia}^{ru})^2 \\ \tilde{x}_{ia}^{ru} &= (U_{ia}^{ru}, V_{ia}^{ru}, W_{ia}^{ru})^T \quad and \quad c_{ia}^{ru}\geq 0. \end{cases} \tag{5.4.38}$$

Then $V_1 \in \mathcal{C}^{2,1}(\mathbb{R}^{3n^2}\times\mathbb{R}_+, \mathbb{R}_+)$, and it satisfies

$$b(||\tilde{x}_{00}^{00}||) \quad\leq\quad V_1(\tilde{x}_{00}^{00}(t)) \leq a(||\tilde{x}_{00}^{00}||) \tag{5.4.39}$$

where

$$
\begin{aligned}
b(||\tilde{x}_{00}^{00}||) &= \min_{1\le r,u\le M, 1\le i\le n_r, 1\le a\le n_u}\left\{\frac{c_{ia}^{ru}}{2+c_{ia}^{ru}}\right\}\sum_{r=1}^{M}\sum_{u=1}^{M}\sum_{i=1}^{n_r}\sum_{a=1}^{n_u}\left[(U_{ia}^{ru})^2+(V_{ia}^{ru})^2+(W_{ia}^{ru})^2\right]\\
a(||\tilde{x}_{00}^{00}||) &= \max_{1\le r,u\le M, 1\le i\le n_r, 1\le a\le n_u}\{c_{ia}^{ru}+2\}\sum_{r=1}^{M}\sum_{u=1}^{M}\sum_{i=1}^{n_r}\sum_{a=1}^{n_u}\left[(U_{ia}^{ru})^2+(V_{ia}^{ru})^2+(W_{ia}^{ru})^2\right].
\end{aligned}
\tag{5.4.40}
$$

Proof: See Lemma 3.4.1.

Remark 5.4.1 *Lemma 5.4.1 shows that the Lyapunov function V defined in (5.4.38) is positive definite((5.4.39)), decrescent and radially unbounded ((5.4.39)) function[34, 59].*

We now state the following lemma

Lemma 5.4.2 *Assume that the hypothesis of Lemma 5.4.1 is satisfied. Define a Lyapunov functional*

$$V = V_1 + V_2, \tag{5.4.41}$$

where V_1 is defined by (5.4.38), and

$$V_2 = 3\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u=1}^{M}\sum_{a=1}^{n_u}\left[\frac{(\rho_a^u)^2}{\mu_{ia}^{ru}}e^{-\delta_a^u T_i^r}\right]\int_{t-T_i^r}^{t}(V_{ia}^{ru}(\theta))^2 d\theta, \tag{5.4.42}$$

Suppose that

$$
\begin{cases}
(\gamma_i^r+\sigma_i^r+\delta_i^r) > \max\left(\left(\sum_{u=1}^{M}\sum_{a=1}^{n_u}\mu_{ia}^{ru}+\sum_{a\neq i}^{n_r}\frac{(\sigma_{ia}^{rr})^2}{\mu_{ii}^{rr}}+\sum_{a\neq r}^{M}\sum_{a=1}^{n_r}\frac{(\gamma_{ia}^{ru})^2}{\mu_{ii}^{rr}}+\frac{3}{2}\mu_{ii}^{rr}\right),\right.\\
\left.\left(\frac{1}{2}\sum_{u=1}^{M}\sum_{a=1}^{n_u}\mu_{ia}^{ru}+\frac{1}{2}\sum_{u\neq r}^{M}\sum_{a=1}^{n_r}\frac{(\gamma_{ia}^{ru})^2}{\mu_{ii}^{rr}}+\frac{1}{2}\sum_{a\neq i}^{n_r}\frac{(\sigma_{ia}^{rr})^2}{\mu_{ii}^{rr}}+\frac{1}{2}\mu_{ii}^{rr}\right)\right), for \quad u=r, a=i,\\
(\rho_{ia}^{rr}+\delta_a^r) > max\left(\left(\frac{(\rho_{ia}^{rr})^2}{\mu_{ia}^{rr}}+\mu_{ii}^{rr}+\mu_{ia}^{rr}\right),\left(\frac{1}{2}\frac{(\rho_{ia}^{rr})^2}{\mu_{ia}^{rr}}+\frac{1}{2}\mu_{ii}^{rr}+\frac{1}{2}\mu_{ia}^{rr}\right)\right), for \quad u=r, a\neq i,\\
(\rho_{ia}^{ru}+\delta_a^u) > \max\left(\left(\frac{(\rho_{ia}^{ru})^2}{\mu_{ia}^{ru}}+\mu_{ii}^{rr}+\mu_{ia}^{ru}\right),\left(\frac{1}{2}\frac{(\rho_{ia}^{ru})^2}{\mu_{ia}^{ru}}+\frac{1}{2}\mu_{ii}^{rr}+\frac{1}{2}\mu_{ia}^{ru}\right)\right), for \quad u\neq r.
\end{cases}
$$

Furthermore, let

$$\mathfrak{M}_{ia}^{ru} = \begin{cases} \max\left(\frac{1}{\delta_i^r}\log\left(\frac{\mu_{ii}^{rr}}{2\left((\gamma_i^r+\sigma_i^r+\delta_i^r)-\left[\sum_{u=1}^{M}\sum_{a=1}^{n_u}\mu_{ia}^{ru}+\sum_{a\neq i}^{n_r}\frac{(\sigma_{ia}^{rr})^2}{\mu_{ii}^{rr}}+\sum_{a\neq r}^{M}\sum_{a=1}^{n_r}\frac{(\gamma_{ia}^{ru})^2}{\mu_{ii}^{rr}}+\frac{3}{2}\mu_{ii}^{rr}\right]\right)}\right),\right. \\ \left.\frac{1}{\delta_i^r}\log\left(\frac{\mu_{ii}^{rr}}{2\left((\gamma_i^r+\sigma_i^r+\delta_i^r)-\left[\frac{1}{2}\sum_{u=1}^{M}\sum_{a=1}^{n_u}\mu_{ia}^{ru}+\frac{1}{2}\sum_{u\neq r}^{M}\sum_{a=1}^{n_r}\frac{(\gamma_{ia}^{ru})^2}{\mu_{ii}^{rr}}+\frac{1}{2}\sum_{a\neq i}^{n_r}\frac{(\sigma_{ia}^{rr})^2}{\mu_{ii}^{rr}}+\frac{1}{2}\mu_{ii}^{rr}\right]\right)}\right)\right), \\ for \quad u=r, i=a \\ \max\left(\frac{1}{\delta_a^r}\log\left(\frac{\mu_{ia}^{rr}}{2\left((\rho_{ia}^{rr}+\delta_a^r)-\left[\frac{(\rho_{ia}^{rr})^2}{\mu_{ia}^{rr}}+\mu_{ii}^{rr}+\mu_{ia}^{rr}\right]\right)}\right),\right. \\ \left.\frac{1}{\delta_a^r}\log\left(\frac{\mu_{ia}^{rr}}{2\left((\rho_{ia}^{rr}+\delta_a^r)-\left[\frac{1}{2}\frac{(\rho_{ia}^{rr})^2}{\mu_{ia}^{rr}}+\frac{1}{2}\mu_{ii}^{rr}+\frac{1}{2}\mu_{ia}^{rr}\right]\right)}\right)\right), \\ for \quad u=r, a\neq i \\ \max\left(\frac{1}{\delta_a^u}\log\left(\frac{\mu_{ia}^{ru}}{2\left((\rho_{ia}^{ru}+\delta_a^u)-\left[\frac{(\rho_{ia}^{ru})^2}{\mu_{ia}^{ru}}+\mu_{ii}^{rr}+\mu_{ia}^{ru}\right]\right)}\right),\right. \\ \left.\frac{1}{\delta_a^u}\log\left(\frac{\mu_{ia}^{ru}}{2\left((\rho_{ia}^{ru}+\delta_a^u)-\left[\frac{1}{2}\frac{(\rho_{ia}^{ru})^2}{\mu_{ia}^{ru}}+\frac{1}{2}\mu_{ii}^{rr}+\frac{1}{2}\mu_{ia}^{ru}\right]\right)}\right)\right), \\ for \quad u\neq r, \end{cases}$$

and

$$\mathfrak{V}_{ia}^{ru} = \begin{cases} \frac{\frac{1}{2}\sum_{u=1}^{M}\sum_{a=1}^{n_u}\mu_{ia}^{ru}+\frac{1}{2}\sum_{v=1}^{M}\sum_{b=1}^{n_v}\beta_{iib}^{rrv}(S_{ii}^{rr*}\mu_{ii}^{rr}+\mu_{ii}^{rr})+\frac{1}{2}d_{ii}^{rr}}{\rho_i^r+\gamma_i^r+\sigma_i^r+\delta_i^r+d_i^r}, for \quad a=i, u=r \\ \frac{\frac{1}{2}\mu_{ii}^{rr}+\frac{1}{2}\sum_{v=1}^{M}\sum_{b=1}^{n_v}\beta_{aib}^{rrv}(S_{ia}^{rr*}\mu_{ia}^{rr}+\mu_{ia}^{rr})+\frac{1}{2}d_{ai}^{rr}}{\rho_a^r+\rho_{ia}^{rr}+\delta_a^r+d_a^r}, for \quad a\neq i, u=r \\ \frac{\frac{1}{2}\mu_{ii}^{rr}+\frac{1}{2}\sum_{v=1}^{M}\sum_{b=1}^{n_v}\beta_{aib}^{urv}(S_{ii}^{ru*}\mu_{ia}^{ru}+\mu_{ia}^{ru})+\frac{1}{2}d_{ai}^{ur}}{\rho_a^u+\rho_{ia}^{ru}+\delta_a^u+d_a^u}, for \quad u\neq r. \end{cases} \tag{5.4.43}$$

for some suitably defined positive number μ_{ia}^{ru}, depending on δ_a^u, for all $r,u \in I^r(1,M)$, $i \in I(1,n)$ and $a \in I_i^r(1,n_r)$. Assume that $\mathfrak{V}_{ia}^{ru} < 1$ and $T_i^r \geq \max_{1\leq r,u\leq M, 1\leq i\leq n_r, 1\leq a\leq n_u}(\mathfrak{M}_{ia}^{ru})$. There exist positive numbers ϕ_{ia}^{ru}, ψ_{ia}^{ru} and φ_{ia}^{ru} such that the differential operator LV associated with Ito-Doob type stochastic system (5.2.1)-(5.2.3)satisfies the following inequality

$$LV(\tilde{x}_{00}^{00}) \leq \sum_{r=1}^{M}\sum_{i=1}^{n_r}\Big[-[\phi_{ii}^{rr}(U_{ii}^{rr})^2+\psi_{ii}^{rr}(V_{ii}^{rr})^2+\varphi_{ii}^{rr}(W_{ii}^{rr})^2]$$
$$-\sum_{a\neq i}^{n_r}[\phi_{ia}^{rr}(U_{ia}^{rr})^2+\psi_{ia}^{rr}(V_{ia}^{rr})^2+\varphi_{ia}^{rr}(W_{ia}^{rr})^2]$$
$$-\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}[\phi_{ia}^{ru}(U_{ia}^{rr})^2+\psi_{ia}^{ru}(V_{ia}^{ru})^2+\varphi_{ia}^{ru}(W_{ia}^{ru})^2]\Big]. \tag{5.4.44}$$

Moreover,

$$LV(\tilde{x}_{00}^{00}) \leq -cV_1(\tilde{x}_{00}^{00}) \tag{5.4.45}$$

where a positive constant c is defined by

$$c=\frac{\min_{1\leq r,u\leq M,1\leq i\leq n_r,1\leq a\leq n_u}\{\phi_{ia}^{ru},\psi_{ia}^{ru},\varphi_{ia}^{ru}\}}{\max_{1\leq r,u\leq M,1\leq i\leq n_r,1\leq a\leq n_u}\{C_{ia}^{ru}+2\}} \tag{5.4.46}$$

Proof:

The computation of differential operator[34, 59] applied to the Lyapunov function V_1 in (5.4.38) with respect to the large-scale system of Ito-Doob type stochastic differential equation (5.2.1)-(5.2.3) is as follows:

$$LV_1(\tilde{x}_{00}^{00})=\sum_{r=1}^{M}\sum_{i=1}^{n_r}\left[LV_1(\tilde{x}_{ii}^{rr})+\sum_{j\neq i}^{n_r}LV_1(\tilde{x}_{ij}^{rr})+\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}LV_1(\tilde{x}_{ia}^{ru})\right], \tag{5.4.47}$$

where,

$$LV_1(\tilde{x}_{ii}^{rr}) = 2\sum_{u=1}^{M}\sum_{a=1}^{n_u}[(1+C_{ii}^{rr})\rho_{ia}^{ru}V_{ia}^{ru}V_{ii}^{rr}+\rho_{ia}^{ru}U_{ia}^{ru}U_{ii}^{rr}+\rho_{ia}^{ru}V_{ia}^{ru}U_{ii}^{rr}+\rho_{ia}^{ru}U_{ia}^{ru}V_{ii}^{rr}$$
$$+\rho_{ia}^{ru}W_{ia}^{ru}W_{ii}^{rr}]+2\rho_i^rV_{ii}^{rr}(t-T_i^r)U_{ii}^{rr}e^{-\delta_i^rT_i^r}+2\rho_i^rV_{ii}^{rr}(t-T_i^r)V_{ii}^{rr}e^{-\delta_i^rT_i^r}$$
$$+2\rho_i^rV_{ii}^{rr}(t-T_i^r)W_{ii}^{rr}e^{-\delta_i^rT_i^r}+2\rho_i^rV_{ii}^{rr}W_{ii}^{rr}$$
$$-2[(\rho_i^r+d_i^r)+2(\gamma_i^r+\sigma_i^r+\delta_i^r)]V_{ii}^{rr}U_{ii}^{rr}-2(\gamma_i^r+\sigma_i^r+\delta_i^r)(U_{ii}^{rr})^2$$
$$-2[(c_{ii}^{rr}+1)\rho_i^r+2(c_{ii}^{rr}+1)(\gamma_i^r+\sigma_i^r+\delta_i^r+d_i^r)](V_{ii}^{rr})^2-2(\gamma_i^r+\sigma_i^r+\alpha_i^r+\delta_i^r)(W_{ii}^{rr})^2$$
$$+2c_{ii}^{rr}\sum_{u=1}^{M}\sum_{a=1}^{n_u}\beta_{iia}^{rru}(S_{ii}^{rr*}+U_{ii}^{rr})V_{ai}^{ur}V_{ii}^{rr}+c_{ii}^{rr}\sum_{u=1}^{M}\sum_{a=1}^{n_u}(v_{iia}^{rru})^2(S_{ii}^{rr*}+U_{ii}^{rr})^2(V_{ai}^{ur})^2,$$
$$for \quad u=r,a=i \tag{5.4.48}$$

$$
\begin{aligned}
\sum_{a\neq i}^{n_r} LV_1(\tilde{x}_{ia}^{rr}) &= \sum_{a\neq r}^{n_r}\{2(1+c_{ia}^{rr})\sigma_{ia}^{rr}V_{ia}^{rr}V_{ii}^{rr}+2\sigma_{ia}^{rr}U_{ia}^{rr}U_{ii}^{rr}+2\sigma_{ia}^{rr}V_{ia}^{rr}U_{ii}^{rr}+2\sigma_{ia}^{rr}U_{ia}^{rr}V_{ii}^{rr}+2\sigma_{ia}^{rr}W_{ia}^{rr}W_{ii}^{rr}\\
&+2\rho_a^r V_{ia}^{rr}(t-T_i^r)U_{ia}^{rr}e^{-\delta_a^r T_i^r}+2\rho_a^r V_{ia}^{rr}(t-T_i^r)V_{ia}^{rr}e^{-\delta_a^r T_i^r}\\
&-2\rho_a^r V_{ia}^{rr}(t-T_i^r)W_{ia}^{rr}e^{-\delta_a^r T_i^r}-2[(c_{ia}^{rr}+1)\rho_a^r+2(c_{ia}^{rr}+1)(\rho_{ia}^{rr}+\delta_a^r)](V_{ia}^{rr})^2\\
&-2(\rho_{ia}^{rr}+\delta_a^r)(U_{ia}^{rr})^2-2(\rho_{ia}^{rr}+\delta_a^r)(W_{ia}^{rr})^2+2\rho_a^r V_{ia}^{rr}W_{ia}^{rr}\\
&-2[(\rho_a^r+d_a^r)+2(\rho_{ia}^{rr}+\delta_a^r)]V_{ia}^{rr}U_{ia}^{rr}\}+2\sum_{a\neq i}^{n_r}c_{ia}^{rr}\sum_{v=1}^{M}\sum_{b=1}^{n_v}\beta_{aib}^{rrv}(S_{ia}^{rr*}+U_{ia}^{rr})V_{ba}^{vr}V_{ia}^{rr}\\
&+\sum_{a\neq i}^{n_r}c_{ia}^{rr}\sum_{v=1}^{M}\sum_{b=1}^{n_v}(v_{aib}^{rrv})^2(S_{ia}^{rr*}+U_{ia}^{rr})^2(V_{ba}^{vr})^2\ ,for\quad u=r,\quad a\neq i \qquad (5.4.49)
\end{aligned}
$$

$$
\begin{aligned}
\sum_{u\neq r}^{M}\sum_{a=1}^{n_r} LV_1(\tilde{x}_{ia}^{ru}) &= \sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\{2(1+c_{ia}^{ru})\gamma_{ia}^{ru}V_{ia}^{ru}V_{ii}^{rr}+2\gamma_{ia}^{ru}U_{ia}^{ru}U_{ii}^{rr}+2\gamma_{ia}^{ru}V_{ia}^{ru}U_{ii}^{rr}+2\gamma_{ia}^{ru}U_{ia}^{ru}V_{ii}^{rr}\\
&+2\gamma_{ia}^{ru}W_{ia}^{ru}W_{ii}^{rr}+2\rho_a^u V_{ia}^{ru}(t-T_i^r)U_{ia}^{ru}e^{-\delta_a^u T_i^r}+2\rho_a^u V_{ia}^{ru}(t-T_i^r)V_{ia}^{ru}e^{-\delta_a^u T_i^r}\\
&-2\rho_a^u V_{ia}^{ru}(t-T_i^r)W_{ia}^{ru}e^{-\delta_a^u T_i^r}-2[(c_{ia}^{ru}+1)\rho_a^u+2(c_{ia}^{ru}+1)(\rho_{ia}^{ru}+\delta_a^u+d_a^u)](V_{ia}^{ru})^2\\
&-2(\rho_{ia}^{ru}+\delta_a^u)(U_{ia}^{ru})^2-2(\rho_{ia}^{ru}+\alpha_a^u+\delta_a^u)(W_{ia}^{rr})^2+2\rho_a^u V_{ia}^{ru}W_{ia}^{ru}\\
&-2[(\rho_a^u+d_a^u)+2(\rho_{ia}^{ru}+\delta_a^u)]V_{ia}^{ru}U_{ia}^{ru}\}+2\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}c_{ia}^{ru}\sum_{v=1}^{M}\sum_{b=1}^{n_v}\beta_{aib}^{urv}(S_{ia}^{ru*}+U_{ia}^{ru})V_{ba}^{vu}V_{ia}^{ru}\\
&+\sum_{u\neq r}^{M}\sum_{a=1}^{n_r}c_{ia}^{ru}\sum_{v=1}^{M}\sum_{b=1}^{n_v}(v_{aib}^{urv})^2(S_{ia}^{ru*}+U_{ia}^{ru})^2(V_{ba}^{vu})^2, for\quad u\neq r \qquad (5.4.50)
\end{aligned}
$$

By using (5.3.31) and the algebraic inequality

$$
2ab \leq \frac{a^2}{g(c)}+b^2 g(c) \qquad (5.4.51)
$$

where $a,b,c \in \mathbb{R}$, and the function g is such that $g(c)>0$. The sixth term in (5.4.48)-(5.4.50) is estimated as follows:

$$
\begin{aligned}
2\sum_{v=1}^{M}\sum_{b=1}^{n_v}c_{ii}^{rr}\beta_{iib}^{rrv}(S_{ii}^{rr*}+U_{ii}^{rr})V_{bi}^{vr}V_{ii}^{rr} &\leq \sum_{v=1}^{M}\sum_{b=1}^{n_v}c_{ii}^{rr}\beta_{iib}^{rrv}(S_{ii}^{rr*}g_i^r(\delta_i^r)+g_i^r(\delta_i^r))(V_{ii}^{rr})^2\\
&+ \sum_{v=1}^{M}\sum_{b=1}^{n_v}c_{ii}^{rr}\beta_{iib}^{rrv}\left(\frac{S_{ii}^{rr*}}{g_i^r(\delta_i^r)}+\frac{\bar{B}^2}{g_i^r(\delta_i^r)}\right)(V_{bi}^{vr})^2
\end{aligned}
$$

$$2\sum_{a\neq r}^{n_r}\sum_{v=1}^{M}\sum_{b=1}^{n_v} c_{ia}^{rr}\beta_{aib}^{rrv}(S_{ia}^{rr*}+U_{ia}^{rr})V_{ba}^{vr}V_{ia}^{rr} \leq \sum_{a\neq r}^{n_r}\sum_{v=1}^{M}\sum_{b=1}^{n_v} c_{ia}^{rr}\beta_{aib}^{rrv}(S_{ia}^{rr*}g_i^r(\delta_a^r)+g_i^r(\delta_a^r))(V_{ia}^{rr})^2 + \sum_{a\neq r}^{n_r}\sum_{v=1}^{M}\sum_{b=1}^{n_v} c_{ia}^{rr}\beta_{aib}^{rrv}\left(\frac{S_{ia}^{rr*}}{g_i^r(\delta_a^r)}+\frac{\bar{B}^2}{g_i^r(\delta_a^r)}\right)(V_{bi}^{vr})^2$$

and

$$2\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\sum_{v=1}^{M}\sum_{b=1}^{n_v} c_{ia}^{ru}\beta_{aib}^{urv}(S_{ia}^{ru*}+U_{ia}^{ru})V_{ba}^{vu}V_{ia}^{ru} \leq \sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\sum_{v=1}^{M}\sum_{b=1}^{n_v} c_{ia}^{ru}\beta_{aib}^{urv}(S_{ia}^{ru*}g_i^r(\delta_a^u)+g_i^r(\delta_a^u))(V_{ia}^{ru})^2 + \sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\sum_{v=1}^{M}\sum_{b=1}^{n_v} c_{ia}^{ru}\beta_{aib}^{urv}\left(\frac{S_{ia}^{ru*}}{g_i^r(\delta_a^u)}+\frac{\bar{B}^2}{g_i^r(\delta_a^u)}\right)(V_{ba}^{vu})^2 \quad (5.4.52)$$

From (5.4.48-(5.4.52), (5.4.47) and repeated usage of (5.3.31)and inequality (5.4.51) coupled with some algebraic manipulations and simplifications, we have the following inequality

$$\begin{aligned}
LV_1(\tilde{x}_{00}^{00}) \leq & \sum_{r=1}^{M}\sum_{i=1}^{n_r}\Bigg\{\Bigg[2\sum_{u=1}^{M}\sum_{a=1}^{n_u}\mu_{ia}^{ru}+2\sum_{a\neq i}^{n_r}\frac{(\sigma_{ia}^{rr})^2}{\mu_{ii}^{rr}}+2\sum_{u\neq r}^{M}\sum_{a=1}^{n_r}\frac{(\gamma_{ia}^{ru})^2}{\mu_{ii}^{rr}}+3\mu_{ii}^{rr}+\mu_{ii}^{rr}e^{-\delta_i^r T_i^r} \\
& -2(\gamma_i^r+\sigma_i^r+\delta_i^r)\Bigg](U_{ii}^{rr})^2 \\
& +\Bigg[(2+c_{ii}^{rr})\sum_{u=1}^{M}\sum_{a=1}^{n_u}\mu_{ia}^{ru}+\sum_{a\neq r}^{n_r}(2+c_{ia}^{rr})\frac{(\sigma_{ia}^{rr})^2}{\mu_{ii}^{rr}}+\sum_{u=1}^{M}\sum_{a=1}^{n_u}(2+c_{ia}^{ru})\frac{(\gamma_{ia}^{ru})^2}{\mu_{ii}^{rr}}+\mu_{ii}^{rr}e^{-\delta_i^r T_i^r} \\
& +\frac{(\rho_i^r+d_i^r)^2}{\mu_{ii}^{rr}}+4\frac{(\gamma_i^r+\sigma_i^r+\delta_i^r)^2}{\mu_{ii}^{rr}}+\frac{(\rho_i^r)^2}{\mu_{ii}^{rr}}+c_{ii}^{rr}\sum_{v=1}^{M}\sum_{b=1}^{n_r}\beta_{iib}^{rrv}(S_{ii}^{rr*}\mu_{ii}^{rr}+\mu_{ii}^{rr}) \\
& -2(c_{ii}^{rr}+1)(\rho_i^r+\gamma_i^r+\sigma_i^r+\delta_i^r+d_i^r)\Bigg](V_{ii}^{rr})^2 \\
& +\Bigg[\sum_{u=1}^{M}\sum_{a=1}^{n_u}\mu_{ia}^{ru}+\sum_{a\neq i}^{n_r}\frac{(\sigma_{ia}^{rr})^2}{\mu_{ii}^{rr}}+\sum_{u\neq r}^{M}\sum_{a=1i}^{n_r}\frac{(\gamma_{ia}^{ru})^2}{\mu_{ii}^{rr}}+\mu_{ii}^{rr}e^{-\delta_i^r T_i^r}+\mu_{ii}^{rr} \\
& -2(\gamma_i^r+\sigma_i^r+\delta_i^r)\Bigg](W_{ii}^{rr})^2\Bigg\}
\end{aligned}$$

$$\begin{aligned}
& +\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{a\neq i}^{n_r}\Bigg\{\Bigg[2\frac{(\rho_{ia}^{rr})^2}{\mu_{ia}^{rr}}+2\mu_{ii}^{rr}+2\mu_{ia}^{rr}+\mu_{ia}^{rr}e^{-\delta_a^r T_i^r}-2(\rho_{ia}^{rr}+\delta_a^r)\Bigg](U_{ia}^{rr})^2 \\
& +\Bigg[(2+c_{ii}^{rr})\frac{(\rho_{ia}^{rr})^2}{\mu_{ia}^{rr}}+(2+c_{ia}^{rr})\mu_{ii}^{rr}+\frac{(\rho_a^r+d_a^r)^2}{\mu_{ia}^{rr}}+4\frac{(\rho_{ia}^{rr}+\delta_a^r)^2}{\mu_{ia}^{rr}} \\
& +c_{ia}^{rr}\sum_{v=1}^{M}\sum_{b=1}^{n_r}\beta_{aib}^{rrv}(S_{ia}^{rr*}\mu_{ia}^{rr}+\mu_{ia}^{rr})-2(c_{ia}^{rr}+1)(\rho_a^r+\rho_{ia}^{rr}+\delta_a^r+d_a^r)\Bigg](V_{ia}^{rr})^2 \\
& +\Bigg[\frac{(\rho_{ia}^{rr})^2}{\mu_{ia}^{rr}}+\mu_{ii}^{rr}+\mu_{ia}^{rr}+\mu_{ia}^{rr}e^{-\delta_a^r T_i^r}-2(\rho_{ia}^{rr}+\delta_a^r)\Bigg](W_{ia}^{rr})^2\Bigg\}
\end{aligned}$$

$$+\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\Bigg\{\left[2\frac{(\rho_{ia}^{ru})^2}{\mu_{ia}^{rr}}+2\mu_{ii}^{rr}+2\mu_{ia}^{ru}+\mu_{ia}^{ru}e^{-\delta_a^u T_i^r}-2(\rho_{ia}^{ru}+\delta_a^u)\right](U_{ia}^{ru})^2$$

$$+\Bigg[(2+c_{ii}^{rr})\frac{(\rho_{ia}^{ru})^2}{\mu_{ia}^{ru}}+(2+c_{ia}^{ru})\mu_{ii}^{rr}+\frac{(\rho_a^u+d_a^u)^2}{\mu_{ia}^{ru}}+4\frac{(\rho_{ia}^{ru}+\delta_a^u)^2}{\mu_{ia}^{ru}}$$

$$+c_{ia}^{ru}\sum_{v=1}^{M}\sum_{b=1}^{n_r}\beta_{aib}^{urv}(S_{ia}^{ru*}\mu_{ia}^{ru}+\mu_{ia}^{ru})-2(c_{ia}^{ru}+1)(\rho_a^u+\rho_{ia}^{ru}+\delta_a^u+d_a^u)\Bigg](V_{ia}^{ru})^2$$

$$+\left[\frac{(\rho_{ia}^{ru})^2}{\mu_{ia}^{ru}}+\mu_{ii}^{rr}+\mu_{ia}^{ru}+\mu_{ia}^{ru}e^{-\delta_a^u T_i^r}-2(\rho_{ia}^{ru}+\delta_a^u)\right](W_{ia}^{ru})^2\Bigg\}$$

$$+3\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u=1}^{M}\sum_{a=1}^{n_u}\left[\frac{(\rho_a^u)^2}{\mu_{ia}^{ru}}e^{-\delta_a^u T_i^r}\right](V_{ia}^{ru}(t-T_i^r))^2$$

$$+\sum_{r=1}^{M}\sum_{i=1}^{n_r}c_{ii}^{rr}\sum_{v=1}^{M}\sum_{b=1}^{n_r}\left[\beta_{iib}^{rrv}(\frac{S_{ii}^{rr*}}{\mu_{ii}^{rr}}+\frac{\bar{B}^2}{\mu_{ii}^{rr}})+(v_{iib}^{rrv})^2(S_{ii}^{rr*}+\bar{B})^2\right](V_{bi}^{vr})^2$$

$$+\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{a\neq i}^{n_r}c_{ia}^{rr}\left[\sum_{v=1}^{M}\sum_{b=1}^{n_r}\beta_{aib}^{rrv}(\frac{S_{ia}^{rr*}}{\mu_{ia}^{rr}}+\frac{\bar{B}^2}{\mu_{ia}^{rr}})+(v_{aib}^{rrv})^2(S_{ia}^{rr*}+\bar{B})^2\right](V_{ba}^{vr})^2$$

$$+\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u\neq r}^{M}\sum_{a=1}^{n_r}c_{ia}^{ru}\left[\sum_{v=1}^{M}\sum_{b=1}^{n_r}\beta_{aib}^{urv}(\frac{S_{ia}^{ru*}}{\mu_{ia}^{ru}}+\frac{\bar{B}^2}{\mu_{ia}^{ru}})+(v_{aib}^{urv})^2(S_{ia}^{ru*}+\bar{B})^2\right](V_{ba}^{vu})^2, \tag{5.4.53}$$

where $\mu_{ia}^{ru}=g_i^r(\delta_a^u)$, g_i^r is appropriately defined by (5.4.51). For each $r,u\in I(1,M)$, $i\in I(1,n_r)$ and $a\in I(1,n_u)$, we define the constants d_{ai}^{ur}, ϕ_{ia}^{ru}, ψ_{ia}^{ru} and φ_{ia}^{ru} as follows:

$$d_{ai}^{ur} = \sum_{v=1}^{M}\sum_{b=1}^{n_v}c_{ba}^{vu}\beta_{abi}^{uvr}\left(\frac{S_{ba}^{vu*}+\bar{B}^2}{\mu_{ba}^{vu}}\right)+\sum_{v=1}^{M}\sum_{b=1}^{n_v}c_{ba}^{vu}(v_{abi}^{uvr})^2(S_{ba}^{vu*}+\bar{B})^2 \tag{5.4.54}$$

for some positive numbers c_{ia}^{ru}, for all $r,u\in I^r(1,M)$, $i\in I(1,n)$ and $a\in I_i^r(1,n_r)$.

$$\phi_{ia}^{ru}=\begin{cases} 2(\gamma_i^r+\sigma_i^r+\delta_i^r)(1-\mathfrak{U}_{ia}^{ru}), for \quad u=r, a=i\\ 2(\rho_{ia}^{rr}+\delta_a^r)(1-\mathfrak{U}_{ia}^{ru}), for \quad u=r, a\neq i\\ 2(\rho_{ia}^{ru}+\delta_a^u)(1-\mathfrak{U}_{ia}^{ru}), for \quad u\neq r,\end{cases} \tag{5.4.55}$$

$$\psi_{ia}^{ru}=\begin{cases} 2(\rho_i^r+\gamma_i^r+\sigma_i^r+\delta_i^r+d_i^r)\left[c_{ii}^{rr}(1-\mathfrak{V}_{ii}^{rr})+(1-\frac{1}{2}\mathfrak{E}_{ii}^{rr})\right], for \quad u=r, a=i\\ 2(\rho_a^r+\rho_{ia}^{rr}+\delta_a^r+d_a^r)\left[c_{ia}^{rr}(1-\mathfrak{V}_{ia}^{rr})+(1-\frac{1}{2}\mathfrak{E}_{ia}^{rr})\right], for \quad u=r, a\neq i\\ 2(\rho_a^u+\rho_{ia}^{ru}+\delta_a^u+d_a^u)\left[c_{ia}^{ru}(1-\mathfrak{V}_{ia}^{ru})+(1-\frac{1}{2}\mathfrak{E}_{ia}^{ru})\right], for \quad u\neq r\end{cases} \tag{5.4.56}$$

and

$$\varphi_{ia}^{ru} = \begin{cases} 2(\gamma_i^r + \sigma_i^r + \delta_i^r)(1 - \mathfrak{W}_{ia}^{ru}), for \quad u = r, a = i, \\ 2(\rho_{ia}^{rr} + \delta_a^r)(1 - \mathfrak{W}_{ia}^{ru}), \quad for u = r, a \neq i, \\ 2(\rho_{ia}^{ru} + \delta_a^u)(1 - \mathfrak{W}_{ia}^{ru}), for \quad u \neq r \end{cases} \tag{5.4.57}$$

where $\mathfrak{V}_{ia}^{ru}$ is given in (5.4.43),

$$\mathfrak{U}_{ia}^{ru} = \begin{cases} \frac{\left[\sum_{u=1}^{M}\sum_{a=1}^{n_u}\mu_{ia}^{ru}+\sum_{a\neq i}^{n_r}\frac{(\sigma_{ia}^{rr})^2}{\mu_{ii}^{rr}}+\sum_{a\neq r}^{M}\sum_{a=1}^{n_r}\frac{(\gamma_{ia}^{ru})^2}{\mu_{ii}^{rr}}+\frac{3}{2}\mu_{ii}^{rr}+\frac{1}{2}\mu_{ii}^{rr}e^{-\delta_i^r T_i^r}\right]}{(\gamma_i^r+\sigma_i^r+\delta_i^r)} for \quad u = r, i = a \\ \frac{\left[\frac{(\rho_{ia}^{rr})^2}{\mu_{ia}^{rr}}+\mu_{ii}^{rr}+\mu_{ia}^{rr}+\frac{1}{2}\mu_{ia}^{rr}e^{-\delta_a^r T_i^r}\right]}{(\rho_{ia}^{rr}+\delta_a^r)}, for \quad u = r, a \neq i \\ \frac{\left[\frac{(\rho_{ia}^{ru})^2}{\mu_{ia}^{ru}}+\mu_{ii}^{rr}+\mu_{ia}^{ru}+\frac{1}{2}\mu_{ia}^{ru}e^{-\delta_a^u T_i^r}\right]}{(\rho_{ia}^{ru}+\delta_a^u)}, for \quad u \neq r, \end{cases} \tag{5.4.58}$$

$$\mathfrak{W}_{ia}^{ru} = \begin{cases} \frac{\left[\frac{1}{2}\sum_{u=1}^{M}\sum_{a=1}^{n_u}\mu_{ia}^{ru}+\frac{1}{2}\sum_{u\neq r}^{M}\sum_{a=1}^{n_r}\frac{(\gamma_{ia}^{ru})^2}{\mu_{ii}^{rr}}+\frac{1}{2}\sum_{a\neq i}^{n_r}\frac{(\sigma_{ia}^{rr})^2}{\mu_{ii}^{rr}}+\frac{1}{2}\mu_{ii}^{rr}+\frac{1}{2}\mu_{ii}^{rr}e^{-\delta_i^r T_i^r}\right]}{(\gamma_i^r+\sigma_i^r+\delta_i^r)}, for \quad u = r, a = i, \\ \frac{\left[\frac{1}{2}\frac{(\rho_{ia}^{rr})^2}{\mu_{ia}^{rr}}+\frac{1}{2}\mu_{ii}^{rr}+\frac{1}{2}\mu_{ia}^{rr}+\frac{1}{2}\mu_{ia}^{rr}e^{-\delta_a^r T_i^r}\right]}{(\rho_{ia}^{rr}+\delta_a^r)}, for \quad u = r, a \neq i, \\ \frac{\left[\frac{1}{2}\frac{(\rho_{ia}^{ru})^2}{\mu_{ia}^{ru}}+\frac{1}{2}\mu_{ii}^{rr}+\frac{1}{2}\mu_{ia}^{ru}+\frac{1}{2}\mu_{ia}^{ru}e^{-\delta_a^u T_i^r}\right]}{(\rho_{ia}^{ru}+\delta_a^u)}, for \quad u \neq r \end{cases} \tag{5.4.59}$$

and

$$\mathfrak{E}_{ia}^{ru} = \begin{cases} \left[\frac{2\sum_{u=1}^{M}\sum_{a=1}^{n_u}\mu_{ia}^{ru}+\sum_{a\neq r}^{n_r}(2+c_{ia}^{rr})\frac{(\sigma_{ia}^{rr})^2}{\mu_{ii}^{rr}}+\sum_{u=1}^{M}\sum_{a=1}^{n_u}(2+c_{ia}^{ru})\frac{(\gamma_{ia}^{ru})^2}{\mu_{ii}^{rr}}+\mu_{ii}^{rr}e^{-\delta_i^r T_i^r}}{(\rho_i^r+\gamma_i^r+\sigma_i^r+\delta_i^r+d_i^r)}\right. \\ \left. +\frac{\frac{(\rho_i^r+d_i^r)^2}{\mu_{ii}^{rr}}+4\frac{(\gamma_i^r+\sigma_i^r+\delta_i^r)^2}{\mu_{ii}^{rr}}+\frac{(\rho_i^r)^2}{\mu_{ii}^{rr}}+3\frac{(\rho_i^r)^2}{\mu_{ii}^{rr}}e^{-\delta_i^r T_i^r}}{(\rho_i^r+\gamma_i^r+\sigma_i^r+\delta_i^r+d_i^r)}\right], for \quad u = r, a = i, \\ \left[\frac{(2+c_{ii}^{rr})\frac{(\rho_{ia}^{rr})^2}{\mu_{ia}^{rr}}+2\mu_{ii}^{rr}+\frac{(\rho_a^r+d_a^r)^2}{\mu_{ia}^{rr}}+4\frac{(\rho_{ia}^{rr}+\delta_a^r)^2}{\mu_{ia}^{rr}}+3\frac{(\rho_a^r)^2}{\mu_{ia}^{rr}}e^{-\delta_a^r T_i^r}}{(\rho_a^r+\rho_{ia}^{rr}+\delta_a^r+d_a^r)}\right], for \quad u = r, a \neq i, \\ \left[\frac{(2+c_{ii}^{rr})\frac{(\rho_{ia}^{ru})^2}{\mu_{ia}^{ru}}+2\mu_{ii}^{rr}+\frac{(\rho_a^u+d_a^u)^2}{\mu_{ia}^{ru}}+4\frac{(\rho_{ia}^{ru}+\delta_a^u)^2}{\mu_{ia}^{ru}}+3\frac{(\rho_a^u)^2}{\mu_{ia}^{ru}}e^{-\delta_a^u T_i^r}}{(\rho_a^u+\rho_{ia}^{ru}+\delta_a^u+d_a^u)}\right], for \quad , u \neq r \end{cases}$$

From (5.4.41), (5.4.42), (5.4.53), (5.4.54), the differential operator *LV* [34, 59] applied to the Lyapunov functional (5.4.41), and some further algebraic manipulations we have the following inequality

$$\begin{aligned}
LV(\tilde{x}_{00}^{00}) \quad \leq \quad & \sum_{r=1}^{M}\sum_{i=1}^{n_r} -\left\{[\phi_{ii}^{rr}(U_{ii}^{rr})^2 + \psi_{ii}^{rr})(V_{ii}^{rr})^2\right. \\
& \varphi_{ii}^{rr}(W_{ii}^{rr})^2] + \sum_{a\neq r}^{n_r}[\phi_{ia}^{rr}(U_{ia}^{rr})^2 + \psi_{ia}^{rr}(V_{ia}^{rr})^2 \\
& +\varphi_{ia}^{rr}(W_{ia}^{rr})^2] + \sum_{u\neq r}^{M}\sum_{a=1}^{n_u}[\phi_{ia}^{ru}(U_{ia}^{ru})^2 + \psi_{ia}^{ru}(V_{ia}^{ru})^2 \\
& \left. + \varphi_{ia}^{ru}(W_{ia}^{ru})^2]\right\}. \qquad (5.4.60)
\end{aligned}$$

Under the assumption on T_i^r, it follows that $\mathfrak{U}_{ia}^{ru} \leq 1$ and $\mathfrak{W}_{ia}^{ru} \leq 1, \forall u \in I(1,M), a \in I(1,n_r)$. Moreover, under the assumption on $\mathfrak{V}_{ia}^{ru}$, it is clear that $\phi_{ia}^{ru}, \psi_{ia}^{ru}$ and φ_{ia}^{ru} are positive for suitable choices of the constants $c_{ia}^{ru} > 0$. Thus this proves the inequality (5.4.44). Now, the validity of (5.4.45) follows from (5.4.44) and (5.4.39), that is,

$$LV(\tilde{x}_{00}^{00}) \leq -cV_1(\tilde{x}_{00}^{00}),$$

where $c = \frac{\min_{1\leq r,u\leq M, 1\leq i\leq n_r, 1\leq a\leq n_u}\{\phi_{ia}^{ru}, \psi_{ia}^{ru}, \varphi_{ia}^{ru}\}}{\max_{1\leq r,u\leq M, 1\leq i\leq n_r, 1\leq a\leq n_u}\left\{C_{ia}^{ru}+2\right\}}$. This completes the proof. We now formally state the stochastic stability theorems for the disease free equilibria.

Theorem 5.4.3 *Given $r,u \in I(1,M)$, $i \in I(1,n_r)$ and $a \in I(1,n_u)$. Let us assume that the hypotheses of Lemma 5.4.2 are satisfied. Then the disease free solutions E_{ia}^{ru}, are asymptotically stable in the large. Moreover, the solutions E_{ia}^{ru} are exponentially mean square stable.*

Proof:

From the application of comparison result[34, 59], the proof of stochastic asymptotic stability follows immediately. Moreover, the disease free equilibrium state is exponentially mean square stable. We now consider the following corollary to Theorem 5.4.3.

Corollary 5.4.4 *Let $r \in I(1,M)$ and $i \in I(1,n_r)$. Assume that $\sigma_i^r = \gamma_i^r = 0$, for all $r \in I(1,M)$ and $i \in I(1,n_r)$. Suppose that*

$$\begin{cases}
\delta_i^r > \max\left(\left(\sum_{u=1}^{M}\sum_{a=1}^{n_u}\mu_{ia}^{ru} + \frac{3}{2}\mu_{ii}^{rr}\right), \left(\frac{1}{2}\sum_{u=1}^{M}\sum_{a=1}^{n_u}\mu_{ia}^{ru} + \frac{1}{2}\mu_{ii}^{rr}\right)\right), for \quad u = r, a = i, \\
(\rho_{ia}^{rr} + \delta_a^r) > \max\left(\left(\frac{(\rho_{ia}^{rr})^2}{\mu_{ia}^{rr}} + \mu_{ii}^{rr} + \mu_{ia}^{rr}\right), \left(\frac{1}{2}\frac{(\rho_{ia}^{rr})^2}{\mu_{ia}^{rr}} + \frac{1}{2}\mu_{ii}^{rr} + \frac{1}{2}\mu_{ia}^{rr}\right)\right), for \quad u = r, a \neq i, \\
(\rho_{ia}^{ru} + \delta_a^u) > \max\left(\left(\frac{(\rho_{ia}^{ru})^2}{\mu_{ia}^{ru}} + \mu_{ii}^{rr} + \mu_{ia}^{ru}\right), \left(\frac{1}{2}\frac{(\rho_{ia}^{ru})^2}{\mu_{ia}^{ru}} + \frac{1}{2}\mu_{ii}^{rr} + \frac{1}{2}\mu_{ia}^{ru}\right)\right), for \quad u \neq r.
\end{cases}$$

Furthermore, let

$$\mathfrak{M}^{ru}_{ia}=\begin{cases}\max\left(\frac{1}{\delta^r_i}\log\left(\frac{\mu^{rr}_{ii}}{2\left(\delta^r_i-\left(\sum_{u=1}^M\sum_{a=1}^{n_u}\mu^{ru}_{ia}+\frac{3}{2}\mu^{rr}_{ii}\right)\right)}\right),\right.\\ \left.\frac{1}{\delta^r_i}\log\left(\frac{\mu^{rr}_{ii}}{2\left(\delta^r_i-\left(\frac{1}{2}\sum_{u=1}^M\sum_{a=1}^{n_u}\mu^{ru}_{ia}+\frac{1}{2}\mu^{rr}_{ii}\right)\right)}\right)\right),\\ for\quad u=r,i=a\\ \max\left(\frac{1}{\delta^r_a}\log\left(\frac{\mu^{rr}_{ia}}{2\left((\rho^{rr}_{ia}+\delta^r_a)-\left[\frac{(\rho^{rr}_{ia})^2}{\mu^{rr}_{ia}}+\mu^{rr}_{ii}+\mu^{rr}_{ia}\right]\right)}\right),\right.\\ \left.\frac{1}{\delta^r_a}\log\left(\frac{\mu^{rr}_{ia}}{2\left((\rho^{rr}_{ia}+\delta^r_a)-\left[\frac{1}{2}\frac{(\rho^{rr}_{ia})^2}{\mu^{rr}_{ia}}+\frac{1}{2}\mu^{rr}_{ii}+\frac{1}{2}\mu^{rr}_{ia}\right]\right)}\right)\right),\\ for\quad u=r,a\neq i\\ \max\left(\frac{1}{\delta^u_a}\log\left(\frac{\mu^{ru}_{ia}}{2\left((\rho^{ru}_{ia}+\delta^u_a)-\left[\frac{(\rho^{ru}_{ia})^2}{\mu^{ru}_{ia}}+\mu^{rr}_{ii}+\mu^{ru}_{ia}\right]\right)}\right),\right.\\ \left.\frac{1}{\delta^u_a}\log\left(\frac{\mu^{ru}_{ia}}{2\left((\rho^{ru}_{ia}+\delta^u_a)-\left[\frac{1}{2}\frac{(\rho^{ru}_{ia})^2}{\mu^{ru}_{ia}}+\frac{1}{2}\mu^{rr}_{ii}+\frac{1}{2}\mu^{ru}_{ia}\right]\right)}\right)\right),\\ for\quad u\neq r,\end{cases}$$

and

$$\mathfrak{V}^{ru}_{ia}=\begin{cases}\frac{\frac{1}{2}\sum_{u=1}^M\sum_{a=1}^{n_u}\mu^{ru}_{ia}+\frac{1}{2}\sum_{v=1}^M\sum_{b=1}^{n_v}\beta^{rrv}_{iib}(S^{rr*}_{ii}\mu^{rr}_{ii}+\mu^{rr}_{ii})}{\rho^r_i+\gamma^r_i+\sigma^r_i+\delta^r_i+d^r_i}, for\quad a=i,u=r\\ \frac{\frac{1}{2}\mu^{rr}_{ii}+\frac{1}{2}\sum_{v=1}^M\sum_{b=1}^{n_v}\beta^{rrv}_{aib}(S^{rr*}_{ia}\mu^{rr}_{ia}+\mu^{rr}_{ia})}{\rho^r_a+\rho^{rr}_{ia}+\delta^r_a+d^r_a}, for\quad a\neq i,u=r\\ \frac{\frac{1}{2}\mu^{rr}_{ii}+\frac{1}{2}\sum_{v=1}^M\sum_{b=1}^{n_v}\beta^{urv}_{aib}(S^{ru*}_{ii}\mu^{ru}_{ia}+\mu^{ru}_{ia})}{\rho^u_a+\rho^{ru}_{ia}+\delta^u_a+d^u_a}, for\quad u\neq r.\end{cases}\tag{5.4.61}$$

for some suitably defined positive number μ^{ru}_{ia}, depending on δ^u_a, for all $r,u\in I^r(1,M)$, $i\in I(1,n)$ and $a\in I^r_i(1,n_r)$. The equilibrium state E^{rr}_{ii} is stochastically asymptotically stable provided that $\mathfrak{V}^{ru}_{ia}<1$ and $T^r_i\geq\max_{1\leq r,u\leq M,1\leq i\leq n_r,1\leq a\leq n_u}(\mathfrak{M}^{ru}_{ia})$.

Proof: Follows immediately from the hypotheses of Lemma 5.4.2,(letting $\sigma^r_i=\gamma^r_i=0$), the conclusion of Theorem 5.4.3 and some algebraic manipulations.

Remark 5.4.2 *The presented results about the two-level large scale delayed SIR disease dynamic model depend on the underlying system parameters. In particular, the sufficient conditions are algebraically simple, computationally attractive and explicit in terms of the rate parameters. As a result of this, several scenarios can be discussed and exhibit practical course of action to control the disease. For simplicity, we present an illustration as follows: the conditions of $\sigma^r_i=\gamma^r_i=0,\forall r,i$ in Corollary 5.4.4 signify that the arbitrary site s^r_i is a 'sink' in the context of compartmental* *systems[28, 29] for all other sites in the inter and intra-regional accessible domain. This scenario is displayed in Figure 7.1. The condition* $T_i^r \geq \max_{1\leq r,u\leq M, 1\leq i\leq n_r, 1\leq a\leq n_u}(\mathfrak{M}_{ia}^{ru})$ *is a threshold condition for the immunity delay period of residents of site* s_i^r *in region* C_r*, controlling the stochastic asymptotic stability of the disease free equilibrium. Furthermore, the condition* $\mathfrak{V}_{ia}^{ru} < 1$ *signifies that the magnitude of disease inhibitory processes for example, the magnitude of the recovery process is greater than the disease transmission process. A future detailed study of the disease dynamics in the two scale network dynamic structure for many real life scenarios using the presented two level large-scale delay SIR disease dynamic model will appear elsewhere.*

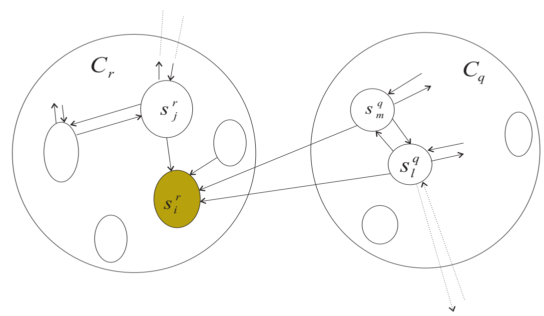

Figure 5.1: Shows that residents of site s_i^r are present only at their home site s_i^r. Hence they isolate every site from their inter and intra regional accessible domain $C(s_i^r)$. Site s_i^r is a 'sink' in the context of the compartmental system[28, 29]. The arrows represent a transport network between any two sites and regions. Furthermore, the dotted lines and arrows indicate connection with other sites and regions.

5.5 Conclusion

The formulated two-scale network delayed epidemic dynamic model characterizes the dynamics of an SIR epidemic in a population with various scale levels created by the heterogeneities in the population. Moreover, the disease dynamics is subject to random environmental perturbations at the disease transmission stage of the disease. Furthermore, the SIR epidemic confers temporary natural immunity to recovered individuals immediately after recovery. This work provides a mathematical and probabilistic algorithmic tool to develop different levels nested type disease transmission rates as well as the variability in the disease diseases transmission process in the framework of the network-centric Ito-Doob type dynamic equations. In addition, the concept of temporary natural immunity delay of human epidemics is developed for the first time in the context of scale-structured human meta-populations.

The model validation results are exhibited and a positively self-invariant set for the dynamic model is defined. Moreover, the globalization of the solution existence is obtained by applying the Lyapunov energy function technique. In addition, using the Lyapunov functional technique, the detailed stochastic asymptotic stability results of the disease free equilibria are also exhibited. Moreover, the system parameter dependent and also temporary delay time threshold values controlling the stochastic asymptotic stability of the disease free equilibrium are also defined. Furthermore, the analysis of the general stochastic delayed dynamic model are exhibited in a controlled quarantine strategy.

The stochastic delayed epidemic dynamic model will be extended to the variability in the mobility, recovery and birth and death processes. A further detailed study of the oscillation of the epidemic process about the ideal endemic equilibrium of the dynamic epidemic model will also appear else where.

6 SIMULATION RESULTS FOR A TWO-SCALE STOCHASTIC NETWORK SIR TEMPORARY DELAYED INFLUENZA EPIDEMIC DYNAMIC MODEL

6.1 Introduction

In this chapter, we extend the influenza stochastic epidemic model studied in Chapter 4 by incorporating the natural immunity delay period of the naturally immune population. Influenza has a short lived and strain-dependent immunity[78]. The population hierarchic structure, the human mobility process, the influenza transmission process, the birth and death processes of the previous example in Chapter 4 are preserved in this example. Moreover, the respective parametric specifications defined in Chapter 4 are also valid in this example. In the following, we describe the influenza recovery process and the acquisition of temporary immunity. We refer the reader to Chapter 4 for the influenza scenario that is presented in this example.

We assume that residents of site s_a^u in region C_u recover from the disease and acquire temporary natural immunity to the specific influenza strain. In the absence of data for the recovery and disease related death processes, we take the recovery and disease mortality rate to be $\rho_a^u = 0.05067$ and $d_a^u = 0.01838, u = 1,2,3; a,i = 1,2,3$ respectively. Furthermore, we assume that the average natural immunity period of recovered residents of all sites in the two scale population structure is the same. In this example, for all residents of site s_i^r in region C_r present at any sites in the intra-regional and interregional accessible domain, we set the natural immunity period $T_i^r = 1, \forall r = 1,2,3, \forall i = 1,2,3$.

6.2 Mathematical Algorithm and Simulation Results

We apply the standard Euler-Maruyama method stochastic approximation scheme[69, 70] to generate the trajectories for the residents of sites s_1^1, s_1^2 and s_1^3 in regions C_1, C_2 and C_3 respectively, for the different population disease classification (S,I,R), and current locations at some sites in the intra and inter-regional accessible domain of the sites. We summarize the Euler-Maruyama method steps to obtain strong solution approximations to a system of stochastic delay differential equations in the following. Given a scalar autonomous stochastic delay differential equation

$$\begin{aligned} dX(t) &= f(X(t),X(t-\tau))dt + g(X(t),X(t-\tau))dW(t), \quad T_0 \leq t \leq T, \\ X(t) &= \varphi_0(t), \quad t \in [T_0-\tau, T_0]. \end{aligned} \tag{6.2.1}$$

where $\varphi_0(t)$ is a measurable random variable on $\mathcal{C}([T_0-\tau,T_0],\mathbb{R})$ and $T_0,T \geq 0$. Let $T_0 = t_0 \leq t_1 \leq t_2 \leq \ldots \leq t_n = T$, be a regular partition of $[T_0,T]$, where $h = t_j - t_{j-1} = \frac{T-T_0}{N}$, $t_j = T_0 + jh, \quad j = 0,\ldots,N$ and N is a positive integer. Furthermore, suppose there exist a positive integer N_τ such that the delay parameter $\tau = N_\tau h$. The Euler-Maruyama method takes the form

$$\begin{aligned} X(t_{j-N_\tau}) &= \varphi_0(t_j-\tau), \quad j - N_\tau \leq 0, \\ X(t_{j+1}) &= X(t_j) + hf(X(t_j),X(t_{j-N_\tau})) + g(X(t_j),X(t_{j-N_\tau}))\Delta W(t_{j+1}), \\ & \quad j = 0,1,\ldots,N-1, \end{aligned} \tag{6.2.2}$$

where $\Delta W(t_{j+1}) = (W(t_{j+1}) - W(t_j)), j = 0,\ldots,N-1$ are an independent Gaussian $N(0,h)$ random variables. Using (6.2.1) as a general form of the equations in the system (5.2.1)-(5.2.3) in the context of this example, the algorithm to execute the Euler-Maruyama method to find the solution process of (6.2.1) consists of the following steps:

Step one: Parameter Specification: The system rate parameters for the epidemic model (5.2.1)-(5.2.3) represented in this example are specified in Section 4.1.

Step two: Initial Conditions: The initial solutions are approximated using $X(t_{j-N_\tau}) = \varphi_0(t_j-\tau), \quad j = 0,\ldots N_\tau$, $X \in \{S^{ru}_{ia}, I^{ru}_{ia}, R^{ru}_{ia}\}$. In this example, we set $\tau = T^r_i = 1, \forall r = 1,2,3, \quad i = 1,2,3$, $T_0 = 0$ and $T = 1$. This implies from the definition of τ and N_τ in (6.2.2) that $N_\tau = N$. Furthermore, from (5.2.4) and (6.2.1), the following convenient initial conditions are used for the simulation process: for $r,u \in I(1,3), i,a \in I(1,3)$, $\varphi^{ru}_{iak} \in \mathcal{C}([-1,0],\mathbb{R}_+), k = 1,2,3$

$$S^{ru}_{ia}(t) = \varphi^{ru}_{ia1}(t) = \begin{cases} 9, for & r = u, i = a \\ 8, for & r = u, i \neq a \\ 7, for & r \neq u, \forall t \in [-1,0] \end{cases}$$

$$I_{ia}^{ru}(t) = \varphi_{ia2}^{ru}(t) = \begin{cases} 6, for & r = u, i = a \\ 4, for & r = u, i \neq a \\ 3, for & r \neq u, \forall t \in [-1,0] \end{cases}$$

and $R_{ia}^{ru}(t) = \varphi_{ia3}^{ru}(t) = 2, \forall t \in [-1,0], \forall r,u,i,a \in I(1,3)$.

Step Three: Generate Brownian Path: The standard Brownian motion or normalized Wierner process $W(t)$ is generated over the time interval $[T_0, T]$. That is, we let $T_0 = \tau_0 \leq \tau_1 \leq \tau_2 \leq \ldots \leq \tau_n = T$, be a regular partition of $[T_0, T]$, where $\delta\tau = \tau_{j+1} - \tau_j = \frac{T - T_0}{N}$, $\tau_j = T_0 + j\delta\tau, \quad j = 0, \ldots, N$ and N is a positive integer. The Brownian motion is generated as the solution to the following difference equation

$$\begin{cases} W(0) = 0, \\ W(\tau_{j+1}) = W(\tau_j) + \ dW(\tau_{j+1}), \quad j = 0, \ldots, N-1. \end{cases} \tag{6.2.3}$$

where $dW(\tau_{j+1}), \quad j = 0, \ldots, N-1$ are the independent $\sqrt{\delta t} N(0,1)$ Gaussian random variables. Furthermore, for this example given that $T_0 = 0$ and $T = 1$, we simulated 500 sample points for the Brownian motion over the interval $[0,1]$.

Step Four: Generate Solution Path for the Susceptible, Infectious and Removal Populations(States): Using (6.2.1) as a general representation of each equation in the system (5.2.1)-(5.2.3), we use the discretization (6.2.2) to find solutions path for each equation in the system. For convenience, we choose $h = R\delta\tau$,[38], where the positive integer $R \geq 1$ and $\delta\tau$ is defined in Step Three. Moreover, from (6.2.2), it follows that

$$\Delta W(t_{j+1}) = W(t_{j+1}) - W(t_j) = W(T_0 + (j+1)R\delta\tau) - W(T_0 + jR\delta\tau) = \sum_{k=jR}^{jR+R-1} dW_{k+1}, \tag{6.2.4}$$

where dW_{k+1} is given by the Brownian path (6.2.3). We choose $R = 1$ for this example. Moreover, from (6.2.2), (6.2.3), and (6.2.4), we obtain trajectories for susceptible, infectious and removal populations of residents of sites s_1^1, s_1^2 and s_1^3 in regions C_1, C_2 and C_3 over the time interval $[0,1]$. The trajectories for the residents of sites s_1^1, s_1^2 and s_1^3 in regions C_1, C_2 and C_3 are exhibited in Figure 6.1, Figure 6.2 and Figure 6.3 respectively.

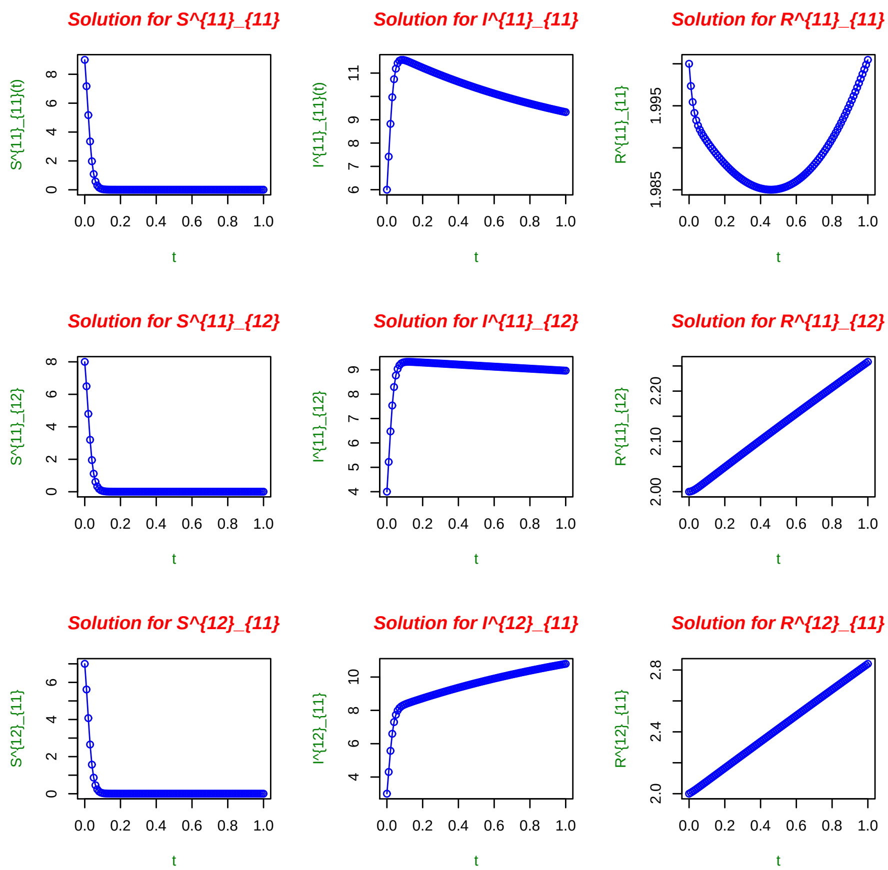

Figure 6.1: Trajectories of the disease classification (S, I, R) for residents of site s_1^1 in region C_1 at their current location in the two-scale spatial patch dynamic structure. See Remark 6.2.1 for more comments on this figure.

Remark 6.2.1 From Figure 6.1, the Figures (a),(b) & (c) represent the trajectories of the different disease classes of residents of site s_1^1 at home. Figures (d),(e) & (f) represent the trajectories of the different disease classes of residents of site s_1^1 visiting site s_2^1 in home region C_1. These two groups of figures are representative of the disease dynamics of influenza affecting the residents of site s_1^1 at the intra-regional level. Figures (g),(h) & (i) represent the trajectories of the different disease classes of residents of site s_1^1 visiting site s_1^2 in region C_2. These figures reflect the behavior of the disease affecting the residents of site s_1^1 at the inter-regional level.

Furthermore, we observe that the trajectories of the susceptible (S) and infectious(I) populations saturate to their equilibrium states. We further note that the trajectory paths are random in character but because of the scale of the pictures presented in this figure, they appear to be smooth.

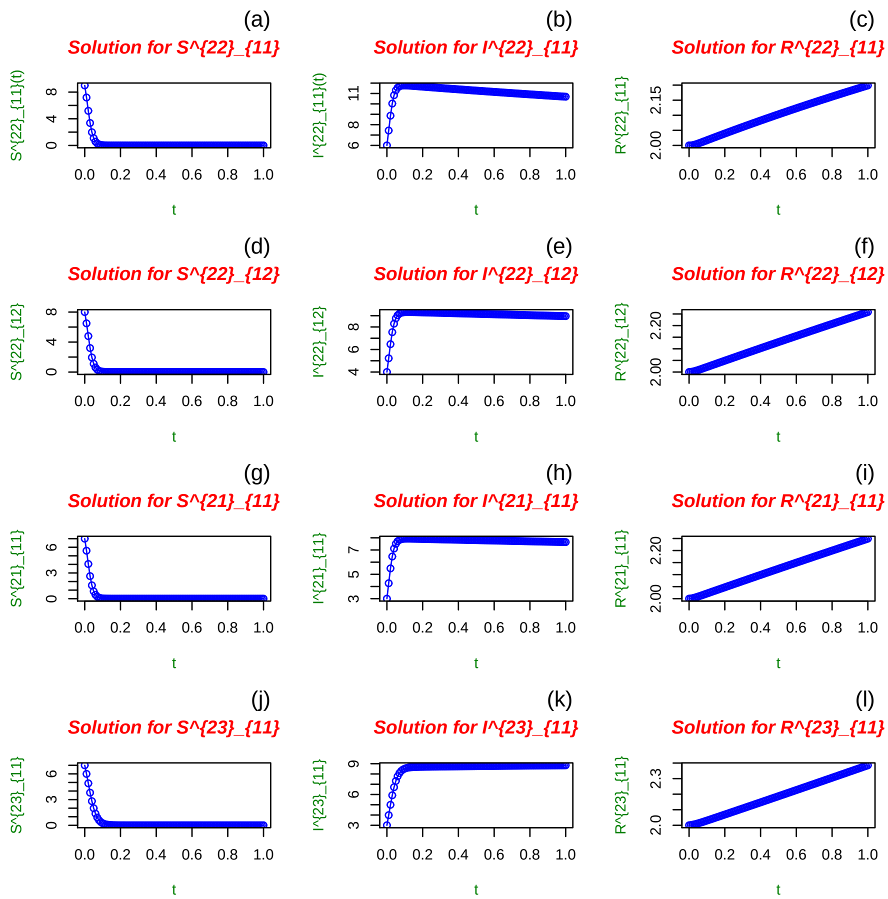

Figure 6.2: Trajectories of the disease classification (S,I,R) for residents of site s_1^2 in region C_2 at their current location in the two-scale spatial patch dynamic structure. See Remark 6.2.1 for more comments on this figure.

Remark 6.2.2 From Figure 6.2, we note that Figures (a),(b) & (c) represent the trajectories of the different disease classes of residents of site s_1^2 at home. Figures (d),(e) & (f) represent the trajectories of the different disease classes of residents of site s_1^2 visiting site s_2^2 in home region C_2. These two groups of figures are representative of the disease dynamics of influenza affecting the residents of site s_1^2 at the intra-regional level. Figures (g),(h) & (i) represent the trajectories of the different disease classes of residents of site s_1^2 visiting site s_1^1 in region C_1. Figures (j),(k) & (l) represent the trajectories of the different disease classes of residents of site s_1^2 visiting site s_1^3 in region C_3. These last two groups of figures reflect the behavior of the disease affecting the residence of site s_1^2 at the inter-regional level. Furthermore, we observe that the trajectories of the susceptible (S) and infectious(I) populations saturate to their equilibrium states. We further note that the trajectory paths are random in character but because of the scale of the pictures presented in this figure, they appear to be smooth.

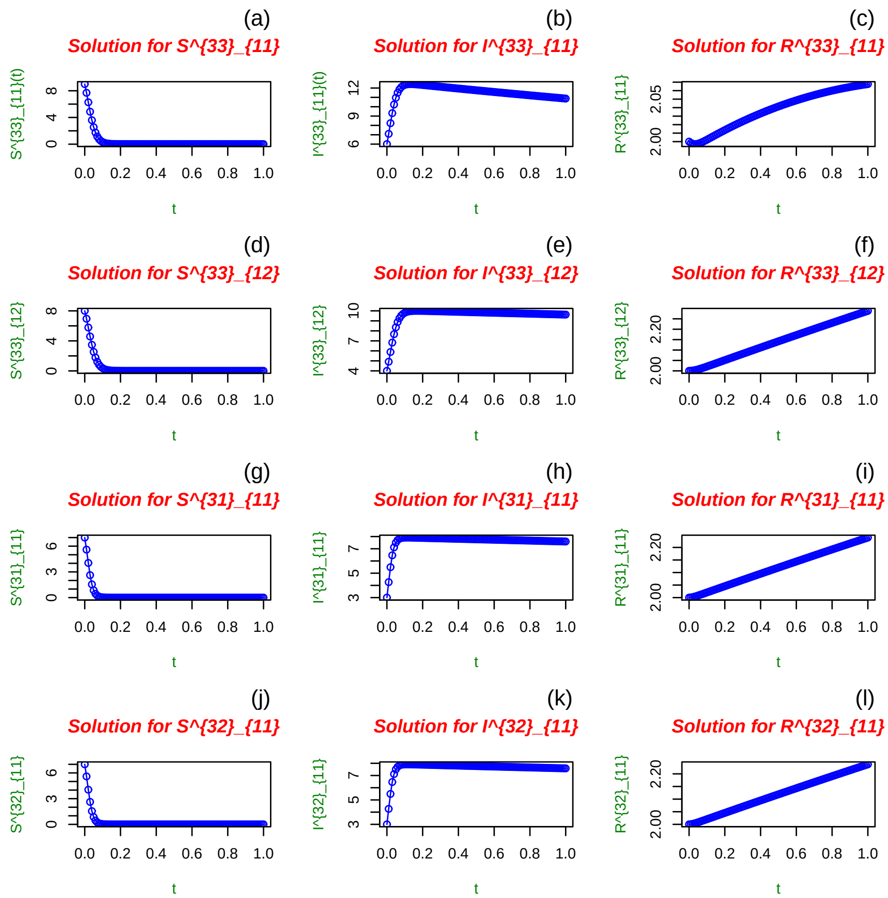

Figure 6.3: Trajectories of the disease classification (S,I,R) for residents of site s_1^3 in region C_3 at their current location in the two-scale spatial patch dynamic structure. See Remark 6.2.3 for more comments on this figure.

Remark 6.2.3 From Figure 6.3, we note that Figures (a),(b) & (c) represent the trajectories of the different disease classes of residents of site s_1^3 at home. Figures (d),(e) & (f) represent the trajectories of the different disease classes of residents of site s_1^3 visiting site s_2^3 in home region C_3. These two groups of figures are representative of the disease dynamics of influenza affecting the residents of site s_1^3 at the intra-regional level. Figures (g),(h) & (i) represent the trajectories of the different disease classes of residents of site s_1^3 visiting site s_1^1 in region C_1. Figures (j),(k) & (l) represent the trajectories of the different disease classes of residents of site s_1^3 visiting site s_1^2 in region C_2. The last two groups of figures reflect the behavior of the disease affecting the residence of site s_1^3 at the inter-regional level. Furthermore, we observe that the trajectories of the susceptible (S) and infectious(I) populations saturate to their equilibrium states. We further note that the trajectory paths are random in character but because of the scale of the pictures presented in this figure, they appear to be smooth.

We make the following comparative remark on the trends in the trajectories of the naturally immune populations represented in Figure 6.1, Figure 6.2 and Figure 6.3, and on the trends of the naturally immune populations represented in Figure 4.1, Figure 4.2 and Figure 4.3

Remark 6.2.4 *We observe significant differences between the trajectories of the naturally immune populations R_{11}^{11} in Figure 6.1, R_{11}^{22} in Figure 6.2 and R_{11}^{33} in Figure 6.3, and the corresponding trajectories of R_{11}^{11} in [Figure 4.1,[66]], R_{11}^{22} in [Figure 4.2,[66]] and R_{11}^{33} in [Figure 4.3,[66]]. The trajectories of the naturally immune populations R_{11}^{11} in Figure 6.1, R_{11}^{22} in Figure 6.2 and R_{11}^{33} in Figure 6.3, exhibit a growth trend in the naturally immune population that initially decreases from the initial state, and finally increases over time. This further exhibit the fact that natural immunity and the fluctuating environment influence the growth trends of the endemic population. Moreover, the trajectory of the naturally immune populations R_{11}^{11} in Figure 6.1 indicates a periodic solution over time with period equal to the length of the immunity period.*

6.3 Conclusion

An influenza stochastic temporary delayed epidemic dynamic model in a two-scale population structure with specific model parameters is implicitly defined as an extension of the influenza epidemic dynamic model studied in Chapter 5. The influenza transmission process at the site level is elaborated. In addition, the Euler-Maruyama stochastic simulation scheme and application process for the two-scale network centric Ito-Doob system of delay stochastic differential equations is explained. Furthermore, simulated trajectories for the different state processes (susceptible, infective, removal) of residents of some sites in the three regions with respect to the current locations in the intra and interregional levels are developed and presented. The simulated findings reveal comparative evolution patterns for the different state processes and current locations over time. Furthermore, there is an oscillatory effect in the trajectory of the naturally immune population. Moreover, the oscillations are more significant at the homesite.

7 GLOBAL ANALYSIS OF A STOCHASTIC TWO-SCALE NETWORK HUMAN EPIDEMIC DYNAMIC MODEL WITH VARYING IMMUNITY PERIOD

7.1 Introduction

In this chapter we extend the epidemic dynamic model with temporary immunity delay studied in Chapter 5 into a more realistic epidemic dynamic model with varying immunity period delay. Generally, the length of the natural immunity period after recovery from the disease varies within the immune population and also for different diseases. This variation is accounted for by the variations in strengths of the immune system of individuals recovering from diseases, and also because individuals in the population exhibit varying immunity responses to different antigens produced by different diseases. Some diseases confer almost life long immunity, and others give only a temporary immunity after recovery. For instance, those who recover from measles acquire life long natural immunity [84]. Influenza has a temporary immunity to the particular disease strain after recovery to the disease.

The epidemic dynamic processes in populations exhibiting varying time disease latency or immunity delay periods are represented by differential equation models with distributed time delays. Several studies[79, 81, 82, 83, 56] incorporating distributed delays describing the effects of disease latency or immunity in the dynamics of human infectious diseases have been done. A mathematical SIR (susceptible-infective-removal) epidemic dynamic model with distributed time delays representing the varying time temporal immunity period in the immune population class is studied by Blyuss and Kyrychko[83]. In their study, the existence of positive solution is exhibited. Furthermore, the global asymptotic stability of the disease free and endemic equilibria are shown by using Lyapunov functional technique. Moreover, they presented numerical simulation results for a special case SIR epidemic with temporal immunity.

The temporal immunity was represented in the epidemic dynamic model by letting the Dirac *delta*-function be the integral kernel or the probability density function of the distributed time delay.

Stochastic models with distributed time delays offer a much better representation of the reality. In [56], a stochastic SIR epidemic dynamic model with distributed time delay is studied. Moreover, the stochastic asymptotic stability of the disease free equilibrium is exhibited by applying the Lyapunov functional technique. Furthermore, in [68, 80] the existence of positive solution process for the stochastic epidemic model is exhibited by applying a Lyapunov energy function method. This work is organized as follows. In Section 7.2, we derive the distributed time acquired immunity delay epidemic dynamic model. In Section 7.3, we present the model validation results of the epidemic model. In Section 7.4, we show the stochastic asymptotic stability of the disease free equilibrium.

7.2 Derivation of the SIR Delayed Stochastic dynamic Model

We assume that for each $r \in I(1,M)$, and $i \in I(1,n_r)$, infectious (I_{ia}^{ru}) residents of site s_i^r in region C_r visiting site s_a^u in region C_u recover from the disease and acquire immunity against the disease immediately after recovery. The recovered individuals further loose immunity and become susceptible to the disease after a period of time $s \in [0,\infty)$, where the immunity period s is an infinite random variable. Using ideas from [83], we incorporate the varying acquired immunity time delay into the epidemic dynamic model (5.2.1)-(5.2.3), by introducing the term $\rho_a^u \int_0^\infty I_{ia}^{ru}(t-s) f_{ia}^{ru}(s) e^{-\delta_a^u s} ds$, where $e^{-\delta_a^u s}$ is the probability that an individual who recovered from disease at an earlier time $t-s$ is still alive at time t. Furthermore, $f_{ia}^{ru}(s)$ is the integral kernel[83] representing the probability density of the time s to loose acquired immunity by residents of site s_i^r in region C_r. The naturally immune individuals were previously infectious at their visiting site s_a^u in region C_u, and the have recovered from disease acquiring temporal natural immunity. Moreover, $\int_0^\infty f_{ia}^{ru}(s) ds = 1$, and $f_{ia}^{ru} \geq 0$. The two level large scale stochastic SIR delayed epidemic dynamic model with varying immunity period delay is as follows:

$$dS^{ru}_{ia} = \begin{cases} [B^r_i + \sum_{k=1}^{n_r} \rho^{rr}_{ik} S^{rr}_{ik} + \sum_{q\neq r}^{M} \sum_{a=1}^{n_q} \rho^{rq}_{ia} S^{rq}_{ia} + \rho^r_i \int_0^\infty I^{rr}_{ii}(t-s) f^{rr}_{ii}(s) e^{-\delta^r_i s} ds \\ -(\gamma^r_i + \sigma^r_i + \delta^r_i) S^{rr}_{ii} - \sum_{u=1}^{M} \sum_{a=1}^{n_u} \beta^{rru}_{iia} S^{rr}_{ii} I^{ur}_{ai}] dt \\ -[\sum_{u=1}^{M} \sum_{a=1}^{n_u} v^{rru}_{iia} S^{rr}_{ii} I^{ur}_{ai} dw^{rru}_{iia}(t)], u=r, a=i \\ [\sigma^{rr}_{ij} S^{rr}_{ii} + \rho^r_j \int_0^\infty I^{rr}_{ij}(t-s) f^{rr}_{ij}(s) e^{-\delta^r_j s} ds - (\rho^{rr}_{ij} + \delta^r_j) S^{rr}_{ij} \\ -\sum_{u=1}^{M} \sum_{a=1}^{n_u} \beta^{rru}_{jia} S^{rr}_{ij} I^{ur}_{aj}] dt - [\sum_{u=1}^{M} \sum_{a=1}^{n_u} v^{rru}_{jia} S^{rr}_{ij} I^{ur}_{aj} dw^{rru}_{jia}(t)], u=r, a=j, j\neq i, \\ [\gamma^{rq}_{il} S^{rr}_{ii} + \rho^q_l \int_0^\infty I^{rq}_{il}(t-s) f^{rq}_{il}(s) e^{-\delta^q_l s} ds - (\rho^{rq}_{il} + \delta^q_l) S^{rq}_{il} \\ -\sum_{u=1}^{M} \sum_{a=1}^{n_u} \beta^{qru}_{lia} S^{rq}_{il} I^{uq}_{al}] dt - [\sum_{u=1}^{M} \sum_{a=1}^{n_u} v^{qru}_{lia} S^{rq}_{il} I^{uq}_{al} dw^{qru}_{lia}(t)], u=q, a=l, q\neq r, \end{cases} \tag{7.2.1}$$

$$dI^{ru}_{ia} = \begin{cases} [\sum_{k=1}^{n_r} \rho^{rr}_{ik} I^{rr}_{ik} + \sum_{q\neq r}^{M} \sum_{a=1}^{n_q} \rho^{rq}_{ia} I^{rq}_{ia} - \rho^r_i I^{rr}_{ii} \\ -(\gamma^r_i + \sigma^r_i + \delta^r_i + d^r_i) I^{rr}_{ii} + \sum_{u=1}^{M} \sum_{a=1}^{n_u} \beta^{rru}_{iia} S^{rr}_{ii} I^{ur}_{ai}] dt \\ +[\sum_{u=1}^{M} \sum_{a=1}^{n_u} v^{rru}_{iia} S^{rr}_{ii} I^{ur}_{ai} dw^{rru}_{iia}(t)], u=r, a=i \\ [\sigma^{rr}_{ij} I^{rr}_{ii} - \rho^r_j I^{rr}_{ij} - (\rho^{rr}_{ij} + \delta^r_j + d^r_j) I^{rr}_{ij} + \sum_{u=1}^{M} \sum_{a=1}^{n_u} \beta^{rru}_{jia} S^{rr}_{ij} I^{ur}_{aj}] dt \\ +[\sum_{u=1}^{M} \sum_{a=1}^{n_u} v^{rru}_{jia} S^{rr}_{ij} I^{ur}_{aj} dw^{rru}_{jia}(t)], u=r, a=j, j\neq i, \\ [\gamma^{rq}_{il} I^{rr}_{ii} - \rho^q_l I^{rq}_{il} - (\rho^{rq}_{il} + \delta^q_l + d^q_l) I^{rq}_{il} \\ +\sum_{u=1}^{M} \sum_{a=1}^{n_u} \beta^{qru}_{lia} S^{rq}_{il} I^{uq}_{al}] dt + [\sum_{u=1}^{M} \sum_{a=1}^{n_u} v^{qru}_{lia} S^{rq}_{il} I^{uq}_{al} dw^{qru}_{lia}(t)], u=q, a=l, q\neq r, \end{cases} \tag{7.2.2}$$

$$R^{ru}_{ia} = \begin{cases} [\sum_{k=1}^{n_r} \rho^{rr}_{ik} R^{rr}_{ik} + \sum_{q\neq r}^{M} \sum_{l=1}^{n_q} \rho^{rq}_{il} R^{rq}_{il} + \rho^r_i I^{rr}_{ii} - \rho^r_i \int_0^\infty I^{rr}_{ii}(t-s) f^{rr}_{ii}(s) e^{-\delta^r_i s} ds \\ -(\gamma^r_i + \sigma^r_i + \delta^r_i) R^{rr}_{ii}] dt, u=r, a=i \\ [\sigma^{rr}_{ij} R^{rr}_{ii} + \rho^r_j I^{rr}_{ij} - \rho^r_j \int_0^\infty I^{rr}_{ij}(t-s) f^{rr}_{ij}(s) e^{-\delta^r_j s} ds \\ -(\rho^{rr}_{ij} + \delta^r_j) R^{rr}_{ij}] dt, u=r, a=j, j\neq i, \\ [\gamma^{rq}_{il} R^{rr}_{ii} + \rho^q_l I^{rq}_{il} - \rho^q_l \int_0^\infty I^{rq}_{il}(t-s) f^{rq}_{il}(s) e^{-\delta^q_l s} ds \\ -(\rho^{rq}_{il} + \delta^q_l) R^{rq}_{il}] dt, u=q, a=l, q\neq r, \end{cases} \tag{7.2.3}$$

where all parameters are previously defined. Furthermore, for each $r \in I(1,M)$, and $i \in I(1,n_r)$, we have the following initial conditions

$$(S^{ru}_{ia}(t), I^{ru}_{ia}(t), R^{ru}_{ia}(t)) = (\varphi^{ru}_{ia1}(t), \varphi^{ru}_{ia2}(t), \varphi^{ru}_{ia3}(t)), t \in [-\infty, t_0],$$
$$\varphi^{ru}_{iak} \in \mathcal{C}([-\infty, t_0], \mathbb{R}_+), \forall k = 1,2,3, \forall r,q \in I(1,M), a \in I(1,n_u), i \in I(1,n_r),$$
$$\varphi^{ru}_{iak}(t_0) > 0, \forall k = 1,2,3, \tag{7.2.4}$$

where $\mathcal{C}([-\infty,t_0],\mathbb{R}_+)$ is the space of continuous functions with the supremum norm

$$||\varphi||_\infty = Sup_{-\infty\leq t\leq t_0}|\varphi(t)|. \tag{7.2.5}$$

and w is a Wierner process. Furthermore, the random continuous functions $\varphi^{ru}_{iak}, k=1,2,3$ are $\mathcal{F}_0 - measurable$, or independent of $w(t)$ for all $t\geq t_0$.

It follows from (3.2.18) and the system (7.2.1)-(7.2.3) that for $i\in I(1,n_r), l\in I^r_i(1,n_q), r\in I(1,M)$ and $q\in I^r(1,M)$,

$$dy^{rq}_{il} = \begin{cases} \left[B^r_i + \sum_{k\neq i}^{n_r}\rho^{rr}_{ik}y^{rr}_{ik} + \sum_{q\neq r}^{M}\sum_{a=1}^{n_q}\rho^{rq}_{ia}y^{rq}_{ia} - (\gamma^r_i+\sigma^r_i+\delta^r_i)y^{rr}_{ii} - d^r_i I^{rr}_{ii}\right]dt, for \quad q=r, l=i \\ \left[\sigma^{rr}_{ij}y^{rr}_{ii} - (\rho^{rr}_{ij}+\delta^r_j)y^{rr}_{ij} - d^r_j I^{rr}_{ij}\right]dt, for \quad q=r, a=j \quad and \quad i\neq j, \\ \left[\gamma^{rq}_{il}y^{rr}_{ii} - (\rho^{rq}_{il}+\delta^q_l)y^{rq}_{il} - d^q_l I^{rq}_{il}\right]dt, for \quad q\neq r, y^{rq}_{il}(t_0)\geq 0, \end{cases} \tag{7.2.6}$$

7.3 Model Validation Results

In the following we state and prove a positive solution process existence theorem for the delayed system (7.2.1)-(7.2.3). We utilize the Lyapunov energy function method[68] to establish the results of this theorem. We observe from (7.2.1)-(7.2.3) that (7.2.3) decouples from the first two equations in the system. Therefore, it suffices to prove the existence of positive solution process for $(S^{ru}_{ia}, I^{ru}_{ia})$. We utilize the notations (3.2.18) and keep in mind that $X^{ru}_{ia} = (S^{ru}_{ia}, I^{ru}_{ia})^T$.

Theorem 7.3.1 *Let $r,u\in I(1,M)$, $i\in I(1,n_r)$ and $a\in I(1,n_u)$. Given any initial conditions (7.2.4) and (7.2.5), there exists a unique solution process $X^{ru}_{ia}(t,w) = (S^{ru}_{ia}(t,w), I^{ru}_{ia}(t,w))^T$ satisfying (7.2.1) and (7.2.2), for all $t\geq t_0$. Moreover, the solution process is positive for all $t\geq t_0$ a.s. That is, $S^{ru}_{ia}(t,w)>0, I^{ru}_{ia(t,w)}>0, \forall t\geq t_0$ a.s.*

Proof:

It is easy to see that the coefficients of (7.2.1) and (7.2.2) satisfy the local Lipschitz condition for the given initial data (7.2.4). Therefore there exist a unique maximal local solution $X^{ru}_{ia}(t,w)$ on $t\in[-\infty,\tau_e(w)]$, where $\tau_e(w)$ is the first hitting time or the explosion time[34]. We show subsequently that $S^{ru}_{ia}(t,w), I^{ru}_{ia}(t,w)>0$ for all $t\in[-\infty,\tau_e(w)]$ almost surely. We define the following stopping time

$$\begin{cases} \tau_+ & = & sup\{t \in (t_0, \tau_e(w)) : S_{ia}^{ru}|_{[t_0,t]} > 0 \quad and \quad I_{ia}^{ru}|_{[t_0,t]} > 0\}, \\ \tau_+(t) & = & \min(t, \tau_+), \quad for \quad t \geq t_0. \end{cases} \tag{7.3.7}$$

and we show that $\tau_+(t) = \tau_e(w)$ a.s. Suppose on the contrary that $P(\tau_+(t) < \tau_e(w)) > 0$. Let $w \in \{\tau_+(t) < \tau_e(w)\}$, and $t \in [t_0, \tau_+(t))$. Define

$$\begin{cases} V(X_{00}^{00}) = \sum_{r=1}^{M} \sum_{i=1}^{n_r} \sum_{u=1}^{M} \sum_{a=1}^{n_u} V(X_{ia}^{ru}), \\ V(X_{ia}^{ru}) = \ln(S_{ia}^{ru}) + \ln(I_{ia}^{ru}), \forall t \leq \tau_+(t). \end{cases} \tag{7.3.8}$$

We rewrite (7.3.8) as follows

$$V(X_{00}^{00}) = \sum_{r=1}^{M} \sum_{i=1}^{n_r} \left[V(X_{ii}^{rr}) + \sum_{j \neq i}^{n_r} V(X_{ij}^{rr}) + \sum_{q \neq r}^{M} \sum_{l=1}^{n_q} V(X_{il}^{rq}) \right], \tag{7.3.9}$$

And (7.3.9) further implies that

$$dV(X_{00}^{00}) = \sum_{r=1}^{M} \sum_{i=1}^{n_r} \left[dV(X_{ii}^{rr}) + \sum_{j \neq i}^{n_r} dV(X_{ij}^{rr}) + \sum_{q \neq r}^{M} \sum_{l=1}^{n_q} dV(X_{il}^{rq}) \right], \tag{7.3.10}$$

where dV is the Ito-Doob differential operator with respect to the system (7.2.1)-(7.2.3). We express the terms on the right-hand-side of (7.3.10) in the following:

Site Level: From (7.3.8) the terms on the right-hand-side of (7.3.10) for the case of $u = r, a = i$

$$\begin{aligned} dV(X_{ii}^{rr}) \quad = \quad & \left[\frac{B_i^r}{S_{ii}^{rr}} + \sum_{k \neq i}^{n_r} \rho_{ik}^{rr} \frac{S_{ik}^{rr}}{S_{ii}^{rr}} + \sum_{q \neq r}^{M} \sum_{l=1}^{n_q} \rho_{ia}^{rq} \frac{S_{ia}^{rq}}{S_{ii}^{rr}} + \frac{\rho_i^r}{S_{ii}^{rr}} \int_0^\infty I_{ii}^{rr}(t-s) f_{ii}^{rr}(s) e^{-\delta_i^r s} ds \right. \\ & \left. -(\gamma_i^r + \sigma_i^r + \delta_i^r) - \sum_{u=1}^{M} \sum_{a=1}^{n_u} \beta_{iia}^{rru} I_{ai}^{ur} - \frac{1}{2} \sum_{u=1}^{M} \sum_{a=1}^{n_u} (v_{iia}^{rru})^2 (I_{ai}^{ur})^2 \right] dt \\ & \left[\sum_{k \neq i}^{n_r} \rho_{ik}^{rr} \frac{I_{ik}^{rr}}{S_{ii}^{rr}} + \sum_{q \neq r}^{M} \sum_{l=1}^{n_q} \rho_{ia}^{rq} \frac{I_{ia}^{rq}}{S_{ii}^{rr}} - \rho_i^r - (\gamma_i^r + \sigma_i^r + \delta_i^r + d_i^r) \right. \\ & \left. - \sum_{u=1}^{M} \sum_{a=1}^{n_u} \beta_{iia}^{rru} \frac{S_{ii}^{rr}}{I_{ii}^{rr}} I_{ai}^{ur} - \frac{1}{2} \sum_{u=1}^{M} \sum_{a=1}^{n_u} (v_{iia}^{rru})^2 \frac{(S_{ii}^{rr})^2}{(I_{ii}^{rr})^2} (I_{ai}^{ur})^2 \right] dt \\ & - \sum_{u=1}^{M} \sum_{a=1}^{n_u} v_{iia}^{rru} I_{ai}^{ur} dw_{iia}^{rru}(t) + \sum_{u=1}^{M} \sum_{a=1}^{n_u} v_{iia}^{rru} \frac{S_{ii}^{rr}}{I_{ii}^{rr}} I_{ai}^{ur} dw_{iia}^{rru}(t) \end{aligned} \tag{7.3.11}$$

Intra-regional Level: From (7.3.8) the terms on the right-hand-side of (7.3.10) for the case of $u=r, a=j, j \neq i$

$$
\begin{aligned}
dV(X_{ij}^{rr}) \quad = \quad & \Bigg[\sigma_{ij}^{rr}\frac{S_{ii}^{rr}}{S_{ij}^{rr}} + \frac{\rho_j^r}{S_{ij}^{rr}}\int_0^\infty I_{ij}^{rr}(t-s)f_{ij}^{rr}(s)e^{-\delta_j^r s}ds \\
& -(\rho_{ij}^{rr}+\delta_j^r) - \sum_{u=1}^{M}\sum_{a=1}^{n_u}\beta_{jia}^{rru}I_{aj}^{ur} - \frac{1}{2}\sum_{u=1}^{M}\sum_{a=1}^{n_u}(v_{jia}^{rru})^2(I_{aj}^{ur})^2\Bigg]dt \\
& + \Bigg[\sigma_{ij}^{rr}\frac{I_{ii}^{rr}}{I_{ij}^{rr}} - \rho_j^r - (\rho_{ij}^{rr}+\delta_j^r+d_j^r) \\
& + \sum_{u=1}^{M}\sum_{a=1}^{n_u}\beta_{jia}^{rru}\frac{S_{ij}^{rr}}{I_{ij}^{rr}}I_{aj}^{ur} - \frac{1}{2}\sum_{u=1}^{M}\sum_{a=1}^{n_u}(v_{jia}^{rru})^2\frac{(S_{ij}^{rr})^2}{(I_{ij}^{rr})^2}(I_{aj}^{ur})^2\Bigg]dt \\
& - \sum_{u=1}^{M}\sum_{a=1}^{n_u}v_{jia}^{rru}I_{aj}^{ur}dw_{jia}^{rru}(t) + \sum_{u=1}^{M}\sum_{a=1}^{n_u}v_{jia}^{rru}\frac{S_{ij}^{rr}}{I_{ij}^{rr}}I_{aj}^{ur}dw_{jia}^{rru}(t) \qquad (7.3.12)
\end{aligned}
$$

Regional Level: From (7.3.8) the terms on the right-hand-side of (7.3.10) for the case of $u=q, q\neq r, a=l$,

$$
\begin{aligned}
dV(X_{il}^{rq}) \quad = \quad & \Bigg[\gamma_{il}^{rq}\frac{S_{ii}^{rr}}{S_{iq}^{rq}} + \frac{\rho_l^q}{S_{il}^{rq}}\int_0^\infty I_{il}^{rq}(t-s)f_{il}^{rq}(s)e^{-\delta_l^q s}ds \\
& -(\rho_{il}^{rq}+\delta_l^q) - \sum_{u=1}^{M}\sum_{a=1}^{n_u}\beta_{lia}^{qru}I_{al}^{uq} - \frac{1}{2}\sum_{u=1}^{M}\sum_{a=1}^{n_u}(v_{lia}^{qru})^2(I_{al}^{uq})^2\Bigg]dt \\
& + \Bigg[\gamma_{il}^{rq}\frac{I_{ii}^{rr}}{I_{il}^{rq}} - \rho_l^q - (\rho_{il}^{rq}+\delta_l^q+d_l^q) \\
& + \sum_{u=1}^{M}\sum_{a=1}^{n_u}\beta_{lia}^{qru}\frac{S_{il}^{rq}}{I_{il}^{rq}}I_{al}^{uq} - \frac{1}{2}\sum_{u=1}^{M}\sum_{a=1}^{n_u}(v_{lia}^{qru})^2\frac{(S_{il}^{rq})^2}{(I_{il}^{rq})^2}(I_{al}^{uq})^2\Bigg]dt \\
& - \sum_{u=1}^{M}\sum_{a=1}^{n_u}v_{lia}^{qru}I_{al}^{uq}dw_{lia}^{qru}(t) + \sum_{u=1}^{M}\sum_{a=1}^{n_u}v_{lia}^{qru}\frac{S_{il}^{rq}}{I_{il}^{rq}}I_{al}^{uq}dw_{lia}^{qru}(t) \qquad (7.3.13)
\end{aligned}
$$

It follows from (7.3.11)-(7.3.13), (7.3.10), and (7.3.7) that for $t < \tau_+(t)$,

$$\begin{aligned}
V(X_{00}^{00}(t)) - V(X_{00}^{00}(t_0)) \geq & \sum_{r=1}^{M}\sum_{i=1}^{n_r}\int_{t_0}^{t}\left[\frac{\rho_i^r}{S_{ii}^{rr}}\int_0^\infty I_{ii}^{rr}(t-s)f_{ii}^{rr}(s)e^{-\delta_i^r s}ds - (\gamma_i^r+\sigma_i^r+\delta_i^r)\right.\\
&\left. - \sum_{u=1}^{M}\sum_{a=1}^{n_u}\beta_{iia}^{rru}I_{ai}^{ur} - \frac{1}{2}\sum_{u=1}^{M}\sum_{a=1}^{n_u}(v_{iia}^{rru})^2(I_{ai}^{ur})^2\right]ds\\
&+\sum_{r=1}^{M}\sum_{i=1}^{n_r}\int_{t_0}^{t}\left[-\rho_i^r - (\gamma_i^r+\sigma_i^r+\delta_i^r+d_i^r)\right.\\
&\left. - \sum_{u=1}^{M}\sum_{a=1}^{n_u}\beta_{iia}^{rru}\frac{S_{ii}^{rr}}{I_{ii}^{rr}}I_{ai}^{ur} - \frac{1}{2}\sum_{u=1}^{M}\sum_{a=1}^{n_u}(v_{iia}^{rru})^2\frac{(S_{ii}^{rr})^2}{(I_{ii}^{rr})^2}(I_{ai}^{ur})^2\right]ds\\
&-\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u=1}^{M}\sum_{a=1}^{n_u}\int_{t_0}^{t}v_{iia}^{rru}I_{ai}^{ur}dw_{iia}^{rru}(s)\\
&+\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u=1}^{M}\sum_{a=1}^{n_u}\int_{t_0}^{t}v_{iia}^{rru}\frac{S_{ii}^{rr}}{I_{ii}^{rr}}I_{ai}^{ur}dw_{iia}^{rru}(s)\\
&\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{j\neq i}^{n_r}\int_{t_0}^{t}\left[\frac{\rho_j^r}{S_{ij}^{rr}}\int_0^\infty I_{ij}^{rr}(t-s)f_{ij}^{rr}(s)e^{-\delta_j^r s}ds\right.\\
&\left.-(\rho_{ij}^{rr}+\delta_j^r) - \sum_{u=1}^{M}\sum_{a=1}^{n_u}\beta_{jia}^{rru}I_{aj}^{ur} - \frac{1}{2}\sum_{u=1}^{M}\sum_{a=1}^{n_u}(v_{jia}^{rru})^2(I_{aj}^{ur})^2\right]ds\\
&+\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{j\neq i}^{n_r}\int_{t_0}^{t}\left[-\rho_j^r - (\rho_{ij}^{rr}+\delta_j^r+d_j^r)\right.\\
&\left.-\frac{1}{2}\sum_{u=1}^{M}\sum_{a=1}^{n_u}(v_{jia}^{rru})^2\frac{(S_{ij}^{rr})^2}{(I_{ij}^{rr})^2}(I_{aj}^{ur})^2\right]ds\\
&-\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{j\neq i}^{n_r}\sum_{u=1}^{M}\sum_{a=1}^{n_u}\int_{t_0}^{t}v_{jia}^{rru}I_{aj}^{ur}dw_{jia}^{rru}(s)\\
&+\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{j\neq i}^{n_r}\sum_{u=1}^{M}\sum_{a=1}^{n_u}\int_{t_0}^{t}v_{jia}^{rru}\frac{S_{ij}^{rr}}{I_{ij}^{rr}}I_{aj}^{ur}dw_{jia}^{rru}(s)\\
&+\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{q\neq r}^{M}\sum_{l=1}^{n_q}\int_{t_0}^{t}\left[\frac{\rho_l^q}{S_{il}^{rq}}\int_0^\infty I_{il}^{rq}(t-s)f_{il}^{rq}(s)e^{-\delta_l^q s}ds - (\rho_{il}^{rq}+\delta_l^q)\right.\\
&\left.-\sum_{u=1}^{M}\sum_{a=1}^{n_u}\beta_{lia}^{qru}I_{al}^{uq} - \frac{1}{2}\sum_{u=1}^{M}\sum_{a=1}^{n_u}(v_{lia}^{qru})^2(I_{al}^{uq})^2\right]ds\\
&+\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{q\neq r}^{M}\sum_{l=1}^{n_q}\int_{t_0}^{t}\left[-\rho_l^q - (\rho_{il}^{rq}+\delta_l^q+d_l^q)\right.\\
&\left.-\frac{1}{2}\sum_{u=1}^{M}\sum_{a=1}^{n_u}(v_{lia}^{qru})^2\frac{(S_{il}^{rq})^2}{(I_{il}^{rq})^2}(I_{al}^{uq})^2\right]ds\\
&-\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{q\neq r}^{M}\sum_{l=1}^{n_q}\sum_{u=1}^{M}\sum_{a=1}^{n_u}\int_{t_0}^{t}v_{lia}^{qru}I_{al}^{uq}dw_{lia}^{qru}(s)\\
&+\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{q\neq r}^{M}\sum_{l=1}^{n_q}\sum_{u=1}^{M}\sum_{a=1}^{n_u}\int_{t_0}^{t}v_{lia}^{qru}\frac{S_{il}^{rq}}{I_{il}^{rq}}I_{al}^{uq}dw_{lia}^{qru}(s) \quad (7.3.14)
\end{aligned}$$

Taking the limit on (7.3.14) as $t \to \tau_+(t)$, it follows from (7.3.8) and (7.3.7) that the left-hand-side $V(X_{00}^{00}(t)) - V(X_{00}^{00}(t_0)) \leq -\infty$ (since from (7.3.8) and (7.3.7), $V(X_{ia}^{ru}(\tau_+(t))) = \ln S_{ia}^{ru}(\tau_+(t)) + \ln I_{ia}^{ru}(\tau_+(t)) = -\infty$). This contradicts the finiteness of the right-hand-side of the inequality (7.3.14). Hence $\tau_+(t) = \tau_e(w)$ a.s. We show subsequently that $\tau_e(w) = \infty$.

Let $k > 0$ be a positive integer such that $||\varphi_{00}^{00}||_1 \leq k$, where the vector of initial values $\varphi_{00}^{00} = (\varphi_{ia}^{ru})_{1\leq r,u \leq M, 1\leq i \leq n_r, 1 \leq a \leq n_u} \in \mathbb{R}^{2n^2}$ is defined in (7.2.4). Furthermore, $||.||_1$ is the p-sum norm (3.2.19) for the case of $p = 1$. We define the stopping time

$$\begin{cases} \tau_k = sup\{t \in [t_0, \tau_e) : ||X_{00}^{00}(s)||_1 \leq k, s \in [0,t]\} \\ \tau_k(t) = \min(t, \tau_k). \end{cases} \tag{7.3.15}$$

where from (3.2.19),

$$||X_{00}^{00}(s)||_1 = \sum_{r=1}^{M} \sum_{u=1}^{M} \sum_{i=1}^{n_r} \sum_{a=1}^{n_u} (S_{ia}^{ru}(s) + I_{ia}^{ru}(s)). \tag{7.3.16}$$

It is easy to see that as $k \to \infty$, τ_k increases. Set $\lim_{k\to\infty} \tau_k(t) = \tau_\infty$. Then $\tau_\infty \leq \tau_e$ a.s. We show in the following that: (1.) $\tau_e = \tau_\infty \quad a.s. \Leftrightarrow P(\tau_e \neq \tau_\infty) = 0$, (2.) $\tau_\infty = \infty \quad a.s. \Leftrightarrow P(\tau_\infty = \infty) = 1$.

Suppose on the contrary that $P(\tau_\infty < \tau_e) > 0$. Let $w \in \{\tau_\infty < \tau_e\}$ and $t \leq \tau_\infty$. In the same structure form as (7.3.8) and (7.3.10), define

$$\begin{cases} V_1(X_{00}^{00}) = \sum_{r=1}^{M} \sum_{i=1}^{n_r} \sum_{u=1}^{M} \sum_{a=1}^{n_u} V(X_{ia}^{ru}), \\ V_1(X_{ia}^{ru}) = e^{\delta_a^u t}(S_{ia}^{ru} + I_{ia}^{ru}), \forall t \leq \tau_k(t). \end{cases} \tag{7.3.17}$$

From (7.3.17), using the expression (7.3.10), the Ito-Doob differential dV_1 with respect to the system (7.2.1)-(7.2.3) is given as follows:

Site Level: From (7.3.17), the terms of the right-hand-side of (7.3.10) for the case of $u = r, a = i$

$$\begin{aligned} dV_1(X_{ii}^{rr}) &= e^{\delta_i^r t}\Bigg[B_i^r + \sum_{k\neq i}^{n_r} \rho_{ik}^{rr} S_{ik}^{rr} + \sum_{q\neq r}^{M} \sum_{l=1}^{n_q} \rho_{ia}^{rq} S_{ia}^{rq} + \rho_i^r \int_0^\infty I_{ii}^{rr}(t-s) f_{ii}^{rr}(s) e^{-\delta_i^r s} ds \\ &\quad -(\gamma_i^r + \sigma_i^r) S_{ii}^{rr}\Bigg] dt + e^{\delta_i^r t}\Bigg[\sum_{k\neq i}^{n_r} \rho_{ik}^{rr} I_{ik}^{rr} + \sum_{q\neq r}^{M} \sum_{l=1}^{n_q} \rho_{ia}^{rq} I_{ia}^{rq} - \rho_i^r I_{ii}^{rr} \\ &\quad -(\gamma_i^r + \sigma_i^r + d_i^r) I_{ii}^{rr}\Bigg] dt \end{aligned} \tag{7.3.18}$$

Intra-regional Level: From (7.3.17), the terms of the right-hand-side of (7.3.10) for the case of $u = r, a = j, j \neq i$

$$
\begin{aligned}
dV_1(X_{ij}^{rr}) &= e^{\delta_i^r t}\left[\sigma_{ij}^{rr}S_{ii}^{rr} + \rho_j^r \int_0^\infty I_{ij}^{rr}(t-s)f_{ij}^{rr}(s)e^{-\delta_j^r s}ds - \rho_{ij}^{rr}S_{ij}^{rr}\right]dt \\
&\quad + e^{\delta_j^r t}\left[\sigma_{ij}^{rr}I_{ii}^{rr} + \rho_j^r I_{ij}^{rr} - (\rho_{ij}^{rr} + d_j^r)I_{ij}^{rr}\right]dt \qquad (7.3.19)
\end{aligned}
$$

Regional Level: From (7.3.17), the terms of the right-hand-side of (7.3.10) for the case of $u = q, q \neq r, a = l$

$$
\begin{aligned}
dV_1(X_{il}^{rq}) &= e^{\delta_l^q t}\left[\gamma_{il}^{rq}S_{ii}^{rr} + \rho_l^q \int_0^\infty I_{il}^{rq}(t-s)f_{il}^{rq}(s)e^{-\delta_l^q s}ds - \rho_{il}^{rq}S_{il}^{rq}\right]dt \\
&\quad + e^{\delta_l^q t}\left[\gamma_{il}^{rq}I_{ii}^{rr} + \rho_l^q I_{il}^{rq} - (\rho_{il}^{rq} + d_l^q)I_{il}^{rq}\right]dt \qquad (7.3.20)
\end{aligned}
$$

From (7.3.18)-(7.3.20), (7.3.10), integrating (7.3.10) over $[t_0, \tau]$ leads to the following

$$
\begin{aligned}
& V_1(X_{00}^{00}(\tau)) \\
= \; & V_1(X_{00}^{00}(t_0)) + \sum_{r=1}^{M}\sum_{i=1}^{n_r}\int_{t_0}^{\tau} e^{\delta_i^r s}\left[B_i^r + \sum_{k\neq i}^{n_r}\rho_{ik}^{rr}S_{ik}^{rr} + \sum_{q\neq r}^{M}\sum_{l=1}^{n_q}\rho_{ia}^{rq}S_{ia}^{rq} + \rho_i^r\int_0^\infty I_{ii}^{rr}(t-s)f_{ii}^{rr}(s)e^{-\delta_i^r s}ds\right. \\
& \left. -(\gamma_i^r + \sigma_i^r)S_{ii}^{rr}\right]ds + \sum_{r=1}^{M}\sum_{i=1}^{n_r}\int_{t_0}^{\tau} e^{\delta_i^r s}\left[\sum_{k\neq i}^{n_r}\rho_{ik}^{rr}I_{ik}^{rr} + \sum_{q\neq r}^{M}\sum_{l=1}^{n_q}\rho_{ia}^{rq}I_{ia}^{rq} - \rho_i^r I_{ii}^{rr}\right. \\
& \left. -(\gamma_i^r + \sigma_i^r + d_i^r)I_{ii}^{rr}\right]ds \\
& + \sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{j\neq i}^{n_r}\int_{t_0}^{\tau} e^{\delta_j^r s}\left[\sigma_{ij}^{rr}S_{ii}^{rr} + \rho_j^r\int_0^\infty I_{ij}^{rr}(t-s)f_{ij}^{rr}(s)e^{-\delta_j^r s}ds - \rho_{ij}^{rr}S_{ij}^{rr}\right]ds \\
& + \sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{j\neq i}^{n_r}\int_{t_0}^{\tau} e^{\delta_j^r s}\left[\sigma_{ij}^{rr}I_{ii}^{rr} - \rho_j^r I_{ij}^{rr} - (\rho_{ij}^{rr} + d_j^r)I_{ij}^{rr}\right]ds \\
& + \sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{q\neq r}^{n_r}\sum_{l=1}^{n_q}\int_{t_0}^{\tau} e^{\delta_l^q s}\left[\gamma_{il}^{rq}S_{ii}^{rr} + \rho_l^q\int_0^\infty I_{il}^{rq}(t-s)f_{il}^{rq}(s)e^{-\delta_l^q s}ds - \rho_{il}^{rq}S_{il}^{rq}\right]ds \\
& + \sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{q\neq r}^{n_r}\sum_{l=1}^{n_q}\int_{t_0}^{\tau} e^{\delta_l^q s}\left[\gamma_{il}^{rq}I_{ii}^{rr} - \rho_l^q I_{il}^{rq} - (\rho_{il}^{rq} + d_l^q)I_{il}^{rq}\right]ds \qquad (7.3.21)
\end{aligned}
$$

From (7.3.21), we let $\tau = \tau_k(t)$, where $\tau_k(t)$ is defined in (7.3.15). It is easy to see from (7.3.21), (7.3.15), (7.3.16), and (7.3.17) that

$$
k = ||X_{00}^{00}(\tau_k(t))||_1 \leq V_1(X_{00}^{00}(\tau_k(t))) \qquad (7.3.22)
$$

Taking the limit on (7.3.22) as $k \to \infty$ leads to a contradiction because the left-hand-side of the inequality (7.3.22) is infinite, and the right-hand-side is finite. Hence $\tau_e = \tau_\infty$ a.s. In the following, we show that $\tau_e = \tau_\infty = \infty$ a.s.

We let $w \in \{\tau_e < \infty\}$. Applying some algebraic manipulations and simplifications to (7.3.21), we have the following

$$
\begin{aligned}
& I_{\{\tau_e<\infty\}} V_1(X_{00}^{00}(\tau)) \\
= \; & I_{\{\tau_e<\infty\}} V_1(X_{00}^{00}(t_0)) + I_{\{\tau_e<\infty\}} \sum_{r=1}^{M}\sum_{i=1}^{n_r} \frac{B_i^r}{\delta_i^r}(e^{\delta_i^r \tau}-1) \\
& + I_{\{\tau_e<\infty\}} \sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{q=1}^{M}\sum_{l=1}^{n_q} \int_0^\infty f_{il}^{rq}(t)\left[\rho_l^q \int_{-t}^{t_0} I_{il}^{rq}(s) e^{\delta_l^q s} ds - \rho_l^q \int_{\tau-t}^{\tau} I_{il}^{rq}(s) e^{\delta_l^q s} ds\right] dt \\
& - I_{\{\tau_e<\infty\}} \sum_{r=1}^{M}\sum_{i=1}^{n_r} \int_{t_0}^{\tau}\left[\sigma_i^r e^{\delta_i^r s} - \sum_{j\neq i}^{n_r} \sigma_{ij}^{rr} e^{\delta_j^r s}\right](S_{ii}^{rr}+I_{ii}^{rr}) ds \\
& - I_{\{\tau_e<\infty\}} \sum_{r=1}^{M}\sum_{i=1}^{n_r} \int_{t_0}^{\tau}\left[\gamma_i^r e^{\delta_i^r s} - \sum_{q=1}^{M}\sum_{l=1}^{n_q} \gamma_{il}^{rq} e^{\delta_l^q s}\right](S_{ii}^{rr}+I_{ii}^{rr}) ds \\
& - I_{\{\tau_e<\infty\}} \sum_{r=1}^{M}\sum_{i=1}^{n_r} d_i^r \int_{t_0}^{\tau} I_{ii}^{rr} e^{\delta_i^r s} ds \\
& - I_{\{\tau_e<\infty\}} \sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{j\neq i}^{n_r} d_j^r \int_{t_0}^{\tau} I_{ij}^{rr} e^{\delta_j^r s} ds \\
& - I_{\{\tau_e<\infty\}} \sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{q=1}^{M}\sum_{l=1}^{n_q} d_l^q \int_{t_0}^{\tau} I_{il}^{rq} e^{\delta_l^q s} ds,
\end{aligned}
\tag{7.3.23}
$$

where I_A is the indicator function of the set A.

We recall [30], $\sigma_i^r = \sum_{j\neq i}^{n_r} \sigma_{ij}^{rr}$ and $\gamma_i^r = \sum_{q\neq r}^{M}\sum_{l=1}^{n_q}\gamma_{il}^{rq}$. Hence the fourth and fifth terms on the right-hand-side of (7.3.23) are such that $\left[\sigma_i^r e^{\delta_i^r s} - \sum_{j\neq i}^{n_r}\sigma_{ij}^{rr} e^{\delta_j^r s}\right] \geq 0, \forall \delta_i^r \geq \delta_j^r, j\neq i$ and $\left[\gamma_i^r e^{\delta_i^r s} - \sum_{q=1}^{M}\sum_{l=1}^{n_q}\gamma_{il}^{rq} e^{\delta_l^q s}\right] \geq 0, \forall \delta_i^r \geq \delta_l^q, q\neq r, l \in I(1,n_q)$. We now let $\tau = \tau_k(t)\wedge T$ in (7.3.23), $\exists T>0$, where $\tau_k(t)$ is defined in (7.3.15). The expected value of (7.3.23) is estimated as follows

$$
\begin{aligned}
E\left[I_{\{\tau_e<\infty\}} V_1(X_{00}^{00}(\tau_k(t)\wedge T))\right] \;\; \leq \;\; & V_1(X_{00}^{00}(t_0)) + \sum_{i=1}^{n_r} \frac{B_i^r}{\delta_i^r} e^{\delta_i^r \tau_k(t)\wedge T} \\
& + \sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{q=1}^{M}\sum_{l=1}^{n_q} \int_0^\infty f_{il}^{rq}(t)\left[\rho_l^q \int_{-t}^{t_0} \varphi_{il2}^{rq}(s) e^{\delta_l^q s} ds\right] dt
\end{aligned}
\tag{7.3.24}
$$

Furthermore, from (7.3.16), (7.3.17) and the definition of the indicator function I_A it follows that

$$I_{\{\tau_e<\infty,\tau_k(t)\leq T\}}||X_{00}^{00}(\tau_k(t))||_1 \leq I_{\{\tau_e<\infty\}}V_1(X_{00}^{00}(\tau_k(t)\wedge T)) \tag{7.3.25}$$

It follows from (7.3.24), (7.3.25) and (7.3.15) that

$$\begin{aligned} P(\{\tau_e<\infty,\tau_k(t)\leq T\})k &= E\left[I_{\{\tau_e<\infty,\tau_k(t)\leq T\}}||X_{00}^{00}(\tau_k(t))||_1\right] \\ &\leq E\left[I_{\{\tau_e<\infty\}}V(X_{00}^{00}(\tau_k(t)\wedge T))\right] \\ &\leq V_1(X_{00}^{00}(t_0))+\sum_{i=1}^{n_r}\frac{B_i^r}{\delta_i^r}e^{\delta_i^r T} \\ &\quad +\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{q=1}^{M}\sum_{l=1}^{n_q}\int_0^\infty f_{il}^{rq}(t)\left[\rho_l^q\int_{-t}^{t_0}\varphi_{il2}^{rq}(s)e^{\delta_l^q s}ds\right]dt \end{aligned} \tag{7.3.26}$$

It follows immediately from (7.3.26) that $P(\{\tau_e<\infty,\tau_\infty\leq T\})\to 0$ as $k\to\infty$. Furthermore, since $T<\infty$ is arbitrary, we conclude that $P(\{\tau_e<\infty,\tau_\infty<\infty\})=0$.

Finally, by the total probability principle,

$$\begin{aligned} P(\{\tau_e<\infty\}) &= P(\{\tau_e<\infty,\tau_\infty=\infty\})+P(\{\tau_e<\infty,\tau_\infty<\infty\}) \\ &\leq P(\{\tau_e\neq\tau_\infty\})+P(\{\tau_e<\infty,\tau_\infty<\infty\}) \\ &= 0. \end{aligned} \tag{7.3.27}$$

Thus from (7.3.27), $\tau_e=\tau_\infty=\infty$ a.s. as was required to show.

Remark 7.3.1 *For any $r\in I(1,M)$ and $i\in I(1,n_r)$, Theorem 7.3.1 signifies that the number of residents of site s_i^r of all categories present at home site s_i^r, or visiting intra and inter-regional sites s_j^r and s_l^q respectively, are nonnegative. This implies that the total number of residents of site s_i^r present at home site and also visiting sites in regions in their intra and inter-regional accessible domains[66], given by the sum $N_{i0}^{rr}(t)=\sum_{u=1}^{M}\sum_{a=1}^{n_u}y_{ia}^{ru}$, is nonnegative. Moreover, the total effective population[66], defined by $eff(N_{i0}^{rr})(t)=\sum_{u=1}^{M}\sum_{a=1}^{n_u}y_{ai}^{ur}$, at any site s_i^r in region C_r is also nonnegative at all time $t\geq t_0$.*

The following result defines an upper bound for the solution process of the system (7.2.1)-(7.2.3). We utilize Theorem 7.3.1 to establish this result.

Theorem 7.3.2 *Suppose the hypotheses of Theorem 7.3.1 is satisfied. Let* $\mu = \min_{1\leq u\leq M, 1\leq a\leq n_u}(\delta_a^u)$. *If*

$$\sum_{r=1}^{M}\sum_{u=1}^{M}\sum_{i=1}^{n_r}\sum_{a=1}^{n_u} y_{ia}^{ru}(t_0) \leq \frac{1}{\mu}\sum_{r=1}^{M}\sum_{i=1}^{n_r} B_i^r, \tag{7.3.28}$$

then

$$\sum_{r=1}^{M}\sum_{u=1}^{M}\sum_{i=1}^{n_r}\sum_{a=1}^{n_u} y_{ia}^{ru}(t) \leq \frac{1}{\mu}\sum_{r=1}^{M}\sum_{i=1}^{n_r} B_i^r, \quad for \quad t \geq t_0 \quad a.s. \tag{7.3.29}$$

Proof: See [[66],Lemma 3.2]

Remark 7.3.2 *From Theorem 7.3.1 and Theorem 7.3.2, we conclude that a closed ball* $\bar{\mathfrak{B}}_{R^{3n^2}}(\vec{0};r)$ *in* R^{3n^2} *under the sum norm* $||.||_1$ *centered at the origin* $\vec{0} \in R^{3n^2}$, *with radius* $r = \frac{1}{\mu}\sum_{r=1}^{M}\sum_{i=1}^{n_r} B_i^r$ *is self-invariant with regard to a two-scale network dynamics of human epidemic process (7.2.1)-(7.2.3) that is under the influence of human mobility process[30]. That is,*

$$\bar{\mathfrak{B}}_{R^{3n^2}}(\vec{0};r) = \left\{(S_{ia}^{ru}, I_{ia}^{ru}, R_{ia}^{ru}) : y_{ia}^{ru}(t) \geq 0 \quad and \quad ||x_{00}^{00}||_1 = \sum_{r=1}^{M}\sum_{u=1}^{M}\sum_{i=1}^{n_r}\sum_{a=1}^{n_u} y_{ia}^{ru}(t) \leq \frac{1}{\mu}\sum_{r=1}^{M}\sum_{i=1}^{n_r} B_i^r\right\} \tag{7.3.30}$$

is a positive self-invariant set for system (7.2.1)-(7.2.3). We shall denote

$$\bar{B} \equiv \frac{1}{\mu}\sum_{r=1}^{M}\sum_{i=1}^{n_r} B_i^r \tag{7.3.31}$$

7.4 Existence and Asymptotic Behavior of Disease Free Equilibrium

In this section, we study the existence and the asymptotic behavior of the disease free equilibrium state of the system (7.2.1)-(7.2.3). The disease free equilibrium is obtained by solving the system of algebraic equations obtained by setting the drift and the diffusion parts of the system of stochastic differential equations to zero. In addition, we utilize the conditions that $I = R = 0$ in the event when there is no disease in the population. We summarize the results in the following. For any $r,u \in I(1,M)$, $i \in I(1,n_r)$ and $a \in I(1,n_u)$, let

$$D_i^r = \gamma_i^r + \sigma_i^r + \delta_i^r - \sum_{a=1}^{n_r}\frac{\rho_{ia}^{rr}\sigma_{ia}^{rr}}{\rho_{ia}^{rr}+\delta_a^r} - \sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\frac{\rho_{ia}^{rr}\gamma_{ia}^{ru}}{\rho_{ia}^{ru}+\delta_a^u} > 0. \tag{7.4.32}$$

Furthermore, let $(S^{ru*}_{ia}, I^{ru*}_{ia}, R^{ru*}_{ia})$, be the equilibrium state of the delayed system (7.2.1)-(7.2.3). One can see that the disease free equilibrium state is given by $E^{ru}_{ia} = (S^{ru*}_{ia}, 0, 0)$, where

$$S^{ru*}_{ia} = \begin{cases} \frac{B^r_i}{D^r_i}, \quad for \quad u = r, a = i, \\ \frac{B^r_i}{D^r_i} \frac{\sigma^{rr}_{ij}}{\rho^{rr}_{ij} + \delta^r_j}, \quad for \quad u = r, a \neq i, \\ \frac{B^r_i}{D^r_i} \frac{\gamma^{ru}_{ia}}{\rho^{ru}_{ia} + \delta^u_a}, \quad for \quad u \neq r. \end{cases} \tag{7.4.33}$$

The asymptotic stability property of E^{ru}_{ia} will be established by verifying the conditions of the stochastic version of the Lyapunov second method given in [[34],Theorem 2.4],[59],
and [[34],Theorem 4.4], respectively. In order to study the qualitative properties of (7.2.1)-(7.2.3) with respect to the equilibrium state $(S^{ru*}_{ia}, 0, 0)$, first, we use the change of variable that shifts the equilibrium to the origin. For this purpose, we use the following transformation:

$$\begin{cases} U^{ru}_{ia} &= S^{ru}_{ia} - S^{ru*}_{ia} \\ V^{ru}_{ia} &= I^{ru}_{ia} \\ W^{ru}_{ia} &= R^{ru}_{ia}. \end{cases} \tag{7.4.34}$$

By employing this transformation, system (7.2.1)-(7.2.3) is transformed into the following forms

$$dU^{rq}_{il} = \begin{cases} \big[\sum_{q=1}^{M}\sum_{a=1}^{n_q} \rho^{rq}_{ia} U^{rq}_{ia} + \rho^r_i \int_0^\infty V^{rr}_{ii}(t-s) f^{rr}_{ii}(s) e^{-\delta^r_i s} ds \\ -(\gamma^r_i + \sigma^r_i + \delta^r_i) U^{rr}_{ii} - \sum_{u=1}^{M}\sum_{a=1}^{n_u} \beta^{rru}_{iia}(S^{rr*}_{ii} + U^{rr}_{ii}) V^{ur}_{ai}\big] dt \\ -\big[\sum_{u=1}^{M}\sum_{a=1}^{n_u} \nu^{rru}_{iia}(S^{rr*}_{ii} + U^{rr}_{ii}) V^{ur}_{ai} dw^{rru}_{iia}(t)\big], for \quad q = r, l = i \\ \big[\sigma^{rr}_{ij} U^{rr}_{ii} + \rho^r_j \int_0^\infty V^{rr}_{ij}(t-s) f^{rr}_{ij}(s) e^{-\delta^r_j s} ds - (\rho^{rr}_{ij} + \delta^r_j) U^{rr}_{ij} \\ -\sum_{u=1}^{M}\sum_{a=1}^{n_u} \beta^{rru}_{jia}(S^{rr*}_{ij} + U^{rr}_{ij}) V^{ur}_{aj}\big] dt \\ -\big[\sum_{u=1}^{M}\sum_{a=1}^{n_u} \nu^{rru}_{jia}(S^{rr*}_{ij} + U^{rr}_{ij}) V^{ur}_{aj} dw^{rru}_{jia}(t)\big], for \quad q = r, l = j, j \neq i, \\ \big[\gamma^{rq}_{il} U^{rr}_{ii} + \rho^q_l \int_0^\infty V^{rq}_{il}(t-s) f^{rq}_{il}(s) e^{-\delta^q_l s} ds - (\rho^{rq}_{il} + \delta^q_l) U^{rq}_{il} \\ -\sum_{u=1}^{M}\sum_{a=1}^{n_u} \beta^{qru}_{lia} S^{rq}_{il} I^{uq}_{al}\big] dt - \big[\sum_{u=1}^{M}\sum_{a=1}^{n_u} \nu^{qru}_{lia}(S^{rq*}_{il} + U^{rq}_{il}) V^{uq}_{al} dw^{qru}_{lia}(t)\big], for \quad q \neq r, \end{cases} \tag{7.4.35}$$

$$dV_{il}^{rq}=\begin{cases}\big[\sum_{q=1}^{M}\sum_{a=1}^{n_q}\rho_{ia}^{rq}V_{ia}^{rq}-(\rho_i^r+\gamma_i^r+\sigma_i^r+\delta_i^r+d_i^r)W_{ii}^{rr}\\ +\sum_{u=1}^{M}\sum_{a=1}^{n_u}\beta_{iia}^{rru}(S_{ii}^{rr*}+U_{ii}^{rr})V_{ai}^{ur}\big]dt\\ +\big[\sum_{u=1}^{M}\sum_{a=1}^{n_u}\nu_{iia}^{rru}(S_{ii}^{rr*}+U_{ii}^{rr})V_{ai}^{ur}dw_{iia}^{rru}(t)\big],for\quad q=r,l=i\\ \big[\sigma_{ij}^{rr}V_{ii}^{rr}-(\rho_j^r+\rho_{ij}^{rr}+\delta_j^r+d_j^r)V_{ij}^{rr}+\sum_{u=1}^{M}\sum_{a=1}^{n_u}\beta_{jia}^{rru}(S_{ij}^{rr*}+U_{ij}^{rr})V_{aj}^{ur}\big]dt\\ +\big[\sum_{u=1}^{M}\sum_{a=1}^{n_u}\nu_{jia}^{rru}(S_{ij}^{rr*}+U_{ij}^{rr})V_{aj}^{ur}dw_{jia}^{rru}(t)\big],for\quad q=r,l=j,j\neq i,\\ \big[\gamma_{il}^{rq}V_{ii}^{rr}-(\rho_l^q+\rho_{il}^{rq}+\delta_l^q+d_l^q)V_{il}^{rq}\\ \sum_{u=1}^{M}\sum_{a=1}^{n_u}\beta_{lia}^{qru}(S_{il}^{rq*}+U_{il}^{rq})V_{al}^{uq}\big]dt+\big[\sum_{u=1}^{M}\sum_{a=1}^{n_u}\nu_{lia}^{qru}(S_{il}^{rq*}+U_{il}^{rq})V_{al}^{uq}dw_{lia}^{qru}(t)\big],\\ for\quad q\neq r,\end{cases}\tag{7.4.36}$$

and

$$dW_{il}^{rq}=\begin{cases}\big[\sum_{q=1}^{M}\sum_{l=1}^{n_q}\rho_{il}^{rq}W_{il}^{rq}+\rho_i^rV_{ii}^{rr}-\rho_i^r\int_0^\infty V_{ii}^{rr}(t-s)f_{ii}^{rr}(s)e^{-\delta_i^rs}ds-(\gamma_i^r+\sigma_i^r+\delta_i^r)W_{ii}^{rr}\big]dt,\\ for\quad q=r,l=i\\ \big[\sigma_{ij}^{rr}W_{ii}^{rr}+\rho_j^rV_{ij}^{rr}-\rho_j^r\int_0^\infty V_{ij}^{rr}(t-s)f_{ij}^{rr}(s)e^{-\delta_j^rs}ds-(\rho_{ij}^{rr}+\delta_j^r)W_{ij}^{rr}\big]dt,\\ for\quad q=r,l=j,j\neq i\\ \big[\gamma_{il}^{rq}W_{ii}^{rr}+\rho_l^qV_{il}^{rq}-\rho_l^q\int_0^\infty V_{il}^{rq}(t-s)f_{il}^{rq}(s)e^{-\delta_l^qs}ds-(\rho_{il}^{rq}+\delta_l^q)W_{il}^{rq}\big]dt,for\quad q\neq r\end{cases}\tag{7.4.37}$$

We state and prove the following lemmas that would be useful in the proofs of the stability results.

Lemma 7.4.1 *Let $V_1:\mathbb{R}^{3n^2}\times\mathbb{R}_+\to\mathbb{R}_+$ be a function defined by*

$$\begin{cases}V_1(\tilde{x}_{00}^{00}) &=& \sum_{r=1}^{M}\sum_{u=1}^{M}\sum_{i=1}^{n_r}\sum_{a=1}^{n_u}V(\tilde{x}_{ia}^{ru}),\\ V_1(\tilde{x}_{ia}^{ru}) &=& (S_{ia}^{ru}-S_{ia}^{ru*}+I_{ia}^{ru})^2+c_{ia}^{ru}(I_{ia}^{ru})^2+(R_{ia}^{ru})^2\\ \tilde{x}_{00}^{00} &=& (U_{ia}^{ru},V_{ia}^{ru},W_{ia}^{ru})^T\quad and\quad c_{ia}^{ru}\geq 0.\end{cases}\tag{7.4.38}$$

Then $V_1\in\mathcal{C}^{2,1}(\mathbb{R}^{3n^2}\times\mathbb{R}_+,\mathbb{R}_+)$, and it satisfies

$$b(||\tilde{x}_{00}^{00}||)\quad\leq\quad V_1(\tilde{x}_{00}^{00}(t))\leq a(||\tilde{x}_{00}^{00}||)\tag{7.4.39}$$

where

$$b(||\tilde{x}_{00}^{00}||)\quad=\quad\min_{1\leq r,u\leq M,1\leq i\leq n_r,1\leq a\leq n_u}\left\{\frac{c_{ia}^{ru}}{2+c_{ia}^{ru}}\right\}\sum_{r=1}^{M}\sum_{u=1}^{M}\sum_{i=1}^{n_r}\sum_{a=1}^{n_u}\left[(U_{ia}^{ru})^2+(V_{ia}^{ru})^2+(W_{ia}^{ru})^2\right],$$

and

$$a(||\tilde{x}_{00}^{00}||) \quad = \quad \max_{1\leq r,u\leq M, 1\leq i\leq n_r, 1\leq a\leq n_u} \{c_{ia}^{ru}+2\} \sum_{r=1}^{M}\sum_{u=1}^{M}\sum_{i=1}^{n_r}\sum_{a=1}^{n_u} \left[(U_{ia}^{ru})^2+(V_{ia}^{ru})^2+(W_{ia}^{ru})^2\right]. \tag{7.4.40}$$

Proof: See [(Chapter3,Lemma 7.4.1) or ([66], Lemma 4.1)].

Remark 7.4.1 *Lemma 7.4.1 shows that the Lyapunov function V defined in (7.4.38) is positive definite, decrescent and radially unbounded ((7.4.39)) function[34, 59].*

We now state the following lemma

Lemma 7.4.2 *Assume that the hypothesis of Lemma 7.4.1 is satisfied. Define a Lyapunov functional*

$$V = V_1 + V_2, \tag{7.4.41}$$

where V_1 is defined by (7.4.38), and

$$V_2 = 3\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u=1}^{M}\sum_{a=1}^{n_u}\left[\frac{(\rho_a^u)^2}{\mu_{ia}^{ru}}\int_0^\infty\left(f_{ia}^{rr}(s)e^{-2\delta_a^r s}\int_{t-s}^{t}(V_{ia}^{ru}(\theta))^2 d\theta\right)ds\right], \tag{7.4.42}$$

Furthermore, let

$$\mathfrak{U}_{ia}^{ru} = \begin{cases} \dfrac{\left[\sum_{u=1}^{M}\sum_{a=1}^{n_u}\mu_{ia}^{ru}+\sum_{a\neq i}^{n_r}\frac{(\sigma_{ia}^{rr})^2}{\mu_{ii}^{rr}}+\sum_{a\neq r}^{M}\sum_{a=1}^{n_r}\frac{(\gamma_{ia}^{ru})^2}{\mu_{ii}^{rr}}+2\mu_{ii}^{rr}\right]}{(\gamma_i^r+\sigma_i^r+\delta_i^r)} for \quad u=r, i=a \\ \dfrac{\left[\frac{(\rho_{ia}^{rr})^2}{\mu_{ia}^{rr}}+\mu_{ii}^{rr}+\frac{3}{2}\mu_{ia}^{rr}\right]}{(\rho_{ia}^{rr}+\delta_a^r)}, for \quad u=r, a\neq i \\ \dfrac{\left[\frac{(\rho_{ia}^{ru})^2}{\mu_{ia}^{ru}}+\mu_{ii}^{rr}+\frac{3}{2}\mu_{ia}^{ru}\right]}{(\rho_{ia}^{ru}+\delta_a^u)}, for \quad u\neq r, \end{cases} \tag{7.4.43}$$

$$\mathfrak{V}_{ia}^{ru} = \begin{cases} \dfrac{\frac{1}{2}\sum_{u=1}^{M}\sum_{a=1}^{n_u}\mu_{ia}^{ru}+\frac{1}{2}\sum_{v=1}^{M}\sum_{b=1}^{n_v}\beta_{iib}^{rrv}(S_{ii}^{rr*}\mu_{ii}^{rr}+\mu_{ii}^{rr})+\frac{1}{2}d_{ii}^{rr}}{\rho_i^r+\gamma_i^r+\sigma_i^r+\delta_i^r+d_i^r}, for \quad a=i, u=r \\ \dfrac{\frac{1}{2}\mu_{ii}^{rr}+\frac{1}{2}\sum_{v=1}^{M}\sum_{b=1}^{n_v}\beta_{aib}^{rrv}(S_{ia}^{rr*}\mu_{ia}^{rr}+\mu_{ia}^{rr})+\frac{1}{2}d_{ai}^{rr}}{\rho_a^r+\rho_{ia}^{rr}+\delta_a^r+d_a^r}, for \quad a\neq i, u=r \\ \dfrac{\frac{1}{2}\mu_{ii}^{rr}+\frac{1}{2}\sum_{v=1}^{M}\sum_{b=1}^{n_v}\beta_{aib}^{urv}(S_{ii}^{ru*}\mu_{ia}^{ru}+\mu_{ia}^{ru})+\frac{1}{2}d_{ai}^{ur}}{\rho_a^u+\rho_{ia}^{ru}+\delta_a^u+d_a^u}, for \quad u\neq r. \end{cases} \tag{7.4.44}$$

and

$$\mathfrak{W}_{ia}^{ru} = \begin{cases} \frac{\left[\frac{1}{2}\sum_{u=1}^{M}\sum_{a=1}^{n_u}\mu_{ia}^{ru}+\frac{1}{2}\sum_{u\neq r}^{M}\sum_{a=1}^{nr}\frac{(\gamma_{ia}^{ru})^2}{\mu_{ii}^{rr}}+\frac{1}{2}\sum_{a\neq i}^{nr}\frac{(\sigma_{ia}^{rr})^2}{\mu_{ii}^{rr}}+\mu_{ii}^{rr}\right]}{(\gamma_i^r+\sigma_i^r+\delta_i^r)}, for \quad u=r, a=i, \\ \frac{\left[\frac{1}{2}\frac{(\rho_{ia}^{rr})^2}{\mu_{ia}^{rr}}+\frac{1}{2}\mu_{ii}^{rr}+\mu_{ia}^{rr}\right]}{(\rho_{ia}^{rr}+\delta_a^r)}, for \quad u=r, a\neq i, \\ \frac{\left[\frac{1}{2}\frac{(\rho_{ia}^{ru})^2}{\mu_{ia}^{ru}}+\frac{1}{2}\mu_{ii}^{rr}+\mu_{ia}^{ru}\right]}{(\rho_{ia}^{ru}+\delta_a^u)}, for \quad u\neq r \end{cases} \tag{7.4.45}$$

for some suitably defined positive number μ_{ia}^{ru}, depending on δ_a^u, for all $r,u \in I^r(1,M)$, $i \in I(1,n)$ and $a \in I_i^r(1,n_r)$. Assume that $\mathfrak{U}_{ia}^{ru} \leq 1$, $\mathfrak{V}_{ia}^{ru} < 1$ and $\mathfrak{W}_{ia}^{ru} \leq 1$. There exist positive numbers ϕ_{ia}^{ru}, ψ_{ia}^{ru} and φ_{ia}^{ru} such that the differential operator LV associated with Ito-Doob type stochastic system (7.2.1)-(7.2.3)satisfies the following inequality

$$\begin{aligned} LV(\tilde{x}_{00}^{00}) \quad \leq \quad & \sum_{r=1}^{M}\sum_{i=1}^{n_r}\Big[-[\phi_{ii}^{rr}(U_{ii}^{rr})^2+\psi_{ii}^{rr}(V_{ii}^{rr})^2+\varphi_{ii}^{rr}(W_{ii}^{rr})^2] \\ & -\sum_{a\neq i}^{n_r}[\phi_{ia}^{rr}(U_{ia}^{rr})^2+\psi_{ia}^{rr}(V_{ia}^{rr})^2+\varphi_{ia}^{rr}(W_{ia}^{rr})^2] \\ & -\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}[\phi_{ia}^{ru}(U_{ia}^{rr})^2+\psi_{ia}^{ru}(V_{ia}^{ru})^2+\varphi_{ia}^{ru}(W_{ia}^{ru})^2]\Bigg]. \end{aligned} \tag{7.4.46}$$

Moreover,

$$LV(\tilde{x}_{00}^{00}) \quad \leq \quad -cV_1(\tilde{x}_{00}^{00}) \tag{7.4.47}$$

where a positive constant c is defined by

$$c = \frac{\min_{1\leq r,u\leq M, 1\leq i\leq n_r, 1\leq a\leq n_u}\{\phi_{ia}^{ru}, \psi_{ia}^{ru}, \varphi_{ia}^{ru}\}}{\max_{1\leq r,u\leq M, 1\leq i\leq n_r, 1\leq a\leq n_u}\{C_{ia}^{ru}+2\}} \tag{7.4.48}$$

Proof:

The computation of differential operator[34, 59] applied to the Lyapunov function V_1 in (7.4.38) with respect to the large-scale system of Ito-Doob type stochastic differential equation (7.2.1)-(7.2.3) is as follows:

$$LV_1(\tilde{x}_{00}^{00}) = \sum_{r=1}^{M}\sum_{i=1}^{n_r}\left[LV_1(\tilde{x}_{ii}^{rr})+\sum_{j\neq i}^{n_r}LV_1(\tilde{x}_{ij}^{rr})+\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}LV_1(\tilde{x}_{ia}^{ru})\right], \tag{7.4.49}$$

where,

$$
\begin{aligned}
LV_1(\tilde{x}_{ii}^{rr}) \quad = \quad & 2\sum_{u=1}^{M}\sum_{a=1}^{n_u}[(1+C_{ii}^{rr})\rho_{ia}^{ru}V_{ia}^{ru}V_{ii}^{rr}+\rho_{ia}^{ru}U_{ia}^{ru}U_{ii}^{rr}+\rho_{ia}^{ru}V_{ia}^{ru}U_{ii}^{rr}+\rho_{ia}^{ru}U_{ia}^{ru}V_{ii}^{rr} \\
& +\rho_{ia}^{ru}W_{ia}^{ru}W_{ii}^{rr}]+2\rho_i^rU_{ii}^{rr}\int_0^\infty V_{ii}^{rr}(t-s)f_{ii}^{rr}(s)e^{-\delta_i^r s}ds+2\rho_i^rV_{ii}^{rr}\int_0^\infty V_{ii}^{rr}(t-s)f_{ii}^{rr}(s)e^{-\delta_i^r s}ds \\
& -2\rho_i^rW_{ii}^{rr}\int_0^\infty V_{ii}^{rr}(t-s)f_{ii}^{rr}(s)e^{-\delta_i^r s}ds-2\rho_i^rV_{ii}^{rr}W_{ii}^{rr} \\
& -2[(\rho_i^r+d_i^r)+2(\gamma_i^r+\sigma_i^r+\delta_i^r)]V_{ii}^{rr}U_{ii}^{rr}-2(\gamma_i^r+\sigma_i^r+\delta_i^r)(U_{ii}^{rr})^2 \\
& -2[(c_{ii}^{rr}+1)\rho_i^r+2(c_{ii}^{rr}+1)(\gamma_i^r+\sigma_i^r+\delta_i^r+d_i^r)](V_{ii}^{rr})^2-2(\gamma_i^r+\sigma_i^r+\alpha_i^r+\delta_i^r)(W_{ii}^{rr})^2 \\
& +2c_{ii}^{rr}\sum_{u=1}^{M}\sum_{a=1}^{n_u}\beta_{iia}^{rru}(S_{ii}^{rr*}+U_{ii}^{rr})V_{ai}^{ur}V_{ii}^{rr}+c_{ii}^{rr}\sum_{u=1}^{M}\sum_{a=1}^{n_u}(v_{iia}^{rru})^2(S_{ii}^{rr*}+U_{ii}^{rr})^2(V_{ai}^{ur})^2, \\
& for \quad u=r, a=i
\end{aligned}
\tag{7.4.50}
$$

$$
\begin{aligned}
\sum_{a\neq i}^{n_r}LV_1(\tilde{x}_{ia}^{rr}) \quad = \quad & \sum_{a\neq r}^{n_r}\{2(1+c_{ia}^{rr})\sigma_{ia}^{rr}V_{ia}^{rr}V_{ii}^{rr}+2\sigma_{ia}^{rr}U_{ia}^{rr}U_{ii}^{rr}+2\sigma_{ia}^{rr}V_{ia}^{rr}U_{ii}^{rr}+2\sigma_{ia}^{rr}U_{ia}^{rr}V_{ii}^{rr}+2\sigma_{ia}^{rr}W_{ia}^{rr}W_{ii}^{rr} \\
& +2\rho_a^rU_{ia}^{rr}\int_0^\infty V_{ia}^{rr}(t-s)f_{ia}^{rr}(s)e^{-\delta_a^r s}ds+2\rho_a^rV_{ia}^{rr}\int_0^\infty V_{ia}^{rr}(t-s)f_{ia}^{rr}(s)e^{-\delta_a^r s}ds \\
& -2\rho_a^rW_{ia}^{rr}\int_0^\infty V_{ia}^{rr}(t-s)f_{ia}^{rr}(s)e^{-\delta_a^r s}ds-2[(c_{ia}^{rr}+1)\rho_a^r+2(c_{ia}^{rr}+1)(\rho_{ia}^{rr}+\delta_a^r)](V_{ia}^{rr})^2 \\
& -2(\rho_{ia}^{rr}+\delta_a^r)(U_{ia}^{rr})^2-2(\rho_{ia}^{rr}+\delta_a^r)(W_{ia}^{rr})^2+2\rho_a^rV_{ia}^{rr}W_{ia}^{rr} \\
& -2[(\rho_a^r+d_a^r)+2(\rho_{ia}^{rr}+\delta_a^r)]V_{ia}^{rr}U_{ia}^{rr}\}+2\sum_{a\neq i}^{n_r}c_{ia}^{rr}\sum_{v=1}^{M}\sum_{b=1}^{n_v}\beta_{aib}^{rrv}(S_{ia}^{rr*}+U_{ia}^{rr})V_{ba}^{vr}V_{ia}^{rr} \\
& +\sum_{a\neq i}^{n_r}c_{ia}^{rr}\sum_{v=1}^{M}\sum_{b=1}^{n_v}(v_{aib}^{rrv})^2(S_{ia}^{rr*}+U_{ia}^{rr})^2(V_{ba}^{vr})^2\,, for \quad u=r, \quad a\neq i
\end{aligned}
\tag{7.4.51}
$$

$$
\begin{aligned}
\sum_{u\neq r}^{M}\sum_{a=1}^{n_r}LV_1(\tilde{x}_{ia}^{ru}) \quad = \quad & \sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\{2(1+c_{ia}^{ru})\gamma_{ia}^{ru}V_{ia}^{ru}V_{ii}^{rr}+2\gamma_{ia}^{ru}U_{ia}^{ru}U_{ii}^{rr}+2\gamma_{ia}^{ru}V_{ia}^{ru}U_{ii}^{rr}+2\gamma_{ia}^{ru}U_{ia}^{ru}V_{ii}^{rr} \\
& +2\gamma_{ia}^{ru}W_{ia}^{ru}W_{ii}^{rr}+2\rho_a^uU_{ia}^{ru}\int_0^\infty V_{ia}^{ru}(t-s)f_{ia}^{ru}(s)e^{-\delta_a^u s}ds \\
& +2\rho_a^uV_{ia}^{ru}\int_0^\infty V_{ia}^{ru}(t-s)f_{ia}^{ru}(s)e^{-\delta_a^u s}ds-2\rho_a^uW_{ia}^{ru}\int_0^\infty V_{ia}^{ru}(t-s)f_{ia}^{ru}(s)e^{-\delta_a^u s}ds \\
& -2[(c_{ia}^{ru}+1)\rho_a^u+2(c_{ia}^{ru}+1)(\rho_{ia}^{ru}+\delta_a^u+d_a^u)](V_{ia}^{ru})^2-2(\rho_{ia}^{ru}+\delta_a^u)(U_{ia}^{ru})^2 \\
& -2(\rho_{ia}^{ru}+\alpha_a^u+\delta_a^u)(W_{ia}^{rr})^2+2\rho_a^uV_{ia}^{ru}W_{ia}^{ru}-2[(\rho_a^u+d_a^u)+2(\rho_{ia}^{ru}+\delta_a^u)]V_{ia}^{ru}U_{ia}^{ru}\}
\end{aligned}
$$

$$+2\sum_{u\neq r}^{M}\sum_{a=1}^{n_u} c_{ia}^{ru}\sum_{v=1}^{M}\sum_{b=1}^{n_v}\beta_{aib}^{urv}(S_{ia}^{ru*}+U_{ia}^{ru})V_{ba}^{vu}V_{ia}^{ru}$$

$$+\sum_{u\neq r}^{M}\sum_{a=1}^{n_r} c_{ia}^{ru}\sum_{v=1}^{M}\sum_{b=1}^{n_v}(v_{aib}^{urv})^2(S_{ia}^{ru*}+U_{ia}^{ru})^2(V_{ba}^{vu})^2, for \quad u\neq r \tag{7.4.52}$$

By using (7.3.31) and the algebraic inequality

$$2ab \leq \frac{a^2}{g(c)}+b^2g(c) \tag{7.4.53}$$

where $a,b,c\in\mathbb{R}$, and the function g is such that $g(c)>0$. The fourteenth term in (7.4.50)-(7.4.52) is estimated as follows:

$$\begin{aligned}
2\sum_{v=1}^{M}\sum_{b=1}^{n_v} c_{ii}^{rr}\beta_{iib}^{rrv}(S_{ii}^{rr*}+U_{ii}^{rr})V_{bi}^{vr}V_{ii}^{rr} &\leq \sum_{v=1}^{M}\sum_{b=1}^{n_v} c_{ii}^{rr}\beta_{iib}^{rrv}(S_{ii}^{rr*}g_i^r(\delta_i^r)+g_i^r(\delta_i^r))(V_{ii}^{rr})^2\\
&+\sum_{v=1}^{M}\sum_{b=1}^{n_v} c_{ii}^{rr}\beta_{iib}^{rrv}\left(\frac{S_{ii}^{rr*}}{g_i^r(\delta_i^r)}+\frac{\bar{B}^2}{g_i^r(\delta_i^r)}\right)(V_{bi}^{vr})^2\\
2\sum_{a\neq r}^{n_r}\sum_{v=1}^{M}\sum_{b=1}^{n_v} c_{ia}^{rr}\beta_{aib}^{rrv}(S_{ia}^{rr*}+U_{ia}^{rr})V_{ba}^{vr}V_{ia}^{rr} &\leq \sum_{a\neq r}^{n_r}\sum_{v=1}^{M}\sum_{b=1}^{n_v} c_{ia}^{rr}\beta_{aib}^{rrv}(S_{ia}^{rr*}g_i^r(\delta_a^r)+g_i^r(\delta_a^r))(V_{ia}^{rr})^2\\
&+\sum_{a\neq r}^{n_r}\sum_{v=1}^{M}\sum_{b=1}^{n_v} c_{ia}^{rr}\beta_{aib}^{rrv}\left(\frac{S_{ia}^{rr*}}{g_i^r(\delta_a^r)}+\frac{\bar{B}^2}{g_i^r(\delta_a^r)}\right)(V_{bi}^{vr})^2
\end{aligned}$$

and

$$\begin{aligned}
2\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\sum_{v=1}^{M}\sum_{b=1}^{n_v} c_{ia}^{ru}\beta_{aib}^{urv}(S_{ia}^{ru*}+U_{ia}^{ru})V_{ba}^{vu}V_{ia}^{ru} &\leq \sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\sum_{v=1}^{M}\sum_{b=1}^{n_v} c_{ia}^{ru}\beta_{aib}^{urv}(S_{ia}^{ru*}g_i^r(\delta_a^u)+g_i^r(\delta_a^u))(V_{ia}^{ru})^2\\
&+\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\sum_{v=1}^{M}\sum_{b=1}^{n_v} c_{ia}^{ru}\beta_{aib}^{urv}\left(\frac{S_{ia}^{ru*}}{g_i^r(\delta_a^u)}+\frac{\bar{B}^2}{g_i^r(\delta_a^u)}\right)(V_{ba}^{vu})^2
\end{aligned} \tag{7.4.54}$$

Furthermore, by using $Cauchy-Swartz$ and $Hölder$ inequalities and (7.4.53), the sixth, seventh and eighth terms in (7.4.50)-(7.4.52) are estimated as follows:

$$2\rho_a^u A_{ia}^{ru}\int_0^\infty V_{ia}^{ru}(t-s)f_{ia}^{ru}(s)e^{-\delta_a^u s}ds \leq \frac{(\rho_a^u)^2}{\mu_{ia}^{ru}}\int_0^\infty (V_{ia}^{ru}(t-s))^2 f_{ia}^{ru}(s)e^{-2\delta_a^u s}ds+\mu_{ia}^{ru}(A_{ia}^{ru})^2,$$
$$\forall r,u\in I(1,M), i\in I(1,n_r), a\in I(1,n_u), A_{ia}^{ru}\in\{U_{ia}^{ru},V_{ia}^{ru},W_{ia}^{ru}\}. \tag{7.4.55}$$

From (7.4.50)-(7.4.54), (7.4.49), repeated usage of (7.3.31) and inequality (7.4.53) coupled with some algebraic manipulations and simplifications, we have the following inequality

$$
\begin{aligned}
LV_1(\tilde{x}^{00}_{00}) \quad \leq \quad & \sum_{r=1}^{M}\sum_{i=1}^{n_r}\Bigg\{\Bigg[2\sum_{u=1}^{M}\sum_{a=1}^{n_u}\mu_{ia}^{ru}+2\sum_{a\neq i}^{n_r}\frac{(\sigma_{ia}^{rr})^2}{\mu_{ii}^{rr}}+2\sum_{u\neq r}^{M}\sum_{a=1}^{n_r}\frac{(\gamma_{ia}^{ru})^2}{\mu_{ii}^{rr}}+4\mu_{ii}^{rr} \\
& -2(\gamma_i^r+\sigma_i^r+\delta_i^r)\Bigg](U_{ii}^{rr})^2 \\
& +\Bigg[(2+c_{ii}^{rr})\sum_{u=1}^{M}\sum_{a=1}^{n_u}\mu_{ia}^{ru}+\sum_{a\neq r}^{n_r}(2+c_{ia}^{rr})\frac{(\sigma_{ia}^{rr})^2}{\mu_{ii}^{rr}}+\sum_{u=1}^{M}\sum_{a=1}^{n_u}(2+c_{ia}^{ru})\frac{(\gamma_{ia}^{ru})^2}{\mu_{ii}^{rr}}+\mu_{ii}^{rr} \\
& +\frac{(\rho_i^r+d_i^r)^2}{\mu_{ii}^{rr}}+4\frac{(\gamma_i^r+\sigma_i^r+\delta_i^r)^2}{\mu_{ii}^{rr}}+\frac{(\rho_i^r)^2}{\mu_{ii}^{rr}}+c_{ii}^{rr}\sum_{v=1}^{M}\sum_{b=1}^{n_r}\beta_{iib}^{rrv}(S_{ii}^{rr*}\mu_{ii}^{rr}+\mu_{ii}^{rr}) \\
& -2(c_{ii}^{rr}+1)(\rho_i^r+\gamma_i^r+\sigma_i^r+\delta_i^r+d_i^r)\Bigg](V_{ii}^{rr})^2 \\
& +\Bigg[\sum_{u=1}^{M}\sum_{a=1}^{n_u}\mu_{ia}^{ru}+\sum_{a\neq i}^{n_r}\frac{(\sigma_{ia}^{rr})^2}{\mu_{ii}^{rr}}+\sum_{u\neq r}^{M}\sum_{a=1i}^{n_r}\frac{(\gamma_{ia}^{ru})^2}{\mu_{ii}^{rr}}+2\mu_{ii}^{rr}-2(\gamma_i^r+\sigma_i^r+\delta_i^r)\Bigg](W_{ii}^{rr})^2\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
& +\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{a\neq i}^{n_r}\Bigg\{\Bigg[2\frac{(\rho_{ia}^{rr})^2}{\mu_{ia}^{rr}}+2\mu_{ii}^{rr}+3\mu_{ia}^{rr}-2(\rho_{ia}^{rr}+\delta_a^r)\Bigg](U_{ia}^{rr})^2 \\
& +\Bigg[(2+c_{ii}^{rr})\frac{(\rho_{ia}^{rr})^2}{\mu_{ia}^{rr}}+(2+c_{ia}^{rr})\mu_{ii}^{rr}+\frac{(\rho_a^r+d_a^r)^2}{\mu_{ia}^{rr}}+4\frac{(\rho_{ia}^{rr}+\delta_a^r)^2}{\mu_{ia}^{rr}}+\mu_{ia}^{rr} \\
& +c_{ia}^{rr}\sum_{v=1}^{M}\sum_{b=1}^{n_r}\beta_{aib}^{rrv}(S_{ia}^{rr*}\mu_{ia}^{rr}+\mu_{ia}^{rr})-2(c_{ia}^{rr}+1)(\rho_a^r+\rho_{ia}^{rr}+\delta_a^r+d_a^r)\Bigg](V_{ia}^{rr})^2 \\
& +\Bigg[\frac{(\rho_{ia}^{rr})^2}{\mu_{ia}^{rr}}+\mu_{ii}^{rr}+2\mu_{ia}^{rr}-2(\rho_{ia}^{rr}+\delta_a^r)\Bigg](W_{ia}^{rr})^2\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
& +\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\Bigg\{\Bigg[2\frac{(\rho_{ia}^{ru})^2}{\mu_{ia}^{rr}}+2\mu_{ii}^{rr}+3\mu_{ia}^{ru}-2(\rho_{ia}^{ru}+\delta_a^u)\Bigg](U_{ia}^{ru})^2 \\
& +\Bigg[(2+c_{ii}^{rr})\frac{(\rho_{ia}^{ru})^2}{\mu_{ia}^{ru}}+(2+c_{ia}^{ru})\mu_{ii}^{rr}+\frac{(\rho_a^u+d_a^u)^2}{\mu_{ia}^{ru}}+4\frac{(\rho_{ia}^{ru}+\delta_a^u)^2}{\mu_{ia}^{ru}}+\mu_{ia}^{ru} \\
& +c_{ia}^{ru}\sum_{v=1}^{M}\sum_{b=1}^{n_r}\beta_{aib}^{urv}(S_{ia}^{ru*}\mu_{ia}^{ru}+\mu_{ia}^{ru})-2(c_{ia}^{ru}+1)(\rho_a^u+\rho_{ia}^{ru}+\delta_a^u+d_a^u)\Bigg](V_{ia}^{ru})^2 \\
& +\Bigg[\frac{(\rho_{ia}^{ru})^2}{\mu_{ia}^{ru}}+\mu_{ii}^{rr}+2\mu_{ia}^{ru}-2(\rho_{ia}^{ru}+\delta_a^u)\Bigg](W_{ia}^{ru})^2\Bigg\} \\
& +3\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u=1}^{M}\sum_{a=1}^{n_u}\frac{(\rho_a^u)^2}{\mu_{ia}^{ru}}\int_0^\infty (V_{ia}^{ru}(t-s))^2 f_{ia}^{ru}(s)e^{-2\delta_a^u s}ds
\end{aligned}
$$

$$+\sum_{r=1}^{M}\sum_{i=1}^{n_r} c_{ii}^{rr} \sum_{v=1}^{M}\sum_{b=1}^{n_r}\left[\beta_{iib}^{rrv}(\frac{S_{ii}^{rr*}}{\mu_{ii}^{rr}}+\frac{\bar{B}^2}{\mu_{ii}^{rr}})+(v_{iib}^{rrv})^2(S_{ii}^{rr*}+\bar{B})^2\right](V_{bi}^{vr})^2$$
$$+\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{a\neq i}^{n_r} c_{ia}^{rr}\left[\sum_{v=1}^{M}\sum_{b=1}^{n_r}\beta_{aib}^{rrv}(\frac{S_{ia}^{rr*}}{\mu_{ia}^{rr}}+\frac{\bar{B}^2}{\mu_{ia}^{rr}})+(v_{aib}^{rrv})^2(S_{ia}^{rr*}+\bar{B})^2\right](V_{ba}^{vr})^2$$
$$+\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u\neq r}^{M}\sum_{a=1}^{n_r} c_{ia}^{ru}\left[\sum_{v=1}^{M}\sum_{b=1}^{n_r}\beta_{aib}^{urv}(\frac{S_{ia}^{ru*}}{\mu_{ia}^{ru}}+\frac{\bar{B}^2}{\mu_{ia}^{ru}})+(v_{aib}^{urv})^2(S_{ia}^{ru*}+\bar{B})^2\right](V_{ba}^{vu})^2, \tag{7.4.56}$$

where $\mu_{ia}^{ru}=g_i^r(\delta_a^u)$, g_i^r is appropriately defined by (7.4.53). For each $r,u\in I(1,M)$, $i\in I(1,n_r)$ and $a\in I(1,n_u)$, using (7.4.43), (7.4.44) and (7.4.45), we define the constants d_{ai}^{ur}, ϕ_{ia}^{ru}, ψ_{ia}^{ru} and φ_{ia}^{ru} as follows:

$$d_{ai}^{ur} = \sum_{v=1}^{M}\sum_{b=1}^{n_v} c_{ba}^{vu}\beta_{abi}^{uvr}\left(\frac{S_{ba}^{vu*}+\bar{B}^2}{\mu_{ba}^{vu}}\right)+\sum_{v=1}^{M}\sum_{b=1}^{n_v} c_{ba}^{vu}(v_{abi}^{uvr})^2(S_{ba}^{vu*}+\bar{B})^2 \tag{7.4.57}$$

for some positive numbers c_{ia}^{ru}, for all $r,u\in I^r(1,M)$, $i\in I(1,n)$ and $a\in I_i^r(1,n_r)$.

$$\phi_{ia}^{ru}=\begin{cases} 2(\gamma_i^r+\sigma_i^r+\delta_i^r)(1-\mathfrak{U}_{ia}^{ru}), for \quad u=r, a=i \\ 2(\rho_{ia}^{rr}+\delta_a^r)(1-\mathfrak{U}_{ia}^{ru}), for \quad u=r, a\neq i \\ 2(\rho_{ia}^{ru}+\delta_a^u)(1-\mathfrak{U}_{ia}^{ru}), for \quad u\neq r, \end{cases} \tag{7.4.58}$$

$$\psi_{ia}^{ru}=\begin{cases} 2(\rho_i^r+\gamma_i^r+\sigma_i^r+\delta_i^r+d_i^r)\left[c_{ii}^{rr}(1-\mathfrak{V}_{ii}^{rr})+(1-\frac{1}{2}\mathfrak{E}_{ii}^{rr})\right], for \quad u=r, a=i \\ 2(\rho_a^r+\rho_{ia}^{rr}+\delta_a^r+d_a^r)\left[c_{ia}^{rr}(1-\mathfrak{V}_{ia}^{rr})+(1-\frac{1}{2}\mathfrak{E}_{ia}^{rr})\right], for \quad u=r, a\neq i \\ 2(\rho_a^u+\rho_{ia}^{ru}+\delta_a^u+d_a^u)\left[c_{ia}^{ru}(1-\mathfrak{V}_{ia}^{ru})+(1-\frac{1}{2}\mathfrak{E}_{ia}^{ru})\right], for \quad u\neq r \end{cases} \tag{7.4.59}$$

and

$$\varphi_{ia}^{ru}=\begin{cases} 2(\gamma_i^r+\sigma_i^r+\delta_i^r)(1-\mathfrak{W}_{ia}^{ru}), for \quad u=r, a=i, \\ 2(\rho_{ia}^{rr}+\delta_a^r)(1-\mathfrak{W}_{ia}^{ru}), \quad for u=r, a\neq i, \\ 2(\rho_{ia}^{ru}+\delta_a^u)(1-\mathfrak{W}_{ia}^{ru}), for \quad u\neq r \end{cases} \tag{7.4.60}$$

where $\mathfrak{U}_{ia}^{ru}, \mathfrak{V}_{ia}^{ru}, \mathfrak{W}_{ia}^{ru}$ are given in (7.4.43), (7.4.44), (7.4.45) and

$$\mathfrak{E}_{ia}^{ru} = \begin{cases} \left[\frac{2\sum_{u=1}^{M}\sum_{a=1}^{n_u}\mu_{ia}^{ru} + \sum_{a\neq r}^{n_r}(2+c_{ia}^{rr})\frac{(\sigma_{ia}^{rr})^2}{\mu_{ii}^{rr}} + \sum_{u=1}^{M}\sum_{a=1}^{n_u}(2+c_{ia}^{ru})\frac{(\gamma_{ia}^{ru})^2}{\mu_{ii}^{rr}} + \mu_{ii}^{rr}}{(\rho_i^r+\gamma_i^r+\sigma_i^r+\delta_i^r+d_i^r)} \right. \\ \left. + \frac{\frac{(\rho_i^r+d_i^r)^2}{\mu_{ii}^{rr}} + 4\frac{(\gamma_i^r+\sigma_i^r+\delta_i^r)^2}{\mu_{ii}^{rr}} + \frac{(\rho_i^r)^2}{\mu_{ii}^{rr}} + 3\frac{(\rho_i^r)^2}{\mu_{ii}^{rr}}\int_0^\infty f_{ii}^{rr}(s)e^{-2\delta_i^r s}ds}{(\rho_i^r+\gamma_i^r+\sigma_i^r+\delta_i^r+d_i^r)} \right], for \quad u=r, a=i, \\ \left[\frac{(2+c_{ii}^{rr})\frac{(\rho_{ia}^{rr})^2}{\mu_{ia}^{rr}} + 2\mu_{ii}^{rr} + \frac{(\rho_a^r+d_a^r)^2}{\mu_{ia}^{rr}} + 4\frac{(\rho_{ia}^{rr}+\delta_a^r)^2}{\mu_{ia}^{rr}} + \mu_{ia}^{rr} + 3\frac{(\rho_a^r)^2}{\mu_{ia}^{rr}}\int_0^\infty f_{ia}^{rr}(s)e^{-2\delta_a^r s}ds}{(\rho_a^r+\rho_{ia}^{rr}+\delta_a^r+d_a^r)} \right], for \quad u=r, a\neq i, \\ \left[\frac{(2+c_{ii}^{rr})\frac{(\rho_{ia}^{ru})^2}{\mu_{ia}^{ru}} + 2\mu_{ii}^{rr} + \frac{(\rho_a^u+d_a^u)^2}{\mu_{ia}^{ru}} + 4\frac{(\rho_{ia}^{ru}+\delta_a^u)^2}{\mu_{ia}^{ru}} + \mu_{ia}^{ru} + 3\frac{(\rho_a^u)^2}{\mu_{ia}^{ru}}\int_0^\infty f_{ia}^{ru}(s)e^{-2\delta_a^u s}ds}{(\rho_a^u+\rho_{ia}^{ru}+\delta_a^u+d_a^u)} \right], for \quad , u\neq r \end{cases}$$

From (7.4.41), (7.4.42), (7.4.56), (7.4.57), the differential operator LV [34, 59] applied to the Lyapunov functional (7.4.41), and some further algebraic manipulations we have the following inequality

$$\begin{aligned} LV(\tilde{x}_{00}^{00}) \leq & \sum_{r=1}^{M}\sum_{i=1}^{n_r} -\{[\phi_{ii}^{rr}(U_{ii}^{rr})^2 + \psi_{ii}^{rr})(V_{ii}^{rr})^2 \\ & \varphi_{ii}^{rr}(W_{ii}^{rr})^2] + \sum_{a\neq r}^{n_r}[\phi_{ia}^{rr}(U_{ia}^{rr})^2 + \psi_{ia}^{rr}(V_{ia}^{rr})^2 \\ & +\varphi_{ia}^{rr}(W_{ia}^{rr})^2] + \sum_{u\neq r}^{M}\sum_{a=1}^{n_u}[\phi_{ia}^{ru}(U_{ia}^{ru})^2 + \psi_{ia}^{ru}(V_{ia}^{ru})^2 \\ & + \varphi_{ia}^{ru}(W_{ia}^{ru})^2]\}. \end{aligned} \tag{7.4.61}$$

Under the assumptions on $\mathfrak{U}_{ia}^{ru}$, $\mathfrak{V}_{ia}^{ru}$ and $\mathfrak{W}_{ia}^{ru}$, it is clear that $\phi_{ia}^{ru}, \psi_{ia}^{ru}$ and φ_{ia}^{ru} are positive for suitable choices of the constants $c_{ia}^{ru} > 0$. Thus this proves the inequality (7.4.46). Now, the validity of (7.4.47) follows from (7.4.46) and (7.4.39), that is,

$$LV(\tilde{x}_{00}^{00}) \leq -cV_1(\tilde{x}_{00}^{00}),$$

where $c = \frac{\min_{1\leq r,u\leq M, 1\leq i\leq n_r, 1\leq a\leq n_u}\{\phi_{ia}^{ru}, \psi_{ia}^{ru}, \varphi_{ia}^{ru}\}}{\max_{1\leq r,u\leq M, 1\leq i\leq n_r, 1\leq a\leq n_u}\{C_{ia}^{ru}+2\}}$. This completes the proof.
We now formally state the stochastic stability theorems for the disease free equilibria.

Theorem 7.4.3 *Given $r,u \in I(1,M)$, $i \in I(1,n_r)$ and $a \in I(1,n_u)$. Let us assume that the hypotheses of Lemma 7.4.2 are satisfied. Then the disease free solutions E_{ia}^{ru}, are asymptotically stable in the large. Moreover, the solutions E_{ia}^{ru} are exponentially mean square stable.*

Proof:

From the application of comparison result[34, 59], the proof of stochastic asymptotic stability follows immediately. Moreover, the disease free equilibrium state is exponentially mean square stable. We now consider the following corollary to Theorem 7.4.3.

Corollary 7.4.4 *Let $r \in I(1,M)$ and $i \in I(1,n_r)$. Assume that $\sigma_i^r = \gamma_i^r = 0$, for all $r \in I(1,M)$ and $i \in I(1,n_r)$.*

$$\mathfrak{U}_{ia}^{ru} = \begin{cases} \dfrac{\frac{1}{\delta_i^r}}{\frac{1}{\left[\sum_{u=1}^{M}\sum_{a=1}^{n_u}\mu_{ia}^{ru}+2\mu_{ii}^{rr}\right]}} for \quad u=r, i=a \\ \dfrac{\left[\frac{(\rho_{ia}^{rr})^2}{\mu_{ia}^{rr}}+\mu_{ii}^{rr}+\frac{3}{2}\mu_{ia}^{rr}\right]}{(\rho_{ia}^{rr}+\delta_a^r)}, for \quad u=r, a\neq i \\ \dfrac{\left[\frac{(\rho_{ia}^{ru})^2}{\mu_{ia}^{ru}}+\mu_{ii}^{rr}+\frac{3}{2}\mu_{ia}^{ru}\right]}{(\rho_{ia}^{ru}+\delta_a^u)}, for \quad u\neq r, \end{cases} \tag{7.4.62}$$

$$\mathfrak{V}_{ia}^{ru} = \begin{cases} \dfrac{\frac{1}{2}\sum_{u=1}^{M}\sum_{a=1}^{n_u}\mu_{ia}^{ru}+\frac{1}{2}\sum_{v=1}^{M}\sum_{b=1}^{n_v}\beta_{iib}^{rrv}(S_{ii}^{rr*}\mu_{ii}^{rr}+\mu_{ii}^{rr})+\frac{1}{2}d_{ii}^{rr}}{\rho_i^r+\delta_i^r+d_i^r}, for \quad a=i, u=r \\ \dfrac{\frac{1}{2}\mu_{ii}^{rr}+\frac{1}{2}\sum_{v=1}^{M}\sum_{b=1}^{n_v}\beta_{aib}^{rrv}(S_{ia}^{rr*}\mu_{ia}^{rr}+\mu_{ia}^{rr})+\frac{1}{2}d_{ai}^{rr}}{\rho_a^r+\rho_{ia}^{rr}+\delta_a^r+d_a^r}, for \quad a\neq i, u=r \\ \dfrac{\frac{1}{2}\mu_{ii}^{rr}+\frac{1}{2}\sum_{v=1}^{M}\sum_{b=1}^{n_v}\beta_{aib}^{urv}(S_{ii}^{ru*}\mu_{ia}^{ru}+\mu_{ia}^{ru})+\frac{1}{2}d_{ai}^{ur}}{\rho_a^u+\rho_{ia}^{ru}+\delta_a^u+d_a^u}, for \quad u\neq r. \end{cases} \tag{7.4.63}$$

and

$$\mathfrak{W}_{ia}^{ru} = \begin{cases} \dfrac{\frac{1}{\delta_i^r}}{\frac{1}{\left[\frac{1}{2}\sum_{u=1}^{M}\sum_{a=1}^{n_u}\mu_{ia}^{ru}+\mu_{ii}^{rr}\right]}}, for \quad u=r, a=i, \\ \dfrac{\left[\frac{1}{2}\frac{(\rho_{ia}^{rr})^2}{\mu_{ia}^{rr}}+\frac{1}{2}\mu_{ii}^{rr}+\mu_{ia}^{rr}\right]}{(\rho_{ia}^{rr}+\delta_a^r)}, for \quad u=r, a\neq i, \\ \dfrac{\left[\frac{1}{2}\frac{(\rho_{ia}^{ru})^2}{\mu_{ia}^{ru}}+\frac{1}{2}\mu_{ii}^{rr}+\mu_{ia}^{ru}\right]}{(\rho_{ia}^{ru}+\delta_a^u)}, for \quad u\neq r \end{cases} \tag{7.4.64}$$

The equilibrium state E_{ii}^{rr} is stochastically asymptotically stable provided that $\mathfrak{U}_{ia}^{ru}, \mathfrak{W}_{ia}^{ru} \leq 1$ and $\mathfrak{V}_{ia}^{ru} < 1$, for all $u \in I^r(1,M)$ and $a \in I_i^r(1,n_u)$.

Proof: Follows immediately from the hypotheses of Lemma 7.4.2,(letting $\sigma_i^r = \gamma_i^r = 0$), the conclusion of Theorem 7.4.3 and some algebraic manipulations.

Remark 7.4.2 *The presented results about the two-level large scale delayed SIR disease dynamic model depend on the underlying system parameters. In particular, the sufficient conditions are algebraically simple, computationally attractive and explicit in terms of the rate parameters. As a result of this, several scenarios can be discussed and exhibit practical course of action to control the disease. For simplicity, we present an illustration as follows: the conditions of $\sigma_i^r = \gamma_i^r = 0, \forall r, i$ in Corollary 7.4.4 signify that the arbitrary site s_i^r is a 'sink' in the context of compartmental* *systems[28, 29] for all other sites in the inter and intra-regional accessible domain. This scenario is displayed in Figure 7.1. The conditions* $\mathfrak{U}_{ii}^{rr} \leq 1$ *and* $\mathfrak{W}_{ii}^{rr} \leq 1$ *exhibit that the average life span is smaller than the joint average life span of individuals in the intra and inter-regional accessible domain of site* s_i^r*. Furthermore, the conditions* $\mathfrak{V}_{ia}^{ru} < 1, \forall u \in I(1,M), a \in I(1,n_r)$*, and* $\mathfrak{U}_{ia}^{ru} \leq 1, \mathfrak{W}_{ia}^{ru} \leq 1 \forall u = r, a \neq i,$ *and* $\forall u \neq r, a \in I(1,n_r)$*, signify that the magnitude of disease inhibitory processes for example, the magnitude of the recovery process is greater than the disease transmission process. A future detailed study of the disease dynamics in the two scale network dynamic structure for many real life scenarios using the presented two level large-scale delay SIR disease dynamic model will appear elsewhere.*

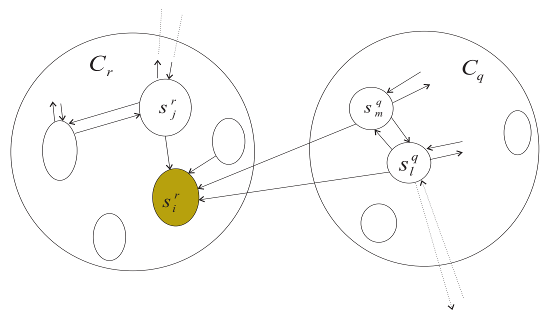

Figure 7.1: Shows that residents of site s_i^r are present only at their home site s_i^r. Hence they isolate every site from their inter and intra regional accessible domain $C(s_i^r)$. Site s_i^r is a 'sink' in the context of the compartmental system[28, 29]. The arrows represent a transport network between any two sites and regions. Furthermore, the dotted lines and arrows indicate connection with other sites and regions.

Remark 7.4.3 *The stochastic delayed epidemic model (7.2.1)-(7.2.3) is a general representation of infection acquired immunity delay in a two-scale network population structure. The stochastic delayed epidemic model with temporary immunity period (5.2.1)-(5.2.3) studied in Chapter 5 is a special case of (7.2.1)-(7.2.3) when we let the probability density function of the immunity period,* $f_{ia}^{ru}(s) = \delta(s - T_i^r), \forall r, u \in I(1,3), \forall i, a \in I(1,3)$*, where* δ *is the Dirac* δ*-function[83].*

7.5 Conclusion

The developed two-scale network delayed epidemic dynamic model characterizes the dynamics of an SIR epidemic in a population with various scale levels created by the heterogeneities in the population. Moreover, the disease dynamics is subject to random environmental perturbations at the disease transmission stage of the disease. Furthermore, the SIR epidemic confers varying time acquired immunity to recovered individuals immediately after recovery. This work provides a mathematical and probabilistic algorithmic tool to develop different levels nested type disease transmission rates, the variability in the disease transmission process as well as the distributed time delay in the framework of the network-centric Ito-Doob type dynamic equations. In addition, the concept of distributed natural immunity time delay is explored for the first time in the context of complex scale-structured type human meta-populations.

The model validation results are developed and a positively self-invariant set for the dynamic model is defined. Moreover, the globalization of the positive solution process existence is established by applying an energy function method. In addition, using the Lyapunov functional technique, the detailed stochastic asymptotic stability results of the disease free equilibria are also exhibited in this Chapter. Moreover, the system parameter dependent threshold values controlling the stochastic asymptotic stability of the disease free equilibrium are also defined. Furthermore, the analysis of the general stochastic dynamic model are illustrated in a controlled quarantine strategy. We note, further detail study of the stochastic SIR human epidemic dynamic model with varying immunity period for two scale network dynamic populations with underlying different real life human mobility patterns will appear in our future study.

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