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On Algorithmic Fractional Packings of Hypergraphs

by

Jill S. Dizona

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
Department of Mathematics & Statistics
College of Arts and Sciences
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Table of Contents

List of Figures

Abstract

Let F_0 be a fixed k-uniform hypergraph, and let H be a given k-uniform hypergraph on n vertices. An F_0-*packing* of H is a family $\mathscr{F}$ of edge-disjoint copies of F_0 which are subhypergraphs in H. Let $\nu_{F_0}(H)$ denote the maximum size $|\mathscr{F}|$ of an F_0-packing $\mathscr{F}$ of H. It is well-known that computing $\nu_{F_0}(H)$ is NP-hard for nearly any choice of F_0.

In this thesis, we consider the special case when F_0 is a linear hypergraph, that is, when no two edges of F_0 overlap in more than one vertex. We establish for $\zeta > 0$ and $n \geq n_0(\zeta)$ sufficiently large, an algorithm which, in time polynomial in n, constructs an F_0-packing $\mathscr{F}$ of H of size $|\mathscr{F}| \geq \nu_{F_0}(H) - \zeta n^k$.

A central direction in our proof uses so-called fractional F_0-packings of H which are known to approximate $\nu_{F_0}(H)$. The driving force of our argument, however, is the use and development of several tools within the theory of hypergraph regularity.

Chapter 1

Introduction

Extremal problems are among the most natural and interesting in the subject of combinatorics. Roughly speaking, extremal problems concern optimizing a fixed parameter over a class of discrete structures sharing a fixed condition. Over the last century, extremal combinatorics has enjoyed much activity and development, and many results obtained here are rather deep and difficult. In recent years, much of the development in extremal combinatorics has consisted of trying to extend results known for graphs to their corresponding problems for hypergraphs. Such is the case with this thesis, although we will include, in Chapter 1.2, a brief overview on other uses and applications of hypergraphs.

A *hypergraph* H is a family of subsets from a fixed set of vertices V. Sometimes it is written that $H = (V, E)$, where E is the family of subsets, which are somtimes called edges (or hyperedges). If all edges of H have the same cardinality k, then we say $H = H^{(k)}$ is a *k-uniform hypergraph*. When $k = 2$, a 2-uniform hypergraph is, more simply, a graph. We often consider when H is a *linear* k-uniform hypergraph. This means that no two edges of H meet in more than one vertex. In Figure 1 below, we illustrate the well-studied *Fano Plane* F, which is a linear 3-uniform hypergraph with 7 vertices and 7 edges.

In other words, the Fano Plane F depicted above has vertex set $V = \{a, b, c, d, e, f, g\}$ and edge set $E = \{\{a, b, c\}, \{a, d, g\}, \{a, e, f\}, \{b, d, f\}, \{b, e, g\}, \{c, d, e\}, \{c, f, g\}\}$.

A well-studied area of extremal graph theory concerns so-called packings by fixed subgraphs. We investigate this problem, more generally, for k-uniform hypergraphs and continue now with the formal definition.

Definition 1.1 **(F_0-packing)** *For fixed hypergraphs F_0 and H, let $\binom{H}{F_0}$ denote the collection of all copies of F_0 which are subhypergraphs in H. An F_0-packing of H is a family $\mathscr{F} \subseteq \binom{H}{F_0}$ which is*

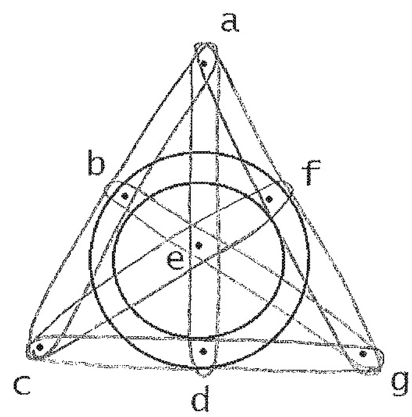

Figure 1.: Fano Plane F

pairwise edge-disjoint. (That is to say, if $F_1, F_2 \in \mathscr{F}$ are distinct, then no edge of F_1 is an edge of F_2.)

In Figure 2, we give two examples of a K_3-packing $\mathscr{F}$ of a (2-uniform) graph H.

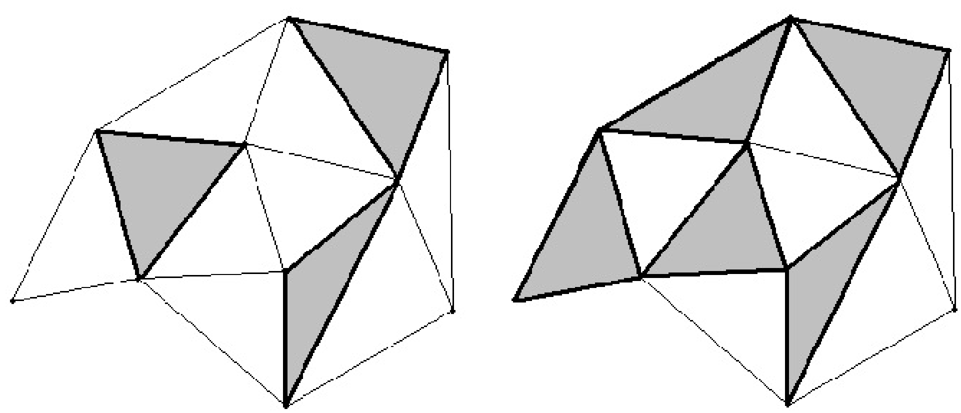

Figure 2.: Two K_3-Packings of H

For k-uniform hypergraphs F_0 and H, let $\nu_{F_0}(H)$ denote the maximum size $|\mathscr{F}|$ of an F_0-packing $\mathscr{F}$ in H. Already in the case of graphs ($k = 2$), when F_0 has a component with 3 edges (as in Figure 2), computing $\nu_{F_0}(H)$ is NP-hard (see Dor and Tarsi [7]). In this thesis, we utilize the concept of fractional F_0-packings (defined below) to approximate the parameter $\nu_{F_0}(H)$.

Definition 1.2 (Fractional F_0-packing) *For k-uniform hypergraphs F_0 and H, a function ψ : $\binom{H}{F_0} \to [0,1]$ is a fractional F_0-packing of H if for each fixed edge $e \in H$,*

$$\sum \left\{\psi(F) : \ F \in \tbinom{H}{F_0} \text{ satisfying } e \in F\right\} \leq 1.$$

In Figure 3, we give a fractional K_3-packing to the graph H from Figure 2.

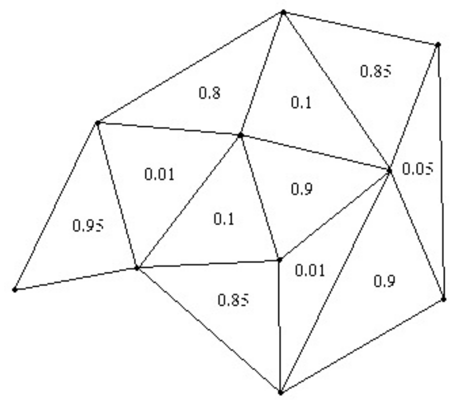

Figure 3.: Fractional K_3-Packings of H

We define the *size* of a fractional F_0-packing ψ of H by $|\psi| = \sum_{F \in \binom{H}{F_0}} \psi(F)$. Note that, for the fractional K_3-packing in Figure 3, $|\psi| = 5.52$. We denote by $\nu^*_{F_0}(H)$ the maximum size $|\psi|$ of a fractional F_0-packing ψ of H, i.e.

$$\nu^*_{F_0}(H) = \max\{|\psi| : \ \psi \text{ is a fractional } F_0\text{-packing of } H\}.$$

Calculating $\nu^*_{F_0}(H)$ is a linear programming problem, and hence computable in polynomial time (polynomial in $|V(H)|$). More strongly, constructing the fractional F_0-packing ψ^* of H that attains the size $|\psi^*| = \nu^*_{F_0}(H)$ is also a linear programming problem, and hence constructable in polynomial time. While not the focus of this thesis, we include a small collection of examples where $\nu_{F_0}(H)$ and $\nu^*_{F_0}(H)$ are computed. Some examples are verified to be correct using Mathematica [31], and another example will include a proof of correctness.

The parameter $\nu^*_{F_0}(H)$ can be used to approximate $\nu_{F_0}(H)$. Indeed, an F_0-packing $\mathscr{F}$ of H can be viewed as a fractional F_0-packing ψ of H (assign $\psi(F) = 1$ for all $F \in \mathscr{F}$ and $\psi(F) = 0$ otherwise). Thus, $\nu_{F_0}(H) \leq \nu^*_{F_0}(H)$. The following result, considered by several authors, shows that these parameters are, in fact, quite close.

Theorem 1.3 ([14, 15, 26, 32]) *For all k-uniform hypergraphs F_0 and all $\zeta > 0$, there exists $N_0 = N_0(F_0, \zeta)$ so that for any k-uniform hypergraph H on $n > N_0$ vertices,*

$$\nu^*_{F_0}(H) - \nu_{F_0}(H) \leq \zeta n^k.$$

In regards to Theorem 1.3, note that $\nu^*_{F_0}(H) \leq \frac{|H|}{|F_0|} < n^k$ since if ψ^* is a fractional F_0-packing ψ^* of H with $|\psi^*| = \nu^*_{F_0}(H)$, then

$$|F_0|\nu^*_{F_0}(H) = \sum_{F \in \binom{H}{F_0}} \sum_{e \in F} \psi^*(F) = \sum_{e \in H} \sum_{F \ni e} \psi^*(F) \leq |H|.$$

Thus, for any constant $c > 0$, the function $\nu_{F_0}(H)$ can be asymptotically estimated in polynomial time on the class of all k-uniform hypergraphs H for which $\nu^*_{F_0}(H) \geq c|V(H)|^k$.

Theorem 1.3 was initiated by Haxell and Rödl [14], who proved it in the case of graphs ($k = 2$). Yuster [32] gave an alternative (and perhaps simpler) proof. Haxell, Nagle, and Rödl [15] later proved Theorem 1.3 for 3-uniform hypergraphs. Most recently, Rödl, Schacht, Siggers and Tokushige [26] proved Theorem 1.3 for general $k \geq 2$.

In the case of graphs, Haxell and Rödl [14] proved the following stronger version of Theorem 1.3 which guarantees the existence of an algorithm which produces a nearly-optimal F_0-packing.

Theorem 1.4 (Haxell and Rödl [14]) *For all graphs F_0 and all $\zeta > 0$, there exists an integer $N_0 = N_0(F_0, \zeta)$ and an algorithm which, for a given graph H on $n > N_0$ vertices, constructs in time polynomial in n an F_0-packing $\mathscr{F}$ in H of size $|\mathscr{F}| \geq \nu^*_{F_0}(H) - \zeta n^2$. In particular, $|\mathscr{F}| \geq \nu_{F_0}(H) - \zeta n^2$.*

The aim of this thesis is to extend the algorithm of Theorem 1.4 to include when F_0 is a linear k-uniform hypergraph and H is a given k-uniform hypergraph.

Theorem 1.5 (Main Result) *For all linear k-uniform hypergraphs F_0 and for every $\zeta > 0$, there exists an integer $N_0 = N_0(F_0, \zeta)$ and an algorithm which, for a given k-uniform hypergraph H on $n > N_0$ vertices, constructs in time polynomial in n, an F_0-packing $\mathscr{F}$ in H of size $|\mathscr{F}| \geq \nu^*_{F_0}(H) - \zeta n^k$. In particular, $|\mathscr{F}| \geq \nu_{F_0}(H) - \zeta n^k$.*

Let us emphasize an important consideration concerning Theorems 1.4 and 1.5. The algorithms there are deterministic and polynomial but non-implementable. Indeed, the size of $N_0 = N_0(F_0, \zeta)$, for reasons we indicate later in the Introduction, must be taken as a tower function of height polynomial in $1/\zeta$ (see Theorem 1.6 later in this Chapter).

One would like to remove the constraint in Theorem 1.5 that F_0 is a linear hypergraph i.e. where F_0 would be allowed to an arbitrary hypergraph. (For $k = 2$, every simple graph F_0 is a linear graph, and so this constraint was irrelevant for Haxell and Rödl.) The methods of this thesis will not allow us to remove the assumption that F_0 is linear, and we explain this point in the Concluding Remarks (Chapter 8). We believe that one could remove the linearity of F_0 when $k = 3$, and we indicate how such a proof would proceed in the Concluding Remarks.

Our proof of Theorem 1.5 depends heavily on tools from the theory of hypergraph regularity, just as Haxell and Rödl's proof of Theorem 1.4 depended heavily on tools from the theory of graph regularity. The tools for graphs are easier to understand than the tools for hypergraphs. We therefore proceed with the following itinerary:

- In Chapter 1.1 of the Introduction, we introduce (largely for motivation) the tools from graph regularity (most notably, Szemerédi's Regularity Lemma) that Haxell and Rödl used to prove Theorem 1.4. (Recall Chapter 1.2 will discuss real-world applications of hypergraphs, and Chapter 1.3 will feature a few calculations of the parameters $\nu_{F_0}(H)$ and $\nu^*_{F_0}(H)$.

- In Chapter 2, we present hypergraph analogues of the graph regularity tools presented in Chapter 1.1. Two of our tools (Lemmas 2.4 and 2.6) are new and may be of independent interest.

- In Chapter 3, we prove our Main Result, Theorem 1.5.

- In Chapters 4 and 5, we prove Lemmas 2.4 and 2.6, respectively. These proofs are technically involved.

- In Chapters 6 and 7, we prove a pair of lemmas (Lemmas 2.7 and 2.8) regarding fractional packing concepts. (These lemmas will also be introduced in Chapter 2.2.)

- In Chapter 8, we give concluding remarks where we will discuss our dependency on the linearity of F_0 as well as discuss the polynomial in the running time of Theorem 1.5.

- In Chapter 9, we provide a few standard inequalities we use in this thesis. (Namely, we give a version of the Cauchy-Schwarz Inequality, the Markov Inequality and a version of the Chernoff Inequality.)

1.1 Graph Regularity Concepts and Tools

The goal for the remainder of the Introduction is to motivate tools from hypergraph regularity by their graph analogues. The most important tool below is the celebrated *Szemerédi Regularity Lemma*, which we now prepare to state.

Szemerédi's Regularity Lemma hinges on the following concept of an *ε-regular pair* (A, B). For a graph $G = (V, E)$ and disjoint sets $A, B \subseteq V$, write $E(A, B)$ for the set of edges $\{a, b\} \in E$ with $a \in A$ and $b \in B$, and write $e(A, B) = |E(A, B)|$. The *density* of (A, B) is $d(A, B) = \frac{e(A,B)}{|A||B|}$. Let $\varepsilon > 0$ be given. We say that the pair (A, B) is *ε-regular* if $|d(A, B) - d(A', B')| < \varepsilon$ holds for all $A' \subseteq A$ and $B' \subseteq B$ whenever $|A'| > \varepsilon|A|$ and $|B'| > \varepsilon|B|$. In other words, the density of (A, B) is "evenly distributed" among all "sizable" pairs of subsets $A' \subseteq A$ and $B' \subseteq B$. In Figure 4 below, we demonstrate the concept of ε-regularity for two vertex sets A and B. We say a partition

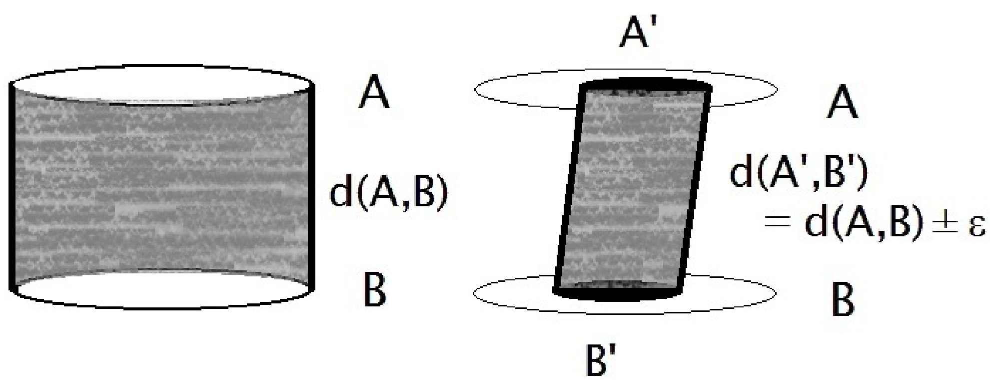

Figure 4.: ε-Regular Pair (A, B)

$V = V(G) = V_0 \cup V_1 \cup \cdots \cup V_t$ is *ε-regular* if all but $\varepsilon\binom{t}{2}$ pairs (V_i, V_j) are ε-regular. Szemerédi's Regularity Lemma is now given as follows.

Theorem 1.6 (Szemerédi's Regularity Lemma [29]) *Let $\varepsilon > 0$ be given and let t_0 be a positive integer. There exist positive integers $N_0 = N_0(\varepsilon, t_0)$ and $T_0 = T_0(\varepsilon, t_0)$ so that any graph $G = (V, E)$ with $|V| = n \geq N_0$ admits an ε-regular vertex partition $V = V_0 \cup V_1 \cup \cdots \cup V_t$ with $t_0 \leq t \leq T_0$ and $|V_1| = \cdots = |V_t|$ and $|V_0| < \varepsilon n$.*

Figure 5 illustrates how Szemerédi's Regularity Lemma partitions the vertices of a given graph G

into t parts (most of which are ε-regular). Note that Szemerédi's Regularity Lemma ensures that all

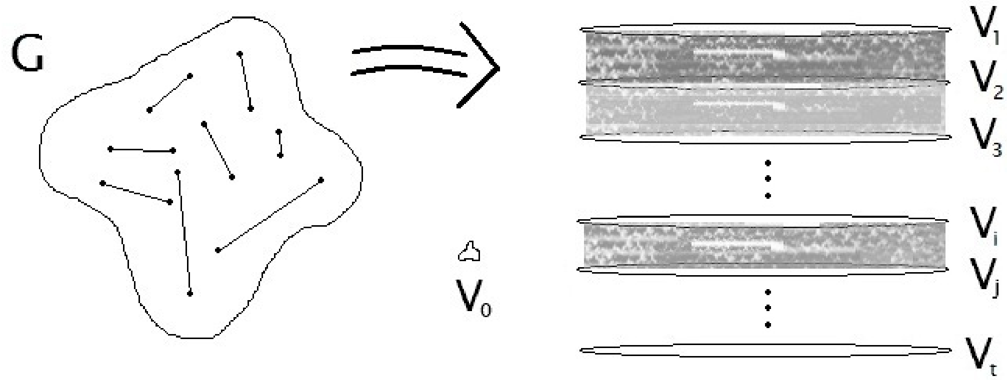

Figure 5.: ε-Regular Partition $V = V_0 \cup V_1 \cup \cdots \cup V_t$

large graphs G can be decomposed into a bounded number (independent of $|V(G)|$) of subgraphs, almost all of which consist of uniformly distributed edges. Szemerédi's formulated and proved the Regularity Lemma in the context of it his famous *Density Theorem* (that all subsets of integers of positive upper density contain arithmetic progressions of arbitrary length (a longstanding conjecture of Erdös and Turán)). Since his initial application, the Regularity Lemma went on to be one of the most powerful tools in all of graph theory. We direct the reader to the survey papers [21, 22] for some of these applications.

It is worth noting that (with $t_0 = 1$) the parameter $T_0 = T_0(\varepsilon)$ is known, from the work of Gowers [12], to be required as a tower function of height polynomial in $1/\varepsilon$. Any meaningful application of Szemerédi's Regularity Lemma needs n to be considerably larger than T_0 (so that each class V_i $(1 \leq i \leq t)$ has significant size $|V_i| \geq \left\lfloor \frac{n(1-\varepsilon)}{t} \right\rfloor > \frac{n}{2T_0}$). Thus, the input graphs $G = (V, E)$ for Szemerédi's Regularity Lemma, which are otherwise arbitrary, must be at least of very large order $N_0 = N_0(\varepsilon, t_0)$ (which is necessarily at least a tower of height polynomial in $1/\varepsilon$). The situation for k-uniform hypergraphs is only worse.

Haxell and Rödl's proof of Theorem 1.4 uses an algorithmic version of Szemerédi's Regularity Lemma (Szemerédi's original proof did not immediately yield an algorithm). This algorithm was found nearly twenty years later by Alon, Duke, Lefmann, Rödl, Yuster [1].

Theorem 1.7 (Algorithmic Graph Regularity Lemma) *In the context of Theorem 1.6, the*

ε-regular partition $V = V_0 \cup V_1 \cup \cdots \cup V_t$ of $V(G)$ can be produced in time $O(n^{2.376})$.

While we don't need it here, the complexity $O(n^{2.376})$ in Theorem 1.7 was later improved to $O(n^2)$ by Kohayakawa, Rödl and Thoma [20]. In this thesis, we shall employ a hypergraph extension of Theorems 1.6 and 1.7 due to Czygrinow and Rödl [5]. This result is given as Theorem 2.2 in Chapter 2.

Another key ingredient of Haxell and Rödl's proof of Theorem 1.4 is a so-called Graph Packing Lemma. We present this statement now in order to draw an analogy to it later. The Graph Packing Lemma roughly asserts that whenever a graph G is given with "sufficiently regular parts", its edges may be "packed" by a family of copies of any fixed subgraph F_0.

Theorem 1.8 (Graph Packing Lemma) *For all integers f and for all $d_0, \mu > 0$, there exists $\varepsilon > 0$ so that the following holds.*

Let F_0, a graph with $V(F_0) = [f] = \{1, ..., f\}$, be given, and let G be an f-partite graph with vertex partition $V_1 \cup \cdots \cup V_f$, where $|V_1| = \cdots = |V_f| = m$ is sufficiently large. Suppose G has the property that for all $1 \le i < j \le f$, $G[V_i, V_j]$ is ε-regular with density $d \ge d_0$, if $\{i, j\} \in F_0$ and $G[V_i, V_j] = \emptyset$, otherwise.

Then, in time polynomial in m, one may construct a family $\mathscr{F}$ of pairwise edge-disjoint copies of F_0 in G which cover all but μm^2 edges of G.

An important part of this thesis will be to develop a hypergraph analogue of Theorem 1.8 suitable for the context of the Czygrinow-Rödl Regularity Lemma. We state this result as Lemma 2.4 in Chapter 2, and prove it in Chapter 4.

A final regularity tool needed in the Haxell and Rödl proof of Theorem 1.4 is a so-called Graph Slicing Lemma. We present a generalization of their lemma in order to draw an analogy to it later. (Note, in the following statement, if we only wished to guarantee the existence of the promised partition, rather than to construct it, the desired "slices" would be an easy consequence of the Chernoff Inequality.)

Theorem 1.9 (Graph Slicing Lemma) *For all positive d_0 and ε', there exists an $\varepsilon > 0$ so that the following holds.*

Let G be an ε-regular bipartite graph with vertex partition $A \cup B$, where $|A| = |B|$ is sufficiently large. Suppose, moreover, that $d_0 \le p_1, ..., p_s$ satisfy $\sum_{i=1}^{s} p_i \le d_G(A, B)$.

Then, there exists an algorithm which, in time $O(|A||B|)$, partitions $G = G_0 \cup G_1 \cup \cdots \cup G_s$, where each G_i ($1 \le i \le s$) is ε'-regular with density $d_{G_i}(A, B) = p_i \pm \varepsilon'$.

In this thesis, we shall prove a hypergraph analogue of Theorem 1.9 suitable for the context of the Czygrinow-Rödl Regularity Lemma. We shall state this result as Lemma 2.6 in Chapter 2, and prove it in Chapter 5.

In addition to the regularity tools indicated above, we also require a pair of lemmas regarding fractional packing concepts. These lemmas are stated as Lemmas 2.7 and 2.8 in Chapter 2, and are proven in Chapters 6 and 7, respectively.

1.2 Applications of Hypergraphs

Hypergraphs, viewed as families of sets, trace their origins to set theory. Extremal hypergraph theory (the flavor of problems considered in this thesis) traces its origins to seminal works of Erdös, and to Ramsey in the early part of the last century. More recently, hypergraph theory is growing in its number and reach of applications to real-world problems, particularly in computer science, software and other computer engineering, molecular biology and related areas. We briefly describe a few models of hypergraphs to scientific applications, and otherwise refer the Reader to the excellent survey of Bretto [4] on this topic.

In chemistry, a *molecular hypergraph* $H = (V, E)$ is one where V denotes a set of individual atoms and E denotes subsets of atoms with polycentric bonds. Such structures generalize the concept of molecular graphs and can be used to describe some chemical structures. In telecommunications, hypergraphs $H = (V, E)$ can be used to model cellular mobile communication systems. Here, a vertex $v \in V$ represents a cell, and an edge of E represents a group of cells all of which cannot use a channel simultaneously. (For a significant study of this topic, see [24].) In parallel data structures, hypergraphs $H = (V, E)$ provide an effective means of modelling parallel data structure. Here, V represents a collection of data elements. A hyperedge from E will represent a *template*, which is a group of data elements to be processed in parallel. Many other applications of hypergraphs can be found in computer science, to areas such as database schemes and constraint satisfaction problems (see [4] for details).

We close by briefly describing a final model of hypergraphs which seems quite relevant to topics studied in this thesis. A *partition* of a hypergraph $H = (V, E)$ is a set partition of either V or E (and if V is partitioned, it naturally induces a partition of E). Hyperedges belonging to a common class of the partition of E (induced or otherwise) can be viewed as verifying a fixed property associated with that class. Such applications were considered by Alpert and Kahng [2] for VLSI design and Mobasher et al. [23] in data mining. Since our work so extensively employs hypergraph partitions, we wonder if some of our current work could be useful in these other settings.

1.3 Computing $\nu_{F_0}(H)$ and $\nu^*_{F_0}(H)$ for a Few Small Graphs

In this section, we consider two principle examples, one for graphs and one for hypergraphs, where explicit values of $\nu_{F_0}(H)$ and $\nu^*_{F_0}(H)$ are computed. We first discuss graphs.

We begin by setting $F_0 = K_3$, the triangle. For an integer $i \geq 3$, define G_i to be the *pinwheel*, which consists of a star on $i+1$ vertices with a cycle C_i traversing the leaves. (see Figure 6 for illustrations of G_3, G_4 and G_5).

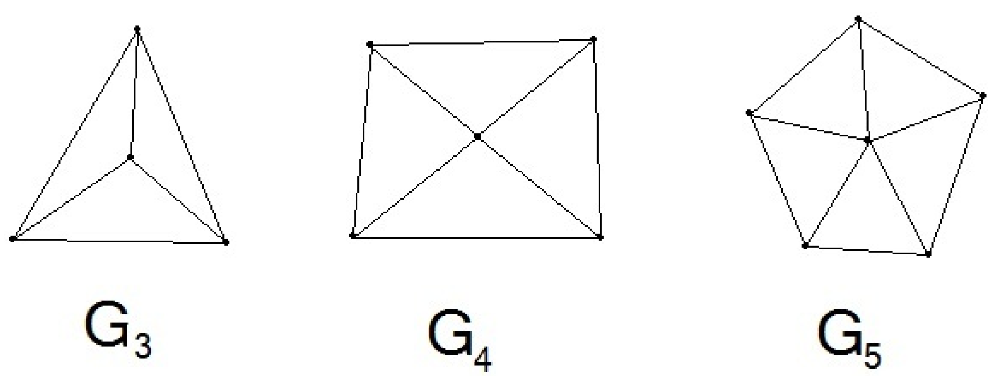

Figure 6.: Pinwheel for $i = 3, 4$ & 5

Using Mathematica [31], one may compute

$$\nu^*_{K_3}(G_3) = 2,\ \nu^*_{K_3}(G_4) = 2,\ \nu^*_{K_3}(G_5) = 2.5 \tag{1.1}$$

and clearly,

$$\nu_{K_3}(G_3) = 1,\ \nu_{K_3}(G_4) = 2,\ \nu_{K_3}(G_5) = 2.$$

To produce the values in (1.1) with Mathematica, let us consider, for example $\nu^*_{K_3}(G_4)$. Write the triangles of G_4 as F_1, F_2, F_3, F_4. One wishes to maximize the quantity

$$\psi(F_1) + \psi(F_2) + \psi(F_3) + \psi(F_4)$$

subject to $0 \leq \psi(F_j) \leq 1$ ($1 \leq j \leq 4$), and

$$\psi(F_j) + \psi(F_{j+1}) \leq 1$$

where $j, j+1$ are taken modulo 4. This is, however, a built-in function of Mathematica. From this empirical evidence, one wonders if $\nu^*_{K_3}(G_i) = i/2$ whenever $i > 3$. (It is clearly the case that $\nu_{K_3}(G_i) = \lfloor 1/2 \rfloor$ whenever $i \geq 3$.) We hope to return to such questions in the near future.

Next, for our hypergraph example, for integers $t > k \geq 2$, let $K_t = K_t^{(k)}$ denote the complete k-uniform hypergraph on t vertices. Set $F_0 = K_{k+1}$ (the k-*simplex*) and $H = K_{k+2}$. Then,

$$\nu^*_{K_{k+1}}(K_{k+2}) = 1 + \frac{k}{2} > 1 = \nu_{K_{k+1}}(K_{k+2}).$$

Proof. Clearly, any K_{k+1}-packing $\mathscr{F}$ of K_{k+2} can consist of at most 1 element. Indeed, if $F \neq F' \in \mathscr{F}$, then $|V(F) \cap V(F')| = k$, which in K_{k+2} means $V(F) \cap V(F')$ spans an edge. Thus, $\nu_{K_{k+1}}(K_{k+2}) = 1$.

Let us now observe that $\nu^*_{K_{k+1}}(K_{k+2}) \geq 1 + (k/2)$. Indeed, let ψ^*_0 be defined by $\psi^*_0(F) = 1/2$ for every copy F of K_{k+1} in K_{k+2}. Since each k-tuple of K_{k+2} belongs to precisely two copies of K_{k+1}, ψ^*_0 is a fractional K_{k+1}-packing of K_{k+2}. Moreover, since K_{k+2} contains $\binom{k+2}{k+1} = k+2$ copies of K_{k+1}, we have $|\psi^*_0| = (k+2)/2 = 1 + (k/2)$ (and $\nu^*_{K_{k+1}}(K_{k+2}) \geq |\psi^*_0|$).

Assume now, on the contrary, that $\nu^*_{K_{k+1}}(K_{k+2}) > 1 + (k/2)$, and suppose ψ^*_1 is a corresponding maximum fractional K_{k+1}-packing of K_{k+2}. Then, for some copy F_1 of K_{k+1} in K_{k+2}, $\psi^*_1(F_1) > 1/2$. (Indeed, otherwise $|\psi^*_1| \leq |\psi^*_0| = 1 + (k/2)$, where ψ^*_0 was from the previous paragraph.) Let us write, for some $0 < \varepsilon \leq 1/2$,

$$\psi^*_1(F_1) = \frac{1}{2} + \varepsilon. \tag{1.2}$$

Observe now that every copy $F \neq F_1$ of K_{k+1} in K_{k+2} now satisfiies

$$\psi^*_1(F) \leq \frac{1}{2} - \varepsilon. \tag{1.3}$$

Indeed, $|V(F) \cap V(F_1)| = k$, and so $V(F) \cap V(F_1)$ spans an edge in K_{k+2} which belongs to precisely two copies of K_{k+1} (namely, F_1 and F). Then (1.3) follows from the fact that ψ^*_1 is a fractional K_{k+1}-packing. Using (1.2) and (1.3), we see

$$|\psi_1^*| = \sum_{F \in \binom{H}{F_0}} \psi_1^*(F) = \frac{1}{2} + \varepsilon + \sum_{F_1 \neq F \in \binom{H}{F_0}} \psi_1^*(F) \leq \frac{1}{2} + \varepsilon + \left(\frac{1}{2} - \varepsilon\right)\left(\binom{k+2}{k+1} - 1\right)$$
$$= \frac{1}{2} + \varepsilon + \left(\frac{1}{2} - \varepsilon\right)(k+1) = 1 + \frac{k}{2} - k\varepsilon < 1 + \frac{k}{2},$$

a contradiction. □

We believe it would be of interest to consider $\nu^*_{K_s}(K_t)$ for more general values of s and t. We hope to return to such problems in the near future.

Chapter 2

Fundamental Tools for Proving Theorem 1.5

As indicated in the Introduction, the main tools for our proof of Theorem 1.5 rely upon hypergraph regularity statements. We now state these tools precisely in Section 2.1. We shall also need some fractional packing tools; we shall give these in Section 2.2. In addition, amongst many of the statements below, we adopt the following notation format: $c_{Lem.\ x}$ will mean the constant c promised by Lemma x. We use this format for ease of referencing in the context of proofs.

2.1 Regularity Concepts

In the Introduction, we formulated bipartite graph density and regularity concepts. We now extend these to the context of k-partite k-uniform hypergraphs. Recall that an ℓ-partite vertex set $V = V_1 \cup \cdots \cup V_\ell$ of a k-uniform hypergraph $H = (V, E)$ is one where each edge of H meets each class V_i $(1 \leq i \leq \ell)$ at most once.

Let H be a k-uniform hypergraph. For nonempty pairwise-disjoint subsets $U_1, ..., U_k \subset V(H)$, write $H[U_1, ..., U_k]$ for the edges of H which intersect each $U_i, 1 \leq i \leq k$. The *density* of $(U_1, ..., U_k)$ is defined as $d(U_1, ..., U_k) = \frac{|H[U_1,...,U_k]|}{|U_1|\cdots|U_k|}$.

Definition 2.1 (**(d, ε)-Regular**) *For $d \in [0, 1]$ and $\varepsilon > 0$, we say $(U_1, ..., U_k)$ is (d, ε)-regular if for all $U_i' \subseteq U_i (1 \leq i \leq k)$ where $|U_i'| > \varepsilon|U_i|$, we have $|d(U_1', ..., U_k') - d| < \varepsilon$. We say $(U_1, ..., U_k)$ is ε-regular if it is (d, ε)-regular for some $d \in [0, 1]$.*

The following regularity lemma for k-uniform hypergraphs, due to Czygrinow and Rödl, extends Theorems 1.6 and 1.7.

Theorem 2.2 (Algorithmic Hypergraph Regularity Lemma [5]) *For all $\varepsilon > 0$ and all positive integers k, ℓ, t_0, there exist $T_0 = T_0(\varepsilon, k, \ell, t_0)$ and $N_0 = N_0(\varepsilon, k, \ell, t_0)$ so that the following holds.*

Let a k-uniform hypergraph H be given with vertex set V, having $|V| = n > N_0$, together with a vertex partition $\Pi = (V_1, ..., V_\ell)$, with $|V_1| \leq \cdots \leq |V_\ell| \leq |V_1| + 1$. One may construct, in time $O(n^{2k-1} \log^2 n)$, a refined vertex partition $\hat{\Pi}$ with the following properties:

1. *$\hat{\Pi}$ has classes V_0 and $V_{ij}, 1 \leq i \leq \ell, 1 \leq j \leq t$, where $t_0 \leq t \leq T_0$, and where*

   (a) for each $1 \leq i \leq \ell$,

   $$V_{i1} \cup \cdots \cup V_{it} \subseteq V_i,$$

   (b)

   $$|V_{11}| = \cdots = |V_{\ell t}| \quad \textit{and} \quad |V_0| < \varepsilon n;$$

2. *All but $\varepsilon\binom{\ell}{k}t^k$ many k-tuples $(V_{i_1 j_1}, ..., V_{i_k j_k}), with\ 1 \leq i_1 < \cdots < i_k \leq \ell, 1 \leq j_1, ..., j_k \leq t$, are ε-regular.*

Moveover, any k-tuple $(V_{i_1 j_1}, ..., V_{i_k j_k})$, with $1 \leq i_1 < \cdots < i_k \leq \ell, 1 \leq j_1, ..., j_k \leq t$, that is not ε-regular will be labeled as such.

A nonconstructive version of Theorem 2.2 was known prior to [5] and was given by Frankl and Rödl [8]. Similar to Theorem 1.7, Theorem 2.2 concerns a deterministic polynomial time algorithm which, due to the parameters T_0 and N_0, is non-implementable.

The next important result we need for the proof of Theorem 1.5 is an extension of the Graph Packing Lemma to k-uniform hypergraphs. We begin by first giving the following context which considers an appropriate collection of "dense and regular blocks" from Theorem 2.2.

Setup 2.3 (Packing Setup) *Let F_0 be a linear k-uniform hypergraph with vertex set $V(F_0) = [f] = \{1, ..., f\}$, and let G be an f-partite k-uniform hypergraph with vertex partition $V_1 \cup \cdots \cup V_f$ satisfying $|V_1| = \cdots = |V_f| = m$. Suppose, moreover, that for some $d_0, \varepsilon > 0$, G has the following property.*

For each $\{i_1, ..., i_k\} \in \binom{[f]}{k}$,

1. *$(V_{i_1}, ..., V_{i_k})$ is (d, ε)-regular (where $d \geq d_0$) if $\{i_1, ..., i_k\} \in F_0$.*

2. *$G[V_{i_1}, ..., V_{i_k}] = \emptyset$, if $\{i_1, ..., i_k\} \notin F_0$.*

In the context of Setup 2.3, a subhypergraph F' of G on vertices $v_1, ..., v_f$ is a *partite-isomorphic copy* of F_0 if $v_i \in V_i$ for all $1 \leq i \leq f$, and $v_i \to i$ is an isomorphism from F' to F_0. We may now give the Hypergraph Packing Lemma.

Lemma 2.4 (Hypergraph Packing Lemma) *Let F_0 be a fixed linear k-uniform hypergraph with $V(F_0) = [f]$, as in Setup 2.3. For all $d_0, \mu > 0$, there exists $\varepsilon_{\text{Lem.2.4}} = \varepsilon_{\text{Lem.2.4}}(d_0, \mu) > 0$ so that the following holds.*

Let G, together with F_0, be given as in Setup 2.3 with m sufficiently large. Then, there exists an algorithm which, in time polynomial in m, constructs an F_0-packing of G covering all but $\mu|G|$ edges of G, and which consists entirely of partite-isomorphic copies of F_0 in G.

We emphasize that, in the proof of Theorem 1.5, the application of Lemma 2.4 is the central "builder" of the promised F_0-packing $\mathscr{F}$ of H.

Remark 2.5 *The conclusion of Lemma 2.4 is false when the assumption of linearity of F_0 is removed. For further discussion, see Chapter 8.*

We prove Lemma 2.4 in Chapter 4.

Our final regularity tool needed for the proof of Theorem 1.5 is an extension of the Graph Slicing Lemma to a "dense and regular" k-partite k-uniform hypergraph.

Lemma 2.6 (Hypergraph Slicing Lemma) *For all $d_0, \varepsilon' > 0$, there exists $\varepsilon_{\text{Lem.2.6}} = \varepsilon_{\text{Lem.2.6}}(d_0, \varepsilon') > 0$ so that the following holds.*

Let G be an ε-regular k-partite k-uniform hypergraph with vertex partition $V_1 \cup \cdots \cup V_k$, where $|V_1| = \cdots = |V_k| = m$ is sufficiently large. Suppose, moreover, that $d_0 \leq p_1, ..., p_s$ are given satisfying $\sum_{i=1}^{s} p_i \leq d_G(V_1, ..., V_k) = D$.

Then, there exists an algorithm which, in time $O(m^k)$, partitions $G = G_0 \cup G_1 \cup \cdots \cup G_s$, where each G_i ($1 \leq i \leq s$) is (p_i, ε')-regular.

We prove Lemma 2.6 in Chapter 5.

2.2 Fractional Packing Concepts

The remaining lemmas which are needed to prove Theorem 1.5 concern fractional packing concepts. In what follows, we use the same notation as the Introduction and begin with the following, which we call the Fractional-Crossing Lemma.

For k-uniform hypergraphs H and F_0, recall $\binom{H}{F_0}$ denotes the set of copies F of F_0 which are subhypergraphs of H. For a partition $\Pi = (V_1, ..., V_l)$ of $V(H)$, let $\binom{H}{F_0}_\Pi \subseteq \binom{H}{F_0}$ denote the subcollection of those copies F which *cross* Π, i.e., for which $|V(F) \cap V_i| \leq 1$ for all $1 \leq i \leq l$.

Lemma 2.7 (Fractional Crossing Lemma) *For all $\mu > 0$ and k-uniform hypergraphs F_0, there exists $L_0 = L_0(\mu, F_0)$ such that the following holds.*

Let H be a k-uniform hypergraph on n vertices, and let ψ be a fractional F_0-packing of H. Then there exists an algorithm which constructs, in time $O(n^2)$, a vertex partition $\Pi = (V_1, ..., V_l)$ of H, $l \leq L_0$, with the following properties:

1. $\lfloor \frac{n}{l} \rfloor \leq |V_i| \leq \lceil \frac{n}{l} \rceil (1 \leq i \leq l)$,
2. $\sum_{F \in \binom{H}{F_0}_\Pi} \psi(F) \geq (1 - \mu)|\psi|$.

We prove Lemma 2.7 in Chapter 6.

We conclude this section with our final tool, which we call the δ-Bounded Lemma. Note, in what follows, we consider weighted hypergraphs.

Let F_0 be a k-uniform hypergraph and let H_0 be an edge-weighted k-uniform hypergraph with weight function $\omega : H_0 \to [0, 1]$. A *fractional* (ω, F_0)*-packing* of H_0 is a function $\hat{\psi} : \binom{H_0}{F_0} \to [0, 1]$ satisfying that for each edge $e \in H_0$,

$$\sum_{F \ni e} \hat{\psi}(F) \leq \omega(e).$$

We say that $\hat{\psi}$ is δ*-bounded* if for each $F \in \binom{H_0}{F_0}$, $\hat{\psi}(F) \in \{0\} \cup [\delta, 1]$. We set $|\hat{\psi}| = \sum_{F \in \binom{H_0}{F_0}} \hat{\psi}(F)$ and $\nu^*_{F_0}(H_0) = \max\{|\hat{\psi}| : \hat{\psi}$ is a fractional (ω, F_0)-packing of $H_0\}$.

Lemma 2.8 (δ-bounded Lemma) *For all k-uniform hypergraphs F_0 and $\xi > 0$, there exists $\delta = \delta_{\text{Lem.2.8}} > 0$ so that the following holds.*

For every edge-weighted k-uniform hypergraph H_0 with weight function ω on r vertices, there exists a δ-bounded fractional (ω, F_0)-packing $\hat{\psi}$ of H_0 such that $|\hat{\psi}| \geq \nu^*_{F_0}(H_0) - \xi r^k$. Moreover, $\hat{\psi}$ can be found by an exhaustive search (as a function of r).

We prove Lemma 2.8 in Chapter 7.

Chapter 3

Proof of the Theorem 1.5

In all that follows, let F_0 be a fixed linear k-uniform hypergraph on f vertices, and let $\zeta > 0$ be given. Our first step in establishing the algorithm of Theorem 1.5 is to generate some auxiliary constants (depending on F_0 and ζ) with respect to which N_0 must be sufficiently large.

3.1 Step I: Defining Constants

Define

$$\gamma = \mu = \xi = \zeta/7. \tag{3.1}$$

With ξ defined above, let

$$\delta = \delta_{\text{Lem.2.8}}(F_0, \xi) > 0 \tag{3.2}$$

be the constant guaranteed by Lemma 2.8. Set

$$d_0 = \delta. \tag{3.3}$$

With μ in (3.1) and d_0 in (3.3), let $\varepsilon_{\text{Lem.2.4}} = \varepsilon_{\text{Lem.2.4}}(d_0, \mu) > 0$ be the constant guaranteed by Lemma 2.4. With d_0 in (3.3) and

$$\varepsilon' \ = \ (d_0\mu)\varepsilon_{\text{Lem.2.4}}, \tag{3.4}$$

let $\varepsilon_{\text{Lem.2.6}} = \varepsilon_{\text{Lem.2.6}}(d_0, \varepsilon') \ > \ 0$ be the constant guaranteed by Lemma 2.6. We define

$$\varepsilon = \min\{\varepsilon_{\text{Lem.2.4}}, \varepsilon_{\text{Lem.2.6}}\}. \tag{3.5}$$

In all that follows, the constant $N_0 = N_0(F_0, \zeta)$ is assumed to be sufficiently large with respect to all constants discussed above.

3.2 Step II: Input H

Now, let H be a given k-uniform hypergraph on $n > N_0(F_0, \zeta)$ vertices. Our goal is to construct the promised F_0-packing $\mathscr{F}$ of H. Our next step, to that end, is to equip H with a fractional F_0-packing $\psi^* : \binom{H}{F_0} \to [0,1]$ which attains the maximum size $|\psi^*| = \nu^*_{F_0}(H)$. Recall, this subroutine is a linear programming problem (as mentioned in the Introduction) and is hence constructible in time polynomial in n.

Our next major steps will be to apply the Fractional-Crossing Lemma (Lemma 2.7) to provide a partition of $V(H)$, and then apply the Algorithmic Hypergraph Regularity Lemma (Theorem 2.2) to refine and regularize H.

3.3 Step III: Crossing and Regularizing H

With $\mu > 0$ given in (3.1), apply Lemma 2.7 to construct, in time $O(n^2)$, a partition $\Pi = (V_1, ..., V_\ell)$ of $V(H)$ $(\ell \leq L_0)$ which satisfies both $\lfloor \frac{n}{\ell} \rfloor \leq |V_i| \leq \lceil \frac{n}{\ell} \rceil$ and

$$|\psi^*_\Pi| \overset{\text{def}}{=} \sum_{F \in \binom{H}{F_0}_\Pi} \psi^*(F) \geq (1-\mu)|\psi^*| \tag{3.6}$$

(see notation of Chapter 1.)

With $\varepsilon > 0$ in (3.5), ℓ given above, and $t_0 = 1$, apply Theorem 2.2 to H and Π to obtain, in time $O(n^{2k-1}\log^2 n)$, a vertex partition $\hat{\Pi}$ with classes $V_0, |V_0| < \varepsilon n$, and V_{ij} $(1 \leq i \leq \ell, 1 \leq j \leq t,$ and $1 \leq t \leq T_0)$ so that both $V_{i1} \cup \cdots \cup V_{it} \subseteq V_i$ $(1 \leq i \leq \ell)$ and $|V_{11}| = \cdots = |V_{\ell t}| = m$. Moreover, $\hat{\Pi}$ has the property that all but $\varepsilon\binom{\ell}{k}t^k$-many k-tuples $(V_{i_1 j_1}, ..., V_{i_k j_k}), 1 \leq i_1 < \cdots < i_k \leq \ell, 1 \leq j_1, ..., j_k \leq t$, are ε-regular.

We now have a regularized hypergraph, but we now need to record which parts of it are of sufficient density.

3.4 Step IV: Recording Density in H

To record densities, we will construct the cluster hypergraph H_0 which will, in fact, be a weighted hypergraph.

To begin, define $V(H_0) = \{u_{ij} : 1 \leq i \leq \ell, 1 \leq j \leq t\}$, and consider the set of all $\binom{\ell}{k}t^k$ many k-tuples of the form $\{u_{i_1 j_1}, ..., u_{i_k j_k}\}$, where $1 \leq i_1 < \cdots < i_k \leq \ell$ and $1 \leq j_1, ..., j_k \leq t$. For each such k-tuple, with d_0 as in (3.3), define the weight function

$$\omega(\{u_{i_1j_1},...,u_{i_kj_k}\}) = \begin{cases} d_H(V_{i_1j_1},...,V_{i_kj_k}) & \text{if } d_H(V_{i_1j_1},...,V_{i_kj_k}) \geq d_0 \\ & \text{and } (V_{i_1j_1},...,V_{i_kj_k}) \text{ is } \varepsilon\text{-regular,} \\ \\ 0 & \text{otherwise.} \end{cases}$$

We define H_0 to consist of all k-tuples above whose edge-weight is nonzero. Together with the weight ω, H_0 is an edge-weighted k-uniform hypergraph on ℓt vertices, and since $\ell \leq L_0$ and $t \leq T_0$, the construction of H_0 is complete in time $O(1)$.

Our next step is to apply the δ-Bounded Lemma (Lemma 2.8) to the maximal fractional F_0-packing ψ^* so that we may then apply the Slicing Lemma (Lemma 2.6) to H.

3.5 Step V: δ-Bounding and Slicing $H_{\hat{\Pi}}$

With $\xi > 0$ given in (3.1), and the edge-weighted k-uniform hypergraph H_0, with weight function ω on $r = \ell t$ vertices constructed above, let $\hat{\psi}$ be a δ-bounded (see (3.2)) fractional (ω, F_0)-packing $\hat{\psi}$ of H_0 satisfying

$$|\hat{\psi}| \geq \nu^*_{F_0}(H_0) - \xi(\ell t)^k, \tag{3.7}$$

as guaranteed by Lemma 2.8. (Recall $\hat{\psi}$ may be constructed greedily in time $O(1)$.) We now use $\hat{\psi}$ to apply Lemma 2.6 to the hypergraph H.

Fix $e = \{u_{i_1j_1},...,u_{i_kj_k}\} \in H_0$. Write $\binom{H_0}{F_0}_e$ for the collection of all $F \in \binom{H_0}{F_0}$ for which $e \in F$, and write $\binom{H_0}{F_0}_e^+$ for those $F \in \binom{H_0}{F_0}$ for which $\hat{\psi}(F) \neq 0$, i.e., $\hat{\psi}(F) \geq \delta = d_0$ (see (3.3)). We shall apply Lemma 2.6 to the hypergraph $H[V_{i_1j_1},...,V_{i_kj_k}]$ with the constants $p_F = \hat{\psi}(F)$, over all $F \in \binom{H_0}{F_0}_e^+$. For that purpose, note that

$$\sum p_F = \sum_{F \in \binom{H_0}{F_0}_e^+} \hat{\psi}(F) \leq \omega(e) \stackrel{\text{def}}{=} d_H(V_{i_1j_1},...,V_{i_kj_k}),$$

as needed. Also, since $\varepsilon \leq \varepsilon_{\text{Lem.2.6}} = \varepsilon_{\text{Lem.2.6}}(d_0, \varepsilon' = \varepsilon_{\text{Lem.2.4}})$ is sufficient for the application of Lemma 2.6, we may now construct, in time $O(m^k)$, a partition

$$H[V_{i_1j_1},...,V_{i_kj_k}] = H_*[V_{i_1j_1},...,V_{i_kj_k}] \cup \bigcup_{F \in \binom{H_0}{F_0}_e^+} H_F[V_{i_1j_1},...,V_{i_kj_k}], \tag{3.8}$$

where $H_F[V_{i_1j_1},...,V_{i_kj_k}]$ is the slice from $H[V_{i_1j_1},...,V_{i_kj_k}]$ corresponding to the copy $F \in \binom{H_0}{F_0}_e^+$. Also, for each $F \in \binom{H_0}{F_0}_e^+$,

$$H_F[V_{i_1j_1},...,V_{i_kj_k}] \text{ is } (p_F = \hat{\psi}(F), \varepsilon' = \varepsilon_{\text{Lem.2.4}})\text{-regular}$$

(see (3.5)).

This concludes our preparations on H_0 and H. We now construct the F_0-packing of H.

3.6 Step VI: Constructing of the F_0-Packing $\mathscr{F}_H$

We now define the promised F_0-packing of H which will largely be obtained by Lemma 2.4. To that end, fix $F \in \binom{H_0}{F_0}^+$, that is, an $F \in \binom{H_0}{F_0}$ for which $\hat{\psi}(F) \neq 0$. Consider the following f-partite (recall $f = |V(F_0)|$) k-uniform subhypergraph $G_F \subseteq H$: set

$$V_{ij} \subseteq V(G_F) \Leftrightarrow u_{ij} \in V(F)$$

for all $1 \leq i \leq \ell$ and $1 \leq j \leq t$. For each edge $e = \{u_{i_1j_1},...,u_{i_kj_k}\} \in F$, recall that $H_F[V_{i_1j_1},...,V_{i_kj_k}]$ is the slice from $H[V_{i_1j_1},...,V_{i_kj_k}]$ corresponding to the copy $F \in \binom{H_0}{F_0}_e^+$ and set

$$H_F[V_{i_1j_1},...,V_{i_kj_k}] \subseteq G_F. \tag{3.9}$$

Otherwise, for $\{u_{i_1j_1},...,u_{i_kj_k}\} \in \binom{V(F)}{k} \setminus F$ (the complement of F), we take

$$G_F[V_{i_1j_1},...,V_{i_kj_k}] = \emptyset.$$

This defines the hypergraph G_F.

We now apply Lemma 2.4, the Packing Lemma, to the hypergraph G_F. Observe that Lemma 2.4 may be applied to G_F since, by construction, each edge $e = \{u_{i_1j_1},...,u_{i_kj_k}\} \in F$ has $G_F[V_{i_1j_1},...,V_{i_kj_k}]$ being $(\hat{\psi}(F), \varepsilon')$-regular, where $\varepsilon' = \varepsilon_{\text{Lem.2.4}}(d_0, \mu)$ was chosen in accordance with Lemma 2.4 (recall $\hat{\psi}(F) \geq d_0$), and otherwise, $G_F[V_{i_1j_1},...,V_{i_kj_k}] = \emptyset$. As such, Lemma 2.4 constructs, in time polynomial in m, an F_0-packing $\mathscr{F}_{G_F}$ of G_F covering all but $\mu|G_F|$ edges of G_F. We define

$$\mathscr{F}_H = \left\{ \mathscr{F}_{G_F} : F \in \binom{H_0}{F_0}^+ \right\}. \tag{3.10}$$

This defines the family $\mathscr{F}_H$ promised in Theorem 1.5. It remains to check several features.

3.7 Verifying that $\mathscr{F}_H$ is an F_0-packing of H

Let us first show that $\mathscr{F}_H$ is a family of edge-disjoint copies of F_0 in H. Indeed, let $F, F' \in \mathscr{F}_H$ be distinct. By construction of $\mathscr{F}_H$ (see (3.10)), $\exists \hat{F}, \hat{F}' \in \binom{H_0}{F_0}^+$ so that $F \in \mathscr{F}_{G_{\hat{F}}}$ and $F' \in \mathscr{F}_{G_{\hat{F}'}}$. If $\hat{F} = \hat{F}'$, then F and F' are edge disjoint by the application of Lemma 2.4. Henceforth, assume $\hat{F} \neq \hat{F}'$, and assume for contradiction that $e \in F \cap F'$. Let $e \in H[V_{i_1 j_1}, ..., V_{i_k j_k}]$ for some $1 \leq i_1 < \cdots < i_k \leq l$, $1 \leq j_1, ..., j_k \leq t$. It then follows that

$$e \in G_{\hat{F}}[V_{i_1 j_1}, ..., V_{i_k j_k}] \cap G_{\hat{F}'}[V_{i_1 j_1}, ..., V_{i_k j_k}],$$

or equivalently (c.f. (3.9)),

$$e \in H_{\hat{F}}[V_{i_1 j_1}, ..., V_{i_k j_k}] \cap H_{\hat{F}'}[V_{i_1 j_1}, ..., V_{i_k j_k}]. \tag{3.11}$$

However, the 'slices' $H_{\hat{F}}[V_{i_1 j_1}, ..., V_{i_k j_k}]$ and $H_{\hat{F}'}[V_{i_1 j_1}, ..., V_{i_k j_k}]$ are classes from a partition of $H[V_{i_1 j_1}, ..., V_{i_k j_k}]$. Therefore, (3.11) implies $H_{\hat{F}}[V_{i_1 j_1}, ..., V_{i_k j_k}] = H_{\hat{F}'}[V_{i_1 j_1}, ..., V_{i_k j_k}]$, and in particular (see (3.9)), $\hat{F} = \hat{F}'$, a contradiction. Therefore, $\mathscr{F}_H$ is an F_0-packing of H.

3.8 Verifying the Running Time

We next observe that $\mathscr{F}_H$ was constructed in polynomial time as is promised by Theorem 1.5. Indeed, to construct H_0 and $\hat{\psi}$, we applied Lemmas 2.7, 2.2 and 2.8, which require time $O(n^k)$, $O(n^{2k-1} \log^2 n)$ and $O(1)$, respectively. To construct the hypergraphs G_F, over $F \in \binom{H_0}{F_0}^+$, we applied Lemma 2.6 (at most $\binom{lt}{k} = O(1)$ times), which requires time $O(m^k)$. Finally, to construct the actual family $\mathscr{F}_H$, we applied Lemma 2.4 (at most $\binom{lt}{f} = O(1)$ times), which requires time polynomial in n. Indeed, the fractional F_0-packing ψ^* of H is constructed in polynomial time by linear programming. This confirms the promised running time of Theorem 1.5. It remains to show that $\mathscr{F}_H$ has the desired size.

3.9 Verifying the Size $|\mathscr{F}_H|$

To complete the proof of Theorem 1.5, we now show that $|\mathscr{F}_H| \geq \nu^*_{F_0}(H) - \zeta n^k$. To that end, observe that it follows from (3.10) that

$$|\mathscr{F}_H| = \sum_{F \in \binom{H_0}{F_0}^+} |\mathscr{F}_{G_F}|.$$

Observe that by Lemma 2.4, for fixed $F \in \binom{H_0}{F_0}^+$, $|\mathscr{F}_{G_F}| \geq (1-\mu)|G_F|/|F_0|$. Indeed, all but $\mu|G_F|$ edges of G_F are covered by copies of F_0 in $\mathscr{F}_{G_F}$, and each such copy covers $|F_0|$ edges of G_F. Thus,

$$|\mathscr{F}_H| \geq (1-\mu)\frac{1}{|F_0|} \sum_{F \in \binom{H_0}{F_0}^+} |G_F|,$$

and so by (3.9)

$$|\mathscr{F}_H| \geq (1-\mu)\frac{1}{|F_0|} \sum_{F \in \binom{H_0}{F_0}^+} \sum \{|H_F[V_{i_1j_1}, ..., V_{i_kj_k}]| : e = \{u_{i_1j_1}, ..., u_{i_kj_k}\} \in F\}. \quad (3.12)$$

For a fixed $e = \{u_{i_1j_1}, ..., u_{i_kj_k}\} \in F \in \binom{H_0}{F_0}^+$,

$$|H_F[V_{i_1j_1}, ..., V_{i_kj_k}]| \geq (\hat{\psi}(F) - \varepsilon')|V_{i_1j_1}| \cdots |V_{i_kj_k}| = (\hat{\psi}(F) - \varepsilon')m^k,$$

since $H_F[V_{i_1j_1}, ..., V_{i_kj_k}]$ is $(\hat{\psi}(F), \varepsilon')$-regular (by the application of Lemma 2.6). This along with (3.12) implies

$$\begin{aligned}
|\mathscr{F}_H| &\geq (1-\mu)\frac{1}{|F_0|} \sum_{F \in \binom{H_0}{F_0}^+} \sum_{e \in F} (\hat{\psi}(F) - \varepsilon')m^k \\
&= (1-\mu)\frac{m^k}{|F_0|} \sum_{F \in \binom{H_0}{F_0}^+} \hat{\psi}(F)\,(1 - \frac{\varepsilon'}{\hat{\psi}(F)})\,|F_0| \\
&\geq (1-\mu)(1 - \frac{\varepsilon'}{\delta})m^k \sum_{F \in \binom{H_0}{F_0}^+} \hat{\psi}(F),
\end{aligned}$$

where $\hat{\psi}(F) \geq \delta$ for all $F \in \binom{H_0}{F_0}^+$ (as provided by the application of Lemma 2.8). Next note that $|\hat{\psi}| \overset{\text{def}}{=} \sum_{F \in \binom{H_0}{F_0}} \hat{\psi}(F) = \sum_{F \in \binom{H_0}{F_0}^+} \hat{\psi}(F)$ since each $F \in \binom{H_0}{F_0} \setminus \binom{H}{F_0}^+$ satisfy $\hat{\psi}(F) = 0$.

Applying (3.7) to the inequality above, we get

$$|\mathscr{F}_H| \geq (1-\mu)(1-\frac{\varepsilon'}{\delta})m^k(\nu^*_{F_0}(H_0) - \xi(lt)^k).$$

We next state a necessary claim which we prove at the end of this chapter.

Claim 3.1 $m^k\nu^*_{F_0}(H_0) \geq \sum_{F \in \binom{H}{F_0}_\Pi} \psi^*(F) - 3\gamma n^k$.

Using Claim 3.1 and $ltm \leq n$,

$$|\mathscr{F}_H| \geq (1-\mu)(1-\frac{\varepsilon'}{\delta})m^k \left(\frac{1}{m^k} \sum_{F \in \binom{H}{F_0}_\Pi} \psi^*(F) - 3\gamma(lt)^k - \xi(lt)^k \right)$$

$$\geq (1-\mu)(1-\frac{\varepsilon'}{\delta}) \left(\sum_{F \in \binom{H}{F_0}_\Pi} \psi^*(F) - 4\gamma n^k \right).$$

Applying (3.6) to the inequality above and (3.1) as needed, we have

$$|\mathscr{F}_H| \geq (1-\mu)(1-\frac{\varepsilon'}{\delta})\left((1-\mu)|\psi^*| - 4\gamma n^k\right) \geq (1-2\mu)(1-\frac{\varepsilon'}{\delta})|\psi^*| - 4\gamma n^k$$

$$\geq |\psi^*| - (2\mu + \frac{\varepsilon'}{\delta})|\psi^*| - 4\gamma n^k \geq |\psi^*| - 7\mu n^k = |\psi^*| - \zeta n^k,$$

where we used $|\psi^*| \leq n^k$ and $\varepsilon' \leq \mu\delta$ (which is a consequence of (3.4).)

This completes the proof of Theorem 1.5.

Proof. [Proof of Claim 3.1]

It suffices to produce a fractional F_0-packing $\psi_0 : \binom{H_0}{F_0} \to [0,1]$ for which $m^k|\psi_0|$ has the lower bound of Claim 3.1. To produce ψ_0, we use the following notation. Define

$$H_{\hat{\Pi}} = \bigcup \{H[V_{i_1j_1}, ..., V_{i_kj_k}] : \{u_{i_1j_1}, ..., u_{i_kj_k}\} \in H_0\}.$$

In other words, $H_{\hat{\Pi}}$ consists of all edges $\{v_{i_1j_1}, ..., v_{i_kj_k}\}$ for which $v_{i_1j_1} \in V_{i_1j_1}$,...,$v_{i_kj_k} \in V_{i_kj_k}$, for some $1 \leq i_1 < \cdots < i_k \leq \ell, 1 \leq j_1, ..., j_k \leq t$, where $H[V_{i_1j_1}, ..., V_{i_kj_k}]$ has density at least $d_0 > 0$ and is ε-regular. Then, the mapping

$$\pi : V(H_{\hat{\Pi}}) \to V(H_0)$$

given by

$$v \mapsto u_{ij} \Longleftrightarrow v \in V_{ij}$$

defines a homomorphism from $H_{\hat{\Pi}}$ to H_0. Moreover, for each $F' \in \binom{H_{\hat{\Pi}}}{F_0}$ that crosses in $H_{\hat{\Pi}}$, we shall call the copy $F = \pi(F') \in \binom{H_0}{F_0}$ (of F_0 in H_0) the H_0-*projection* of F' in H_0.

Now, define the function $\psi_0 : \binom{H_0}{F_0} \to [0,1]$ by setting, for $F \in \binom{H_0}{F_0}$,

$$\psi_0(F) = \frac{1}{m^k} \sum \{\psi^*(F') : F' \in \tbinom{H_{\hat{\Pi}}}{F_0} \text{ has } H_0\text{-projection } F\}. \tag{3.13}$$

We first observe that ψ_0 is a fractional (ω, F_0)-packing of H_0. To that end, fix $e \in H_0$, and observe from (3.13) that

$$\sum_{e \in F \in \binom{H_0}{F_0}} \psi_0(F) = \sum_{e \in F \in \binom{H_0}{F_0}} \frac{1}{m^k} \sum \{\psi^*(F') : F' \in \tbinom{H_{\hat{\Pi}}}{F_0} \text{ has } H_0\text{-projection } F\}$$

$$= \frac{1}{m^k} \sum_{e' \in \pi^{-1}(e)} \sum \{\psi^*(F') : e' \in F' \in \tbinom{H_{\hat{\Pi}}}{F_0} \text{ is crossing}\}$$

$$\leq \frac{1}{m^k} \sum_{e' \in \pi^{-1}(e)} \sum \{\psi^*(F') : e' \in F' \in \tbinom{H}{F_0}\} \leq \frac{1}{m^k} \left|\pi^{-1}(e)\right| = \omega(e),$$

where we used that ψ^* is a fractional F_0-packing of H.

We now show the fractional F_0-packing $\psi_0 : \binom{H_0}{F_0} \to [0,1]$ has the required lower bound (for Claim (3.1)). To that end, consider

$$\left(\sum_{F \in \binom{H}{F_0}_{\Pi}} \psi^*(F) \right) - m^k |\psi_0|.$$

Using (3.13), this expression equals

$$\sum \left\{ \psi^*(F) : F \in \tbinom{H}{F_0}_{\Pi} \text{ such that } F \cap (H \backslash H_{\hat{\Pi}}) \neq \emptyset \right\}. \tag{3.14}$$

To analyze the sum in (3.14), write H_Π to denote those edges $e \in H$ which cross Π. Then, for a fixed term $F \in \binom{H}{F_0}_{\Pi}$ in (3.14), the construction of H_0 and $H_{\hat{\Pi}}$ ensures that there exists an edge $e \in F$ satisfying that $e \in H_\Pi \backslash H_{\hat{\Pi}}$, meaning that

1. $e \cap V_0 \neq \emptyset$, or

2. $e = \{v_{i_1j_1}, \dots, v_{i_kj_k}\}$ for $v_{i_1j_1} \in V_{i_1j_1}, \dots, v_{i_kj_k} \in V_{i_kj_k}, 1 \leq i_1 < \cdots < i_k \leq l$ and $1 \leq j_1, \dots, j_k \leq t$, where

   (a) $d_H(V_{i_1j_1}, \dots, V_{i_kj_k}) < d_0$, or

   (b) $(V_{i_1j_1}, \dots, V_{i_kj_k})$ is not ε-regular.

(Recall $\varepsilon, \delta_0 \leq l$ and $m \leq \lceil \frac{n}{lt} \rceil$.) However, at most

$$\varepsilon n \cdot n^{k-1} + \binom{l}{k} t^k \cdot d_0 m^k + \varepsilon \binom{l}{k} t^k m^k \leq 3\gamma n^k$$

edges $e \in H$ can satisfy the properties above, respectively. Therefore, returning to (3.14), we then have

$$\begin{aligned} &\sum \{\psi^*(F) : F \in \tbinom{H}{F_0}_\Pi \text{ such that } F \cap (H_\Pi \backslash H_{\hat{\Pi}}) \neq \emptyset\} \\ &\leq \sum_{e \in H_\Pi \backslash H_{\hat{\Pi}}} \sum \{\psi^*(F) : e \in F \in \tbinom{H_0}{F_0}\} \leq |H_\Pi \backslash H_{\hat{\Pi}}| \leq 3\gamma n^k, \end{aligned}$$

where the next to last inequality we once more used that ψ^* is a fractional F_0-packing of H. This completes the proof of Claim 3.1.

□

Chapter 4

Proof of the Hypergraph Packing Lemma

Our proof of the Hypergraph Packing Lemma (Lemma 2.4) follows similar lines to the proof of the Graph Packing Lemma (Theorem 1.8) of Haxell and Rödl [14]. (However, the context of hypergraphs makes the following proof quite technical.) The proof is nearly immediate from two technical tools (Theorem 4.1, Lemma 4.2). The first tool is a well-known result of Grable [13] concerning hypergraph packings, which we now define.

A packing $\mathscr{P}$ in a hypergraph P is a family of pairwise disjoint edges. In a hypergraph P and $x \in V(P)$, as usual, let $N_P(x) = \{Q : Q \cup x \in P\}$ denote the neighborhood of x in P, and for $x, x' \in V(P)$, write $N_P(x, x') = N_P(x) \cap N_P(x')$. As well, write $\deg_P(x) = |N_P(x)|$ and $\deg_P(x, x') = |N_P(x, x')|$.

Theorem 4.1 (Grable [13]) *For all integers p and $\lambda > 0$, there exists $\beta = \beta_{\mathrm{Thm.4.1}} > 0$ so that the following holds. Let P be a p-uniform hypergraph with sufficiently large vertex set X satisfying the following properties. For some $\Delta > 0$,*

1. $\deg_P(x) = (1 \pm \beta)\Delta$ *for all* $x \in X$.

2. $\deg_P(x, x') < \frac{\Delta}{(\log |X|)^4}$ *for all distinct* $x, x' \in X$.

Then P has a packing $\mathscr{P}$ which covers all but $\lambda|X|$ vertices of X, and moreover, $\mathscr{P}$ can be constructed in time polynomial in $|X|$.

The second tool (for the proof of Lemma 2.4) we call "The Hypergraph Extension Lemma". (One may wish to recall Setup 2.3.)

Lemma 4.2 (Hypergraph Extension Lemma) *For all integers $f \geq k \geq 2$ and all $d_0, \gamma > 0$, there exists $\delta = \delta_{\mathrm{Lem.4.2}} > 0$ so that the following holds. Let linear k-uniform hypergraph F_0 with vertex set $[f]$ be given, and let G be given as in Setup 2.3 with the constants d_0, $\varepsilon = \delta$ and a* *sufficiently large integer m. Then, there exists $G' \subseteq G$, where $|G'| > (1-\gamma)|G|$, so that for each $\{i_1, \ldots, i_k\} \in F_0$, every $\{v_{i_1}, \ldots, v_{i_k}\} \in G'[V_{i_1}, \ldots, V_{i_k}]$ belongs to within $(1 \pm \gamma)d^{|F_0|-1}m^{f-k}$ many partite-isomorphic copies of F_0 in G'. Moreover, the subhypergraph G' can be found in time $O(m^f)$.*

We shall prove Lemma 4.2 at the end of the chapter, and proceed with the proof of Lemma 2.4.

4.1 Proof of Lemma 2.4

Let F_0, d_0, and $\mu > 0$ be given as in Lemma 2.4. To define the promised constant $\varepsilon_{\text{Lem.2.4}} = \varepsilon_{\text{Lem.2.4}}(d_0, \mu) > 0$, we first consider some auxillary constants. With $p = f = |V(F_0)|$ and $\lambda = \mu/2$, let $\beta = \beta_{\text{Thm.4.1}} > 0$ be the constant guaranteed by Theorem 4.1. With $\gamma = \beta$, let $\delta = \delta_{\text{Lem.4.2}} > 0$ by the constant guaranteed by Lemma 4.2. We set $\varepsilon = \varepsilon_{\text{Lem.4.2}} = \delta$, and take m to be sufficiently large where ever needed.

Now, let G be as in Lemma 2.4. Apply Lemma 4.2 to G to obtain, in time $O(m^f)$, the subhypergraph $G' \subseteq G$ with the properties described there. As in Theorem 4.1, set $X = G'$ and define $P = \binom{G'}{F_0}$. In other words, each vertex of P is an edge of G', and each edge of P is a copy F of F_0 in G'. Note that a packing $\mathscr{P}$ of P corresponds to a F_0-packing of G'.

We now apply Theorem 4.1 to P. Note that, from Lemma 4.2, every vertex $x \in X = V(P) = G'$ satisfies $\deg_P(x) = (1 \pm \gamma)d^{|F_0|-1}m^{f-k}$. Setting $\Delta = d^{|F_0|-1}m^{f-k}$, we see $\deg_P(x) = (1 \pm \gamma)\Delta$. Note that, easily, for each $x \neq x' \in X$, $\deg_P(x, x') \leq m^{f-(k+1)} = O(\frac{1}{m}\Delta)$. Moreover, $|X| = \Theta(m^k)$, so $\deg_P(x, x') < \frac{\Delta}{\log^4 |X|}$.

Thus, by Theorem 4.1, P has a packing $\mathscr{P}$ covering all but $\lambda|X|$ vertices $x \in X$. This corresponds to an F_0-packing $\mathscr{F}$ covering all but $\lambda|G'|$ edges in G'. Together with the edges $G \backslash G'$, the F_0-packing $\mathscr{F}$ covers all but $2\lambda|G| = \mu|G|$ edges of G, as desired. This proves Lemma 2.4.

4.2 Proof of Lemma 4.2

To prove Lemma 4.2, we will use the following seemingly "weaker" version of it.

Lemma 4.3 *For all integers $f \geq k \geq 2$ and all $d_0, \zeta > 0$, there exists $\varepsilon = \varepsilon_{\text{Lem.4.3}} > 0$ so that the following holds. Let a linear k-uniform hypergraph F_0 with vertex set $[f]$ be given, and let G be given as in Setup 2.3 with the constants d_0, ε and a sufficiently large integer m. Then, for* *each* $\{i_1,\dots,i_k\} \in F_0$*, all but* ζm^k *elements* $\{v_{i_1},\dots,v_{i_k}\} \in G[V_{i_1},\dots,V_{i_k}]$ *belong to within* $(1 \pm \zeta)d^{|F_0|-1}m^{f-k}$ *many partite-isomorphic copies of* F_0 *in* G*.*

It is clear that Lemma 4.2 implies Lemma 4.3, but we need the converse to hold, a result which is not immediate (see Remark 4.4).

Remark 4.4 *To form* G'*, it would be natural to delete from* G *all* $|F_0|\zeta m^k$ *many edges which are "bad" in the sense of Lemma 4.3. Then, all remaining edges in* G' *extend to at most* $(1+\zeta)d^{|F_0|-1}m^{f-k}$ *copies* F *of* F_0 *in* G'*. The concern is that each such edge may not extend to at least* $(1-\zeta)d^{|F_0|-1}m^{f-k}$ *copies* F *of* F_0 *in* G' *(on account of deletion).*

We continue now with proving that Lemma 4.3 implies Lemma 4.2 and will prove Lemma 4.3 at the end of the chapter.

Let integers $f \geq k \geq 2$ and $d_0, \gamma > 0$ be given. To define the promised constant $\delta_{\text{Lem 4.2}} = \delta_{\text{Lem 4.2}}(f,k,d_0,\gamma) > 0$, first define an auxiliary constant $\zeta > 0$ to satisfy

$$4f^2 \frac{\sqrt{\zeta}}{d_0^{f^2}} < \gamma. \tag{4.1}$$

Now, let $\varepsilon_{\text{Lem 4.3}} = \varepsilon_{\text{Lem 4.3}}(f,k,d_0,\zeta) > 0$ be the constant guaranteed by Lemma 4.3, and set $\delta = \varepsilon_{\text{Lem 4.3}}$. Let linear k-uniform hypergraph F_0 and G be given as in Setup 2.3 with the constants d_0, δ and a sufficiently large integer m. To begin our proof of Lemma 4.2, we prepare to define the promised hypergraph $G' \subseteq G$, and require the following two considerations.

First, for a fixed $\{i_1,\dots,i_k\} \in F_0$, we shall call an edge $\{v_{i_1},\dots,v_{i_k}\} \in G[V_{i_1},\dots,V_{i_k}]$ a *good edge* if it satisfies the conclusion of Lemma 4.3, that is, it belongs to within $(1 \pm \zeta)d^{|F_0|-1}m^{f-k}$ many partite-isomorphic copies of F_0 in G. Otherwise, we call $\{v_{i_1},\dots,v_{i_k}\}$ a *bad edge*. (Clearly, good and bad edges can be distinguished in time $O(m^f)$.) One step in defining G' is to delete all bad edges from G, across all $\{i_1,\dots,i_k\} \in F_0$. Upon doing so, we shall call this (intermediate) hypergraph $G_1 \subseteq G$, where it follows by Lemma 4.3 that $|G_1| \geq |G| - |F_0|\zeta m^k$. Since $|F_0| \leq \frac{1}{3}\binom{f}{2} < f^2$ (owing to the linearity of F_0), we have that $|G_1| \geq |G| - \zeta f^2 m^k$.

Second, fix $1 \leq i \leq f$ and $i \in \{i_1,\dots,i_k\} = K \in F_0$. We shall call a vertex $v_i \in V_i$ a K*-bad vertex* if v_i belongs to at least $\sqrt{\zeta}m^{k-1}$ bad edges $\{v_{i_1},\dots,v_{i_k}\} \in G[V_{i_1},\dots,V_{i_k}]$. Note that, for K fixed above, at most $\sqrt{\zeta}m$ vertices $v_i \in V_i$ can be K-bad, since otherwise, we'd have ζm^k bad edges within $G[V_{i_1}, \ldots, V_{i_k}]$, contradicting Lemma 4.3. Now, call a vertex $v_i \in V_i$ a *bad vertex* if there exists any $K \in F_0$ for which v_i is a K-bad vertex, and call v_i a *good vertex* otherwise. Since, for any $i \in [f]$, $\deg_{F_0}(i) \leq (f-1)/2 < f$, there are at most $\sqrt{\zeta} fm$ bad vertices $v_i \in V_i$, and hence, at most $\sqrt{\zeta} f^2 m$ bad vertices in all of G.

Now, to define G', we simply induce the hypergraph G_1, defined above, on the good vertices of G (which is clearly constructible in time $O(m^k)$). Since each bad vertex of G can belong to at most $((f-1)/2))m^{k-1} < fm^k$ edges of G_1, we have that

$$|G'| > |G_1| - \sqrt{\zeta} f^2 m^k > |G| - \zeta f^2 m^k - \sqrt{\zeta} f^2 m^k > |G| - 2\sqrt{\zeta} f^2 m^k.$$

Since $|G| \geq |F_0|(d-\varepsilon)m^k > (d_0/2)m^k$, we thus have

$$|G'| > \left(1 - 4f^2 \frac{\sqrt{\zeta}}{d_0}\right)|G| \overset{(4.1)}{>} (1-\gamma)|G|,$$

as promised. We now verify the conclusion of Lemma 4.2.

To that end, fix $\{i_1, \ldots, i_k\} = K \in F_0$, and without loss of generality, assume $\{i_1, \ldots, i_k\} = \{1, \ldots, k\}$. Fix an edge $\{v_1, \ldots, v_k\} \in G'[V_1, \ldots, V_k]$. Let $\text{ext}_{F_0,G}(\{v_1, \ldots, v_k\})$ $(\text{ext}_{F_0,G'}(\{v_1, \ldots, v_k\}))$ denote the number of copies of F_0 in G (in G') containing the edge $\{v_1, \ldots, v_k\}$. Since, by construction, $\{v_1, \ldots, v_k\}$ is a good edge in G,

$$\text{ext}_{F_0,G}(\{v_1, \ldots, v_k\}) = (1 \pm \zeta) d^{|F_0|-1} m^{f-k},$$

and clearly,

$$\text{ext}_{F_0,G'}(\{v_1, \ldots, v_k\}) \leq \text{ext}_{F_0,G}(\{v_1, \ldots, v_k\}) \leq (1+\zeta) d^{|F_0|-1} m^{f-k}. \tag{4.2}$$

It remains to verify that $\text{ext}_{F_0,G'}(\{v_1, \ldots, v_k\})$ isn't too much smaller than $\text{ext}_{F_0,G}(\{v_1, \ldots, v_k\})$. To that end, fix $\{j_1, \ldots, j_k\} = K_1$, with $K_1 \in F_0$ and $K_1 \neq K$. We consider two cases.

Case 1. $(K \cap K_1 = \emptyset)$ In this case, note that

$$\left|(G \setminus G')[V_{j_1}, \ldots, V_{j_k}]\right| \leq 2\sqrt{\zeta} f^2 m^k. \tag{4.3}$$

That is to say, G and G' differ on $V_{j_1} \cup \cdots \cup V_{j_k}$ in at most $2\sqrt{\zeta} f^2 m^k$ edges, every one of which is bad in G, and missing in G'. Fix $\{v_{j_1}, \ldots, v_{j_k}\} \in G \setminus G'$. Clearly, at most m^{f-2k}

partite-isomorphic copies of F_0 in G can contain both $\{v_1, \dots, v_k\}$ and $\{v_{j_1}, \dots, v_{j_k}\}$, and all of these copies are lost from the original $\mathrm{ext}_{F_0,G}(\{v_1, \dots, v_k)-$many copies of F_0 of G containing $\{v_1, \dots, v_k\}$. Thus, (4.3) implies that, over all bad $\{v_{j_1}, \dots, v_{j_k}\} \in (G \setminus G')[V_{j_1}, \dots, V_{j_k}]$, the edge $\{v_1, \dots, v_k\}$ lost at most

$$2\sqrt{\zeta} f^2 m^k \times m^{f-2k} = 2\sqrt{\zeta} f^2 m^{f-k}$$

many copies of F_0 from G for the reason of Case 1.

Case 2. ($K \cap K_1 \neq \emptyset$) Since F_0 is a linear hypergraph, it must be the case that $|K \cap K_1| = 1$. Set $i = K \cap K_1$ and write $v_i \in \{v_1, \dots, v_k\}$ to satisfy $v_i \in V_i$. Fix $\{v_{j_1}, \dots, v_{j_k}\} \in G \setminus G'$ where we assume, for sake of argument, that $v_i \in \{v_{j_1}, \dots, v_{j_k}\}$ (so that the removal of $\{v_{j_1}, \dots, v_{j_k}\}$ from G impacts $\mathrm{ext}_{F_0,G'}(\{v_1, \dots, v_k\})$. Now, by construction, v_i is a good vertex, and hence, a K_1-good vertex. Thus, since $\{v_{j_1}, \dots, v_{j_k}\}$ is bad, it can be one of only at most $\sqrt{\zeta} m^{k-1}$ edges deleted from G which contain v_i. Since $\{v_1, \dots, v_k\}$ and $\{v_{j_1}, \dots, v_{j_k}\}$ constitute $2k - 1$ distict vertices, there can be at most $m^{f-2k+1}-$many copies of F_0 in G containing both these edges. Thus, the goodness of v_i ensures that, over all bad $\{v_{j_1}, \dots, v_{j_k}\} \in G[V_{j_1}, \dots, V_{j_k}]$ containing v_i, the edge $\{v_1, \dots, v_k\}$ lost at most

$$\sqrt{\zeta} m^{k-1} \times m^{f-2k+1} = \sqrt{\zeta} m^{f-k}$$

many copies of F_0 from G for the reason of Case 2.

Over all $\{j_1, \dots, j_k\} = K_1 \in F_0$ distinct from $\{1, \dots, k\} = K \in F_0$, Cases 1 and 2 imply that

$$\begin{aligned}\mathrm{ext}_{F_0,G'}(\{v_1, \dots, v_k\}) &\geq \mathrm{ext}_{F_0,G}(\{v_1, \dots, v_k\}) - \left((|F_0| - 1) \left(2\sqrt{\zeta} f^2 m^{f-k} + \sqrt{\zeta} m^{f-k} \right) \right) \\ &\overset{(4.2)}{\geq} (1 - \zeta) d^{|F_0|-1} m^{f-k} - 3\sqrt{\zeta} f^4 m^{f-k} \geq \left(1 - \zeta - 3f^4 \frac{\sqrt{\zeta}}{d_0^{f^2}} \right) d^{|F_0|-1} m^{f-k} \\ &\overset{(4.1)}{>} (1 - \gamma) d^{|F_0|-1} m^{f-k}.\end{aligned}$$

The above inequality and (4.2) imply that $\mathrm{ext}_{F_0,G'}(\{v_1, \dots, v_k\}) = (1 \pm \gamma) d^{|F_0|-1} m^{f-k}$, which concludes the proof of Lemma 4.2.

4.3 Proof of Lemma 4.3

To prove Lemma 4.3, we shall use the following result from [19] (see also [18]).

Theorem 4.5 (Counting Lemma for Linear Hypergraphs) *For all integers $f_1 \geq k \geq 2$ and all $d_0, \tau > 0$, there exists $\delta = \delta_{\text{Thm.4.5}} > 0$ so that the following holds. Let linear k-uniform hypergraph F_1 with vertex set $[f_1]$ be given, and let G be given as in Setup 2.3 with the constants d_0, $\varepsilon = \delta$ and a sufficiently large integer m. Then, the number of partite-isomorphic copies of F_1 in G, which we write as $\#\{F_1 \subset G\}$, satisfies*

$$\#\{F_1 \subset G\} = (1 \pm \tau) d^{|F_1|} m^{f_1}.$$

Now, let integers $f \geq k \geq 2$ be given and let $d_0, \zeta > 0$ be given. Define auxiliary constant $\tau = \zeta^3/6$. Let $\delta_{\text{Thm.4.5}}(f, k, d_0, \tau) > 0$ be the constant guaranteed by Theorem 4.5 for the parameters $f_1 = f$, k, d_0 and τ. Let $\delta_{\text{Thm.4.5}}(2f - k, k, d_0, \tau) > 0$ be the constant guaranteed by Theorem 4.5 for the parameters $f_1 = 2f - k$, k, d_0 and τ. Let $\varepsilon_0 > 0$ be small enough so that each of the following inequalities holds:

$$(1 + \tau)\left(1 - \varepsilon_0 d_0^{-1}\right)^{-1} \leq 1 + 2\tau \quad \text{and} \quad (1 - \tau)\left(1 + \varepsilon_0 d_0^{-1}\right)^{-1} \geq 1 - 2\tau. \tag{4.4}$$

Define $\varepsilon = \min\{\varepsilon_0, \delta_{\text{Thm.4.5}}(f, k, d_0, \tau), \delta_{\text{Thm.4.5}}(2f - k, k, d_0, \tau)\}$. Let F_0 and G be given as in Setup 2.3 with the constants d_0, ε and a sufficiently large integer m. Throughout this proof, we fix $\{i_1, \ldots, i_k\} \in F_0$, and assume, without loss of generality, that $i_1 = 1, \ldots, i_k = k$.

Define the following hypergraph F_0^2, which will necessarily contain the hypergraph F_0. Let

$$V(F_0^2) = \{1, \ldots, k, k+1, \ldots, f\} \cup \{(k+1)', \ldots, f'\}$$

so that F_0^2 has $2f - k$ vertices. Include every edge of F_0 in F_0^2. To define the remaining edges, suppose $K \in \binom{V(F_0^2)}{k}$ has the form that, for some $1 \leq \ell \leq k$, K includes vertices $\{i_1, \ldots, i_\ell\} \subseteq \{1, \ldots, k\}$, and otherwise, $K \setminus \{i_1, \ldots, i_\ell\} = \{i'_{\ell+1}, \ldots, i'_k\} \subseteq \{(k+1)', \ldots, f'\}$. Then,

$$\text{include } K \in F_0^2 \text{ if and only if, } \{i_1, \ldots, i_\ell, i_{\ell+1}, \ldots, i_k\} \in F_0.$$

In other words, $K = \{i_1, \ldots, i_\ell, i'_{\ell+1}, \ldots, i'_k\} \in F_0^2$ would be a copy of the edge $\{i_1, \ldots, i_k\} \in F_0$. Note that F_0^2 is a linear hypergraph with $2f - k$ vertices and $2|F_0| - 1$ edges.

We also define the following hypergraph G^2, which will necessarily contain the hypergraph G. For $k+1 \le t \le f$, let V_t' be a copy of the class V_t. Let

$$V(G^2) = V_1 \cup \cdots \cup V_k \cup V_{k+1} \cup \cdots \cup V_f \cup V_{k+1}', \ldots, V_f'$$

be a $(2f-k)$-partition. Include every edge of G in G^2. To define the remaining edges, suppose $K \in \binom{V(F_0^2)}{k}$ has the form that, for some $1 \le \ell \le k$, K includes vertices $\{i_1, \ldots, i_\ell\} \subseteq \{1, \ldots, k\}$, and otherwise, $K \setminus \{i_1, \ldots, i_\ell\} = \{i_{\ell+1}', \ldots, i_k'\} \subseteq \{(k+1)', \ldots, f'\}$. Then, let

$$G_K^2 = G^2[V_{i_1}, \ldots, V_{i_\ell}, V_{i_{\ell+1}}', \ldots, V_{i_k}'] \text{ be a copy of } G^2[V_{i_1}, \ldots, V_{i_\ell}, V_{i_{\ell+1}}, \ldots, V_{i_k}],$$

and define

$$G^2 = \bigcup \left\{ G_K^2 : K \in \binom{V(F_0^2)}{k} \right\}.$$

We now make the following observations (see upcoming (4.5) and (4.7)). To begin, $\{v_1, \ldots, v_k\} \in G[V_1, \ldots, V_k]$, and write $\text{ext}_{F_0,G}(\{v_1, \ldots, v_k\})$ for the number of partite-isomorphic copies of F_0 in G containing the edge $\{v_1, \ldots, v_k\}$. (Recall we assume $\{1, \ldots, k\} \in F_0$). Then,

$$\#\{F_0 \subset G\} = \sum \left\{ \text{ext}_{F_0,G}(\{v_1, \ldots, v_k\}) : \{v_1, \ldots, v_k\} \in G[V_1, \ldots, V_k] \right\}.$$

Then, Theorem 4.5 implies that

$$\sum \left\{ \text{ext}_{F_0,G}(\{v_1, \ldots, v_k\}) : \{v_1, \ldots, v_k\} \in G[V_1, \ldots, V_k] \right\} \ge d^{|F_0|} m^f (1-\tau).$$

Since, by the hypothesis of Setup 2.3, we have $|G[V_1, \ldots, V_k]| = (d \pm \varepsilon) m^k$, where $d \ge d_0$, the inequality above implies

$$\begin{aligned} \sum \left\{ \text{ext}_{F_0,G}(\{v_1, \ldots, v_k\}) : \{v_1, \ldots, v_k\} \in G[V_1, \ldots, V_k] \right\} \\ \ge d^{|F_0|-1} m^{f-k} |G[V_1, \ldots, V_k]| (1-\tau) \left(1 + \varepsilon d_0^{-1}\right)^{-1} \\ \overset{(4.4)}{\ge} d^{|F_0|-1} m^{f-k} |G[V_1, \ldots, V_k]| (1-2\tau). \quad (4.5) \end{aligned}$$

Similarly, for $\{v_1, \ldots, v_k\} \in G[V_1, \ldots, V_k] = G^2[V_1, \ldots, V_k]$, write $\text{ext}_{F_0^2,G^2}(\{v_1, \ldots, v_k\})$ for the number of partite-isomorphic copies of F_0^2 in G^2 containing the edge $\{v_1, \ldots, v_k\}$. Then,

$$\#\{F_0^2 \subset G^2\} = \sum \left\{ \text{ext}_{F_0^2,G^2}(\{v_1, \ldots, v_k\}) : \{v_1, \ldots, v_k\} \in G[V_1, \ldots, V_k] \right\},$$

and Theorem 4.5 (applied with $F_1 = F_0^2$) implies that

$$\sum \left\{ \text{ext}_{F_0^2, G^2}(\{v_1, \ldots, v_k\}) : \{v_1, \ldots, v_k\} \in G[V_1, \ldots, V_k] \right\} \leq d^{|F_0^2|} m^{|V(F_0^2)|}(1+\tau). \tag{4.6}$$

However, $|F_0^2| = 2|F_0| - 1$, $|V(F_0^2)| = 2f - k$, and for some fixed $\{v_1, \ldots, v_k\} \in G[V_1, \ldots, V_k]$, we have

$$\text{ext}_{F_0^2, G^2}(\{v_1, \ldots, v_k\}) = \text{ext}^2_{F_0, G}(\{v_1, \ldots, v_k\}).$$

Since $|G[V_1, \ldots, V_k]| = (d \pm \varepsilon)m^k$, the inequality (4.6) implies

$$\begin{aligned} \sum \left\{ \text{ext}^2_{F_0, G}(\{v_1, \ldots, v_k\}) : \{v_1, \ldots, v_k\} \in G[V_1, \ldots, V_k] \right\} \\ \leq d^{2|F_0|} m^{2f-2k} |G[V_1, \ldots, V_k]|(1+\tau) \left(1 - \varepsilon d_0^{-1}\right)^{-1} \\ \stackrel{(4.4)}{\leq} \left(d^{|F_0|} m^{f-k}\right)^2 |G[V_1, \ldots, V_k]|(1+3\tau). \end{aligned} \tag{4.7}$$

Comparing (4.5) and (4.7) and using the Cauchy-Schwarz inequality (see Fact 9.1), we see that all but $6\tau^{1/3}|G[V_1, \ldots, V_k]| \leq \zeta m^k$ elements $\{v_1, \ldots, v_k\} \in G[V_1, \ldots, V_k]$ satisfy the conclusion of Lemma 4.3, as promised.

Chapter 5

Proof of the Hypergraph Slicing Lemma

Our proof of the Hypergraph Slicing Lemma (Lemma 2.6) follows similar lines to the proof of the Graph Slicing Lemma (Theorem 1.9) of Haxell and Rödl [14]. (However, once again the context of hypergraphs makes the following proof quite technical.) We shall use the following statement, which extends one of Haxell and Rödl (Lemma 16 in [14]).

Lemma 5.1 (Miniature Slicing Lemma) *For every real number $0 < \varsigma < 1$ and every integer $s \geq 1$, there exists a sufficiently large $m_0 = m_0(\varsigma, s)$ so that the following holds.*

Let $K[A_1, ..., A_k]$ be the complete k-uniform k-partite hypergraph with vertex partition $A_1 \cup \cdots \cup A_k$, where $|A_1| = \cdots = |A_k| = m_0$. Let $q_1, ..., q_s > 0$ be given so that $q_0 = 1 - \sum_{i=0}^{s} q_i \geq 0$. Then, there exists a partition $K[A_1, ..., A_k] = \mathcal{J}_0 \cup \mathcal{J}_1 \cup \cdots \cup \mathcal{J}_s$ with the following property.

For every $w : \bigcup_{j=1}^{k} A_j \to [0,1]$ with $w(A_j) \overset{def}{=} \sum_{a \in A_j} w(a) \geq \varsigma m_0$ for all $1 \leq j \leq k$ and for every i, $0 \leq i \leq s$, we have

$$(q_i - \varsigma) \prod_{j=1}^{k} w(A_j) \leq \sum_{\{a_1, ..., a_k\} \in \mathcal{J}_i} w(a_1) \cdots w(a_k) \leq (q_i + \varsigma) \prod_{j=1}^{k} w(A_j).$$

We refer to the lemma above as a "Miniature Slicing Lemma" because it concerns slicing a hypergraph of fixed order. (Hence, the slices described above can be constructed by an exhaustive search.)

5.1 Proof of Lemma 2.6

Let $d_0, \varepsilon' > 0$ be given. Set $\varsigma = \frac{\varepsilon'}{(k2^{k+1})}$. For an integer $1 \leq s \leq \lceil 1/d_0 \rceil$, let $m_0(s) = m_0(\varsigma, s)$ be the integer guaranteed by Lemma 5.1. Let $\varepsilon(s) = \frac{(\varepsilon')^{k+1}}{5km_0(s)}$. Now, set
$\varepsilon = \min \varepsilon(s)$, where the minimum is taken over all $1 \leq s \leq \lceil 1/d_0 \rceil$.

Let G be given as in Lemma 2.6. For some integer s, let $p_1, ..., p_s \geq d_0$ be given satisfying that $\sum_{i=1}^{s} p_i \leq d_G(V_1, ..., V_k) \stackrel{\text{def}}{=} D$. Note that $1 \leq s \leq D/d_0 \leq \lceil 1/d_0 \rceil$, as we encountered when previously defining $\varepsilon > 0$.

To define the promised partition $G = G_0 \cup G_1 \cup \cdots \cup G_s$, we make two auxiliary considerations. First, consider the complete k-partite k-uniform hypergraph $K[A_1, ..., A_k]$, where $A_1, ..., A_k$ are arbitrary sets of size $|A_1| = \cdots = |A_k| = m_0$. Next, for each $1 \leq i \leq s$, set $q_i = \frac{p_i}{D}$ and $q_0 = 1 - \sum_{i=1}^{s} q_i$. Then, let $K[A_1, ..., A_k] = \mathcal{J}_0 \cup \mathcal{J}_1 \cup \cdots \cup \mathcal{J}_s$ be the partition guaranteed by Lemma 5.1.

Second, refine the vertex classes $A_1, ..., A_k$ as follows. For each of the sets A_j above, $1 \leq j \leq k$, write $A_j = \{a_{j1}, ..., a_{jm_0}\}$. Next, for each $a_{j\ell_j} \in A_j$ $(1 \leq j \leq k, 1 \leq \ell_j \leq m_0)$, choose a subset $V_{j\ell_j} \subset V_j$ of size

$$|V_{j\ell_j}| = \left\lfloor \frac{m}{m_0} \right\rfloor \stackrel{\text{def}}{=} \hat{m} \qquad \text{so that} \qquad V_j = V_{j0} \cup \bigcup_{a_{j\ell_j} \in A_j} V_{j\ell_j}$$

is a partition. Note that the class V_{j0} is the remainder of size at most $m_0 - 1$ for all $1 \leq j \leq k$.

Now for each $1 \leq j \leq k$, fix a choice $0 \leq \ell_j \leq m_0$ and consider $G[V_{1\ell_1}, ..., V_{k\ell_k}]$. If $\ell_j = 0$ for any $1 \leq j \leq k$, put $G[V_{1\ell_1}, ..., V_{k\ell_k}] \subset G_0$. Otherwise for each $0 \leq i \leq s$, put

$$G[V_{1\ell_1}, ..., V_{k\ell_k}] \subset G_i \Longleftrightarrow \{a_{1\ell_1}, ..., a_{k\ell_k}\} \in \mathcal{J}_i.$$

This defines the partition $G = G_0 \cup G_1 \cup \cdots \cup G_s$ as promised by Lemma 2.6. Note that this partition is constructed in time $O(m^k)$.

It remains to check that each G_i, $1 \leq i \leq s$ is (p_i, ε')-regular. To that end, fix $1 \leq i \leq s$, and for each $1 \leq j \leq k$, let $V_j' \subseteq V_j$ be given with $|V_j'| \geq \varepsilon' m$. We will show that $d_{G_i}(V_1', ..., V_k') = p_i \pm \varepsilon'$.

To that end, we first establish a few 'underlying' considerations. First, for each $1 \leq j \leq k$ and $1 \leq \ell_j \leq m_0$, write

$$V_{j\ell_j}' = V_j' \cap V_{j\ell_j} \text{ and } w(a_{j\ell_j}) \stackrel{\text{def}}{=} \frac{|V_{j\ell_j}'|}{|V_{j\ell_j}|} = \frac{|V_{j\ell_j}'|}{\hat{m}}.$$

Then,

$$w(A_j) \stackrel{\text{def}}{=} \sum_{\ell_j=1}^{m_0} w(a_{j\ell_j}) = \frac{\sum_{\ell_j=1}^{m_0} |V_{j\ell_j}'|}{\hat{m}} = \frac{|V_j' \backslash V_{j0}|}{\hat{m}}. \tag{5.1}$$

Recalling that $|V_{j0}| \leq m_0 - 1$ and $|V_j'| \geq \varepsilon' m$ for all $1 \leq j \leq k$, we have

$$w(A_j) \geq \frac{\varepsilon' m - (m_0 - 1)}{\hat{m}} \geq \varsigma m_0.$$

Second, for $1 \leq j \leq k$ and $1 \leq \ell_j \leq m_0$, we say $a_{j\ell_j}$ is ε-*big* if

$$|V'_{j\ell_j}| > \varepsilon m \qquad \iff \qquad w(a_{j\ell_j}) > \frac{\varepsilon m}{\hat{m}} = \frac{\varepsilon m}{\left\lfloor \frac{m}{m_0} \right\rfloor} > \varepsilon m_0,$$

and ε-*small* otherwise. In addition, for $1 \leq i \leq s$ fixed above, let $\mathcal{J}_i^+$ denote the set of all $\{a_{1\ell_1}, ..., a_{k\ell_k}\} \in \mathcal{J}_i$ for which every $a_{j\ell_j}$ $(1 \leq j \leq k, 1 \leq \ell_j \leq m_0)$ is ε-big, and let $\mathcal{J}_i^- = \mathcal{J}_i \backslash \mathcal{J}_i^+$ denote the set of all $\{a_{1\ell_1}, ..., a_{k\ell_k}\} \in \mathcal{J}_i$ for which some $a_{j\ell_j}$ $(1 \leq j \leq k, 1 \leq \ell_j \leq m_0)$ is ε-small.

Now, for $1 \leq i \leq s$ fixed above,

$$d_{G_i}(V'_1, ..., V'_k) = \frac{|E_{G_i}(V'_1, ..., V'_k)|}{|V'_1| \cdots |V'_k|} = \sum_{\{a_{1\ell_1}, ..., a_{k\ell_k}\} \in \mathcal{J}_i} \frac{|E_G(V'_{1\ell_1}, ..., V'_{k\ell_k})|}{|V'_1| \cdots |V'_k|}$$

$$= \sum_{\{a_{1\ell_1}, ..., a_{k\ell_k}\} \in \mathcal{J}_i^+} \frac{|E_G(V'_{1\ell_1}, ..., V'_{k\ell_k})|}{|V'_1| \cdots |V'_k|} + \sum_{\{a_{1\ell_1}, ..., a_{k\ell_k}\} \in \mathcal{J}_i^-} \frac{|E_G(V'_{1\ell_1}, ..., V'_{k\ell_k})|}{|V'_1| \cdots |V'_k|}. \tag{5.2}$$

Also, since $\frac{|V_{j\ell_j}|\, w(a_{j\ell_j})}{|V'_{j\ell_j}|} = 1$ for each $1 \leq j \leq k$,

$$\sum_{\{a_{1\ell_1}, ..., a_{k\ell_k}\} \in \mathcal{J}_i^+} \frac{|E_G(V'_{1\ell_1}, ..., V'_{k\ell_k})|}{|V'_1| \cdots |V'_k|}$$

$$= \sum_{\{a_{1\ell_1}, ..., a_{k\ell_k}\} \in \mathcal{J}_i^+} \frac{|V_{1\ell_1}| \cdots |V_{k\ell_k}|}{|V'_1| \cdots |V'_k|} w(a_{1\ell_1}) \cdots w(a_{k\ell_k}) \frac{|E_G(V'_{1\ell_1}, ..., V'_{k\ell_k})|}{|V'_{1\ell_1}| \cdots |V'_{k\ell_k}|}. \tag{5.3}$$

Applying this to (5.2), we have

$$d_{G_i}(V'_1, ..., V'_k) = \sum_{\{a_{1\ell_1}, ..., a_{k\ell_k}\} \in \mathcal{J}_i^+} \frac{|V_{1\ell_1}| \cdots |V_{k\ell_k}|}{|V'_1| \cdots |V'_k|} w(a_{1\ell_1}) \cdots w(a_{k\ell_k}) \frac{|E_G(V'_{1\ell_1}, ..., V'_{k\ell_k})|}{|V'_{1\ell_1}| \cdots |V'_{k\ell_k}|}$$

$$+ \sum_{\{a_{1\ell_1}, ..., a_{k\ell_k}\} \in \mathcal{J}_i^-} \frac{|E_G(V'_{1\ell_1}, ..., V'_{k\ell_k})|}{|V'_1| \cdots |V'_k|}. \tag{5.4}$$

We shall now focus on bounding $d_{G_i}(V'_1, ..., V'_k)$ from above. First, note that

$$V_{j\ell_j} = \left\lfloor \frac{m}{m_0} \right\rfloor \leq \frac{|V'_j|}{w(A_j)} \tag{5.5}$$

for each $1 \leq j \leq k$. Also, for any $\{a_{1\ell_1}, ..., a_{k\ell_k}\} \in \mathcal{J}_i^+$, we know by the e-regularity of G that

$$\frac{|E_G(V'_{1\ell_1}, ..., V'_{k\ell_k})|}{|V'_{1\ell_1}| \cdots |V'_{k\ell_k}|} = d_G(V_1, ..., V_k) \pm \varepsilon = D \pm \varepsilon. \tag{5.6}$$

In addition, since w satisfies the conditions of Lemma 5.1, we can conclude that

$$\sum_{\{a_{1\ell_1},...,a_{k\ell_k}\}\in\mathcal{J}_i^+} w(a_{1\ell_1})\cdots w(a_{k\ell_k}) \leq (q_i \pm \varsigma)\prod_{j=1}^{k} w(A_j). \tag{5.7}$$

Applying these to (5.3), we have

$$\begin{aligned}
&\sum_{\{a_{1\ell_1},...,a_{k\ell_k}\}\in\mathcal{J}_i^+} \frac{|E_G(V'_{1\ell_1},...,V'_{k\ell_k})|}{|V'_1|\cdots|V'_k|} \\
&\overset{(5.5)}{\leq} \frac{1}{w(A_1)\cdots w(A_k)} \sum_{\{a_{1\ell_1},...,a_{k\ell_k}\}\in\mathcal{J}_i^+} w(a_{1\ell_1})\cdots w(a_{k\ell_k}) \frac{|E_G(V'_{1\ell_1},...,V'_{k\ell_k})|}{|V'_{1\ell_1}|\cdots|V'_{k\ell_k}|} \\
&\overset{(5.6)}{\leq} \frac{D+\varepsilon}{w(A_1)\cdots w(A_k)} \sum_{\{a_{1\ell_1},...,a_{k\ell_k}\}\in\mathcal{J}_i^+} w(a_{1\ell_1})\cdots w(a_{k\ell_k}) \\
&\overset{(5.7)}{\leq} \frac{D+\varepsilon}{w(A_1)\cdots w(A_k)}(q_i+\varsigma)\prod_{j=1}^{k} w(A_j) = (D+\varepsilon)(q_i+\varsigma)
\end{aligned} \tag{5.8}$$

Alternatively, if $\{a_{1\ell_1},...,a_{k\ell_k}\}\in\mathcal{J}_i^-$, then $|G_i(V'_{1\ell_1},...,V'_{k\ell_k})| < \varepsilon m^k$. Thus,

$$\begin{aligned}
\sum_{1\leq \ell_1,...,\ell_k\leq m_0} \Big\{|E_{G_i}(V'_{1\ell_1},V'_{2\ell_2},...,V'_{k\ell_k})| : \text{ some } |V'_{j\ell_j}| < \varepsilon m\Big\} &< m_0^k \varepsilon m \left\lfloor \frac{m}{m_0}\right\rfloor^{k-1} \\
&\leq m_0\varepsilon m^k.
\end{aligned} \tag{5.9}$$

This gives that

$$\sum_{\{a_{1\ell_1},...,a_{k\ell_k}\}\in\mathcal{J}_i^-} \frac{|E_G(V'_{1\ell_1},...,V'_{k\ell_k})|}{|V'_1|\cdots|V'_k|} \leq \frac{km_0\varepsilon m^k}{|V'_1|\cdots|V'_k|} \leq \frac{km_0\varepsilon m^k}{(\varepsilon' m)^k} = \frac{km_0\varepsilon}{(\varepsilon')^k}$$

where we used that $|V'_j| \geq \varepsilon'|V_j| = \varepsilon' m$ for all $1\leq j\leq k$ in the second inequality. Finally, when this and (5.8) are applied to (5.4) (recall $1\leq i\leq s$ is fixed), we have

$$d_{G_i}(V'_1,...,V'_k) \leq (D+\varepsilon)(q_i+\varsigma) + \frac{km_0\varepsilon}{(\varepsilon')^k}.$$

Using $p_i = Dq_i$, $\varsigma = \frac{\varepsilon'}{(k2^{k+1})}$ and $\varepsilon = \frac{(\varepsilon')^{k+1}}{5km_0}$, we infer

$$d_{G_i}(V'_1,...,V'_k) \leq p_i + \varsigma + 2\varepsilon + \frac{km_0}{(\varepsilon')^k}\varepsilon \leq p_i + \frac{2\varepsilon'}{3} + \frac{\varepsilon'}{5} \leq p_i + \varepsilon',$$

which proves the upper bound.

For the lower bound, we first note that for the fixed $1 \le i \le s$, and using (5.4),

$$d_{G_i}(V_1', ..., V_k') \ge \sum_{\{a_{1\ell_1},...,a_{k\ell_k}\}\in \mathcal{J}_i^+} \frac{|V_{1\ell_1}|\cdots|V_{k\ell_k}|}{|V_1'|\cdots|V_k'|} w(a_{1\ell_1})\cdots w(a_{k\ell_k}) \frac{|E_G(V_{1\ell_1}', ..., V_{k\ell_k}')|}{|V_{1\ell_1}'|\cdots|V_{k\ell_k}'|}.$$

First, applying (5.6) and that $|V_{j\ell_j}| = \left\lfloor \frac{m}{m_0} \right\rfloor$, we have

$$d_{G_i}(V_1', ..., V_k') \ge (D-\varepsilon)\frac{(\lfloor m/m_0 \rfloor)^k}{|V_1'|\cdots|V_k'|} \sum_{\{a_{1\ell_1},...,a_{k\ell_k}\}\in \mathcal{J}_i^+} w(a_{1\ell_1})\cdots w(a_{k\ell_k}). \tag{5.10}$$

Next, we compare

$$\sum_{\{a_{1\ell_1},...,a_{k\ell_k}\}\in \mathcal{J}_i} w(a_{1\ell_1})\cdots w(a_{k\ell_k}) \qquad \text{and} \qquad \sum_{\{a_{1\ell_1},...,a_{k\ell_k}\}\in \mathcal{J}_i^+} w(a_{1\ell_1})\cdots w(a_{k\ell_k}).$$

To that end, if $\{a_{1\ell_1}, ..., a_{k\ell_k}\} \in \mathcal{J}_i^+$, then $|V_{j\ell_j}'| \ge \varepsilon m$ for all $1 \le j \le k$ and $1 \le l_j \le m_0$ which is equivalent to $w(a_{j\ell_j}) = \frac{|V_{j\ell_j}'|}{|V_{j\ell_j}|} = \frac{\varepsilon m}{\left\lfloor \frac{m}{m_0} \right\rfloor} \ge \varepsilon m_0$. Therefore,

$$\sum_{\{a_{1\ell_1},...,a_{k\ell_k}\}\in \mathcal{J}_i^+} w(a_{1\ell_1})\cdots w(a_{k\ell_k}) =$$

$$\sum_{\{a_{1\ell_1},...,a_{k\ell_k}\}\in \mathcal{J}_i} \left\{ w(a_{1\ell_1})\cdots w(a_{k\ell_k}) : \begin{array}{c} w(a_{j\ell_j}) \ge \varepsilon m_0 \text{ for all} \\ 1 \le j \le k \text{ and } 1 \le l_j \le m_0 \end{array} \right\}$$

Conversely,

$$\sum_{\{a_{1\ell_1},...,a_{k\ell_k}\}\in \mathcal{J}_i} \left\{ w(a_{1\ell_1})\cdots w(a_{k\ell_k}) : \begin{array}{c} w(a_{j\ell_j}) < \varepsilon m_0 \text{ for some} \\ 1 \le j \le k \text{ and } 1 \le l_j \le m_0 \end{array} \right\} \le \varepsilon m_0^k$$

which gives that

$$\sum_{\{a_{1\ell_1},...,a_{k\ell_k}\}\in \mathcal{J}_i^+} w(a_{1\ell_1})\cdots w(a_{k\ell_k}) \ge \sum_{\{a_{1\ell_1},...,a_{k\ell_k}\}\in \mathcal{J}_i} w(a_{1\ell_1})\cdots w(a_{k\ell_k}) - \varepsilon m_0^k.$$

Applied to (5.10) and we have

$$d_{G_i}(V_1', ..., V_k') \ge (D-\varepsilon)\frac{(\lfloor m/m_0 \rfloor)^k}{|V_1'|\cdots|V_k'|} \left[\sum_{\{a_{1\ell_1},...,a_{k\ell_k}\}\in \mathcal{J}_i} w(a_{1\ell_1})\cdots w(a_{k\ell_k}) - \varepsilon m_0^k \right].$$

Then applying (5.7), we have

$$d_{G_i}(V_1', ..., V_k') \ge \frac{(D-\varepsilon)(\lfloor m/m_0 \rfloor)^k}{|V_1'|\cdots|V_k'|} \left[(q_i - \varsigma)\prod_{j=1}^k w(A_j) - \varepsilon m_0^k \right].$$

Finally, note that (cf. (5.1))

$$\frac{w(A_j)}{|V'_j|} \geq \frac{|V'_j| - 1}{|V'_j| \left\lfloor \frac{m}{m_0} \right\rfloor} \geq \frac{1}{\lfloor m/m_0 \rfloor}(1 - o(1)),$$

where $o(1) \to 0$ as $m \to \infty$. This, along with $|V'_j| \leq m$ for all $1 \leq j \leq k$, gives

$$d_{G_i}(V'_1, ..., V'_k) \geq (D - \varepsilon) \lfloor m/m_0 \rfloor^k \left[(q_i - \varsigma) \frac{(1 - o(1))}{\lfloor m/m_0 \rfloor^k} - o(1) \right].$$

Then, using $p_i = Dq_i$, $\varsigma = \frac{\varepsilon'}{(k2^{k+1})}$ and $\varepsilon = \frac{(\varepsilon')^{k+1}}{5km_0}$, we infer

$$d_{G_i}(V'_1, ..., V'_k) \geq (D - \varepsilon)(q_i - \varsigma) - o(1) \geq p_i - D - \varepsilon q_i - o(1) \geq p_i - \varepsilon'.$$

5.2 Proof of Lemma 5.1

Let $0 < \varsigma < 1$ and integer $s \geq 1$ be given. We take $m_0 = m_0(\varsigma, s)$ to be sufficinetly large (and argue, in context, that this parameter needs only to depend on k, ς and s and not $q_0, q_1, ..., q_s$ nor w). Let $K[A_1, ..., A_k]$ be the k-uniform k-partite hypergraph with vertex partition $A_1 \cup \cdots \cup A_k$, where $|A_1| = \cdots = |A_k| = m_0$.

Let $q_1, ..., q_s \geq 0$ be given where $q_0 = 1 - \sum_{i=0}^{s} q_i \geq 0$. We shall define the hypergraphs $\mathcal{J}_0, \mathcal{J}_1, ..., \mathcal{J}_s$ by a standard random construction. Let $\mathbb{J}_0, \mathbb{J}_1, ..., \mathbb{J}_s$ be defined by, for each $0 \leq i \leq s$ and $\{a_1, ..., a_k\} \in K[A_1, ..., A_k]$, $\mathbb{P}[\{a_1, ..., a_k\} \in \mathbb{J}_i] = q_i$ (independently). We seek an instance of $\mathbb{J}_0, \mathbb{J}_1, ..., \mathbb{J}_s$ behaving according to the following claim.

Claim 5.2 *For $m_0 = m_0(\varsigma, s)$ sufficiently large, the following holds. For each $0 \leq i \leq s$,*

1. *If $0 \leq q_i \leq \frac{\varsigma^{k+1}}{2s}$, then with probability $1 - \frac{1}{2s}$,*

$$|\mathbb{J}_i| \leq 2sq_i m_0^k. \tag{5.11}$$

2. *If $\frac{\varsigma^{k+1}}{2s} \leq q_i \leq 1$, then with probability $1 - \frac{1}{2s}$, every choice $A'_j \subseteq A_j$, $1 \leq j \leq k$ with $|A'_j| \geq \frac{\varsigma}{2ks} m_0$, satisfies*

$$|\mathbb{J}_i \cap (A'_1 \times \cdots \times A'_k)| = q_i(1 \pm \frac{\varsigma}{2ks})|A'_1| \cdots |A'_k|. \tag{5.12}$$

We defer the proof of Claim 5.2 to the end of the section but mention, for now, that Statement 1 is an immediate consequence of Markov's Inequality (Fact 9.2) and Statement 2 is a straightforward application of the Cheroff Inequality (Fact 9.3).

Now, define $\mathcal{J}_0 \in \mathbb{J}_0, \mathcal{J}_1 \in \mathbb{J}_1, ..., \mathcal{J}_s \in \mathbb{J}_s$ to be instances satisfying the properties in (5.11) and (5.12) for all $0 \le i \le s$. Let a function $w : \cup_{j=1}^k A_j \to [0,1]$ be given satisfying $w(A_j) = \sum_{a \in A_j} w(a) \ge \varsigma m_0$ for all $1 \le j \le k$. For the remainder of the proof fix $0 \le i \le s$. We show

$$(q_i - \varsigma) \prod_{j=1}^k w(A_j) \le \sum_{\{a_1,...,a_k\} \in \mathcal{J}_i} w(a_1) \cdots w(a_k) \le (q_i + \varsigma) \prod_{j=1}^k w(A_j) \tag{5.13}$$

We proceed by considering two cases, the first of which is nearly trivial.

Indeed, for a fixed i ($0 \le i \le s$), assume first that $0 \le q_i \le \frac{\varsigma^{k+1}}{2s}$. Then, there is nothing to show for the lower bound of (5.13). For the upper bound note that

$$\sum_{\{a_1,...,a_k\} \in \mathcal{J}_i} w(a_1) \cdots w(a_k) \le |\mathcal{J}_i| \overset{(5.11)}{\le} 2sq_i m_0^k.$$

By our assumptions of $w(A_j) \ge \varsigma m_0$ for all $1 \le j \le k$ and $q_i \ge 0$, we further conclude

$$\sum_{\{a_1,...,a_k\} \in \mathcal{J}_i} w(a_1) \cdots w(a_k) \le \frac{2sq_i}{\varsigma^k} \prod_{j=1}^k w(A_j) \le \varsigma \prod_{j=1}^k w(A_j) \le (q_i + \varsigma) \prod_{j=1}^k w(A_j),$$

as desired.

For the remainder of the proof, we assume that for fixed $0 \le i \le s$,

$$q_i \ge \frac{\varsigma^{k+1}}{2s}. \tag{5.14}$$

For this more difficult case, we need the following claim.

Claim 5.3 *With w given above and $0 \le i \le s$ fixed above, there exists a function $w_0 : \bigcup_{j=1}^k A_j \to [0,1]$ with the following properties.*

1. *For each $1 \le j \le k$*

$$w_0(A_j) = w(A_j). \tag{5.15}$$

2. *For each $1 \le j \le k$, define $M_{A_j}(w_0) \overset{def}{=} \{a \in A_j : 0 < w_0(a) < 1\}$. Then,*

$$w_0(M_{A_j}(w_0)) \le 1. \tag{5.16}$$

3. For $\bar{w} \in \{w, w_0\}$, define $W_i(\bar{w}) \overset{def}{=} \sum_{\{a_1,...,a_k\}\in\mathcal{J}_i} \bar{w}(a_1)\cdots\bar{w}(a_k)$. Then

$$W_i(w) \le W_i(w_0). \tag{5.17}$$

We defer the proof of Claim 5.3 to the end of the section in favor of finishing the proof of Lemma 5.1.

For the fixed index $0 \le i \le s$ (cf. (5.14), we prove the upper bound of (5.13). To that end, define $S_{A_j} \overset{\text{def}}{=} \{a \in A_j :\ w_0(a) = 1\}$ for all $1 \le j \le k$. Then,

$$\begin{aligned}
\sum_{\{a_1,...,a_k\}\in\mathcal{J}_i} w(a_1)\cdots w(a_k) &\overset{(5.17)}{\le} W_i(w_0) \\
&\le \sum_{\{a_1,...,a_k\}\in\mathcal{J}_i[S_{A_1},...,S_{A_k}]} 1 + \sum_{h=1}^{k} \sum_{a_h\in M_{A_h}(w_0)} w_0(a_h) \prod_{j=1,j\neq h}^{k} \prod_{a_j\in A_j} w_0(a_j) \\
&= |\mathcal{J}_i[S_{A_1},...,S_{A_k}]| + \sum_{h=1}^{k}\left(\prod_{j=1,j\neq h}^{k}\prod_{a_j\in A_j} w_0(a_j)\right)\sum_{a_h\in M_{A_h}(w_0)} w_0(a_h) \\
&= |\mathcal{J}_i[S_{A_1},...,S_{A_k}]| + \sum_{h=1}^{k}\prod_{j=1,j\neq h}^{k}\prod_{a_j\in A_j} w_0(a_j)w_0(a_h) \\
&\overset{(5.16)}{\le} |\mathcal{J}_i[S_{A_1},...,S_{A_k}]| + \sum_{h=1}^{k}\prod_{j=1,j\neq h}^{k}\prod_{a_j\in A_j} w_0(a_j)
\end{aligned}$$

$$\begin{aligned}
&= |\mathcal{J}_i[S_{A_1},...,S_{A_k}]| + \left(\frac{1}{w_0(A_1)} + \cdots + \frac{1}{w_0(A_k)}\right)\prod_{j=1}^{k} w_0(A_j) \\
&\overset{(5.15)}{=} |\mathcal{J}_i[S_{A_1},...,S_{A_k}]| + \left(\sum_{j=1}^{k}\frac{1}{w(A_j)}\right)\prod_{j=1}^{k} w(A_j) \\
&\le |\mathcal{J}_i[S_{A_1},...,S_{A_k}]| + \frac{k}{\varsigma m_0}\prod_{j=1}^{k} w(A_j),
\end{aligned}$$

where the last unequality is due to $w(A_j) \ge \varsigma m_0$ for $1 \le j \le k$. Then, since

$$|S_{A_j}| = w_0(A_j) - w_0(M_{A_j}(w)) \overset{(5.15)}{=} w(A_j) - w_0(M_{A_j}(w)) \overset{(5.17)}{\ge} w(A_j) - 1 \ge \frac{\varsigma m_0}{2ks},$$

we may apply (5.12) from Claim 5.2 to conclude

$$W_i(w) \leq q_i \left(1 + \frac{\varsigma}{2ks}\right) |S_{A_1}| \cdots |S_{A_k}| + \frac{k}{\varsigma m_0} \prod_{j=1}^{k} w(A_j)$$
$$\overset{|S_{A_j}| \leq w(A_j)}{\leq} \left(q_i \left(1 + \frac{\varsigma}{2ks}\right) + \frac{k}{\varsigma m_0}\right) \prod_{j=1}^{k} w(A_j)$$
$$\leq (q_i + \varsigma) \prod_{j=1}^{k} w(A_j),$$

where the last inequality follows with $m_0 = m_0(\varsigma, s)$ sufficiently large (as a function of k, ς and s alone). This proves the upper bound of (5.13).

The lower bound is an easy consequence of the upper bound (which we may now assume holds for all $0 \leq i \leq s$). For $0 \leq i \leq s$ fixed, note that

$$\sum_{\{a_1, \ldots, a_k\} \in \mathcal{J}_i} w(a_1) \cdots w(a_k)$$
$$= \sum_{a_1 \in A_1, \ldots, a_k \in A_k} w(a_1) \cdots w(a_k) - \sum_{h=0, h \neq i}^{s} \sum_{\{a_1, \ldots, a_k\} \in \mathcal{J}_h} w(a_1) \cdots w(a_k)$$
$$\geq \prod_{j=1}^{k} w(A_j) - \sum_{i \neq h, h=0}^{s} q_h \left(1 + \frac{\varsigma}{2ks}\right) \left[\prod_{j=1}^{k} w(A_j)\right]$$
$$\geq \left(1 - \sum_{i \neq h, h=0}^{s} q_h - \frac{\varsigma}{2ks} \sum_{i \neq h, h=0}^{s} q_h\right) \left[\prod_{j=1}^{k} w(A_j)\right]$$
$$\overset{q_h \leq 1}{\geq} \left(q_i - \frac{\varsigma}{2k}\right) \left[\prod_{j=1}^{k} w(A_j)\right] \geq (q_i - \varsigma) \left[\prod_{j=1}^{k} w(A_j)\right],$$

as promised by (5.13).

Proof. [Proof of Claim 5.2] Recall that the index $0 \leq i \leq s$ is fixed.

First, assume that $0 \leq q_i \leq \frac{\varsigma^{k+1}}{2s}$, and to avoid triviality, also assume that $q_i \neq 0$. Then, by the Markov Inequality (Fact 9.2),

$$\mathbb{P}[|\mathbb{J}_i| > 2sq_i m_0^k] \leq \frac{\mathbb{E}[|\mathbb{J}_i|]}{2sq_i m_0^k} = \frac{1}{2s},$$

as promised.

Otherwise, assume $\frac{\varsigma^{k+1}}{2s} \leq q_i \leq 1$, and fix $A'_j \subseteq A_j$, $1 \leq j \leq k$ with $|A'_j| \geq \frac{\varsigma}{2ks} m_0$. By Fact 9.3, we have

$$\mathbb{P}\left[|\mathbb{J}_i[A'_i, ..., A'_k]| \neq (1 \pm \frac{\varsigma}{2ks}) q_i |A'_1| \cdots |A'_k|\right] \leq 2\exp\left\{-\frac{\varsigma^2}{12k^2 s^2} q_i |A'_1| \cdots |A'_k|\right\}$$

$$\leq 2\exp\left\{-\frac{\varsigma^{2k+3}}{24 \cdot 2^k k^{k+2} s^{k+3}} m_0^k\right\}.$$

Over all choices $A'_1 \subseteq A_1, ..., A'_k \subseteq A_k$, we see condition 2 of Claim 5.2 holds with probability

$$1 - 2^{km_0+1} \exp\left\{-\frac{\varsigma^{2k+3}}{24 \cdot 2^k k^{k+2} s^{k+3}} m_0^k\right\} \geq 1 - \frac{1}{2s},$$

where the last inequality holds with $m_0 = m_0(\varsigma, s)$ sufficiently large as a function of k, ς, s. This proves Claim 5.2. □

Proof. [Proof of Claim 5.3] Recall that $0 \leq i \leq s$ and w are fixed.

If w satisfies (5.16), set $w_0 = w$ and we are done. Otherwise, $w(M_{A_j}(w)) > 1$ where, without loss of generality, $j = 1$. We shall alter w to arrive at a function satisfying (5.16).

To that end we shall first define

$$\hat{W}_i(\hat{a}_\iota) \stackrel{\text{def}}{=} \sum \{w(a_2) \cdots w(a_k) : \ \{\hat{a}_\iota, a_2, ..., a_k\} \in \mathcal{J}_i\}. \tag{5.18}$$

Then, without loss of generality, enumerate the vertices $a \in M_{A_1}(w)$ such that $\hat{W}_i(\hat{a}_0) \geq \hat{W}_i(\hat{a}_1) \geq \cdots \geq \hat{W}_i(\hat{a}_\ell)$. Then, find constants $\vartheta_1, ..., \vartheta_\ell > 0$ such that $w(\hat{a}_0) = 1 - \sum_{\iota=1}^{\ell} \vartheta_\iota$ and $w(\hat{a}_\iota) \geq \vartheta_\iota$ for all $1 \leq \iota \leq \ell$. Note that this is possible due to each $a \in M_{A_1}(w)$ having strictly positive weight.

Now define weight function w' by

$$w'(a) = \begin{cases} w(a) & a \in (A_1 \backslash M_{A_1}(w)) \cup \bigcup_{j=2}^k A_j \\ w(a) - \vartheta_\iota & a = \hat{a}_\iota \in M_{A_1}(w) \text{ for } \iota = 1, ..., \ell \\ 1 & a = \hat{a}_0 \in M_{A_1}(w) \end{cases}$$

Note that as w' is defined, $M_{A_1}(w') = \{\hat{a}_1, ..., \hat{a}_\ell\}$.

We now show w', as defined above, satisfies (5.15) and (5.17). To verify (5.15), it suffices to show that $w'(A_1) = w(A_1)$ since $w'(A_j) = w(A_j)$ for all $2 \leq j \leq k$.

Indeed,

$$w'(A_1) = w'(M_{A_1}(w')) + w'(A_1 \backslash M_{A_1}(w')) = \sum_{\iota=1}^{\ell} w'(\hat{a}_\iota) + w'(A_1 \backslash M_{A_1}(w'))$$

$$= \sum_{\iota=1}^{\ell} w'(\hat{a}_\iota) + w'(A_1 \backslash M_{A_1}(w)) + w'(\hat{a}_0).$$

Since $w'(\hat{a}_\iota) = w(\hat{a}_\iota) - \vartheta_\iota$ for all $1 \leq \iota \leq \ell$, $w'(A_1 \backslash M_{A_1}(w)) = w(A_1 \backslash M_{A_1}(w))$ and $w'(\hat{a}_0) = 1$, we conclude

$$w'(A_1) = \sum_{\iota=1}^{\ell} [w(\hat{a}_\iota) - \vartheta_\iota] + w(A_1 \backslash M_{A_1}(w)) + 1 = \sum_{\iota=1}^{\ell} w(\hat{a}_\iota) + w(A_1 \backslash M_{A_1}(w)) + 1 - \sum_{\iota=1}^{\ell} \vartheta_\iota$$

$$= \sum_{\iota=1}^{\ell} w(\hat{a}_\iota) + w(A_1 \backslash M_{A_1}(w)) + w(\hat{a}_0) = w(M_{A_1}(w)) + w(A_1 \backslash M_{A_1}(w)) = w(A_1),$$

as needed.

We next show that w' satisfies (5.17). Indeed,

$$W_i(w') - W_i(w) = \sum_{\{a_1, a_2, \ldots, a_k\} \in \mathcal{J}_i} w'(a_1) \cdots w'(a_k) - \sum_{\{a_1, a_2, \ldots, a_k\} \in \mathcal{J}_i} w(a_1) \cdots w(a_k)$$

$$\overset{(5.18)}{=} \sum_{\iota=0}^{\ell} [w'(\hat{a}_\iota) - w(\hat{a}_\iota)] \, \hat{W}_i(\hat{a}_\iota)$$

$$= (w'(\hat{a}_0) - w(\hat{a}_0)) \hat{W}_i(\hat{a}_0) + \sum_{\iota=1}^{\ell} [w'(\hat{a}_\iota) - w(\hat{a}_\iota)] \, \hat{W}_i(\hat{a}_\iota).$$

Since $w'(\hat{a}_0) = 1$, $w(\hat{a}_0) = 1 - \sum_{\iota=1}^{\ell} \vartheta_\iota$ and $w'(\hat{a}_\iota) = w(\hat{a}_\iota) - \vartheta_\iota$ for all $1 \leq \iota \leq \ell$, we have

$$W_i(w') - W_i(w) = \left(\sum_{\iota=1}^{\ell} \vartheta_\iota \right) \hat{W}_i(\hat{a}_0) - \sum_{\iota=1}^{\ell} \vartheta_\iota \hat{W}_i(\hat{a}_\iota).$$

Recall $\hat{W}_i(\hat{a}_\iota) \leq \hat{W}_i(\hat{a}_0)$ for all $1 \leq \iota \leq \ell$ and so the quantity above is nonegative, proving that $W_i(w) \leq W_i(w')$.

While w' satisfies (5.15) and (5.17), it may be the case that w' does not satisfy (5.16). However, recalling $M_{A_1}(w') = \{\hat{a}_1, ..., \hat{a}_\ell\} = M_{A_1}(w) \backslash \{\hat{a}_0\}$, we have

$$w'(M_{A_1}(w')) = w'(M_{A_1}(w)) - 1.$$

By repeating this process, we arrive at a function $w_0 = w'$ with $w_0(M_{A_1}(w_0)) \leq 1$.

□

Chapter 6

Proof of the Fractional Crossing Lemma

To prove Lemma 2.7, we actually prove a more general statement which we call the "General Crossing Lemma", and for which we now prepare. For a partition $\Pi = (V_i)_{i=1}^{\ell}$ of a set V, we write $\mathcal{C}(s, V, \Pi)$ for the set of subsets of V of size s that are crossing in the partition Π, i.e., $S \in \binom{V}{s}$ satisfies $|S \cap V_i| \leq 1$ for each $1 \leq i \leq \ell$.

Lemma 6.1 (General Crossing Lemma) *Let a positive integer f and a real number $\mu > 0$ be given. Then there exists $L_0 = L_0(f, \mu)$ such that the following holds. For any set V and function $g : \binom{V}{f} \to [0, 1]$, there exists a partition $\Pi = (V_j)_{j=1}^{\ell}$ with $l \leq L_0$ such that*

1. $||V_i| - |V_j|| \leq 1$ *for each i, j,*
2. $\sum_{S \in \mathcal{C}(f,V,\Pi)} g(S) \geq (1 - \mu) \sum_{S \in \binom{V}{f}} g(S)$.

Proof. [Lemma 2.7] This immediately follows by taking $f = |V(F_0)|$, $V = V(H)$, and $g(S) = \sum \left\{ \psi(F) : F \in \binom{H}{F_0} \;\&\; V(F) = S \right\}$. □

6.1 Proof of Lemma 6.1

To prove Lemma 6.1, we use the following result of Haxell and Rödl [14] for partitioning weighted pairs. We give their proof (at the end of the chapter) for completeness.

Lemma 6.2 *Let V be a set and $w : \binom{V}{2} \to [0, \infty)$ be any nonnegative weight function defined on the pairs of elements of V. Let $|V| = n$. Then there is an $O(n^2)$ algorithm which partitions V into subsets V_0 and V_1 such that $\lfloor n/2 \rfloor \leq |V_0|, |V_1| \leq \lceil n/2 \rceil$ and*

$$\sum_{x,y \in V_0} w(x, y) + \sum_{x,y \in V_1} w(x, y) \leq \frac{1}{2} \sum_{x,y \in V} w(x, y).$$

Let positive integer f and $\mu > 0$ be given. To define the constant $L_0 = L_0(f, \mu)$, let integer $c > 0$ satisfy $2^c \geq \binom{f}{2}\mu^{-1}$. Take $L_0 = 2^c$. Now, let set V and function $g : \binom{V}{2} \to [0,1]$ be given where $|V| = n$. If $n \leq L_0$, then the partition of V into singletons clearly satisfies the required conditions, so we may assume $n > L_0$. We next define a weight function w on $\binom{V}{2}$ by $w(x,y) = \sum_{x,y \in S \in \binom{V}{f}} g(S)$, and we apply Lemma 6.2 to V to obtain a partition $V_0 \cup V_1$ such that $\lfloor n/2 \rfloor \leq |V_0|, |V_1| \leq \lceil n/2 \rceil$ and

$$\sum_{x,y \in V_0} w(x,y) + \sum_{x,y \in V_1} w(x,y) \leq \frac{1}{2} \sum_{x,y \in V} w(x,y).$$

Now suppose that $i \geq 1$ and that we have a partition $V = \cup_{\pi \in \{0,1\}^i} V_\pi$ into subsets such that for each π, $\lfloor n/2^i \rfloor \leq |V_\pi| \leq \lceil n/2^i \rceil$ satisfying

$$\sum_{\pi \in \{0,1\}^i} \sum_{x,y \in V_\pi} w(x,y) \leq \frac{1}{2^i} \sum_{x,y \in V} w(x,y).$$

We apply Lemma 6.2 again to each V_π to obtain a partition $V_\pi = V_{\pi_0} \cup V_{\pi_1}$ such that $\lfloor n/2^{i+1} \rfloor \leq |V_{\pi_0}|, |V_{\pi_1}| \leq \lceil n/2^{i+1} \rceil$ and

$$\sum_{x,y \in V_{\pi_0}} w(x,y) + \sum_{x,y \in V_{\pi_1}} w(x,y) \leq \frac{1}{2} \sum_{x,y \in V_\pi} w(x,y).$$

Therefore after c steps we have a partition $V = \cup_{\pi \in \{0,1\}^c} V_\pi$ such that $\lfloor n/2^c \rfloor \leq |V_\pi| \leq \lceil n/2^c \rceil$ for each π, and

$$\sum_{\pi \in \{0,1\}^c} \sum_{x,y \in V_\pi} w(x,y) \leq \frac{1}{2^c} \sum_{x,y \in V} w(x,y) = \frac{1}{2^c} \binom{f}{2} \sum_{S \in \binom{V}{f}} g(S) \leq \mu \sum_{S \in \binom{V}{f}} g(S)$$

where the last equality is due to the definition of w, and the last inequality is due to our choice of c. The partition above is the partition Π promised in Lemma 6.1.

Then note that by our construction of Π,

$$\sum_{S \notin \mathcal{C}(f,V,\Pi)} g(S) \leq \sum_{\pi \in \{0,1\}^c} \sum_{x,y \in V_\pi} w(x,y) \leq \mu \sum_{S \in \binom{V}{f}} g(S),$$

which completes the proof.

6.2 Proof of Lemma 6.2

Let V be a given set of size $|V| = n$, and let $w : \binom{V}{2} \to [0, \infty)$ be a given nonnegative weight function. We produce the desired partition $V_0 \cup V_1$ greedily. First we take two arbitrary elements a and b of V and let $A_1 = \{a\}$ and $B_1 = \{b\}$. Now suppose that $i \geq 1$ and that sets A_i and B_i have been defined, where $|A_i| = |B_i| = i$ and

$$\sum_{x,y \in A_i} w(x,y) + \sum_{x,y \in B_i} w(x,y) \leq \frac{1}{2} \sum_{x,y \in (A_i \cup B_i)} w(x,y) \tag{6.1}$$

Suppose also that $|V \backslash (A_i \cup B_i)| \geq 2$. Then define A_{i+1} and B_{i+1} by taking two arbitrary elements a and b of $V \backslash (A_i \cup B_i)$ and letting $A_{i+1} = A_i \cup \{a\}$ and $B_{i+1} = B_i \cup \{b\}$ if

$$\sum_{v \in A_i} w(a,v) + \sum_{v \in B_i} w(b,v) \leq \sum_{v \in A_i} w(b,v) + \sum_{v \in B_i} w(a,v),$$

and otherwise we let $A_{i+1} = A_i \cup \{b\}$ and $B_{i+1} = B_i \cup \{a\}$. Note that, by our construction, (6.1) still holds.

We continue this procedure until $|V \backslash (A_i \cup B_i)| \leq 1$. If n is odd, then we add the last vertex a to A_i if $\sum_{v \in A_i} w(a,v) \leq \sum_{v \in B_i} w(a,v)$, and otherwise add it to B_i. As (6.1) still holds, we let V_0 and V_1 denote the final sets obtained in this construction which completes the proof.

Chapter 7

Proof of the δ-Bounded Lemma

For our proof of Lemma 2.8, we will need the following result, which we call the "General Bounding Lemma". This statement comes directly from Haxell and Rödl [14], but we include a proof at the end of this chapter for completeness. In what follows, when ϕ is a weighted fractional F-packing of $\mathcal{H}$ where $F = E$ is a single edge, then we more simply say ϕ is a weighted fractional edge-packing of $\mathcal{H}$.

Lemma 7.1 (General Bounding Lemma) *Let $\xi > 0$ and $\rho \in \mathbb{N}$ be given. Then there exists $k_0 = k_0(\xi, \rho)$ such that the following holds.*

Let $\mathcal{H}$ be any ρ-uniform vertex-weighted hypergraph where the weight $\omega(v)$ of each vertex v satisfies $0 \leq \omega(v) \leq 1$, and let ϕ be a fractional edge-packing of $\mathcal{H}$ such that $\phi(E) < 1/k_0$ for every $E \in \mathcal{H}$. Then there exists a $(1/k_0)$-bounded fractional edge-packing $\bar{\phi}$ of $\mathcal{H}$ such that $|\bar{\phi}| \geq |\phi| - \xi m$, where $|V(\mathcal{H})| = m$. Moreover, the function ϕ can be found by an exhaustive search (in time dependent on m).

We now show that Lemma 7.1 implies Lemma 2.8.

Proof. [Proof of Lemma 2.8] Given F_0 and $\xi > 0$, we let $\rho = |E(F_0)|$, and we let $\delta = 1/k_0(\rho, \xi)$ where k_0 is defined as in Lemma 7.1. Let H_0 be a k-uniform edge-weighted hypergraph with r vertices, and let ψ^* be a maximum fractional F_0-packing of H_0. We define a ρ-uniform vertex-weighted hypergraph $\mathcal{H}$ as follows. The vertex set of $\mathcal{H}$ is the edge set of H_0, i.e. $V(\mathcal{H}) = \{v_e : e \in E(H_0)\}$. A set of ρ vertices of $\mathcal{H}$ forms an edge of $\mathcal{H}$ if and only if the corresponding ρ edges form a copy of F_0 in H_0. The weight $w(v_e)$ of a vertex v_e is the weight in H_0 of the corresponding edge e. Set $m = |V(\mathcal{H})| = |E(H_0)|$. Then ψ^* immediately renders a fractional edge-packing ϕ^* of $\mathcal{H}$, such that

$$|\psi^*| = |\phi^*| = \nu^*_{F_0}(H_0). \tag{7.1}$$

We first modify $\mathcal{H}$ by setting aside (for the moment) edges E for which $\phi^*(E) \geq \delta$. Let $\Delta_0 = \{E \in \mathcal{H} : \phi^*(E) \geq \delta\}$, and define a new vertex-weighted hypergraph $\mathcal{H}'$ with $\mathcal{H}' = \mathcal{H} \backslash \Delta_0$ (where $V(\mathcal{H}') = V(\mathcal{H})$). We now weight the vertices of $\mathcal{H}'$ by

$$w_{\mathcal{H}'}(v) = w_{\mathcal{H}}(v) - \sum_{v \in E \in \Delta_0} \phi^*(E), \tag{7.2}$$

for a fixed $v \in V(\mathcal{H}')$. We also define a weighted fractional edge-packing ϕ' on $\mathcal{H}'$ by $\phi'(E) = \phi^*(E)$ for each $E \in \mathcal{H}'$. Note also that

$$|\phi'| = |\phi^*| - \sum_{E \in \Delta_0} \phi^*(E), \tag{7.3}$$

that $w_{\mathcal{H}'}(v) \in [0,1]$ since $w_{\mathcal{H}'}(v) \leq w_{\mathcal{H}}(v) \leq 1$, and that the definition of ϕ^* being a (weighted) fractional edge-packing ensures that

$$\sum_{v \in E \in \mathcal{H}} \phi^*(E) \leq w_{\mathcal{H}}(v). \tag{7.4}$$

Then ϕ' is in fact a weighted fractional edge-packing of $\mathcal{H}'$ since for $v \in V(\mathcal{H}')$ we have

$$\begin{aligned} \sum_{E \ni v} \phi'(E) &= \sum_{v \in E \in \mathcal{H}} \phi^*(E) - \sum_{v \in E \in \Delta_0} \phi^*(E) \\ &\overset{(7.4)}{\leq} w_{\mathcal{H}}(v) - \sum_{v \in E \in \Delta_0} \phi^*(E) \overset{(7.2)}{=} w_{\mathcal{H}'}(V). \end{aligned}$$

Finally, for every $E \in \mathcal{H}'$ we have $\phi'(E) < \delta$ by the construction of $\mathcal{H}'$.

Then, since $\delta = 1/k_0(\rho, \xi)$, by Lemma 7.1 there exists a δ-bounded fractional edge-packing $\bar{\phi}$ of $\mathcal{H}'$ such that

$$|\bar{\phi}| \geq |\phi'| - \xi m, \tag{7.5}$$

where $m = |V(\mathcal{H}')| = |V(\mathcal{H})| = |E(H_0)|$.

We now return the set of edges Δ_0 and define the fractional edge-packing $\hat{\psi}$ of $\mathcal{H}$ as follows: Let

$$\hat{\psi}(E) = \begin{cases} \phi^*(E) & \text{if } E \in \Delta_0 \\ \bar{\phi}(E) & \text{if } E \in \mathcal{H}'. \end{cases}$$

Then $\hat{\psi}$ is δ-bounded by construction. Also,

$$|\hat{\psi}| = \sum_{E \in \Delta_0} \phi^*(E) + |\bar{\phi}| \overset{(7.5)}{\geq} \sum_{E \in \Delta_0} \phi^*(E) + |\phi'| - \xi m$$

$$\stackrel{(7.3)}{=} |\phi^*| - \xi m \stackrel{(7.1)}{\geq} \nu^*_{F_0}(H_0) - \xi r^k$$

where $m = |E(H_0)| \leq r^k$. Since $\hat{\psi}$ corresponds to a weighted fractional (w, F_0)-packing of H_0, the result follows. □

7.1 Proof of Lemma 7.1

Let $\xi > 0$ and $\rho \in \mathbb{N}$ be given. We define the promised constant k_0 in terms of the following considerations. Let h be large enough so that the following hold:

$$h \geq 400/\xi^4, \tag{7.6}$$

$$2\exp(-\xi^2\sqrt{h}/300) < 1/h^2, \tag{7.7}$$

$$\sum_{x=\lceil 7\sqrt{h}\rceil}^{\infty} x \, \exp\{-x\} < 1/h. \tag{7.8}$$

Then we set $k_0 = \lceil h(1 + \xi/10)\rceil$.

With the constants ξ, ρ, k_0 above, let $\mathcal{H}$ be given as in Lemma 7.1. We begin by removing from $\mathcal{H}$ vertices that have small values of $\phi(v)$. In general, for a vertex v, let $\phi(v) = \sum_{E \ni v} \phi(E)$. For a fractional edge-packing f of a hypergraph $\mathcal{J}$ on m vertices we will say that a vertex $v \in V(\mathcal{J})$ is *f-small* if $f(v) < \xi|f|/10m$. We shall remove small vertices one by one from $\mathcal{H}$ as follows. Let $\mathcal{H}_0 = \mathcal{H}$ and $\phi_0 = \phi$. Now for $i \geq 0$, if there are no ϕ_i-small vertices in $V(\mathcal{H}_i)$ then set $\phi' = \phi_i$, $V' = V(\mathcal{H}_i)$, $\mathcal{H}' = \mathcal{H}_i$ and stop. Otherwise let x be a ϕ_i-small vertex. Then set $V(\mathcal{H}_{i+1}) = V(\mathcal{H}_i)\backslash\{x\}$, let $\mathcal{H}_{i+1} = \mathcal{H}_i\backslash\{E \in \mathcal{H} : x \in E\}$ and define ϕ_{i+1} on $\mathcal{H}_{i+1}$ by $\phi_{i+1}(E) = \phi_i(E)$ for each $E \in \mathcal{H}_{i+1}$.

When this process is completed, we have $\mathcal{H}'$ and ϕ' that satisfy the following properties:

$$|\phi'| \geq |\phi| - \sum_{v \in V\backslash V'} \phi(v) \geq |\phi| - m\xi|\phi|/(10m) \geq |\phi|(1 - \xi/10), \tag{7.9}$$

and since we may assume, without loss of generality, that $|\phi| > \xi m$,

$$\phi'(v) \geq \xi|\phi|/10m \stackrel{(7.6)}{\geq} 1/\sqrt{h} \text{ for every } v \in V'. \tag{7.10}$$

Now we let $\tilde{\mathcal{H}}$ be a random subset of $\mathcal{H}'$ where each $E \in \mathcal{H}'$ is chosen randomly and independently with probability $p_E = h\phi'(E) < 1$. (Indeed, $k_0 > h$ and $\phi'(E) = \phi(E) < 1/k_0$ by hypothesis.) Let $\tilde{V} = V' = V(\tilde{\mathcal{H}})$. Now for each vertex $v \in \tilde{V}$ we have $\mathbb{E}[d_{\tilde{\mathcal{H}}}(v)] = \sum_{E \ni v} h\phi'(E) =$

$h\phi'(v)$, where we use $d_{\tilde{\mathcal{H}}}(v)$ to denote the number of edges of $\tilde{\mathcal{H}}$ that contain v. Therefore by the Chernoff Inequality (see Fact 9.3)

$$\mathbb{P}[|d_{\tilde{\mathcal{H}}}(v) - h\phi'(v)| > \xi/10h\phi'(v)] < 2\exp(-\xi^2 h\phi'(v)/300)$$
$$\stackrel{(7.10)}{<} 2\exp(-\xi^2\sqrt{h}/300) \stackrel{(7.7)}{<} 1/h^2. \tag{7.11}$$

We shall call a vertex v ***big*** if $d_{\tilde{\mathcal{H}}}(v) > (1+\xi/10)h\phi'(v)$ and ***thin*** if $d_{\tilde{\mathcal{H}}}(v) < (1-\xi/10)h\phi'(v)$. Let B and T denote the sets of big and thin vertices, respectively. Then we have

$$\mathbb{E}[|B|] = \sum_{v\in\tilde{V}} \mathbb{P}[d_{\tilde{\mathcal{H}}} > (1+\xi/10)h\phi'(v)] \stackrel{(7.11)}{<} \sum_{v\in\tilde{V}} 1/h^2 < m/h^2$$

and

$$\mathbb{E}[|T|] = \sum_{v\in\tilde{V}} \mathbb{P}[d_{\tilde{\mathcal{H}}} > (1-\xi/10)h\phi'(v)] < m/h^2.$$

Next we check that the number of edges incident to big vertices is small. For a vertex $v \in \tilde{V}$ define

$$M_v = \begin{cases} d_{\tilde{\mathcal{H}}}(v) & \text{if } (1+\xi/10)h\phi'(v) \le d_{\tilde{\mathcal{H}}}(v) < 7h\phi'(v) \\ 0 & \text{otherwise.} \end{cases}$$

Let $M = \sum_{v\in\tilde{V}} M_v$. Then note that M gives a ρ multiple of the number of edges of $\tilde{\mathcal{H}}$ incident to 'moderately big' vertices, that is, big vertices with degree less than $7h\phi'(v)$. Then we have

$$\mathbb{E}[M] = \sum_{v\in\tilde{V}} \mathbb{E}[M_v] = \sum_{v\in\tilde{V}} (d_{\tilde{\mathcal{H}}}(v)\,\mathbb{P}[(1+\xi/10)h\phi'(v) \le d_{\tilde{\mathcal{H}}}(v) < 7h\phi'(v)])$$
$$\stackrel{(7.11)}{<} \sum_{v\in\tilde{V}} d_{\tilde{\mathcal{H}}}(v)(1/h^2) < \sum_{v\in\tilde{V}} 7h\phi'(v)(1/h^2) \le \frac{7}{h}\sum_{v\in\tilde{V}} w(v) < \frac{7m}{h} \tag{7.12}$$

due to the definition of 'moderately big' vertices and ϕ' being a fractional packing.

Now define

$$R_v = \begin{cases} d_{\tilde{\mathcal{H}}}(v) & \text{if } d_{\tilde{\mathcal{H}}}(v) \ge 7h\phi'(v), \\ 0 & \text{otherwise.} \end{cases}$$

Let $R = \sum_{v\in\tilde{V}} R_v$, so R gives a ρ multiple of the number of edges incident to 'really big' vertices, that is, big vertices with degree at least $7h\phi'(v)$. Let $\tilde{V_R}$ be the set of vertices $V \in \tilde{V}$ for which $R_v \neq 0$. Then, $R = \sum_{v\in\tilde{V_R}} R_v$ and so

$$\mathbb{E}[R] = \sum_{v\in\tilde{V_R}} \mathbb{E}[R_v] = \sum_{v\in\tilde{V_R}} \mathbb{E}[d_{\tilde{\mathcal{H}}}(v)].$$

Note that $d_{\tilde{\mathcal{H}}}(v)$ is a random variable with possible values $\lceil 7h\phi'(v)\rceil$, $\lceil 7h\phi'(v)\rceil + 1,\ \dots, m$, for each $v \in \tilde{V_R}$. Thus,

$$\mathbb{E}[R] = \sum_{v\in\tilde{V_R}} \sum_{x=\lceil 7h\phi'(v)\rceil}^{\infty} x\, \mathbb{P}[d_{\tilde{\mathcal{H}}}(v) = x].$$

Since $7h\phi'(v) \geq 7\sqrt{h}$ by (7.10), the Chernoff Inequality (see Fact 9.3) yields

$$\mathbb{E}[R] \leq \sum_{v\in\tilde{V_R}} \sum_{x=\lceil 7h\phi'(v)\rceil}^{\infty} x\ \exp\{-x\} \overset{(7.8)}{<} \frac{1}{h}\,|\tilde{V_R}| \leq \frac{m}{h}. \tag{7.13}$$

Therefore from (7.12) and (7.13) the number of edges incident with big vertices m_B satisfies $\mathbb{E}[m_B] < 15m/h$. Then, since $\mathbb{E}[|T|] < m/h^2$, there exists some $\tilde{\mathcal{H}}_0$ such that $m_B \leq 45m/h$ and $|T| \leq 3m/h^2$.

Let $\tilde{\mathcal{H}}_1$ be the hypergraph formed by removing all edges from $\tilde{\mathcal{H}}_0$ that are incident to big vertices, and note that by construction $\tilde{\mathcal{H}}_1$ has the property that $d_{\tilde{\mathcal{H}}_1}(v) \leq (1+\xi/10)h\phi'(v)$ for every vertex v. Define $\bar{\phi}$ by setting

$$\bar{\phi}(E) = \begin{cases} 1/[h(1+\xi/10)] & \text{if } E \in \tilde{\mathcal{H}}_1 \\ 0 & \text{otherwise.} \end{cases}$$

Then $\bar{\phi}$ is a fractional packing of $\mathcal{H}$ since for $v \in V$ we have

$$\sum_{E\ni v} \bar{\phi}(E) = d_{\tilde{\mathcal{H}}_1}(v)/[h(1\ +\ \xi/10)] \leq \phi'(v).$$

Also

$$\begin{aligned}
|\bar{\phi}| &\geq \frac{1}{h(1+\xi/10)}\left[\frac{1}{\rho}\sum_{v\in\tilde{V}} d_{\tilde{\mathcal{H}}_1}(v)\right] = \frac{1}{h(1+\xi/10)}\left[\frac{1}{\rho}\sum_{v\in\tilde{V}} d_{\tilde{\mathcal{H}}_0}(v) - m_B\right] \\
&\geq \frac{1}{h(1+\xi/10)}\left[\frac{1}{\rho}\sum_{v\in\tilde{V}\setminus T} d_{\tilde{\mathcal{H}}_0}(v) - m_B\right] \\
&\geq \frac{1}{h(1+\xi/10)}\left[\frac{1}{\rho}\sum_{v\in\tilde{V}\setminus T} (1-\xi/10)h\phi'(v) - m_B\right] \\
&\geq \frac{1}{h(1+\xi/10)}\left[\frac{h(1-\xi/10)}{\rho}\sum_{v\in\tilde{V}} \phi'(v) - \frac{h(1-\xi/10)}{\rho}|T| - m_B\right] \\
&\geq \frac{(1-\xi/10)}{(1+\xi/10)}|\phi'| - \frac{3m(1-\xi/10)}{h^2(1+\xi/10)\rho} - \frac{45m}{h^2(1+\xi/10)}
\end{aligned}$$

$$\geq |\phi'| - \frac{\xi|\phi'|}{5} - \frac{3m}{h^2\rho} - \frac{45m}{h^2} \overset{(7.6)}{\geq} |\phi'| - \frac{\xi|\phi'|}{5} - \frac{3m}{\rho}\left(\frac{\xi^4}{400}\right)^2 - 45m\left(\frac{\xi^4}{400}\right)^2$$

$$\geq |\phi'| - \frac{\xi m}{2}\left(\frac{2|\phi'|}{5m} + \frac{3\xi^7}{80000\rho} + \frac{45\xi^7}{80000}\right) \geq |\phi'| - \xi m/2.$$

Therefore $|\bar{\phi}| \geq |\phi| - \xi m$ by (7.9). Finally, since $k_0 = \lceil h(1+\xi/10) \rceil$, we see that $\bar{\phi}$ is the stated $(1/k_0)$-bounded fractional edge-packing of $\mathcal{H}$.

Chapter 8

Concluding Remarks

Recall, in Theorem 1.5, we assume F_0 is a linear hypergraph, and we stressed, in the Introduction, that this assumption is irremovable in our approach. We stated, in Remark 2.5, that the conclusion of the Packing Lemma (Lemma 2.4) is false when F_0 is not linear. This is due more generally to the fact that the conclusion of the Counting Lemma for Linear Hypergraphs (Theorem 4.5) is also false when F_0 is not linear. Let us now explain this problem in detail for 3-uniform hypergraphs.

Return to Setup 2.3 and let $f = 4$, $k = 3$, $F_0 = K_4^{(3)}$ (which removes the linearity as here every 2 hyperedges share 2 vertices). The following example demonstrates that, in this setting, the concept of ε-regularity is not a strong enough condition to conclude any Counting Lemma (and hence, no Packing Lemma).

Example 8.1 *Let $V_1 \cup V_2 \cup V_3 \cup V_4$ be a partition with $|V_1| = |V_2| = |V_3| = |V_4| = m$. For each $1 \leq i < j \leq 4$, let $\vec{\mathbb{K}}[V_i, V_j]$ be an orientation of the complete bipartite graph $K[V_i, V_j]$ selected uniformly at random. In other words, independently for each $\{v_i, v_j\} \in K[V_i, V_j]$, let $(v_i, v_j) \in \vec{\mathbb{K}}[V_i, V_j]$ with probability $\frac{1}{2}$, and $(v_j, v_i) \in \vec{\mathbb{K}}[V_i, V_j]$ otherwise. Now, define the edges of the 3-uniform hypergraph $\mathbb{H}$ to consist of all oriented triangles obtained above.*

Figure 7 illustrates the construction of the hyperedges of $\mathbb{H}$ for Example 8.1. Clearly, $\mathbb{H}$ contains no copies of $K_4^{(3)}$. (In fact, it contains no 3 hyperedges on 4 points.) A routine application of the Chernoff Inequality (Fact 9.3) shows that, with a probability which tends to 1 (as $m \to \infty$), each $\mathbb{H}[V_i, V_j, V_k]$ is $o(1)$-regular with density $\frac{1}{4} + o(1)$ ($1 \leq i < j < k \leq 4$) where $o(1) \to 0$ as $m \to \infty$.

To overcome the problem in Example 8.1, one needs a stronger notion of hypergraph regularity. Such concepts are available, and are due to Frankl-Rödl-Skokan [8, 27] and to Gowers [10, 11]. These authors established various *Strong Hypergraph Regularity Lemmas*. These lemmas are

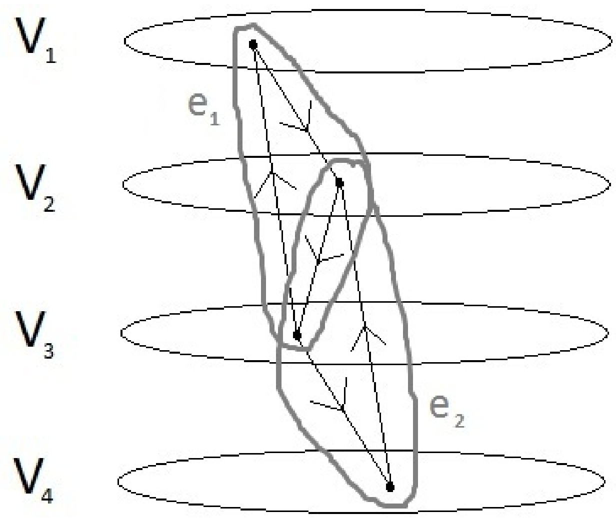

Figure 7.: Example 8.1

"strong" in the sense that they afford Counting Lemmas (and hence Packing Lemmas). (These proofs, by Nagle, Rödl and Schacht and by Gowers, are each well over fifty pages.) However, like Szemerédi's Regularity Lemma prior to [1], the Strong Hypergraph Regularity Lemmas have no current algorithmic versions. (Current work of Nagle, Rödl, Schacht seeks to resolve this issue. For partial progress, see [6, 15, 16, 25].)

On a related note, the partial progress mentioned above should allow one to prove a version of Theorem 1.5 with $k = 3$ for arbitrary hypergraphs F_0. In particular, Haxell, Nagle and Rödl [16] proved an Algorithmic Strong Hypergraph Regularity Lemma for 3-uniform hypergraphs as well as a corresponding Counting Lemma. With work, their Counting Lemma should imply a corresponding Packing Lemma. Thus, one could run their Algorithmic Strong Hypergraph Regularity Lemma in place of Czygrinow and Rödl's and employ their Packing Lemma in place of ours. Then, suitably adjusting our remaining tools, one should be able to prove Theorem 1.5 with $k = 3$ and F_0 arbitrary. (However, this work would be quite technical and lengthly and would result in its own project.)

Next recall, in Theorem 1.5, that we only commit to polynomial running time and not to any specific polynomial of n. This is largely due to Theorem 4.1, Grable's Algorithm, where no specific polynomial was given. Our proof suggests that Theorem 4.1 contributes running time $O(n^{k|F_0|})$

(where $f = |V(F_0)|$) to our algorithm in Theorem 1.5, but we have not yet proven this. We hope to return to this issue in the future and now state the following conjecture.

Conjecture 8.2 *For any linear k-uniform F_0 on f vertices, where f is suitably larger than k, the running time of Theorem 1.5 is at most $O(n^{k|F_0|})$.*

Note that Theorem 2.2 (The Algorithmic Hypergraph Regularity Lemma) contributes a complexity of $O(n^{2k-1} \log^2 n)$. Note also that the running time of the linear programming aspect of computing $\nu^*_{F_0}(H)$, as well as constructing the maximum fractional F_0-packing ψ^* of H, must be considered. Such results can be accomplished in time $O(n^k \log^c n)$, where $c = c(F_0)$, is a function of F_0. (The Algorithmic Hypergraph Regularity Lemma has higher complexity.) Thus, the complexity of Theorem 1.5 should be dominated by Grable's Algorithm and should run in time $O(n^{k|F_0|})$. We hope to return to these issues in the future.

Chapter 9

Background Material

We now state, for reference, several well-known results which were used in one or more of the previous proofs. We begin with a version of the well-known Cauchy-Schwarz Inequality. Recall that for $a_1, ..., a_t \geq 0$, the Cauchy-Schwarz Inequality asserts that $\sum_{i=1}^{t} a_i^2 \geq \frac{\left(\sum_{i=1}^{t} a_i\right)^2}{t}$,where equality holds if and only if $a_1 = \cdots = a_t$. The following extension thereof states that if, in the Cauchy-Schwarz Inequality, one has "near equality", then "most" of the a_i's are all "nearly the same".

Fact 9.1 (Cauchy-Schwarz Inequality (see, e.g. [28])) *For $a_1, \ldots, a_t \geq 0$ and $\tau \geq 0$, suppose*

1. $\sum_{i=1}^{t} a_i \geq (1 - \tau)at$*, and*
2. $\sum_{i=1}^{t} a_i^2 \leq (1 + \tau)a^2 t$*.*

Then, for all but $2\tau^{1/3}t$ terms $1 \leq i \leq t$, we have $a_i = a(1 \pm 2\tau^{1/3})$.

We continue by stating the well-known Markov Inequality.

Fact 9.2 (Markov Inequality (see, e.g. [3])) *If X is any nonnegative random variable and $a > 0$, then*

$$\mathbb{P}[|X| \geq a] \leq \frac{\mathbb{E}[X]}{a}.$$

Finally, we state two formulations of a high-deviation inequality that both hail from the Chernoff Inequality.

Fact 9.3 (Chernoff Inequality (see, e.g. [3, 17])) *Let $X_1, ..., X_n$ be independent Bernoulli random variable with $p_i = \mathbb{P}[X_i = 1]$ and define $X = \sum_i X_i$.*

1. Then, for any $0 < \delta < \frac{3}{2}$,

$$\mathbb{P}[X \neq (1 \pm \delta)\mathbb{E}[X]] \leq 2 \exp\left\{-\frac{\delta^2\, \mathbb{E}[X]}{3}\right\}.$$

2. Then, if $x \geq 7\, \mathbb{E}[X],\ \mathbb{P}[X \geq x] \leq \exp\{-x\}$.

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