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Wronskian, Grammian and Pfaffian Solutions to Nonlinear Partial Differential Equations

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Wronskian, Grammian and Pfaffian Solutions to Nonlinear Partial Differential Equations

by

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A dissertation submitted in partial fulfillment
of the requirements for the degree of
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Dedication

To my mother, wife, sons, brothers, sisters and friends
for supporting me in all my endeavors.

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Abstract

It is significantly important to search for exact soliton solutions to nonlinear partial differential equations (PDEs) of mathematical physics. Transforming nonlinear PDEs into bilinear forms using the Hirota differential operators enables us to apply the Wronskian and Pfaffian techniques to search for exact solutions for a (3+1)-dimensional generalized Kadomtsev-Petviashvili (KP) equation with not only constant coefficients but also variable coefficients under a certain constraint

$$(u_t + \alpha_1(t)u_{xxy} + 3\alpha_2(t)u_xu_y)_x + \alpha_3(t)u_{ty} - \alpha_4(t)u_{zz} + \alpha_5(t)(u_x + \alpha_3(t)u_y) = 0.$$

However, bilinear equations are the nearest neighbors to linear equations, and expected to have some properties similar to those of linear equations. We have explored a key feature of the linear superposition principle, which linear differential equations have, for Hirota bilinear equations, while intending to construct a particular sub-class of N-soliton solutions formed by linear combinations of exponential traveling waves. Applications are given for the (3+1) dimensional KP, Jimbo-Miwa (JM) and BKP equations, thereby presenting their particular N-wave solutions. An opposite question is also raised and discussed about generating Hirota bilinear equations possessing the indicated N-wave solutions, and two illustrative examples are presented.

Using the Pfaffianization procedure, we have extended the generalized KP equation to a generalized KP system of nonlinear PDEs. Wronskian-type Pfaffian and Gramm-type Pfaffian solutions of the resulting Pfaffianized system have been presented. Our results and computations basically depend on Pfaffian identities given by Hirota and Ohta. The Plücker relation and the Jacobi identity for determinants have also been employed.

A (3+1)-dimensional JM equation has been considered as another important example in soliton theory,

$$u_{yt} - u_{xxy} - 3(u_xu_y)_x + 3u_{xz} = 0.$$

Three kinds of exact soliton solutions have been given: Wronskian, Grammian and Pfaffian solutions. The Pfaffianization procedure has been used to extend this equation as well.

Within Wronskian and Pfaffian formulations, soliton solutions and rational solutions are usually expressed as some kind of logarithmic derivatives of Wronskian and Pfaffian type determinants and the determinants involved are made of functions satisfying linear systems of differential equations. This connection between nonlinear problems and linear ones utilizes linear theories in solving soliton equations.

Bäcklund transformations are another powerful approach to exact solutions of nonlinear equations. We have computed different classes of solutions for a (3+1)-dimensional generalized KP equation based on a bilinear Bäcklund transformation consisting of six bilinear equations and containing nine free parameters.

A variable coefficient Boussinesq (vcB) model in the long gravity water waves is one of the examples that we are investigating,

$$u_t + \alpha_1(t)u_{xy} + \alpha_2(t)(uw)_x + \alpha_3(t)v_x = 0,$$

$$v_t + \beta_1(t)(wv_x + 2vu_y + uv_y) + \beta_2(t)(u_xw_y - (u_y)^2) + \beta_3(t)v_{xy} + \beta_4(t)u_{xyy} = 0,$$

where $w_x = u_y$. Double Wronskian type solutions have been constructed for this (2+1)-dimensional vcB model.

Chapter 1

Introduction

Nonlinear partial differential equations arise in various subjects of mathematical physics and engineering, including fluid dynamics, plasma physics, quantum field theory, nonlinear wave propagation and nonlinear fiber optics. Nonlinear wave equations and the soliton concept have introduced remarkable achievements in the field of applied sciences [1]-[5].

In general, it is a very hard to find exact solutions to nonlinear partial differential equations, including soliton equations. Moreover, there is almost no general technique or algorithm that works for all equations, and usually each particular equation has to be studied as a separate problem.

However, in the past six decades, many powerful and systematic methods have been developed to obtain exact solutions for nonlinear differential equations, which play an important role in understanding various qualitative and quantitative features of nonlinear phenomena, and such methods include the inverse scattering method, the Darboux transformation, the Bäcklund transformation, the Hirota direct method, the Wronskian and Pfaffian techniques [6]-[21].

In this chapter, we present an overview of soliton theory and its historical background. Then we outline the organization of the dissertation.

1.1 Historical Perspective

Solitons were first accidentally observed by J. Scott Russell in 1834 [22, 23] while he was riding his horse along a canal near Edinburgh. He did extensive experiments in a laboratory scale wave tank in order to study this phenomenon more carefully. Included amongst Russell's results are the following:

1. He observed solitary waves, which are long, shallow, water waves of permanent shape, and so he concluded that they exist.
2. The speed of propagation, v , of a solitary wave in a channel of uniform depth h is given by

$v^2 = g(h + \eta)$, where η is the amplitude of the wave and g is the force due to gravity. For more details of his discovery, see [24].

In 1876, J. V. Boussinesq proposed another theory of shallow water waves, which agreed with what Russell observed [25, 26]; he derived a one dimensional nonlinear evolution equation, which named after him, in order to obtain his result. The existence of the solitary wave was first corroborated by the equation derived by Dierderik Johannes Korteweg and Gustav de Vries [27], now known as the KdV equation,

$$\frac{\partial \eta}{\partial \tau} = \frac{3}{2} \sqrt{\frac{g}{h}} \frac{\partial}{\partial \xi} \left(\frac{1}{2} \eta^2 + \frac{2}{3} \alpha \eta + \frac{1}{3} \sigma \frac{\partial^2 \eta}{\partial \xi^2} \right), \quad \sigma = \frac{1}{3} h^3 - \frac{Th}{(\rho g)}, \quad (1.1)$$

where η is the surface elevation of the wave above the equilibrium level h , α is a small arbitrary constant related to the uniform motion of the liquid, g is the gravitational constant, T is the surface tension and ρ is the density. Equation (1.1) may be brought into nondimensional form by making the transformation

$$t = \frac{1}{2} \sqrt{\frac{g}{h\sigma}} \tau, \quad x = -\sigma^{-\frac{1}{2}} \xi, \quad u = \frac{1}{2} \eta + \frac{1}{3} \alpha.$$

Hence, we can obtain

$$u_t + 6uu_x + u_{xxx} = 0. \quad (1.2)$$

In 1955, Fermi, Pasta and Ulam employed numerical methods to solve Newton's equations of motion for a one-dimensional series of similar masses connected by springs [28]. Zabusky and Kruskal [2] were inspired by those studies and they studied and analyzed the KdV equation which had been originally arisen from the Fermi, Pasta and Ulam work. They discovered that the solitary waves preserve their shape and velocity even after collisions. They called such waves 'solitons'.

The first exact soliton solution for the KdV equation was given by Gardner, Greene, Kruskal and Miura in 1967 [6]. They reduced the nonlinear problem to a well known Sturm-Liouville eigenvalue problem, and then they developed a new method for solving the initial value problem of the KdV equation, which is well known as the inverse scattering transform (IST) method. The IST is a well-developed mathematical theory which can be used to solve the initial value problems for a limited class of evolution equations. However, it is very difficult to establish an appropriate inverse scattering problem depending on the existence of an infinite number of independent conservation laws for an evolution equation. A generalization of their results was made by Lax in 1968 [29] and

he introduced a Lax pair concept.

In 1971, Ryogo Hirota developed an ingenious method for obtaining the exact multisoliton solution of the KdV equation and derived an explicit expression of the N -soliton solution [30]. His method consisted of transforming a nonlinear evolution equation into a bilinear equation through the dependent variable transformation [31]. The bilinear equation thus obtained can be solved by employing a perturbation method, which was shown to be applicable to a large class of nonlinear evolution equations such as the modified Korteweg-de Vries (mKdV) [32], sine-Gordon (sG) [33, 36], nonlinear Schrödinger (NLS) [34] and Toda lattice (TL) [35] equations.

The Hirota direct method, in 1984, enabled Tajiri to obtain N -soliton solutions of two and three dimensional nonlinear Klein-Gordon (KG) equations [37] and Higgs field (H) equation [38]. In 1987, Hietarinta published four papers regarding to the searching for integrable partial differential equations from the bilinear form of KdV, mKdV, sG, and NLS equations. In his investigations, he used computer algebra software to check the condition for existence of three soliton solutions, and he discovered many new integrable bilinear equations [39]-[42].

Satsuma discovered that the soliton solutions of the KdV equation could be expressed in terms of Wronskian determinants in 1979 [43]. Later, in 1983, Freeman and Nimmo found that the Kadomtsev-Petviashvili (KP) equation in its bilinear form could be written as a determinantal identity [15]. Following these notable achievements, Wronskian solutions of other equations, for example, the Boussinesq [44], sine-Gordon [45], nonlinear Schrödinger [46] and Davey-Stewartson [46] equations, were subsequently obtained. On the other hand, Nakamura was the first to consider soliton solutions of the KP equation in Grammian form. He noted that the Grammian determinant is related to the determinant with integral entries often used in the IST [47].

In 1989, Hirota described some properties of Pfaffians, which can be defined by the property that the square of a Pfaffian is the determinant of an antisymmetric matrix, and showed that the derivatives of the Pfaffians of special elements are represented by the sum of the Pfaffians. Using these properties, he proved that the KP equations, having B-type, can be reduced to the identity of Pfaffians [48]. In 1991, Hirota and Ohta developed a procedure for generalizing nonlinear evolution equations from the Kadomtsev-Petviashvili hierarchy to produce coupled systems of equations. This procedure is now called Pfaffianization [49].

1.2 Traveling Waves and Solitons

One of the main interesting properties of the KdV equation is the existence of permanent wave solutions, including solitary wave solutions.

Definition 1.1 [50] *A solitary wave solution of a partial differential equation*

$$L(x, t, u) = 0, \quad (1.3)$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}$ are temporal and spatial variables and the dependent variable $u \in \mathbb{R}$ is a traveling wave solution of the form

$$u(x, t) = f(x - \gamma t) = f(z), \quad (1.4)$$

whose transition is from one constant asymptotic state as $z \rightarrow -\infty$ to (possibly) another constant asymptotic state as $z \rightarrow \infty$. (Note that some definitions of solitary waves require the constant asymptotic states to be equal, often to zero.)

Definition 1.2 [50] *A soliton is a solitary wave which asymptotically preserves its shape and velocity upon nonlinear interaction with other solitary waves, or more generally, with another (arbitrary) localized disturbance.*

The physical definition of a wave is a disturbance that transmits energy from one place to another. The simplest wave propagation equation is given by

$$u_{tt} = v_0^2 u_{xx}, \quad (1.5)$$

where $u(x, t)$ represents the amplitude of the wave, and v_0 is the speed of the wave. The general d'Alembert's solution is

$$u(x, t) = f(x - v_0 t) + g(x + v_0 t), \quad (1.6)$$

where f and g are arbitrary functions which represent the right and the left propagating waves respectively. Since the one-dimensional wave equation can be factorized as

$$\left(\frac{\partial}{\partial t} - v_0 \frac{\partial}{\partial x}\right) \left(\frac{\partial}{\partial t} + v_0 \frac{\partial}{\partial x}\right) u(x, t) = 0, \quad (1.7)$$

let us consider the simpler form,

$$\left(\frac{\partial}{\partial t} + v_0 \frac{\partial}{\partial x}\right) u(x, t) = 0, \quad (1.8)$$

which has the right moving wave solution,

$$u(x, t) = f(x - v_0 t). \quad (1.9)$$

If we assume that u is periodic, then the most fundamental solution is the plane wave solution,

$$u(x, t) = \exp[i(\omega t - kx)]. \quad (1.10)$$

Substitution this solution in the simple wave equation (1.8), we get the relationship between the wave number k and the angular frequency ω which is given by $\omega = v_0 k$. This is called a *dispersion relation*, which, in this case, is linear. Such kind of waves that governed by linear dispersion relations are called *nondispersive* waves. The shape of these waves do not change as the wave propagate.

Adding a third order spatial derivative, which is the dispersion term, to the equation (1.8) gives the linear dispersive equation

$$u_t + v_0 u_x + u_{xxx} = 0. \quad (1.11)$$

Assuming that the equation (1.11) has the plane wave solution (1.10), then its dispersion relation is given by the following nonlinear relation in k ,

$$\omega = v_0 k - k^3. \quad (1.12)$$

Therefore, the wave propagates at the velocity

$$v_p(k) = \frac{\omega}{k} = v_0 - k^2. \quad (1.13)$$

Since the velocity varies with k , the wave spreads out as it travels. This shows that linear dispersive waves do not preserve their original shape. Now let us consider the following nonlinear nondispersive wave equation,

$$u_t + v(u)u_x = 0. \quad (1.14)$$

This equation is nonlinear wave equation in which the speed $v(u)$ depends on the amplitude u .

Equation (1.14) has the formal solution

$$u(x, t) = f(x - v(u)t), \quad (1.15)$$

and if $v = v(u)$ is an increasing function in u , then this formula tells us that a wave travels faster as its amplitude increases. This means that the top of the wave will move faster than the base of

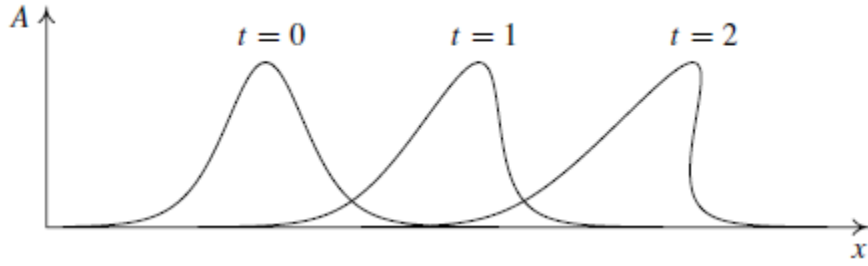


Figure 1.: Steepening of a solitary wave. A wave which is symmetrical at $t = 0$ steepens and breaks because of the dependence of the wave speed on its amplitude.

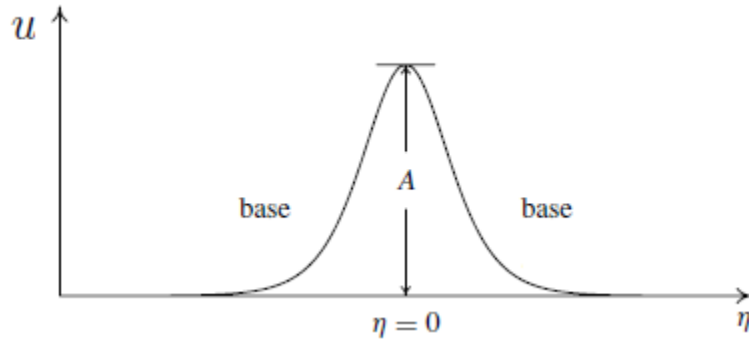


Figure 2.: Approximation of a solitary wave at its top and base.

the wave, and the wave will steepen (and eventually break). Thus, a non-linear non-dispersive wave will exhibit steepening and does not remain invariant like a soliton, see Figure 1.

We have seen from the above examples that neither linear dispersive solitary wave nor nonlinear nondispersive solitary wave can exist. We would like to investigate the influence that nonlinearity together with dispersion have on the wave behavior. For this end, we consider the well known KdV equation in its standard form,

$$u_t + 6uu_x + u_{xxx} = 0. \quad (1.16)$$

Assume that the solitary wave, shown in Figure 2, exists and it is symmetrical around the point of the maximum amplitude A . We can approximate u by the function u_{top} in the neighborhood of $\eta = 0$, which can be a quadratic in η and the dispersion term u_{xxx} will be zero. Therefore, the

function u_{top} satisfies the equation (1.14) with $v(u) = 6u$,

$$u_t + 6uu_x = 0 \quad (1.17)$$

On the other hand, at the base of the wave, the nonlinear term can be neglected because u is very small, and so the approximation of the solution at the base, denoted by u_{base} , satisfies the linear differential equation

$$u_t + u_{xxx} = 0. \quad (1.18)$$

Hence, the phase velocity is given by

$$v_p(k) = \frac{\omega}{k} = -k^2. \quad (1.19)$$

Therefore, the top and the bottom of the wave do not move at the same speed. But this contradicts our assumption that the above wave is a solitary wave. This contradiction comes from the nonlinearity of the KdV equation and the superposition principle of waves no longer valid. So we need to express the base of the wave in term of exponentially decaying solutions

$$u(x, t) = e^{\pm\eta}, \text{ where } \eta = px - \Omega t. \quad (1.20)$$

From the equation (1.18), we obtain the nonlinear dispersion relation,

$$\Omega = p^3. \quad (1.21)$$

The velocities at the top, v_{top} , and at the base, v_{base} , are given by

$$v_{top} = 6A, \quad v_{base} = \frac{\Omega}{p} = p^2. \quad (1.22)$$

Hence, in order to get a solitary wave that travels without changing its shape, at least, these two velocities should be coincide, which happens if and only if p and A satisfy the relation

$$6A = p^2. \quad (1.23)$$

From the above discussion, we can say that a wave equation having soliton solutions has both nonlinearity and dispersion.

To obtain traveling wave solutions of the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0, \quad (1.24)$$

where $u = u(x, t)$ is a differentiable function and $u(x, t)$ along with its derivatives tends to zero as $|x| \rightarrow \infty$, we seek a solution in the form $f(x - \gamma t)$. By substitution this form in the KdV equation, then integrating and multiplying by $2f'$, we get

$$(f')^2 = \gamma f^2 + 2f^3, \quad (1.25)$$

which is an ordinary differential equation with an explicit solution

$$f(z) = \frac{\gamma}{2} \operatorname{sech}^2 \frac{\sqrt{\gamma}}{2} z, \quad (1.26)$$

where $z = x - \gamma t$. Hence the KdV equation has the following one-soliton solution

$$u(x, t) = \frac{\gamma}{2} \operatorname{sech}^2 \frac{\sqrt{\gamma}}{2} (x - \gamma t) \quad (1.27)$$

$$= 2 \frac{\partial^2}{\partial x^2} \log(1 + e^{\sqrt{\gamma}(x - \gamma t)}). \quad (1.28)$$

One of the restrictions in the application of the KdV equation as a practical model for water waves, is that the KdV equation is strictly only (1+1)-dimensional, whereas the surface is two-dimensional. A two dimensional generalization of the KdV equation is the Kadomtsev-Petviashvili (KP) equation

$$(u_t + 6uu_x + u_{xxx})_x + 3\sigma^2 u_{yy} = 0, \quad (1.29)$$

where $\sigma^2 = \pm 1$. For more details about the physical derivation of the KP equation as a model for surface water, we refer to [50, 51].

1.3 Dissertation Outline

This dissertation is organized as follows. In Chapter two, we will provide a brief introduction to the Hirota perturbation method, where we present basic properties of the Hirota differential operators and their applications in transforming nonlinear partial differential equations into the Hirota bilinear form. The concrete examples that we will discuss are: the KdV, vcKP and JM equations and the Boussinesq system. Then we are going to discuss the Ma and Fan superposition principle for bilinear Hirota equations.

In Chapter three, we will introduce a new generalization for the KP equation with variable coefficients. Wronskian and Grammian solutions will be formulated for this generalized equation

after transforming it into a bilinear form. Furthermore, we will present an extension of the vcKP equation that results in a nonlinear system of bilinear differential equations with two kind of solutions: Wronski-type form solutions and Gram-type form solutions. Our results basically depend on Pfaffian identities provided in the first section of this chapter. In the last section, we use the bilinear Bäcklund transformation to present exponential and rational traveling wave solutions to the (3+1)-dimensional generalized KP equation.

The fourth chapter will be about another nonlinear partial differential equation, named after Jimbo and Miwa. Three kind of solutions will be constructed. Particular solutions will be given along with their figures in three dimensional plots and two dimensional contour plots. The Pfaffianization procedure will be used to extend this equation to a nonlinear system with two kind of exact solutions of Wronski-type and Gram-type. The variable coefficients JM equation will be discussed in the last section of this chapter.

In Chapter five, a double Wronskian determinant solution will be formulated for a new generalized Boussinesq system with time dependent coefficients. Indeed, we will show that this system will be transformed into a bilinear system and then we are going to verify that each equation in this system will be reduced to different forms of the Plücker relation.

Finally, conclusions and remarks will be given in the sixth chapter.

Chapter 2

Hirota Bilinear Equations

Although the Inverse Scattering Transform (IST) method is one of the powerful tools used to solve many initial value problems for nonlinear evolution equations, the transform is not easy to deal with and it needs strong assumptions and difficult analysis. On the other hand, one can find a traveling wave solution to many equations by a simple substitution, which often reduces the equation to an ordinary differential equation. The Hirota direct method lies within these two extremes. In this chapter, we give an introduction to the Hirota method and we are going to discuss a linear superposition principle applying to Hirota bilinear equations, which recently established by W. X. Ma and others.

2.1 The Hirota D-Operators

Definition 2.1 *Let S be the space of differentiable functions and $M \in \mathbb{N}$. Then the Hirota D-operator $D : S \times S \rightarrow S$ is defined by [12]*

$$\begin{aligned} & [D_{x_1}^{m_1} D_{x_2}^{m_2} \cdots D_{x_M}^{m_M}] f \cdot g \\ &= (\partial_{x_1} - \partial_{\acute{x}_1})^{n_1} \cdots (\partial_{x_M} - \partial_{\acute{x}_M})^{n_M} f(x_1, \cdots, x_M) g(\acute{x}_1, \cdots, \acute{x}_M) \Big|_{\acute{x}_1=x_1, \cdots, \acute{x}_M=x_M} \\ &= \partial_{\acute{x}_1}^{n_1} \cdots \partial_{\acute{x}_M}^{n_M} f(x_1 + \acute{x}_1, \cdots, x_M + \acute{x}_M) g(x_1 - \acute{x}_1, \cdots, x_M - \acute{x}_M) \Big|_{\acute{x}_1=\cdots=\acute{x}_M=0}, \quad (2.1) \end{aligned}$$

where n_1, \cdots, n_M are arbitrary nonnegative integers and x_1, \cdots, x_M are independent variables.

Let us list the following simple examples

$$D_x f \cdot g = f_x g - f g_x, \quad (2.2)$$

$$D_x D_t f \cdot g = f_{xt} g - f_x g_t - f_t g_x + f g_{xt}, \quad (2.3)$$

$$D_x^2 f \cdot g = f_{xx} g - 2f_x g_x + f g_{xx}, \quad (2.4)$$

$$D_x^3 f \cdot g = f_{xxx} g - f_{xx} g_x + f_x g_{xx} - f g_{xxx}, \quad (2.5)$$

$$\begin{aligned} & \vdots \\ D_x^n f \cdot g &= \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\partial^{(n-k)} f}{\partial x^{(n-k)}} \frac{\partial^k g}{\partial x^k}. \end{aligned} \quad (2.6)$$

It is useful to notice that the formulae for the D-operators in terms of derivatives are almost the same as those for normal derivatives of products. The only difference is that the signs in front of the terms having an odd degree of derivatives on the second function is negative.

The following properties of the D-operators can be derived from Definition 2.1

$$D_x^n D_t^m f \cdot g = D_t^m D_x^n f \cdot g = D_x^{n-1} D_t^m D_x f \cdot g, \quad (2.7)$$

$$D_x^n f \cdot 1 = \frac{\partial^n f}{\partial x^n}, \quad (2.8)$$

$$D_x^n f \cdot g = (-1)^n D_x^n g \cdot f, \quad (2.9)$$

$$D_x^n f \cdot f = 0 \quad \text{if } n \text{ is odd,} \quad (2.10)$$

$$D_x f \cdot g = 0 \quad \text{if } f \text{ is scalar multiple of } g, \quad (2.11)$$

$$D_x(D_x f \cdot g) \cdot h + D_x(D_x h \cdot f) \cdot g + D_x(D_x g \cdot h) \cdot f = 0. \quad (2.12)$$

Remark 2.2 Writing $D_x f \cdot g$ as $[f, g]$, we see that the identity (2.12) can be written as the Jacobi identity

$$[[f, g], h] + [[h, f], g] + [[g, h], f] = 0, \quad (2.13)$$

which indicates one connection between the D-operators and Lie algebras. A deep connection between bilinear equations written in terms of the D-operators and Lie algebras was discovered by Sato, Date, Kashiwara, Jimbo and Miwa [52]-[55].

Let $a(x)$ and $b(x)$ be two arbitrary differentiable functions in all orders in x , and δ be a parameter. Then, by Taylor expansions of $a(x + \delta)$ and $b(x - \delta)$ with respect to δ , we have

$$\begin{aligned} a(x + \delta)b(x - \delta) &= \left(\sum_{k=0}^{\infty} \frac{a^{(k)}(x)\delta^k}{k!} \right) \left(\sum_{j=0}^{\infty} \frac{b^{(j)}(x)(-\delta)^j}{j!} \right) \\ &= \sum_{n=0}^{\infty} \sum_{i=0}^n (-1)^i \binom{n}{i} \frac{a^{(n-i)}(x)b^{(i)}(x)\delta^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{\delta^n}{n!} D_x^n a(x) \cdot b(x) \\ &= e^{\delta D_x} a(x) \cdot b(x). \end{aligned}$$

Hence, we may define the D-operators by the exponential identity

$$e^{\delta D_x} a(x) \cdot b(x) = e^{\delta \partial_{\hat{x}}} a(x + \hat{x}) \cdot b(x - \hat{x})|_{\hat{x}=0} \quad (2.14)$$

$$= a(x + \delta)b(x - \delta). \quad (2.15)$$

For an exponential function, the relation

$$D_x^n e^{p_1 x} \cdot e^{p_2 x} = (p_1 - p_2)^n e^{(p_1 + p_2)x} \quad (2.16)$$

holds, where p_1 and p_2 are real numbers. In the case of normal derivatives, we have

$$\partial_x^n (e^{p_1 x} e^{p_2 x}) = (p_1 + p_2)^n e^{(p_1 + p_2)x}, \quad (2.17)$$

from which we obtain

$$D_x^n e^{p_1 x} \cdot e^{p_2 x} = \left(\frac{p_1 - p_2}{p_1 + p_2} \right) \partial_x^n (e^{p_1 x} e^{p_2 x}). \quad (2.18)$$

Generally speaking, if P is a polynomial in D_{x_1}, \dots, D_{x_M} , then

$$P(D_{x_1}, \dots, D_{x_M}) e^{\eta_1} \cdot e^{\eta_2} = \frac{P(p_{11} - p_{21}, \dots, p_{1M} - p_{2M})}{P(p_{11} + p_{21}, \dots, p_{1M} + p_{2M})} P(\partial_{x_1}, \dots, \partial_{x_M}) e^{\eta_1 + \eta_2}, \quad (2.19)$$

where $\eta_i = p_{i1}x_1 + \dots + p_{iM}x_M$, for $i = 1, 2$. This formula is useful in the expression for the two soliton solution of the bilinear equation

$$P(D_{x_1}, \dots, D_{x_M}) f \cdot f = 0. \quad (2.20)$$

In the next proposition, we formulate a very important identity, called *the exchange formula* [12, 10], which is the most useful when deriving Bäcklund transformations, transformation between solutions of a pair of differential equations, as will be seen in the third chapter of the dissertation.

Proposition 2.3 [12] *Let $a, b, c,$ and d be differentiable functions in $x,$ and let $\alpha, \beta,$ and γ be arbitrary parameters. Then the following identity holds*

$$e^{\alpha D_x} \left[e^{\beta D_x} a \cdot b \right] \cdot \left[e^{\gamma D_x} c \cdot d \right] = e^{(\frac{\beta-\gamma}{2}) D_x} \left[e^{(\alpha+\frac{\beta+\gamma}{2}) D_x} a \cdot d \right] \cdot \left[e^{(-\alpha+\frac{\beta+\gamma}{2}) D_x} c \cdot b \right]. \quad (2.21)$$

In particular, we have

$$e^{\delta D_x} ab \cdot cd = \left(e^{\delta D_x} a \cdot c \right) \left(e^{\delta D_x} b \cdot d \right) \quad (2.22)$$

$$= \left(e^{\delta D_x} a \cdot d \right) \left(e^{\delta D_x} b \cdot c \right). \quad (2.23)$$

The following formulae can be obtained by equating terms of the same order in δ on both sides of the above exchange formulae (2.22) and (2.23):

$$D_x ab \cdot c = a_x bc + a D_x b \cdot c, \quad (2.24)$$

$$D_x^2 ab \cdot cd = bd D_x^2 a \cdot c + 2(D_x a \cdot c)(D_x b \cdot d) + ac D_x^2 b \cdot d, \quad (2.25)$$

$$D_x^3 ac \cdot bc = c^2 D_x^3 a \cdot c + 3(D_x a \cdot b)(D_x c \cdot c), \quad (2.26)$$

$$D_x^n e^{px} a(x) \cdot e^{px} b(x) = e^{2px} D_x^n a(x) \cdot b(x), \quad (2.27)$$

where p is a constant parameter. For more properties, details and generalization of the Hirota D-operators the reader is referred to [12, 10, 56, 57].

2.2 Bilinearization of Nonlinear Partial Differential Equations

In this section, we discuss different types of transformations which map nonlinear PDEs into bilinear forms, which is the first step in the Hirota direct method. Then we consider some concrete examples that will be discussed in details in the coming chapters.

Definition 2.4 *We say that a nonlinear partial differential equation has a bilinear form if it can be written in the form*

$$\sum_{i,j=1}^n P_{ij}^m(D) f_i \cdot f_j = 0, \quad m = 1, \dots, r, \quad (2.28)$$

for some positive integers n, r and linear operators $P_{ij}^m(D)$. Here f_k are new dependent variables and D is vector of the Hirota operators.

Moreover, it is easy to prove the following proposition by using the properties of the D-operators in the previous section.

Proposition 2.5 *Let P be a polynomial in the Hirota operator D , and f, g are differentiable functions. Then the following hold:*

$$(a) \quad P(D)f \cdot g = P(-D)g \cdot f, \quad (2.29)$$

$$(b) \quad P(D)f \cdot 1 = P(\partial)f, \quad (2.30)$$

$$(c) \quad \text{if } P(D)a \cdot a = 0, \text{ where } a \text{ is any nonzero constant, then } P(0, \dots, 0) = 0. \quad (2.31)$$

Remark 2.6 *Since*

$$[D_{x_1}^{m_1} D_{x_2}^{m_2} \dots D_{x_M}^{m_M}]f \cdot f = 0 \quad \text{if} \quad \sum_{i=1}^M m_i \quad \text{is an odd number.} \quad (2.32)$$

Hence, we may assume that P is even.

Next we present some examples to describe transformations from nonlinear PDEs to bilinear PDEs.

Example 2.1 Let us start from the KdV equation in its standard form,

$$u_t + 6uu_x + u_{xxx} = 0. \quad (2.33)$$

The first transformation defined by

$$u = \frac{a}{b} \quad (2.34)$$

is called a rational transformation. From (2.15) we can see the identity

$$e^{\delta \frac{\partial}{\partial x}} \left(\frac{a}{b} \right) = \frac{e^{(\delta D_x)a \cdot b}}{\cosh(\delta D_x)b \cdot b}. \quad (2.35)$$

Expanding both sides of (2.35) with respect to the parameter δ and collecting terms in powers of δ , we obtain the formulae which express derivatives of $u = a/b$ in terms of the D-operators:

$$\frac{\partial}{\partial x} \frac{a}{b} = \frac{D_x a \cdot b}{b^2}, \quad (2.36)$$

$$\frac{\partial^2}{\partial x^2} \frac{a}{b} = \frac{D_x^2 a \cdot b}{b^2} - \frac{a D_x^2 b \cdot b}{b^2}, \quad (2.37)$$

$$\frac{\partial^3}{\partial x^3} \frac{a}{b} = \frac{D_x^3 a \cdot b}{b^2} - 3 \frac{D_x a \cdot b D_x^2 b \cdot b}{b^2}, \quad (2.38)$$

...

Setting $u = G/F$ and making use of the above formulae, The KdV equation may be written as

$$\frac{D_t G \cdot F}{F^2} + 6 \frac{G D_x G \cdot F}{F^2} + \frac{D_x^3 G \cdot F}{F^2} - 3 \frac{D_x G \cdot F D_x^2 F \cdot F}{F^2} = 0. \quad (2.39)$$

Multiplying by F^4 on both sides and rearranging the terms, we get

$$[(D_t + D_x^3)G \cdot F]F^2 + 3[D_x G \cdot F][2GF - D_x^2 F \cdot F] = 0. \quad (2.40)$$

Therefore, if we introduce an arbitrary function λ , the above equation may be decoupled into the bilinear form

$$(D_t + D_x^3)G \cdot F = 3\lambda D_x G \cdot F, \quad (2.41)$$

$$D_x^2 F \cdot F - 2GF = \lambda F^2. \quad (2.42)$$

An other kind of transformations is the logarithmic one:

$$u = 2(\log f)_{xx}. \quad (2.43)$$

A fundamental formula related to this transformation is

$$2 \cosh\left(\delta \frac{\partial}{\partial x}\right) \log f(x) = \log[\cosh(\delta D_x) f(x) \cdot f(x)]. \quad (2.44)$$

Expanding the above formula with respect to δ and collecting terms in powers of δ , we have

$$2 \frac{\partial^2}{\partial x^2} \log f = \frac{D_x^2 f \cdot f}{f^2}, \quad (2.45)$$

$$2 \frac{\partial^2}{\partial x \partial t} \log f = \frac{D_x D_t f \cdot f}{f^2}, \quad (2.46)$$

$$2 \frac{\partial^4}{\partial x^4} \log f = \frac{D_x^4 f \cdot f}{f^2} - 3 \left(\frac{D_x^2 f \cdot f}{f^2} \right)^2, \quad (2.47)$$

...

The KdV equation may be integrated to give

$$w_t + 3w_x^2 + w_{xxx} = c, \quad (2.48)$$

where $u = w_x$ and c is a constant of integration. Next, by using the dependent variable transformation

$$w = 2(\log f)_x, \quad \text{which is equivalent to} \quad u = 2(\log f)_{xx}. \quad (2.49)$$

From the above identities, the KdV equation gives

$$\frac{D_x D_t f \cdot f}{f^2} + 3 \left(\frac{D_x^2 f \cdot f}{f^2} \right)^2 + \frac{D_x^4 f \cdot f}{f^2} - 3 \left(\frac{D_x^2 f \cdot f}{f^2} \right)^2 = c, \quad (2.50)$$

hence the bilinear form of the KdV equation is

$$D_x (D_t + D_x^3) f \cdot f = c f^2. \quad (2.51)$$

In the above expression, the operator $D_t + D_x^2$ corresponds to the linear part of the KdV equation $\partial_t + \partial_x^3$.

Example 2.2 We consider the (3+1)-dimensional Jimbo-Miwa (JM) equation [54]

$$u_{yt} - u_{xxx} - 3(u_x u_y)_x + 3u_{xz} = 0, \quad (2.52)$$

Through the dependent variable transformation

$$u = 2(\log f)_x, \quad (2.53)$$

and integrating with respect to x , taking the constant of integration to be zero, then use the D-operator properties, the JM equation gives

$$\begin{aligned} & \frac{D_y D_t f \cdot f}{f^2} - \frac{D_x^3 D_y f \cdot f}{f^2} + 3 \left(\frac{D_x^2 f \cdot f}{f^2} \right) \left(\frac{D_x D_y f \cdot f}{f^2} \right) - 3 \left(\frac{D_x^2 f \cdot f}{f^2} \right) \left(\frac{D_x D_y f \cdot f}{f^2} \right) \\ & + 3 \frac{D_x D_z f \cdot f}{f^2} = 0. \end{aligned} \quad (2.54)$$

By Multiplying by f^2 , we get the bilinear form of the JM equation:

$$D_y D_t f \cdot f - D_x^3 D_y f \cdot f + 3 D_x D_z f \cdot f = 0. \quad (2.55)$$

Example 2.3 We consider the following (3+1)-dimensional nonlinear equation:

$$(u_t + \alpha_1(t) u_{xxy} + 3\alpha_2(t) u_x u_y)_x + \alpha_3(t) u_{ty} - \alpha_4(t) u_{zz} + \alpha_5(t) (u_x + \alpha_3(t) u_y) = 0, \quad (2.56)$$

where α_i , $1 \leq i \leq 5$, are nonzero arbitrary analytic functions in t . When $\alpha_i \equiv 1$ for $1 \leq i \leq 5$, $\alpha_5 \equiv 0$ and $x = y$, the equation (2.56) is reduced to the KP equation, and so we call it a generalized vcKP.

Through the dependent variable transformation

$$u = 2 \frac{\alpha_1(t)}{\alpha_2(t)} (\ln f)_x, \quad (2.57)$$

the above (3+1)-dimensional generalized vcKP equation is mapped into a Hirota bilinear equation

$$(\alpha_1(t)D_x^3D_y + D_tD_x + \alpha_3(t)D_tD_y - \alpha_4(t)D_z^2)f \cdot f = 0, \quad (2.58)$$

under the constraint:

$$\alpha_1(t) = C_0\alpha_2(t)e^{-\int \alpha_5(t)dt}, \quad (2.59)$$

where $C_0 \neq 0$ is an arbitrary constant.

Indeed, by transformation (2.57) the vcKP equation (2.56) gives

$$\begin{aligned} &\alpha_1(t)(\ln f)_{xxxxy} + 6C_0\alpha_2(t)e^{-\int \alpha_5(t)dt}[(\ln f)_{xx}(\ln f)_{xy}]_x + (\ln f)_{txx} \\ &+ \alpha_3(t)(\ln f)_{txy} - \alpha_4(t)(\ln f)_{xzz} = 0. \end{aligned} \quad (2.60)$$

By integrating with respect to x and taking the constant of integration to be zero, we get

$$\begin{aligned} &\alpha_1(t)\frac{D_x^3D_yf \cdot f}{2f^2} - \frac{3\alpha_1(t)}{2}\left(\frac{D_x^2f \cdot f}{f^2}\right)\left(\frac{D_xD_yf \cdot f}{f^2}\right) + 6\alpha_1(t)\left(\frac{D_x^2f \cdot f}{2f^2}\right)\left(\frac{D_xD_yf \cdot f}{2f^2}\right) \\ &+ \frac{D_xD_t f \cdot f}{2f^2} + \alpha_3(t)\frac{D_yD_t f \cdot f}{2f^2} - \alpha_4(t)\frac{D_z^2f \cdot f}{2f^2} = 0, \end{aligned} \quad (2.61)$$

from which the equation (2.56) can be written in the bilinear form (2.58).

Example 2.4 This example will be the (2+1) Ablowitz-Kaup-Newell-Segur (AKNS) system with variable coefficients [58]

$$p_t + a(t)\left(\frac{1}{2}p_{xy} - qp^2\right) = 0, \quad (2.62a)$$

$$q_t - a(t)\left(\frac{1}{2}q_{xy} - pq^2\right) = 0. \quad (2.62b)$$

Under the following rational transformations,

$$p = \frac{g}{f}, \quad q = \frac{h}{f}, \quad (2.63)$$

the system (2.62) gives

$$\frac{D_tg \cdot f}{f^2} + \frac{a(t)}{2}\left(\frac{D_xD_yg \cdot f}{f^2} - \frac{g}{f}\frac{D_xD_yf \cdot f}{f^2}\right) - a(t)\frac{hg^2}{f^3} = 0, \quad (2.64a)$$

$$\frac{D_th \cdot f}{f^2} - \frac{a(t)}{2}\left(\frac{D_xD_yh \cdot f}{f^2} - \frac{h}{f}\frac{D_xD_yf \cdot f}{f^2}\right) + a(t)\frac{h^2g}{f^3} = 0. \quad (2.64b)$$

From the above system, we can get the bilinear system for the system (2.62)

$$D_x D_y (f \cdot f) + 2gh = 0, \quad (2.65a)$$

$$(D_t + \frac{1}{2}a(t)D_x D_y)g \cdot f = 0, \quad (2.65b)$$

$$(D_t - \frac{1}{2}a(t)D_x D_y)h \cdot f = 0, \quad (2.65c)$$

2.3 The Hirota Direct Method

The Hirota direct method is a powerful tool for solving a wide class of nonlinear evolution equations. In this method, nonlinear equations are first transformed into bilinear equations through dependent variable transformations. These bilinear equations are then used to construct N-soliton solutions by employing a perturbation method.

Consider the bilinearized KdV equation

$$D_x (D_t + D_x^3) f \cdot f = 0. \quad (2.66)$$

Expand f with respect to a small parameter ϵ to obtain

$$f = 1 + \sum_{n=1}^{\infty} f_n \epsilon^n. \quad (2.67)$$

Substituting the above expansion formulae of f into the bilinear equation and arranging it at each order of ϵ , we have

$$\epsilon : D_x (D_t + D_x^3) (f_1 \cdot 1 + 1 \cdot f_1) = 0, \quad (2.68a)$$

$$\epsilon^2 : D_x (D_t + D_x^3) (f_2 \cdot 1 + f_1 \cdot f_1 + 1 \cdot f_2) = 0, \quad (2.68b)$$

$$\epsilon^3 : D_x (D_t + D_x^3) (f_3 \cdot 1 + f_2 \cdot f_1 + f_1 \cdot f_2 + 1 \cdot f_3) = 0, \quad (2.68c)$$

$$\epsilon^3 : D_x (D_t + D_x^3) (f_4 \cdot 1 + f_3 \cdot f_1 + f_2 \cdot f_2 + f_1 \cdot f_3 + 1 \cdot f_4) = 0, \quad (2.68d)$$

...

Using (2.8), the equation of order ϵ is equivalent to

$$\frac{\partial}{\partial x} \left(\frac{\partial}{\partial t} + \frac{\partial^3}{\partial x^3} \right) f_1 = 0. \quad (2.69)$$

The solution of the above linear differential equation (2.69) that describes a solitary wave (one-soliton) is given by

$$f_1 = e^{\eta_1}, \quad (2.70)$$

where $\eta_1 = k_1x + \omega_1t + \eta_1^0$, with a constant η_1^0 and $\omega_1(k_1) = -k_1^3$.

Hence the one-soliton solution to the KdV equation (2.33) equation is given by

$$u(x, t) = 2 \frac{\partial^2}{\partial x^2} \log(1 + e^{k_1x - k_1^3t}), \quad (2.71)$$

which coincides with the solution we computed in section (1.2).

To find the two-soliton solution, which describes the interaction of two single solitons, we choose the solution to the linear differential equation (2.69) to be

$$f_1 = e^{\eta_1} + e^{\eta_2}, \quad (2.72)$$

where $\eta_i = k_ix + \omega_it + \eta_i^0$, with a constant η_i^0 and $\omega_i(k_i) = -k_i^3$ for $i = 1, 2$. The equation of order ϵ^2 is

$$2 \frac{\partial}{\partial x} \left(\frac{\partial}{\partial t} + \frac{\partial^3}{\partial x^3} \right) f_2 = -D_x(D_t + D_x^3) f_1 \cdot f_1. \quad (2.73)$$

Substituting (2.72) into the right hand side of (2.73), we have, from the property of the D-operators (2.19),

$$\begin{aligned} D_x(D_t + D_x^3) f_1 \cdot f_1 &= D_x(D_t + D_x^3)(e^{\eta_1} + e^{\eta_2}) \cdot (e^{\eta_1} + e^{\eta_2}) \\ &= 2D_x(D_t + D_x^3)e^{\eta_1} \cdot e^{\eta_2} \\ &= 2(k_1 - k_2)[\omega_1 - \omega_2 + (k_1 - k_2)^3]e^{\eta_1 + \eta_2}. \end{aligned} \quad (2.74)$$

Equation (2.73) has a solution of the form

$$f_2 = a_{12}e^{\eta_1 + \eta_2}, \quad (2.75)$$

where, using (2.19), the coefficient a_{12} is given by

$$\begin{aligned} a_{12} &= -\frac{2(k_1 - k_2)[\omega_1 - \omega_2 + (k_1 - k_2)^3]}{2(k_1 + k_2)[\omega_1 + \omega_2 + (k_1 + k_2)^3]} \\ &= \left(\frac{k_1 - k_2}{k_1 + k_2} \right)^2. \end{aligned} \quad (2.76)$$

Substitution the expression for f_1 and f_2 given above into the linear differential equation of order ϵ^3 , we obtain

$$\begin{aligned} 2\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} + \frac{\partial^3}{\partial x^3}\right)f_3 &= -D_x(D_t + D_x^3)(f_2 \cdot f_1 + f_1 \cdot f_2) \\ &= -2k_2(\omega_2 + k_2^3)e^{2\eta_1 + \eta_2} - 2k_1(\omega_1 + k_1^3)e^{\eta_1 + 2\eta_2}. \end{aligned} \quad (2.77)$$

The right hand side of the above equation is zero because of the nonlinear dispersion relation $\omega_i(k_i) = -k_i^3$ for $i = 1, 2$. Hence we may choose $f_3 = 0$. In the same way, we may choose $f_n = 0$ for $n \geq 4$. Substitution f_n for $n = 1, 2, \dots$ into the the perturbation expansion of f , we get

$$f = 1 + \epsilon(e^{\eta_1} + e^{\eta_2}) + \epsilon^2 a_{12} e^{\eta_1 + \eta_2}. \quad (2.78)$$

Since each η_i is given by

$$\eta_i = k_i x + \omega_i t + \eta_i^0, \quad (2.79)$$

any positive ϵ can be absorbed into the constants η_i^0 . Hence

$$u = 2\frac{\partial^2}{\partial x^2} \log(1 + (e^{\eta_1} + e^{\eta_2}) + a_{12} e^{\eta_1 + \eta_2}) \quad (2.80)$$

gives the two-soliton solution to the KdV equation (2.33). In a way similar to the above treatment and writing

$$a_{ij} = e^{A_{ij}}, \quad (2.81)$$

we obtain the following N-soliton solution to the bilinear KdV equation

$$f = \sum' \exp \left[\sum_{i=1}^N \mu_i \eta_i + \sum_{i < j}^{(N)} A_{ij} \mu_i \mu_j \right], \quad (2.82)$$

where the first sum \sum' means a summation over all possible combinations of $\mu_1 = 0, 1, \mu_2 = 0, 1, \dots, \mu_N = 0, 1$, and the sum $\sum_{i < j}^{(N)}$ means a summation over all possible pairs (i, j) chosen from the set $\{1, 2, \dots, N\}$, with the condition that $i < j$.

We next consider a bilinear equation of the form

$$P(D_{x_1}, \dots, D_{x_M})f \cdot f = 0, \quad (2.83)$$

where P is a polynomial in D_{x_1}, \dots, D_{x_M} and satisfies the condition $P(\mathbf{0}) = 0$. We call this kind of equations *KdV-type bilinear equations*. The distinguishing feature of a KdV-type bilinear equation is that it has just one dependent variable f . Let us introduce the following vector notations,

$$\begin{aligned}\mathbf{D} &= (D_{x_1}, \dots, D_{x_M}), \\ \mathbf{x} &= (x_1, \dots, x_M), \\ \mathbf{k}_i &= (k_{i1}, \dots, k_{iM}).\end{aligned}$$

For all KdV-type bilinear equations,

$$P(\mathbf{D})f \cdot f = 0, \quad (2.84)$$

having N-soliton solutions, the N-soliton solutions f have the form (2.82), where

$$\eta_i = \mathbf{k}_i \cdot \mathbf{x} + \eta_i^0, \quad \eta_i^0 \text{ is constant} \quad (2.85)$$

$$P(\mathbf{k}_i) = 0, \quad (2.86)$$

and the phase shift a_{ij} is given by

$$a_{ij} = e^{A_{ij}} = -\frac{P(\mathbf{k}_i - \mathbf{k}_j)}{P(\mathbf{k}_i + \mathbf{k}_j)}, \quad (2.87)$$

provided that the following identity holds

$$\sum' P\left(\sum_{i=1}^N \rho_i \mathbf{k}_i\right) \prod_{i < j}^{(N)} P(\rho_i \mathbf{k}_i - \rho_j \mathbf{k}_j) \rho_i \rho_j = 0, \quad (2.88)$$

where the summation \sum' is taken over all possible combinations of $\rho_1 = 0, 1, \dots, \rho_N = 0, 1$. This is called the Hirota condition [39]-[42],[59]-[61].

2.4 Hirota Bilinear Equations with Linear Subspaces of Solutions

We would, in this section, like to explore when Hirota bilinear equations can possess linear subspaces of exponential traveling wave solutions. The involved exponential wave solutions may or may not satisfy the corresponding dispersion relation. The theory will explore that Hirota bilinear equations share some common characteristics with linear equations, which explains, to some extent, why Hirota bilinear equations can be solved analytically. Based on the Hirota bilinear formulation,

we will present a condition which is both sufficient and necessary for guaranteeing the applicability of the linear superposition principle for exponential waves [62, 63].

Interestingly, multivariate polynomials whose zeros form a vector space can generate the desired Hirota bilinear equations which possess given linear subspaces of solutions. However, it is still an open question to us how to judge when a multivariate polynomial possesses one and only one real zero point. The (3 + 1)-dimensional KP and BKP equations will be covered as special cases of the computed illustrative examples. The contents of this section are from the references [62, 63].

2.4.1 Linear Superposition Principle

Let P be a polynomial in M variables satisfying

$$P(0, \dots, 0) = 0, \quad (2.89)$$

which means that the constant term of P is zero. The corresponding Hirota bilinear equation reads

$$P(D_{x_1}, \dots, D_{x_M})f \cdot f = 0. \quad (2.90)$$

Using Remark 2.6, we may assume that P is an even polynomial, i.e.,

$$P(-x_1, \dots, -x_M) = P(x_1, \dots, x_M). \quad (2.91)$$

Let $N \in \mathbb{N}$ be fixed and introduce the following N wave variables:

$$\eta_i = k_{i1}x_1 + \dots + k_{iM}x_M, \quad 1 \leq i \leq N, \quad (2.92)$$

and N exponential wave functions:

$$f_i = e^{\eta_i} = e^{k_{i1}x_1 + \dots + k_{iM}x_M}, \quad 1 \leq i \leq N, \quad (2.93)$$

where the k_{ji} 's are all constants. Using the bilinear identity (2.19)

$$P(D_{x_1}, \dots, D_{x_M})e^{\eta_i} \cdot e^{\eta_j} = P(k_{1i} - k_{1j}, \dots, k_{Mi} - k_{Mj})e^{\eta_i + \eta_j}, \quad (2.94)$$

it follows directly from (2.89) that every exponential wave functions f_i , $1 \leq i \leq N$, gives a solution to the introduced Hirota bilinear equation (2.90).

Next, we consider a linear combination

$$f = \sum_{i=1}^N \varepsilon_i f_i = \sum_{i=1}^N \varepsilon_i e^{\eta_i}, \quad (2.95)$$

where ε_i for $1 \leq i \leq N$ are all arbitrary constants. A natural question here is when this linear combination will still tell a solution to the Hirota bilinear equation (2.90).

To answer this question, we make the following computation by using (2.89), (2.91) and (2.94):

$$\begin{aligned}
P(D_{x_1}, \dots, D_{x_M})f \cdot f &= \sum_{i=1, j=1}^N \varepsilon_i \varepsilon_j P(D_{x_1}, \dots, D_{x_M})e^{\eta_i} \cdot e^{\eta_j} \\
&= \sum_{i=1, j=1}^N \varepsilon_i \varepsilon_j P(k_{1i} - k_{1j}, \dots, k_{Mi} - k_{Mj})e^{\eta_i + \eta_j} \\
&= \sum_{1 \leq i < j \leq N} \varepsilon_i \varepsilon_j [P(k_{1i} - k_{1j}, \dots, k_{Mi} - k_{Mj}) \\
&\quad + P(k_{1j} - k_{1i}, \dots, k_{Mi} - k_{Mj})]e^{\eta_i + \eta_j} \\
&= \sum_{1 \leq i < j \leq N} 2\varepsilon_i \varepsilon_j P(k_{1i} - k_{1j}, \dots, k_{Mi} - k_{Mj})e^{\eta_i + \eta_j}.
\end{aligned}$$

This computation will play a key role in furnishing the linear superposition principle for the exponential waves e^{η_i} , $1 \leq i \leq N$.

It now follows that a linear combination function f defined by (2.95) solves the Hirota bilinear equation (2.90) if and only if the condition of

$$P(k_{1i} - k_{1j}, \dots, k_{Mi} - k_{Mj}) = 0, \quad 1 \leq i < j \leq N, \quad (2.96)$$

is satisfied. The condition (2.96) gives us a big system of nonlinear algebraic equations on the wave related numbers k_{ij} 's, as soon as the polynomial P is given. We will see that higher dimensional cases have more opportunities for us to get solutions for the variables k_{ij} 's, because there are more parameters to be determined in the resulting system of algebraic equations (2.96). The above analysis yields to the following criterion for the linear superposition principle.

Theorem 2.7 [63] *Let $P(x_1, \dots, x_M)$ be an even polynomial which satisfies $P(0, \dots, 0) = 0$ and the N wave variables η_i , $1 \leq i \leq N$, be defined by $\eta_i = k_{i1}x_1 + \dots + k_{iM}x_M$, $1 \leq i \leq N$, where the k_{ij} 's are all constants. Then any linear combination of e^{η_i} , $1 \leq i \leq N$, solves the Hirota bilinear equation $P(D_{x_1}, \dots, D_{x_M})f \cdot f = 0$ if and only if the following condition holds:*

$$P(k_{1i} - k_{1j}, \dots, k_{Mi} - k_{Mj}) = 0, \quad 1 \leq i < j \leq N.$$

This theorem informs us exactly when a linear superposition of exponential wave solutions can still solve a given Hirota bilinear equation, and it describes the interrelation between Hirota bilinear equations and the linear superposition principle for exponential waves. It also paves a way of constructing N-wave solutions to Hirota bilinear equations. The system (2.96) actually is a resonance condition we need to handle (see [64] for resonance of 2-solitons). Once we obtain a solution of the wave related numbers k_{ij} 's by solving the algebraic system (2.96), we can tell an N -wave solution, formed by (2.95), to the considered Hirota bilinear equation.

Next we list two (3+1)-dimensional examples with (3+1)-dimensional variables:

$$\eta_i = k_i x + l_i y + m_i z + \omega_i t, \quad 1 \leq i \leq N, \quad (2.97)$$

in order to present a working idea about what kind of related numbers could exist. More general examples will be created later in Subsection 2.4.2.

Example 2.5 Let $P = P(x, y, z, t)$ be the following polynomial

$$P(x, y, z, t) = x^3 y - tx + ty - z^2. \quad (2.98)$$

The corresponding required condition (2.96) now is

$$P(k_i - k_j, l_i - l_j, m_i - m_j, \omega_i - \omega_j) \quad (2.99)$$

$$= k_i^3 l_i - k_i^3 l_j - 3k_i^2 k_j l_i + 3k_i^2 k_j l_j + 3k_i k_j^2 l_i - 3k_i k_j^2 l_j - k_j^3 l_i + k_j^3 l_j + \omega_i l_i \quad (2.100)$$

$$- \omega_i l_j - \omega_j l_i + \omega_j l_j - \omega_i k_i + \omega_i k_j + \omega_j k_i - \omega_j k_j - m_i^2 + 2m_i m_j - m_j^2 = 0, \quad (2.101)$$

and the resulting Hirota bilinear equation becomes

$$(D_x^3 D_y - D_t D_x + D_t D_y - D_z^2) f \cdot f = 0, \quad (2.102)$$

namely,

$$(f_{xxxy} - f_{tx} + f_{ty} - f_{zz})f - 3f_{xxy}f_x + 3f_{xy}f_{xx} - f_y f_{xxx} + f_t f_x - f_t f_y + f_z^2 = 0. \quad (2.103)$$

Under the transformation $u = (\log f)_x$, this equation is mapped into

$$u_{xxxy} + 3(u_x u_y)_x - u_{tx} + u_{ty} - u_{zz} = 0. \quad (2.104)$$

Based on the linear superposition principle for exponential waves in Theorem 2.7, solving the above system on the wave related numbers tells an N-wave solution to the nonlinear equation (2.104):

$$u = 2(\log f)_x, \quad f = \sum_{i=1}^N \epsilon_i f_i = \sum_{i=1}^N \epsilon_i e^{k_i x - (1/3)a^2 k_i y + a k_i^2 z - 4[a^2/(a^2+3)]k_i^3 t}, \quad (2.105)$$

where the ϵ_i 's and k_i 's are all arbitrary constants. Each exponential wave f_i in the solution f satisfies the corresponding nonlinear dispersion relation, i.e., we have

$$P(k_i, l_i, m_i, \omega_i) = 0, \quad 1 \leq i \leq N. \quad (2.106)$$

Example 2.6 Let P be the following polynomial

$$P(x, y, z, t) = ty - x^3 y + 3x^2 + 3z^2. \quad (2.107)$$

The corresponding required condition (2.96) now is

$$P(k_i - k_j, l_i - l_j, m_i - m_j, \omega_i - \omega_j) \quad (2.108)$$

$$= \omega_i l_i - \omega_i l_j - \omega_j l_i + \omega_j l_j - k_i^3 l_i + k_i^3 l_j + 3k_i^2 k_j l_i - 3k_i^2 k_j l_j - 3k_i k_j^2 l_i \quad (2.109)$$

$$+ 3k_i k_j^2 l_j + k_j^3 l_i - k_j^3 l_j + 3m_i^2 - 6m_i m_j + 3m_j^2 + 3k_i^2 - 6k_i k_j + 3k_j^2 = 0, \quad (2.110)$$

and the resulting Hirota bilinear equation becomes

$$(D_t D_y - D_x^3 D_y + 3D_x^2 + 3D_z^2) f \cdot f = 0, \quad (2.111)$$

namely,

$$(f_{ty} - f_{xxx} y + 3f_{xx} + 3f_{zz})f - f_t f_y + f_y f_{xxx} + 3f_{xy} f_x - 3f_{xy} f_{xx} - 3f_x^2 - 3f_z^2 = 0. \quad (2.112)$$

Under the transformation $u = (\log f)_x$, this equation is mapped into

$$u_{ty} - u_{xxx} y - 3(u_x u_y)_x + 3u_{xx} + 3u_{zz} = 0. \quad (2.113)$$

Based on the linear superposition principle for exponential waves in Theorem 2.7, solving the above system on the wave related numbers engenders an N-wave solution to the nonlinear equation (2.113):

$$u = 2(\log f)_x, \quad f = \sum_{i=1}^N \epsilon_i f_i = \sum_{i=1}^N \epsilon_i e^{k_i x - (1+a^2)k_i^{-1} y + a k_i^2 z + k_i^3 t}, \quad (2.114)$$

where the ϵ_i 's and k_i 's are all arbitrary constants. However, each exponential wave f_i in the solution f doesn't satisfy the corresponding nonlinear dispersion relation, i.e., we have

$$P(k_i, l_i, m_i, \omega_i) \neq 0, \quad 1 \leq i \leq N. \quad (2.115)$$

It is also direct to prove that

$$P(D_x, D_y, D_z, D_t)(e^\xi f) \cdot (e^\eta g) \quad (2.116)$$

$$= e^{\xi+\eta} P(D_x + k_1 - k_2, D_y + l_1 - l_2, D_z + m_1 - m_2, D_t - \omega_1 + \omega_2) f \cdot g, \quad (2.117)$$

where $\xi = k_1 x + l_1 y + m_1 z - \omega_1 t$, $\eta = k_2 x + l_2 y + m_2 z - \omega_2 t$, and P is a polynomial in the indicated variables. Taking

$$\xi = \eta = \eta_0 = k_0 x + l_0 y + m_0 z - \omega_0 t,$$

the above identity yields

$$P(D_x, D_y, D_z, D_t)(e^{\eta_0} f) \cdot (e^{\eta_0} g) = e^{2\eta_0} P(D_x, D_y, D_z, D_t) f \cdot g. \quad (2.118)$$

Therefore, we can get a new class of multiple exponential wave solutions by $f' = e^{\eta_0} f$, where f is an original multiple exponential wave solution like any one in (2.105) and (2.114); and such solutions form a new linear subspace of solutions and thus there exist infinitely many subspaces of solutions.

2.4.2 Bilinear Equations with Given Linear Subspaces of Solutions

Taking one of the wave variables η_i , $1 \leq i \leq N$, to be a constant, say, taking

$$\eta_{i_0} = \epsilon_{i_0}, \quad \text{i.e., } k_{j i_0} = 0, \quad 1 \leq j \leq M, \quad (2.119)$$

where $1 \leq i_0 \leq N$ is fixed, the N-wave solution condition (2.96) requires that all other wave related numbers have to satisfy the dispersion relation of the Hirota bilinear equation (2.90):

$$P(k_{1i}, \dots, k_{Mi}) = 0, \quad 1 \leq i \leq N, \quad i \neq i_0. \quad (2.120)$$

The resulting solution presents a specific class of N-soliton solutions by the Hirota perturbation technique, truncated at the second-order perturbation term.

Combining the dispersion relation (2.120) with the N-wave solution condition (2.96) leads to the following sufficient condition on P for the corresponding Hirota bilinear equation (2.90) to satisfy the linear superposition principle for exponential waves:

$$P(\mathbf{k}) = P(\mathbf{l}) = 0 \Rightarrow P(\mathbf{k} - \mathbf{l}) = 0, \quad (2.121)$$

where \mathbf{k} and \mathbf{l} are two M dimensional vectors. Let us list a property of zeros of such multivariate polynomials as follows.

Lemma 2.8 [63] *Let $P(x_1, \dots, x_M)$ be a real (or complex) multivariate polynomial (which could contain terms of both even and odd degree). Suppose P has at least one zero and satisfies that if*

$$P(\mathbf{k}) = P(\mathbf{l}) = 0, \text{ then } P(\mathbf{k} - \mathbf{l}) = 0.$$

Then all zeros of the polynomial P form a real (or complex) vector space.

Given an n -dimensional linear subspace V_n of \mathbb{R}^M , let us introduce a constant matrix $A = (a_{ij})_{M' \times M}$ of rank n for any $M' \in \mathbb{N}$ such that the solution space of a linear system:

$$A\mathbf{x} = 0, \quad \mathbf{x} = (x_1, \dots, x_M)^T, \quad (2.122)$$

defines the n -dimensional subspace V_n . Let $Q(y_1, \dots, y_{M'})$ be a multivariate polynomial in $\mathbf{y} = (y_1, \dots, y_{M'})^T$ and posses only one zero: $\mathbf{y} = \mathbf{y}_0$. Then

$$P(x_1, \dots, x_M) = Q((A\mathbf{x} + \mathbf{y}_0)^T) \quad (2.123)$$

presents a multivariate polynomial which obviously satisfies the property (2.121), and the resulting Hirota bilinear equation possesses the linear subspace of exponential wave solutions determined by

$$f = \sum_{i=1}^N \epsilon_i e^{k_{1i}x_1 + \dots + k_{Mi}x_M}, \quad N \geq 1, \quad (2.124)$$

where $A(k_{1i} - k_{1j}, \dots, k_{Mi} - k_{Mj})^T = 0$, $1 \leq i \neq j \leq N$, and the ϵ_i 's are all arbitrary constants. While generating illastrative examples of such Hirota bilinear equations, one remaining question is how to determine if a multivariate polynomial has one and only one real zero point. This is a more difficult problem than Hilbert's 17th problem.

Now we list what we have analyzed in the following theorem.

Theorem 2.9 [63] Let $M, M', n \in \mathbb{N}$, and $A = (a_{ij})_{M'M}$ be a constant matrix of rank n . Suppose that $Q(y_1, \dots, y_{M'})$ be a multivariate polynomial in $\mathbf{y} = (y_1, \dots, y_{M'})^T$ and posses only one real zero: $\mathbf{y} = \mathbf{y}_0$. Then the zeros of the multivariate polynomial

$$P(x_1, \dots, x_M) = Q((A\mathbf{x} + \mathbf{y}_0)^T), \quad \mathbf{x} = (x_1, \dots, x_M)^T \quad (2.125)$$

form an n -dimensional linear subspace, and the corresponding Hirota bilinear equation

$$P(D_{x_1}, \dots, D_{x_M})f \cdot f = Q((AD_{\mathbf{x}} + \mathbf{y}_0)^T)f \cdot f = 0 \quad (2.126)$$

possesses a linear subspace of solutions determined by

$$f = \sum_{i=1}^N \epsilon_i e^{k_{1i}x_1 + \dots + k_{Mi}x_M}, \quad N \geq 1, \quad (2.127)$$

where $A(k_{1i} - k_{1j}, \dots, k_{Mi} - k_{Mj})^T = 0$, $1 \leq i \neq j \leq N$, and the ϵ_i 's are all arbitrary constants.

In what follows, we present two illustrative examples to shed light on the algorithm in Theorem 2.9.

Example 2.7 This example has

$$Q(y_1, y_2) = y_1^2 + y_2^2, \quad \mathbf{y}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \end{bmatrix}, \quad \mathbf{x} = (x, y, z, t)^T. \quad (2.128)$$

Then the associated multivariate polynomial reads

$$P(x, y, z, t) = x^2 - 2tx + t^2 + y^2 + 4yz + 4z^2, \quad (2.129)$$

and the corresponding Hirota bilinear equation is defined by

$$(D_x^2 - 2D_x D_t + D_t^2 + D_y^2 + 4D_y D_z + 4D_z^2)f \cdot f = 0. \quad (2.130)$$

This bilinear equation possesses the linear subspace of solutions given by

$$f = \sum_{i=1}^N \epsilon_i f_i = e^{k_0 x + l_0 y + m_0 z - \omega_0 t} \sum_{i=1}^N \epsilon_i e^{-\omega_i x - 2m_i y + m_i z - \omega_i t}, \quad N \geq 1, \quad (2.131)$$

where the ϵ_i 's, m_i 's and ω_i 's are all arbitrary constants but k_0, l_0, m_0 and x_0 are arbitrary fixed constants. Evidently, all exponential waves f_i in the solution f satisfy the corresponding nonlinear dispersion relation iff the wave function $e^{k_0 x + l_0 y + m_0 z - \omega_0 t}$ satisfies the corresponding nonlinear dispersion relation.

Example 2.8 This example has

$$Q(y_1, y_2) = (y_1 + 1)^2 + y_2^4, \mathbf{y}_0 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, A = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 2 & 1 \end{bmatrix}, \mathbf{x} = (x, y, z, t)^T. \quad (2.132)$$

Then the associated multivariate polynomial reads

$$P(x, y, z, t) = y^2 - 2yt + 25t^2 + x^4 + 8x^3z + 4tx^3 + 24x^2z^2 + 24tx^2z + 6t^2x^2 + 32xz^3 + 48txz^2, \quad (2.133)$$

and the corresponding Hirota bilinear equation is given by

$$(D_y^2 - 2D_yD_t + 25D_t^2 + D_x^4 + 8D_x^3D_z + 4D_x^3D_t) \quad (2.134)$$

$$+ 24D_x^2D_z^2 + 24D_tD_x^2D_z + 6D_x^2D_t^2 + 32D_xD_z^3 + 48D_xD_tD_z^2) f \cdot f = 0. \quad (2.135)$$

This bilinear equation possesses the linear subspace of solutions determined by

$$f = \sum_{i=1}^N \epsilon_i f_i = e^{k_0x+l_0y+m_0z-\omega_0t} \sum_{i=1}^N \epsilon_i e^{(\omega_i-2m_i)x-\omega_iy+m_iz-\omega_it}, \quad N \geq 1, \quad (2.136)$$

where the ϵ_i 's, m_i 's and ω_i 's are all arbitrary constants but k_0, l_0, m_0 and x_0 are arbitrary fixed constants. Evidently, all exponential wave f_i in the solution f satisfy the corresponding nonlinear dispersion relation iff the wave function $e^{k_0x+l_0y+m_0z-\omega_0t}$ satisfies the corresponding nonlinear dispersion relation.

Chapter 3

Wronskian and Pfaffian Solutions to (3+1)-Dimensional Generalized Soliton Equations of KP Type

Wronskian and Pfaffian formulations are a common feature for soliton equations, and lead to a powerful tool to construct exact solutions to soliton equations [15]-[21]. The techniques have been applied to many soliton equations such as the KdV, MKdV, NLS, KP, BKP and sin-Gordon equations. Within Wronskian and Pfaffian formulations, solitons are usually expressed as some kind of logarithmic derivatives of Wronskian type and Pfaffian type determinants and the determinants involved are made of eigenfunctions satisfying linear partial differential equations. This connection between nonlinear problems and linear ones utilizes linear theories in solving soliton equations.

In this chapter, we would like to study Pfaffians, their relation with determinants, Pfaffian expansion formulae and Pfaffian identities. Then Wronskian and Pfaffian solutions will be formulated for a (3+1)-dimensional generalized KP equation with variable coefficients. In order to verify our results we will follow the following procedures: transform a nonlinear partial differential equation into a bilinear form, then rewrite the bilinear equation using Wronskians and Pfaffians, and finally confirm that the bilinear equation is nothing but Pfaffian identities.

In the third section, the (3+1)-dimensional generalized KP equation will be extended to a system of nonlinear partial differential equations. This procedure is called Pfaffianization [65]. Wronskian-type and Gramm-type Pfaffian solutions of the resulting Pfaffianized system will be constructed. Two kinds of Pfaffian identities are the basis of our analysis.

In the last section, we will consider the constant coefficients case of the considered (3+1)-dimensional generalized KP equation. A bilinear Bäcklund Transformation will be presented for a (3+1)-dimensional generalized KP equation, which consists of six bilinear equations and involves nine arbitrary parameters. Two classes of exponential and rational traveling wave solutions with arbitrary wave numbers are computed, based on the proposed bilinear Bäcklund transformation [12, 67, 66].

3.1 Pfaffians

The determinant of a skew-symmetric matrix $A = \det(a_{ij})_{1 \leq i, j \leq m}$ can always be written as the square of a polynomial in the matrix entries [69]. This polynomial is called the Pfaffian of the matrix, denoted by $Pf(A)$. The term Pfaffian was introduced by Cayley (1852) who named it after Johann Friedrich Pfaff [70]. The Pfaffian is nonvanishing only for $2n \times 2n$ skew-symmetric matrices, in which case it is a polynomial of degree n .

We are going to use the following notation for a Pfaffian of order n

$$Pf(A) = (1, 2, \dots, 2n). \quad (3.1)$$

For example, for $n = 1$,

$$\begin{vmatrix} 0 & a_{12} \\ -a_{12} & 0 \end{vmatrix} = a_{12}^2 \equiv (1, 2)^2, \quad (3.2)$$

$$\begin{vmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{vmatrix} = (a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})^2 \equiv (1, 2, 3, 4)^2. \quad (3.3)$$

Therefore, a second-order Pfaffian given by $(1, 2, 3, 4)$ is expanded as

$$(1, 2, 3, 4) = (1, 2)(3, 4) - (1, 3)(2, 4) + (1, 4)(2, 3), \quad (3.4)$$

where $(j, k) = a_{ij}$ for $j < k$. It should be noted that from the skew-symmetric property $a_{kj} = -a_{jk}$, we have

$$(k, j) = -(j, k).$$

In general, a Pfaffian $(1, 2, \dots, 2n)$ can be expanded as

$$\begin{aligned} (1, 2, \dots, 2n) &= (1, 2)(3, 4, \dots, 2n) - (1, 3)(2, 4, 5, \dots, 2n) \\ &\quad + (1, 4)(2, 3, 5, \dots, 2n) - \dots + (1, 2n)(2, 3, \dots, 2n-1) \\ &= \sum_{j=2}^{2n} (-1)^j (1, j)(2, 3, \dots, \hat{j}, \dots, 2n), \end{aligned} \quad (3.5)$$

where \hat{j} means that index j is omitted. Repeating the expansion (3.5), we arrive at the summation of products of first-order Pfaffians [68]:

$$(1, 2, \dots, 2n) = \sum'_P (-1)^P (j_1, j_2)(j_3, j_4)(j_5, j_6) \cdots (j_{2n-1}, j_{2n}), \quad (3.6)$$

where the sum notation \sum'_P means the sum over all possible combinations of pairs selected from $\{1, 2, \dots, 2n\}$ which satisfy

$$j_1 < j_2, j_3 < j_4, j_5 < j_6, \dots, j_{2n-1} < j_{2n},$$

$$j_1 < j_3 < j_5 < \dots < j_{2n-1},$$

and $(-1)^P$ has the value 1 or -1 if the sequence j_1, j_2, \dots, j_{2n} is an even or odd permutation respectively.

3.1.1 Pfaffian Expression for General Determinants and Wronskians

We have already defined the Pfaffian through the determinant of a $2n \times 2n$ skew-symmetric matrix. Conversely, an n th-order determinant,

$$B \equiv \det(b_{jk})_{1 \leq j, k \leq n}, \quad (3.7)$$

can be expressed as an n th-order Pfaffian,

$$B = (1, 2, \dots, n, n^*, \dots, 2^*, 1^*), \quad (3.8)$$

where the Pfaffian entries $(j, k), (j^*, k^*), (j, k^*)$ are defined by

$$(j, k) = 0, \quad (j^*, k^*) = 0, \quad (j, k^*) = b_{jk}. \quad (3.9)$$

For example, if $n = 2$, we have

$$\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} = (1, 2, 2^*, 1^*) \quad (3.10)$$

Now let us consider an n th-order Wronskian determinant $W(f_1, f_2, \dots, f_n)$ which is defined by

$$W(f_1, f_2, \dots, f_n) \equiv \det \left(\frac{\partial^{j-1}}{\partial x^{j-1}} f_i \right)_{1 \leq i, j \leq n}. \quad (3.11)$$

It can be expressed as an n th-order Pfaffian [48]:

$$W(f_1, f_2, \dots, f_n) = (d_0, d_1, d_2, \dots, d_{n-1}, n, \dots, 3, 2, 1), \quad (3.12)$$

where the Pfaffian entries (i, j) , (d_k, i) , and (d_k, d_l) are defined by

$$(i, j) = 0, \quad (d_k, i) \equiv f_i^{(k)}, \quad (d_k, d_l) \equiv 0, \quad (3.13)$$

for $i, j = 1, 2, \dots, n$ and $k, l = 0, 1, \dots, n-1$, and $f_i^{(k)}$ stands for the k -th derivative of f_i with respect to x .

For example, if $n = 2$, we have

$$\begin{vmatrix} f_1^{(0)} & f_1^{(1)} \\ f_2^{(0)} & f_2^{(1)} \end{vmatrix} = (d_0, d_1, 2, 1) \quad (3.14)$$

$$= -(d_0, 2)(d_1, 1) + (d_0, 1)(d_1, 2) \quad (3.15)$$

$$= f_1^{(0)} f_2^{(1)} - f_2^{(0)} f_1^{(1)}. \quad (3.16)$$

3.1.2 Pfaffian Expression of Jacobi Identities for Determinants

The Jacobi identity for determinants is stated in the following proposition [71]

Proposition 3.1 [10] *Let D be the determinant of the n th-order matrix A . Then*

$$DD \begin{pmatrix} i & j \\ k & l \end{pmatrix} = D \begin{pmatrix} i \\ k \end{pmatrix} D \begin{pmatrix} i \\ l \end{pmatrix} - D \begin{pmatrix} i \\ l \end{pmatrix} D \begin{pmatrix} j \\ k \end{pmatrix}, \quad i < j, k < l, \quad (3.17)$$

where

$D \begin{pmatrix} i \\ k \end{pmatrix}$ is obtained by eliminating j th row and k th column from D , and

$D \begin{pmatrix} i & j \\ k & l \end{pmatrix}$ is obtained by eliminating the i th and j th rows and the k th and l th columns from the determinant D .

For example, take $n = 3$ and $i = k = 1$, $j = l = 2$, we get

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{vmatrix} a_{33} \end{vmatrix} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}.$$

Employing the Pfaffian expressions given in the previous subsection, the terms of the Jacobi identity (3.17) can be expressed by

$$D = (1, 2, \dots, n, n^*, \dots, 2^*, 1^*) \quad (3.18)$$

$$D \begin{pmatrix} i \\ k \end{pmatrix} = (1, 2, \dots, \hat{i}, \dots, n, n^*, \dots, \hat{k}^*, \dots, 2^*, 1^*) \quad (3.19)$$

$$D \begin{pmatrix} i & j \\ k & l \end{pmatrix} = (1, 2, \dots, \hat{i}, \dots, \hat{j}, \dots, n, n^*, \dots, \hat{k}^*, \dots, \hat{l}^*, \dots, 2^*, 1^*), \quad (3.20)$$

where the Pfaffian entries are defined by the same way as in (3.9). For more details and the proof for the Jacobi identity in its determinant and Pfaffian forms, the reader is referred to [10] and [72], respectively.

3.1.3 Pfaffian Identities and Expansion Formulae

In [73], Y. Ohta introduced a simple proof to the following identity

$$\begin{aligned} & \sum_{j=0}^M (-1)^j (b_0, b_1, \dots, \hat{b}_j, \dots, b_M) (b_j, c_0, c_1, \dots, c_N) \\ &= \sum_{k=0}^N (-1)^k (b_0, b_1, \dots, b_M, c_k) (c_0, c_1, \dots, \hat{c}_k, \dots, c_N), \end{aligned} \quad (3.21)$$

which can be proved by using expansion formula (3.5). Expanding the second Pfaffian on the left hand side with respect to b_j and expanding the first Pfaffian on the right hand side with respect to c_k , we obtain

$$\begin{aligned} & \sum_{j=0}^M (-1)^j \sum_{k=0}^N (-1)^k (b_0, b_1, \dots, \hat{b}_j, \dots, b_M) (b_j, c_k) (c_0, c_1, \dots, \hat{c}_k, \dots, c_N) \\ &= \sum_{k=0}^N (-1)^k \sum_{j=0}^M (-1)^j (b_0, b_1, \dots, \hat{b}_j, \dots, b_M) (b_j, c_k) (c_0, c_1, \dots, \hat{c}_k, \dots, c_N). \end{aligned} \quad (3.22)$$

The above equality is easily obtained by interchanging the sums over j and k .

In order to get Pfaffian identities, let us take the following two special cases.

Case I:

Taking $M = 2n$, $N = 2n + 2m - 2$ (m is odd) and the characters b_j, c_k as follows

$$\begin{aligned} b_0 &= a_1, b_1 = 1, b_2 = 2, \dots, b_M = b_{2n} = 2n, \\ c_0 &= a_2, c_1 = a_3, c_2 = a_4, \dots, c_{2m-2} = a_{2m}, \\ c_{2m-1} &= 1, c_{2m} = 2, c_{2m+1} = 3, \dots, c_N = c_{2n+2m-2} = 2n, \end{aligned}$$

we get the following Pfaffian identity

$$\begin{aligned} &(1, 2, \dots, 2n)(a_1, a_2, \dots, a_{2m}, 1, 2, \dots, 2n) \\ &= \sum_{s=2}^{2m} (-1)^s (a_1, a_s, 1, 2, \dots, 2n)(a_2, a_3, \dots, \hat{a}_s, \dots, a_{2m}, 1, 2, \dots, 2n). \end{aligned} \quad (3.23)$$

For example, in the case $m = 2$, the identity (3.23) can be written as

$$\begin{aligned} &(1, 2, \dots, 2n)(a_1, a_2, a_3, a_4, 1, 2, \dots, 2n) \\ &= (a_1, a_2, 1, 2, \dots, 2n)(a_3, a_4, 1, 2, \dots, 2n) \\ &\quad - (a_1, a_3, \dots, 2n)(a_2, a_4, 1, 2, \dots, 2n) \\ &\quad + (a_1, a_4, \dots, 2n)(a_2, a_3, 1, 2, \dots, 2n). \end{aligned}$$

Case II:

Taking $M = 2n - 2$, $N = 2n + m - 1$ (m is odd) and the characters b_j, c_k as follows

$$\begin{aligned} b_0 &= 1, b_1 = 2, b_2 = 3, \dots, b_M = b_{2n-2} = 2n - 1, \\ c_0 &= a_1, c_1 = a_2, c_2 = a_3, \dots, c_{2m-1} = a_m, \\ c_m &= 1, c_{m+1} = 2, c_{m+2} = 3, \dots, c_N = c_{2n+m-1} = 2n, \end{aligned}$$

we get the following Pfaffian identity

$$\begin{aligned} &(1, 2, \dots, 2n)(a_1, a_2, \dots, a_m, 1, 2, \dots, 2n - 1) \\ &= \sum_{j=1}^m (-1)^j (a_j, 1, 2, \dots, 2n - 1)(a_1, a_2, a_3, \dots, \hat{a}_j, \dots, a_m, 1, 2, \dots, 2n). \end{aligned} \quad (3.24)$$

For example, in the case $m = 3$, the identity (3.24) can be written as

$$\begin{aligned} &(1, 2, \dots, 2n)(a_1, a_2, a_3, 1, 2, \dots, 2n - 1) \\ &= (a_1, 1, 2, \dots, 2n)(a_2, a_3, 1, 2, \dots, 2n) \\ &\quad - (a_2, \dots, 2n)(a_1, a_3, 1, 2, \dots, 2n) \\ &\quad + (a_3, \dots, 2n)(a_1, a_2, 1, 2, \dots, 2n). \end{aligned}$$

The above identities will play a crucial role in our studying the vcKP and JM equations, and we will prove that the bilinear forms of those equations are nothing but Pfaffian identities.

In order to find the derivative formulae for Pfaffians, we need the following expansion formulae for Pfaffians [48]:

Proposition 3.2 [48] *If $(a_1, a_2) = 0$, then the Pfaffian $(a_1, a_2, 1, 2, \dots, 2n)$ can be extended in two different ways:*

$$(i) \quad (a_1, a_2, 1, 2, \dots, 2n) = \sum_{1 \leq j < k \leq 2n} (-1)^{j+k-1} (a_1, a_2, j, k) \times (1, 2, \dots, \hat{j}, \dots, \hat{k}, \dots, 2n), \quad (3.25)$$

$$(ii) \quad (a_1, a_2, 1, 2, \dots, 2n) = \sum_{j=2}^{2n} (-1)^j [(a_1, a_2, 1, j)(2, 3, \dots, \hat{j}, \dots, 2n) + (1, j)(a_1, a_2, 2, 3, \dots, \hat{j}, \dots, 2n)]. \quad (3.26)$$

Proof :[48] To prove identity (3.25), expanding the Pfaffian $(a_1, a_2, 1, 2, \dots, 2n)$ with respect to a_1 and then with respect to a_2 , we get

$$\begin{aligned} (a_1, a_2, 1, 2, \dots, 2n) &= \sum_{j=1}^{2n} \sum_{k=1}^{2n} (-1)^{j+k} (a_1, j)(a_2, k)(1, 2, \dots, \hat{j}, \dots, \hat{k}, \dots, 2n) \\ &= \sum_{1 \leq j < k \leq 2n} (-1)^{j+k} [(a_1, j)(a_2, k) - (a_1, k)(a_2, j)] \\ &\quad \times (1, 2, \dots, \hat{j}, \dots, \hat{k}, \dots, 2n). \end{aligned}$$

By the condition $(a_1, a_2) = 0$, we see that the right hand side of the last equality is equivalent to the right hand side of (3.25).

To prove the second expansion formula (3.26), we expand the Pfaffian $(a_1, a_2, 1, 2, \dots, 2n)$ with respect to 1, and then we get the following

$$\begin{aligned} (a_1, a_2, 1, 2, \dots, 2n) &= (1, a_1)(a_2, 2, \dots, 2n) - (1, a_2)(a_1, 2, \dots, 2n) \\ &\quad + \sum_{j=2}^{2n} (-1)^j (1, j)(a_1, a_2, 2, 3, \dots, \hat{j}, \dots, 2n). \end{aligned}$$

Next, we expand the Pfaffians $(a_1, 2, 3, \dots, 2n)$ and $(a_2, 2, 3, \dots, 2n)$ to get

$$\begin{aligned} (a_1, a_2, 1, 2, \dots, 2n) &= (1, a_1) \sum_{j=2}^{2n} (-1)^j (a_2, j)(2, 3, \dots, \hat{j}, \dots, 2n) \\ &\quad - (1, a_2) \sum_{j=2}^{2n} (-1)^j (a_1, j)(2, 3, \dots, \hat{j}, \dots, 2n) \\ &\quad + \sum_{j=2}^{2n} (-1)^j (1, j)(a_1, a_2, 2, 3, \dots, \hat{j}, \dots, 2n). \end{aligned}$$

Using the condition $(a_1, a_2) = 0$, the right hand side of the above equality will be

$$\sum_{j=2}^{2n} (-1)^j [(a_1, a_2, 1, j)(2, 3, \dots, \hat{j}, \dots, 2n) + (1, j)(a_1, a_2, 2, 3, \dots, \hat{j}, \dots, 2n)], \quad (3.27)$$

which is the right hand side of (3.25). \square

Replacing the Pfaffian $(1, 2, \dots, 2n)$ by $(b_1, b_2, \dots, 2n)$, the first expansion formula (3.25) can be generalized to the following

$$\begin{aligned} (a_1, a_2, b_1, b_2, 1, 2, \dots, 2n) &= \sum_{j=1}^{2n} \sum_{k=j+1}^{2n} (-1)^{j+k-1} (a_1, a_2, j, k) \\ &\quad \times (b_1, b_2, 1, 2, \dots, \hat{j}, \dots, \hat{k}, \dots, 2n), \end{aligned} \quad (3.28)$$

where $(a_j, b_k) = 0$ for $j, k = 1, 2$.

Lemma 3.3 [10] *If the x -derivative of a Pfaffian entry (i, j) is expressed as follows*

$$\frac{\partial}{\partial x}(i, j) = (a_0, b_0, i, j), \quad (a_0, b_0) = 0, \quad (3.29)$$

$$\text{then} \quad (3.30)$$

$$\frac{\partial}{\partial x}(1, 2, \dots, 2n) = (a_0, b_0, 1, 2, \dots, 2n). \quad (3.31)$$

Proof :[10] We proceed by induction. For $n = 1$, the statement is true by assumption (3.29).

Assume that (3.31) is true for $n - 1$. Then

$$\begin{aligned} \frac{\partial}{\partial x}(1, 2, \dots, 2n) &= \frac{\partial}{\partial x} \sum_{j=1}^{2n} (-1)^{j-1} (1, j)(2, 3, \dots, \hat{j}, \dots, 2n) \\ &= \sum_{j=1}^{2n} (-1)^{j-1} \left[\left(\frac{\partial}{\partial x}(1, j) \right) (2, 3, \dots, \hat{j}, \dots, 2n) \right. \\ &\quad \left. + (i, j) \frac{\partial}{\partial x}(2, 3, \dots, \hat{j}, \dots, 2n) \right], \end{aligned}$$

Using the condition (3.29) and the induction assumption, the right hand side of the above equality equals to

$$\begin{aligned} & \sum_{j=1}^{2n} (-1)^{j-1} [(a_0, b_0, 1, j)(2, 3, \dots, \hat{j}, \dots, 2n) \\ & + (i, j)(a_0, b_0, 2, 3, \dots, \hat{j}, \dots, 2n)] = (a_0, b_0, 1, 2, \dots, 2n), \end{aligned}$$

where the last equality is by the expansion formula (3.26). \square

In the next lemma, we are going to give a derivative of the Pfaffian $(a_0, b_0, 1, 2, \dots, 2n)$ with respect to another variable y .

Lemma 3.4 [10] *If the y -derivative of a Pfaffian entry (i, j) and the Pfaffian (a_0, b_0, i, j) can be expressed as follows*

$$\frac{\partial}{\partial y}(i, j) \equiv (a_1, b_1, i, j), \quad (3.32a)$$

$$\frac{\partial}{\partial y}(a_0, b_0, i, j) \equiv (a_2, b_0, i, j) + (a_0, b_2, i, j), \quad (3.32b)$$

then

$$\begin{aligned} \frac{\partial}{\partial y}(a_0, b_0, 1, 2, \dots, 2n) &= (a_2, b_0, 1, 2, \dots, 2n) + (a_0, b_2, 1, 2, \dots, 2n) \\ &+ (a_0, b_0, a_1, b_1, 1, 2, \dots, 2n), \end{aligned} \quad (3.33)$$

where $(a_i, a_j) = (a_i, b_j) = (b_i, b_j) = 0$ for $i, j = 0, 1, 2$.

Proof :[10] Using the expansion formula (3.25), we have

$$\begin{aligned} \frac{\partial}{\partial y}(a_0, b_0, 1, 2, \dots, 2n) &= \sum_{1 \leq i < j \leq 2n} (-1)^{i+j-1} \left[\frac{\partial}{\partial y}(a_0, b_0, i, j) \right. \\ &\times (1, 2, \dots, \hat{i}, \dots, \hat{j}, \dots, 2n) + (a_0, b_0, i, j) \\ &\left. \times \frac{\partial}{\partial y}(1, 2, \dots, \hat{i}, \dots, \hat{j}, \dots, 2n) \right]. \end{aligned} \quad (3.34)$$

Using the condition (3.32) and Lemma 3.4, we have

$$\begin{aligned} \frac{\partial}{\partial y}(a_0, b_0, 1, 2, \dots, 2n) &= \sum_{1 \leq i < j \leq 2n} (-1)^{i+j-1} \left\{ [(a_2, b_0, i, j) + (a_0, b_2, i, j)] \right. \\ &\times (1, 2, \dots, \hat{i}, \dots, \hat{j}, \dots, 2n) + (a_0, b_0, i, j) \\ &\left. \times (a_1, b_1, 1, 2, \dots, \hat{i}, \dots, \hat{j}, \dots, 2n) \right\}. \end{aligned} \quad (3.35)$$

Finally, by the expansion formulae (3.25) and (3.28), we can see that the right hand side of the last equality equals to the right hand side of (3.33), which completes the proof. \square

3.2 Determinant Solutions for a (3+1)-Dimensional Generalized KP Equation with Variable Coefficients

Recently, Wronskian and Grammian solutions, nonsingular and singular soliton solutions and a Bäcklund transformation in bilinear form to a (3+1)-dimensional generalized KP equation

$$u_{xxxy} + 3(u_x u_y)_x + u_{tx} + u_{ty} - u_{zz} = 0$$

has been presented in [21],[74] and [66], respectively. This equation can be written in the Hirota bilinear form and reduced to the KP equation if taking $y = x$, but does not belong to a class of generalized KP and Boussinesq equations [76]

$$(u_{x_1 x_1 x_1} - 6u u_{x_1})_{x_1} + \sum_{i,j=1}^M a_{ij} u_{x_i x_j} = 0, \quad a_{ij} = \text{constant}, \quad M \in \mathbb{N}.$$

In this section, we would like to consider the following generalized KP equation with variable coefficients (vcKP):

$$(u_t + \alpha_1(t)u_{xxy} + 3\alpha_2(t)u_x u_y)_x + \alpha_3(t)u_{ty} - \alpha_4(t)u_{zz} + \alpha_5(t)(u_x + \alpha_3(t)u_y) = 0,$$

where $\alpha_i, i = 1, 2, 3, 4, 5$, are nonzero arbitrary analytic functions in t . Under a certain constraint, we will show that this generalized vcKP equation has a class of Wronskian solutions and a class of Grammian solutions, with all generating functions for matrix entries satisfying a linear system of partial differential equations. The Plücker relation and the Jacobi identity for determinants are the tools to establish the corresponding Wronskian and Grammian formulations.

3.2.1 Wronskian Formulation

Let us introduce the following helpful notation

$$\begin{aligned} |\widehat{N-j-1, i_1, \dots, i_j}| &= |\Phi^{(0)}, \Phi^{(1)}, \dots, \Phi^{(N-j-1)}, \Phi^{(i_1)}, \dots, \Phi^{(i_j)}| \\ &= \det(\Phi^{(0)}, \Phi^{(1)}, \dots, \Phi^{(N-j-1)}, \Phi^{(i_1)}, \dots, \Phi^{(i_j)}), \quad 1 \leq j \leq N-1, \end{aligned} \quad (3.36)$$

where i_1, \dots, i_j are non-negative integers, and the vectors of functions $\Phi^{(j)}$ are defined by

$$\Phi^{(j)} = (\phi_1^{(j)}, \phi_2^{(j)}, \dots, \phi_N^{(j)})^T, \phi_i^{(j)} = \frac{\partial^j}{\partial x^j} \phi_i. \quad (3.37)$$

A Wronskian determinant is given by

$$W(\phi_1, \phi_2, \dots, \phi_N) = |\widehat{N-1}|. \quad (3.38)$$

We also use the assumption for convenience that if $i < 0$, the column vector $\Phi^{(i)}$ does not appear in the determinant $\det(\dots, \Phi^{(i)}, \dots)$.

We consider the following (3+1)-dimensional nonlinear equation:

$$(u_t + \alpha_1(t)u_{xxy} + 3\alpha_2(t)u_x u_y)_x + \alpha_3(t)u_{ty} - \alpha_4(t)u_{zz} + \alpha_5(t)(u_x + \alpha_3(t)u_y) = 0, \quad (3.39)$$

where $\alpha_i, i = 1, 2, 3, 4, 5$, are nonzero arbitrary analytic functions in t . When $\alpha_i \equiv 1$ for $i = 1, 2, 3, 4, \alpha_5 \equiv 0$ and $x = y$, the equation (3.39) is reduced to the KP equation, and so we call it a generalized vcKP equation. The KP equation was also generalized by constructing decomposition of (2+1)-dimensional equations into (1+1)-dimensional equations [75].

Through the dependent variable transformation

$$u = 2 \frac{\alpha_1(t)}{\alpha_2(t)} (\ln f)_x, \quad (3.40)$$

the above (3+1)-dimensional generalized vcKP equation is mapped into a Hirota bilinear equation

$$(\alpha_1(t)D_x^3 D_y + D_t D_x + \alpha_3(t)D_t D_y - \alpha_4(t)D_z^2) f \cdot f = 0, \quad (3.41)$$

under the constraint:

$$\alpha_1(t) = C_0 \alpha_2(t) e^{-\int \alpha_5(t) dt}, \quad (3.42)$$

where $C_0 \neq 0$ is an arbitrary constant and D_x, D_y, D_z and D_t are Hirota bilinear differential operators [10, 12].

Equivalently, we have

$$\begin{aligned} & (\alpha_1(t)f_{xxxxy} + f_{tx} + \alpha_3(t)f_{ty} - \alpha_4(t)f_{zz})f - 3\alpha_1(t)f_{xxy}f_x + 3\alpha_1(t)f_{xy}f_{xx} \\ & - \alpha_1(t)f_y f_{xxx} - f_t f_x - \alpha_3(t)f_t f_y + \alpha_4(t)(f_z)^2 = 0. \end{aligned} \quad (3.43)$$

In the next theorem, we would like to present a system of three linear partial differential equations for which the N -th order Wronskian determinant solves the generalized Hirota bilinear vckP equation (3.41).

Theorem 3.5 *Let a set of functions $\phi_i = \phi_i(x, y, z, t)$ satisfy the following linear partial differential equations:*

$$\phi_{i,y} = -\frac{a^2\alpha_4(t)}{3\alpha_1(t)}\phi_{i,x}, \quad \phi_{i,z} = a\phi_{i,xx}, \quad \phi_{i,t} = \beta(t)\phi_{i,xxx}, \quad (3.44)$$

with $1 \leq i \leq N$, and

$$\beta(t) = \frac{4a^2\alpha_1(t)\alpha_4(t)}{3\alpha_1(t) - a^2\alpha_3(t)\alpha_4(t)},$$

where a is an arbitrary nonzero constant, $\frac{\alpha_4}{\alpha_1}$ is an arbitrary constant and $\frac{3}{a^2\alpha_3(t)}$ does not equal to the constant $\frac{\alpha_4}{\alpha_1}$ for all values of t . Then the Wronskian determinant $f_N = |\widehat{N-1}|$ defined by (3.38) solves the (3+1)-dimensional generalized bilinear vckP equation (3.41).

Proof: Using the following equality and (3.44)

$$\sum_{k=1}^N |A|_{lk} = \sum_{i,j=1}^N A_{ij} \frac{\partial^l a_{ij}}{\partial x^l},$$

where $A = (a_{ij})_{N \times N}$, and $|A|_{lk}$ denotes the determinant resulting from $|A|$ with its k th column differentiated l times with respect to x , whereas A_{ij} denotes the co-factor of a_{ij} , we can compute various derivatives of the Wronskian determinant $f_N = |\widehat{N-1}|$ with respect to the variables x, y, z, t :

$$\begin{aligned} f_{N,x} &= |\widehat{N-2}, N|, \\ f_{N,xx} &= |\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|, \\ f_{N,xxx} &= |\widehat{N-4}, N-2, N-1, N| + 2|\widehat{N-3}, N-1, N+1| + |\widehat{N-2}, N+2|, \\ f_{N,y} &= -\frac{a^2\alpha_4(t)}{3\alpha_1(t)} |\widehat{N-2}, N|, \end{aligned}$$

$$\begin{aligned}
f_{N,xy} &= -\frac{a^2\alpha_4(t)}{3\alpha_1(t)}(|\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|), \\
f_{N,xyx} &= -\frac{a^2\alpha_4(t)}{3\alpha_1(t)}(|\widehat{N-4}, N-2, N-1, N| + 2|\widehat{N-3}, N-1, N+1| + |\widehat{N-2}, N+2|), \\
f_{N,xyxy} &= -\frac{a^2\alpha_4(t)}{3\alpha_1(t)}(|\widehat{N-5}, N-3, N-2, N-1, N| + 3|\widehat{N-4}, N-2, N-1, N+1| \\
&\quad + 2|\widehat{N-3}, N, N+1| + 3|\widehat{N-3}, N-1, N+2| + |\widehat{N-2}, N+3|), \\
f_{N,z} &= a(|\widehat{N-2}, N+1| - |\widehat{N-3}, N-1, N|), \\
f_{N,zz} &= a^2(-|\widehat{N-4}, N-2, N-1, N+1| + 2|\widehat{N-3}, N, N+1| + |\widehat{N-2}, N+3| \\
&\quad + |\widehat{N-5}, N-3, N-2, N-1, N| - |\widehat{N-3}, N-1, N+2|), \\
f_{N,t} &= \beta(t)(|\widehat{N-4}, N-2, N-1, N| - |\widehat{N-3}, N-1, N+1| + |\widehat{N-2}, N+2|), \\
f_{N,tx} &= \beta(t)(|\widehat{N-5}, N-3, N-2, N-1, N| - |\widehat{N-3}, N, N+1| + |\widehat{N-2}, N+3|), \\
f_{N,ty} &= -\frac{a^2\alpha_4(t)\beta(t)}{3\alpha_1(t)}(|\widehat{N-5}, N-3, N-2, N-1, N| \\
&\quad - |\widehat{N-3}, N, N+1| + |\widehat{N-2}, N+3|).
\end{aligned}$$

{ In computing the derivative $f_{N,ty}$ we have used the condition that $\frac{\alpha_4}{\alpha_1}$ is an arbitrary constant}.

In the above expressions, the column $\Phi^{(N-5)}$ does not appear if $N < 5$, as we assumed before since $N - 5 < 0$. Therefore, we can now compute that

$$\begin{aligned}
\alpha_1(t)f_{N,xyxy} + f_{N,tx} + \alpha_3(t)f_{N,ty} - \alpha_4(t)f_{N,zz} &= -4a^2\alpha_4(t)|\widehat{N-3}, N, N+1|, \\
-3\alpha_1(t)f_{N,xyx}f_{N,x} - \alpha_1(t)f_{N,y}f_{N,xxx} - f_{N,t}f_{N,x} - \alpha_3(t)f_{N,t}f_{N,y} &= \\
4a^2\alpha_4(t)|\widehat{N-2}, N||\widehat{N-3}, N-1, N+1|, \\
3\alpha_1(t)f_{N,xy}f_{N,xx} + \alpha_4(t)(f_{N,z})^2 &= -4a^2\alpha_4(t)|\widehat{N-3}, N-1, N||\widehat{N-2}, N+1|,
\end{aligned}$$

Furthermore, we obtain that

$$\begin{aligned}
&(\alpha_1(t)D_x^3D_y + D_tD_x + \alpha_3(t)D_tD_y - \alpha_4(t)D_z^2)f_N \cdot f_N \\
&= 2(\alpha_1(t)f_{N,xyxy} + f_{N,tx} + \alpha_3(t)f_{N,ty} - \alpha_4(t)f_{N,zz})f_N - \alpha_1(t)(6f_{N,xyx}f_{N,x} \\
&\quad - 6f_{N,xy}f_{N,xx} + 2f_{N,y}f_{N,xxx}) - 2f_{N,t}f_{N,x} - 2\alpha_3(t)f_{N,t}f_{N,y} + 2\alpha_4(t)(f_{N,z})^2 \\
&= -8a^2\alpha_4(t)(|\widehat{N-1}||\widehat{N-3}, N, N+1| - |\widehat{N-2}, N||\widehat{N-3}, N-1, N+1| \\
&\quad + |\widehat{N-3}, N-1, N||\widehat{N-2}, N+1|) = 0.
\end{aligned}$$

This last equality is nothing but the Plücker relation for determinants:

$$|B, A_1, A_2||B, A_3, A_4| - |B, A_1, A_3||B, A_2, A_4| + |B, A_1, A_4||B, A_2, A_3| = 0,$$

where B denotes an $N \times (N - 2)$ matrix, and A_i , $1 \leq i \leq 4$, are four N -dimensional column vectors. Therefore, we have shown that $f = f_N$ solves the (3+1)-dimensional generalized Hirota bilinear vcKP equation (3.41), under the condition (3.44). \square

The condition (3.44) is a linear system of partial differential equations. It has an exponential-type function solution:

$$\phi_i = \sum_{j=1}^p d_{ij} e^{\eta_{ij}}, \quad \eta_{ij} = k_{ij}x - \frac{a^2 \alpha_4(t)}{3\alpha_1(t)} k_{ij}y + ak_{ij}^2 z + k_{ij}^3 h(t), \quad (3.45)$$

where

$$h(t) = \int \beta(t) dt \quad (3.46)$$

and d_{ij}, k_{ij} are free parameters and p is an arbitrary natural number.

3.2.2 Grammian Formulation

Let us now introduce the following Grammian determinant

$$f_N = \det(a_{ij})_{1 \leq i, j \leq N}, \quad a_{ij} = c_{ij} + \int^x \phi_i \psi_j dx, \quad c_{ij} = \text{constant}, \quad (3.47)$$

with ϕ_i and ψ_j satisfying

$$\phi_{i,y} = -\frac{a^2 \alpha_4(t)}{3\alpha_1(t)} \phi_{i,x}, \quad \phi_{i,z} = a \phi_{i,xx}, \quad \phi_{i,t} = \beta(t) \phi_{i,xxx}, \quad 1 \leq i \leq N, \quad (3.48)$$

$$\psi_{i,y} = -\frac{a^2 \alpha_4(t)}{3\alpha_1(t)} \psi_{i,x}, \quad \psi_{i,z} = -a \psi_{i,xx}, \quad \psi_{i,t} = \beta(t) \psi_{i,xxx}, \quad 1 \leq i \leq N, \quad (3.49)$$

where $\beta, \alpha_1, \alpha_3, \alpha_4$ and a are as in Theorem 3.5.

Theorem 3.6 *Let ϕ_i and ψ_j satisfy (3.48) and (3.49), respectively. Then the Grammian determinant $f_N = \det(a_{ij})_{1 \leq i, j \leq N}$ defined by (3.47) solves the (3+1)-dimensional generalized bilinear vcKP equation (3.41).*

Proof: Let us express the Grammian determinant f_N by means of a Pfaffian as

$$f_N = (1, 2, \dots, N, N^*, \dots, 2^*, 1^*), \quad (3.50)$$

where $(i, j^*) = a_{ij}$ and $(i, j) = (i^*, j^*) = 0$.

To compute derivatives of the entries a_{ij} and the Grammian f_N , we introduce new Pfaffian entries

$$(d_n, j^*) = \frac{\partial^n}{\partial x^n} \psi_j, \quad (d_n^*, i) = \frac{\partial^n}{\partial x^n} \phi_i, \quad (d_m, d_n^*) = (d_n, i) = (d_m^*, j^*) = 0, \quad m, n \geq 0, \quad (3.51)$$

as usual. In terms of these new entries, derivatives of the entries $a_{ij} = (i, j^*)$ are given, upon using (3.48) and (3.49), by

$$\begin{aligned} \frac{\partial}{\partial x} a_{ij} &= \phi_i \psi_j = (d_0, d_0^*, i, j^*), \\ \frac{\partial}{\partial y} a_{ij} &= \int^x (\phi_{i,y} \psi_j + \phi_i \psi_{j,y}) dx \\ &= -\frac{a^2 \alpha_4(t)}{3\alpha_1(t)} \int^x (\phi_{i,x} \psi_j + \phi_i \psi_{j,x}) dx \\ &= -\frac{a^2 \alpha_4(t)}{3\alpha_1(t)} \phi_i \psi_j \\ &= -\frac{a^2 \alpha_4(t)}{3\alpha_1(t)} (d_0, d_0^*, i, j^*), \\ \frac{\partial}{\partial z} a_{ij} &= \int^x (\phi_{i,z} \psi_j + \phi_i \psi_{j,z}) dx \\ &= a \int^x (\phi_{i,xx} \psi_j - \phi_i \psi_{j,xx}) dx \\ &= a(\phi_{i,x} \psi_j - \phi_i \psi_{j,x}) \\ &= a[-(d_1, d_0^*, i, j^*) + (d_0, d_1^*, i, j^*)], \\ \frac{\partial}{\partial t} a_{ij} &= \int^x (\phi_{i,t} \psi_j + \phi_i \psi_{j,t}) dx \\ &= \beta(t) \int^x (\phi_{i,xxx} \psi_j + \phi_i \psi_{j,xxx}) dx \\ &= \beta(t)(\phi_{i,xx} \psi_j - \phi_{i,x} \psi_{j,x} + \phi_i \psi_{j,xx}) \\ &= \beta(t)[(d_2, d_0^*, i, j^*) - (d_1, d_1^*, i, j^*) + (d_0, d_2^*, i, j^*)]. \end{aligned}$$

Then we can develop differential rules for Pfaffians as in [10], and compute various derivatives of the Grammian determinant $f_N = \det(a_{ij})$ with respect to the variables x, y, z, t as follows:

$$\begin{aligned}
f_{N,x} &= (d_0, d_0^*, \bullet), \\
f_{N,xx} &= (d_1, d_0^*, \bullet) + (d_0, d_1^*, \bullet), \\
f_{N,xxx} &= (d_2, d_0^*, \bullet) + 2(d_1, d_1^*, \bullet) + (d_0, d_2^*, \bullet), \\
f_{N,y} &= -\frac{a^2\alpha_4(t)}{3\alpha_1(t)}(d_0, d_0^*, \bullet), \\
f_{N,xy} &= -\frac{a^2\alpha_4(t)}{3\alpha_1(t)}[(d_1, d_0^*, \bullet) + (d_0, d_1^*, \bullet)], \\
f_{N,xyx} &= -\frac{a^2\alpha_4(t)}{3\alpha_1(t)}[(d_2, d_0^*, \bullet) + 2(d_1, d_1^*, \bullet) + (d_0, d_2^*, \bullet)], \\
f_{N,xyxy} &= -\frac{a^2\alpha_4(t)}{3\alpha_1(t)}[(d_3, d_0^*, \bullet) + 3(d_2, d_1^*, \bullet) + 2(d_0, d_0^*, d_1, d_1^*, \bullet) + 3(d_1, d_2^*, \bullet) + (d_0, d_3^*, \bullet)], \\
f_{N,z} &= a[-(d_1, d_0^*, \bullet) + (d_0, d_1^*, \bullet)], \\
f_{N,zz} &= a^2[(d_3, d_0^*, \bullet) - (d_2, d_1^*, \bullet) + 2(d_0, d_0^*, d_1, d_1^*, \bullet) - (d_1, d_2^*, \bullet) + (d_0, d_3^*, \bullet)], \\
f_{N,t} &= \beta(t)[(d_2, d_0^*, \bullet) - (d_1, d_1^*, \bullet) + (d_0, d_2^*, \bullet)], \\
f_{N,tx} &= \beta(t)[(d_3, d_0^*, \bullet) - (d_0, d_0^*, d_1, d_1^*, \bullet) + (d_0, d_3^*, \bullet)], \\
f_{N,ty} &= -\frac{a^2\alpha_4(t)\beta(t)}{3\alpha_1(t)}[(d_3, d_0^*, \bullet) - (d_0, d_0^*, d_1, d_1^*, \bullet) + (d_0, d_3^*, \bullet)],
\end{aligned}$$

where the abbreviated notation \bullet denotes the list of indices $1, 2, \dots, N, N^*, \dots, 2^*, 1^*$ common to each Pfaffian. Under the conditions on $\alpha_1, \alpha_3, \alpha_4$ and a , we can now compute that

$$\begin{aligned}
&\alpha_1(t)f_{N,xyx} + f_{N,tx} + \alpha_3(t)f_{N,ty} - \alpha_4(t)f_{N,zz} = -4a^2\alpha_4(t)(d_0, d_0^*, d_1, d_1^*, \bullet), \\
&\quad - 3\alpha_1(t)f_{N,xy}f_{N,x} - \alpha_1(t)f_{N,y}f_{N,xxx} - f_{N,t}f_{N,x} - \alpha_3(t)f_{N,t}f_{N,y} = \\
&\quad 4a^2\alpha_4(t)(d_0, d_0^*, \bullet)(d_1, d_1^*, \bullet), \\
&\quad 3\alpha_1(t)f_{N,xy}f_{N,xx} + \alpha_4(t)(f_{N,z})^2 = -4a^2\alpha_4(t)(d_1, d_0^*, \bullet)(d_0, d_1^*, \bullet),
\end{aligned}$$

and further obtain that

$$\begin{aligned}
&(\alpha_1(t)D_x^3D_y + D_tD_x + \alpha_3(t)D_tD_y - \alpha_4(t)D_z^2)f_N \cdot f_N \\
&= 2(\alpha_1(t)f_{N,xyx} + f_{N,tx} + \alpha_3(t)f_{N,ty} - \alpha_4(t)f_{N,zz})f_N - 2\alpha_1(t)(3f_{N,xy}f_{N,x} \\
&\quad - 3f_{N,y}f_{N,xx} + f_{N,y}f_{N,xxx}) - 2f_{N,t}f_{N,x} - 2\alpha_3(t)f_{N,t}f_{N,y} + 2\alpha_4(t)(f_{N,z})^2 \\
&= -8a^2\alpha_4(t)[(\bullet)(d_0, d_0^*, d_1, d_1^*, \bullet) - (d_0, d_0^*, \bullet)(d_1, d_1^*, \bullet) + (d_1, d_0^*, \bullet)(d_0, d_1^*, \bullet)] = 0.
\end{aligned}$$

This last equality is nothing but the Jacobi identity for determinants. Therefore, we have shown that $f_N = \det(a_{ij})_{1 \leq i, j \leq N}$ defined by (3.47) solves the (3+1)-dimensional generalized Hirota bilinear vcKP equation (3.41) under the condition of (3.48) and (3.49). \square

The systems (3.48) and (3.49) have solutions

$$\phi_i = \sum_{j=1}^p d_{ij} e^{\eta_{ij}}, \quad \eta_{ij} = k_{ij}x - \frac{a^2 \alpha_4(t)}{3\alpha_1(t)} k_{ij}y + ak_{ij}^2 z + k_{ij}^3 h(t), \quad (3.52)$$

$$\psi_j = \sum_{i=1}^q e_{ji} e^{\zeta_{ji}}, \quad \zeta_{ji} = l_{ji}x - \frac{a^2 \alpha_4(t)}{3\alpha_1(t)} l_{ji}y - al_{ji}^2 z + l_{ji}^3 h(t), \quad (3.53)$$

where

$$h(t) = \int \beta(t) dt, \quad (3.54)$$

d_{ij} , e_{ji} , k_{ij} and l_{ji} are free parameters and p, q are two arbitrary natural numbers.

3.3 Pfaffian Solutions to a Generalized KP System with Variable Coefficients

Since Pfaffians generalize determinants, it is natural to ask if there exists Pfaffian solutions complementing Wronskian and Grammian solutions. The answer is very positive. Pfaffian solutions to the BKP equation were constructed for the first time by Hirota [48]. However, Pfaffians may not solve given nonlinear equations, and so, one needs to generalize the given equations to some coupled equations. The procedure for doing this [65] is now called Pfaffianization [77]-[79], and two kinds of Pfaffian solutions, Wronski-type and Gramm-type Pfaffian solutions, can be often constructed while doing Pfaffianization [80]-[83].

In this section, we would like to apply the Pfaffianization procedure to the (3+1)-dimensional generalized vcKP equation (3.39). Our construction for Wronski-type and Gramm-type Pfaffian solutions are totally based on two Pfaffian identities (3.24) and (3.25).

3.3.1 Pfaffianization and Wronski-type Pfaffian Solutions

Let us consider again the (3+1)-dimensional generalized KP equation with variable coefficients:

$$(u_t + \alpha_1(t)u_{xxy} + 3\alpha_2(t)u_x u_y)_x + \alpha_3(t)u_{ty} - \alpha_4(t)u_{zz} + \alpha_5(t)(u_x + \alpha_3(t)u_y) = 0. \quad (3.55)$$

Through the dependent variable transformation

$$u = 2 \frac{\alpha_1(t)}{\alpha_2(t)} (\ln f)_x, \quad (3.56)$$

the above (3+1)-dimensional generalized vcKP equation is mapped into a Hirota bilinear equation

$$(\alpha_1(t)D_x^3D_y + D_tD_x + \alpha_3(t)D_tD_y - \alpha_4(t)D_z^2)f \cdot f = 0, \quad (3.57)$$

under the constraint:

$$\alpha_1(t) = C_0\alpha_2(t)e^{-\int \alpha_5(t)dt}, \quad (3.58)$$

where $C_0 \neq 0$ is an arbitrary constant and D_x, D_y, D_z and D_t are Hirota bilinear differential operators [10, 12]. The equation (3.57) precisely presents

$$\begin{aligned} &(\alpha_1(t)f_{xxxxy} + f_{tx} + \alpha_3(t)f_{ty} - \alpha_4(t)f_{zz})f - 3\alpha_1(t)f_{xxy}f_x + 3\alpha_1(t)f_{xy}f_{xx} \\ &- \alpha_1(t)f_yf_{xxx} - f_tf_x - \alpha_3(t)f_tf_y + \alpha_4(t)(f_z)^2 = 0. \end{aligned} \quad (3.59)$$

Now consider Wronski-type Pfaffian solution

$$f_N = (1, 2, \dots, 2N),$$

whose Pfaffian entries satisfy the following linear condition:

$$\frac{\partial}{\partial x}(i, j) = (i + 1, j) + (i, j + 1), \quad (3.60a)$$

$$\frac{\partial}{\partial y}(i, j) = -\frac{a^2\alpha_4(t)}{3\alpha_1(t)}[(i + 1, j) + (i, j + 1)], \quad (3.60b)$$

$$\frac{\partial}{\partial z}(i, j) = a[(i + 2, j) + (i, j + 2)], \quad (3.60c)$$

$$\frac{\partial}{\partial t}(i, j) = \beta(t)[(i + 3, j) + (i, j + 3)], \quad (3.60d)$$

where

$$\beta(t) = \frac{4a^2\alpha_1(t)\alpha_4(t)}{3\alpha_1(t) - a^2\alpha_3(t)\alpha_4(t)}, \quad (3.61)$$

and a could be any real number which satisfies

$$3\alpha_1(t) - a^2\alpha_3(t)\alpha_4(t) \neq 0, \text{ for all values of } t. \quad (3.62)$$

We deduce the following differential rules for the Wronski-type Pfaffian under the condition

(3.60):

$$\frac{\partial}{\partial x}(i_1, i_2, \dots, i_{2N}) = \sum_{k=1}^{2N} (i_1, i_2, \dots, i_k + 1, \dots, i_{2N}), \quad (3.63a)$$

$$\frac{\partial}{\partial y}(i_1, i_2, \dots, i_{2N}) = -\frac{a^2 \alpha_4(t)}{3\alpha_1(t)} \sum_{k=1}^{2N} (i_1, i_2, \dots, i_k + 1, \dots, i_{2N}), \quad (3.63b)$$

$$\frac{\partial}{\partial z}(i_1, i_2, \dots, i_{2N}) = a \sum_{k=1}^{2N} (i_1, i_2, \dots, i_k + 2, \dots, i_{2N}), \quad (3.63c)$$

$$\frac{\partial}{\partial t}(i_1, i_2, \dots, i_{2N}) = \beta(t) \sum_{k=1}^{2N} (i_1, i_2, \dots, i_k + 3, \dots, i_{2N}). \quad (3.63d)$$

In the next theorem, we present a Wronski-type Pfaffian solution to a coupled system for the generalized vcKP equation under a certain condition on the variable coefficients.

Theorem 3.7 *Let a and α_1, α_2 satisfy the conditions (3.62) and (3.58), respectively. If the Pfaffian entries (i, j) satisfy the conditions (3.60), and $\frac{\alpha_4}{\alpha_1}$ is an arbitrary constant, then*

$$u = 2 \frac{\alpha_1(t)}{\alpha_2(t)} (\ln f)_x, \quad v = g/f, \quad w = h/f, \quad (3.64)$$

where

$$f = f_N = (1, 2, \dots, 2N),$$

$$g = g_N = (1, 2, \dots, 2N - 2),$$

$$h = h_N = (1, 2, \dots, 2N + 2),$$

solve the following system of nonlinear equations

$$(u_t + \alpha_1(t)u_{xxy} + 3\alpha_2(t)u_x u_y)_x + \alpha_3(t)u_{ty} - \alpha_4(t)u_{zz} + \alpha_5(t)(u_x + \alpha_3(t)u_y) = -8a^2 \frac{\alpha_4(t)\alpha_1(t)}{\alpha_2(t)} (vw)_x, \quad (3.65a)$$

$$\frac{2}{\beta(t)}v_t + 3\frac{\alpha_2(t)}{\alpha_1(t)}u_x v_x + v_{xxx} + \frac{3}{a}(v_{xz} + \frac{\alpha_2(t)}{\alpha_1(t)}vu_z) = 0, \quad (3.65b)$$

$$\frac{2}{\beta(t)}w_t + 3\frac{\alpha_2(t)}{\alpha_1(t)}u_x w_x + w_{xxx} - \frac{3}{a}(w_{xz} + \frac{\alpha_2(t)}{\alpha_1(t)}wu_z) = 0. \quad (3.65c)$$

Proof: Under the dependent variable transformation given in (3.64), the system (3.65) can be mapped into the following system of bilinear equations:

$$(\alpha_1(t)D_x^3D_y + D_tD_x + \alpha_3(t)D_tD_y - \alpha_4(t)D_z^2)f \cdot f = -8a^2\alpha_4(t)gh, \quad (3.66a)$$

$$(D_x^3 + \frac{2}{\beta(t)}D_t + \frac{3}{a}D_xD_z)g \cdot f = 0, \quad (3.66b)$$

$$(D_x^3 + \frac{2}{\beta(t)}D_t - \frac{3}{a}D_xD_z)h \cdot f = 0. \quad (3.66c)$$

Based on the differential rules in (3.63), we can compute various derivatives of the Pfaffian $f_N = (1, 2, \dots, 2N)$ with respect to the variables x, y, z and t :

$$f_{N,x} = (1, 2, \dots, 2N-1, 2N+1),$$

$$f_{N,xx} = (1, 2, \dots, 2N-1, 2N+2) + (1, 2, \dots, 2N-2, 2N, 2N+1),$$

$$f_{N,xxx} = (1, 2, \dots, 2N-1, 2N+3) + 2(1, 2, \dots, 2N-2, 2N, 2N+2) \\ + (1, 2, \dots, 2N-3, 2N-1, 2N, 2N+1),$$

$$f_{N,y} = -\frac{a^2\alpha_4(t)}{3\alpha_1(t)}(1, 2, \dots, 2N-1, 2N+1),$$

$$f_{N,xy} = -\frac{a^2\alpha_4(t)}{3\alpha_1(t)}[(1, 2, \dots, 2N-1, 2N+2) + (1, 2, \dots, 2N-2, 2N, 2N+1)],$$

$$f_{N,xyy} = -\frac{a^2\alpha_4(t)}{3\alpha_1(t)}[(1, 2, \dots, 2N-1, 2N+3) + 2(1, 2, \dots, 2N-2, 2N, 2N+2) \\ + (1, 2, \dots, 2N-3, 2N-1, 2N, 2N+1)],$$

$$f_{N,xyyy} = -\frac{a^2\alpha_4(t)}{3\alpha_1(t)}[(1, 2, \dots, 2N-1, 2N+4) + 2(1, 2, \dots, 2N-2, 2N+1, 2N+2) \\ + 3(1, 2, \dots, 2N-2, 2N, 2N+3) + 3(1, 2, \dots, 2N-3, 2N-1, 2N, 2N+2) \\ + (1, 2, \dots, 2N-4, 2N-2, 2N-1, 2N, 2N+1)],$$

$$f_{N,z} = a[(1, 2, \dots, 2N-1, 2N+2) - (1, 2, \dots, 2N-2, 2N, 2N+1)],$$

$$f_{N,zz} = a^2[(1, 2, \dots, 2N-1, 2N+4) + 2(1, 2, \dots, 2N-2, 2N+1, 2N+2) \\ - (1, 2, \dots, 2N-3, 2N-1, 2N, 2N+2) - (1, 2, \dots, 2N-2, 2N, 2N+3) \\ + (1, 2, \dots, 2N-4, 2N-2, 2N-1, 2N, 2N+1)],$$

$$f_{N,t} = \beta(t)[(1, 2, \dots, 2N-1, 2N+3) - (1, 2, \dots, 2N-2, 2N, 2N+2) \\ + (1, 2, \dots, 2N-3, 2N-1, 2N, 2N+1)],$$

$$\begin{aligned}
f_{N,tx} &= \beta(t)[(1, 2, \dots, 2N-1, 2N+4) - (1, 2, \dots, 2N-2, 2N+1, 2N+2) \\
&\quad + (1, 2, \dots, 2N-4, 2N-2, 2N-1, 2N, 2N+1)], \\
f_{N,ty} &= -\frac{a^2\alpha_4(t)\beta(t)}{3\alpha_1(t)}[(1, 2, \dots, 2N-1, 2N+4) \\
&\quad - (1, 2, \dots, 2N-2, 2N+1, 2N+2) \\
&\quad + (1, 2, \dots, 2N-4, 2N-2, 2N-1, 2N, 2N+1)].
\end{aligned}$$

{ In the last derivative we used the condition that $\frac{\alpha_4}{\alpha_1}$ is an arbitrary constant}.

It is easy to verify the following relations:

$$\alpha_1(t)\left(\frac{-a^2\alpha_4(t)}{3\alpha_1(t)}\right) + \beta(t) + \alpha_3(t)\left(\frac{-a^2\alpha_4(t)\beta(t)}{3\alpha_1(t)}\right) - a^2\alpha_4(t) = 0, \quad (3.67a)$$

$$3\alpha_1(t)\left(\frac{-a^2\alpha_4(t)}{3\alpha_1(t)}\right) + a^2\alpha_4(t) = 0. \quad (3.67b)$$

Therefore, using the relations (3.67), we have

$$\begin{aligned}
&\alpha_1(t)f_{N,xxxy} + f_{N,tx} + \alpha_3(t)f_{N,ty} - \alpha_4(t)f_{N,zz} \\
&\quad = -4a^2\alpha_4(t)(1, 2, \dots, 2N-2, 2N+1, 2N+2), \\
&-3\alpha_1(t)f_{N,xy}f_{N,x} - \alpha_1(t)f_{N,y}f_{N,xx} - f_{N,t}f_{N,x} - \alpha_3(t)f_{N,t}f_{N,y} \\
&\quad = 4a^2\alpha_4(t)(1, 2, \dots, 2N-1, 2N+1)(1, 2, \dots, 2N-2, 2N, 2N+2), \\
&3\alpha_1(t)f_{N,xy}f_{N,xx} + \alpha_4(t)(f_{N,z})^2 \\
&\quad = -4a^2\alpha_4(t)(1, 2, \dots, 2N-1, 2N+2)(1, 2, \dots, 2N-2, 2N, 2N+1),
\end{aligned}$$

and further obtain

$$\begin{aligned}
&(\alpha_1(t)D_x^3D_y + D_tD_x + \alpha_3(t)D_tD_y - \alpha_4(t)D_z^2)f_N \cdot f_N \\
&= 2(\alpha_1(t)f_{N,xxxy} + f_{N,tx} + \alpha_3(t)f_{N,ty} - \alpha_4(t)f_{N,zz})f_N - \alpha_1(t)(6f_{N,xy}f_{N,x} \\
&\quad - 6f_{N,xy}f_{N,xx} + 2f_{N,y}f_{N,xx}) - 2f_{N,t}f_{N,x} - 2\alpha_3(t)f_{N,t}f_{N,y} + 2\alpha_4(t)(f_{N,z})^2 \\
&= -8a^2\alpha_4(t)[(1, 2, \dots, 2N)(1, 2, \dots, 2N-2, 2N+1, 2N+2) \\
&\quad - (1, 2, \dots, 2N-1, 2N+1)(1, 2, \dots, 2N-2, 2N, 2N+2) \\
&\quad + (1, 2, \dots, 2N-1, 2N+2)(1, 2, \dots, 2N-2, 2N, 2N+1)] \\
&= -8a^2\alpha_4(t)(1, 2, \dots, 2N-2)(1, 2, \dots, 2N+2).
\end{aligned}$$

The last equality is gotten by employing a Pfaffian identity of type (3.24).

But the second equation in the bilinear system (3.66) is equivalent to the Pfaffian identity of type (3.25), indeed:

$$\begin{aligned}
& (D_x^3 + \frac{2}{\beta(t)}D_t + \frac{3}{a}D_xD_z)g_N \cdot f_N \\
&= (g_{N,xxx} + \frac{2}{\beta(t)}g_{N,t} + \frac{3}{a}g_{N,xz})f_N - (3g_{N,xx} + \frac{3}{a}g_{N,z})f_{N,x} \\
&\quad + g_{N,x}(3f_{N,xx} - \frac{3}{a}f_{N,z}) - g_N(f_{N,xxx} + \frac{2}{\beta(t)}f_{N,t} - \frac{3}{a}f_{N,xz}) \\
&= 6[(1, 2, \dots, 2N-3, 2N+1)(1, 2, \dots, 2N-3, 2N-2, 2N-1, 2N) \\
&\quad - (1, 2, \dots, 2N-3, 2N)(1, 2, \dots, 2N-3, 2N-2, 2N-1, 2N+1) \\
&\quad + (1, 2, \dots, 2N-3, 2N-1)(1, 2, \dots, 2N-3, 2N-2, 2N, 2N+1) \\
&\quad - (1, 2, \dots, 2N-3, 2N-2)(1, 2, \dots, 2N-3, 2N-1, 2N, 2N+1)] \\
&= 0.
\end{aligned} \tag{3.68}$$

Similarly, the Pfaffian identity of type (3.24) gives:

$$\begin{aligned}
& (1, 2, \dots, 2N-1, 2N+3)(1, 2, \dots, 2N-1, 2N, 2N+1, 2N+2) \\
&\quad - (1, 2, \dots, 2N-1, 2N+2)(1, 2, \dots, 2N-1, 2N, 2N+1, 2N+3) \\
&\quad + (1, 2, \dots, 2N-1, 2N+1)(1, 2, \dots, 2N-1, 2N, 2N+2, 2N+3) \\
&\quad - (1, 2, \dots, 2N-1, 2N)(1, 2, \dots, 2N-1, 2N+1, 2N+2, 2N+3) \\
&= 0,
\end{aligned} \tag{3.69}$$

which is equivalent to

$$(D_x^3 + \frac{2}{\beta(t)}D_t - \frac{3}{a}D_xD_z)h_N \cdot f_N = 0.$$

Therefore we have shown that $f = f_N, g = g_N$ and $h = h_N$ solve the system (3.66) under the conditions (3.62), (3.60) and $\frac{\alpha_4}{\alpha_1}$ is an arbitrary constant, which implies that u, v , and w solve the system of nonlinear differential equations (3.65) and this completes the proof of the theorem. \square

In particular, one can choose the following Pfaffian entries

$$(i, j) = \sum_{k=1}^M (\phi_k^{(i)} \psi_k^{(j)} - \phi_k^{(j)} \psi_k^{(i)}), \tag{3.70}$$

with $M \in \mathbb{N}$ being arbitrary and ϕ_k and ψ_k satisfying

$$\phi_{k,y} = -\frac{a^2\alpha_4(t)}{3\alpha_1(t)}\phi_k^{(1)}, \quad \phi_{k,z} = a\phi_k^{(2)}, \quad \phi_{k,t} = \beta(t)\phi_k^{(3)}, \quad (3.71)$$

$$\psi_{k,y} = -\frac{a^2\alpha_4(t)}{3\alpha_1(t)}\psi_k^{(1)}, \quad \psi_{k,z} = a\psi_k^{(2)}, \quad \psi_{k,t} = \beta(t)\psi_k^{(3)}, \quad (3.72)$$

where $\phi^{(i)}$ and $\psi^{(i)}$ are the i -th derivatives of ϕ and ψ with respect to x , respectively. It is easy to see that all (i, j) satisfy the condition (3.60). Examples of the functions ϕ_k and ψ_k can be the following:

$$\phi_i = \sum_{j=1}^p d_{ij}e^{\eta_{ij}}, \quad \eta_{ij} = k_{ij}x - \frac{a^2\alpha_4(t)}{3\alpha_1(t)}k_{ij}y + ak_{ij}^2z + k_{ij}^3h(t), \quad (3.73)$$

$$\psi_j = \sum_{i=1}^q e_{ji}e^{\zeta_{ji}}, \quad \zeta_{ji} = l_{ji}x - \frac{a^2\alpha_4(t)}{3\alpha_1(t)}l_{ji}y + al_{ji}^2z + l_{ji}^3h(t), \quad (3.74)$$

where

$$h(t) = \int \beta(t)dt, \quad (3.75)$$

and d_{ij}, e_{ji}, k_{ij} and l_{ji} are free parameters, and p, q are two arbitrary natural numbers.

3.3.2 Gramm-type Pfaffian Solutions

In this section, we would like to discuss another class of Pfaffian solutions, Gramm-type Pfaffian solutions, for the Pfaffianized (3+1)-dimensional generalized vcKP system (3.66), which could be introduced as

$$f = f_N = (1, 2, \dots, 2N), \quad (3.76a)$$

$$g = g_N = (c_1, c_0, 1, 2, \dots, 2N), \quad (3.76b)$$

$$h = h_N = (d_0, d_1, 1, 2, \dots, 2N), \quad (3.76c)$$

where the Pfaffian entries are defined by

$$\begin{cases} (i, j) = c_{ij} + \int^x (\phi_i\psi_j - \phi_j\psi_i)dx, & c_{ij} = -c_{ji}, \quad c_{ij} = \text{constants}, \\ (d_n, i) = \frac{\partial^n}{\partial x^n}\phi_i, & (c_n, i) = \frac{\partial^n}{\partial x^n}\psi_i, \\ (d_m, d_n) = (c_m, c_n) = (c_m, d_n) = 0, \end{cases} \quad (3.77)$$

where the lower limit in the above integration is chosen so that the functions ϕ_i, ψ_i and their derivatives are zero at the lower limit.

Theorem 3.8 Let ϕ_i and ψ_i satisfy the following condition

$$\phi_{i,y} = -\frac{a^2\alpha_4(t)}{3\alpha_1(t)}\phi_{i,x}, \quad \phi_{i,z} = a\phi_{i,xx}, \quad \phi_{i,t} = \beta(t)\phi_{i,xxx}, \quad (3.78a)$$

$$\psi_{i,y} = -\frac{a^2\alpha_4(t)}{3\alpha_1(t)}\psi_{i,x}, \quad \psi_{i,z} = -a\psi_{i,xx}, \quad \psi_{i,t} = \beta(t)\psi_{i,xxx}, \quad (3.78b)$$

where the constant a satisfies the condition (3.62) and $\frac{\alpha_4}{\alpha_1}$ is an arbitrary constant. Then f_N, g_N and h_N defined by (3.76) and (3.77) solve the Pfaffianized system (3.66).

Proof: Based on the Pfaffian entries defined by (3.77) and the condition (3.78), we can compute the derivatives of the Pfaffian entries with respect to x, y, z, t :

$$\begin{aligned} \frac{\partial}{\partial x}(i, j) &= \phi_i\psi_j - \phi_j\psi_i = (c_0, d_0, i, j), \\ \frac{\partial}{\partial y}(i, j) &= \int^x (\phi_{i,y}\psi_j + \phi_i\psi_{j,y} - \phi_{j,y}\psi_i - \phi_j\psi_{i,y})dx \\ &= -\frac{a^2\alpha_4(t)}{3\alpha_1(t)} \int^x (\phi_{i,x}\psi_j + \phi_i\psi_{j,x} - \phi_{j,x}\psi_i - \phi_j\psi_{i,x})dx \\ &= -\frac{a^2\alpha_4(t)}{3\alpha_1(t)}(\phi_i\psi_j - \phi_j\psi_i) \\ &= -\frac{a^2\alpha_4(t)}{3\alpha_1(t)}(c_0, d_0, i, j), \\ \frac{\partial}{\partial z}(i, j) &= \int^x (\phi_{i,z}\psi_j + \phi_i\psi_{j,z} - \phi_{j,z}\psi_i - \phi_j\psi_{i,z})dx \\ &= a \int^x (\phi_{i,xx}\psi_j + \phi_i\psi_{j,xx} - \phi_{j,xx}\psi_i - \phi_j\psi_{i,xx})dx \\ &= a(\phi_{i,x}\psi_j - \phi_i\psi_{j,x} - \phi_{j,x}\psi_i + \phi_j\psi_{i,x}) \\ &= a[(c_0, d_1, i, j) - (c_1, d_0, i, j)], \\ \frac{\partial}{\partial t}(i, j) &= \int^x (\phi_{i,t}\psi_j + \phi_i\psi_{j,t} - \phi_{j,t}\psi_i - \phi_j\psi_{i,t})dx \\ &= \beta(t) \int^x (\phi_{i,xxx}\psi_j + \phi_i\psi_{j,xxx} - \phi_{j,xxx}\psi_i - \phi_j\psi_{i,xxx})dx \\ &= \beta(t)(\phi_{i,xx}\psi_j - \phi_{j,xx}\psi_i - \phi_{i,x}\psi_{j,x} + \phi_{j,x}\psi_{i,x} + \phi_i\psi_{j,xx} - \phi_j\psi_{i,xx}) \\ &= \beta(t)[(c_0, d_2, i, j) - (c_1, d_1, i, j) + (c_2, d_0, i, j)]. \end{aligned}$$

Now we can develop differential rules for Pfaffians, and compute various derivatives of the Gramm-type Pfaffians $f_N = (1, 2, \dots, 2N)$ with respect to the variables x, y, z, t as follows:

$$\begin{aligned}
f_{N,x} &= (c_0, d_0, \bullet), \\
f_{N,xx} &= (c_0, d_1, \bullet) + (c_1, d_0, \bullet), \\
f_{N,xxx} &= (c_0, d_2, \bullet) + 2(c_1, d_1, \bullet) + (c_2, d_0, \bullet), \\
f_{N,y} &= -\frac{a^2\alpha_4(t)}{3\alpha_1(t)}(c_0, d_0, \bullet), \\
f_{N,xy} &= -\frac{a^2\alpha_4(t)}{3\alpha_1(t)}[(c_0, d_1, \bullet) + (c_1, d_0, \bullet)], \\
f_{N,xyx} &= -\frac{a^2\alpha_4(t)}{3\alpha_1(t)}[(c_0, d_2, \bullet) + 2(c_1, d_1, \bullet) + (c_2, d_0, \bullet)], \\
f_{N,xyxy} &= -\frac{a^2\alpha_4(t)}{3\alpha_1(t)}[(c_0, d_3, \bullet) + 3(c_1, d_2, \bullet) + 2(c_0, d_0, c_1, d_1, \bullet) + 3(c_2, d_1, \bullet) \\
&\quad + (c_3, d_0, \bullet)], \\
f_{N,z} &= a[(c_0, d_1, \bullet) - (c_1, d_0, \bullet)], \\
f_{N,zz} &= a^2[(c_0, d_3, \bullet) - (c_1, d_2, \bullet) + 2(c_0, d_0, c_1, d_1, \bullet) - (c_2, d_1, \bullet) + (c_3, d_0, \bullet)], \\
f_{N,t} &= \beta(t)[(c_0, d_2, \bullet) - (c_1, d_1, \bullet) + (c_2, d_0, \bullet)], \\
f_{N,tx} &= \beta(t)[(c_0, d_3, \bullet) - (c_0, d_0, c_1, d_1, \bullet) + (c_3, d_0, \bullet)], \\
f_{N,ty} &= -\frac{a^2\alpha_4(t)}{3\alpha_1(t)}\beta(t)[(c_0, d_3, \bullet) - (c_0, d_0, c_1, d_1, \bullet) + (c_3, d_0, \bullet)],
\end{aligned}$$

where the abbreviated notation \bullet denotes the list of indices $1, 2, \dots, 2N$ common to each Pfaffian.

Using the relations (3.67), we can compute

$$\begin{aligned}
&\alpha_1(t)f_{N,xyxy} + f_{N,tx} + \alpha_3(t)f_{N,ty} - \alpha_4(t)f_{N,zz} = -4a^2\alpha_4(t)(c_0, d_0, c_1, d_1, \bullet), \\
&-3\alpha_1(t)f_{N,xyx}f_{N,x} - \alpha_1(t)f_{N,y}f_{N,xxx} - f_{N,t}f_{N,x} - \alpha_3(t)f_{N,t}f_{N,y} \\
&= 4a^2\alpha_4(t)(c_0, d_0, \bullet)(c_1, d_1, \bullet), \\
&3\alpha_1(t)f_{N,xy}f_{N,xx} + \alpha_4(t)(f_{N,z})^2 = -4a^2\alpha_4(t)(c_1, d_0, \bullet)(c_0, d_1, \bullet),
\end{aligned}$$

and further obtain

$$\begin{aligned}
& (\alpha_1(t)D_x^3D_y + D_tD_x + \alpha_3(t)D_tD_y - \alpha_4(t)D_z^2)f_N \cdot f_N \\
&= 2(\alpha_1(t)f_{N,xxx} + f_{N,t} + \alpha_3(t)f_{N,ty} - \alpha_4(t)f_{N,zz})f_N - \alpha_1(t)(6f_{N,xy}f_{N,x} \\
&\quad - 6f_{N,xy}f_{N,xx} + 2f_{N,y}f_{N,xxx}) - 2f_{N,t}f_{N,x} - 2\alpha_3(t)f_{N,t}f_{N,y} + 2\alpha_4(t)(f_{N,z})^2 \\
&= -8a^2\alpha_4(t)[(\bullet)(c_0, d_0, c_1, d_1, \bullet) - (c_0, d_0, \bullet)(c_1, d_1, \bullet) + (c_1, d_0, \bullet)(c_0, d_1, \bullet)] \\
&= 8a^2\alpha_4(t)(c_0, c_1, \bullet)(d_0, d_1, \bullet).
\end{aligned}$$

The last equality is gotten by employing a Pfaffian identity of type (3.24).

Similarly, one can show that

$$\begin{aligned}
& (D_x^3 + \frac{2}{\beta(t)}D_t + \frac{3}{a}D_xD_z)g_N \cdot f_N \\
&= (g_{N,xxx} + \frac{2}{\beta(t)}g_{N,t} + \frac{3}{a}g_{N,xz})f_N - (3g_{N,xx} + \frac{3}{a}g_{N,z})f_{N,x} \\
&\quad + g_{N,x}(3f_{N,xx} - \frac{3}{a}f_{N,z}) - g_N(f_{N,xxx} + \frac{2}{\beta(t)}f_{N,t} - \frac{3}{a}f_{N,xz}) \\
&= 6[(\bullet)(c_2, c_1, c_0, d_0, \bullet) - (c_2, c_1, \bullet)(c_0, d_0, \bullet) \\
&\quad + (c_2, c_0, \bullet)(c_1, d_0, \bullet) - (c_1, c_0, \bullet)(c_2, d_0, \bullet)] \\
&= 0.
\end{aligned}$$

The last equality is nothing but the Pfaffian identity of type (3.24). By interchanging c and d in the above equation, we can verify that f_N, h_N solve the third equation in the system (3.66).

Therefore, we have shown that f_N, g_N and h_N defined by (3.76) solve the Pfaffianized (3+1)-dimensional bilinear generalized vcKP system (3.66) under the conditions in the theorem. \square

Since the system (3.78) is linear, examples of generating functions for the Pfaffian entries can be easily computed as follows:

$$\phi_i = \sum_{j=1}^p d_{ij}e^{\eta_{ij}}, \quad \eta_{ij} = k_{ij}x - \frac{a^2\alpha_4(t)}{3\alpha_1(t)}k_{ij}y + ak_{ij}^2z + k_{ij}^3h(t), \quad (3.79)$$

$$\psi_j = \sum_{i=1}^q e_{ji}e^{\zeta_{ji}}, \quad \zeta_{ji} = l_{ji}x - \frac{a^2\alpha_4(t)}{3\alpha_1(t)}l_{ji}y + al_{ji}^2z + l_{ji}^3h(t), \quad (3.80)$$

where

$$h(t) = \int \beta(t)dt \quad (3.81)$$

and d_{ij}, e_{ji}, k_{ij} and l_{ji} are free parameters and p, q are two arbitrary natural numbers.

3.4 Bilinear Bäcklund Transformation of a (3+1)-Dimensional Generalized KP Equation

Bäcklund transformations are another powerful approach to solutions of nonlinear equations, and they can be written in the Hirota bilinear form when an equation under consideration has a bilinear form [12, 67]. For example, the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0, \quad (3.82)$$

which can be written as

$$D_x(D_t + D_x^3)f \cdot f = 0, \quad (3.83)$$

under $u = 2(\ln f)_{xx}$, D_r being a Hirota bilinear operator [10], has the bilinear Bäcklund transformation [12]:

$$\begin{cases} (D_x^2 - \lambda)f' \cdot f = 0, \\ (D_t + 3\lambda D_x + D_x^3)f' \cdot f = 0. \end{cases} \quad (3.84)$$

This means that f solves the bilinear KdV equation (3.83) if and only if f' solves the bilinear KdV equation (3.83). The (2+1)-dimensional generalized KdV equation, i.e., the KP equation

$$(-4u_t + u_{xxx} + 6uu_x)_x + 3u_{yy} = 0, \quad (3.85)$$

which can be written as

$$(-4D_x D_t + 3D_y^2 + D_x^4)f \cdot f = 0, \quad (3.86)$$

under $u = 2(\ln f)_{xx}$, has the bilinear Bäcklund transformation [10, 84]:

$$\begin{cases} (D_y - D_x^2)f' \cdot f = 0, \\ (3D_y D_x - 4D_t + D_x^3)f' \cdot f = 0. \end{cases} \quad (3.87)$$

Such bilinear Bäcklund transformations also connect with Lax pairs and generate the modified soliton equations [84, 85].

In this section, we would like to study the (3+1)-dimensional generalized KP equation [62]:

$$u_{xxxy} + 3(u_x u_y)_x + u_{tx} + u_{ty} - u_{zz} = 0,$$

which can be written in the Hirota bilinear form

$$(D_x^3 D_y + D_t D_x + D_t D_y - D_z^2)f \cdot f = 0,$$

under $u = 2(\ln f)_x$. This equation was discussed for the first time in a study on the linear superposition principle for exponential waves [62].

We will generate a bilinear Bäcklund transformation for the above (3+1)-dimensional generalized KP equation, which consists of six equations and contains nine arbitrary parameters. The exchange formula for Hirota bilinear operators are the basis for manipulating the necessary interchanges in deriving the bilinear Bäcklund transformation. Exponential and rational traveling wave solutions with arbitrary wave numbers are computed by applying the proposed bilinear Bäcklund transformation.

3.4.1 Bilinear Bäcklund Transformation

We consider the following (3+1)-dimensional generalized KP equation:

$$u_{xxxy} + 3(u_x u_y)_x + u_{tx} + u_{ty} - u_{zz} = 0. \quad (3.88)$$

Under the dependent variable transformation

$$u = 2(\ln f)_x, \quad (3.89)$$

the above (3+1)-dimensional nonlinear equation is put into a Hirota bilinear equation

$$(D_x^3 D_y + D_t D_x + D_t D_y - D_z^2) f \cdot f = 0, \quad (3.90)$$

where D_x, D_y, D_z and D_t are Hirota bilinear differential operators [10, 12]. This exactly gives

$$(f_{xxxy} + f_{tx} + f_{ty} - f_{zz})f - 3f_{xy}f_x + 3f_{xy}f_{xx} - f_y f_{xxx} - f_t f_x - f_t f_y + (f_z)^2 = 0.$$

We would like to generate a bilinear Bäcklund transformation for the (3+1)-dimensional generalized bilinear KP equation (3.90).

Let us assume that we have another solution f' to the generalized bilinear KP equation (3.90):

$$(D_x^3 D_y + D_t D_x + D_t D_y - D_z^2) f' \cdot f' = 0, \quad (3.91)$$

and introduce a key function

$$P = [(D_x^3 D_y + D_t D_x + D_t D_y - D_z^2) f' \cdot f'] f^2 - [(D_x^3 D_y + D_t D_x + D_t D_y - D_z^2) f \cdot f] f'^2. \quad (3.92)$$

If $P = 0$, then f solves the generalized bilinear KP equation (3.90) if and only if f' solves the generalized bilinear KP equation (3.90). Therefore, if we can obtain, from $P = 0$ by interchanging

the dependent variables f and f' , a system of bilinear equations that guarantees $P = 0$:

$$B_i(D_t, D_x, D_y, D_z)f' \cdot f = 0, \quad 1 \leq i \leq M,$$

where the B_i 's are polynomials in the indicated variables and M is a natural number depending on the complexity of the equation, then this system gives us a bilinear Bäcklund transformation for the generalize bilinear KP equation (3.90).

Let us now start to explore what those bilinear equations could be. First we want to list three exchange identities for Hirota bilinear operators:

$$(D_t D_x a \cdot a)b^2 - (D_t D_x b \cdot b)a^2 = 2D_x(D_t a \cdot b) \cdot ba, \quad (3.93)$$

$$(D_t D_y a \cdot a)b^2 - (D_t D_y b \cdot b)a^2 = 2D_y(D_t a \cdot b) \cdot ba, \quad (3.94)$$

$$\begin{aligned} & 2(D_x^3 D_y a \cdot a)b^2 - 2(D_x^3 D_y b \cdot b)a^2 \\ &= D_x[(3D_x^2 D_y a \cdot b) \cdot ba + (3D_x^2 a \cdot b) \cdot (D_y b \cdot a) + (6D_x D_y a \cdot b) \cdot (D_x b \cdot a)] \\ &+ D_y[(D_x^3 a \cdot b) \cdot ba + (3D_x^3 a \cdot b) \cdot (D_x b \cdot a)]. \end{aligned} \quad (3.95)$$

The first and second identities can be found in [10], and the third one can be obtained from the coefficient of ε^1 , while taking the independent variable transformation $D_x \rightarrow D_x + \varepsilon D_y$ for

$$(D_x^4 a \cdot a)b^2 - (D_x^4 b \cdot b)a^2 = 2D_x[(D_x^3 a \cdot b) \cdot ba + (3D_x^2 a \cdot b) \cdot (D_x b \cdot a)],$$

which is the known identity in [10]. All these identities come from the general exchange formula (see [10] for details). Now from the first identity (3.93) or the second identity (3.94), we can easily obtain

$$(D_z^2 a \cdot a)b^2 - (D_z^2 b \cdot b)a^2 = 2D_z(D_z a \cdot b) \cdot ba, \quad (3.96)$$

$$D_r(D_s a \cdot b) \cdot ba = D_s(D_r a \cdot b) \cdot ba, \quad (3.97)$$

by taking $x = t = z$ and noting $D_r D_s g \cdot g = D_s D_r g \cdot g$.

Then, it can be proved that $P = 0$ if we take

$$\left\{ \begin{array}{l} B_1 f' \cdot f \equiv (3D_x^2 D_y + 4D_t + \lambda_1 D_y + 4\lambda_8 D_z + \lambda_2) f' \cdot f = 0, \\ B_2 f' \cdot f \equiv (D_x^3 + 4D_t - \lambda_1 D_x + 4\lambda_9 D_z + \lambda_3) f' \cdot f = 0, \\ B_3 f' \cdot f \equiv (3D_x^2 + \lambda_4 D_y + \lambda_6) f' \cdot f = 0, \\ B_4 f' \cdot f \equiv (3D_x^2 + \lambda_5 D_x - \lambda_6) f' \cdot f = 0, \\ B_5 f' \cdot f \equiv (D_x D_y + \lambda_7 D_x) f' \cdot f = 0, \\ B_6 f' \cdot f \equiv (D_z + \lambda_8 D_x + \lambda_9 D_y) f' \cdot f = 0, \end{array} \right. \quad (3.98)$$

where nine arbitrary parameters have been introduced successfully. This system presents a bilinear Bäcklund transformation for the (3+1)-dimensional generalized KP equation (3.90).

Actually, by using the exchange identities (3.93)-(3.96), we can carry out the following conversion:

$$\begin{aligned} 2P &= [2(D_x^3 D_y f' \cdot f') f^2 - 2(D_x^3 D_y f \cdot f) f'^2] + [2(D_t D_x f' \cdot f') f^2 - 2(D_t D_x f \cdot f) f'^2] \\ &\quad + [2(D_t D_y f' \cdot f') f^2 - 2(D_t D_y f \cdot f) f'^2] - [2(D_z^2 f' \cdot f') f^2 - 2(D_z^2 f \cdot f) f'^2] \\ &= \{ D_x [(3D_x^2 D_y f' \cdot f) \cdot f f' + (3D_x^2 f' \cdot f) \cdot (D_y f \cdot f') + (6D_x D_y f' \cdot f) \cdot (D_x f \cdot f')] \\ &\quad + D_y [(D_x^3 f' \cdot f) \cdot f f' + (3D_x^2 f' \cdot f) \cdot (D_x f \cdot f')] \} \\ &\quad + 4D_x (D_t f' \cdot f) \cdot f f' + 4D_y (D_t f' \cdot f) \cdot f f' - 4D_z (D_z f' \cdot f) \cdot f f' \\ &= D_x (3D_x^2 D_y f' \cdot f + \lambda_1 D_y f' \cdot f + \lambda_2 f' f) \cdot f f' \\ &\quad + D_x (3D_x^2 f' \cdot f + \lambda_4 D_y f' \cdot f + \lambda_6 f' f) \cdot (D_y f \cdot f') \\ &\quad + D_x (6D_x D_y f' \cdot f + 6\lambda_7 D_x f' \cdot f) \cdot (D_x f \cdot f') \\ &\quad + D_y (D_x^3 f' \cdot f - \lambda_1 D_x f' \cdot f + \lambda_3 f' f) \cdot f f' \\ &\quad + D_y (3D_x^2 f' \cdot f + \lambda_5 D_x f' \cdot f - \lambda_6 f' f) \cdot (D_x f \cdot f') \\ &\quad + 4D_x (D_t f' \cdot f) \cdot f f' + 4D_y (D_t f' \cdot f) \cdot f f' \\ &\quad - 4D_z (D_z f' \cdot f + \lambda_8 D_x f' \cdot f + \lambda_9 D_y f' \cdot f) \cdot f f' \\ &\quad + 4D_x (\lambda_8 D_z f' \cdot f) \cdot f f' + 4D_y (\lambda_9 D_z f' \cdot f) \cdot f f' \\ &= D_x (B_1 f' \cdot f) \cdot f f' + D_y (B_2 f' \cdot f) \cdot f f' + D_x (B_3 f' \cdot f) \cdot (D_y f \cdot f') \\ &\quad + D_y (B_4 f' \cdot f) \cdot (D_x f \cdot f') + 6D_x (B_5 f' \cdot f) \cdot (D_x f \cdot f') - 4D_z (B_6 f' \cdot f) \cdot f f'. \end{aligned}$$

In the above deduction, the coefficients of $\lambda_2, \lambda_3, \lambda_4, \lambda_5$ and λ_7 are zero because of $D_r g \cdot g = 0$, and the coefficients of $\lambda_1, \lambda_6, \lambda_8$ and λ_9 are zero because of (3.97). This shows that (3.98) provides a bilinear Bäcklund transformation for the (3+1)-dimensional generalized bilinear KP equation (3.90).

3.4.2 Traveling Wave Solutions

Let us take a simple solution $f = 1$ to the (3+1)-dimensional generalized KP equation (3.90), which is transformed into the original variable u as $u = 2(\ln f)_x = 0$. Noting that

$$D_r^n g \cdot 1 = \frac{\partial^n}{\partial r^n} g, \quad n \geq 1,$$

the bilinear Bäcklund transformation (3.98) associated with $f = 1$ gives rise to a system of linear partial differential equations

$$\left\{ \begin{array}{l} 3f'_{xxy} + 4f'_t + \lambda_1 f'_y + 4\lambda_8 f'_z + \lambda_2 f' = 0, \\ f'_{xxx} + 4f'_t - \lambda_1 f'_x + 4\lambda_9 f'_z + \lambda_3 f' = 0, \\ 3f'_{xx} + \lambda_4 f'_y + \lambda_6 f' = 0, \\ 3f'_{xx} + \lambda_5 f'_x - \lambda_6 f' = 0, \\ f'_{xy} + \lambda_7 f'_x = 0, \\ f'_z + \lambda_8 f'_x + \lambda_9 f'_y = 0. \end{array} \right. \quad (3.99)$$

Let us first consider a class of exponential wave solutions

$$f' = 1 + \varepsilon e^{kx+ly+mz-\omega t}, \quad (3.100)$$

where ε, k, l, m and ω are constants to be determined. Upon selecting

$$\lambda_2 = 0, \quad \lambda_3 = 0, \quad \lambda_6 = 0, \quad (3.101)$$

a direct computation yields

$$m = -(\lambda_8 k + \lambda_9 l), \quad \omega = \frac{k^3 l - (\lambda_8 k + \lambda_9 l)^2}{k + l}, \quad (3.102)$$

and

$$\lambda_1 = \frac{k^3 - 3k^2 l + 4\lambda_8^2 k - 4\lambda_8 \lambda_9 (k - l) - 4\lambda_9^2 l}{k + l}, \quad \lambda_4 = -\frac{3k^2}{l}, \quad \lambda_5 = -3k, \quad \lambda_7 = -l. \quad (3.103)$$

Therefore, we obtain a class of exponential wave solutions to the (3+1)-dimensional generalized bilinear KP equation (3.90):

$$f' = 1 + \varepsilon \exp[kx + ly - (\lambda_8 k + \lambda_9 l)z - \frac{k^3 l - (\lambda_8 k + \lambda_9 l)^2}{k + l} t], \quad (3.104)$$

where $\varepsilon, k, l, \lambda_8$ and λ_9 are arbitrary constants; and $u = 2(\ln f')_x$ solves the (3+1)-dimensional generalized KP equation (3.88).

Let us second consider a class of first-order polynomial solutions

$$f' = kx + ly + mz - \omega t, \quad (3.105)$$

where ε, k, l, m and ω are constants to be determined. Similarly upon selecting

$$\lambda_i = 0, \quad 2 \leq i \leq 7, \quad (3.106)$$

a direct computation tells that the system (3.99) becomes

$$\begin{cases} l\lambda_1 + 4m\lambda_8 - 4\omega = 0, \\ -k\lambda_1 + 4m\lambda_9 - 4\omega = 0, \\ k\lambda_8 + l\lambda_9 + m = 0. \end{cases} \quad (3.107)$$

Evidently, this system needs a necessary but not sufficient (see the last chapter for a counterexample) condition

$$(k + l)\omega + m^2 = 0 \quad (3.108)$$

for the existence of λ_1, λ_8 and λ_9 . Under this condition (3.108), it is direct to check that f' defined by (3.105) solves the (3+1)-dimensional generalized bilinear KP equation (3.90), and so,

$$u = 2(\ln f')_x = \frac{2k}{kx + ly + mz - \omega t} \quad (3.109)$$

presents a class of rational solutions to the (3+1)-dimensional generalized KP equation (3.88).

Chapter 4

Wronskian and Pfaffian Solutions to a (3+1)-Dimensional Generalized Jimbo-Miwa Equation

In 1983, Jimbo and Miwa introduced and studied the following nonlinear partial differential equation [54]:

$$u_{xxxy} + 3(u_x u_y)_x - u_{yt} - 3u_{xz} = 0, \quad (4.1)$$

which describes a (3+1)-dimensional wave in physics. Interestingly, this equation has Wronskian, Grammian and Pfaffian solutions. In this chapter, we will not only formulate these solutions, but also extend this equation to a system of nonlinear partial differential equations by applying what we called the Pfaffianization procedure. Finally, we are going to generalize this equation to an equation with time dependent coefficients.

4.1 Wronskian Formulation

Under the dependent variable transformation

$$u = 2(\ln f)_x, \quad (4.2)$$

the above (3+1)-dimensional generalized JM equation (4.1) is mapped into a Hirota bilinear equation

$$(D_x^3 D_y - D_t D_y - 3D_x D_z) f \cdot f = 0, \quad (4.3)$$

where D_x, D_y, D_z and D_t are Hirota bilinear differential operators [10, 12]. Equivalently, we have

$$(f_{ty} - f_{xxxy} + 3f_{xz})f - f_t f_y + f_{xx} f_y + 3f_{xy} f_x - 3f_{xx} f_{xy} - 3f_x f_z = 0.$$

In the next theorem, we present a sufficient condition under which the Wronskian determinant solves the bilinear generalized JM equation (4.3).

Theorem 4.1 *Let a set of functions $\phi_i = \phi_i(x, y, z, t)$ satisfy the following condition:*

$$\phi_{i,y} = \phi_{i,xx}, \phi_{i,t} = -2\phi_{i,xxx}, \phi_{i,z} = \phi_{i,xxxx}, 1 \leq i \leq N. \quad (4.4)$$

Then the Wronskian determinant $f_N = |\widehat{N-1}|$ defined by (3.38) solves the (3+1)-dimensional bilinear generalized JM equation (4.3).

Proof: Under the condition (4.4), we can compute various derivatives of the Wronskian determinant $f_N = |\widehat{N-1}|$ with respect to the variables x, y, z, t .

It is not hard to obtain that

$$\begin{aligned} f_{N,x} &= |\widehat{N-2}, N|, \\ f_{N,xx} &= |\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|, \\ f_{N,xxx} &= |\widehat{N-4}, N-2, N-1, N| + 2|\widehat{N-3}, N-1, N+1| + |\widehat{N-2}, N+2|, \\ f_{N,y} &= |\widehat{N-2}, N+1| - |\widehat{N-3}, N-1, N|, \\ f_{N,t} &= -2(|\widehat{N-2}, N+2| - |\widehat{N-3}, N-1, N+1| + |\widehat{N-4}, N-2, N-1, N|), \\ f_{N,z} &= -|\widehat{N-5}, N-3, N-2, N-1, N| + |\widehat{N-2}, N+3| \\ &\quad + |\widehat{N-4}, N-2, N-1, N+1| - |\widehat{N-3}, N-1, N+2|, \\ f_{N,xy} &= |\widehat{N-2}, N+2| - |\widehat{N-4}, N-2, N-1, N|, \\ f_{N,xz} &= -|\widehat{N-6}, N-4, N-3, N-2, N-1, N| + |\widehat{N-4}, N-2, N, N+1| \\ &\quad - |\widehat{N-3}, N, N+2| + |\widehat{N-2}, N+4|, \\ f_{N,yt} &= -2(|\widehat{N-5}, N-3, N-2, N-1, N+1| - |\widehat{N-4}, N-2, N, N+1| \\ &\quad + |\widehat{N-2}, N+4| + |\widehat{N-3}, N, N+2| - |\widehat{N-3}, N-1, N+3| \\ &\quad - |\widehat{N-6}, N-4, N-3, N-2, N-1, N|), \\ f_{N,xy} &= |\widehat{N-3}, N-1, N+2| + |\widehat{N-2}, N+3| - |\widehat{N-4}, N-2, N-1, N+1| \\ &\quad - |\widehat{N-5}, N-3, N-2, N-1, N|, \\ f_{N,xxx} &= -2|\widehat{N-5}, N-3, N-2, N-1, N+1| + |\widehat{N-3}, N, N+2| + |\widehat{N-2}, N+4| \\ &\quad - |\widehat{N-4}, N-2, N, N+1| + |\widehat{N-6}, N-4, N-3, N-2, N-1, N|. \end{aligned}$$

So we can compute now

$$\begin{aligned}
(D_x^3 D_y - D_t D_y - 3D_x D_z) f_N \cdot f_N &= 12[|\widehat{N-4}, N-2, N, N+1| |\widehat{N-1}| \\
&- |\widehat{N-4}, N-2, N-1, N+1| |\widehat{N-4}, N-3, N-2, N| \\
&+ |\widehat{N-4}, N-2, N-1, N| |\widehat{N-4}, N-3, N-2, N+1|] \\
&- 12[|\widehat{N-3}, N, N+2| |\widehat{N-3}, N-2, N-1| \\
&- |\widehat{N-3}, N-1, N+2| |\widehat{N-3}, N-2, N| \\
&+ |\widehat{N-3}, N-2, N+2| |\widehat{N-3}, N-1, N|] = 0.
\end{aligned}$$

The last equality is nothing but the Plücker relation for determinants. Therefore, we have shown that $f = f_N$ solves the (3+1)-dimensional bilinear generalized JM equation (4.3), under the condition (4.4). \square

The condition (4.4) is a linear system of partial differential equations. It has an exponential-type function solution:

$$\phi_i = \sum_{j=1}^p d_{ij} e^{\eta_{ij}}, \quad \eta_{ij} = k_{ij}x - 2k_{ij}^2y + k_{ij}^4z + k_{ij}^3t, \quad (4.5)$$

where d_{ij} and k_{ij} are free parameters and p is an arbitrary natural number.

4.2 Grammian Formulation

Let us now introduce the following Grammian determinant

$$f_N = \det(a_{ij})_{1 \leq i, j \leq N}, \quad a_{ij} = c_{ij} + \int^x \phi_i \psi_j dx, \quad c_{ij} = \text{constant}, \quad (4.6)$$

with ϕ_i and ψ_j satisfying

$$\phi_{i,y} = \phi_{i,xx}, \quad \phi_{i,z} = \phi_{i,xxxx}, \quad \phi_{i,t} = -2\phi_{i,xxx}, \quad 1 \leq i \leq N, \quad (4.7)$$

$$\psi_{i,y} = -\psi_{i,xx}, \quad \psi_{i,z} = -\psi_{i,xxxx}, \quad \psi_{i,t} = -2\psi_{j,xxx}, \quad 1 \leq j \leq N. \quad (4.8)$$

Theorem 4.2 *Let ϕ_i and ψ_j satisfy (4.7) and (4.8), respectively. Then the Grammian determinant $f_N = \det(a_{ij})_{1 \leq i, j \leq N}$ defined by (4.6) solves the (3+1)-dimensional bilinear generalized JM equation (4.3).*

Proof: Let the Grammian determinant f_N be written by means of a Pfaffian as

$$f_N = (1, 2, \dots, N, N^*, \dots, 2^*, 1^*), \quad (4.9)$$

where $(i, j^*) = a_{ij}$ and $(i, j) = (i^*, j^*) = 0$.

To compute derivatives of the entries a_{ij} and the Grammian f_N , we introduce new Pfaffian entries

$$(d_n, j^*) = \frac{\partial^n}{\partial x^n} \psi_j, \quad (d_n^*, i) = \frac{\partial^n}{\partial x^n} \phi_i, \quad (d_m, d_n^*) = (d_n, i) = (d_m^*, j^*) = 0, \quad m, n \geq 0, \quad (4.10)$$

as usual. In terms of these new entries, derivatives of the entries $a_{ij} = (i, j^*)$ are given, upon using (4.7) and (4.8), by

$$\begin{aligned} \frac{\partial}{\partial x} a_{ij} &= (d_0, d_0^*, i, j^*), \\ \frac{\partial}{\partial y} a_{ij} &= -(d_1, d_0^*, i, j^*) + (d_0, d_1^*, i, j^*), \\ \frac{\partial}{\partial t} a_{ij} &= (d_2, d_0^*, i, j^*) - (d_1, d_1^*, i, j^*) + (d_0, d_2^*, i, j^*), \\ \frac{\partial}{\partial z} a_{ij} &= (d_0, d_3^*, i, j^*) - (d_3, d_0^*, i, j^*) + (d_2, d_1^*, i, j^*) - (d_1, d_2^*, i, j^*). \end{aligned}$$

By using Lemmas 3.3 and 3.4, we can develop differential rules for Pfaffians, and compute the required derivatives of the Grammian determinant $f_N = \det(a_{ij})$ with respect to the variables x, y, z, t as follows:

$$\begin{aligned} f_{N,x} &= (d_0, d_0^*, \bullet), \\ f_{N,xx} &= (d_1, d_0^*, \bullet) + (d_0, d_1^*, \bullet), \\ f_{N,xxx} &= (d_2, d_0^*, \bullet) + 2(d_1, d_1^*, \bullet) + (d_0, d_2^*, \bullet), \\ f_{N,y} &= -(d_1, d_0^*, \bullet) + (d_0, d_1^*, \bullet), \\ f_{N,t} &= -2[(d_2, d_0^*, \bullet) - (d_1, d_1^*, \bullet) + (d_0, d_2^*, \bullet)], \\ f_{N,z} &= (d_0, d_3^*, \bullet) - (d_3, d_0^*, \bullet) + (d_2, d_1^*, \bullet) - (d_1, d_2^*, \bullet), \end{aligned}$$

$$\begin{aligned}
f_{N,xy} &= (d_0, d_2^*, \bullet) - (d_2, d_0^*, \bullet), \\
f_{N,xy} &= (d_0, d_3^*, \bullet) - (d_3, d_0^*, \bullet) + (d_1, d_2^*, \bullet) - (d_2, d_1^*, \bullet), \\
f_{N,xz} &= (d_0, d_4^*, \bullet) - (d_4, d_0^*, \bullet) + (d_2, d_1^*, d_0, d_0^*, \bullet) - (d_1, d_2^*, d_0, d_0^*, \bullet), \\
f_{N,yt} &= -2[(d_0, d_4^*, \bullet) - (d_4, d_0^*, \bullet) + (d_3, d_1^*, \bullet) - (d_1, d_3^*, \bullet) \\
&\quad + (d_2, d_0^*, d_0, d_1^*, \bullet) - (d_0, d_2^*, d_1, d_0^*, \bullet)], \\
f_{N,xxxy} &= (d_0, d_4^*, \bullet) - (d_4, d_0^*, \bullet) - 2(d_3, d_1^*, \bullet) + 2(d_1, d_3^*, \bullet) \\
&\quad - (d_0, d_2^*, d_1, d_0^*, \bullet) + (d_2, d_0^*, d_0, d_1^*, \bullet),
\end{aligned}$$

where the abbreviated notation \bullet denotes the list of indices $1, 2, \dots, N, N^*, \dots, 2^*, 1^*$ common to each Pfaffian.

By substituting f_N into the left hand side of the bilinear equation (4.3) and utilizing the above derivatives, we get

$$\begin{aligned}
(D_x^3 D_y - D_t D_y - 3D_x D_z) f_N \cdot f_N &= 12[(d_0, d_0^*, d_2, d_1^*, \bullet)(\bullet) \\
&\quad - (d_0, d_0^*, \bullet)(d_2, d_1^*, \bullet) + (d_0, d_1^*, \bullet)(d_2, d_0^*, \bullet)] \\
&\quad - 12[(d_0, d_0^*, d_1, d_2^*, \bullet)(\bullet) \\
&\quad - (d_0, d_0^*, \bullet)(d_1, d_2^*, \bullet) + (d_0, d_2^*, \bullet)(d_1, d_0^*, \bullet)] = 0.
\end{aligned}$$

The last equality comes by the Jacobi identity for determinants. Therefore, we have shown that $f_N = \det(a_{ij})_{1 \leq i, j \leq N}$ defined by (4.6) solves the (3+1)-dimensional bilinear generalized JM equation (4.3), under the condition of (4.7) and (4.8). \square

The systems (4.7) and (4.8) have solutions

$$\phi_i = \sum_{j=1}^p d_{ij} e^{\eta_{ij}}, \quad \eta_{ij} = k_{ij}x + k_{ij}^2 y + k_{ij}^4 z - 2k_{ij}^3 t, \quad (4.11)$$

$$\psi_j = \sum_{i=1}^q e_{ji} e^{\zeta_{ji}}, \quad \zeta_{ji} = l_{ji}x + l_{ji}^2 y + l_{ji}^4 z - 2l_{ji}^3 t, \quad (4.12)$$

where d_{ij} , e_{ji} , k_{ij} and l_{ji} are free parameters and p, q are two arbitrary natural numbers.

4.3 Pfaffian Formulation

Let us introduce the following Pfaffian

$$f_N = (1, 2, \dots, 2N), \quad (4.13)$$

with Pfaffian entries

$$(i, j) = c_{ij} + \int_{-\infty}^x D_x \phi_i \cdot \phi_j dx, \quad i, j = 1, 2, \dots, 2N, \quad (4.14)$$

where $c_{ij} (= -c_{ji}$ for $i \neq j$) are constants, D_x is the Hirota D-operator and all ϕ_i , $1 \leq i \leq 2N$, satisfy the following linear system of differential equations

$$\phi_{i,y} = a \int_{-\infty}^x \phi_i(x) dx, \quad \phi_{i,z} = a \phi_{i,x}, \quad \phi_{i,t} = \phi_{i,xxx}, \quad 1 \leq i \leq 2N, \quad (4.15)$$

where a is nonzero parameter.

Theorem 4.3 *Let $\phi_i(x, y, z, t)$, $1 \leq i \leq 2N$, satisfy (4.15). Then the Pfaffian defined by (4.13) solves the bilinear generalized JM equation (4.3) and the function $u = 2(\ln f_N)_x$ solves the (3+1)-dimensional generalized JM equation (4.1).*

Proof: In order to compute derivatives of the Pfaffian entries (i, j) and the Pfaffian f_N , we introduce the following new Pfaffian entries

$$(d_n, i) = \frac{\partial^n \phi_i}{\partial x^n}, \quad (d_{-n}, i) = \frac{\partial^n \phi_i}{\partial x_{-1}^n}, \quad \text{for } n \geq 0, \quad (4.16)$$

where

$$\frac{\partial^n \phi_i}{\partial x_{-1}^n} = \int_{-\infty}^x \int_{-\infty}^x \cdots \int_{-\infty}^x \phi_i(x) dx dx \cdots dx. \quad (4.17)$$

By using the definition of the Pfaffian entries (4.14) and the linear condition (4.15), we have

$$\begin{aligned} \frac{\partial}{\partial x}(i, j) &= \phi_j \phi_{i,x} - \phi_i \phi_{j,x} = (d_0, d_1, i, j), \\ \frac{\partial}{\partial y}(i, j) &= \frac{\partial}{\partial x_{-1}} \int_{-\infty}^x \phi_j \phi_{i,x} - \phi_i \phi_{j,x} \\ &= a[\phi_i \phi_{j,x_{-1}} - \phi_j \phi_{i,x_{-1}}] = a(d_{-1}, d_0, i, j^*), \\ \frac{\partial}{\partial z}(i, j) &= a(d_0, d_1, i, j), \\ \frac{\partial}{\partial t}(i, j) &= \phi_j \phi_{i,xxx} - \phi_i \phi_{j,xxx} - 2(\phi_{j,x} \phi_{i,xx} - \phi_{i,x} \phi_{j,xx}) \\ &= (d_0, d_3, i, j) - 2(d_1, d_2, i, j). \end{aligned}$$

By using Lemmas 3.3 and 3.4, we can develop differential rules for Pfaffians, and compute the required derivatives of the Pfaffian $f_N = (1, 2, \dots, 2N)$ defined by (4.13) with respect to the variables x, y, z, t as follows:

$$f_{N,x} = (d_0, d_1, \bullet), \quad (4.18)$$

$$f_{N,xx} = (d_0, d_2, \bullet), \quad (4.19)$$

$$f_{N,xxx} = (d_1, d_2, \bullet) + (d_0, d_3, \bullet), \quad (4.20)$$

$$f_{N,y} = a(d_{-1}, d_0, \bullet), \quad (4.21)$$

$$f_{N,t} = (d_0, d_3, \bullet) - 2(d_1, d_2, \bullet), \quad (4.22)$$

$$f_{N,z} = a(d_0, d_1, \bullet), \quad (4.23)$$

$$f_{N,xy} = a(d_{-1}, d_1, \bullet), \quad (4.24)$$

$$f_{N,xz} = a(d_0, d_2, \bullet), \quad (4.25)$$

$$f_{N,xy} = a[(d_{-1}, d_2, \bullet) + (d_0, d_1, \bullet)], \quad (4.26)$$

$$f_{N,yt} = a[(d_{-1}, d_3, \bullet) - (d_0, d_2, \bullet) - 2(d_{-1}, d_0, d_1, d_2, \bullet)], \quad (4.27)$$

$$f_{N,xxxy} = a[(d_{-1}, d_3, \bullet) + 2(d_0, d_2, \bullet) + (d_{-1}, d_0, d_1, d_2, \bullet)], \quad (4.28)$$

where the abbreviated notation \bullet denotes the list of indices $1, 2, \dots, 2N$ common to each Pfaffian.

Using the above derivatives, we can see

$$(f_{N,yt} - f_{N,yxxx} + 3f_{N,xz})f_N = -3a(d_{-1}, d_0, d_1, d_2, \bullet)(\bullet), \quad (4.29)$$

$$f_{N,xxx}f_{N,y} - f_{N,t}f_{N,y} = 3a(d_{-1}, d_0, \bullet)(d_1, d_2, \bullet) \quad (4.30)$$

$$-3f_{N,xx}f_{N,yx} = -3a(d_0, d_2, \bullet)(d_{-1}, d_1, \bullet) \quad (4.31)$$

$$3f_{N,yxx}f_{N,x} - 3f_x f_z = 3a(d_0, d_1, \bullet)(d_{-1}, d_2, \bullet). \quad (4.32)$$

By substituting f_N into the left hand side of the bilinear generalized JM equation (4.3), we get

$$\begin{aligned} (D_x^3 D_y - D_t D_y - 3D_x D_z)f_N \cdot f_N &= -6[(d_{-1}, d_0, d_1, d_2, \bullet)(\bullet) \\ &\quad - (d_{-1}, d_0, \bullet)(d_1, d_2, \bullet) \\ &\quad + (d_0, d_2, \bullet)(d_{-1}, d_1, \bullet)] \\ &\quad + (d_0, d_1, \bullet)(d_{-1}, d_2, \bullet)] = 0. \end{aligned}$$

The last equality is nothing but the Pfaffian identity (3.24). Therefore we have shown that the Pfaffian f_N defined by (4.13) satisfying the condition (4.15) solves the bilinear generalized JM equation (4.3). \square

The system (4.15) has solutions in the form

$$\phi_i = \sum_{j=1}^p d_{ij} e^{\eta_{ij}}, \quad \eta_{ij} = k_{ij}x + ak_{ij}^2y + ak_{ij}^4z - 2k_{ij}^3t + \eta_{ij}^0, \quad (4.33)$$

where d_{ij}, k_{ij} and η_{ij}^0 are free parameters and p is arbitrary natural number. In particular, we have the following specific solutions

$$\phi_i = e^{\eta_i}, \quad \eta_i = k_i x + ak_i^2 y + ak_i^4 z - 2k_i^3 t + \eta_i^0, \quad (4.34)$$

where k_i and η_i^0 are free parameters and a is arbitrary nonzero parameter. Hence we obtain

$$(i, j) = c_{ij} + \frac{k_i - k_j}{k_i + k_j} \phi_i \phi_j. \quad (4.35)$$

Let us consider the two-soliton and three soliton solutions for the bilinear generalized JM equation (4.3). For the two-soliton solution, we may choose $c_{12} = c_{34} = 1$, $c_{13} = c_{14} = c_{23} = c_{24} = 0$. Then we have

$$\begin{aligned} f_2 &= (1\ 2)(3\ 4) - (1\ 3)(2\ 4) + (1\ 4)(2\ 3) \\ &= 1 + \frac{k_1 - k_2}{k_1 + k_2} e^{\eta_1 + \eta_2} + \frac{k_3 - k_4}{k_3 + k_4} e^{\eta_3 + \eta_4} \\ &\quad + \frac{(k_1 - k_2)(k_1 - k_3)(k_1 - k_4)(k_2 - k_3)(k_2 - k_4)(k_3 - k_4)}{(k_1 + k_2)(k_1 + k_3)(k_1 + k_4)(k_2 + k_3)(k_2 + k_4)(k_3 + k_4)} e^{\eta_1 + \eta_2 + \eta_3 + \eta_4}. \end{aligned}$$

Setting

$$\theta_i = \eta_i + \eta_{i+1} + \delta_i, \quad \text{where} \quad e^{\delta_i} = \frac{k_i - k_{i+1}}{k_i + k_{i+1}}, \quad (4.36)$$

we can write f_2 as

$$f_2 = 1 + e^{\theta_1} + e^{\theta_3} + k_{12}^{34} e^{\theta_1 + \theta_3}, \quad (4.37)$$

where

$$k_{ij}^{lm} = \frac{(k_i - k_l)(k_i - k_m)(k_j - k_l)(k_j - k_m)}{(k_i + k_l)(k_i + k_m)(k_j + k_l)(k_j + k_m)}. \quad (4.38)$$

Similarly, we can write the three-soliton solution for the bilinear generalized JM equation (4.3). Let us choose $c_{12} = c_{34} = c_{56} = 1$, otherwise $c_{ij} = 0$, then we can rewrite f_3 as

$$f_2 = 1 + e^{\theta_1} + e^{\theta_3} + e^{\theta_5} + k_{12}^{34} e^{\theta_1 + \theta_3} + k_{12}^{56} e^{\theta_1 + \theta_5} + k_{34}^{56} e^{\theta_3 + \theta_5} + k_{123}^{456} e^{\theta_1 + \theta_3 + \theta_5}, \quad (4.39)$$

where

$$k_{ijp}^{lmn} = k_{ij}^{pl} k_{ij}^{mn} k_{pl}^{mn}. \quad (4.40)$$

Now putting $k_{ij}^{lm} = e^{K_{ij}^{lm}}$, we get the following formula for the N-soliton solution to the bilinear generalized JM equation (4.3):

$$f_N = \sum^I e^{(\sum_{i=1}^N \mu_{2i-1} \theta_{2i-1} + \sum_{i < j < l < m}^{(2N)} K_{ij}^{lm} \mu_i \mu_l)}, \quad (4.41)$$

where \sum^I is the sum taken over all possible combinations of $\mu_1 = 0, 1, \mu_2 = 0, 1, \dots, \mu_{2N} = 0, 1$, and $\sum_{i < j < l < m}^{(2N)}$ is the sum taken over all i, j, l, m ($i < j < l < m$) chosen from $\{1, 2, \dots, 2N\}$.

Furthermore, the (3+1)-dimensional generalized JM equation (4.1) has the N-soliton solution

$$u = (\ln f_N)_x. \quad (4.42)$$

The following three Figures of three dimensional plots and two dimensional contour plots show the Pfaffian solutions defined by (4.42) on the indicated specified regions, with certain values chosen for the parameters. In the contour plots, we can see the interaction regions of the involved soliton solutions.

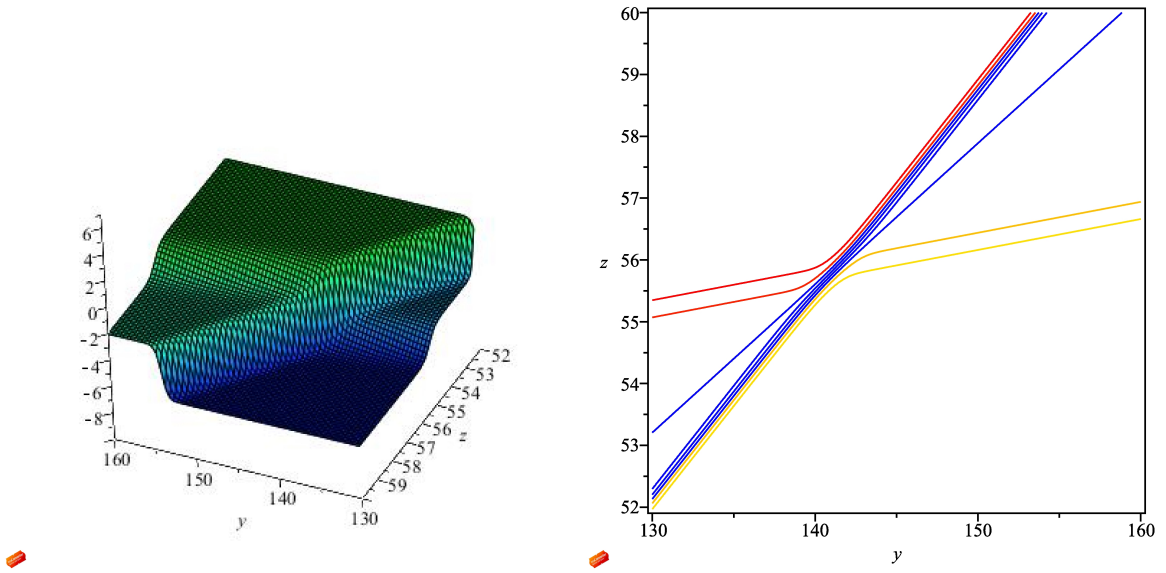


Figure 3.: Three-soliton solution : $k_1 = 2, k_2 = 3, k_3 = 4, k_4 = 7, k_5 = 1, k_6 = -5, a = \frac{1}{5}, x = 4, t = 3$.

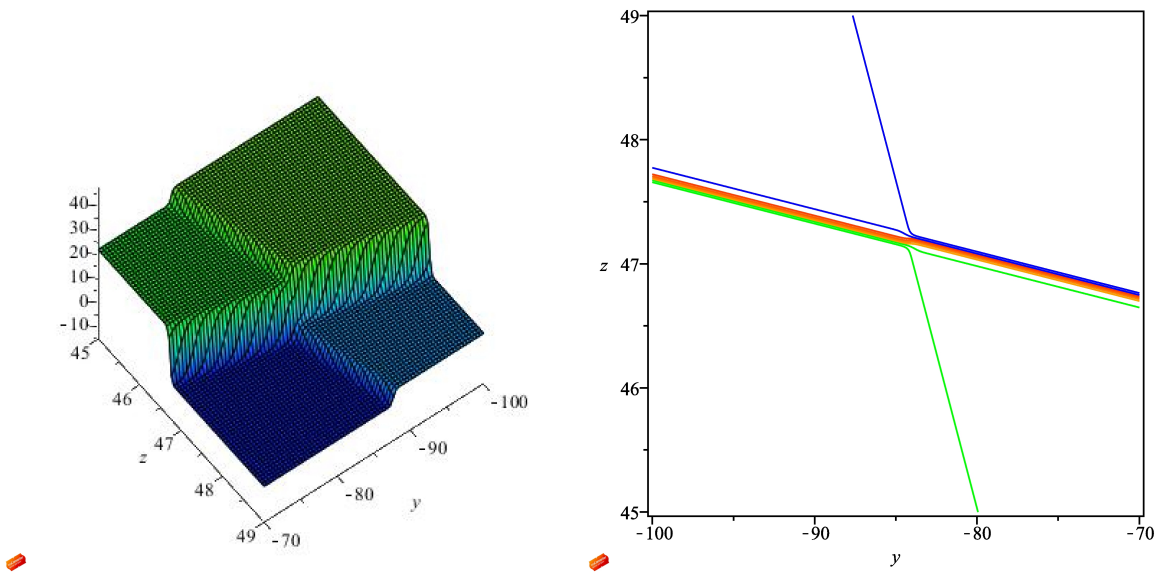


Figure 4.: Three-soliton solution : $k_1 = 1, k_2 = 2, k_3 = 3, k_4 = 4, k_5 = 5, k_6 = 6, a = \frac{-1}{5}, x = 5, t = 7$.

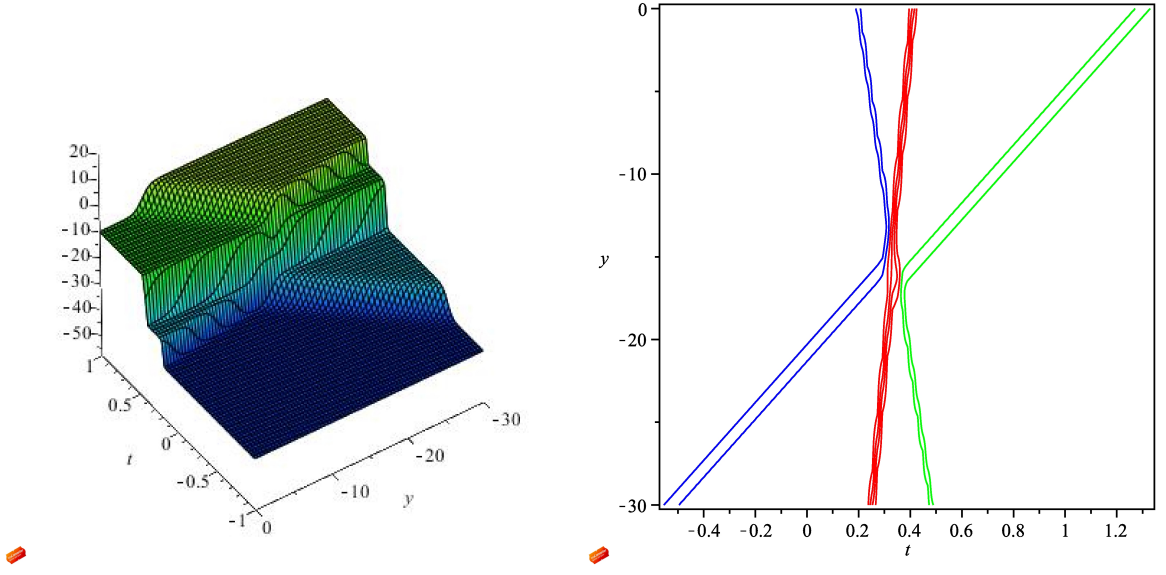


Figure 5.: Three-soliton solution : $k_1 = -5, k_2 = -4, k_3 = -3, k_4 = -2, k_5 = 1, k_6 = -6, a = \frac{-5}{12}, x = -2, z = 3$.

4.4 Pfaffian Solutions to a (3+1)-Dimensional Pfaffianized Jimbo-Miwa System

In this section, we would like to apply Pfaffianization procedure to the bilinear generalized JM equation (4.3) which is similar to what we did for the generalized vcKP equation in the previous chapter. Wronski-type and Grammian type Pfaffian solutions will be formulated and both of these solutions are totally based on the two Pfaffian identities (3.24) and (3.25).

4.4.1 Pfaffianization and Wronski-type Pfaffian Solutions

Consider a Wronski-type Pfaffian solutions

$$f_N = (1, 2, \dots, 2N),$$

whose Pfaffian entries satisfy the following linear condition:

$$\frac{\partial}{\partial x}(i, j) = (i + 1, j) + (i, j + 1), \quad (4.43a)$$

$$\frac{\partial}{\partial y}(i, j) = (i + 2, j) + (i, j + 2), \quad (4.43b)$$

$$\frac{\partial}{\partial t}(i, j) = -2[(i + 3, j) + (i, j + 3)], \quad (4.43c)$$

$$\frac{\partial}{\partial z}(i, j) = (i + 4, j) + (i, j + 4). \quad (4.43d)$$

We deduce the following differential rules for the Wronski-type Pfaffian under the condition (4.43):

$$\frac{\partial}{\partial x}(i_1, i_2, \dots, i_{2N}) = \sum_{k=1}^{2N} (i_1, i_2, \dots, i_k + 1, \dots, i_{2N}), \quad (4.44a)$$

$$\frac{\partial}{\partial y}(i_1, i_2, \dots, i_{2N}) = \sum_{k=1}^{2N} (i_1, i_2, \dots, i_k + 2, \dots, i_{2N}), \quad (4.44b)$$

$$\frac{\partial}{\partial t}(i_1, i_2, \dots, i_{2N}) = -2 \sum_{k=1}^{2N} (i_1, i_2, \dots, i_k + 3, \dots, i_{2N}), \quad (4.44c)$$

$$\frac{\partial}{\partial z}(i_1, i_2, \dots, i_{2N}) = \sum_{k=1}^{2N} (i_1, i_2, \dots, i_k + 4, \dots, i_{2N}). \quad (4.44d)$$

In the next theorem, we present a Wronski-type Pfaffian solution to a coupled system for the (3+1)-dimensional generalized JM equation under the sufficient condition (4.43).

Theorem 4.4 *Let the Pfaffian entries (i, j) satisfy the conditions (4.43). Then*

$$u = 2(\ln f)_x, \quad v = g/f, \quad w = h/f, \quad (4.45)$$

where

$$f = f_N = (1, 2, \dots, 2N), \quad (4.46)$$

$$g = g_N = (1, 2, \dots, 2N - 2), \quad (4.47)$$

$$h = h_N = (1, 2, \dots, 2N + 2), \quad (4.48)$$

solve the following Pfaffianized JM system of nonlinear equations

$$u_{xxy} + 3u_{xx}u_y + 3u_xu_{xy} - u_{yt} - 3u_{xz} + 12(wv_x - vw_x)_x = 0, \quad (4.49a)$$

$$-v_t + 3u_xv_x + v_{xxx} + 3v_{xy} + 3vu_y = 0, \quad (4.49b)$$

$$-w_t + 3u_xw_x + w_{xxx} - 3w_{xy} - 3wu_y = 0. \quad (4.49c)$$

Proof: Under the dependent variable transformation given by (4.45), the Pfaffianized JM system (4.49) can be mapped into the following system of bilinear equations:

$$(D_x^3 D_y - D_y D_t - 3D_x D_z) f \cdot f + 12D_x g \cdot h = 0, \quad (4.50a)$$

$$(D_x^3 - D_t + 3D_x D_y) g \cdot f = 0, \quad (4.50b)$$

$$(D_x^3 - D_t - 3D_x D_y) h \cdot f = 0. \quad (4.50c)$$

Based on the differential rules (4.44), we can compute various derivatives of the Pfaffian $f_N = (1, 2, \dots, 2N)$ with respect to the variables x, y, z and t :

$$f_{N,x} = (1, 2, \dots, 2N-1, 2N+1),$$

$$f_{N,xx} = (1, 2, \dots, 2N-1, 2N+2) + (1, 2, \dots, 2N-2, 2N, 2N+1),$$

$$f_{N,xxx} = (1, 2, \dots, 2N-1, 2N+3) + 2(1, 2, \dots, 2N-2, 2N, 2N+2) \\ + (1, 2, \dots, 2N-3, 2N-1, 2N, 2N+1),$$

$$f_{N,y} = (1, 2, \dots, 2N-1, 2N+2) - (1, 2, \dots, 2N-2, 2N, 2N+1),$$

$$f_{N,t} = -2[(1, 2, \dots, 2N-1, 2N+3) - (1, 2, \dots, 2N-2, 2N, 2N+2) \\ + (1, 2, \dots, 2N-3, 2N-1, 2N, 2N+1)],$$

$$f_z = -(1, 2, \dots, 2N-4, 2N-2, 2N-1, 2N, 2N+1) + (1, 2, \dots, 2N-1, 2N+4) \\ + (1, 2, \dots, 2N-3, 2N-1, 2N, 2N+2) - (1, 2, \dots, 2N-2, 2N, 2N+3),$$

$$f_{xy} = (1, 2, \dots, 2N-1, 2N+3) - (1, 2, \dots, 2N-3, 2N-1, 2N, 2N+1),$$

$$f_{xz} = -(1, 2, \dots, 2N-5, 2N-3, 2N-2, 2N-1, 2N, 2N+1) \\ + (1, 2, \dots, 2N-3, 2N-1, 2N+1, 2N+2) \\ + (1, 2, \dots, 2N-2, 2N+1, 2N+3) + (1, 2, \dots, 2N-1, 2N+5),$$

$$f_{yt} = -2[(1, 2, \dots, 2N-4, 2N-2, 2N-1, 2N, 2N+2) \\ - (1, 2, \dots, 2N-3, 2N-1, 2N+1, 2N+2) + (1, 2, \dots, 2N-1, 2N+5) \\ + (1, 2, \dots, 2N-2, 2N+1, 2N+3) - (1, 2, \dots, 2N-2, 2N, 2N+4) \\ - (1, 2, \dots, 2N-5, 2N-3, 2N+1)],$$

$$\begin{aligned}
f_{xxy} &= (1, 2, \dots, 2N-2, 2N, 2N+3) + (1, 2, \dots, 2N-1, 2N+4) \\
&\quad - (1, 2, \dots, 2N-4, 2N-2, 2N-1, 2N, 2N+1) \\
&\quad + (1, 2, \dots, 2N-3, 2N-1, 2N, 2N+2), \\
f_{xxx} &= (1, 2, \dots, 2N-5, 2N-3, 2N+1) \\
&\quad + (1, 2, \dots, 2N-2, 2N+1, 2N+3) + (1, 2, \dots, 2N-1, 2N+5) \\
&\quad - (1, 2, \dots, 2N-3, 2N-1, 2N+1, 2N+2) \\
&\quad - 2(1, 2, \dots, 2N-4, 2N-2, 2N-1, 2N, 2N+2).
\end{aligned}$$

Therefore, by substituting the above derivatives in the left hand side of the first equation in the bilinear Pfaffianized JM system (4.50), we obtain

$$\begin{aligned}
&(D_x^3 D_y - D_y D_t - 3D_x D_z) f \cdot f + 12D_x g \cdot h = \\
&-12[(1, 2, \dots, 2N-3, 2N-1, 2N+1, 2N+2)(1, 2, \dots, 2N) \\
&- (1, 2, \dots, 2N-3, 2N-1, 2N, 2N+2)((1, 2, \dots, 2N-1, 2N+1) \\
&+ (1, 2, \dots, 2N-3, 2N-1, 2N, 2N+1)(1, 2, \dots, 2N-1, 2N+2) \\
&- (1, 2, \dots, 2N+2)((1, 2, \dots, 2N-3, 2N-1))] \\
&-12[(1, 2, \dots, 2N-2, 2N+1, 2N+3)(1, 2, \dots, 2N) \\
&- (1, 2, \dots, 2N-2, 2N, 2N+3)((1, 2, \dots, 2N-1, 2N+1) \\
&+ (1, 2, \dots, 2N-1, 2N+3)(1, 2, \dots, 2N-2, 2N, 2N+1) \\
&- (1, 2, \dots, 2N-2)((1, 2, \dots, 2N+1, 2N+3))] = 0.
\end{aligned}$$

The last equality is gotten by employing the Pfaffian identity of type (3.24).

But the second equation in the bilinear Pfaffianized JM system (4.50) is equivalent to the Pfaffian identity of type (3.25), indeed:

$$\begin{aligned}
&(D_x^3 - D_t + 3D_x D_y) g_N \cdot f_N \\
&= (g_{N,xxx} - g_{N,t} + 3g_{N,xy}) f_N - (3g_{N,xx} + 3g_{N,y}) f_{N,x} \\
&\quad + g_{N,x} (3f_{N,xx} - 3f_{N,y}) - g_N (f_{N,xxx} - f_{N,t} - 3f_{N,xy})
\end{aligned}$$

$$\begin{aligned}
&= 6[(1, 2, \dots, 2N - 3, 2N + 1)(1, 2, \dots, 2N - 3, 2N - 2, 2N - 1, 2N) \\
&\quad - (1, 2, \dots, 2N - 3, 2N)(1, 2, \dots, 2N - 3, 2N - 2, 2N - 1, 2N + 1) \\
&\quad + (1, 2, \dots, 2N - 3, 2N - 1)(1, 2, \dots, 2N - 3, 2N - 2, 2N, 2N + 1) \\
&\quad - (1, 2, \dots, 2N - 3, 2N - 2)(1, 2, \dots, 2N - 3, 2N - 1, 2N, 2N + 1)] = 0.
\end{aligned}$$

Similarly, the Pfaffian identity of type (3.25) gives

$$\begin{aligned}
&(1, 2, \dots, 2N - 1, 2N + 3)(1, 2, \dots, 2N - 1, 2N, 2N + 1, 2N + 2) \\
&\quad - (1, 2, \dots, 2N - 1, 2N + 2)(1, 2, \dots, 2N - 1, 2N, 2N + 1, 2N + 3) \\
&\quad + (1, 2, \dots, 2N - 1, 2N + 1)(1, 2, \dots, 2N - 1, 2N, 2N + 2, 2N + 3) \\
&\quad - (1, 2, \dots, 2N - 1, 2N)(1, 2, \dots, 2N - 1, 2N + 1, 2N + 2, 2N + 3) = 0,
\end{aligned}$$

which is equivalent to

$$(D_x^3 - D_t - 3D_x D_y)h_N \cdot f_N = 0.$$

Therefore we have shown that $f = f_N, g = g_N$ and $h = h_N$ solve the bilinear Pfaffianized JM system (4.50) under the condition (4.43), which implies that u, v , and w solve the Pfaffianized JM system of nonlinear differential equations (4.49), and this completes the proof of the theorem. \square

In particular, one can choose the following Pfaffian entries

$$(i, j) = \sum_{k=1}^M (\phi_k^{(i)} \psi_k^{(j)} - \phi_k^{(j)} \psi_k^{(i)}), \quad (4.51)$$

with $M \in \mathbb{N}$ being arbitrary and ϕ_k and ψ_k satisfying

$$\phi_{k,y} = \phi_k^{(2)}, \quad \phi_{k,t} = -2\phi_k^{(3)}, \quad \phi_{k,z} = \phi_k^{(4)}, \quad (4.52)$$

$$\psi_{k,y} = \psi_k^{(2)}, \quad \psi_{k,t} = -2\psi_k^{(3)}, \quad \psi_{k,z} = \psi_k^{(4)}, \quad (4.53)$$

where $\phi_k^{(i)}$ and $\psi_k^{(i)}$ are the i -th derivatives of ϕ_k and ψ_k with respect to x respectively. It is easy to see that all (i, j) satisfy the condition (4.43). Examples of such functions ϕ_k and ψ_k can be the following:

$$\phi_i = \sum_{j=1}^p d_{ij} e^{\eta_{ij}}, \quad \eta_{ij} = k_{ij}x + k_{ij}^2 y + ak_{ij}^4 z - 2k_{ij}^3 t, \quad (4.54)$$

$$\psi_j = \sum_{i=1}^q e_{ji} e^{\zeta_{ji}}, \quad \zeta_{ji} = l_{ji}x + l_{ji}^2 y + al_{ji}^4 z - 2l_{ji}^3 t, \quad (4.55)$$

where d_{ij} , e_{ji} , k_{ij} and l_{ji} are free parameters and p, q are two arbitrary natural numbers.

4.4.2 Gramm-type Pfaffian Solutions

In this section, we would like to discuss another class of Pfaffian solutions, Gramm-type Pfaffian solutions for the (3+1)-dimensional Pfaffianized JM system (4.49), which could be introduced as

$$f = f_N = (1, 2, \dots, 2N), \quad (4.56a)$$

$$g = g_N = (c_1, c_0, 1, 2, \dots, 2N), \quad (4.56b)$$

$$h = h_N = (d_0, d_1, 1, 2, \dots, 2N), \quad (4.56c)$$

with the Pfaffian entries being defined by

$$\left\{ \begin{array}{l} (i, j) = c_{ij} + \int^x (\phi_i \psi_j - \phi_j \psi_i) dx, \quad c_{ij} = -c_{ji}, \quad c_{ij} = \text{constants}, \\ (d_n, i) = \frac{\partial^n}{\partial x^n} \phi_i, \quad (c_n, i) = \frac{\partial^n}{\partial x^n} \psi_i, \\ (d_m, d_n) = (c_m, c_n) = (c_m, d_n) = 0, \end{array} \right. \quad (4.57)$$

where the lower limit in the above integration is chosen so that the functions ϕ_i, ψ_i and their derivatives are zero at the lower limit.

Theorem 4.5 *Let ϕ_i, ψ_i satisfy the following condition*

$$\phi_{i,y} = \phi_{i,xx}, \quad \phi_{i,z} = \phi_{i,xxx}, \quad \phi_{i,t} = -2\phi_{i,xxx}, \quad (4.58a)$$

$$\psi_{i,y} = -\psi_{i,xx}, \quad \psi_{i,z} = -\psi_{i,xxx}, \quad \psi_{i,t} = -2\psi_{i,xxx}. \quad (4.58b)$$

Then f_N, g_N and h_N defined by (4.56) and (4.66) solve the (3+1)-dimensional bilinear Pfaffianized JM system (4.50).

Proof: Based on the Pfaffian entries defined by (4.66) and the condition (4.58), we compute derivatives of the Pfaffian entries with respect to x, y, z, t :

$$\begin{aligned}
\frac{\partial}{\partial x}(i, j) &= \phi_i \psi_j - \phi_j \psi_i = (c_0, d_0, i, j), \\
\frac{\partial}{\partial y}(i, j) &= \int^x (\phi_{i,y} \psi_j + \phi_i \psi_{j,y} - \phi_{j,y} \psi_i - \phi_j \psi_{i,y}) dx \\
&= \int^x (\phi_{i,xx} \psi_j + \phi_i \psi_{j,xx} - \phi_{j,xx} \psi_i - \phi_j \psi_{i,xx}) dx \\
&= \phi_{i,x} \psi_j - \phi_i \psi_{j,x} - \phi_{j,x} \psi_i + \phi_j \psi_{i,x} \\
&= (c_0, d_1, i, j) - (c_1, d_0, i, j), \\
\frac{\partial}{\partial t}(i, j) &= \int^x (\phi_{i,t} \psi_j + \phi_i \psi_{j,t} - \phi_{j,t} \psi_i - \phi_j \psi_{i,t}) dx \\
&= -2 \int^x (\phi_{i,xxx} \psi_j + \phi_i \psi_{j,xxx} - \phi_{j,xxx} \psi_i - \phi_j \psi_{i,xxx}) dx \\
&= -2[\phi_{i,xx} \psi_j - \phi_{j,xx} \psi_i - \phi_{i,x} \psi_{j,x} + \phi_{j,x} \psi_{i,x} + \phi_i \psi_{j,xx} - \phi_j \psi_{i,xx}] \\
&= -2[(c_0, d_2, i, j) - (c_1, d_1, i, j) + (c_2, d_0, i, j)], \\
\frac{\partial}{\partial z}(i, j) &= \int^x (\phi_{i,z} \psi_j + \phi_i \psi_{j,z} - \phi_{j,z} \psi_i - \phi_j \psi_{i,z}) dx \\
&= \int^x (\phi_{i,xxxx} \psi_j + \phi_i \psi_{j,xxxx} - \phi_{j,xxxx} \psi_i - \phi_j \psi_{i,xxxx}) dx \\
&= \phi_{i,xxx} \psi_j - \phi_{j,xxx} \psi_i - \phi_i \psi_{j,xxx} + \phi_j \psi_{i,xxx} + \phi_{i,x} \psi_{j,xx} - \phi_{j,x} \psi_{i,xx} \\
&\quad + \phi_{j,xx} \psi_{i,x} - \phi_{i,xx} \psi_{j,x} \\
&= (c_0, d_3, i, j) - (c_3, d_0, i, j) + (c_2, d_1, i, j) - (c_1, d_2, i, j).
\end{aligned}$$

Now we can develop differential rules for the Pfaffians, and compute various derivatives of the Gramm-type Pfaffians $f_N = (1, 2, \dots, 2N)$ with respect to the variables x, y, z, t as follows:

$$\begin{aligned}
f_{N,x} &= (c_0, d_0, \bullet), \\
f_{N,xx} &= (c_0, d_1, \bullet) + (c_1, d_0, \bullet), \\
f_{N,xxx} &= (c_0, d_2, \bullet) + 2(c_1, d_1, \bullet) + (c_2, d_0, \bullet), \\
f_{N,y} &= (c_0, d_1, \bullet) - (c_1, d_0, \bullet), \\
f_{N,t} &= -2[(c_0, d_2, \bullet) - (c_1, d_1, \bullet) + (c_2, d_0, \bullet)],
\end{aligned}$$

$$\begin{aligned}
f_{N,z} &= (c_0, d_3, \bullet) - (c_3, d_0, \bullet) + (c_2, d_1, \bullet) - (c_1, d_0, \bullet), \\
f_{N,ty} &= -2[(c_0, d_4, \bullet) - (c_4, d_0, \bullet) + (c_3, d_1, \bullet) - (c_1, d_3, \bullet) \\
&\quad + (c_2, d_0, c_0, d_1, \bullet) - (c_0, d_1, c_1, d_0, \bullet)], \\
f_{N,xy} &= (c_0, d_2, \bullet) - (c_2, d_0, \bullet), \\
f_{N,xz} &= (c_0, d_4, \bullet) - (c_4, d_0, \bullet) + (c_2, d_1, c_0, d_0, \bullet) - (c_1, d_2, c_0, d_0, \bullet), \\
f_{N,xy} &= (c_0, d_3, \bullet) - (c_3, d_0, \bullet) + (c_1, d_2, \bullet) - (c_2, d_1, \bullet), \\
f_{N,xxx} &= (c_0, d_4, \bullet) - (c_4, d_0, \bullet) - 2(c_3, d_1, \bullet) + 2(c_1, d_3, \bullet) \\
&\quad - (c_0, d_2, c_1, d_0, \bullet) + (c_2, d_0, c_0, d_1, \bullet),
\end{aligned}$$

where the abbreviated notation \bullet denotes the list of indices $1, 2, \dots, 2N$ common to each Pfaffian. By substituting the above derivatives in the left hand side of the first equation in the bilinear Pfaffianized JM system (4.50), we can compute

$$\begin{aligned}
&(D_x^3 D_y + 2D_y D_t - 3D_x D_z) f \cdot f + 12D_x g \cdot h = \\
&12[(c_0, d_0, c_2, d_1, \bullet)(\bullet) - (c_0, d_0, \bullet)(c_2, d_1, \bullet) \\
&+ (c_0, c_2, \bullet)(d_0, d_1, \bullet) - (c_0, d_1, \bullet)(d_0, c_2, \bullet)] \\
&- 12[(c_0, d_0, c_1, d_2, \bullet)(\bullet) - (c_0, d_0, \bullet)(c_1, d_2, \bullet) \\
&+ (c_0, c_1, \bullet)(d_0, d_2, \bullet) - (c_0, d_2, \bullet)(d_0, c_1, \bullet)] = 0.
\end{aligned}$$

The last equality is gotten by employing the Pfaffian identity of type (3.24).

Similarly, one can show that

$$\begin{aligned}
&(D_x^3 - D_t + 3D_x D_y) g_N \cdot f_N \\
&= (g_{N,xxx} - g_{N,t} + 3g_{N,xz}) f_N - (3g_{N,xx} + 3g_{N,z}) f_{N,x} \\
&\quad + g_{N,x}(3f_{N,xx} - 3f_{N,z}) - g_N(f_{N,xxx} - f_{N,t} - 3f_{N,xz}) \\
&= 6[(\bullet)(c_2, c_1, c_0, d_0, \bullet) - (c_2, c_1, \bullet)(c_0, d_0, \bullet) \\
&\quad + (c_2, c_0, \bullet)(c_1, d_0, \bullet) - (c_1, c_0, \bullet)(c_2, d_0, \bullet)] = 0.
\end{aligned}$$

The last equality is nothing but the Pfaffian identity of type (3.24). By interchanging c and d in the above equation, we can verify that f_N and h_N solve the third equation in the bilinear Pfaffianized JM system (4.50).

Therefore, we have shown that f_N, g_N and h_N defined by (4.56) solve the (3+1)-dimensional bilinear Pfaffianized JM system (4.50) under the conditions in the theorem. \square

Since the system (4.58) is linear, examples of the generating functions for the Pfaffian entries can be easily computed as follows:

$$\phi_i = \sum_{j=1}^p d_{ij} e^{\eta_{ij}}, \quad \eta_{ij} = k_{ij}x + k_{ij}^2 y + k_{ij}^4 z - 2k_{ij}^3 t, \quad (4.59)$$

$$\psi_j = \sum_{i=1}^q e_{ji} e^{\zeta_{ji}}, \quad \zeta_{ji} = l_{ji}x - l_{ji}^2 y - l_{ji}^4 z - 2l_{ji}^3 t, \quad (4.60)$$

where d_{ij}, e_{ji}, k_{ij} and l_{ji} are free parameters and p, q are two arbitrary natural numbers.

4.4.3 Jimbo-Miwa Equation with Variable Coefficients

In this section, we are going to consider the following Jimbo-Miwa (JM) equation with variable coefficients

$$\alpha_1(t)u_{xxxxy} + 3\alpha_2(t)(u_x u_y)_x - u_{yt} - 3\alpha_3(t)u_{xz} + 2\alpha_4(t)u_y = 0, \quad (4.61)$$

where $\alpha_i, i = 1, 2, 3, 4$, are nonzero arbitrary smooth functions. Through the dependent variable transformation

$$u = 2 \frac{\alpha_1(t)}{\alpha_2(t)} (\ln f)_x, \quad (4.62)$$

the above (3+1)-dimensional equation is mapped into a Hirota bilinear equation

$$(\alpha_1(t)D_x^3 D_y - D_y D_t - 3\alpha_3(t)D_x D_z) f \cdot f = 0, \quad (4.63)$$

under the constraint:

$$\alpha_1(t) = C_0 \alpha_2(t) e^{-\int \alpha_4(t) dt}, \quad (4.64)$$

where $C_0 \neq 0$ is an arbitrary constant. The equation (4.63) can be extended to the following system

$$(\alpha_1(t)D_x^3 D_y - D_y D_t - 3\alpha_3(t)D_x D_z) f \cdot f + 12a\alpha_3(t)D_x g \cdot h = 0, \quad (4.65a)$$

$$(D_x^3 - \frac{1}{\alpha_1(t)} D_t + \frac{3}{a} D_x D_y) g \cdot f = 0, \quad (4.65b)$$

$$(D_x^3 - \frac{1}{\alpha_1(t)} D_t - \frac{3}{a} D_x D_y) h \cdot f = 0. \quad (4.65c)$$

As in Theorem 4.4, it can be verified that the above Pfaffianized system has the following Wronski-type Pfaffian solutions

$$f_N = (1, 2, \dots, 2N), \quad g_N = (1, 2, \dots, 2N - 2), \quad h_N = (1, 2, \dots, 2N + 2),$$

whose Pfaffian entries satisfy

$$\begin{aligned} \frac{\partial}{\partial x}(i, j) &= (i + 1, j) + (i, j + 1), \\ \frac{\partial}{\partial y}(i, j) &= \frac{a\alpha_3(t)}{\alpha_1(t)}[(i + 2, j) + (i, j + 2)], \\ \frac{\partial}{\partial t}(i, j) &= -2\alpha_1(t)[(i + 3, j) + (i, j + 3)], \\ \frac{\partial}{\partial z}(i, j) &= a[(i + 4, j) + (i, j + 4)], \end{aligned}$$

where a is an arbitrary nonzero constant.

Similarly, as in Theorem 4.5, one can prove that the above bilinear Pfaffianized system has the following Gramm-type Pfaffian solutions

$$f_N = (1, 2, \dots, 2N), \quad g_N = (c_1, c_0, 1, 2, \dots, 2N), \quad h_N = (d_0, d_1, 1, 2, \dots, 2N),$$

where the Pfaffian entries are defined by

$$\left\{ \begin{array}{l} (i, j) = c_{ij} + \int^x (\phi_i \psi_j - \phi_j \psi_i) dx, \quad c_{ij} = -c_{ji}, \quad c_{ij} = \text{constants}, \\ (d_n, i) = \frac{\partial^n}{\partial x^n} \phi_i, \quad (c_n, i) = \frac{\partial^n}{\partial x^n} \psi_i, \\ (d_m, d_n) = (c_m, c_n) = (c_m, d_n) = 0, \end{array} \right. \quad (4.66)$$

where the lower limit in the above integration is chosen so that the functions ϕ_i, ψ_i and their derivatives are zero at the lower limit, with ϕ_i and ψ_i satisfying

$$\phi_{i,y} = \frac{a\alpha_3(t)}{\alpha_1(t)} \phi_{i,xx}, \quad \phi_{i,t} = -2\alpha_1(t) \phi_{i,xxx}, \quad \phi_{i,z} = a\phi_{i,xxxx}, \quad (4.67a)$$

$$\psi_{i,y} = -\frac{a\alpha_3(t)}{\alpha_1(t)} \psi_{i,x}, \quad \psi_{i,t} = -2\alpha_1(t) \psi_{i,xxx}, \quad \psi_{i,z} = -a\psi_{i,xxxx}. \quad (4.67b)$$

Examples of such Wronski-type and Gramm-type Pfaffian entries have been given as in the previous sections.

Chapter 5

Double Wronskian Solutions for a (2+1)-Dimensional Boussinesq System with Variable Coefficients

The Wronskian technique has been applied to many soliton equations such as the KdV, MKdV, NLS, derivative NLS, KP, sine-Gordon and sinh-Gordon equations. Within Wronskian formulations, soliton solutions and rational solutions are usually expressed as some kind of logarithmic derivatives of Wronskian type determinants and the determinants involved are made of eigenfunctions satisfying linear systems of differential equations. This connection between nonlinear problems and linear ones utilizes linear theories in solving soliton equations. In Refs.[86] and [46], the notion of double Wronskians was presented. In 1983, Nimmo proved that the NLS equation has the double Wronskian solution [16].

In view of some variants of Boussinesq systems studied in [87] and [88], we consider, in this chapter, the following (2+1)-dimensional system of nonlinear equations:

$$u_t + \alpha_1(t)u_{xy} + \alpha_2(t)(uw)_x + \alpha_3(t)v_x = 0, \quad (5.1a)$$

$$v_t + \beta_1(t)(wv_x + 2vu_y + uv_y) + \beta_2(t)(u_xw_y - (u_y)^2) + \beta_3(t)v_{xy} + \beta_4(t)u_{xyy} = 0, \quad (5.1b)$$

where $w_x = u_y$ and construct double Wronskian solutions under a certain constraint on the variable coefficients.

When $y = x$, the system (5.1) is reduced to the following variable coefficients variant Boussinesq model in the long gravity water waves:

$$u_t + \alpha_1(t)u_{xx} + \alpha_2(t)(u^2)_x + \alpha_3(t)v_x = 0, \quad (5.2a)$$

$$v_t + 2\beta_1(t)(uv)_x + \beta_3(t)v_{xx} + \beta_4(t)u_{xxx} = 0, \quad (5.2b)$$

In [91], the authors applied the symmetry method based on the Fréchet derivative of the differential operators to deduce Lie symmetries of the reduced system (5.2). In their investigation, the authors

of [88] used the double Wronskian technique in order to explore multi solitonic solutions for the reduced system (5.2).

Model (5.2) has been derived for describing the nonlinear and dispersive long gravity waves traveling in two horizontal directions in shallow water with varying depth [89]. Yao and Li [90] used a direct algebraic method to construct some traveling wave solutions for the reduced system (5.2) with $\alpha_3 = 2\alpha_2 = 2\beta_1 \equiv 1$ and $\alpha_1, \beta_3, \beta_4$ are arbitrary constants. In [92], Zhang used the homogeneous balance method to deduce the multi solitary wave solutions when $2\alpha_2 = \alpha_3 = 2\beta_1 = \beta_4 \equiv 1$, and $\alpha_1 = \beta_3 \equiv 0$. The Kupershmidt equations [92, 93] and Levi equations and Whitham-Broer-Kaup shallow water model [94]-[96] are also special cases of model (5.2). Multiple soliton-like solutions for the following (2+1)-dimensional dispersive long wave equations were constructed in [97]

$$u_{ty} + v_{xx} + u_x u_y + u u_{xy} = 0, \quad (5.3a)$$

$$v_t + (uv + u + u_{xy})_x = 0. \quad (5.3b)$$

In our investigation, we are going to use the double Wronskian technique to explore an exact N-soliton solution for the (2+1)-dimensional variable coefficients system (5.1) after transforming the general system into a bilinear form using the Hirota D-operators.

5.1 Transformations and Bilinear Form

Under the following dependent variable transformations:

$$u = -\frac{1}{2}(\ln p)_x, \quad (5.4a)$$

$$v = \frac{1}{2}b(\ln p)_{xy} - pq, \quad (5.4b)$$

the general system (5.1) can be transformed into the system

$$p_t + a(t)\left(\frac{1}{2}p_{xy} - qp^2\right) = 0, \quad (5.5a)$$

$$q_t - a(t)\left(\frac{1}{2}q_{xy} - pq^2\right) = 0. \quad (5.5b)$$

It is not hard to verify that if p and q are solutions for the system (5.5), then u and v defined by (5.4) are solutions for the general system (5.1) under the following constraint:

$$\begin{aligned}\beta_1(t) &= \alpha_2(t) = 2\alpha_3(t) = -a(t), \beta_2(t) = ba(t), \\ \alpha_1(t) &= -\beta_3(t) = \frac{a(t)}{2}(1-b), \beta_4(t) = b\frac{a(t)}{2}(b-2),\end{aligned}\quad (5.6)$$

where $a(t)$ is an arbitrary smooth function and b is a nonzero arbitrary constant.

Under the following rational transformations,

$$p = \frac{g}{f}, \quad q = \frac{h}{f}, \quad (5.7)$$

the system (5.5) is transformed into the following bilinear form,

$$D_x D_y f \cdot f + 2gh = 0, \quad (5.8a)$$

$$(D_t + \frac{1}{2}a(t)D_x D_y)g \cdot f = 0, \quad (5.8b)$$

$$(D_t - \frac{1}{2}a(t)D_x D_y)h \cdot f = 0, \quad (5.8c)$$

where D_x, D_y and D_t are Hirota bilinear differential operators.

5.2 Double Wronskian Solutions

For the sake of convenience, we adopt Freeman and Nimmo's notation for the double Wronskian determinant [15, 16],

$$W^{N+1, M+1}(\phi; \psi) = \det(\phi, \partial\phi, \dots, \partial^N\phi; \psi, \partial\psi, \dots, \partial^M\psi) = |\hat{N}; \hat{M}|,$$

where $\phi = (\phi_1(x), \phi_2(x), \dots, \phi_{N+M+2})^T$ and $\psi = (\psi_1(x), \psi_2(x), \dots, \psi_{N+M+2})^T$.

In the next theorem, we present a double Wronskian solution for the bilinear system (5.8) under a sufficient condition defined by a system of linear partial differential equations.

Theorem 5.1 *Let ϕ and ψ satisfy the following linear system*

$$\phi_x = -K\phi, \quad \phi_y = \phi_x, \quad \phi_t = -A(t)\phi_{xx}, \quad (5.9a)$$

$$\psi_x = K\phi, \quad \psi_y = \psi_x, \quad \psi_t = B(t)\psi_{xx}, \quad (5.9b)$$

where $K = (k_{ij})_{(N+M+2) \times (N+M+2)}$ is a matrix with arbitrary constant entries k_{ij} , $A(t) = (a_{ij}(t))_{(N+M+2) \times (N+M+2)}$ is a matrix with arbitrary smooth function entries and $\sum_{i=1}^{N+M+2} a_{ii}(t) = a(t)$, but $B(t) = (b_{ij}(t))_{(N+M+2) \times (N+M+2)}$ is a matrix with entries defined to be

$$b_{ij}(t) = \begin{cases} a_{ij}(t), & \text{if } i = j, \\ -a_{ij}(t), & \text{if } i \neq j. \end{cases}$$

Then the following double Wronskian determinants

$$f = W^{N+1, M+1}(\phi; \psi), \quad g = 2W^{N+2, M}(\phi; \psi) \quad \text{and} \quad h = -2W^{N, M+2}(\phi; \psi)$$

solve the bilinear system (5.8).

In order to prove this theorem, we need the following lemmas [98].

Lemma 5.2 Let B be an $N \times (N - 2)$ matrix, and a, b, c and d represent N -dimensional column vectors. Then

$$|B, a, b||B, c, d| - |B, a, c||B, b, d| + |B, a, d||B, c, b| = 0. \quad (5.10)$$

Lemma 5.3 Suppose Ξ is an $N \times N$ matrix with the column vector set of Ξ_j , Ω is an $N \times N$ operator matrix with the column vector set of Ω_j where each entry Ω_{js} is an operator. Then we have

$$\sum_{j=1}^N |\Omega_j * \Xi| = \sum_{j=1}^N |(\Omega^T)_j * \Xi^T|, \quad (5.11)$$

where for any N -dimensional column vectors A_j and B_j , we define

$$A_j \circ B_j = (A_{1j}B_{1j}, A_{2j}B_{2j}, \dots, A_{Nj}B_{Nj})^T, \quad (5.12)$$

and

$$|A_j * \Xi| = |\Xi_1, \dots, \Xi_{j-1}, A_j \circ \Xi_j, \Xi_{j+1}, \dots, \Xi_N|. \quad (5.13)$$

Let ϕ and ψ satisfy the conditions (5.9) and set

$$\Xi = W^{N+1, M+1}(\phi; \psi), \quad \Omega = (\Omega_{ij})_{(N+M+2) \times (N+M+2)}, \quad (5.14)$$

where

$$\Omega_{ij} = \begin{cases} -\partial_x & \text{if } 1 \leq i \leq N + M + 2; 1 \leq j \leq N + 1 \\ \partial_x & \text{if } 1 \leq i \leq N + M + 2; N + 2 \leq j \leq N + M + 2. \end{cases}$$

Then, by Lemma 5.3, we have

$$\begin{aligned}
& \sum_{j=1}^{N+M+2} \left| \begin{array}{cccccc} \phi_1 & \cdots & \partial_x^N \phi_1; \psi_1 & \cdots & \partial_x^M \psi_1 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ -\partial_x \phi_j & \cdots & -\partial_x^{N+1} \phi_j; \partial_x \psi_j & \cdots & \partial_x^{M+1} \psi_j \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \phi_{N+M+2} & \cdots & \partial_x^N \phi_{N+M+2}; \psi_{N+M+2} & \cdots & \partial_x^M \psi_{N+M+2} \end{array} \right| \\
& = -|\widehat{N-1}, N+1; \widehat{M}| + |\widehat{N}; \widehat{M-1}, M+1|.
\end{aligned} \tag{5.15}$$

By the conditions (5.9), the left hand side of the above equality is equal to

$$\begin{aligned}
& \sum_{j=1}^{N+M+2} \left| \begin{array}{cccccc} \phi_1 & \cdots & \partial_x^N \phi_1; \psi_1 & \cdots & \partial_x^M \psi_1 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \sum_{l=1}^{N+M+2} k_{jl} \phi_l & \cdots & \sum_{l=1}^{N+M+2} k_{jl} \partial_x^N \phi_l; \sum_{l=1}^{N+M+2} k_{jl} \psi_l & \cdots & \sum_{l=1}^{N+M+2} k_{jl} \partial_x^M \psi_l \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \phi_{N+M+2} & \cdots & \partial_x^N \phi_{N+M+2}; \psi_{N+M+2} & \cdots & \partial_x^M \psi_{N+M+2} \end{array} \right| \\
& = \sum_{j=1}^{N+M+2} k_{jj} |\widehat{N}; \widehat{M}|.
\end{aligned}$$

Hence, we have the following identity

$$\text{tr}K|\widehat{N}; \widehat{M}| = -|\widehat{N-1}, N+1; \widehat{M}| + |\widehat{N}; \widehat{M-1}, M+1|. \tag{5.16}$$

In order to simplify the notations and save space, we are going to use the following notations for

Wronskian determinants:

$$\begin{aligned}
d_1 &:= |\widehat{N}; \widehat{M}|, & d_2 &:= |\widehat{N-1}, N+1; \widehat{M}|, & d_3 &:= |\widehat{N}; \widehat{M-1}, M+1|, \\
d_4 &:= |\widehat{N-2}, N, N+1; \widehat{M}|, & d_5 &:= |\widehat{N-1}, N+2; \widehat{M}|, \\
d_6 &:= |\widehat{N-1}, N+1; \widehat{M-1}, M+1|, & d_7 &:= |\widehat{N}; \widehat{M-2}, M, M+1|, \\
d_8 &:= |\widehat{N}; \widehat{M-1}, M+2|, & d_9 &:= |\widehat{N+1}; \widehat{M-1}|, \\
d_{10} &:= |\widehat{N}, N+2; \widehat{M-1}|, & d_{11} &:= |\widehat{N+1}; \widehat{M-2}, M|, \\
d_{12} &:= |\widehat{N-1}, N+1, N+2; \widehat{M-1}|, & d_{13} &:= |\widehat{N}, N+3; \widehat{M-1}|, \\
d_{14} &:= |\widehat{N}, N+2; \widehat{M-2}, M|, & d_{15} &:= |\widehat{N+1}; \widehat{M-3}, M-1, M|, \\
d_{16} &:= |\widehat{N+1}; \widehat{M-2}, M+1|, & d_{17} &:= |\widehat{N-1}; \widehat{M+1}|.
\end{aligned}$$

Noticing the following

$$\begin{aligned}
|\widehat{N}; \widehat{M}|(\operatorname{tr}K)^2|\widehat{N}; \widehat{M}| &= (\operatorname{tr}K|\widehat{N}; \widehat{M}|)^2, \\
(\operatorname{tr}K)^2|\widehat{N+1}; \widehat{M-1}||\widehat{N}; \widehat{M}| &= \operatorname{tr}K|\widehat{N+1}; \widehat{M-1}|\operatorname{tr}K|\widehat{N}; \widehat{M}|, \\
(\operatorname{tr}K)^2|\widehat{N}; \widehat{M}||\widehat{N+1}; \widehat{M-1}| &= \operatorname{tr}K|\widehat{N+1}; \widehat{M-1}|\operatorname{tr}K|\widehat{N}; \widehat{M}|,
\end{aligned}$$

and using the identity (5.16), we get the following three identities:

$$(d_4 + d_5 - 2d_6 + d_7 + d_8)d_1 = (d_2 - d_3)^2, \quad (5.17)$$

$$(d_{12} + d_{13} - 2d_{14} + d_{15} + d_{16})d_1 = (d_{10} - d_{11})(d_2 - d_3), \quad (5.18)$$

$$(d_4 + d_5 - 2d_6 + d_7 + d_8)d_9 = (d_2 - d_3)(d_{10} - d_{11}). \quad (5.19)$$

Now we are ready to prove Theorem 5.1.

Proof of Theorem 5.1: Under the linear conditions (5.9) we can compute the following derivatives

$$\begin{aligned}
f_x &= d_2 + d_3, & f_{xy} &= d_4 + d_5 + 2d_6 + d_7 + d_8, & g_x &= 2d_{10} + 2d_{11} \\
f_t &= -a(t)(-d_4 + d_5 + d_7 - d_8), & g_t &= 2a(t)(d_{12} - d_{13} - d_{15} - d_{16}) \\
g_{xy} &= 2d_{12} + 2d_{13} + 4d_{14} + 2d_{15} + 2d_{16}.
\end{aligned}$$

Using the identity (5.17) and Lemma 5.2, we can verify the first equation in the bilinear system

(5.8), indeed:

$$\begin{aligned}
D_x D_y (f \cdot f) + 2gh &= 2f f_{xy} - 2f_x f_y + 2gh \\
&= 2d_1(d_4 + d_5 + 2d_6 + d_7 + d_8) - 2(d_2 + d_3)^2 - 8d_9 d_{17} \\
&= 2(d_2 - d_3)^2 + 8d_1 d_6 - 2(d_2 + d_3)^2 - 8d_9 d_{17} \\
&= -8(d_2 d_3 - d_1 d_6 + d_9 d_{17}) = 0.
\end{aligned}$$

In order to prove the second equation in the bilinear system (5.8), we need to employ the identities (5.18) and (5.19) to compute that

$$\begin{aligned}
(g_t + \frac{1}{2}a(t)g_{xx})f &= a(t)d_1(3d_{12} - d_{13} - d_{15} + 3d_{16} + 2d_{14}) \\
&= 4a(t)d_1(d_{12} + d_{16}) - a(t)(d_{12} + d_{13} + d_{15} + d_{16} - 2d_{14}) \\
&= 4a(t)d_1(d_{12} + d_{16}) - a(t)(d_{10} - d_{11})(d_2 - d_3), \\
(\frac{1}{2}a(t)f_{xx} - f_t)g &= a(t)d_9(-d_4 + 3d_5 + 2d_6 + 3d_7 - d_8) \\
&= 4a(t)d_9(d_5 + d_7) - a(t)d_9(d_4 + d_5 - 2d_6 + d_7 + d_8) \\
&= 4a(t)d_9(d_5 + d_7) - a(t)(d_{10} - d_{11})(d_2 - d_3).
\end{aligned}$$

Hence the second equation in (5.8) reduces to

$$(D_t + \frac{1}{2}a(t)D_x D_y)g \cdot f = 4a(t)([d_1 d_{12} - d_2 d_{12} + d_9 d_5] + [d_1 d_{16} - d_3 d_{11} + d_9 d_7]),$$

and again by Lemma 5.2 the right hand side of the above equality is zero. Similarly the third equation in (5.8) can be verified. Therefore, we have shown that the double Wronskian determinants f , g and h solve the bilinear system (5.8) under the linear condition (5.9). \square

5.3 Soliton Solutions in Double Wronskian Form

In this section, we are going to give soliton solutions obtained from the double Wronskian solutions. The linear system (5.9), given in Theorem 5.1, has the following solution

$$\phi = e^{-(Kx+Ky+K^2 \int A(t)dt)} C, \quad \psi = e^{Kx+Ky+K^2 \int B(t)dt} D, \quad (5.20)$$

where $C^T = (C_1, C_2, \dots, C_{N+M+2})$ and $D^T = (D_1, D_2, \dots, D_{N+M+2})$ are arbitrary real constant vectors.

If $K = \text{diag}(k_1, k_2, \dots, k_{N+M+2})$ and $A(t) = \text{diag}(a_1(t), a_2(t), \dots, a_{N+M+2}(t))$, then we have

$$\phi_j = C_j e^{-\eta_j}, \quad \psi_j = D_j e_j^\eta, \quad j = 1, 2, \dots, N + M + 2, \quad (5.21)$$

where $\eta_j = k_j x + k_j y + k_j^2 \int a_j(t) dt$.

Taking $N = M = 0$, $C_1 = D_1 = D_2 = 1$ and $C_2 = -1$, we have the one-soliton solution

$$u = -\frac{1}{2} \left(\ln \frac{(k_2 - k_1) e^{-(\eta_1 + \eta_2)}}{\cosh(\eta_2 - \eta_1)} \right)_x,$$

$$v = \frac{1}{2} b \left(\ln \frac{k_2 - k_1}{\cosh(\eta_2 - \eta_1)} \right)_{xy} - (k_2 - k_1)^2 \text{sech}^2(\eta_2 - \eta_1).$$

By taking $M = 1$, $N = 0$ and $C_i = D_i = 1$, $i = 1, 2, 3$, we get the two-soliton solution

$$u = -\frac{1}{2} \left(\ln \frac{(k_2 - k_3) e^{\eta_1 - \eta_2 - \eta_3} - (k_1 - k_3) e^{-\eta_1 + \eta_2 - \eta_3} + (k_1 - k_2) e^{-\eta_1 - \eta_2 + \eta_3}}{(k_3 - k_2) e^{-\eta_1 + \eta_2 + \eta_3} - (k_3 - k_1) e^{\eta_1 - \eta_2 + \eta_3} + (k_2 - k_1) e^{\eta_1 + \eta_2 - \eta_3}} \right)_x,$$

$$v = \frac{1}{2} b \left(\ln \frac{(k_2 - k_3) e^{\eta_1 - \eta_2 - \eta_3} - (k_1 - k_3) e^{-\eta_1 + \eta_2 - \eta_3} + (k_1 - k_2) e^{-\eta_1 - \eta_2 + \eta_3}}{(k_3 - k_2) e^{-\eta_1 + \eta_2 + \eta_3} - (k_3 - k_1) e^{\eta_1 - \eta_2 + \eta_3} + (k_2 - k_1) e^{\eta_1 + \eta_2 - \eta_3}} \right)_{xy}$$

$$+ 4\kappa \frac{(k_2 - k_3) e^{2\eta_1} - (k_1 - k_3) e^{2\eta_2} + (k_1 - k_2) e^{2\eta_3}}{\left((k_3 - k_2) e^{-\eta_1 + \eta_2 + \eta_3} - (k_3 - k_1) e^{\eta_1 - \eta_2 + \eta_3} + (k_2 - k_1) e^{\eta_1 + \eta_2 - \eta_3} \right)^2},$$

where $\kappa = \prod_{1 \leq i < j \leq 3} (k_j - k_i)$.

Chapter 6

Conclusions and Remarks

In this dissertation, we have used the Wronskian and Pfaffian techniques to formulate exact solutions to a few generalized soliton equations. In order to give more general results, we have extended the well known equations and systems like the KP and JM equations and Boussinesq system to higher dimensions with variable coefficients. Besides that, we have also extended the first two equations to nonlinear Pfaffianized systems by using the Pfaffianization procedure and gave different types of Pfaffian solutions.

Under a certain constraint on the variable coefficients, we have verified that the (3+1)-dimensional generalized vcKP equation

$$(u_t + \alpha_1(t)u_{xxy} + 3\alpha_2(t)u_xu_y)_x + \alpha_3(t)u_{ty} - \alpha_4(t)u_{zz} + \alpha_5(t)(u_x + \alpha_3(t)u_y) = 0$$

has two classes of exact determinant solutions. One has been formulated in Wronskian determinant form and the other, in Grammian determinant form. Indeed, we have shown that the above vcKP equation was reduced to the Plücker relation for determinants and the Jacobi identity for determinants in the cases of the obtained Wronskian and Grammian determinant solutions. In our solutions, there is a free parameter a which satisfies

$$3\alpha_1(t) - a^2\alpha_3(t)\alpha_4(t) \neq 0, \text{ for all values of } t.$$

Theorems 3.5 and 3.6 present the main results on these Pfaffian solutions.

We remark that in order to get more solutions to the above generalized vcKP equation, we have tried to replace arbitrary constants with arbitrary functions in t . But we faced a problem with a compatibility condition of the system of the linear differential equations (3.44). It is unavoidable that $\frac{\alpha_4}{\alpha_1}$ must be a constant. Actually, while computing the derivative $f_{N,ty}$ without this condition, the term

$$\frac{a^2}{3} \frac{d}{dt} \left(\frac{\alpha_4}{\alpha_1} \right) | \widehat{N-2}, N |$$

would appear and the vcKP equation could not be reduced to the Plücker relation for determinants or the Jacobi identity for determinants, not only that but also $f_{N,ty} \neq f_{N,yt}$.

In particular, if we put $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 \equiv 1$ and $\alpha_5 \equiv 0$, then we will get an equivalent solution to the one given in Theorem 2.1 in [21] with a condition on the parameter a , which accepts any real number except $\pm\sqrt{3}$ for a .

On the other hand, if we choose $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 \equiv -1$ and $\alpha_5 \equiv 0$, then we will have the equation

$$u_{xxxy} + 3(u_x u_y)_x - u_{tx} + u_{ty} - u_{zz} = 0.$$

Note here that the coefficient of the term u_{tx} is -1 . Using Theorem 3.5, one can get the following Wronskian solution

$$u = 2(\ln f_N)_x, \quad f_N = W(\phi_1, \phi_2, \dots, \phi_N),$$

where

$$\phi_i = \sum_{j=1}^p d_{ij} e^{\eta_{ij}}, \quad \eta_{ij} = k_{ij}x - \frac{1}{3}a^2 k_{ij}y + a k_{ij}^2 z - \frac{4a^2}{a^2 + 3} k_{ij}^3 t,$$

d_{ij} and k_{ij} are free parameters, and p is an arbitrary natural number. There are not any restrictions on our parameter a here.

The (3+1)-dimensional generalized vcKP has been extended to the following system of nonlinear differential equations:

$$\begin{aligned} (u_t + \alpha_1(t)u_{xxy} + 3\alpha_2(t)u_x u_y)_x + \alpha_3(t)u_{ty} - \alpha_4(t)u_{zz} \\ + \alpha_5(t)(u_x + \alpha_3(t)u_y) &= -8a^2 \frac{\alpha_4(t)\alpha_1(t)}{\alpha_2(t)} (vw)_x, \\ \frac{2}{\beta(t)}v_t + 3\frac{\alpha_2(t)}{\alpha_1(t)}u_x v_x + v_{xxx} + \frac{3}{a}(v_{xz} + \frac{\alpha_2(t)}{\alpha_1(t)}vu_z) &= 0, \\ \frac{2}{\beta(t)}w_t + 3\frac{\alpha_2(t)}{\alpha_1(t)}u_x w_x + w_{xxx} - \frac{3}{a}(w_{xz} + \frac{\alpha_2(t)}{\alpha_1(t)}wu_z) &= 0, \end{aligned}$$

which we call the Pfaffianized (3+1)-dimensional vcKP system. Through the dependent variable transformation

$$u = 2\frac{\alpha_1(t)}{\alpha_2(t)}(\ln f)_x, \quad v = g/f, \quad w = h/f,$$

and under the constraint:

$$\alpha_1(t) = C_0 \alpha_2(t) e^{-\int \alpha_5(t) dt},$$

this extension has been mapped into the following bilinear Pfaffianized form:

$$\begin{aligned} (\alpha_1(t) D_x^3 D_y + D_t D_x + \alpha_3(t) D_t D_y - \alpha_4(t) D_z^2) f \cdot f &= -8a^2 \alpha_4(t) gh, \\ (D_x^3 + \frac{2}{\beta(t)} D_t + \frac{3}{a} D_x D_z) g \cdot f &= 0, \\ (D_x^3 + \frac{2}{\beta(t)} D_t - \frac{3}{a} D_x D_z) h \cdot f &= 0. \end{aligned}$$

Theorem 3.7 presents the following Wronski-type Pfaffian solutions for the above bilinear Pfaffianized system:

$$f_N = (1, 2, \dots, 2N), \quad g_N = (1, 2, \dots, 2N - 2), \quad h_N = (1, 2, \dots, 2N + 2),$$

whose Pfaffian entries satisfy

$$\begin{aligned} \frac{\partial}{\partial x}(i, j) &= (i + 1, j) + (i, j + 1), \\ \frac{\partial}{\partial y}(i, j) &= -\frac{a^2 \alpha_4(t)}{3\alpha_1(t)} [(i + 1, j) + (i, j + 1)], \\ \frac{\partial}{\partial z}(i, j) &= a[(i + 2, j) + (i, j + 2)], \\ \frac{\partial}{\partial t}(i, j) &= \beta(t)[(i + 3, j) + (i, j + 3)], \end{aligned}$$

where

$$\beta(t) = \frac{4a^2 \alpha_1(t) \alpha_4(t)}{3\alpha_1(t) - a^2 \alpha_3(t) \alpha_4(t)}.$$

Another type of Pfaffian solutions, called Gramm-type Pfaffian solutions, has been given in Theorem 3.8 as follows

$$f_N = (1, 2, \dots, 2N), \quad g_N = (c_1, c_0, 1, 2, \dots, 2N), \quad h_N = (d_0, d_1, 1, 2, \dots, 2N),$$

where the Pfaffian entries are defined by

$$\begin{aligned} (i, j) &= c_{ij} + \int^x (\phi_i \psi_j - \phi_j \psi_i) dx, \quad c_{ij} = -c_{ji}, \quad c_{ij} = \text{constants}, \\ (d_n, i) &= \frac{\partial^n}{\partial x^n} \phi_i, \quad (c_n, i) = \frac{\partial^n}{\partial x^n} \psi_i, \\ (d_m, d_n) &= (c_m, c_n) = (c_m, d_n) = 0, \end{aligned}$$

with ϕ_i and ψ_i satisfying

$$\begin{aligned}\phi_{i,y} &= -\frac{a^2\alpha_4(t)}{3\alpha_1(t)}\phi_{i,x}, & \phi_{i,z} &= a\phi_{i,xx}, & \phi_{i,t} &= \beta(t)\phi_{i,xxx}, \\ \psi_{i,y} &= -\frac{a^2\alpha_4(t)}{3\alpha_1(t)}\psi_{i,x}, & \psi_{i,z} &= -a\psi_{i,xx}, & \psi_{i,t} &= \beta(t)\psi_{i,xxx}.\end{aligned}$$

Examples of such Wronski-type and Gramm-type Pfaffian entries have been presented.

In both results, Theorem 3.7 and Theorem 3.8, it is unavoidable that $\frac{\alpha_4}{\alpha_1}$ must be constant. Actually, without this condition we will get $f_{N,ty} \neq f_{N,yt}$.

We remark that the resulting system contains a free parameter and this characteristic implies that Pfaffianization does not have the uniqueness property.

Since the Pfaffianization procedure depends on the the two kinds of Pfaffian identities mentioned in the first section of chapter three, an interesting question for us is whether there exist other kinds of Pfaffian identities which can be used to formulate new kinds of Pfaffian solutions for nonlinear partial differential equations.

We have computed a bilinear Bäcklund transformation for the (3+1)-dimensional generalized KP equation

$$u_{xxy} + 3(u_x u_y)_x + u_{tx} + u_{ty} - u_{zz} = 0.$$

The facts used in our construction are the exchange identities for Hirota bilinear operators. The obtained bilinear Bäcklund transformation consists of six bilinear equations and involves nine arbitrary parameters. It is therefore a pretty large system, which in turn implies that the above (3+1)-dimensional generalized KP equation should have diverse solutions. Indeed, two classes of exponential and rational traveling wave solutions with arbitrary wave numbers have been constructed from the proposed bilinear Bäcklund transformation.

It is interesting to note that the condition (3.108) has a solution

$$k = l = m = 0, \omega \neq 0,$$

but this makes it impossible to solve (3.107). Therefore, the corresponding function

$$f' = -\omega t$$

provides a solution for the generalized bilinear KP equation (3.90), but it is not generated from the bilinear Bäcklund transformation (3.98) associated with $f = 1$. It is actually a limit solution of the presented polynomial solutions.

We remark that the above (3+1)-dimensional generalized KP equation possesses linear subspaces of exponential wave solutions [62, 63]. This shows a nice integrability property that nonlinear equations normally do not possess. One can also get some nonlinear superposition formulas of solutions generated from the proposed bilinear Bäcklund transformation [84, 99], but it is hard to prove that the resulting functions are solutions due to a large number of different equations involved in the Bäcklund transformation. To overcome this complexity, one should find a bilinear Bäcklund transformation consisting of a small number of bilinear equations. However, it is a very difficult challenge for us to get a bilinear Bäcklund transformation defined by a system of less than six equations, for example, two or three equations for the above (3+1)-dimensional generalized KP equation. Some new specific exchange identities must be developed for use in merging terms resulted from $P = 0$. There might also be other equations different from $P = 0$ which one can begin with to formulate bilinear Bäcklund transformations.

Similar to what have been done in Chapter 4, we have computed Wronskian and Gramian solutions to the (3+1)-dimensional nonlinear equation of Jimbo-Miwa (JM) type

$$u_{xxxx} + 3(u_x u_y)_x - u_{yt} - 3u_{xz} = 0.$$

These solutions have been presented in Theorems 4.1 and 4.2. Interestingly this equation has also Pfaffian solutions, presented in Theorem 4.3, which says that

$$u = 2(\ln f_N)_x, \quad f_N = (1, 2, \dots, 2N),$$

where the entries of the Pfaffian f_N are defined by

$$(i, j) = (j, i) = c_{ij} + \int_{-\infty}^x D_x \phi_i \cdot \phi_j dx, \quad i, j = 1, 2, \dots, 2N,$$

with $c_{ij} (= -c_{ji} \text{ for } i \neq j)$ are constants, D_x is the Hirota D-operator and all ϕ_i , $1 \leq i \leq 2N$, satisfy the following linear system of differential equations

$$\phi_{i,y} = a \int_{-\infty}^x \phi_i(x) dx, \quad \phi_{i,z} = a \phi_{i,x}, \quad \phi_{i,t} = \phi_{i,xxx},$$

where a is a nonzero parameter, solves the above (3+1)-dimensional JM equation. Examples of the Pfaffian solutions were made, along with a few plots of particular solutions.

We have also made the following extension

$$\begin{aligned}
u_{xxy} + 3u_{xx}u_y + 3u_xu_{xy} - u_{yt} - 3u_{xz} + 12(wv_x - vw_x)_x &= 0, \\
-v_t + 3u_xv_x + v_{xxx} + 3v_{xy} + 3vu_y &= 0, \\
-w_t + 3u_xw_x + w_{xxx} - 3w_{xy} - 3wu_y &= 0,
\end{aligned}$$

for the above (3+1)-dimensional JM equation by using the Pfaffianization procedure [48]. In Theorem 4.4 we have presented our main result on Wronski-type Pfaffian solutions, which says that the above (3+1)-dimensional JM system of nonlinear equations has the following solutions

$$u = 2(\ln f)_x, \quad v = g/f, \quad w = h/f,$$

where

$$\begin{aligned}
f &= f_N = (1, 2, \dots, 2N), \\
g &= g_N = (1, 2, \dots, 2N - 2), \\
h &= h_N = (1, 2, \dots, 2N + 2),
\end{aligned}$$

with the Pfaffian entries (i, j) satisfy the conditions

$$\begin{aligned}
\frac{\partial}{\partial x}(i, j) &= (i + 1, j) + (i, j + 1), \\
\frac{\partial}{\partial y}(i, j) &= (i + 2, j) + (i, j + 2), \\
\frac{\partial}{\partial t}(i, j) &= -2[(i + 3, j) + (i, j + 3)], \\
\frac{\partial}{\partial z}(i, j) &= (i + 4, j) + (i, j + 4).
\end{aligned}$$

The second result has been presented in Theorem 4.5, which says that the above (3+1)-dimensional JM system of nonlinear equations has the following Gram-type Pfaffian solutions

$$u = 2(\ln f)_x, \quad v = g/f, \quad w = h/f,$$

where

$$\begin{aligned}
f &= f_N = (1, 2, \dots, 2N), \\
g &= g_N = (c_1, c_0, 1, 2, \dots, 2N), \\
h &= h_N = (d_0, d_1, 1, 2, \dots, 2N),
\end{aligned}$$

and the Pfaffian entries are defined by

$$\begin{aligned}(i, j) &= c_{ij} + \int^x (\phi_i \psi_j - \phi_j \psi_i) dx, \quad c_{ij} = -c_{ji}, \quad c_{ij} = \text{constants}, \\(d_n, i) &= \frac{\partial^n}{\partial x^n} \phi_i, \quad (c_n, i) = \frac{\partial^n}{\partial x^n} \psi_i, \\(d_m, d_n) &= (c_m, c_n) = (c_m, d_n) = 0,\end{aligned}$$

where ϕ_i s and ψ_i s satisfy the following linear conditions

$$\begin{aligned}\phi_{i,y} &= \phi_{i,xx}, \quad \phi_{i,z} = \phi_{i,xxx}, \quad \phi_{i,t} = -2\phi_{i,xxx}, \\ \psi_{i,y} &= -\psi_{i,xx}, \quad \psi_{i,z} = -\psi_{i,xxx}, \quad \psi_{i,t} = -2\psi_{i,xxx}.\end{aligned}$$

On the other hand, the (3+1)-dimensional vcJM equation

$$\alpha_1(t)u_{xxxxy} + 3\alpha_2(t)(u_x u_y)_x - u_{yt} - 3\alpha_3(t)u_{xz} + 2\alpha_4(t)u_y = 0$$

has been transformed into the following bilinear form

$$(\alpha_1(t)D_x^3 D_y - D_y D_t - 3\alpha_3(t)D_x D_z) f \cdot f = 0,$$

under the constraint:

$$\alpha_1(t) = C_0 \alpha_2(t) e^{-\int \alpha_4(t) dt},$$

where $C_0 \neq 0$ is an arbitrary constant. This equation is extended to the following Pfaffianized JM system

$$\begin{aligned}(\alpha_1(t)D_x^3 D_y - D_y D_t - 3\alpha_3(t)D_x D_z) f \cdot f + 12a\alpha_3(t)D_x g \cdot h &= 0, \\ (D_x^3 - \frac{1}{\alpha_1(t)}D_t + \frac{3}{a}D_x D_y) g \cdot f &= 0, \\ (D_x^3 - \frac{1}{\alpha_1(t)}D_t - \frac{3}{a}D_x D_y) h \cdot f &= 0.\end{aligned}$$

Wronski-type and Gramm-type Pfaffian solutions have been given to the above bilinear Pfaffianized JM system as we did for the constant coefficient JM equation.

The last result in this dissertation is given in Theorem 5.1, in which we introduced the following new system of nonlinear partial differential equations which could be considered as a generalization of a well known Boussinesq system:

$$\begin{aligned}u_t + \alpha_1(t)u_{xy} + \alpha_2(t)(uw)_x + \alpha_3(t)v_x &= 0, \\ v_t + \beta_1(t)(wv_x + 2vu_y + uv_y) + \beta_2(t)(u_x w_y - (u_y)^2) + \beta_3(t)v_{xy} + \beta_4(t)u_{xyy} &= 0,\end{aligned}$$

where $w_x = u_y$. Under the following dependent variable transformations:

$$u = -\frac{1}{2}(\ln p)_x,$$

$$v = \frac{1}{2}b(\ln p)_{xy} - pq,$$

the above (2+1)-dimensional Boussinesq system is transformed into the (2+1)-dimensional AKNS system

$$p_t + a(t)\left(\frac{1}{2}p_{xy} - qp^2\right) = 0,$$

$$q_t - a(t)\left(\frac{1}{2}q_{xy} - pq^2\right) = 0,$$

under the following constraint:

$$\beta_1(t) = \alpha_2(t) = 2\alpha_3(t) = -a(t), \beta_2(t) = ba(t),$$

$$\alpha_1(t) = -\beta_3(t) = \frac{a(t)}{2}(1-b), \beta_4(t) = b\frac{a(t)}{2}(b-2),$$

where $a(t)$ is an arbitrary smooth function and b is a nonzero arbitrary constant.

Under the following rational transformations,

$$p = \frac{g}{f}, \quad q = \frac{h}{f},$$

the above AKNS system has been transformed into the following bilinear form

$$D_x D_y (f \cdot f) + 2gh = 0,$$

$$(D_t + \frac{1}{2}a(t)D_x D_y)g \cdot f = 0,$$

$$(D_t - \frac{1}{2}a(t)D_x D_y)h \cdot f = 0,$$

where D_x, D_y and D_t are Hirota bilinear differential operators. A double Wronskian exact solution has been formulated for the above bilinear system given by

$$f = W^{N+1, M+1}(\phi; \psi), \quad g = 2W^{N+2, M}(\phi; \psi) \quad \text{and} \quad h = -2W^{N, M+2}(\phi; \psi),$$

where ϕ and ψ satisfy the following linear system

$$\phi_x = -K\phi, \quad \phi_y = \phi_x, \quad \phi_t = -A(t)\phi_{xx},$$

$$\psi_x = K\phi, \quad \psi_y = \psi_x, \quad \psi_t = B(t)\psi_{xx},$$

with $K = (k_{ij})_{(N+M+2) \times (N+M+2)}$ being a matrix with arbitrary constant entries k_{ij} ,
 $A(t) = (a_{ij}(t))_{(N+M+2) \times (N+M+2)}$ is a matrix with arbitrary smooth function entries and
 $\sum_{i=1}^{N+M+2} a_{ii}(t) = a(t)$, but $B(t) = (b_{ij}(t))_{(N+M+2) \times (N+M+2)}$ whose entries defined to be

$$b_{ij}(t) = \begin{cases} a_{ij}(t), & \text{if } i = j, \\ -a_{ij}(t), & \text{if } i \neq j. \end{cases}$$

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