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A study on the correlation between a star's Rayleigh-Taylor characteristic timescale and stellar wind activity

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Abstract

In this paper, we investigate the correlations between a star's internal dynamics due to the Rayleigh-Taylor instability and episodes of stellar wind activity, using both a theoretical model and observational data from the NOAA.% \cite{NOAA}. Besides its relevance as an astrophysics problem, this study is also informative for models of climate change which include secular perturbations in the Sun's internal dynamics, as a potential source of solar activity variability.

Keywords

Stellar wind, Rayleigh-Taylor instability, Fluid dynamics

A STUDY ON THE CORRELATION BETWEEN A STAR'S RAYLEIGH-TAYLOR CHARACTERISTIC TIMESCALE AND STELLAR WIND ACTIVITY

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Abstract. In this paper, we investigate the correlations between a star's internal dynamics due to the Rayleigh-Taylor instability and episodes of stellar wind activity, using both a theoretical model and observational data from the NOAA.Besides its relevance as an astrophysics problem, this study is also informative for models of climate change which include secular perturbations in the Sun's internal dynamics, as a potential source of solar activity variability.

Contents

1. Introduction

In this article, we explore the hypothesis under which the stellar wind activity is mainly due to fluctuations in the star's density distribution associated with the Rayleigh-Taylor instability. A fundamental mechanism characterizing the unstable distribution of fluid density where gravitational attraction allows for denser strata to sit atop less dense fluid layers, the Rayleigh-Taylor instability is a plausible universal model for the stellar wind flares and matter ejection. We review the classical mechanics model for planetary motion and relate it to a two-layer Rayleigh-Taylor star model, in order to test the hypothesis that the latter could be a primary cause for the former.

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2. An overview of planetary dynamics and stability analysis of the Rayleigh-Taylor instability

2.1. **Kepler's laws and planetary dynamics.** Following Newton, we show that Kepler's laws (derived empirically from observations of planetary motion made over decades by Danish astronomer Tycho Brahe) can be derived using only two postulates: Newton's Second Law of mechanics and (independently) his Universal Attraction Law:

$$-G\frac{mM}{r^3}\vec{r} = \vec{F} = m\vec{a}, \tag{2.1}$$

where $\vec{F}$ is the force of gravitational attraction exerted on a point mass m, of position $\vec{r}$ and acceleration $\vec{a}$, by another mass M, fixed for convenience at the origin, and G is the Newton gravitational constant.

Remark 2.1. This set-up ignores Newton's 3rd law, according to which the second mass M would feel a reaction force $-\vec{F}$ (hence it cannot be stationary); this is a minor detail in the sense that any two-body system m, M can be reduced to the simplified version above, when the origin is set at the center of mass of the system. In the limit $m/M \to 0$ (relevant for the case of planets (m) and Sun (M) in our system, or stellar mass ejected by solar winds (m) and star (M)), the center of mass can be approximated by the position of the large mass M.

Denote by $\vec{r}, \vec{v} = \frac{d\vec{r}}{dt}$ the position and velocity vectors of the point mass m ("planet") and note that, since the force between the two points (planet and Sun) is *central*, $m\vec{a} = \vec{F} \parallel \vec{r}$, the angular momentum $\vec{L} = m\vec{r} \times \vec{v}$ is conserved:

$$\dot{\vec{L}} = m\dot{\vec{r}} \times \vec{v} + \vec{r} \times \vec{F} = m\vec{v} \times \vec{v} + \vec{r} \times \vec{F} = 0. \tag{2.2}$$

But this means that the plane of motion (determined at all times by the vectors $\vec{v}, \vec{a} \parallel \vec{r}$), is fixed in time. Therefore, the motion is a *planar curve*, taking place in a *fixed* plane of motion (of normal direction given by $\vec{L}(t = 0)$). We have therefore obtained the following result:

Theorem 2.1 (Kepler's third law)**.** *From the formula for planar area swept by the position vector and Eq.* (2.2)*, we find that the rate at which the area is swpt, $dA/dt = L/2m$ is a constant for any body in planetary motion.*

Switching now to polar coordinates in the plane of motion, $\vec{r} = r[\cos(\theta)\hat{\imath} + \sin(\theta)\hat{\jmath}]$, and introducing the moving frame unit vectors

$$\hat{e}_r = \cos(\theta)\hat{\imath} + \sin(\theta)\hat{\jmath}, \quad \hat{e}_\theta = -\sin(\theta)\hat{\imath} + \cos(\theta)\hat{\jmath},$$

we can write the velocity and acceleration vectors as

$$\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta, \quad \vec{a} = [\ddot{r} - r\dot{\theta}^2]\hat{e}_r + [2\dot{r}\dot{\theta} + r\ddot{\theta}]\hat{e}_\theta.$$

In this notation, the angular momentum becomes

$$\vec{L} = mr^2\dot{\theta}(\hat{e}_r \times \hat{e}_\theta), \quad ||\vec{L}|| \equiv L_0 = mr^2\dot{\theta} = \text{const.}$$

Therefore, we obtain that

$$0 = \dot{L}_0 = r(2\dot{r}\dot{\theta} + r\ddot{\theta}),$$

which yields for the acceleration $\vec{a} = [\ddot{r} - r\dot{\theta}^2]\hat{e}_r$, consistent with the assumption $\vec{a} \parallel \vec{r}$ of Eq. (2.1). We also note the useful (chain rule) identities

$$\dot{\theta} = \frac{L_0}{r^2}, \qquad \frac{d}{dt} = \frac{L_0}{r^2}\frac{d}{d\theta}, \qquad \frac{dr}{dt} = -L_0\frac{d(1/r)}{d\theta}. \tag{2.3}$$

2.1.1. *Dynamics.* Using Newton's Law of Universal Attraction, the planet's radial acceleration can be expressed as

$$\frac{-GM}{r^2} = \ddot{r} - \frac{L_0^2}{m^2r^3},$$

leading to (by using the equation (2.3))

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \left(\frac{m}{L_0}\right)^2 GM.$$

When a new variable r_0 is introduced, by defining

$$\left(\frac{m}{L_0}\right)^2 GM \equiv \frac{1}{r_0},$$

$$\frac{L_0^2}{m} = GMmr_0,$$

the final expression for the trajectory of the planet becomes (in polar coordinates)

$$\frac{1}{r} = \frac{1 + e\cos\theta}{r_0}, \quad e \in \mathbb{R}, \tag{2.4}$$

which represents a conic section in the plane. Imposing now for the total energy the bounded-trajectory constraint $TE = KE + PE < 0$,

$$\frac{mv^2}{2}\bigg|_{\theta=0} = \frac{L_0^2}{2mr^2}\bigg|_{\theta=0} = \frac{GMm}{r}\cdot\frac{1+e}{2} < \frac{GMm}{r}\bigg|_{\theta=0},$$

it follows that $e \in [0, 1)$, therefore any planetary trajectory is an ellipse. Values of $e \geq 1$ correspond to unbounded trajectories (parabola $e = 1$, hyperbola $e > 1$), and therefore describe single-passage comets. Thus,

Theorem 2.2 (Kepler's first law)**.** *From Eq.* (2.4)*, it follows that all the planetary (i.e., bounded) trajectories are ellipses.*

From Kepler's first law and Eq.(2.4), we have (a, b are the semi-axes)

$$\left(\frac{\pi ab}{T}\right)^2 = \left(\frac{L_0}{2m}\right)^2 = \frac{GMr_0}{4} \Rightarrow \frac{T^2 r_0}{(ab)^2} = \frac{(2\pi)^2}{GM},$$

where T is the period of motion. Since $r_0 = a(1-e^2)$ and $b^2 = a^2(1-e^2)$:

Theorem 2.3 (Kepler's second law)**.** *For all the planets in the same solar system (i.e. given solar mass M), $\frac{T^2}{a^3} = \frac{(2\pi)^2}{GM}$ is the same for all planets.*

2.1.2. *Total energy.* Now that the trajectory of the planet is known to be described by an ellipse, the energy in the system must be considered.

To asses the total energy in the system, the position vector characterizing the trajectory is obtained from the previous calculations describing the ellipse to produce the square of the velocity $v^2 = (\frac{L_0}{mr_0})^2(1 + 2e\cos\theta + e^2)$.

When $(\frac{L_0}{mr_0})^2$ is substituted for $\frac{GM}{r_0}$ it becomes clear that an interval of negative values is available for the total energy, because, as the sum of kinetic and potential energies,

$$E = \frac{mv^2}{2} - \frac{GMm}{r} = \frac{GMm}{r_0} \cdot \frac{(e^2-1)}{2}, \ e \in [0, 1),$$

which may take any negative value from the interval $\left[-\frac{1}{2}\left(\frac{GMm}{L_0}\right)^2, 0\right)$.

This shows that, given the values of the constants of motion (kinetic momentum and total energy) and the initial position of the objects, one can find uniquely the elliptic trajectory that describes planetary motion in this approximation (two-body problem).

Remark 2.2. In a more general case, if a point is moving on the ellipse

$$\vec{R}(\theta) = \langle a\cos(\theta), b\sin(\theta)\rangle, \ \theta \in [0, 2\pi), \ a \geq b > 0,$$

with (non-constant) speed $v = \frac{d\ell}{dt}$ in such a way that its acceleration is $\vec{a}(t) = f(r)\widehat{e}_r$, where $\vec{r}(t) = r(t)\widehat{e}_r$ is the position vector of the point relative to a fixed point on the major semi-axis, then the following hold:

i) the angular momentum $\vec{L} = \vec{r} \times \vec{v}$ is a constant of motion;
ii) if κ_0 denotes the curvature of the ellipse at $\theta = 0, \pi$, then

$$f(r) = -\frac{\kappa_0 L^2}{r^2} = -\frac{aL_0^2}{b^2} \cdot \frac{1}{r^2}.$$

2.2. **Projectile motion beyond the parabolic approximation.**

2.2.1. *Escape velocity.* Modeling a star by a total mass M uniformly distributed on a spherical solid of radius R, we consider the projectile motion of a point mass $m \ll M$, launched from the surface of the star, with initial speed v_0, at an angle $\theta \in [0, \frac{\pi}{2}]$ above the local tangent plane. Let h be the highest altitude reached along the point's trajectory. We will find the first escape velocity for the mass M.

Starting from Newton's Universal Gravitational Attraction Law,

$$m\ddot{\vec{r}} = \vec{F} = -G\frac{Mm}{r^2}\widehat{e_r} = \vec{\nabla}\left(G\frac{Mm}{r}\right)$$

where $\vec{r} = r\widehat{e_r}$ is the position vector of the point mass m relative to the center of the Earth (this equation already assumes that $m/M \to 0$), we find

$$\frac{d}{dt}\left(\frac{m||\dot{\vec{r}}||^2}{2}\right) = m\ddot{\vec{r}}\cdot\dot{\vec{r}} = \vec{F}\cdot\dot{\vec{r}} = \vec{\nabla}\left(G\frac{Mm}{r}\right)\cdot\dot{\vec{r}} = \frac{d}{dt}\left(G\frac{Mm}{r}\right),$$

so the point mass will have speed v at some distance r away from the center of the star, such that

$$\frac{m}{2}\left(v^2 - w^2\right) = GMm\left(\frac{1}{r} - \frac{1}{R}\right) \Rightarrow$$
$$w^2 - \frac{2GM}{R} = \left[v^2 - \frac{2GM}{r}\right] = \text{const.}$$

From §2.1.1, we know that the trajectory never intersects again the surface of the star when

$$v^2 - \frac{2GM}{r} = \frac{GM}{r}(e-1) \geq 0 \Rightarrow w^2 = \frac{2GM}{R}.$$

Since the force exerted by the star on the object on the surface is given by

$$F_g = \frac{GMm}{R^2} = mg,$$

we can identify the quantity $\frac{GM}{R^2} \equiv g$ to the star's *the gravitational acceleration.* Therefore, the escape velocity is given by

$$w^2 = 2gR \Rightarrow w = \sqrt{2gR}.$$

2.2.2. *Maximum vertical position.* Denoting $\alpha = \frac{v_0}{w}$ and $\eta = \frac{h}{R}$, we will find $\eta(\alpha, \theta)$ and discuss the special cases $\alpha = 1$ (escape trajectory), $\theta = \frac{\pi}{2}$ (vertical motion), and $\alpha \ll 1$.

Using the formula for the escape velocity, we find

$$\alpha^2 = \frac{v_0^2}{w^2} = \frac{v_0^2}{2Rg} = \frac{\frac{mv_0^2}{2}}{\frac{GMm}{R}},$$

so we conclude that the parameter α^2 represents the ratio of kinetic energy to absolute value of the gravitational potential energy of the object.

From the general elliptical trajectory solution presented in §2.1.1, we have the parameters ($v_\parallel$ is the speed at the highest altitude point)

$$\frac{L_0}{m} = Rv_0 \cos\theta = Rw\alpha\cos\theta = (R+h)v_\parallel,$$
$$r_0 = \frac{(L_0/m)^2}{GM} = 2R\alpha^2\cos^2\theta = \frac{v_0^2}{g}\cos^2\theta,$$

where the polar form of the trajectory, with the center of the Earth at one of the focal points, is given by

$$r(\varphi) = \frac{r_0}{1 - e\cos\varphi}, \quad R = r(\varphi_0) = \frac{r_0}{1 - e\cos\varphi_0}, \quad R+h = r(0) = \frac{r_0}{1-e}$$

Conservation of total energy (found in the previous solution) and angular momentum (c.f. §2.1.1) yield

$$\frac{v_\parallel^2}{2} - \frac{GM}{R+h} = \frac{v_0^2}{2} - \frac{GM}{R} \Rightarrow (1-\alpha^2)(\eta^2 + 2\eta) - \eta - \alpha^2\sin^2\theta = 0.$$

For $\alpha \neq 1$, the equation has two real roots, and we choose the positive one:

$$\eta = \frac{-1 + 2\alpha^2 + \sqrt{1 - 4\alpha^2(1-\alpha^2)\cos^2\theta}}{2(1-\alpha^2)},$$
$$1 + \eta = \frac{1 + \sqrt{1 - 4\alpha^2(1-\alpha^2)\cos^2\theta}}{2(1-\alpha^2)}$$

The special limits are obtained as $\eta(\alpha, \frac{\pi}{2}) = \frac{\alpha^2}{1-\alpha^2}$, which for small values $\alpha \ll 1$ has the expansion (and correction to the parabolic approximation)

$$\eta = \alpha^2 + \alpha^4 + \alpha^6 + \ldots \Rightarrow h \simeq \frac{v_0^2}{2g} + \frac{v_0^4}{4Rg^2} + \ldots,$$

and for generic values of the launch angle θ,

$$\eta(\alpha^2 \to 1^-) = \frac{1}{1-\alpha^2} - \sin^2\theta + \ldots \to \infty,$$

$$\eta(\alpha^2 \to 0) \simeq \alpha^2 \sin^2\theta = \frac{v_0^2}{2gR}\sin^2\theta.$$

2.2.3. *Eccentricity and degenerate trajectories.* For the case of an elliptical (that is, bound) trajectory, we can now compute the eccentricity $e(\alpha, \theta)$ and discuss the special cases $\alpha \to 1^-$ (escape trajectory) and $\theta \to \frac{\pi}{2}^-$ (vertical motion).

From the previous solution, $1 - e = \frac{r_0}{R(1+\eta)} = \frac{2\alpha^2\cos^2\theta}{1+\eta}$, so

$$e = 1 - \frac{4\alpha^2(1-\alpha^2)\cos^2\theta}{1+\sqrt{1-4\alpha^2(1-\alpha^2)\cos^2\theta}} = \sqrt{1-4\alpha^2(1-\alpha^2)\cos^2\theta}$$

For both $\alpha \to 1^-$ (escape trajectory) and $\theta \to \frac{\pi}{2}^-$, we find $e \to 1^-$. In the first case the elliptical trajectory crosses into its parabolic limit, while in the second it degenerates into the focal segment.

2.2.4. *Second focal point.* For the case of an elliptical trajectory, we next find the position of the second focal point of the trajectory (the center of the star is the other one), by finding its distance to the center of the star, $d(e, \eta)$, and the angle φ_0 made by the major semi-axis of the ellipse with the local radial direction at the launch point (Note that $2R\varphi_0$ is the range of the projectile motion).

We compute $d = 2ae = 2r_0\frac{e}{1-e^2} = R\frac{e}{1-\alpha^2} = R\frac{\sqrt{1-4\alpha^2(1-\alpha^2)\cos^2\theta}}{1-\alpha^2}$, which yields for the position of the second focal point relative to the surface of the star the altitude

$$\delta = d - R = R\left[\frac{\sqrt{1-4\alpha^2(1-\alpha^2)\cos^2\theta}}{1-\alpha^2} - 1\right]$$

We find that the second focal point is on the surface of the star when

$$4(1-\alpha^2)\cos^2\theta = 1\cos^2\theta - \frac{1}{2} = \alpha^2\left(\cos^2\theta - \frac{1}{4}\right),$$

therefore

$$\alpha^2 = \frac{2-4\cos^2\theta}{1-4\cos^2\theta}, \quad \theta \in \left[0, \frac{\pi}{4}\right],$$

or equivalently,

$$(1-\alpha^2)\cos^2\theta = \frac{1}{2} - \frac{\alpha^2}{4}$$

For the angle,

$$\cos\varphi_0 = \frac{1-2\alpha^2\cos^2\theta}{e} = \frac{1-2\alpha^2\cos^2\theta}{\sqrt{1-4\alpha^2(1-\alpha^2)\cos^2\theta}},$$

so in the limit $\alpha \ll 1$ (parabolic approximation), the range is given by

$$2R\varphi_0 \simeq 2R\alpha^2\sin(2\theta) + \ldots \simeq \frac{v_0^2}{g}\sin(2\theta)$$

2.2.5. *Trajectory beyond the parabolic approximation.* Finally, we find the first correction to the parabolic solution for the projectile motion.

Around the highest-altitude point, for $e \to 1, \eta \ll 1$, the trajectory in local Cartesian coordinates is given by

$$x(\varphi) = R\varphi, \quad y(\varphi) = r(\varphi) - R = \frac{r_0}{1-e\cos(\frac{x}{R})} - R,$$

$$y = h - \frac{e(R+h)}{2(1-e)R^2}x^2 + \frac{e^2(R+h)}{4(1-e)^2R^4}x^4 + \ldots \simeq y_{par}(x) + \frac{e^2(R+h)}{4(1-e)^2R^4}x^4,$$

where the first two terms give the parabolic approximation. The curvature of the trajectory at the highest altitude point is then

$$\kappa_0 = \frac{e(R+h)}{(1-e)R^2} = \frac{er_0}{(1-e)^2R^2} = \frac{v_0^2\cos^2\theta}{R^2g}\frac{e}{(1-e)^2}$$

2.3. **Linear analysis of the stellar Rayleigh-Taylor instability.** We model the Rayleigh-Taylor instability by two layers of fluids of densities $\rho_1 < \rho_2$, where the first layer (of density ρ_1) is at the bottom and the second is at the top, in constant gravitational field of acceleration g. Assume the top layer, at equilibrium, extends vertically from coordinate $y = h_1$ to $y = h_1 + h_2$, and the bottom layer is from $y = 0$ to $y = h_1$, then introducing the field variable $\zeta(x,t) = y(x,t) - h_1$ to represent the interface between the two fluids, we can isolate the first-order perturbation for the Lagrange function of the system,

$$L = \iint \mathcal{L}dA, \quad \mathcal{L} = \left[K\frac{\rho_1+\rho_2}{2}\left(\frac{\partial\zeta}{\partial t}\right)^2 - \frac{(\rho_1-\rho_2)g}{2}\zeta^2\right],$$

with K a constant that depends on the actual geometry of the set-up. The Euler-Lagrange equations then become

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\zeta}}\right) = \frac{\partial\mathcal{L}}{\partial\zeta} \Rightarrow K(\rho_1+\rho_2)\ddot{\zeta} = (\rho_2-\rho_1)g\zeta,$$

which means that the growth of the interface instability is exponential,

$$\zeta(t) \sim e^{\lambda t}, \quad \lambda = \sqrt{\frac{g}{K}\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}}$$

The main conclusion of this analysis is that the Rayleigh-Taylor instability has a characteristic timescale,

$$\tau = \frac{1}{\lambda} \sim \sqrt{\frac{1}{g}\frac{\rho_2 + \rho_1}{\rho_2 - \rho_1}},$$

which can be related to the analysis in the previous section by noting that, under the assumption of incompressibility of the fluid layers, the launching speed at the surface of the star, as result of the growth of the interface is

$$\dot{\zeta}^2 = \frac{e^{2\frac{t}{\tau}}}{\tau^2}$$

This expression allows now to compare this quantity to the escape velocity found in §2.2.1 and test the hypothesis that the Rayleigh-Taylor model of the star is correlated with the observed stellar wind dynamics.

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