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Application of Calculus in the Elastic Curve (Deflection)

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Application of Calculus in the Elastic Curve (Deflection)

Abstract

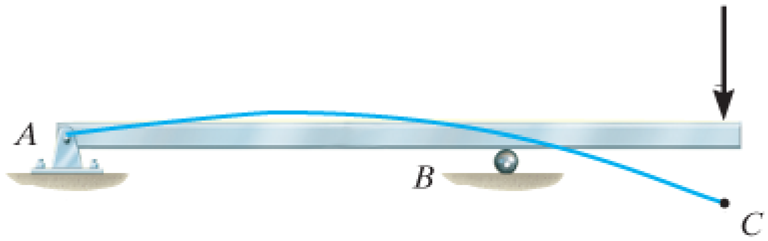
Beams are extremely important structure members that are used in nearly every application where significant loads need to be supported. In mechanical and civil engineering being able to appropriately design beams to withstand the expected-to-be loads is a foundational skill. Proper beam design is what keeps large machines from failing and buildings from collapsing. One important aspect of beam design to consider is the deflection of the beam due to the applied loads. It is often useful to determine the deflection along any given point in the beam in order to make sure the loaded beam does not displace any brittle materials that may be connected to or around it such as concrete. In this paper, we investigate how to determine the displacement of due to deflection at one point along an overhanging beam

Keywords

beam, displacement, deflection, elastic curve

PROBLEM STATEMENT

In an overhanging beam supported at two points by a pin and roller respectively,



determine the displacement of the overhanging end of the beam if the beam is loaded at its overhanging end.

Figure 1: Example of an overhanging beam loaded at its end

MOTIVATION

Within the mechanical and civil engineering disciplines engineers will often face problems that require designing beams, columns or shafts. These beams will naturally need to be able to support loads. Furthermore, in the real world these beams will be expected to deform very minimally when the expected load is applied. This is typically because beams will often have much more brittle materials connected to them, such as concrete which can only allow for minimal deflection before fracture occurs. Thus when designing a beam, it is critical to be able to calculate the amount of deformation along it at any given point. This can be done through calculus using the elastic curve.

MATHEMATICAL DESCRIPTION AND SOLUTION APPROACH

In order to begin determining the displacement of point C in (see Figure 1), some information will need to be known:

1. 2D dimensions of the beam in question
2. Force of all loads applied to the beam
3. Support reaction force(s) at the pin connection

These values are typically obtained by measurement but in the case of the problem suggested, the values are given and are shown below.

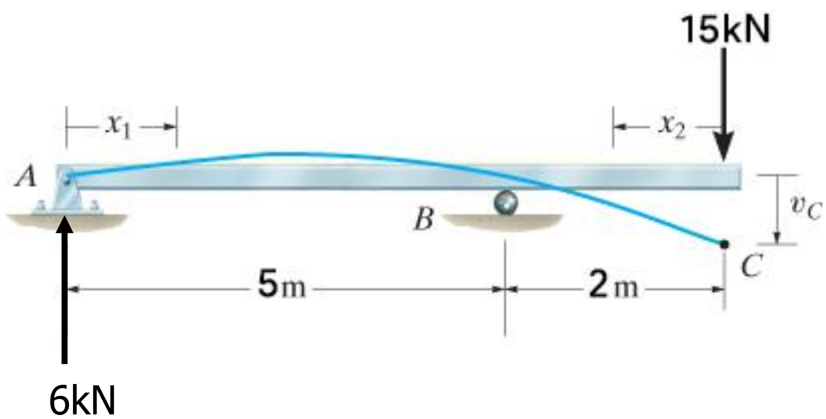


Figure 2: Overhanging beam with correct given information

It should be noted that two other pieces of information would need to be known in to come to a final real number in the end of the calculation.

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1. 3D dimensions of the beam
2. Young's modulus of elasticity for the beam's material.

The 3D dimensions are needed to calculate the *I* value which is a constant and the Young's modulus is another constant specific to the beam's material.

In the following calculation *I* and *E* will be used to represent these constants. Calculating the *I* value is not relevant to calculus in this context and calculating E from scratch is typically beyond the scope of undergraduate engineering courses.

Elastic Curve:

First the beam can be visually 'split' in two separate pieces with the reaction forces and moments accounted for like so:

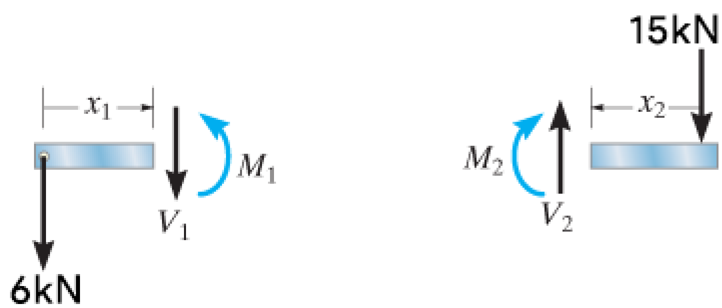

Figure 3: Above beam in two halves

*The 6 kN value is the given pin reaction y-direction force.

*It can be assumed there are no horizontal reactions since there are no horizontal loads

Due to the beam's loading, we are going to consider the x coordinate x_1 and x_2 as:

$0 \leq x_1 < 5m$

$0 \leq x_2 < 2m$

Moment Functions

$M_1 = -6x_1$

$M_2 = -15x_2$

Slope/Elastic Curve

We apply the equation:

$$EI\frac{d^2v}{dx^2} = Mx$$

To both x_1 and x_2

$$EI\frac{d^2v_1}{dx_1^2} = -6x_1$$

(I) $$EI\frac{dv_1}{dx_1} = -3x_1^2 + C_1$$

(II) $$EIv_1 = -x_1^3 + C_1x_1 + C_2$$

$$EI\frac{d^2v_2}{dx_2^2} = -15x_2$$

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(III) $EI\frac{dv_2}{dx_2} = \frac{-15x_2^2}{2} + C_3$

(IV) $EIv_2 = \frac{-5x_2^3}{2} + C_3x + C_4$

There are now four constants of integration that can be found by using three boundary conditions:

$v_1 = 0\ at\ x_1 = 0$

$v_1 = 0\ at\ x_1 = 5\ m$

$v_2 = 0\ at\ x_2 = 2\ m$

and one continuity equation:

$\frac{dv_1}{dx_1} = \frac{-dv_2}{-dx_2}\ at\ x = 5\ m\ \ and\ x = 2\ m$

When we apply the conditions we have:

$v_1 = 0\ at\ x_1\ =\ 0;\quad 0 = 0 + 0 + C_2$

$v_1 = 0\ at\ x_1 = 5\ m;\quad 0 = (5)^3 + C_1(5) + C_2$

$v_2 = 0\ at\ x_2 = 2m;\quad 0 = \frac{-5(2)^3}{2} + C_3(2) + C_4$

$\frac{dv_1}{dx_1}\ \ x = 5\ m = \frac{dv_2}{dx_2}\ \ x = 2\ m;\quad (-3(5)^2 + C_1) = (\frac{-15(2)^2}{2} + C_3)$

Solving the equations gives:

$C_1 = -25 \quad C_2 = 0 \quad C_3 = -70 \quad C_4 = 160$

Plug in C_3 and C_4 values into equation '(IV)'

$$EIv_2 = \frac{-5x_2^3}{2} - 70x_2 + 160$$

To solve for the displacement at point C (v_c), set $x_2 = 0$ and obtain:

$$v_c = \frac{160kN*m^3}{EI}$$

DISCUSSION

For the hypothetical problem I was given, the results were as expected. The result is now a useful equation that is used to determine the deflection at point C on the specific beam as shown in figure 1. The equation has significant utility in its form with the constants E and *I* because it is now a simple process to recheck the deflection at point C if the beam's material or cross-sectional area changes. Therefore, if the current material's elastic modulus is determined to cause too much deflection, it is now possible to swap materials and rerun the numbers in seconds. In the field of mechanical/civil engineering having equations like this on hand one can save significant time when designing and evaluating solutions to beam/shaft requirements. In the real world however, beams can easily be in complicating loading scenarios in which it may take significant time to resolve the entire calculation for v_c if changes are made to the beam.

CONCLUSIONS AND RECOMMENDATIONS

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Engineering mechanics of materials is one of the oldest and broadest fields within mechanical/civil engineering. When talking specifically about beam deflection, problems can become vastly more complicated very easily. For example, if shear stress is present on a beam, then there will be an additional deflection caused by shear alone. This is not accounted for in this project as it is beyond the scope of a 2nd year engineering undergraduate. If this project is repeated or expanded upon the additional deflection caused by shear stress should be accounted for. Rerepeating the project with more complicated loading scenarios such as non-linear distributed loads would be a way to bring more calculus into the problem as well.

NOMENCLATURE

kN: kilonewtons (1000 newtons force)

E: Young's Modulus of Elasticity constant

I: Area Moment of Inertia

EI: Flexural rigidity Constant - product of area moment of inertia and elastic modulus.

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