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Moment of Inertia in Applied Calculus

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Abstract

This paper demonstrates the usefulness of calculus in structural/continuum mechanics. Calculus in structural/continuum mechanics is used to calculate mass, volume, centre of mass, moment of inertia and in solutions of differential equations. In this paper, we will use calculus to calculate moment of inertia. The area moment of inertia of a surface measures the resistance to deflection of the cross section to bending or buckling. Moment of inertia is used by engineers of inertia to determine the state of stress in a cross section and the amount of inertia. It represents a mathematical concept that depends on the size and shape of the cross section of the element. The cross section's bending axis is also the central axis, so the ability to maintain the centroid of the shape is closely related to the moment of inertia.

Keywords

mass, center of mass, moment of inertia, solid cylindrical disk

PROBLEM STATEMENT

Moment of inertia is important parameter in rotation dynamics and structural mechanics. In structural mechanics, it determines stiffness of the object. In rotation dynamics, it determines resistance to change in angular velocity. For continuous objects, moment of inertia calculation requires integration over region of interest.

In this paper, we use calculus to calculate Moment of Inertia of a cylindrical disk with a cylindrical rod passing through the centre and demonstrate its advantages in structural mechanics and rotation dynamics.

The solid cylindrical disk and rod has a constant mass density. Cylindrical disk has a radius of r_a and length L and cylindrical rod has radius r_b and length l. The disk is placed at a height of h from the base of rod. The mass, centre of mass and three principal moments of inertia are to be calculated by analytical methods.

1. MOTIVATION

Invention of calculus was a direct result consequence of studying velocity and accelerations of objects. Problems in physics prompted the development of analytical methods in the 16th century; the central issue in physics was the study of motion.

The expansion of trade and the ensuing Age of Exploration required improvements in navigation techniques, which depended heavily on the development of astronomy and thus the invention of calculus was a necessity [1].

Calculus consists mainly of two parts, differential calculus and integral calculus. Differential calculus deals with the calculation of instantaneous rate of changes [2]. While integral calculus deals with the calculation of the whole by the sum of an infinite number of small elements [3].

Integral calculus deals with magnitude and gross values such as length, area, and volume. The Riemann integral is the simplest definition of an integral commonly used in physics and mathematics [4]. Along with differentiation, integration is the fundamental object of calculus. Integral calculus deals with finding the sum of infinitesimal parts of the contents of a continuous region.

2. MATHEMATICAL DESCRIPTION

In structural/continuum mechanics, calculus is used to calculate mass, volume, center of mass, moment of inertia and in solutions of differential equations. In this paper, we use calculus to calculate moment of inertia. The governing equations of continuum or structural mechanics include mass conservation [5] and Cauchy momentum equation [5] which are given as,

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{u}) = 0 \tag{2.1}$$

$$\frac{\partial(\rho \mathbf{u})}{\partial t} + \nabla.(\rho \mathbf{u} \otimes \mathbf{u}) = \nabla.\boldsymbol{\sigma} + \rho \mathbf{f} \tag{2.2}$$

The dynamics of continuous bodies require the use of integration for calculation of number of important system parameter one of which is moment of inertia which is defined as [6], (2.3)

$$I_{ij} = \int\int\int_V \rho(\mathbf{r})(r^2\delta_{ij} - x_i x_j)dV$$

where I_{ij} is the i_{th} and j_{th} components of moment of inertia of the body.

The area moment of inertia of a surface measures the resistance to deflection of the cross section to bending or buckling. The bending axis of the cross section is also the central axis, so the ability to maintain the centroid of the shape is closely related to the moment of inertia.

The calculus is used in structural mechanics to determine the forces in complex geometries and shapes of structural elements. Calculus is also required for structural calculations related to seismic design. It is also used to determine lateral soil pressures and slope stability in complex situations.

The beam deflection formula is a variation of the general formula that allows for variations, and can be adapted for loads and multiple beam segments. The more complex equations are the more difficult calculation is. Simplifying equation can save a lot of time and effort. The beam bending is governed by Euler-Bernoulli equation [7] which is given as (2.4)

$$EI\frac{d^2w}{dx^2} = M$$

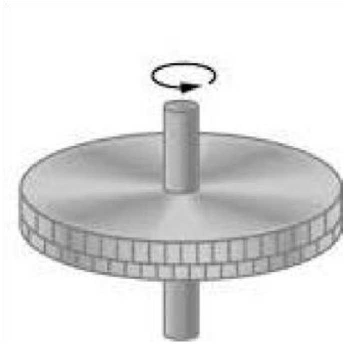
where I is area moment of inertia of cross section, E is Young’s modulus and M is applied moment on the beam.

The above equation can be used under boundary conditions to obtain solutions for buckling. The Euler-Bernoulli equation is not the most accurate equation for beams. More accurate equations are available but they are more complex to solve. The analytical and numerical methods have to be applied to yield solutions for the above equations.

3. CALCULATIONS / SOLUTION APPROACH

The solid cylindrical disk and rod have a constant mass density. Cylindrical disk has a radius of r_a and length L and cylindrical rod has radius r_b and length l. The disk is placed at a height of h from the base of rod. The mass is defined as

$$m = \int_V \rho dV$$



(3.1)

Calculating mass is important as it allows us to write moment of inertia in terms of mass as compared to in other parameters such as radius and lengths. The integral can be expanded to

$$m = 2\pi\left(\int_0^L \int_{r_b}^{r_a} \rho r dr dz + \int_0^l \int_0^{r_b} \rho r dr dz\right) \quad (3.2)$$

and after evaluating the integral we end up with

$$m = \rho\pi\left(r_b^2 l + (r_a^2 - r_b^2)L\right) \quad (3.3)$$

The center of mass is important as it greatly simplifies the evaluation of the moment of inertia about center of mass. The center of mass is defined by the following equation.

$$y_{cm} = \frac{1}{m}\int_V \rho y dV \tag{3.4}$$

The *y* component of center of mass is given as

$$y_{cm} = 2\pi(\int_0^L \int_{r_b}^{r_a} \rho r y dr dz + \int_0^l \int_0^{r_b} \rho r y dr dz) \tag{3.5}$$

which can be simplified to obtain the following expression for vertical component of center of mass

$$y_{cm} = \frac{\rho\pi}{m_g}(\frac{r_b^2 l^2}{2} + (r_a^2 - r_b^2)L(h + \frac{L}{2})) \tag{3.6}$$

The moment of inertia about first principal axis is defined as

$$I_{p1} = \int_V \rho(y^2 + z^2)dV \tag{3.7}$$

which can be expanded to following equation after putting the limits of inertia

$$I_{p1} = \int_0^h \int_0^{r_b} \rho\pi(r^2 + 2z^2)rdrdz + \int_h^{h+L} \int_0^{r_a} \rho\pi(r^2 + 2z^2)rdrdz + \int_{h+L}^l \int_0^{r_b} \rho\pi(r^2 + 2z^2)rdrdz$$

which can be solved to obtain the following expression for the moment of inertia

$$I_{p1} = \rho\pi(\frac{hr_b^4}{4} + \frac{h^3 r_b^2}{3} + \frac{Lr_a^4}{4} + \frac{r_a^2}{3}((h+L)^3 - h^3) + \frac{r_b^4}{4}(l - L - h) + \frac{r_b^2}{3}(l^3 - (h+L)^3))$$

The moment of inertia about second principal axis is defined as

$$I_{p2} = \int_V \rho(x^2 + z^2)dV \tag{3.8}$$

which can be expanded to following equation after putting the limits of inertia

$$I_{p2} = \int_0^h \int_0^{r_b} \rho\pi(r^2 + 2z^2)rdrdz + \int_h^{h+L} \int_0^{r_a} \rho\pi(r^2 + 2z^2)rdrdz + \int_{h+L}^l \int_0^{r_b} \rho\pi(r^2 + 2z^2)rdrdz$$

which can be solved to obtain the following expression for the moment of inertia

$$I_{p2} = \rho\pi(\frac{hr_b^4}{4} + \frac{h^3 r_b^2}{3} + \frac{Lr_a^4}{4} + \frac{r_a^2}{3}((h+L)^3 - h^3) + \frac{r_b^4}{4}(l - L - h) + \frac{r_b^2}{3}(l^3 - (h+L)^3))$$

From the above expressions we can see that moments of inertia about both first and second principal axis are same. This is due to symmetry of the problem. The first and second principal moment of inertia about center of mass can then be evaluated using parallel-axis theorem.

$$I_{p1} = I_{p2} = I_{p1} - m_g y_{cm}^2 \quad (3.9)$$

In similar manner the principal moment of inertia about third axis is defined as

$$I_{p3} = \int_V \rho(x^2 + y^2)dV \quad (3.10)$$

which can be expanded to obtain following expression

$$I_{p3} = 2\pi(\int_0^L \int_{r_b}^{r_a} \rho r^3 dr dz + \int_0^l \int_0^{r_b} \rho r^3 dr dz) \quad (3.11)$$

which can be solved to obtain the following expression for the moment of inertia

$$I_{p3} = \frac{1}{2}\rho\pi(r_a^4 L + r_b^4(l - L)) \quad (3.12)$$

Moment of inertia about three principal moment for cylindrical disk-rod system is evaluated. This gives us insight into the extent of resistance of the system to bending and buckling. Also it is important to note that the mass, center of mass and moment of inertia are volume integrals. The integrations are done in cylindrical coordinate system as the problem has cylindrical structure. Thus integration is easier in cylindrical coordinate system than Cartesian coordinate system. Also from the above expressions for moment of inertia it is apparent that the expressions become large and calculations become tedious really quickly which makes it necessary to employ a numerical technique for more complex geometries, shapes and structures [8].

4. DISCUSSION

We calculated the mass of the cylindrical disk, as it is a continuous object and thus a volume integral has to be evaluated to obtain mass. To perform integration the density is assumed to be constant. The volume integral is evaluated over the region of interest.

The center of mass is also evaluated in a similar way to that of mass. Three principal moments of inertia are evaluated by definition. The two of them are same because of symmetry of the problem. To ease the problem I_{p1} and I_{p2} are evaluated first at base of cylindrical rod and then at center of mass using parallel axis theorem. The final principal moment of inertia is evaluated by definition of moment of inertia. All the integrations are performed in cylindrical coordinate system.

Moment of inertia is important as it gives the measure of resistance to deflection of the cross section to bending or buckling. Details of buckling and bending can be obtained by the use of Euler-Bernoulli equation. For more complex geometries integration becomes harder and harder so use of numerical techniques to compute integrals is useful and necessary which will allow deeper understanding of bending and buckling behaviour.

5. CONCLUSION AND RECOMMENDATIONS

In this paper, we solved for mass, center of mass and moment of inertia of solid cylindrical disk. The technique of integral calculus is applied. We discussed applications of calculus in structural/continuum mechanics and analysed the behaviour of objects under stress.

As recommendation, the solutions of bending and buckling can be obtained for better understanding of behaviour of object/structures under stress. Also for more complex geometries the numerical schemes can be employed to obtain accurate solutions.

NOMENCLATURE

Following symbols are used in the text to mean following quantities:

- P density kg/m^3
- **u** Element Velocity m/s
- σ Stress Tensor N/m^2
- **f** Body force per unit mass m/s^2

- I Moment of inertia kgm^2
- E Young's modulus N/m^2
- M Applied Moment *Nm*
- m mass *kg*
- V Volume m^3
- r_a Radius of disk *m*
- r_b Radius of rod *m*
- *L* Length/width of disk *m*
- *l* Length/width of rod *m*
- *h* height of disk from base of rod *m*

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