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Fundamental Transversals on the Complexes of Polyhedra

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Fundamental Transversals on the Complexes of Polyhedra

by

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DEDICATION

To my favorite Basset Hound Bernie, and my Parents.

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ABSTRACT

We present a formal description of 'Face Fundamental Transversals' on the faces of the Complexes of polyhedra (meaning three-dimensional polytopes). A *Complex* of a polyhedron is the collection of the vertex points of the polyhedron, line segment edges and polygonal faces of the polyhedron. We will prove that for the faces of any 3-dimensional complex of a polyhedron under face adjacency relations, that a 'Face Fundamental Transversal' exists, and it is a union of the connected orbits of faces that are intersected exactly once. While exploring the problem of finding a face fundamental transversal, we have found a partial result for edges that are incident to faces in a face fundamental transversal. Therefore we will present this partial result, as The Edge Transversal Proposition 1. We will also discuss a few conjectures that arose out this proposition. In order to reach our approaches we will first discuss some history of polyhedra, group theory, and incorporate a little crystallography, as this will appeal to various audiences.

1 INTRODUCTION

Polyhedra have been an interest of humans for over 4,000 years, and it is a fast growing research field. Right now, you may be asking yourself, "Why is studying polyhedra important?" Studying polyhedra is of great importance, whether it is for the admiration of their aesthetical beauty, the understanding of their structures, or the knowledge to educate others and ourselves to think more abstractly. Certainly most polyhedra are aesthetically beautiful, and some people have studied polyhedra just for this reason alone. For example, some people collect polyhedral figures because they are enthralled with the beauty of the shapes. The importance of understanding their structures is that it can be used for designing bridges and buildings that stay up i.e, polyhedral stability [16].

From the aesthetical point of view of polyhedra, symmetry is naturally a link that is related to it. This is because the most commonly known polyhedra are proportionally balanced, and this reflects their beauty. While observing polyhedral objects that are symmetrically beautiful, humans may be curious to know why and how the object became what it is. Symmetry is a tool or a guide when searching or discovering new kinds of polyhedra [16]. For example, Archimedes most likely used symmetrical variations when he discovered his solids [16]. This can be understood by the fact that eleven of his solids can be obtained from the Platonic solids by truncating symmetrically [16]. We study the symmetries of polyhedra to help us understand their structures. In order to study the symmetries of polyhedra, we should first go through some of the background information that is needed.

This thesis is about 3-dimensional polyhedra. In two dimensions, a $polygon's$ boundary consists of finitely many line segments, which are it's *edges*.

Figure 1.1: An Edge of a Polygon

In three dimensions it is a *polyhedron* denoted as P, which is a region, whose boundary consists of finitely many polygons, which are the $faces$ of the polyhedron.

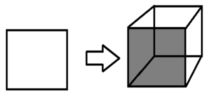

Figure 1.2: A Face of a Polyhedron.

When two faces of the polyhedron meet, the shared boundary of these two faces forms an *edge* of a polyhedron.

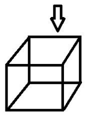

Figure 1.3: An Edge of a Polyhedron

A point of intersection of three or more edges or faces of a polyhedron is known as a *vertex* [9].

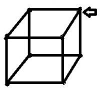

Figure 1.4: A Vertex of a Polyhedron

In Part I of this thesis, we will present a brief historical time line of polyhedra, starting with the first archaeological evidence of it dating 4,000 years ago, up to present day polyhedra, including plenty of pictures. Within the time line, we will include other aspects such as group theory, and a little crystallography. For instance, using group theory and geometry, we find that isometries of polyhedra provide a way to look at the transformations of polyhedra. In this thesis we will concentrate on isometries. An *isometry* is a point to point transformation of the entire space onto itself that preserves distances. This leads us into further discussion of how symmetries became a way to study polyhedra [3].

We will be using the word *symmetry* in a nonstandard way in this thesis. We will say that a *symmetry* on a polyhedron is an isometry that maps the polyhedron to itself, and can include transformations like rotations, reflections, and inversions. A *symmetry* is an isometry on a polyhedron which leaves the polyhedron invariant. A polyhedron P is the maximal cell in the complex of P, so the symmetry group is denoted as $Sym(P)$. $Sym(P)$ induces permutations of the cells of the complex of P, and this permutation group will be denoted as $\mathbf{G}$. Notice that $\mathbf{G}$ is isomorphic to $Sym(P)$, and in the theoretical section, we will only discuss $\mathbf{G}$.

The orbit of a face F in $\mathcal{F}$ [1] is the set of elements of $\mathcal{F}$ to which F can be moved by the elements of $\mathbf{G}$. The orbit of F under $\mathbf{G}$ is denoted by $\mathbf{G}(F) = \{g(F) | g \in \mathbf{G}\}$. This definition is also applied to the edges also with a slight difference. Where we let $\mathcal{E}$ be the set of all edges with coradjacency relations of the complex of P.

[1] *$\mathcal{F}$ is the set of all faces with adjacency relations of the complex of P.*

A coradjacency relation occurs when two edges share a common face in the face fundamental transversal. These are generic representations of the orbits of faces, edges, as there may be ones that are labeled differently. We will see this later on in the thesis. While using geometric group theoretic methods as a group acts transitively on a polyhedron, the orbits of these cells mentioned above are invariant subsets of the polyhedron.

In Part II of this thesis we will explore a generalization of [5]'s concept of a 'fundamental transversal' on the faces of a complex of P. Where mentioned in Proposition 1.2.6 [5], they describe a fundamental transversal as a **G** - transversal in a set X is a subset S, which meets or intersects each orbit of vertices and edges of a connected graph exactly once and there is a bijective mapping from this subset which is contained in X to the quotient X/ G. Generalizing [5]'s concept of a transversal, we will use their concept in regard to a connected complex of a Polyhedron. The only part that we don't consider is the quotient space, but there is a bijective mapping from the subset contained in the complex to the complex itself.

The *CW- Complex* was introduced by J.H.C Whitehead. In this thesis we will use a variant concept of this and apply it to what we call *complexes*. A *complex* is defined as the vertex points of the polyhedron, line segment edges and polygonal faces. It may be obtained by defining the *k-skeleton*,[2] inductively as follows. We start with the 0-dimensional cells (vertices), then add the 1-dimensional cells (edges), and then the 2-dimensional cells (faces).

The following is an example of how a *complex* of a Cube is obtained. We begin with the 0-dimensional cells, which are the vertices.

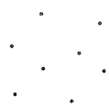

Figure 1.5: The 0-dimensional *k-skeleton* of the Cube, denoted as $\text{skel}_0(Cube)$.

Next we start to add the 1-dimensional cells (edges) to the 0-dimensional cells.

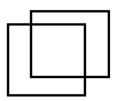

Figure 1.6: Starting to add the 1-dimensional cells (edges) that are incident to the 0-dimensional cells, denoted as $\text{skel}_1(Cube)$.

Notice in the next figure, that we are adding the edges that connect all of the vertices.

[2] *In geometry, a k*-skeleton *of polyhedron P (represented as* $skel_k(P)$*) consists of all polyhedral cells in its boundary of dimension up to k [17].*

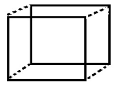

Figure 1.7: Adding the rest of the 1-dimensional cells (edges) that are incident to the 0-dimensional cells.

Now we have all of the edges connected to the vertices.

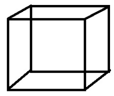

Figure 1.8: The $\text{skel}_1(Cube)$

Last we attach the 2-dimensional cells (faces) to the 1-dimensional cells, and we obtain the faces of the Cube.

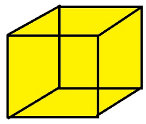

Figure 1.9: The $\text{skel}_2(Cube)$ and *complex* of a Cube, where the faces are colored yellow.

In this thesis following Dicks and Dunwoody's [5] proposition, [3] we will call a connected sub-complex intersecting each orbit exactly once a 'fundamental transversal'. We will have several types of fun-

[3] *If G is a graph, where G = <V,E>, and G is connected, then there exists a subset S (of edges and vertices) contained in G such that S is a transversal in G and S induces a connected subgraph of G that intersects each orbit of vertices and edges exactly once.*

damental transversals, such as 'face fundamental transversals' on the faces of *complexes* of polyhedra. We have also found a partial result of an edge fundamental transversal that is incident to faces in the face fundamental transversal and is connected under coradjacency [4] relations denoted as CRH.

Lastly, we will conclude the thesis with a chapter on the concept of a 'fundamental transversal domain', where a fundamental transversal domain is a set, such that every orbit of a cell of the complex of P is intersected exactly once, and induces a connected subgraph of the complex of P. We will present two conjectures, where in the second conjecture the idea of a 'flag' is used in a slightly different manner. In the last section we will show some crystal net images courtesy of Micheal O'Keefe, and their fundamental transversal domains.

[4] *A Coradjacency relation occurs when two edges share a common face in the face fundamental transversal.*

PART I POLYHEDRA AND PICTURES

2 HISTORY AND BACKGROUND

Geometry is derived from the Greek word "geometria", which means "to measure the earth." Human beings have been using the concepts of geometry since civilization began. The earliest known polyhedra are carved stone spheres that are approximately three inches in diameter dating back to 2000 BC, which have been found in Scotland. Some are carved with lines corresponding to the edges of regular polyhedra [6]. Half of them have 6 knobs, while others range from 3 to 160 knobs.

Figure 2.1: The Neolithic Polyhedra, The Five Regular Solids. Image Courtesy of George W. Hart

None of the solids in the figure above have 12 bumps, so none of them can be a dodecahedron [6]. Well over 400 of these stone balls are known, and their material varies from easily carved sandstone and serpentine to difficult hard granite and quartzite [2]. We don't know what their function was, but they may have been symbols of authority, or for use in fortune telling [2].

Near the year 2900 BC, the first Egyptian Pyramid may have been constructed. Knowledge of geometry was very important for the Egyptians to build pyramids, which then consisted of a square base and triangular faces [12]. Today, what we know from the Egyptian geometry came from two sources, which are the *Rhind Mathematical Papyrus*, and the *Moscow Mathematical Papyrus* [9]. The first known milestone in polyhedra's history is the "Volume of the Truncated Pyramid", which came from the *Moscow Mathematical Papyrus*. The second milestone in polyhedra's history is the "Volume of a Pyramid". Because of length and time, we will only mention the two Egyptian milestones, without investigating their full details.

Before the Greeks knew of polyhedra, the Etruscans knew of some of the regular polyhedra, which can be defined as polyhedra whose faces are congruent regular polygons which are assembled in the same way around each vertex [1]. Let us turn to the time of Greek Antiquity, where the regular polyhedra had a considerable influence on the Greeks. The Greeks were great geometers, inventors, and discoverers of the most popular known polyhedra. For instance, the famous Pythagoras of Samos who was, according to legend, the inventor of the regular dodecahedron. The next Greek mathematician Theaetetus of Athena (415 - 369 BC), discovered the regular octahedron and icosahedron, also some argue that he was the first to construct the five regular polyhedra [13]. Theaetetus, may have looked at the collection of solids not as isolated objects, but as seeing them as part of a theory [9]. One very important Greek geometer, Euclid, is sometimes known as "the father of modern geometry," because of his book *The Elements*, in which he described the regular polyhedra, and he proved that there can't exist more than five of these solids [13].

Around 350 BC the Greek philosopher Plato wrote *Timaeus*, which presents the elements earth, fire, air, and water as a mathematical construction in which the cube, tetrahedron, octahedron, and icosahedron are represented as the shapes of the atoms of the elements [1]. The last Platonic solid, otherwise known as the dodecahedron, was considered Plato's model for the whole universe [1].

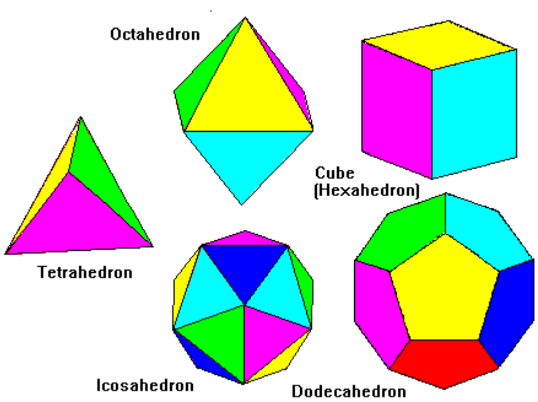

Figure 2.2: The Five Platonic Solids. Image Courtesy of Steve Dutch.

Archimedes of Syracuse (about 287 - 212 BC) was a Greek mathematician who was responsible for the geometric solids that are named after him, the Archimedean Solids. A Greek mathematician, Pappus of Alexandria, wrote a book called *Synagoge* or *Collection*, which has the first known mention of the thirteen Archimedean Solids, in which Pappus attributes them to Archimedes. The Archimedean solids are described in Pappus's narration,

> "Although many solid figures having all kinds of surfaces can be conceived, those which appear to be regulary formed are most deserving of attention. Those include not only the godlike Plato, that is the tetrahedron, cube, octahedron, icosahedron and fifthly the dodecahedron, but also the solids, thirteen in number, which were discovered by Archimedes and are contained by equilateral and equiangular, but not similar polygons [15]."

In today's description, an Archimedean solid is a highly symmetric, $semi-regular$ convex polyhedron, i.e, it's faces are all regular polygons and the vertices they meet are identical. They are distinct from the Platonic solids, which are composed of only one type of polygon meeting in identical vertices [15]. There are seven of the thirteen Archimedean solids that are derived from the Platonic solids by a process known as *truncation*. Truncation is an operation, where one cuts off the corners of the polyhedron, thus creating a new face in place of each vertex that was cut off. All of the solids with the name "truncated" in front of them have been derived from cutting the corners of the Platonic solids; there are five of them. The other two are the Snubs.

The thirteen Archimedean solids are as follows: Top Row: Truncated Cube, Cuboctahedron, Truncated Octahedron, Great Rhombicuboctahedron, Lesser Rhombicuboctahedron. Second Row: Truncated Dodecahedron, Icosidodecahedron, Truncated Icosahedron, Great Rhombicosidodecahedron, Lesser Rhombicosidodecahedron. Third Row: Snub Cube, Snub Dodecahedron, Truncated Tetrahedron.

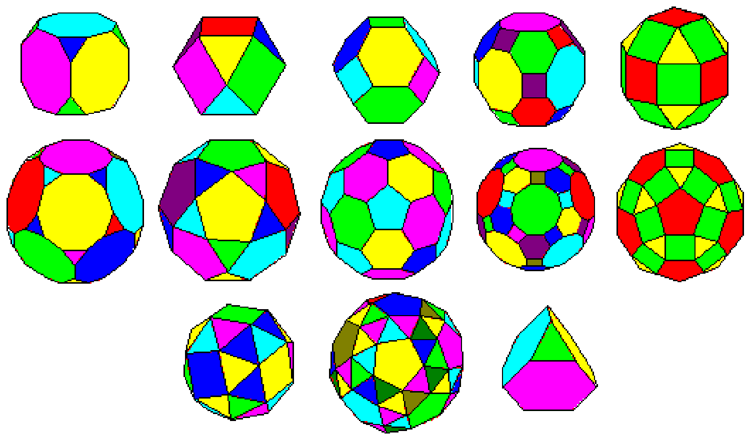

Figure 2.3: The Archimedean Solids. Image Courtesy of Steve Dutch.

There were different types and kinds of polyhedral objects that were made during the time that Archimedes lived, up through the time of Pappus [9]. The exact origins of the objects are not known. The solids may be dated from the Roman period [9].

There is no known account of polyhedra from the time of Pappus until the Renaissance. During the Renaissance, Albrecht Dürer (1471 - 1528), although most know him as a painter, invented the notion of the net of a polyhedron [9]. Commonly people think of nets, as being made by unfolding a planar piece of cardboard along lines and joining the edges of the figure, then the ‘net’ becomes a polyhedron [9].

The next mathematician we will discuss is Johannes Kepler (1571 - 1630). Kepler may have been the next person after Pappus to write about the Archimedean solids, in his book *Harmonices Mundi*. Kepler cleaned up Pappus’s loose definition of the solids and gave a proof that there are precisely thirteen of them (Book Two, *De Congruentia Figurarum Harmonicarum*, proposition XXXVII). He also provided the solids with their modern names [15]. Kepler also realized that star polygons could be used to build star polyhedra, which have pentagrams as faces.

Kepler discovered two star polyhedra, which were the small stellated dodecahedron and the great stellated dodecahedron, and whose regularity (pentagrammic faces) escaped him [13]. In 1619, he gave the first example of facetting, which is the process of removing parts of a polygon, polyhedron or polytope, without creating any new vertices. Moreover, any two vertices may be joined by a line. Typically, this line will be contained inside the polyhedron. If several such lines connect in a planar circuit, they form a complete polygon, or facet, inside the polyhedron. Again, several such facets may connect to form a complete polyhedron inside the original one. The original is the base polyhedron, and the new one is a facetting [7].

Kepler's first example of facetting was the stella octangula. He was also responsible for defining the prisms, anti-prisms, and the non-convex solids [7]. A non-convex solid is defined as a solid that has non-convex faces, meaning that the faces, which are polygons, are non-convex. A convex polygon is a polygon that if any line segment joining any two points on the polygon stays inside itself.

During the Contemporary Age (1789 to the present), there were a few mathematicians who made significant contributions to studying both polyhedra and group theory. We will now intertwine both the history of polyhedra and group theory. The first mathematician whom we will discuss is Louis Poinsot (1777 - 1859), who wrote an important work on polyhedra in 1809 entitled *Polygons and Polyhedra*. Poinsot discovered four new non-Platonic regular polyhedra,two of which appear in Kepler's work of 1619, but Poinsot was unaware of this. He also discovered the non-convex regular polyhedra, where the last two are duals of Kepler's polyhedra [15]. The following two figures are the Kepler-Poinsot Polyhedra.

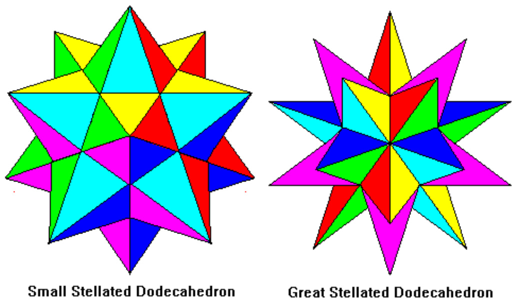

Figure 2.4: The Kepler-Poinsot Solids Part 1. Image Courtesy of Steve Dutch.

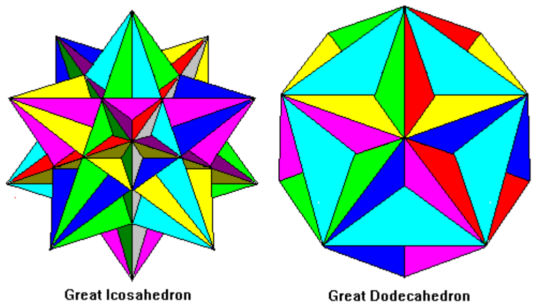

Figure 2.5: The Kepler-Poinsot Solids Part 2. Image Courtesy of Steve Dutch.

Moving further ahead, we will now discuss some of the history of group theory and how some mathematicians used group theory to explore polyhedra and the birth of crystallography. There are three known historical roots of group theory and they are: the theory of algebraic equations, number theory, and lastly geometry. There were three mathematicians who are considered the early researchers in group theory. They were Joseph-Louis Lagrange (January 25, 1736 - April 10, 1813), Niels Henrik Abel (August 5, 1802 - April 6, 1829), and Evariste Galois (October 25, 1811 - May 31, 1832). Abel and Galois's work in early group theory was more directed towards solving polynomials of degree four or higher.

In the late 18th century, the earliest study of groups possibly began with the work of an Italian-born mathematician named Joseph-Louis Lagrange (January 25, 1736 - April 10, 1813). Lagrange's work may have been somewhat isolated, so then publications by Galois and another mathematician Baron Augustin-Louis Cauchy (1789 - 1857), in 1846, are probably referred to as the beginning of group theory. Which brings us to the next mathematician we will discuss. Baron Augustin-Louis Cauchy (1789 - 1857), a French mathematician, defined the concept of regularity for a polyhedron in terms of the equivalence of its faces and edges. He also proved there exist only four non-convex regular polyhedra (Kepler-Poinsot) and there are only nine regular polyhedra[1] [9].

In 1849 Auguste Bravais (1811 - 1863), was a French naval officer and scientist, published his *Memoire sur les polyedres de forme symetrique*, which was very close to Cauchy's work [8]. In this work he represented a Polyhedron by a finite set of points in space (its vertices), and he defined the symmetry elements of the rotation axis, reflection plane and center of inversion in terms of the spatial arrangement of the points, and he derived all possible combinations of these elements [8]. Cauchy, who was one of the early group theorists, presented Bravais's work to the French Academy Of Sciences, but he did not mention that the analogy between combinations of sym-

[1] *The nine regular polyhedra are the five Platonic Solids and the four Kepler-Poinsot Polyhedra.*

metries and the groups of substitutions that were being studied [8]. It was another French mathematician, Camille Jordan (1838-1922), who was aware of this and used Bravais' work for his fundamental *Memoire sur les groupes de mouvements* [8].

According to [8],

> "In his paper Jordan considered both continuous and discrete groups of 'proper' motions (abstractly, rather than in terms of their action on sets of points) by examining the possible combinations of rotations, screw rotations and translations."

Jordan's work was the basic foundation for the later derivation of the space groups, where a space group is a group of isometries on a Euclidean space.

With the use of groups in geometry, that is the underlying group of symmetries, an influential program known as the "Erlangen Program", was introduced by Felix Klein (April 25, 1849 - June 22, 1925) a German mathematician. Klein was at Erlangen at the time the Erlangen Program was developed. In 1872 the Erlangen Program was introduced; the program is a way to classify geometries by their underlying symmetry groups. The Erlangen Program includes symmetry groups of symmetries besides isometries. It had a huge impact and influence on the mathematics at that time. For instance, Klein suggested to the German mathematician Arthur Morton Schoenflies (1853-1928), that by studying transformation groups, he could extend Jordan's work by adding improper motions to the discrete groups of proper motions [8]. Schoenflies also proved that a 'group of motions in three-dimensional space, which maps a regular system of points onto itself necessarily contains a subgroup generated by three independent translations [8].'

E.S. Fedorov (1853-1919), a mineralogist and crystallographer, was responsible for the completion of the enumeration of groups of motions in three dimensions [8]. Fedorov was responsible for deriving the space groups *ab initio*, where he had adapted a form of analytic geometry to the study of regular systems of points [8]. Fedorov's results helped to lay the theoretical foundations for modern crystallography. Fedorov wrote a two-part paper, *Symmetry of Regular Systems of Figures*, which was published in 1891. In this paper he proved that there are exactly 17 distinct wallpaper groups, where a wallpaper group is a two-dimensional symmetry group [4].

The wallpaper groups could also be described as the Euclidean plane isometries. A Euclidean plane isometry is obtained in a manner of transforming the plane so that it preserves distance. Now there are four different types of Euclidean plane isometries such as: translations, rotations, reflections, and glide reflections. The set of Euclidean plane isometries form a group under composition, such as, the Euclidean group in two dimensions, which is generated by reflections so that every element of the Euclidean group can be composed of at most three distinct reflections.

In this thesis, we are mainly sticking to three-dimensional polyhedra, and as we are reading along, we can see that the derivation of the wallpaper groups also helped to classify the space groups much better. Space groups are the symmetry groups that are relevant in a crystal lattice [2] along with a translation element. There exists 230 space groups in three-dimensions, with 11 pairs of mirror images.

The German mathematician, Ludwig Georg Elias Moses Bieberbach (December 4, 1886 - September 1, 1982) proved that every three-dimensional space group that fills space (i.e., that admits a finite fundamental region) contains a subgroup generated by three independent translations (where every translation of that finite fundamental region intersects the orbit of the origin) [8]. His theorem generalized the modern approach to space groups in any dimension [8].

Since a space group is a group of isometries on a Euclidean space, we can easily see that dealing with our three-dimensional polyhedra in this thesis there is a connection between them. What is the connection? The connection is that a symmetry group of a polyhedron is

[2] *A simple three-dimensional network of three sets of evenly spaced parallel lines whose points of intersection are called the crystal lattice, or space lattice [17].*

a point group, where a point group [3] is a group of isometries leaving a point fixed. In this thesis we are dealing with the three-dimensional point groups.

We will resume our polyhedra historical time line. In 1858, Joseph Bertrand (March 11, 1822 - April 5, 1900), a French mathematician, derived the regular star polyhedra by facetting the icosahedron and dodecahedron [7]. Bertrand used the term *etoile*, which means starry or stellated, to give the star polyhedra their names. One year later in 1859, Arthur Cayley translated *etoile* to stellated, to give the Kepler-Poinsot polyhedra the names by which they are generally known today [7].

The Catalan Solids are named after Eugene Charles Catalan (May 30, 1814 - February 14, 1894), a French and Belgian mathematician. The Catalan solids were first described in 1865. They were all convex and are face transitive (lie in the same symmetry orbit), but not vertex transitive (there is only one orbit of vertices). Their duals which are the Archimedean solids, are vertex transitive but not face transitive. More formally, a dual of a polyhedron P being the polyhedron resulting from choosing a vertex in each face of P and using the vertices to define the dual polyhedron. The faces of Catalan solids are not regular polygons, whereas the Platonic and Archimedean solid faces are.

[3] *The meaning of a point group varies throughout literature, and in this thesis we will refer to a point group as defined above.*

The following solids are the Archimedean solids with their corresponding dual Catalan solids.

- Archimedean Solids: cuboctahedron, great rhombicosidodecahedron, great rhombicuboctahedron, icosidodecahedron, small rhombicosidodecahedron, small rhombicuboctahedron, snub cube (laevo), snub dodecahedron (laevo), truncated cube, truncated dodecahedron, truncated icosahedron, truncated octahedron, truncated tetrahedron.

- Catalan Dual Solids: rhombic dodecahedron, disdyakis triacontahedron, disdyakis dodecahedron, rhombic triacontahedron, deltoidal hexecontahedron, deltoidal icositetrahedron, pentagonal icositetrahedron (dextro), pentagonal hexecontahedron (dextro), small triakis octahedron, triakis icosahedron, pentakis dodecahedron, tetrakis hexahedron, triakis tetrahedron

Ludwig Schläfli (January 15, 1814-1895), a Swiss geometer, was one of the key figures in developing the notion of higher dimensional spaces. He wrote $Theorie\ der\ vielfachen\ Kontinuitat$, in which he defined what he called 'polyschemes', which are now called polytopes. Polytopes are basically higher dimensional analogues to polygons and polyhedra. Schläfl developed their theory, and also the higher dimensional version of Euler's formula. He described the regular polytopes, and found that there are six in dimension four and three in all the higher dimensions. He invented a combinatory notation for polyhedra and discovered polytopes, although the name polytopes was given by a woman named Alicia Boole Scott (June 8, 1860 - December 17, 1940). Victor Schlegel (1843-1905), a German mathematician invented what is known as a Schlegel diagram. A Schlegel diagram is a projection of a polyhedron on a plane, and is a visual assistance to comprehend the connectivity of the edges. For example, the following figure represents what a Schlegal Diagram of a Square Pyramid looks like.

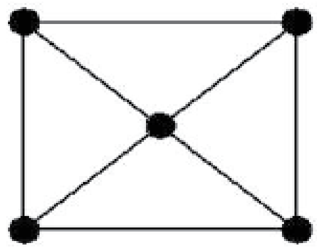

Figure 2.6: The 5-Wheel Graph, the projection of the Square Pyramid on a Plane.

Ernst Steinitz (June 13, 1871 - September 29, 1928) was a German mathematician who in 1916 developed a combinatorial characterization of convex three-dimensional polyhedra [18]. He is responsible for Steinitz's Theorem, which was published as a 1934 book, *Vorlesungen uber die Theorie der Polyeder unter Einschluss der Elemente der Topologie*, by Hans Rademacher. Steinitz's theorem for polyhedra states that every convex polyhedron forms a 3-

connected planar graph, and every 3-connected planar graph can be represented as the graph of a convex polyhedron [18]. This theorem is a very important result for 3-polytopes. In geometry, a *k-skeleton* of a polyhedron P (represented as $\text{skel}_k(P)$) consists of all polyhedral cells in its boundary of dimension up to k [17]. Thus the 1-skeletons of convex polyhedra are exactly the 3-connected planar graphs.

Let us turn quickly to mention a mathematician whose notion of a *CW-Complex* follows from an attempt to generalize the notion of a simplicial complex. J.H.C Whitehead (November 11, 1904 - May 8, 1904), a British mathematician, was one of the founders of homotopy theory. Whitehead used his concept of a CW-Complex, which is defined as a type of topological space, to meet the needs of homotopy theory. We will not discuss homotopy theory here, yet we will use his idea of a CW-Complex in the generic sense of a structure of one. There are some geometers who have picked up on this notion of a *complex*, and have adapted it.

In this thesis our definition of a complex is a variant of the geometer's definition. The *k-skeleton* of a complex is the union of the cells whose dimension is at most k.[4] A complex of a polyhedron, denoted as $C(P)$, is a poset, which consists of the vertices, edges, faces, and interior of P, ordered by inclusion. A complex can be

[4] *Notice the slight difference in the definition of a k-skeleton mentioned on the previous page versus here. Essentially they are similar, but here we are discussing the k-skeleton of a complex of P.*

found by defining the *k-skeleton* inductively. We start with the 0-dimensional cells (vertices), then add the 1-dimensional cells (edges), and then the 2-dimensional cells (faces). Hence *complexes* provide a rather practical use and representation of the building blocks (cells) of polytopes.

We continue on with our polyhedra history timeline. Patrick du Val (1903 -1987) was a British mathematician who invented a notation for the stellations of a polytope [13] where, according to Coxeter, "in order to stellate a polytope, we have to extend its faces symmetrically until they again form a polyhedron. To investigate all possibilities, we consider the set of lines in which the plane of a particular face would be cut by all the other faces (sufficiently extended), and try to select regular polygons bounded by sets of these lines." Mathematician Jeffery C.P. Miller established the rules which define a stellation. He along with H.S.M. Coxeter discovered the twelve last non-convex semi-regular polytopes [13].

Which brings us to a mathematician who made a major contribution to the study of polytopes, H.S.M Coxeter. Coxeter (1907-2003), is was one of the foremost mathematicians of his generation, he is noted for his study of polytopes, which inspired the drawings of M.C. Escher and influenced the architecture of R. Buckminster Fuller [14]. He was best known for his work on regular polytopes and higher-dimensional geometries. *Regular Polytopes* was originally written in 1947, but it was updated and then republished in 1963 and in 1973. In *Regular Polytopes*, Coxeter also named the nine regular polytopes as the five platonic solids and the four Kepler-Poinsot polytopes. Coxeter made important contributions to non-Euclidean geometry, discrete groups, and combinatorial theory [14]. Coxeter also combined some algebra with geometric techniques in what is know as 'polyhedral group theory', that also led to many results that are in several branches of mathematics as well [9].

After H.S.M. Coxeter, there were a few mathematicians who made contributions to studying polytopes. One of these mathematicians was Norman W. Johnson, who was a student of Coxeter's. Johnson described and cataloged the Johnson solids in 1966. There are ninety-two Johnson solids, which are any convex polytopes with regular faces that is not a platonic, Archimedean solid, or a prism or anti-prisms [3]. In other words, there is no requirement that each of the faces is the same polygon, or that they meet at a common vertex. Since there are 92 of these solids, we have an image of all of them on the next page.

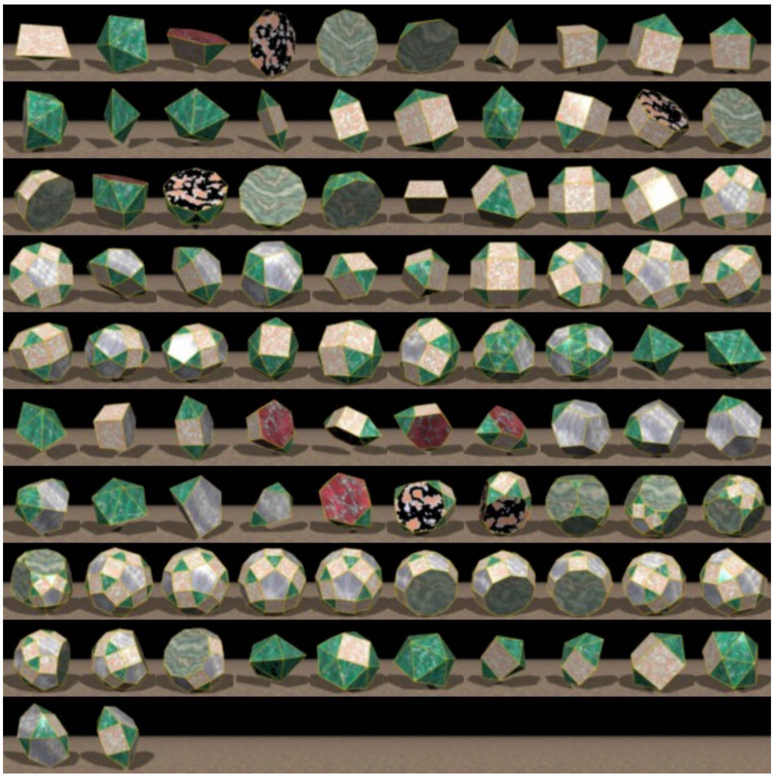

Figure 2.7: The 92 Johnson Solids. Image Courtesy of Vladimir Bulatov.

We will now discuss a person who has made considerable contributions to polyhedra as well as other areas related to it. Branko Grünbaum (born 1929) is a Croatian-born mathematician. He brought the work of Steinitz back to life in 1962 after he realized that he could render Steinitz's work in graph theory. He made it possible to use combinatorial theory on all of the three-dimensional polytopes of the plane [9]. In 1967 he published *Convex Polytopes*, which is a long account of the combinatorial theory of polytopes [9]. In 1977 he published an article *Regular Polyhedra – Old and New*, where he extended the work of Coxeter. In the article he presented a very general way to examine regular polyhedra [9].

Lastly we will conclude this historical timeline with an area of mathematics known as Geometric Group Theory, that evolved from group theoretic concepts and the study of geometric objects. These geometric objects in this thesis will be polyhedra, more specifically finite polyhedra. For the last twenty years or more, Geometric Group Theory has become a new area of mathematics that studies the connections of both algebraic and geometric spaces in which finitely generated groups act on [17]. When groups act on polyhedra, there is an algebraic and geometric connection between the geometric objects and the algebraic groups.[5]

Two concepts that we have noticed, have come out of the area of Geometric Group Theory. These concepts are at the foundation of the ideas that we have expanded upon. The first concept is discussed in Warren Dicks and M.J. Dunwoody's 1989 book *Groups Acting on Graphs*. They discuss combinatorial graphs or two-dimensional graphs in their book, but their nomenclature is a little different than the nomenclature in this thesis. We will not use Dicks and Dunwoody's nomenclature, but describe it as it pertains to our expanded ideas. Dicks and Dunwoody's approach says that there is a connected set of the orbits of edges and vertices of G, such that it has a spanning tree, and intersects every orbit exactly once. They let these connected sets be called **G** transversals, where **G** is a group of symme-

[5] *Geometric Group Theory is very related to computational group theory, algebraic topology, and many more areas of mathematics.*

tries of a graph G. To further explain their approach, first there exists some symmetry of **G** that acts on G, then a **G**-transversal, induces a spanning tree of G and intersects each orbit exactly once. Naturally, this **G**-transversal is a subset of G. Dick's and Dunwoody's proposition 1.2.6 can be found in [5] for additional reading.

The next concept is from John Meier's *Groups, Graphs, and Trees* [11]. He proceeds similarly as Dicks and Dunwoody did, except he calls these connected sets of orbits 'fundamental domains'. Note, the use of the word fundamental domain, is not the usual definition that most mathematicians are commonly familiar with. Meier's approach is also a constructive one. He denotes his fundamental domain F, as a subset of a connected graph $\mathcal{G}$, where a group **G** acts on $\mathcal{G}$.[6] His approach is an inductive process, which is starting with a single vertex and $F_0 = \{single - vertex\}$.

Then keep adding distinct vertices to F until it is large enough, so that its image under the action of **G** covers the entire graph, so we get $F =$ union of all the F_n's. This just means that you start with a single vertex, when you find another distinct vertex is a different orbit, you then add that vertex, and the edges that are shared by these vertices are also part of F. You would keep adding distinct vertices and edges until you found a large enough subset F, that will cover the graph. Meier's method is very similar to that of Coxeter's

[6] *Here* **G** *is in isolation, meaning it's not acting on the underlying space. It is purely a group of permutations of cells of the complex that preserves the structure of the complex.*

for finding fundamental regions in a polytope.[7] The similarity is that both of these men, only use the midpoint of an edge, not the whole edge in part of their constructions.

[7] *In Coxeter's Regular Polytopes book.*

PART II FUNDAMENTAL TRANSVERSALS

3 FACE FUNDAMENTAL TRANSVERSALS

In this part of the thesis, using the polyhedra that we have shown and discussed, we will show our developed ideas that have come from the concepts of Dicks and Dunwoody [5], John Meier [11] [1], and use what Gregory L. McColm has done in [10]. In the chapter 3 we intend to show that a face fundamental transversal is an adjacency connected set of the orbits of faces of the $C(P)$ that intersects each orbit of a face exactly once. In chapter 4 we explore an extension of the face fundamental transversal, to incorporate edges. In chapter 5 we will discuss our concept of a fundamental transversal domain on the complexes of polyhedra. Also, we will discuss a few conjectures that have been developed. In chapter 5 the concept of a fundamental transversal domain will be applied to some crystal nets, where visual examples are provided.

[1] *The only concept from Meier we will be using is a similar inductive process.*

We now formally define the terms we used more colloquially in the previous sections.

Definition 3.0.1. *We define the d-dimensional polytopes as follows, for d=0,1,2,3. When d=0 it is a point or a* vertex*. When d=1, it is a line segment or an edge, whose boundary consists of two endpoints or vertices. When d=2, it is a* polygon*, which is a bounded region in a plane whose boundary consists of finitely many 1-polytopes, which are its edges. When d=3 it is a polyhedron P, which is a region whose boundary consists of finitely many polygons, which are the* faces *of the polyhedron.*

Definition 3.0.2. *Given a polytope P, the* complex *of P, denoted as $C(P)$, is the poset whose elements are P and the elements of complexes of polytopes whose union is the boundary of P, ordered by inclusion. These elements are called the* cells *of P.*

To show visually what a $C(P)$ looks like and how it is formed by the elements of complexes of polytopes whose union is the boundary of P, we have provided two figures on the next page that will clear up any confusion in our definitions. The polyhedron we will use for an example is the Cube.

The following two figures show how a complex of a Cube is the poset of the cells of the Cube.

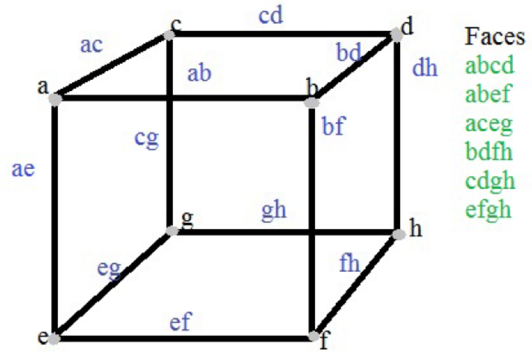

Figure 3.1: The vertices, edges, and faces of a Cube.

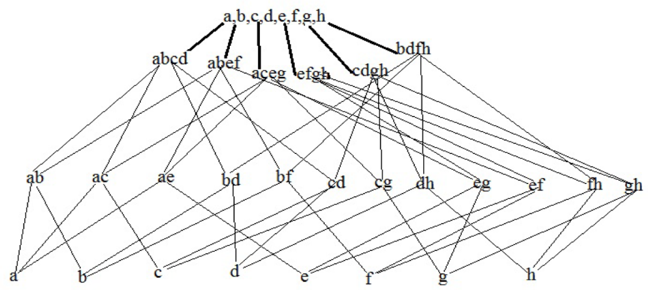

Figure 3.2: The Poset of the Cube.

Notice in the above figure that the vertices are connected to edges and the edges are connected to the faces. The figure above shows that a vertice is the boundary of an edge, and likewise an edge is a boundary of a face.

Definition 3.0.3. *An adjacency relation, denoted as H, is defined to be a relation between two adjacent cells of dimension* d*, when their intersection is a cell of dimension* $d-1$.

Definition 3.0.4. *A coadjacency relation, denoted as CH, is defined to be a relation between two cells of the same dimension* d*, when they both are subsets of the boundary of a cell of dimension* $d+1$.

Definition 3.0.5. *An automorphism is an isomorphism under adjacency and coadjacency relations, from $C(P)$ to itself. The group of symmetries of the underlying space of $C(P)$, whose restrictions are automorphisms, is denoted as $\mathbf{G}$, mapping the complex to itself while preserving its structure. Recall from the introduction section that we decided to have $Sym(P)$ be isomorphic to $\mathbf{G}$.*

Definition 3.0.6. *The orbit of a face F in $\mathcal{F}$, (where $\mathcal{F}$ is the set of all faces under adjacency relations of $C(P)$), is the set of elements of $\mathcal{F}$ to which F can be moved by the elements of $\mathbf{G}$. The orbit of F under $\mathbf{G}$ is denoted by $\mathbf{G}(F) = \{g(F) | g \in \mathbf{G}\}$.*[2]

Example 3.0.1.

For any of the Platonic solids, we will have one orbit each of faces, edges and vertices.

[2] *Given a group of automorphisms $\mathbf{G}$ on $C(P)$, if there are faces in $C(P)$ say $F_1, F_2 \in \mathcal{F}$ such that there exists some $h \in \mathbf{G}$ with $h(F_1) = F_2$. Then write $F_1 \sim F_2$, and we say that F_1 is in the same orbit as F_2.*

Definition 3.0.7. *Let* **G** *be the group of symmetries of* $C(P)$. *A* *face fundamental transversal on the faces of* $C(P)$ *is a set* T *such that:*

- *T has exactly one face from each orbit of faces, and*
- *The subgraph of $C(P)$ induced by T is connected under H.*

We will now introduce our first theorem and its proof.

Theorem 1. *Every Polyhedron's Complex contains an H-connected Face Fundamental Transversal.*

Proof. We will follow the construction in [5], and adapt the concepts of [11] and [10] to prove our Theorem for the faces of a $C(P)$. Let $\mathcal{F}$ be the set of faces with adjacency relation H. We start by choosing any $F \in \mathcal{F}$. Then let T_0=$\{F\}$, so that T_0 intersects $\mathbf{G}(F)$ exactly once and is connected. Now assume that T_n is contained in $\mathcal{F}$, where T_n intersects each orbit of faces at most once and is a connected subgraph of the orbits of faces of the $C(P)$. If T_n does intersect each orbit of faces, then we are finished. However, if not, we choose a face, say $F^* \in \mathbf{G}(F^*)$ where $\mathbf{G}(F^*) \cap T_n = \emptyset$, such that there exists a H-path from a face in T_n, say $F \in T_n$ to F^*.

Go down this H-path of faces until we find a face, say F', where F' is the first face on this H-path such that $\mathbf{G}(F') \cap T_n = \emptyset$. So now denote its predecessor face on that path as pF', and pF' is in an orbit $\mathbf{G}(pF')$, where $\mathbf{G}(pF') \cap T_n \neq \emptyset$. See the figure below for a visual aid.

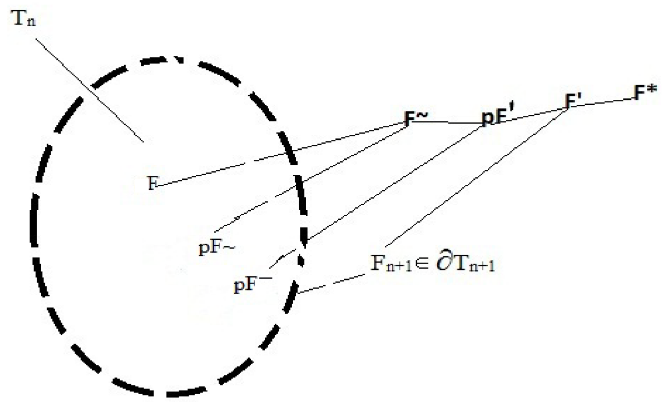

Figure 3.3: An H-path from a face F in T_n to F^*, with F' the first face on the H-path, whose orbit does not intersect T_n.

Since pF' is adjacent to F' and $\mathbf{G}(pF') \cap T_n \neq \emptyset$, then pF' will be equivalent to some other face from the same orbit in T_n; let this face be $pF^- \in T_n$. Hence there will be an automorphism $h \in \mathbf{G}$, such that $h(pF') = pF^-$: $pF' \sim pF^-$. Then $h(F') = F_{n+1}$, for some F_{n+1} in the same orbit as F', and F_{n+1} is adjacent to pF^-. Thus $F_{n+1} \notin T_n$, but F_{n+1} will be adjacent to $pF^- \in T_n$. Therefore, $F_{n+1} \in \partial T_{n+1}$ and by construction, $F_{n+1} \sim F'$.

Since the predecessor face pF^- was adjacent to F_{n+1} and the only new addition is a face in a new orbit, then we have that $T_{n+1} = T_n \cup \{F_{n+1}\}$, which still intersects each orbit at most once and the subgraph induced by T_{n+1} is connected as T_n was. We continue until there are no more unrepresented orbits, and with each iteration T_0, T_n, T_{n+1},..., T_∞, there will be a connected subgraph extended by adding a face adjacent to it. Thus we will eventually get all of the faces of $C(P)$, because with each iteration, we are adding another face adjacent to the previous iteration, and there are finitely many orbits. Therefore, the polyhedron's complex contains an H-connected face fundamental transversal, $T_\infty = \bigcup_n T_{n+1}$. □

Let us look at a few examples so that the concept of a face fundamental transversal is clear. The first example is the easiest one to understand.

Example 3.0.2.

The Cube is one of the Platonic Solids, so it has one orbit of faces.

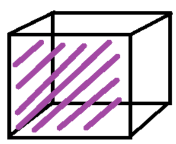

Figure 3.4: A face fundamental transversal on C(Cube).

The next example is just a slightly more difficult than the first example.

Example 3.0.3.

If we look at the Truncated Tetrahedron,[3] there are 2 orbits of faces. So we can start at one of the triangular faces, and denote this as F, then let T_0=$\{F\}$. Then following the previous proof, we can assume that T_1 is contained in $\mathcal{F}$, where T_1 intersects each orbit of faces at most once and is a connected subgraph of the orbits of faces of the $C(P)$. From T_1, we get T_2, since we already know that T_1 existed. Following the previous proof, we go down a H- relation path until we come to a face in an unrepresented orbit, so here that would be one of the hexagonal faces. We know the predecessor face is in an orbit already represented, so this new face must be added, and we get T_2 where $T_2 = T_1 \cup \{F_2\}$. Hence we have an H-connected face fundamental transversal.

The last two examples we will show each have 2 orbits of faces. The face fundamental transversal of these examples is not that difficult to find by hand. The construction in the proof should provide all the tools that are necessary.

[3] *This is a Archimedean Solid.*

Example 3.0.4.

This next example is the Truncated Cube. The Truncated Cube has 2 orbits of faces.

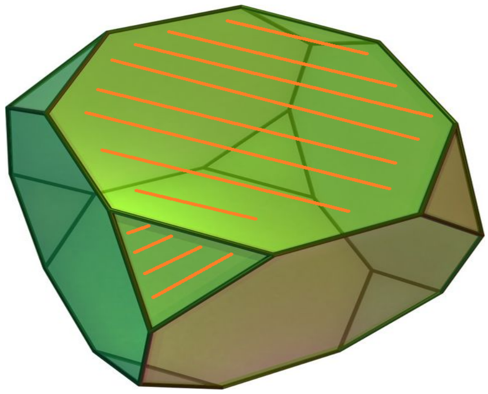

Figure 3.5: A face fundamental transversal on C(Truncated Cube). Image from Wikipedia Commons, licensed under Creative Commons, Attribution - ShareAlike License.

Example 3.0.5.

The last example is the Quartz Crystal. The Quartz Crystal has 2 orbits of faces.

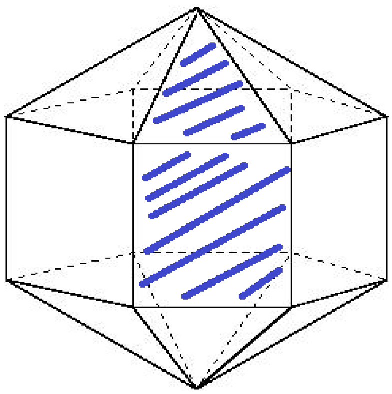

Figure 3.6: A face fundamental transversal on C(Quartz Crystal). Image Courtesy of Gregory L. McColm. Spring 2010

4 EDGE FUNDAMENTAL TRANSVERSALS

Recall from the last chapter that we defined an *automorphism* as an isomorphism under the adjacency and coadjacency relations, from $C(P)$ to itself. The group of symmetries of the underlying space of $C(P)$, whose restrictions are automorphisms of $C(P)$, is denoted $\mathbf{G}$. In this chapter we are using the notion of coadjacency with a restriction. Here the restriction is that the two cells of dimension d will be the edges and they are the shared boundary of a cell of dimension $d+1$, which is a face in the face fundamental transversal. So the definition of coadjacency is now used for the cells that are the shared boundary of a cell in a face fundamental transversal and so we will change it a little for our partial result.

Two edges are called *coradjacent* if they share a common face in the face fundamental transversal. It is required that every edge we obtain in the transversal be incident to two faces in different orbits in the face fundamental transversal. We will define this more formally in this section. To obtain this edge fundamental transversal, we begin by choosing an edge.[1] If there is only one orbit of edges, then we are done. On the other hand, if there is more than one orbit of edges, then we would keep collecting edges until we find our edge fundamental transversal. We need to describe the two types of edges that are required to find the rest of the edges for our edge fundamental transversal.

We define the type of an edge in a represented orbit to be a *predecessor edge*. Naturally, from our assumption above this edge will be coradjacent to another edge. This edge that is coradjacent to the predecessor edge will be an edge that is in an unrepresented orbit. We define this edge to be the successor of the predecessor edge, so we will call it a *successor edge*. We are looking for these edges to be coradjacent to one another and connected. By connected, we mean that there will be a path between each pair of edges. Thus every successor edge shares a face with the predecessor edge, and the edges will be coradjacent to each another. In exploring this partial result, we have found that this works for many the Johnson solids, and especially the types of solids that do not have another face from the same orbit adjacent to it. Let us define a few things to describe the concept we are trying to express.

[1] *Every edge is in an orbit.*

Definition 4.0.8. *A coradjacency relation, denoted as CRH, is a relation between two edges that share a common face in the face fundamental transversal.*

Definition 4.0.9. *The orbit of an edge E in $\mathcal{E}$ [2] is the set of elements of $\mathcal{E}$ to which E can be moved by the elements of $\mathbf{G}$. The orbit of E under $\mathbf{G}$ is denoted by $\mathbf{G}(E) = \{g(E)|g \in \mathbf{G}\}$.*

Definition 4.0.10. *A CRH-path is a path along the coradjacent edges of $C(P)$, where each edge is coradjacent to its successor.*

Definition 4.0.11. *An interior edge is an edge that is incident to two faces in T.*

Definition 4.0.12. *A CRH-connected relation is a relation between each edge that is incident to two different faces in the face fundamental transversal, is coradjacent to another edge in a different orbit, and there exists a path from one edge to another that share a common face in the face fundamental transversal.*

[2] *$\mathcal{E}$ is the set of all edges of the $C(P)$ with coradjacency relation CRH.*

Definition 4.0.13. *A edge fundamental transversal of the edges of $C(P)$, incident to faces in the face fundamental transversal, is a set Q such that:*

- *Q has exactly one edge from each orbit of edges, and each edge is interior.*
- *Each edge is intersected exactly once.*
- *The subgraph of $C(P)$ induced by Q is contained in the boundary of the face fundamental transversal, and is connected.*

We can now introduce our edge transversal proposition 1.

The Edge Transversal Proposition 1. *If an H-connected face fundamental transversal T has a set of interior edges that intersects each orbit at least once, then there exists a CRH-connected edge fundamental transversal whose edges are incident to the faces in the face fundamental transversal.*

Proof. Let $\mathcal{E}$ be the set of all edges with coradjacency relation CRH. Given an H-connected face fundamental transversal T that has a set of interior edges incident to all of it's faces, we let an edge $E \in \mathcal{E}$ be incident to two faces in the face fundamental transversal T. Start by choosing this edge $E \in \mathcal{E}$, where E is incident to say $F^w, F^v \in T$, then we let Q_0=$\{E\}$. Q_0 is the first iteration of the edge fundamental transversal.

Assume that Q_n intersects each orbit of edges at most once, and each edge is incident to two faces in T, and Q_n is CRH-connected. If Q_n intersects each orbit of edges that are incident to at least two faces in T, and is CRH-connected, then we are done. However, if Q_n does not then we continue in the following manner. For some orbit of edges, we would choose an edge incident to two faces in T and is an interior edge, say $E^* \in \mathcal{E}$, where $\mathbf{G}(E^*) \cap Q_n = \emptyset$, such that there exists an CRH-path from E to E^* along the face fundamental transversal T.

Go down this CRH-path from E to E^* until we find an edge say E', where E' is the first edge on this CRH-path, such that $\mathbf{G}(E') \cap T_n = \emptyset$. E''s predecessor edge pE' is in an orbit intersecting Q_n : $\mathbf{G}(pE') \cap Q_n \neq \emptyset$. Hence, since pE' is an interior edge, this means that there does not exist another edge in the same orbit as pE' that is contained in T. So we have that $pE' \in Q_n$. Let the two faces (with any generic representation, here we will call it F^d and F^u), be incident to E', such that $F^d, F^u \in T$, and let F^u be the face that is incident to both E' and pE'. Since $E' \notin Q_n$, then $E' \in \partial Q_{n+1}$. Thus the only addition is E', which is an edge in a new orbit so we must have that $Q_{n+1} = Q_n \cup \{E'\}$. Q_{n+1} still intersects each orbit at most once, and the coradjacency subgraph induced by it, is CRH-connected as Q_n was.

Therefore, we continue until there are no more unrepresented orbits of edges that are coradjacent to one another along T. In each iteration, Q_0, Q_n, Q_{n+1},..., Q_∞, along T, there will be a CRH-connected subgraph of edges extended by adding a coradjacent edge to it, that is incident to two faces in T. Hence we will obtain all of the edges, since each edge in an iteration is incident to at least two faces in the face fundamental transversal, and each edge is coradjacent to another edge in another iteration. Therefore, there exists a CRH-connected edge fundamental transversal $Q_\infty = \bigcup_n Q_{n+1}$ of edges that are incident to the faces of T.

□

Next we will show some examples of the types of polyhedra that hold for the edge transversal proposition. These are all solids from the Johnson solids. The first five examples are from the Cupola family. The last example will be a type of prism. Note that in the Cupolas, that we will be including either the back face or the bottom face in the face fundamental transversal. In each example it will be noted that either the back or the bottom face is included. For instance, in the first example we will present the Triangular Cupola, and we will include the back face in the face fundamental transversal.

Example 4.0.6.

The first example is the Triangular Cupola. It has 4 orbits of faces and 4 orbits of edges. We are including the back face.

Figure 4.1: The Triangular Cupola. Image from Wikipedia Commons, licensed under Creative Commons, Attribution - ShareAlike License.

Notice in the following figure that every orbit of an edge is incident to 2 different faces in the face fundamental transversal. The back face is in the face fundamental transversal.

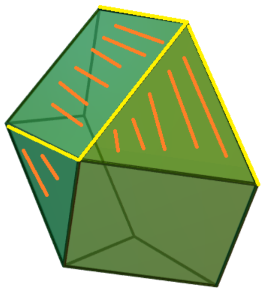

Figure 4.2: A edge fundamental transversal incident to a face fundamental transversal on C(Triangular Cupola). Image from Wikipedia Commons, licensed under Creative Commons, Attribution - ShareAlike License.

Example 4.0.7.

The next example is the Square Cupola. It also has 4 orbits of faces and 4 orbits of edges. Here we are including the bottom face.

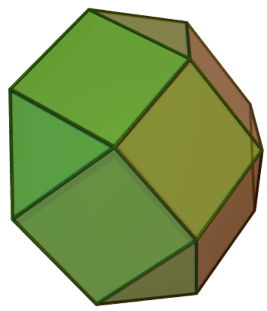

Figure 4.3: The Square Cupola. Image from Wikipedia Commons, licensed under Creative Commons, Attribution - ShareAlike License.

Notice here the bottom face is a face in the face fundamental transversal.

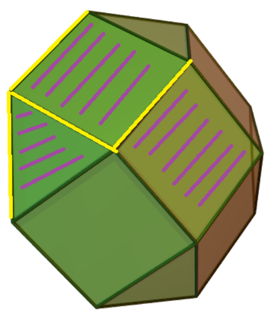

Figure 4.4: A edge fundamental transversal incident to a face fundamental transversal on C(Square Cupola). Image from Wikipedia Commons, licensed under Creative Commons, Attribution - ShareAlike License.

Example 4.0.8.

The Pentagonal Cupola. This solid also has 4 orbits of faces and 4 orbits of edges.

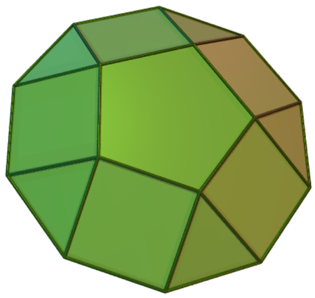

Figure 4.5: The Pentagonal Cupola. Image from Wikipedia Commons, licensed under Creative Commons, Attribution - ShareAlike License.

Again, the bottom face is in the face fundamental transversal.

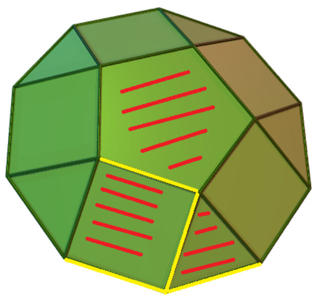

Figure 4.6: An edge fundamental transversal incident to a face fundamental transversal on C(Pentagonal Cupola). Image from Wikipedia Commons, licensed under Creative Commons, Attribution - ShareAlike License.

Example 4.0.9.

The Elongated Pentagonal Cupola. This solid has 6 orbits of faces and 7 orbits of edges.

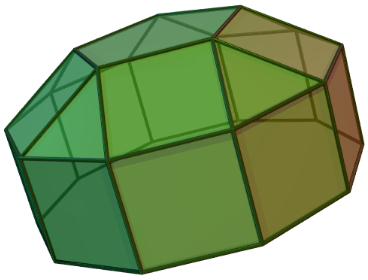

Figure 4.7: Elongated Pentagonal Cupola. Image from Wikipedia Commons, licensed under Creative Commons, Attribution - ShareAlike License.

As before, the bottom face is in the face fundamental transversal.

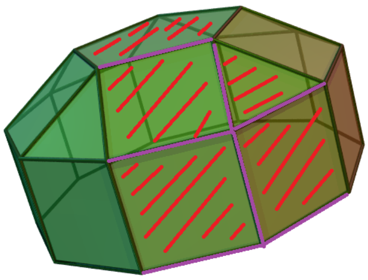

Figure 4.8: An edge fundamental transversal incident to a face fundamental transversal on C(Elongated Pentagonal Cupola). Image from Wikipedia Commons, licensed under Creative Commons, Attribution - ShareAlike License.

Example 4.0.10.

The Elongated Triangular Cupola. This solid also has 6 orbits of faces and 7 orbits of edges.

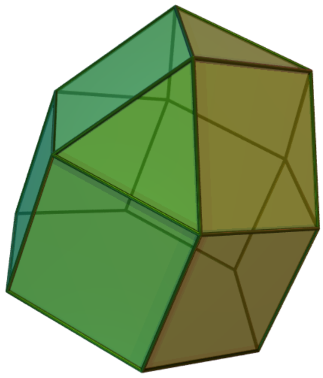

Figure 4.9: Elongated Triangular Cupola. Image from Wikipedia Commons, licensed under Creative Commons, Attribution - ShareAlike License.

As in the other cupolas, the bottom face is in the face fundamental transversal.

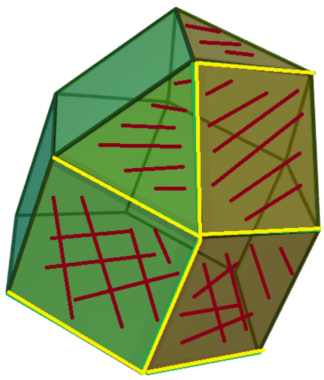

Figure 4.10: An edge fundamental transversal incident to a face fundamental transversal on C(Elongated Triangular Cupola). Image from Wikipedia Commons, licensed under Creative Commons, Attribution - ShareAlike License.

Example 4.0.11.

The Augmented Pentagonal Prism. This solid has 5 orbits of faces and 6 orbits of edges.

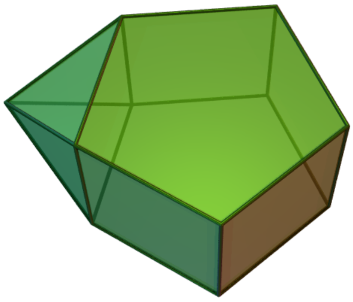

Figure 4.11: Augmented Pentagonal Prism. Image from Wikipedia Commons, licensed under Creative Commons, Attribution - ShareAlike License.

Notice here that the edge fundamental transversal is half of the solid.

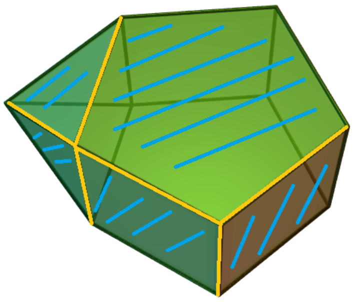

Figure 4.12: An edge fundamental transversal incident to a face fundamental transversal on C(Augmented Pentagonal Prism). Image from Wikipedia Commons, licensed under Creative Commons, Attribution - ShareAlike License.

Next, we will present some examples of polyhedra that don't satisfy the conditions for the proposition.

Example 4.0.12.

If we take the Archimedian solids, the Rhombicuboctahedron and the Truncated Cuboctahedron, then the first solid has three orbits of faces, and three orbits of edges. So the face fundamental transversal will consist of the three different orbits of faces, connected under H-relations. If we start with choosing one of the edges, say the one in between the two square faces, then an edge that is coradjacent to it, then the proposition holds, but we don't get all of the orbits of edges. For instance, since we started with choosing the edge that is between the two square faces, where this edge is incident to at least 2 different faces in the face fundamental transversal, then we choose an edge that is coradjacent to it, say the one between the triangular face and square face, then after this the next edge we choose is not incident to at least 2 different faces in the face fundamental transversal. So we see that we don't get all of the orbits of edges, thus the proposition doesn't hold.

The same idea is applied to the Truncated Cuboctahedron. The Truncated Cuboctahedron has two different orbits of faces and edges, so the face fundamental transversal will consist of the two different orbits of faces, connected under H. Again, let us start by choosing an

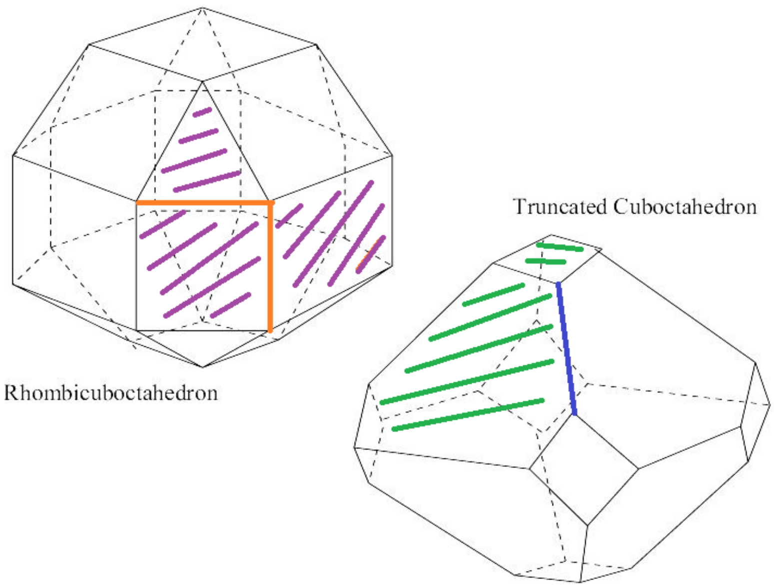

Figure 4.13: Examples of Polyhedra, where the edge proposition doesn't hold, i.e, the Rhombicuboctahedron and the Truncated Cuboctahedron. Image courtesy of Gregory L. McColm Spring 2009.

edge, say the edge that is between the two larger faces. Then, already we can see that the proposition doesn't hold, because the edge we chose is not incident to at least 2 faces in the face fundamental transversal. Notice in the figure that the edges that are outlined show why the proposition doesn't hold, i.e, every edge isn't incident to 2 different faces in the face fundamental transversal.

Next, we will present a few more examples of polyhedra that do not satisfy the proposition.

The following polyhedron was created during the spring semester of 2010. The Caramel.

Example 4.0.13.

The following figure is a polyhedron that was created, while eating a sundae. It has 6 orbits of faces, 6 orbits of edges, and 4 orbits of vertices. If you wish to include the interior faces, then this polyhedron will have 7 orbits of faces, 6 orbits of edges, and 4 orbits of vertices.

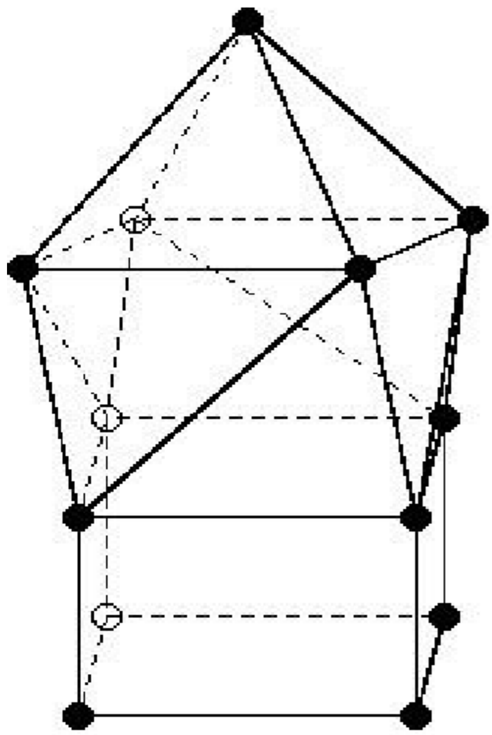

Figure 4.14: The Caramel. Image Courtesy of Gregory L. McColm.

On the next page we will show what the edge fundamental transversal will look like on the C(Caramel).

The following figure shows what the edge fundamental transversal looks like on the C(Caramel). Notice that every edge is not incident to at least 2 different faces in the face fundamental transversal. Thus the proposition doesn't hold for this polyhedron.

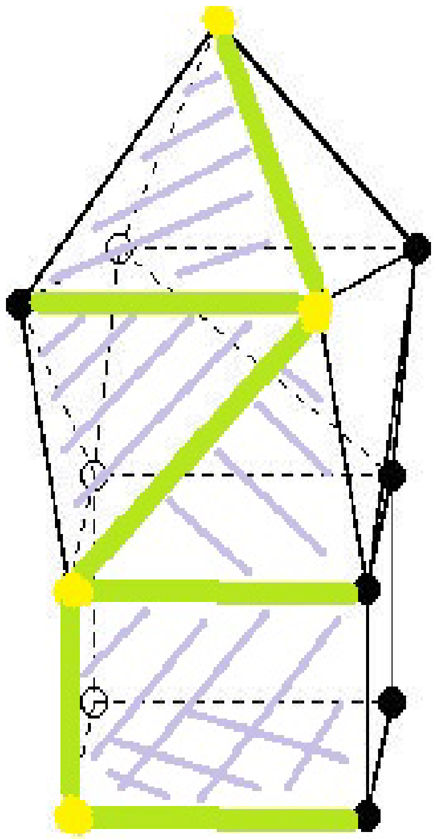

Figure 4.15: A edge fundamental transversal on C(Caramel). Image is a combination of Gregory L. McColm's previous picture and Joy D'Andrea's.

Our next example is a solid that is part of the Dipyramid family of the Johnson Solids. Every edge is not incident to 2 faces in the face fundamental transversal, and there are some faces adjacent to other faces in the same orbit.

Example 4.0.14.

The Elongated Triangular Dipyramid. It has 4 orbits of faces and 6 orbits of edges.

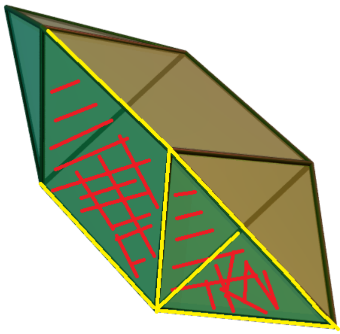

Figure 4.16: The Elongated Triangular Dipyramid. Image from Wikipedia Commons, licensed under Creative Commons, Attribution - ShareAlike License.

Example 4.0.15.

The Pentagonal Pyramid. It has 2 orbits of faces and 2 orbits of edges.[3]

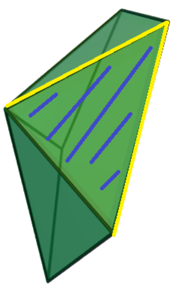

Figure 4.17: The Pentagonal Pyramid. Image from Wikipedia Commons, licensed under Creative Commons, Attribution - ShareAlike License.

[3] *We are including the back face.*

The last example is a polyhedron called Sorcha. This was made in the spring of 2010. The name was given to this polyhedron because it is similar in resemblance of the Helmet worn by Sorcha in the movie *Willow*. Sorcha has a lot of symmetry about itself.

Example 4.0.16.

Sorcha. It has 10 orbits of faces, 15 orbits of edges, and 6 orbits of vertices. If you would like to include the interior faces, then there will be 13 orbits of faces, 15 orbits of edges, and 6 orbits of vertices.

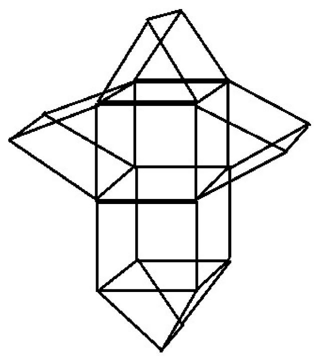

Figure 4.18: Sorcha

Sorcha's face and edge fundamental transversal's are exactly half of the entire solid. We will see this in the next figure.

The next image is of the edge fundamental transversal on C(Sorcha). It appears to hold for the proposition, but notice the edge at the very top, it is the only edge that isn't incident to two different faces in the face fundamental transversal.

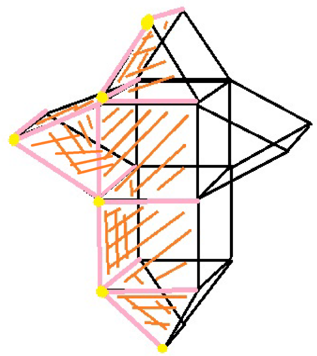

Figure 4.19: A edge fundamental transversal on $C(Sorcha)$

5 FUNDAMENTAL TRANSVERSAL DOMAINS

We have come to a critical chapter of the thesis, although this is a small chapter. Everything from the previous chapters made it possible to get here. Throughout the thesis we did not address certain issues, such as an edge fundamental transversal being CH-connected, or incident to vertices of $C(P)$. We didn't discuss whether or not the vertices were incident to the faces in the face fundamental transversal, or that there exists a vertex fundamental transversal.

Let's start off by describing what we would like to happen. We are looking to obtain a connected set of representatives of the orbits of the cells of $C(P)$. The cells we will be referring to are the faces, edges, and vertices of $C(P)$. Recall in the face fundamental transversal chapter that we defined a face fundamental transversal as follows.

Definition 5.0.14. *Let* **G** *be the group of symmetries of* $C(P)$*. A* $face\ fundamental\ transversal$ *on the faces of* $C(P)$ *is a set* T *such that:*

- *T has exactly one face from each orbit of faces, and*
- *The subgraph of $C(P)$ induced by T is connected under H.*

In this chapter we define an edge fundamental transversal as.

Definition 5.0.15. *Let* **G** *be the group of symmetries of* $C(P)$*. A edge fundamental transversal on the edges of* $C(P)$ *is a set* Q *such that:*

- *Q has exactly one edge from each orbit of edges, and*
- *The subgraph of* $C(P)$ *induced by* Q *is connected under* CH*.*

This definition is the same for a vertex fundamental transversal except we replace Q with L, and edges with vertices.

Definition 5.0.16. *A Fundamental Transversal Domain is a connected union of the face, edge, and vertex fundamental transversals of* $C(P)$*.*

Determining the existence of these fundamental transversal domains was originally the problem we were trying to solve for this thesis. However, this idea may be more difficult than we thought. We found many counterexamples to the formal description we were trying to present. What we have discovered is two conjectures for a fundamental transversal domain. The first conjecture is using the concepts of a H-connected face fundamental transversal, and CH-connected edge and vertex fundamental transversals.

Conjecture 1. *For every polyhedron's complex, there exists a H-connected face, CH-connected edge, and CH-connected vertex fundamental transversals, such that each intersected orbit of a vertex is incident to an intersected orbit of an edge, and each intersected orbit of an edge is incident to a intersected orbit of a face.*

We need to define a few things to approach our next conjecture. In the face fundamental transversal section, we defined a *complex* of a polyhedron $C(P)$, as a poset of cells of P. We define a $flag$ in a *complex* as a subset of a fundamental transversal domain, where a flag is a maximal chain of the poset. A flag consists of a vertex, that is contained in an edge, and that edge is contained in a face from the $C(P)$. We can find a flag by choosing a vertex, then an edge that is incident to the vertex, and then a face that is incident to that edge.

We are looking to find all flags such that a fundamental transversal domain is a union of these flags. This seemed like a good approach at first, but what happens, when the *Complex* is huge or infinite, how do we find all of the flags then? This idea may still work, but it will require some computer applications, such as Maple, SAGE, Mathematica, and so on.

Conjecture 2. *There exists a connected fundamental transversal domain that is the union of flags of* $C(P)$*.*

We close the thesis with fundamental transversal domains on what we call *related objects*. Here related objects are things such as 'nets' of different crystals, or more simple, *crystal nets*. The crystal nets shown here are as follows; PCU, DIA, HEX and AST. The images are courtesy of Reticular Chemistry Structure Resource, initiated by Micheal O'Keefe. Some of these crystal nets have 1 orbit of a cell in each dimension, and some have 2 orbits of faces. Notice in the nets that what appears to look like a face is actually the central cell. The faces are clear in these nets. We will color the faces in, to show what they are in our fundamental transversal domains. Also, we will line draw the central cell, shade in the orbits of vertices, and outline the orbits of edges.

The first crystal net we will show is PCU. PCU has 1 orbit of faces, edges, and vertices, and also a central cell orbit. Thus it's fundamental transversal domain will be easy to find and connect.

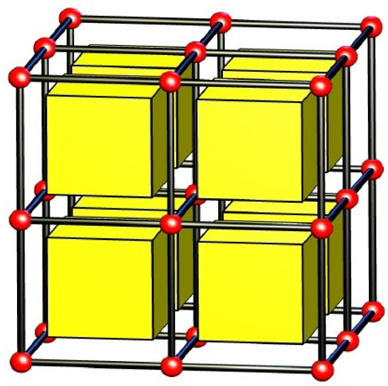

Figure 5.1: PCU. Image Courtesy of Micheal O'Keefe.

The next figure depicts what a fundamental transversal domain on PCU will look like. Here we start with the central cell, connect it to the 1 orbit of vertices, then connect the 1 orbit of vertices to the 1 orbit of edges, and lastly, connect the 1 orbit of edges to the 1 orbit of faces. Hence, we have our visually connected fundamental transversal domain on PCU.

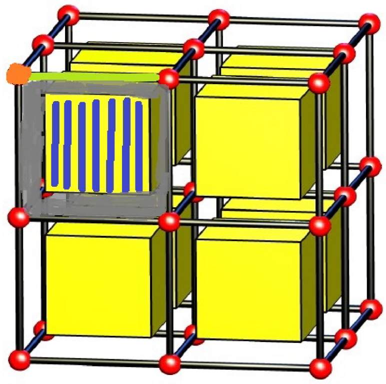

Figure 5.2: A Fundamental Transversal Domain on PCU. Image Courtesy of Micheal O'Keefe.

The next crystal net we will show is DIA. DIA is an interesting looking crystal net, with its energetic color and its intriguing design. It has 1 orbit of faces, 1 orbit of edges and vertices, and a central cell orbit. Here again, a fundamental transversal domain on DIA will be easy to find and connect.

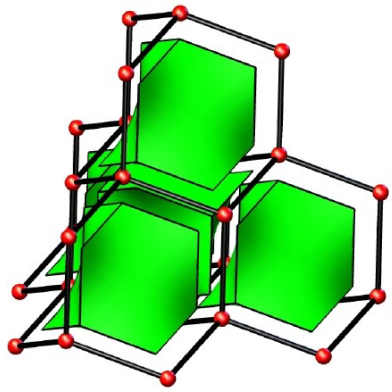

Figure 5.3: DIA. Image Courtesy of Micheal O'Keefe.

The next figure depicts what a fundamental transversal domain on DIA will look like. Here we start with the central cell, connect it to the 1 orbit of faces, then connect the 1 orbit of faces to the 1 orbit of edges, and lastly, connect the 1 orbit of edges to the 1 orbit of vertices. Hence, we have our visually connected fundamental transversal domain on DIA.

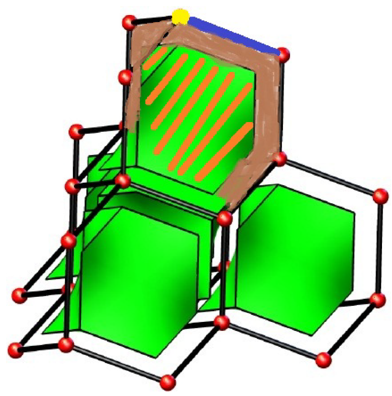

Figure 5.4: A Fundamental Transversal Domain on DIA. Image Courtesy of Micheal O'Keefe

Next, we will look at some crystal nets that may have more than 1 orbit of faces, edges, and vertices. The following crystal net we will show of this nature is Hex. It has 2 orbits of faces and edges, and 1 orbit of vertices, and a central cell orbit. It's fundamental transversal domain, shouldn't be that hard to figure out and connect.

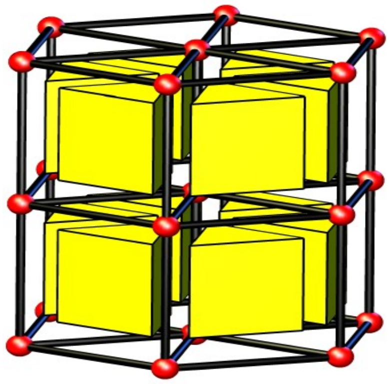

Figure 5.5: HEX. Image Courtesy of Micheal O'Keefe.

The next figure shows what a fundamental transversal domain on Hex will look like. Following the previous directions, we perform the same process. We start off by finding the central cell, connect it to both orbits of faces, which are the square and triangular face. Then connect the orbits of faces to the orbits of edges, say the one on the left of the square face, and the one that intersects the 2 orbits of faces, lastly connect the orbits of edges to the 1 orbit of vertices, on the left. Hence we have our visually represented fundamental transversal domain on HEX on the next page.

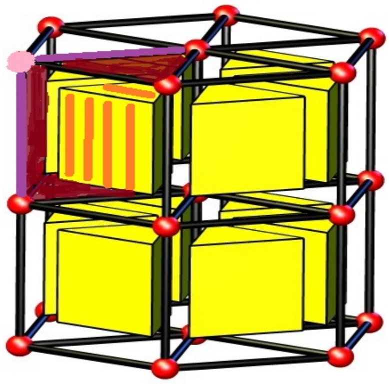

Figure 5.6: A Fundamental Transversal Domain on HEX. Image Courtesy of Micheal O'Keefe.

The last example we will present is a crystal net of AST. AST has 2 orbits of faces, edges, and vertices, and a central cell orbit. The fundamental transversal domain should be just a little more slightly harder to find.

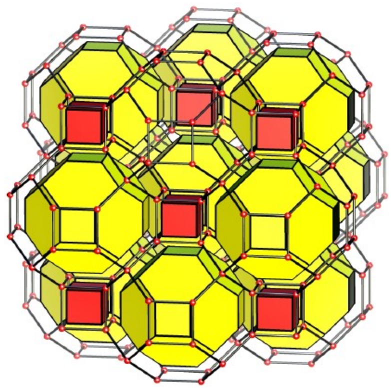

Figure 5.7: AST. Image Courtesy of Micheal O'Keefe.

The figure on the next page depicts what a fundamental fundamental transversal on AST, will look like. Like before, we start with the central cell, connect it to the 2 orbits of faces, say the hexagonal face and the square face. Then connect the orbits of faces to the 2 orbits of edges, here that will be the edge that intersects the 2 orbits of faces, and the other edge is incident to the square face. The last thing we do, is connect the orbits of edges to the 2 orbits of vertices, which are the vertex that is incident to both faces, and the other one is incident to the square face and adjacent to the other vertex.

Notice in the following figure that every orbit of a cell is connected to each other and to the central cell. Thus we have obtained our visually represented fundamental transversal domain on AST. It may be hard to see the fundamental transversal domain on AST right away. Look to the far right side at the top of AST, this is where the fundamental transversal domain is.

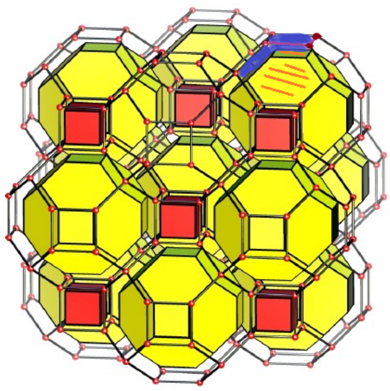

Figure 5.8: A Fundamental Transversal Domain on AST. Image Courtesy of Micheal O'Keefe.

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8. The Quartz Crystal. Image drawn by Gregory L. McColm Spring 2010.
9. The Rhombicubactahedron and Truncated Tetrahedron. Image drawn by Gregory L. McColm. Spring 2009.

10. The Caramel. Image drawn by Gregory L. McColm Spring 2010

11. PCU, DIA, HEX, and AST. Image Courtesy of Micheal O'Keefe. Reticular Chemistry Structure Resource.
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