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Calculating the Electromotive Force through a Loop

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Calculating the Electromotive Force through a Loop

Abstract

This is a short article which utilizes the equations of Faraday's law, Ampere's law, and Ohm's law. We manipulate the formulas to solve a practical application in which the electromotive force must be calculated. With the successful calculation of the electromotive force, we can use this technique in larger applications.

Keywords

electromotive force, magnetic field, Ohm's law, Faraday's law, Ampere's law, Lenz's law, closed loop integral, surface integral

Problem Statement

A magnetic field strength $B(t) = 8t^2 - 5t + 4$ is directed out of a plane of a circular loop of wire with radius *r* and a light bulb part of the loop. Find the electromotive force that is being generated as a function of time *t*. Also calculate the current going through the 66Ω light bulb as a function of time.

Motivation

From wind turbines to automobile alternators to even the small fan that cools the inside of your computer, all electric motors have one thing in common: Their operating principles are governed by Faraday's law, Lenz's law, Ampere's law. These principles have made it possible to calculate the electromotive force (*emf*), or voltage, induced by a changing magnetic field. Every day we go through life and see examples of this yet we do not understand the magnitude of its importance. Without this discovery, science and technology today would be unrecognizable.

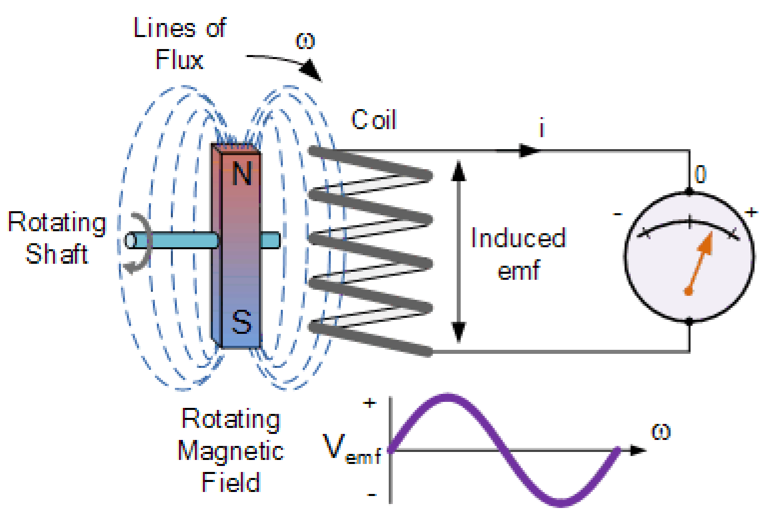

Figure 1: Electromotive Force is being induced in a coil of wire by a changing magnetic field.

MATHEMATICAL DESCRIPTION AND SOLUTION APPROACH

We shall first approach the problem by defining the equations needed to solve such a problem:

1) $V = IR$
2) $\varepsilon = \oint_c \vec{E} \cdot dl = -N\frac{d\Phi_B}{dt}$
3) $\Phi_B = \iint_S \vec{B} \cdot \overrightarrow{dA}$

Equation (1) is Ohm's law that explains how if we multiply the current I by the resistance R, the result is the amount of voltage in a loop. Now, voltage can also be referred to as the amount of electromotive force (*emf*) that a loop contains. Equation (2) is Faraday's law which expresses that the electromotive force induced, ε, is equal to the closed loop integral of the vector dot product of the electric field, $\vec{E}$, and the "contour vector". That result is also equal to the negative magnitude of the number of loops in a coil, N, multiplied by the rate of change in magnetic flux over time, $\frac{d\Phi_B}{dt}$. Equation (3), Ampere's law, states that the magnetic flux, Φ_B, is equal to the surface integral of the vector dot product of the magnetic field, $\vec{B}$, and the area of the infinitesimal surface element. The references to these laws and equations (1), (2), and (3) are given below.

Part 1

If we were to take equation (2):

$$\varepsilon = \oint_c \vec{E} \cdot dl = -N\frac{d\Phi_B}{dt}$$

We do not need the first half of the equation since we are simply looking for the *emf* so:

$$\varepsilon = -N\frac{d\Phi_B}{dt}$$

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If we substitute equation (3) into what we already have, we get:

$$\varepsilon = -N\frac{d}{dt}\iint_S \vec{B}\cdot\overrightarrow{dA}$$

Since in this scenario we only have 1 loop of wire N would equal 1:

$$\varepsilon = -\frac{d}{dt}\iint_S \vec{B}\cdot\overrightarrow{dA}$$

Since $\vec{B}$ is constant with respect to $\overrightarrow{dA}$ it can be removed from the integral

$$\varepsilon = -\frac{d\vec{B}}{dt}\iint_S \overrightarrow{dA}$$

Based on calculus, if we take the surface integral of $\overrightarrow{dA}$, we would just be left with the area A, therefore

$$\varepsilon = -A\frac{d\vec{B}}{dt}$$

In this problem, we are given the magnetic field strength and the area of a circle is πr^2:

$$\varepsilon = -\pi r^2\frac{d}{dt}(8t^2 - 5t + 4)$$

After taking the derivative of the magnetic field strength value, we are left with the final result of:

$$\varepsilon = -\pi r^2(16t - 5)$$

Part 2

If we take equation (1) then we recall that the induced *emf* is also referred to as the voltage, we can replace "*V*" with our result that we obtained from Part 1:

$$-\pi r^2(16t - 5) = IR$$

And finally by rearranging the equation to solve for "I" and substituting the resistance that was given we obtain:

$$\frac{-\pi r^2(16t - 5)}{66} = I$$

Discussion

Our goal was to answer two main questions; the first one was to calculate the electromotive force based on the parameters defined. Despite not arriving at a numerical conclusion, we can still use this process to calculate the electromotive force for any circular loop exposed to that magnetic field strength. The negative sign in the equation is a part of Lenz's law that states "the direction of the current induced in a conductor by a changing magnetic field is such that the magnetic field created by the induced current opposes the initial changing magnetic field" (Wikipedia). The author would also like to emphasize the importance of taking the derivative of the magnetic flux with respect to time because that describes how an electromotive force will only be induced when the magnetic field is changing.

The second question was to use the resistance provided and formulate an equation to give us the current passing through the loop at any given moment in time. Having already calculated the voltage, or induced electromotive force, the rest of the problem was just plugging in values

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and solving for the current. The most difficult part was merely understanding the relationship between electromotive force and voltage.

Conclusions and Recommendations

Based on the analysis conducted, we can apply these principals to a wide range of more complex problems. Ohm's law, equation (1), demonstrates the relationship between the current, resistance, and the voltage, in this case electromotive force. Lenz's law demonstrates the direction of the opposing magnetic field created by the induced electromotive force. Lenz's law does not mention anything about the magnitude of the result but merely the direction. Faraday's law, equation (2), gives us the ability to calculate the electromotive force induced in a loop. Ampere's law, equation (3), allows us to calculate the magnetic flux of the system.

Despite a person's discontent with calculus, it will forever be intertwined with the world of engineering. This paper being a small example of such! These laws build the foundation of classical electromagnetism and electrical engineers apply these concepts more often than not to shape this world into the enigma that it has become.

For someone doing a similar project the author would recommend to choose a more specific application that would conclude with a numerical value. Using that process one could find the optimal level of efficiency and quantify their results of a particular circuit.

Nomenclature

Symbol	Description	Units
$\boldsymbol{V}$	Voltage	Volts, V
$\boldsymbol{I}$	Current	Amps, A
$\boldsymbol{R}$	Resistance	Ohms, Ω
$\boldsymbol{\varepsilon}$	Induced electromotive force (emf)	Volts, V
$\oint_c \dots$	Closed loop integral	-
$\vec{\boldsymbol{E}}$	Electric Field strength	Newtons per coulomb, $\frac{N}{C}$ Or Volts per Meter, $\frac{V}{m}$
$\boldsymbol{dl}$	an infinitesimal vector element of the considered contour	-
N	Number of loops	-
$\frac{\boldsymbol{d\Phi_B}}{\boldsymbol{dt}}$	Rate of Change in Magnetic flux over a period of time	Webbers per sec, $\frac{Wb}{s}$
$\boldsymbol{\Phi_B}$	Magnetic flux	Webbers, Wb
$\iint_S \dots$	Surface integral	-
$\vec{\boldsymbol{B}}$	Magnetic field	Teslas, T
$\overrightarrow{\boldsymbol{dA}}$	area of an infinitesimal patch of surface	Square meters, m^2

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Ampère never utilized the field concept in any of his works; cf. *Assis, André Koch Torres; Chaib, J. P. M. C; Ampère, André-Marie (2015). Ampère's electrodynamics: analysis of the meaning and evolution of Ampère's force between current elements, together with a complete translation of his masterpiece: Theory of electrodynamic phenomena, uniquely deduced from experience. Montreal, QC: Apeiron. ch. 15 p. 221. ISBN 978-1-987980-03-5.* The "Ampère circuital law" is thus more properly termed the "Ampère–Maxwell law." It is named after Ampère because of his contributions to understanding electric current. Maxwell does not take Ampère's force law as a starting point in deriving any of his equations, although he mentions Ampère's force law in his *A Treatise on Electricity and Magnetism*vol. 2, part 4, ch. 2 (§§502-527) & 23 (§§845-866).

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