
2019

Integrating the Equation of Hydrostatics for Groundwater under Non-isothermal Conditions

Megan A. Levine
University of South Florida

Advisors:

Arcadii Grinshpan, Mathematics and Statistics
Thomas Juster, School of Geosciences

Problem Suggested By: Thomas Juster

Follow this and additional works at: <https://digitalcommons.usf.edu/ujmm>



Part of the [Mathematics Commons](#)

UJMM is an open access journal, free to authors and readers, and relies on your support:

[Donate Now](#)

Recommended Citation

Levine, Megan A. (2019) "Integrating the Equation of Hydrostatics for Groundwater under Non-isothermal Conditions," *Undergraduate Journal of Mathematical Modeling: One + Two*: Vol. 10: Iss. 1, Article 6.

DOI: <https://doi.org/10.5038/2326-3652.10.1.4911>

Available at: <https://digitalcommons.usf.edu/ujmm/vol10/iss1/6>

Integrating the Equation of Hydrostatics for Groundwater under Non-isothermal Conditions

Abstract

Groundwater is an important area of study for several scientific fields and is relevant to many problems in our everyday life. One way the study of groundwater has become different in this age of technology is through modeling. Modeling is used in numerous fields to visualize complex problems and view inaccessible areas, such as under the Earth. An essential part of groundwater modeling is to know water pressure at any depth underground. The difficulty in calculating this is that there are several variables that affect water pressure, all of which are affected by one another and all are calculated by a different equation. In this paper we combine and integrate the Equation of Hydrostatics, the Geothermal Gradient equation, and the Coefficient of Thermal Expansion equation to develop a single formula for water pressure that already accounts for the change in temperature and density as you get deeper in the Earth. Another important component of groundwater modeling is that it has to be relevant to the location that you are making a model of. This is why the geothermal gradient and thermal expansion coefficients are important; they are specific to a region. The presented formula makes groundwater modeling easier, faster, more accurate and more geographically relevant, thereby making groundwater modeling even more useful in many fields.

Keywords

groundwater modeling, hydrostatics, geothermal gradient, thermal expansion coefficient

Creative Commons License



This work is licensed under a [Creative Commons Attribution-NonCommercial-Share Alike 4.0 License](https://creativecommons.org/licenses/by-nc-sa/4.0/).

PROBLEM STATEMENT

Derive an equation that can be used to calculate the pressure of groundwater as depth increases under hydrostatic conditions when the density varies as a function of temperature.

MOTIVATION

From environmental sciences and physics to geology and engineering, modeling is an extremely useful tool. Modeling can help to visualize a complicated problem, allowing us to look a large area at once, display an area not usually visually accessible, such as underground, and allow us to test different scenarios quickly. One field in which modeling is quickly becoming more and more important is hydrogeology. With accurate models, we can deduce information about the water we do not easily have access to. We can tell how much water can be held in a certain area, if the water is moving or stagnant, what direction it is travelling, how quickly it is moving. We can tell how much water is accessible to us, possibly predicting sinkholes, water resources planning, and even knowing if the ground is safe to build on. It is useful information for many fields, such as environmental sciences, conservation, industrial building, and agriculture.

Groundwater modeling helps us to actually visualize what is happening to water below our feet that we cannot usually see. In order to use groundwater models, one should be extremely accurate and specific to the geographic region. One of the critical variables in making groundwater models is the pressure on a particle of water at any point in the ground. Solving for the pressure can then play a role in larger equations which are used in modeling. We can integrate and combine different equations, such as the geothermal gradient, to derive a new equation for calculating the pressure specific to any given location of interest. In order to make it

accurate, we need to account for several different variables, such as density, temperature, and pressure. Different equations are required because the pressure changes as a function of density, however, density will change as a function of temperature, and temperature changes as a function of depth. We want to condense all of these functions to create a formula for pressure as a function of depth. So we need to start at the end, solving for temperature, and work our way back to pressure. This paper strives to use calculus tools, such as integration, to create a single equation that is both accurate and geographically relevant, so that it can be used in groundwater modeling. With a single equation, all variables will be accounted for, meaning it will be much easier and faster to construct accurate groundwater models.

MATHEMATICAL DESCRIPTION AND SOLUTION APPROACH

Important Equations

Geothermal Gradient

$$T = T_o - C_g \cdot Z \quad \text{or} \quad T = T_o + C_g \cdot D \quad (1)$$

In order to account for all of the variables in a single equation, we need to start from the last one and work back towards pressure. The first step is the geothermal gradient coefficient. The equation simply states that the temperature at any point in the Earth (T) will increase at a constant rate as we go deeper in the Earth. In equation (1), Z is a negative number because it is depth (D) or negative elevation; T_o in this equation represents the temperature at the surface of the Earth. The rate at which the temperature will increase is determined by location. In South Florida, for example, the geothermal gradient constant (C_g) is about 25°C per kilometer, but one can always plug in the coefficient for a different region. This equation is solving for temperature as a function of depth. It will be important later when we start combining equations.

Coefficient of Thermal Expansion Equation

$$\alpha = -\frac{1}{\rho} \cdot \frac{d\rho}{dT} \quad (2)$$

This equation also looks more complicated than it really is. It simply means that as temperature increases, density (ρ) will decrease. Logically, this makes sense because as water heats up, there are more gas bubbles causing water to expand, meaning the water is going to become less dense. Again, the rate at which density will decrease is determined by geographic conditions. This is what α represents, the correlation between the increasing temperature and the decreasing density. We will integrate our final formula to get density as a function of temperature. Because we need our equation to be geographically specific, α will be a known constant for the region of interest. Instead, we want to solve for ρ . So we rearrange equation (2) accordingly:

$$\frac{d\rho}{\rho} = -\alpha dT \quad (3)$$

Equation of Hydrostatics

$$\nabla P = -\rho g \vec{k} \quad (4)$$

The equation of hydrostatics (4) is a vector equation that solves for pressure in a three-dimensional space. It factors in how the water pressure is changing in two directions: horizontally (x, y) and vertically (z). For our purpose, we can assume that the water pressure does not change when one moves horizontally along either the x or y -axes. The word “static” means showing little or no change. In this case, it means along the x and y -axes. In the (x, y) plane, vector \vec{k} in (4) is made up of two components (i, j), both are equal to zero because they are the horizontal directions and we can simplify equation (4) accordingly. We are only interested in how pressure changes with depth, vertically, which is the z -axis. Remember that z is negative

because it is below the surface. When we negate \vec{k} in (4) and put P as a function of depth (Z) we get an equation that only factors in the change along the vertical axis, which is what we need:

$$\frac{dP}{dZ} = -\rho \cdot g \quad (5)$$

Integrating to Combine Formulas

The reason we integrate these formulas is that, essentially, integration is a summation. When we are trying to find the pressure on a water particle under the Earth's surface, a majority of that comes from the weight of everything on top of it. So, we need to calculate and account for the changing weights of thin sections of material above the water particle. However, the weight of the overlying material is constantly changing as you go deeper into the Earth. That is where each equation is able to account for the changing conditions.

The first thing to do is to integrate our manipulated coefficient of thermal expansion equation to get density as a function of temperature. From (3) we have:

$$\int \frac{d\rho}{\rho} = - \int \alpha dT,$$

$$\ln \rho = -\alpha T + C \quad (6)$$

We use a boundary condition of $\rho = \rho_0 = 1,000 \text{ Kg/m}^3$ and $T = T_0 = 15^\circ\text{C}$ and solve (6) for C : $C = \ln \rho_0 + \alpha T_0$.

We plug the above formula for C back into equation (6)

$$\ln \rho = -\alpha T + \ln \rho_0 + \alpha T_0 \quad (7)$$

and solve (7) for ρ to find density as a function of temperature:

$$\rho = \rho_0 e^{-\alpha(T-T_0)} \quad (8)$$

Now we apply the geothermal gradient equation (1). Because equation (1) is solving for T , we can plug it into our equation (8) to get:

$$\rho = \rho_0 e^{-\alpha(T_0 - (Cg \cdot Z) - T_0)} = \rho_0 e^{(\alpha \cdot Cg \cdot Z)} \quad (9)$$

Now we have an equation that solves for density as a function of depth.

Because equation (9) solves for density, we can plug it into the equation of hydrostatics (5). To simplify the equation we will let θ represent (αCg) . Then we apply equations (5) and (9) ending up with:

$$\frac{dP}{dZ} = -\rho_0 g e^{(\theta Z)} \quad (10)$$

Now we integrate (10) with respect to Z so that we have an equation that solves for pressure as a function of depth:

$$P = -\frac{\rho_0 \cdot g}{\theta} \cdot e^{\theta Z} + C \quad (11)$$

We use another boundary condition to solve (11) for the new constant C . As pressure is zero at the surface ($P=0$ when $Z=0$), we have:

$$0 = -\frac{\rho_0 \cdot g}{\theta} + C \rightarrow C = \frac{\rho_0 \cdot g}{\theta}$$

We plug this formula for the new constant C back into equation (11) and get

$$P(Z) = \frac{\rho_0 \cdot g}{\theta} \cdot (1 - e^{\theta Z}), \text{ where } \theta = \alpha Cg \quad (12)$$

This is our final equation that solves for pressure as a function of depth.

DISCUSSION

The objective of this paper is to create a single equation that would solve for water pressure as depth into the Earth increases while accounting for changes in both temperature and

density. The equation is also needed to be able to be geographically specific, so that groundwater models are accurate to the area they are modeling. That is exactly what we ended up with. The final equation is (12). Surface pressure will normally be used for ρ_o because it is always known. The locally specific variables (Cg and α) are grouped together in the single variable θ so that this formula can be easily applied to any location. The author is extremely happy with this formula result because it is not as overcomplicated as the author thought it would be, and it accomplishes its two objectives; accounting for all the variables at once and being geographically adaptable. The result is both expected and unexpected because we did think it was possible to create such an equation, but we did not think it would be a relatively simple equation. We thought trying to make it geographically adaptable would make the equation too complicated, so the author is pleasantly surprised. This equation will make creating groundwater models much easier and more accurate because it can be done with one equation. It also makes the overall mathematics required to create an accurate model simpler allowing more people can create models. This means we can understand what is happening to the water underground faster and more accurately. Another possible implication of the creation of this formula is how it can be adapted to other fields. A possible use in geology could show how overall pressure is changing under the Earth because it would be relatively simple to account for the rock pressure under the surface. This could potentially also be applied to calculating the pressure as you get deeper in the ocean, which is important to scuba divers, submarines, oceanographers, and industrial companies who put oil pipes and cable lines along the bottom of the ocean. One can also calculate air pressure when getting higher in the atmosphere. There are so many more possible applications of this formula in a wide range of disciplines. See Appendix for some related graphs.

CONCLUSIONS AND RECOMMENDATIONS

The strategy applied to acquire the main equation (12) is effective in being both geographically relevant and inclusive of different variables. This strategy can be adapted and applied to different specific problems. Water pressure is a fundamental component for creating equation (12) and other groundwater models. If this process is to be redone with a different field in mind, another person might factor in salinity to make a formula that can be applied to salt water. Another way someone might change this process is to focus on one region and put the constants for that area directly into the equation, whereas the author wanted the formula to be more general. We would love to continue this work to see if we can also create an equation for overall pressure on a solid particle in a rock mass under the surface. All in all, we think this equation can be extended, manipulated, and adapted in numerous ways to be applicable to almost any field.

NOMENCLATURE

Symbol	Meaning	Units
Z	Negative Elevation	Km
T	Temperature	°C, °F, K
T_o	Known Temperature (usually at the surface)	°C, °F, K
P	Pressure	Psi, Pa, bar
P_o	Known Pressure (usually at the surface)	Psi, Pa, bar
ρ	Density	Kg/m ³
ρ_o	Known Density (usually at the surface)	Kg/m ³
α	Alpha	°C ⁻¹
D	Depth	Km
∇P	Pressure gradient	
g	Gravity	m/s ²
C_g	Geothermal Gradient Coefficient	°C/Km

REFERENCES

- Beddows, C. (2016, March 13). What is Groundwater Modelling and Why is it Important? | SciTech Connect. Retrieved from <http://scitechconnect.elsevier.com/what-is-groundwater-modelling-why-important/>
- Brott, C. A., Blackwell, D. D., & Morgan, P. (1981). Continuation of heat flow data: A method to construct isotherms in geothermal areas. *Geophysics*, 46(12), 1732-1744. doi:10.1190/1.1441180
- Cohen, J. M. (n.d.). Water Density and Temperature. Retrieved from http://www.4physics.com/phy_demo/Galileo_thermometer/galileo-thermometer-d.html
- Carolyn B. Dowling, Klaus Neumann, Lee Florea (editors). *Geothermal Energy: An Important Resource*, The Geological Society of America, Special Paper 519, 2016.
- "Geothermal gradient." *A Dictionary of Earth Sciences*. . Retrieved April 30, 2019 from Encyclopedia.com: <https://www.encyclopedia.com/science/dictionaries-thesauruses-pictures-and-press-releases/geothermal-gradient>
- Ghodoosipour, B. (2014). Simplified Groundwater Modeling Approach for Quaternary Deposits in Laxemar-Simpevarp, Sweden. *World Environmental and Water Resources Congress 2014*. doi:10.1061/9780784413548.034
- Alok Krishna Gupta, Somnath Dasgupta (editors). *Physics and Chemistry of the Earth's Interior: Crust, Mantle and Core*, Indian National Science Academy, Springer, 2009.
- Hepler, L. G. (1969). Thermal expansion and structure in water and aqueous solutions. *Canadian Journal of Chemistry*, 47(24), 4613-4617. doi:10.1139/v69-762
- In, G. (1970, January 01). Geothermal Gradient. Retrieved from <http://www.geologyin.com/2014/12/geothermal-gradient.html>
- Kinzelbach, W. (n.d.). Groundwater Modelling - asian-gwadi.org. Retrieved from <http://www.asian-gwadi.org/wp-content/uploads/2017/09/5.Groundwater-Modelling.pdf>
- Laprise, R. (1992). The Euler Equations of Motion with Hydrostatic Pressure as an Independent Variable. *Monthly Weather Review*, 120(1), 197-207. doi:10.1175/1520-0493(1992)120<207:0.co;2
- Martinez, K. (n.d.). The Use of a Geographic Information System in Groundwater Modeling. *Surface Water Hydrology Department of Civil Engineering The University of Texas at Austin*.
- Non-Hydrostatic Groundwater. (n.d.). Retrieved from <https://www.observationalhydrology.com/non-hydrostatic-groundwater.html>
- Ramsey, A.S. *Hydrostatics*, Cambridge at the University Press, 1961.
- Seifert, D., Sonnenborg, T. O., Scharling, P., & Hinsby, K. (2007). Use of alternative conceptual models to assess the impact of a buried valley on groundwater vulnerability. *Hydrogeology Journal*, 16(4), 659-674. doi:10.1007/s10040-007-0252-3

James Stewart, Troy Day. *Biocalculus: Calculus, Probability, and Statistics for the Life Sciences*, Cengage Learning, 2016.

Scientific modelling. (2018, November 13). Retrieved from <https://www.sciencelearn.org.nz/resources/575-scientific-modelling>

The Hydrostatic Equation [PPT]. (n.d.). no author listed

Sokolova, T. S. (n.d.). Calculated coefficient of volumetric thermal expansion. Retrieved from https://www.researchgate.net/figure/Calculated-coefficient-of-volumetric-thermal-expansion-of-a-Al2O3-solid-line-in_fig2_308601110

Willis, R., & Yeh, W.W.G.. *Groundwater systems planning and management*. United States.

Water - Specific Volume. (n.d.). Retrieved from https://www.engineeringtoolbox.com/water-specific-volume-weight-d_661.html

APPENDIX





