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Bicycle speed through different gears with drag

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Bicycle speed through different gears with drag

Abstract

When riding a bicycle, the rider experiences two forces: the force of the road that acts upon the rear wheel while pedaling, and an opposing force that exerts on the bicycle/cyclist, called aerodynamic drag force (drag). When adjusting Newton's equation in the second law of motion, the instantaneous change in velocity is equal to the object's net force divided by its mass. The corresponding differential equation has not been solved analytically. With the bicycle starting at rest at $t=0$, we solved the equation numerically for any time t by using what is known as Euler's Method to plot the velocity as a function of time of the bicycle through gear shifts.

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PROBLEM STATEMENT

The goal of this paper is to plot the velocity of a bike rider with drag force as a function of time given the needed parameters and the pedal force supplied by the rider. The velocity will start from zero - meaning the bike is initially at rest. Velocity will be plotted through three gear changes.

MOTIVATION

When an analytical solution to the original differential equation cannot be obtained, numerical analysis, in particular Euler's Method can be used. A bicycle shifting through gears with drag is a good illustration to show Euler's Method. With the bicycle starting at rest, its velocity can be plotted with respect to time.

MATHEMATICAL DESCRIPTION AND SOLUTION APPROACH

We remind the reader of Newton's second law of motion, since it will be used in the solution of the problem, and it will play a central role. The second law of motion states that the net external force of an object is proportional to its mass times its acceleration:

$$\sum F = Ma \quad (1)$$

In our case, the only two forces we will be considering in (1) are f_{Road} and f_{drag} , where f_{Road} is the force of the road on the rear wheel while the cyclist pedals, and f_{drag} is the drag force exerted on the cyclist. Rearranging Newton's equation we have:

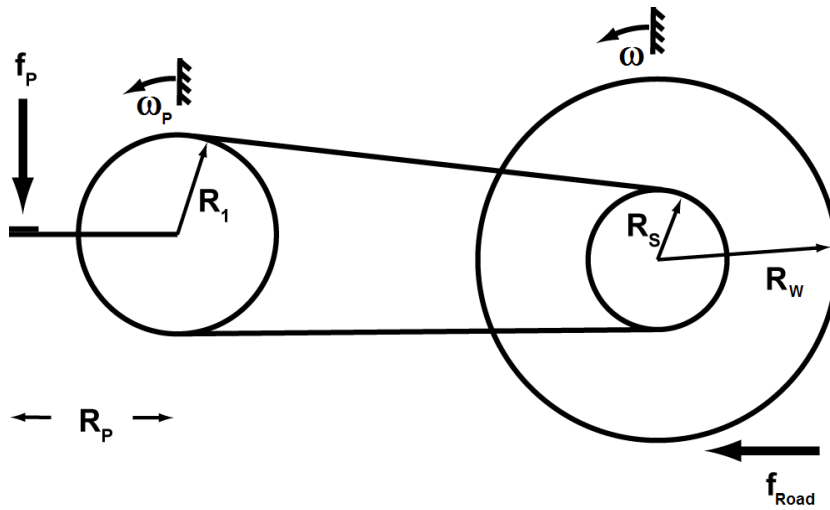
$$Ma = f_{Road} - f_{drag} \quad (2)$$

We will relate the force acting on the bicycle/cyclist system to its velocity by means of equation (2). Since the derivative of velocity with respect to time is acceleration, we can substitute it in (2) to have the following equation:

$$M \frac{dv}{dt} = f_{Road} - f_{drag} \quad (3)$$

In (3), t is time and M is the *combined* mass of the cyclist and the cycle in kg.

Equation (3) needs to be in terms of the known parameters. Consider the diagram below:



A torque balance around the front sprocket results in [5]:

$$f_p R_p = R_1 f_c \quad (4)$$

In (4), R_1 is the front sprocket radius, R_p is the pedal radius, f_p is the force exerted by the cyclist in Newtons, and f_c (not shown) is the force of the chain on the front sprocket. For the sake of simplicity moments of inertia of the front sprocket and rear wheel are ignored in this analysis.

A torque balance in the rear wheel results in [5]:

$$f_c R_s = R_w f_{Road} \quad (5)$$

In (5), R_s is the radius of the rear sprocket, which changes through shifts. R_w is the radius of the rear wheel. Equation (4) and (5) imply that:

$$f_{road} = \frac{f_p R_p R_s}{R_1 R_w} \quad (6)$$

The drag force on the cyclist can be modeled by

$$f_{drag} = \rho A C_D v^2 / 2, \quad (7)$$

where ρ is the density of air (1.25 kg/m³ obtained from engineersedge.com [1]), A is the area of the cyclist and bike normal to the direction of travel, C_D is the drag coefficient and v is the velocity of the bicycle [2, 3].

There is also a relationship between velocity v of the bike and angular velocity ω of the rear wheel and between the angular velocity of the rear wheel and the angular velocity of the front sprocket ω_p . These are:

$$v = \omega R_s, \quad \omega R_s = \omega_p R_1 \quad (8)$$

We can substitute equations (4) – (8) into our differential equation (3) to obtain

$$\frac{dv}{dt} = \frac{f_p R_p R_s}{M R_1 R_w} - \frac{\rho A C_D v^2}{2M} \quad (9)$$

The velocity can now be calculated as a function of time through three gear changes, starting in first gear from rest at $t=0$. We need a reasonable angular velocity ω_p which will be used for each shift. Assume that the velocity v will be continuous across the shift and that ω_p and R_s will change instantaneously. Assume a constant pedal force f_p is 100 N and specify the mass M , area A , and the required radii.

Now we apply Euler's method [4]. To this purpose, we recall the definition of derivative of a function $f(t)$:

$$\left(\frac{df}{dt} \right)_{t=t_0} = f'(t_0) = \lim_{\Delta t \rightarrow 0} \frac{f(t_0 + \Delta t) - f(t_0)}{\Delta t}$$

We arrange this equation as follows: $f(t_1) \approx f(t_0) + \Delta t \cdot f'(t_0)$,

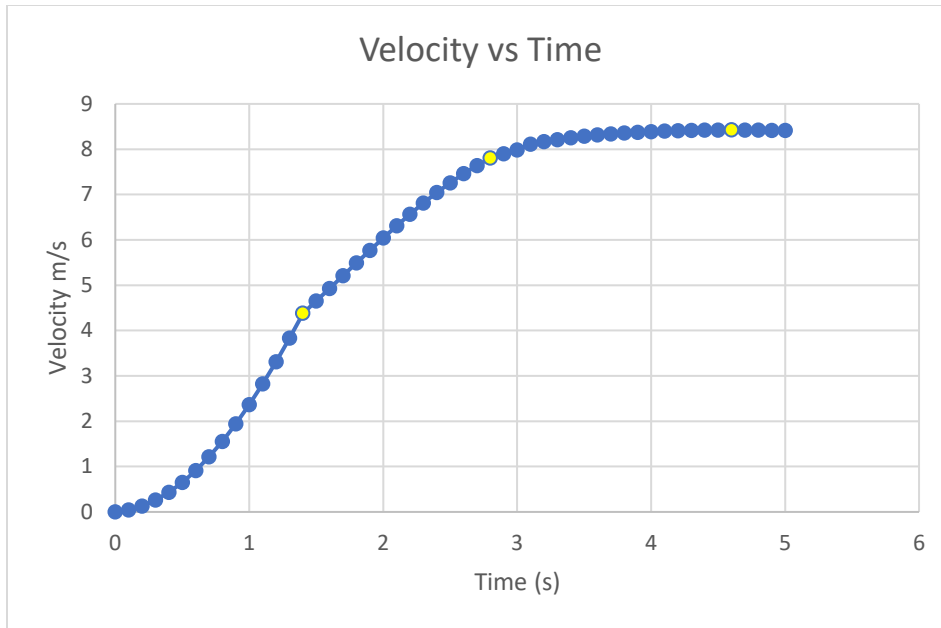
where $f(t_1)$ is just $f(t_0 + \Delta t)$. The equation above is only an approximation, but we know that as Δt gets smaller, our approximation becomes more accurate. We can imagine the procedure for finding $f(t_2) \approx f(t_1) + \Delta t \cdot f'(t_1)$ where $t_2 = t_1 + \Delta t$ and so on until we have every numerical answer for all the times needed in the function.

In our problem, we know the value of $v(t_0) = 0 \text{ m/s}$ and we also have our equation for v' (Equation 9). We will be starting at $t=0$ and use 0.1 seconds for Δt .

Finally, using Excel to our advantage, computations can be done by applying the procedure above.

DISCUSSION

For this trial we used our personal bicycle and measured all the needed values including our mass and the cross-sectional area of the bicycle and the rider himself. Starting in first gear from rest, a constant pedal force of one hundred Newtons we have shifted three times to end the trial in fourth gear. We have shifted at 15 radians/sec for all three shifts during a five second interval. Then the results on Excel before plotting Velocity vs. Time graph were calculated (see figure below):



Each highlighted point represents a gear shift. Velocity did not change across the shift, but ω_p and R_s changed instantaneously. Thus it is shown that a differential equation that does not have an analytical solution can be solved through numerical analysis.

CONCLUSION AND RECOMMENDATIONS

As one can see, Euler's method is a powerful approximation tool. We were able to numerically solve for velocity in all points of time needed to plot the bicycle speed through different gears with drag. With the results, ideas such as how much drag impacts a rider can be studied. It is also possible to apply similar methods to study optimization problems related to the speed of the bicycle in relation to changing radii of components, as well as how much speed is possible to maintain by decreasing the area normal to the drag force.

NOMENCLATURE

Symbol	Description	Unit	Value
f_p	Force of the pedal	Newtons	100
R_p	Radius of the pedal or crank	Meters	0.15
R_s	Radius of 1 st gear sprocket	Meters	0.09
$R_{s_{2nd}}$	Radius of second gear	Meters	0.05
$R_{s_{3rd}}$	Radius of 3 rd gear	Meters	0.0461
$R_{s_{4th}}$	Radius of 4 th gear	Meters	0.0458
M	Mass of bike and rider	Kilograms	120
R_1	Radius of front chainring	Meters	0.07
R_w	Radius of the rear wheel	Meters	0.37

ρ	Density of air(at sea level)	$\frac{kg}{m^3}$	1.25
A	Area of the rider and bicycle against drag force	m^2	0.75
C_d	Drag coefficient	unitless	0.88
v	Velocity of the bicycle and the rider	m/s	
ω_p	Rotational speed of the pedal	Radians/sec	15
ω	Rotational speed of the rear wheel	Radians/sec	Undesignated
f_c	Force of the chain	m	Undesignated

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