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Optimal Discrete-in-Time Inventory Control of a Single Deteriorating Product with Partial Backlogging

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Optimal Discrete-in-Time Inventory Control of a Single Deteriorating Product with

Partial Backlogging

by

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Dedication

To my beloved parents

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Abstract

The implicit assumption in conventional inventory models is that the stored products maintain the same utility forever, i.e., they can be stored for an infinite period of time without losing their value or characteristics. However, generally speaking, almost all products experience some sort of deterioration over time. Some products have very small deterioration rates, and henceforth the effect of such deterioration can be neglected. Some products may be subject to significant rates of deterioration. Fruits, vegetables, drugs, alcohol and radioactive materials are examples that can experience significant deterioration during storage. Therefore the effect of deterioration must be explicitly taken into account in developing inventory models for such products.

In most existing deteriorating inventory models, time is treated as a continuous variable, which is not exactly the case in practice. In real-life problems time factor is always measured on a discrete scale only, i.e. in terms of complete units of days, weeks, etc. In this research, we present several discrete-in-time inventory models and identify optimal ordering policies for a single deteriorating product by minimizing the expected overall costs over the planning horizon. The various conditions have been considered, e.g. periodic review, time-varying deterioration rate, waiting-time-dependent partial backlogging, time-dependent demand, stochastic demand etc. The objective of our research is two-fold: (a) To obtain optimal order quantity and useful insights for the inventory control of a single deteriorating product over a discrete time horizon with

deterministic demand, variable deterioration rates and waiting-time-dependent partial backlogging ratios; (b) To identify optimal ordering policy for a single deteriorating product over a finite horizon with stochastic demand and partial backlogging. The explicit ordering policy will be developed for some special cases.

Through computational experiments and sensitivity analysis, a thorough and insightful understanding of deteriorating inventory management will be achieved.

Chapter 1

Introduction

1.1 Background

Nowadays, for most successful, well-organized businesses, inventory control systems play a critical role in ensuring that adequate inventories are on hand to satisfy their customer demand. In general, the inventories can be classified into the following four categories (Nahmias, 2001):

(1) Raw materials. The raw materials are the resources required in the production or processing activity of a firm.

(2) Components. The components correspond to items that have not yet reached completion in the production process. Sometimes components are referred to as subassemblies.

(3) Work-in-process. This is the inventory either waiting in the system for processing or being processed. Work-in-process inventories include component inventories and may include some raw materials inventories as well.

(4) Finished goods. The finished goods are also known as end items, which are the final products of the production process.

As inventories are expensive and need careful control, a fundamental question arises: Why do organizations hold inventories? Generally speaking, the main reason why inventories are held is to provide a buffer between uncertain supply and demand. For example, if a customer places an order and there are no items available immediately, it is very likely that the customer will go somewhere else and may never return. So the uncertainty of external demand is the most important reason to hold inventories. There are also some other reasons. For instance, if the fixed setup/ordering costs are high, it would be economical to produce/order a relatively large quantity and store them for future use. Also, sometimes if the price of a product increases over a short period of time, then it will be more economical to buy a large amount of the product at current price and put into storage than to pay a higher price in future.

Since it is necessary to hold some inventories on hand, then it is essential to manage inventories economically. Inventory control consists of all the activities and procedures used to ensure the right amount of products is held in stock (Waters, 1992). A frequent objective of inventory control is to provide a moderate amount of inventories at minimum cost, so inventory control often relies on a tradeoff between conflicting costs. The relevant costs that are considered in most inventory systems are as follows (Nahmias, 2001).

(1) Fixed Order Cost. It is incurred independent of the size of the order as long as the order quantity is not zero.

(2) Unit Purchasing Cost. It is incurred on a per-unit basis.

(3) Inventory Holding Cost. It is also known as the inventory carrying cost, which is the sum of all costs that are proportional to the amount of inventory physically on hand. Some of the components of the holding cost include (i) Cost of providing the physical space to store the items. (ii) Taxes and insurance. (iii) Opportunity cost of alternative investment. The inventory holding cost ($/unit/year) is usually

measured by the product of unit purchasing cost and annual interest rate. The interest rate is an aggregated term comprised of some components like cost of capital, taxes and insurance, and cost of storage.

(4) Backlogging Cost. It includes whatever bookkeeping and/or delay costs that might be involved and “loss-of-goodwill” cost. (Orders that cannot be filled immediately are held on the books until the next shipment arrives.)

(5) Penalty Cost for Lost Sales. It includes the lost profit that would have been made from the sale and “loss-of-goodwill” cost.

As one can see, the "loss-of-goodwill" cost is included in either backlogging or penalty cost, and is a measure of customer unsatisfaction. Estimation of the “loss-of-goodwill” cost can be very difficult in practice.

By using cost minimization as an optimization criterion, the following two questions which reflect the fundamental problem of inventory control can be answered.

(1) When should an order be placed?

(2) How many should be ordered?

Regarding the time of ordering, there are two distinct inventory systems with different timing of replenishment: a periodic review system and continuous review system. A periodic review system allows inventory levels to be checked at discrete times periodically, and the order size is subject to change according to the variation in demand in each period. This system is often used in supermarkets, where stocks are reviewed at the end of each day and any sold units are replaced. A continuous review system allows the level of inventory to be monitored continuously and an order is placed whenever the inventory decreases to a specified level. The time between two consecutive orders is subject to change according to the variation in demand over time. Minimizing the total costs of the inventory system can yield the optimal quantity of ordering.

1.2 Inventory Control Models

1.2.1 Classification Criteria

The inventory control models in the literature can be classified according to the following criteria (Ravindran, 2008).

(1) Stocking location: Single stocking location (single location models) and more than one stocking location (multi-echelon inventory models).

(2) Supply process lead times: Deterministic lead time and stochastic lead time.

(3) Demand: Deterministic demand and stochastic demand. The stochastic demand can be stationary or nonstationary. Stationary stochastic demand means all demand parameters are constant over time. If the parameters change over time, then the demand is said to be nonstationary.

(4) Capacities: Uncapacitated inventory models and capacitated inventory models.

(5) Number of items: Single product inventory models and multiple product inventory models.

(6) Sourcing options: Single sourcing and multiple sourcing.

In this proposal, we are considering single stocking location, zero lead time, deterministic/stochastic demand, uncapacitated, single product, and single sourcing inventory models. One commonly used criterion is demand pattern. Based on this, the inventory control models can be classified as deterministic inventory models and stochastic inventory models.

1.2.2 Deterministic Inventory Models

Deterministic inventory models assume that the demand is fixed and known. Two most famous models of this kind are presented in the following section.

1.2.2.1 The Economic Order Quantity (EOQ) Model

The EOQ model lays the foundation for all inventory models. It is the most important analysis of inventory control and describes the important trade-off between fixed order cost and holding cost. The first reference to the EOQ model is by Harris (1913) but this model was popularized by Wilson (1934).

In this model, the demand per unit time is assumed to be a known constant R. A constant fixed cost is incurred whenever an order is placed. An order of quantity Q is placed whenever the on-hand inventory becomes zero and the replenishment time is assumed to be zero. The unit purchasing cost c is constant and known. The on-hand inventory is charged with a constant holding cost h per unit per unit time. Shortages are not allowed. The objective is to determine Q so as to minimize the total average cost. The relationship between order and on-hand inventory can be depicted in Figure 1.1, where T (= Q/R) is called the cycle time. Without loss of generality, it is assumed that the initial inventory is zero.

There are two cost components: ordering cost and inventory holding cost. Since all cycle are identical, it is only necessary to derive these costs in a cycle. The ordering cost consists of a fixed order cost K and purchasing cost cQ. The average inventory in a cycle is $Q/2$. Therefore, the total cost per unit time, $C(Q)$, is

$$C(Q) = \frac{K + cQ}{T} + \frac{hQ}{2} = \frac{K + cQ}{Q / R} + \frac{hQ}{2} = \frac{KR}{Q} + Rc + \frac{hQ}{2}$$

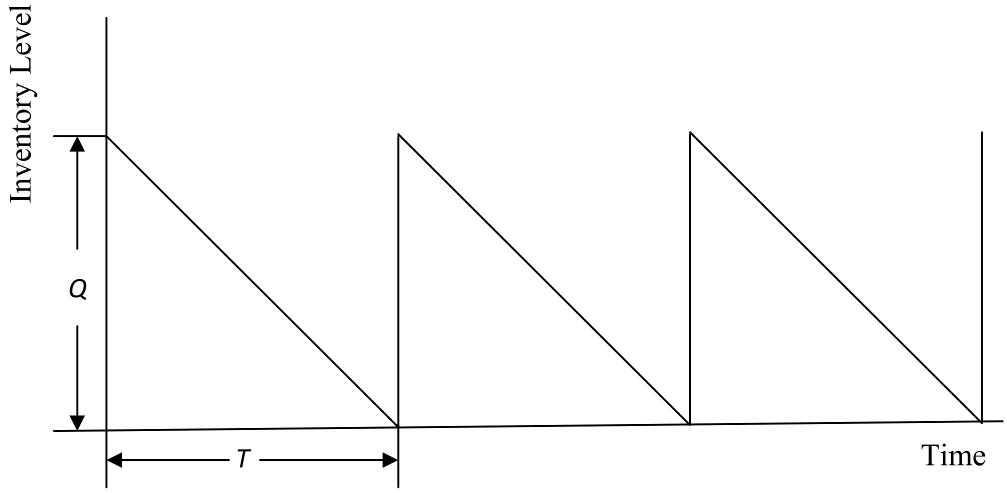


Figure 1.1 Inventory Levels for the EOQ Model

In $C(Q)$, only ordering quantity Q is a decision variable. The first and second order derivatives of $C(Q)$ are given by

$$C'(Q) = \frac{-KR}{Q^2} + \frac{h}{2}$$

$$\text{and } C''(Q) = \frac{2KR}{Q^3} > 0 \text{ for } Q > 0.$$

Since $C''(Q) > 0$, $C(Q)$ is a convex function of Q, for $Q > 0$. The optimal value of Q occurs where $C'(Q) = 0$. So the optimal value of Q can be derived as

$$Q^* = \sqrt{\frac{2KR}{h}}.$$

The Q^* is known as the economic order quantity. One of the main strengths of the EOQ model is that the average total cost increase only slightly for any order quantity Q close to EOQ Q^*. As a matter of fact, $C(\alpha Q^*) = [\frac{1}{2}(\alpha + \frac{1}{\alpha})]\, C(Q^*)$. For example, $\alpha = 3/2$ (or 2/3), $C(\alpha Q^*) = 1.08C(Q^*)$. That is, when the order quantity Q is either 50% higher than the EOQ or the EOQ is 50% higher than the order quantity, the resulting average total cost is only about 8% above the optimal cost. This is very important when there is uncertainty in data, or some other factors prevent the calculated EOQ from the true optimal value, a close estimation will yield good results too (Waters, 1992).

1.2.2.2 The Wagner-Whitin Model

Wagner and Whitin (1958) consider a finite planning horizon, discrete time dynamic lot sizing problem for a single product. Demand is known but varying for each period, i.e. time-varying demand. There are no shortages and backorders. No capacity constraints are considered. A positive fixed order cost is incurred each time an order is placed. In each period, you have to decide if an order should be placed and how many to order. The objective is to find the optimal order quantity in each period so that the total costs over the planning horizon are minimized.

Notations are as follows.

(1) N = number of periods in the planning horizon.

(2) K_t = fixed order cost in period t.

(3) h_t = inventory holding cost per unit remaining at the end of period t.

(4) d_t = demand in period t.

(5) M = a large number.

(6) Q_t = order quantity in period t.

(7) y_t = binary variable.

(8) I_t = inventory remaining at the end of period t. I_0 denotes the initial inventory.

The mixed integer programming (MIP) model can be formulated as follows.

Minimize $\sum_{t=1}^{N} K_t y_t + \sum_{t=1}^{N} h_t I_t$

Subject to $I_{t-1} + Q_t = D_t + I_t$

$My_t \geq Q_t$ for all *t*.

$Q_t, I_t \geq 0$ for all *t*.

y_t binary for all *t*.

The follows are the main conclusions drawn from this dynamic lot size model (Ravindran, 2008; Wagner and Whitin, 1958).

(1) Inventory is held over a period if and only if the ordering costs are bigger than the holding costs.

(2) Replenishment occurs only when the inventory level goes to zero (zero inventory ordering property). Consequently, order quantity must cover demand over an integer number of periods.

(3) $Q_{t-1} I_t = 0$ for all *t*.

(4) If the ending inventory of one period is positive, then this ending inventory level is at least the next period's demand. The maximum amount is all remaining period's demand.

(5) If the ending inventory of one period is zero, then an order must be placed in the next period. The lower bound of this order quantity is equal to the next period's demand and the upper bound is the total demand of all remaining periods.

1.2.3 Stochastic Inventory Models

Stochastic inventory models assume that the demand follows some known distributions.

1.2.3.1 One-Period Stochastic Inventory Model

This model is originated in terms of a newsboy who must decide how many units (newspapers) to buy at the beginning of a day before selling them on a street corner during the day. The problem arises because the customer demand is uncertain. If the newsboy buys too many, he will end up with unsold stock of newspapers which is valueless at the end of the day. If he buys too few, he will have unsatisfied demand which could have yielded a profit. The newsboy's objective is to determine a proper number of units to buy so as to maximize his total profit per day.

Notations and assumptions are as follows.

(1) c_o is the cost of unit inventory that is left at the end of the period (overage cost).

(2) c_u is the cost of unit unsatisfied demand (underage cost).

(3) The demand D is a continuous nonnegative random variable with a density function $\phi(x)$ and a cumulative distribution function $\Phi(x)$.

(4) The decision variable Q is the amount of units ordered at the beginning of the period.

(5) $C(Q, D)$ is the total cost incurred at the end of the period.

If Q units are purchased and the demand is D, then the leftover at the end of the period will be $Q - D$ if $Q > D$, and the unsatisfied demand will be $D - Q$ if Q < D. Hence, the total overage and underage cost can be written as

$$C(Q, D) = c_o \max(0, Q - D) + c_u \max(0, D - Q).$$

Then the expected cost function is

$$C(Q) = E(C(Q,D)) = c_o \int_0^{\infty} \max(0, Q-x)\phi(x)dx + c_u \int_0^{\infty} \max(0, x-Q)\phi(x)$$

$$= c_o \int_0^{Q} (Q-x)\phi(x)dx + c_u \int_0^{\infty} (x-Q)\phi(x)dx$$

Applying the Leibniz's rule, we can obtain the first and second order derivatives of the expected cost function $C(Q)$ as

$$C'(Q) = c_o \int_0^{Q} 1\phi(x)dx + c_u \int_0^{\infty} (-1)\phi(x)dx = c_o\Phi(Q) - c_u(1-\Phi(Q))$$

$$\text{and } C''(Q) = (c_o + c_u)\phi(Q) \geq 0 \text{ for all } Q \geq 0.$$

Since $C''(Q) > 0$, it follows that $C(Q)$ is a convex function of Q. The optimal value of Q occurs where $C'(Q) = 0$. So the optimal value of Q can be computed by

$$\Phi(Q^*) = \frac{c_u}{c_o + c_u}.$$

The right-hand side of the above equation is called critical ratio, and such an optimal solution is called a critical ratio solution. Since c_u and c_o are both positive, then this critical ratio is strictly between 0 and 1. So the above equation is always solvable for a continuous demand distribution.

1.2.3.2 Multi-Period Stochastic Inventory Model

In this model, it assumes that the system will be run for a finite number of periods. Demand in each period is stochastic. Any remaining inventory left at the end of one period can be used in the following period. An inventory holding cost will be incurred for any positive leftover stock. If demand exceeds the on-hand inventory, the excess amount will be backlogged, which means the customers are willing to wait for the next replenishment. A penalty cost will be incurred for any backlogging amount. The system is under periodic review, i.e. the inventory is checked at the beginning of each period and a decision is made on how many to order.

If the model does not include a fixed order cost, it will be classified as periodic-review stochastic inventory model. If the model does include a fixed order cost that is incurred for any non-zero order quantity, it will be classified as stochastic lot-sizing inventory model.

A base stock policy is found to be optimal for periodic-review stochastic inventory model. The decision rule for this policy is as follows: if the inventory is below the base stock, then you should order up to this base stock level; if the inventory is above the base stock, then you order nothing. A (s, S) policy is proved to be optimal for stochastic lot-sizing inventory model. The decision rule for this policy is as following: if the inventory level is below s, then you should order up to S; otherwise do nothing.

Those two types of models will be discussed in detail in Chapter 2.

1.3 Inventory Control for Deteriorating Items

As previous sections indicate, one implicit assumption in most existing inventory models is that products can be stored indefinitely to meet the future demands. However, in general, almost all items deteriorate to a certain extend over time. If the rate of deterioration is small and negligible, its effect can be ignored. Nevertheless, there are many products in the real world that are subject to a significant rate of deterioration. For example, some commonly used products like fruits, vegetables, meat, foodstuffs, perfumes, drugs, alcohol, gasoline, radioactive substances, photographic films, electronic components, etc., can experience significant deterioration. Hence, the impact of product deterioration should be considered explicitly in developing inventory models for those types of products.

In general, deterioration is defined as decay, damage, spoilage, evaporation, obsolesce or loss of utility of an item such that it cannot be used for its original purpose. The inventory models dealing with deterioration can be classified into two categories (Dave, 1985). The first category includes models with age-dependent ongoing deterioration (i.e., the items have fixed life-times). Such models are referred to as perishable inventory models. The second category consists of models with age-independent ongoing deterioration (i.e., the products have random life-times). Such models are called deteriorating inventory models. Milk, fish and blood are examples of category one, while perfume, alcohol and gasoline are examples of category two.

It was Van Zyl (1964) who first started the research work of perishable products inventory with a life-time of two periods. This work was extended by Fries (1975) and Nahmias (1975) independently by considering life-time of m ($\geq$ 2) periods. Then some researchers (e.g. Ishii et al., 1981; Nose et al., 1981, 1984) considered this perishable inventory model with a non-zero lead time. An excellent review of such perishable inventory models is given by Nahmias (1982).

Ghare and Schrader (1963) were the first to start the research of deteriorating product inventory by developing a model with exponential deterioration of inventory. Covert and Philip (1973) and Philip (1974) studied models under the assumption that the time of deterioration of an item follows a Weibull distribution. Since then, many researchers have devoted to the deteriorating inventory issues. Dave (1981) studied an inventory system with finite planning horizon, multiple replenishments, constant deterioration rate and linearly changing demand rate. Hariga (1996) considered a finite planning horizon inventory model with constant deterioration rate and time-varying demand. Wu (2001) studied a generalized EOQ model with Weibull distribution deterioration, ramp type demand rate and variable backlogging ratio. Deng (2005) generalized Wu's model by eliminating an unnecessary condition to ensure the existence of the minimum solution. Interested readers can refer to Raafat (1991), Shah and Shah (2000), and Goyal and Giri (2001) for excellent reviews of such models.

For the deteriorating inventory problems, there is little literature dealing with discrete time and stochastic demand, so we conducted a discrete-in-time inventory research with various demand patterns to mitigate this gap.

1.4 Dissertation Outline

The dissertation is organized as follows.

Chapter 2 reviews the existing literature that is relevant to the dissertation research.

In chapter 3, a deterministic inventory model is developed for a single deteriorating product in which we consider time as a discrete variable, constant cycle time, constant demand rate, constant deterioration rate (deterioration is a constant fraction of on-hand inventory), complete backlogging and the time point at which the inventory level goes to zero is a non-integer. The sensitivity analysis is conducted to provide some insights about deterministic inventory control for deteriorating items.

In chapter 4, a deterministic inventory model is developed for a single deteriorating product in which we consider time as a discrete variable, constant cycle time, time-varying demand pattern, variable deterioration rate (deterioration is a fixed but various in each period fraction of on hand inventory), waiting-time-dependent partial backlogging ratio, and the time point at which the inventory level goes to zero is an integer. The sensitivity analysis is conducted through computational experiments and some insights are provided.

In chapter 5, a periodic-review stochastic inventory model is developed for a single deteriorating product in which we consider finite planning horizon, periodic review, independent and identically distributed demand (i.i.d.) in each period, constant deterioration rate (deterioration is constant fraction of ending inventory), and constant partial backlogging ratio. The costs considered are unit purchasing cost, unit holding cost, unit backlogging cost, and unit penalty cost. We test whether the base stock level policy applies, and if it does, under what conditions. The explicit ordering policy is developed for a special case.

In chapter 6, a stochastic lot-sizing inventory model is developed for a single deteriorating product in which we consider finite planning horizon, periodic review, independent and identically distributed demand in each period, constant deterioration rate (deterioration is a constant fraction of ending inventory), and constant partial backlogging ratio. The costs considered are fixed order cost, unit purchasing cost, unit holding cost, unit backlogging cost, and unit penalty cost. We test whether the (s, S) policy still holds, and if it does, under what conditions.

In chapter 7, a mixed integer programming (MIP) model is built for a more general case, with fixed order costs, non-stationary stochastic demands and service-level constraints. Some numerical examples are presented and solved by programming in

C++/CPLEX to provide explicit ordering policies and comparisons with non-deteriorating case.

Chapter 8 concludes the dissertation and discusses possible future work.

Chapter 2

Literature Review

2.1 Introduction

There has been tremendous research work done for deteriorating inventory control problems during recent decades. A general literature review of deteriorating inventory models is given in section 2.2. As mentioned in Chapter 1, our research includes adding deterioration and partial backlogging to traditional periodic-review stochastic inventory model and stochastic lot-sizing model, so the periodic-review stochastic inventory control problems and the stochastic lot-sizing problems are reviewed in sections 2.3, 2.4 and 2.5 respectively.

2.2 Deteriorating Inventory Models

As mentioned in Chapter 1, these types of models deal with products that have random life times, i.e. the amount of deterioration is a function of on-hand inventory level. Ghare and Schrader (1963) were the first to start the analysis of deteriorating inventory problems by developing an EOQ model with constant rate of deterioration. They formulated that the differential equation describing the inventory level $I(t)$ over a cycle is

$$\frac{dI(t)}{dt} + \theta I(t) = -d \ ,$$

where θ is the constant deterioration rate and d is the constant demand rate. The optimal inventory cycle time T^* is determined by the following equation

$$\frac{d\theta c}{2} - \frac{K}{T^2} + dhc + dhc\theta T = 0 .$$

where c is the unit purchasing cost, K is the fixed order cost, and h is the inventory holding cost. The optimal order quantity Q^* is then

$$Q^* = dT^* + \frac{d\theta T^{*2}}{2} .$$

The first EOQ model with varying rate of deterioration was developed by Covert and Philip (1973). They assume that the time to deterioration of an item follows a two-parameter Weibull distribution. The Weibull density function they use is

$$f(x) = \alpha\beta t^{\beta-1} \exp(-\alpha t^{\beta}) ,$$

where $\alpha > 0$ is the scale parameter, $\beta > 0$ is the shape parameter, and $t > 0$ is the time to deterioration. This function can stand for a decreasing, constant or increasing rate of deterioration. Using same notations as the previous model, the differential equation describing the inventory level over a cycle is

$$\frac{dI(t)}{dt} + \alpha\beta t^{\beta-1} I(t) = -d .$$

The optimal cycle time T^* is obtained by solving the following equation

$$cd\sum_{n=1}^{\infty} \frac{\alpha^n n\beta T^{(n\beta-1)}}{(n\beta+1)n!} + \frac{hd \exp(\alpha T^{\beta})}{2} - \frac{K}{T^2} = 0 .$$

The optimal order quantity is then

$$Q^* = d\sum_{n=0}^{\infty} \frac{\alpha^n T^{*(n\beta+1)}}{n!(n\beta+1)} .$$

The computational procedure for obtaining the optimal cycle time is given by the authors.

Since then, many researchers have devoted to the deteriorating inventory issues and numerous models that consider the effect of deterioration were developed. Dave and Patel (1981) studied an inventory system with finite planning horizon, multiple replenishments but equal order cycles, constant deterioration rate, no shortages and linearly changing demand rate. The objective is to find the optimal number of replenishments. Sachan (1984) extended Dave and Patel's (1981) model by allowing shortages and complete backlogging. He also corrected some approximation errors for their model. Bahari-Kashani (1989) extended Dave and Patel's (1981) model by dropping off the assumption that the planning horizon is divided into multiple equal ordering cycles, i.e. the replenishment cycle length can be varying. A heuristic method was developed to find the near optimal solution. Pal etc. (1993) developed an inventory model with infinite planning horizon, constant deterioration rate, deterministic demand rate which is a known function of the instantaneous inventory level, and no shortages. The objective is to determine the optimal cycle length. Chakrabarty etc. (1998) extended Covert and Philip's (1973) model by considering three-parameter Weibull distribution deterioration, shortages and time-varying demand. They provided a procedure to find the optimal cycle time. Wu (2000) considered a deteriorating inventory model with fixed cycle length, single replenishment per cycle, time-varying demand, Weibull distribution deterioration, and complete backlogging. The objective is to find the optimal time point at which inventory falls to zero, and hence identify the optimal order quantity. There are two excellent review papers describing such models developed before 2001 in detail -- Raafat (1991) and Goyal and Giri (2001).

More recently, some inventory models for deteriorating items take into consideration of the effect of partial backlogging. Wu (2002) extended Wu's (2000) model by considering that the shortages are partially backlogged and the partial backlogging ratio is dependent on the length of the waiting-time until the next replenishment. The inventory system developed is illustrated by Figure 2.1.

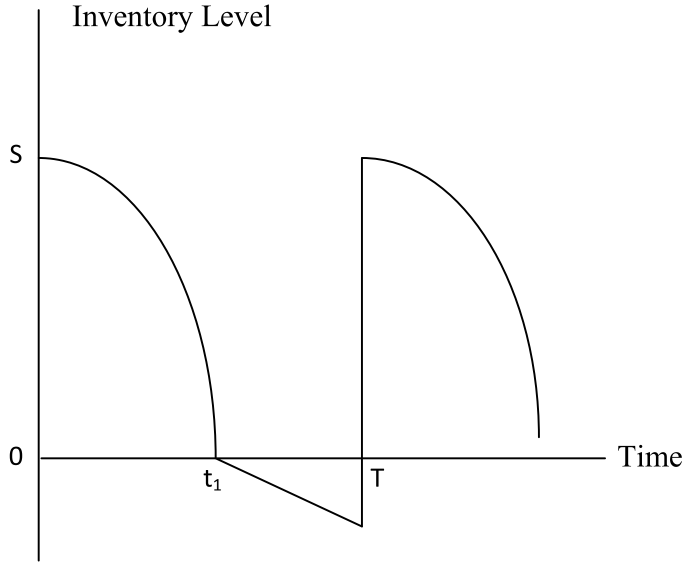


Figure 2.1 Wu's (2002) Inventory System

The differential equations describing inventory level over time are formulated as follows.

$$\frac{dI(t)}{dt} + \alpha\beta t^{\beta-1} I(t) = -D(t), \; 0 \le t \le t_1,$$

where $\theta(t) = \alpha\beta t^{\beta-1}$ is Weibull distribution deterioration, $D(t)$ is the time-varying demand rate, and t_1 is the time point at which inventory level falls to zero.

$$\frac{dI(t)}{dt} = \frac{-D(t)}{1+\delta(T-t)}, \; t_1 \le t \le T.$$

where $\dfrac{1}{1+\delta(T-t)}$ is the partial backlogging ratio which decreases as waiting time (T-t) increases. The optimal t_1^* was found by minimizing the total costs per unit time $C(t_1)$. Unfortunately, it is very difficult to show that $C(t_1)$ is a convex function for all t_1 and the calculation of t_1^* is not very straight-forward.

Teng etc. (2005) establish a deteriorating EOQ model in which the demand rate is a function of the on-hand inventory and no shortages are allowed. There are three possible cases for this inventory problem and they establish the necessary and sufficient conditions for each case. Moreover, an algorithm to determine the optimal replenishment cycle time and ordering quantity is proposed to maximize the total profit. Teng and Chang (2005) establish an EPQ model for deteriorating items when demand rate is function of both stock level and selling price per unit. They provide the necessary conditions to determine an optimal solution that maximizes profits for the EPQ model. Chang etc. (2006) establish a finite horizon EOQ model with deterioration for a retailer to determine its optimal selling price, replenishment number and replenishment schedule. They prove that the optimal replenishment schedule exists and unique. A simple algorithm is provided to find the optimal solutions. Dye etc. (2006) consider an inventory system with non-constant purchase cost, time-varying demand, and partial backlogging rate which linearly depends on the total number of customers in the waiting line. They also provide a simple solution procedure to find the optimal replenishment schedule. Hou (2006) derives a deteriorating inventory model with stock-dependent demand and complete backlogging under inflation and time discounting of money over a finite planning horizon. They show that the total cost function is convex and an algorithm is presented to determine the optimal order quantity. Jolai etc. (2006) derive the optimal production over a finite planning horizon for items that follow a Weibull distribution deterioration with a stock-dependent demand, fixed partial backlogging rate and under inflation. Manna and Chaudhuri (2006) develop an order-level inventory system for deteriorating items with ramp type demand rate, finite production rate and time-dependent deterioration rate. The models of no shortage case and shortage case are discussed. Mandal etc. (2006) consider a deteriorating inventory model with finite demand rate and limited storage space. It is solved by modified geometric programming method and non-linear programming method. Ouyang etc. (2006) establish a general EOQ model for deteriorating items with waiting-time dependent partial backlogging and permissible delay in payments. They mathematically prove that the total cost function is strictly pseudo-convex so that the optimal not only exists but also is unique. Yang (2006) considers an inventory system with constant demand rate, Waiting-time-dependent partial backlogging and two warehouses under inflation. The own warehouse (OW) has a fixed capacity. The rented warehouse (RW) has unlimited capacity. The inventory holding and deterioration costs in RW are higher than those in OW. They prove that the optimal solution not only exists but also is unique. Dye etc. (2007) study an inventory system with constant demand rate, waiting-time-dependent partial backlogging and two warehouses. A rented warehouse is used when the ordering quantity exceeds the capacity of the owned warehouse. They obtain the condition when to rent the warehouse and provide simple solution procedures for finding the maximum total profit per unit time.

Dye (2007) develops a deteriorating inventory model with selling-price-dependent demand rate, time-dependent deterioration rate and waiting-time-dependent partial backlogging. They proved that the optimal replenishment schedule not only exists but also is unique for any given selling price. An algorithm to find the optimal selling price and replenishment schedule for the proposed model is developed. Balkhi and Tadj (2008) establish a generalized economic order quantity model for deteriorating items with time-varying demand, time-varying deterioration rate and waiting-time-dependent partial backlogging. The cost parameters are also assumed to be general functions of time. Necessary and sufficient conditions for a unique optimal solution are derived. Chern etc. (2008) consider a deteriorating inventory system with finite planning horizon, time-varying demand rate and waiting-time-dependent partial backlogging under inflation. They provide an algorithm for determining the optimal replenishment number and schedule. Rong etc. (2008) study a deteriorating inventory system with price-dependent demand, partial/fully backlogging, imprecise lead-time and two warehouses. Holding cost at rented warehouse decreases with the increase of distance from the market place. The optimal solutions are derived by maximizing the average profit. Roy (2008) develops a deteriorating inventory model with selling-price-dependent demand rate, time proportional deterioration rate and time-dependent holding cost. They considered both shortage case and no shortage case. Shah and Acharya (2008) formulate an order-level lot-size inventory model for a time-dependent deterioration and exponentially declining demand. The optimal solutions are obtained through minimizing the total cost per unit time. Lee and Hsu (2009) develop a two-warehouse inventory model for deteriorating items with time-dependent demand and a finite replenishment rate within a finite planning horizon. One approach to determine the number of production cycles and replenishment schedule is developed which permits variation in production cycle times. Shah and Shukla (2009) study a deteriorating inventory model with constant demand rate, constant deterioration rate and waiting-time-dependent partial backlogging. The convexity of the total cost function is shown numerically and the optimal order quantity is obtained through minimizing the total cost. Skouri etc. (2009) consider a deteriorating inventory model with general ramp type demand rate, Weibull deterioration rate and waiting-time-dependent partial backlogging. The optimal replenishment policy is derived under two different replenishment policies: (a) starting with no shortages and (b) starting with shortages. Yang etc. (2010) extended Hou's (2006) model by considering an deteriorating inventory lot-size model under inflation with stock-dependent demand rate, constant deterioration rate, and waiting-time-dependent partial backlogging. They proved that there exists a unique replenishment schedule and a good estimate for finding the optimal replenishment number is provided. Geetha and Uthayakumar (2010) studied an EOQ based model for deteriorating products with permissible delay in payments. They considered constant demand rate, non-instantaneous deterioration, and waiting-time-dependent partial backlogging. The necessary and sufficient conditions of the existence and uniqueness of the optimal solutions are provided. An up-to-date review paper is published recently by Li etc. (2010).

One common feature of the above deteriorating inventory models is that time is treated as a continuous variable, which may not always be the case in practice. For example, in some real-life problems, time may be better treated as a discrete variable and a result of say 6.65891 hours is difficult to measure and confusing. In this regard, some researchers have attempted to study the deteriorating inventory by considering time as a discrete variable. Dave (1978) develops a discrete-in-time EOQ model for deteriorating items with constant demand rate, constant deterioration rate and no shortage. The optimal solution for cycle time is derived.

Dave (1979) establishes a discrete-in-time order-level inventory model for deteriorating items with constant demand rate, constant deterioration rate, complete backlogging and predetermined fixed cycle length. Since time is considered as a discrete variable, the difference equations describing the inventory levels at different time points are formulated as

$$I(t+1)-I(t)=-\theta I(t)-d\,,\ t=0,1,2,\ldots,t_1-1$$

and $I(t+1)-I(t)=-d$, $t=t_1,t_1+1,\ldots,T$

where t_1 is the time at which inventory level goes to zero. The total cost per unit time $C(t_1)$ is derived. Since t_1 is an integer, the necessary and sufficient conditions for $C(t_1)$ to have a global minimum at t_1^* are

$$\Delta C(t_1^*-1)\le 0\le \Delta C(t_1^*)$$

and $\Delta^2 C(t_1)\ge 0$ for all $t_1=0,1,2,\ldots,T$,

where $\Delta C(t_1)=C(t_1+1)-C(t_1)$ and $\Delta^2 C(t_1)=\Delta(\Delta C(t_1))$. The sufficient condition was satisfied and the optimal t_1^* is derived by testing the necessary condition.

Dave and Jaiswal (1980) consider a discrete-in-time probabilistic inventory model for deteriorating items with stationary uniform demand, constant deterioration rate, and no shortages. The optimal solution for cycle time is derived. Dave and Shah (1982) extend Dave and Jaiswal's (1980) model by allowing lead time equal to one scheduling period. The optimal solution for cycle time is derived. Rengarajan and Vartak (1983) extended Dave's (1979) model by allowing time-dependent demand which occurs at the end of each period. They find that the initial stock level is not at all affected by the nature of demand. Dave (1984) generalized Dave and Shah's (1982) model by allowing lead time equal to a fixed constant. The optimal solution for cycle time is derived. Dave (1985) establishes a discrete-in-time deteriorating inventory model with demand rate linearly changing with time, constant deterioration rate, no shortages and finite planning horizon. The optimal replenishment number and schedule are derived. Dave (1987) considers three inventory systems, i.e. EOQ, order-level and order-level lot-size inventory systems, for deteriorating items with finite replenishment rate, constant demand rate and constant deterioration rate. The EOQ does not allow shortage. The order-level system allows shortage but assume the cycle time is a predetermined constant. The order-level lot-size system allows shortage and does not consider the scheduling period as a prescribed constant. Searching procedures for finding the optimal solutions are derived for all three systems. Dave (1988) studies a discrete-in-time deteriorating inventory model with constant demand rate, constant deterioration rate and no shortages under permissible delay in payments. There is no charge if the due amount is paid within this permitted settlement period. The solution procedure for optimal cycle time is provided. Dave (1990) considers a discrete-in-time deteriorating inventory model with stationary uniform and instantaneous demand occurring at the beginning of the scheduling period, constant deterioration rate and no shortages. Three inventory models are developed, i.e. a model with no lead time, a model with a deterministic lead time and a model with a lead time equal to a multiple of the scheduling period. The optimal solution for cycle time is derived for each model. Shah and Shah (1998) consider a discrete-in-time deteriorating inventory model with stationary uniform demand, constant deterioration rate and no shortages under permissible delay in payments. The optimal cycle time is derived by solving a sequence of inequalities. Shah (1998) considers a discrete-in-time deteriorating inventory model with stationary uniform demand, constant deterioration rate and no shortages. The unit purchase cost will increase from a specified future date. The procedure to find the optimal cycle time is provided.

2.3 Periodic-Review Stochastic Inventory Control Problem

In this type of problem, the planning horizon is finite. Demand in each period is stochastic. The system is under periodic review, i.e. the inventory level is checked at the beginning of each period and a decision is made on how many to order. The objective is to determine the optimal ordering policy at the beginning of each period with minimum expected overall purchasing, holding and shortage costs.

Arrow (1958) started this type of research by considering that the demands that arise in successive periods are independent and identically distributed random variables with known distribution functions. Recently, Porteus (2002) reconsidered this problem in his book with stochastic i.i.d demand. The complete proof was provided and the explicit optimal base stock level was computed for some special case. Since our proposed model will be derived based on Porteus's work, the detail review of his model is presented next.

Notations in Porteus' Model are as follows.

(1) c – unit purchasing cost (\$/unit)

(2) c_H – unit holding cost, charged against positive ending inventory (\$/unit)

(3) c_P – unit shortage cost (\$/unit)

(4) D – generic random variable representing demand, which is i.i.d over every period

(5) ϕ – demand density distribution

(6) x – inventory level before ordering (the state of the system)

(7) y – inventory level after ordering (the decision variable)

(8) N – the length of the planning horizon

(9) α – one-period discount factor ($\in (0,1]$)

(10) $x^+ = \max(x,0)$

(11) $x^- = \min(x,0)$

The expected one-period holding and shortage cost function is

$$L(y) = El(y-D)$$

where $l(x) = c_H x^+ + c_P(-x)^+$.

He derived the optimality equations for $1 \le t \le N$ as

$$f_t(x) = \min_{y \ge x}\left\{c(y-x) + L(y) + \alpha\int_0^\infty f_{t+1}(y-D)\phi(D)dD\right\},$$

where $f_{N+1}(x)$ is the terminal value function $v(x)$. Porteus proved that if the terminal value function is convex, then the optimal policy in each period is characterized by a single critical number, which is called base stock policy. That is, order up to the base stock level if the current inventory is less than the base stock and order nothing otherwise. The explicit optimal base stock level was derived for the case that the terminal value function has a slope of –c.

Karlin (1960) considered that the demands in successive periods are not identically distributed. He proved that the optimal ordering policy in each period is a base stock policy, but the base stock level may vary in successive periods. Karlin (1960) studied this dynamic stochastic inventory problem with demand distribution varying over successive periods in a cyclical fashion. The optimal ordering policies are derived for both backlogging and non-backlogging cases. Iglehart (1964) considered the dynamic inventory system with demand distribution possessing a density belonging to either exponential or range family of densities and having an unknown parameter. They applied a Bayesian estimation method to obtain the optimal ordering policies as the amount of demand information varies. Azoury (1985) considered the periodic review inventory problem in which one or more parameters of the demand distribution are unknown but with a known prior distribution chosen from the natural conjugate family. An explicit form of the optimal ordering policy is given. Gavirneni (2004) considered this type of problem with i.i.d demand and fluctuating purchasing cost. He showed that an order up to policy is optimal and proposed a method to predict the effectiveness of myopic heuristics. Bertsimas and Thiele (2006) proposed a general methodology based on robust optimization to study this type of stochastic inventory problem without assuming a specific distribution of the demand. They showed that the structure of the optimal robust policy is of the same base stock character. Levi etc. (2007) considered this type of stochastic inventory problem under the assumption that the explicit demand distributions are not known and that the only information available is a set of independent samples drawn from the true distributions. They described how to compute the optimal policies based only on the observed samples of the demands.

2.4 Stochastic Lot-Sizing Problem

This type of problem is just the periodic-review stochastic inventory control problem with fixed order cost.

It was Scarf (1960) who first established the optimal ordering policy structure for the stochastic lot-sizing problem with independent and identically distributed demands in successive periods. This problem was re-studied by Porteus (2002) and a complete proof was provided. Since our proposed model builds on Porteus's work, the detail review of his model is given below.

Notations in Porteus' Model are as follows.

(1) c – unit purchasing cost (\$/unit)

(2) c_H – unit holding cost, charged against positive ending inventory (\$/unit)

(3) c_P – unit shortage cost (\$/unit)

(4) K – fixed order cost (\$/order)

(5) D – generic random variable representing demand, which is i.i.d over every period

(6) ϕ – demand density distribution

(7) x – inventory level before ordering (the state of the system)

(8) y – inventory level after ordering (the decision variable)

(9) N – the length of the planning horizon

(10) α – one-period discount factor ($\in (0,1]$)

(11) $x^+ = \max(x,0)$

(12) $x^- = \min(x,0)$

The expected one-period holding and shortage cost function is

$$L(y) = El(y - D)$$

where $l(x) = c_H x^+ + c_P(-x)^+$.

The optimality equations for $1 \le t \le N$ was derived as

$$f_t(x) = -cx + \min\left\{G_t(x), \min_{y \ge x}[K + G_t(y)]\right\},$$

where $G_t(y) = cy + L(y) + \alpha \int_0^\infty f_{t+1}(y - D)\phi(D)dD$ and $f_{N+1}(x)$ is the terminal value function $v(x)$. Porteus proved that if the terminal value function is continuous and *K*-convex, then a (*s*, *S*) policy is optimal in each period. That is, order up to inventory level *S* if the current inventory is less than the level *s* and order nothing otherwise.

Schal (1976) generalized Scarf's result by finding some new conditions for the optimality of an (*s*, *S*) policy and a special case without assuming particular demand distributions was obtained. Iyer etc. (1992) analyzed the deterministic (*s*, *S*) inventory problem which is to determine parameters *s* and *S* such that implementing this (*s*, *S*) policy results in the minimum possible total costs given a set of demands for n periods. A polynomial time algorithm for finding an optimal (*s*, *S*) for the deterministic problem was provided. Sox (1997) considered the case in which the demand is random and the costs are non-stationary. He modeled the problem as a mixed integer nonlinear program. An optimal solution algorithm was developed. Then Gallego etc. (2000) studied the finite ordering capacity and they showed that the optimal capacitated policy has an (*s*, *S*)-like structure. Sobel and Zhang (2001) considered that the demands arrive simultaneously from a deterministic source and a random source. The deterministic demand has to be satisfied immediately and demand from a stochastic source can be backlogged if necessary. They proved that a modified (*s*, *S*) policy is optimal assuming that the stochastic demand is satisfied immediately if there is sufficient stock on hand. Dellaert and Melo (2003) considered a stochastic manufacturing system with only partial knowledge on future demand because customers tend to order in advance of their actual needs. A Markov decision model was formulated to find the optimal policy. Two approximate strategies for obtaining near-optimal production lot sizes were proposed. More recently, Ozer and Wei (2004) considered a capacitated production system faced by a manufacturer who has the ability to obtain advance demand information. The capacity constraint is that the number of production periods is limited. Two cases were analyzed. When there is no fixed cost, the optimal policy is of a state-dependent modified base stock policy. When there is positive fixed cost, they analyzed a class of production policies under which the manager is restricted to either producing at full capacity or not at all. Bensoussan etc. (2006) considered the effect of information delay between the current time and the time of the most recent inventory level known to the inventory manager. A constant delay and a random were both discussed. The optimal ordering policy is base stock level when there is no fixed order cost and (s, S) policy when fixed order cost exists.

When there are non-homogeneous stochastic demands and fixed order costs, the computation of the optimal ordering policies are extremely difficult, so a lot of researchers have attempted to develop some efficient computational algorithms to approximate the optimal ordering policies. Most of them are based on heuristic methods (Porteus 1985, Bollapragada and Morton 1999, Levi etc. 2007), while Gavirneni and

Tayur (2001) developed an efficient solution method – Direct Derivative Estimation (DDE) – for computing optimal order-up-to levels for a discrete time non-stationary inventory control model.

2.5 Summary

The following gaps are identified from literature review.

(1) Literatures for discrete-in-time deteriorating inventory models are very limited.

(2) No closed-form solutions for deteriorating inventory problems.

(3) When considering time-varying demand, variable deterioration rate, and waiting-time-dependent partial backlogging, the convexity of total cost function is not proved.

(4) No literatures for multi-period stochastic inventory model with deterioration and partial backlogging.

(5) No literatures for stochastic deteriorating inventory control under service-level constraints.

Our dissertation completes the current literature by filling up all those gaps. The results of this research have the potential to positively influence industrial engineering and management science curricula related to production and inventory control, large scale optimization, optimization modeling, and supply chain management. Furthermore our work could enhance student learning by providing practical examples and by development of case-studies on the design of control systems for deteriorating inventory.

Chapter 3

A Discrete-in-Time Inventory Model with Deterioration and Backlog

There is a single product with a constant rate of demand. Any physical inventory experiences a constant rate of deterioration, and any unsatisfied demand is completely backlogged. A previous model was presented to determine the optimal order-level for a given constant cycle time, where time is treated as a discrete variable. However, the optimal solution derived is only valid under the restriction that the physical inventory level goes to zero at an integer time. This is totally unnecessary from the practical view point. This chapter relaxes this restriction. Furthermore, a closed-form equation is derived to compute the optimal solution, while no closed-form solution was presented in literature (even for the integer-restrictive case). This greatly reduces the computational effort to identify the optimal solutions, and makes sensitivity analysis possible. Some insights are provided through sensitivity analysis.

3.1 Assumptions and Notations

The following assumptions are made.

(1) The cycle has T unit times, where T is a known constant.

(2) The demand rate of R units per unit time is a known constant.

(3) Rate of replenishment is infinite and lead time is zero. The fixed lot-size q raises the inventory at the beginning of each cycle to stock level S.

(4) Shortages are made up immediately after a fresh lot arrives.

(5) There is neither repair nor replacement for the deteriorated items in the inventory during a cycle.

(6) The unit purchasing cost *c*, inventory carrying cost *h* per unit per unit time and the shortage cost *b* per unit per unit time are known and constant.

(7) The deterioration rate θ is a constant (i.e., a constant fraction θ of the on-hand inventory deteriorates per unit time).

(8) The inventory level at any time t within the cycle is denoted as $I(t)$, $0 \le t \le T$.

3.2 Model

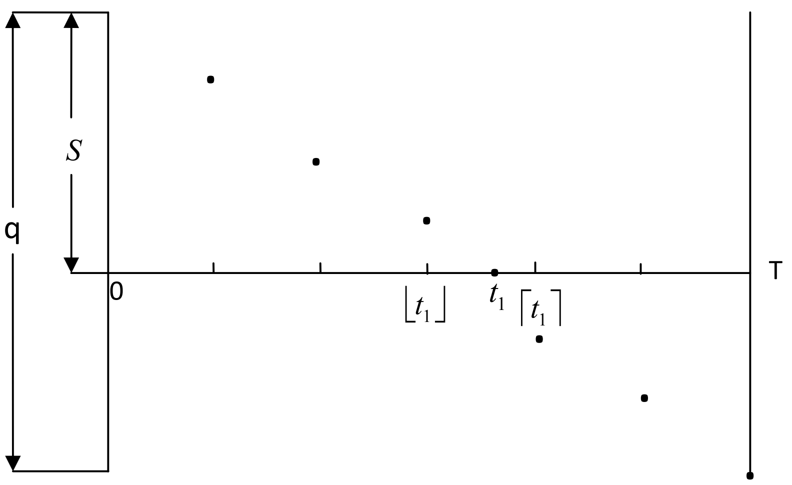


Figure 3.1 Inventory Level of the System

As illustrated in Figure 3.1, the fixed lot-size q raises the inventory at the beginning of a cycle to stock level *S*. This initial inventory is gradually reduced due to both demand and deterioration. At time $t = t_1$, for all $0 \le t_1 \le T$, the inventory level goes to zero, i.e., $I(t_1) = 0$. All demands occurring after time t_1 are fully backlogged and will be fulfilled by the new order from the next cycle. Therefore, the total backlog is equal to $R(T-t_1)$.

Since the lot-size q raises the initial inventory to S during every cycle, we have

$$q = R(T-t_1)+S \ . \tag{3.1}$$

With the depletion of inventory determined by both demand and deterioration, the difference equations describing the inventory level $I(t)$ of the system from time 0 to time t_1 are

$$I(t+1) = I(t) - \theta I(t) - R \ , \ t \in [0, t_1 - 1]. \tag{3.2}$$

Equation (3.2) can be rewritten as

$$\Delta I(t) + \theta I(t) = -R, \ t \in [0, t_1 - 1] \ , \tag{3.3}$$

where

$$\Delta I(t) = I(t+1) - I(t) \ . \tag{3.4}$$

Since there are no units held in inventory from time t_1 to time T , the deterioration will not exist during this time span. Hence, the difference equations describing the inventory level $I(t)$ of the system from time t_1 to time T are

$$I(t+1) = I(t) - R, \ t \in [t_1, T-1] \ , \tag{3.5}$$

which can be rewritten as

$$\Delta I(t) = -R, t \in [t_1, T-1] \ . \tag{3.6}$$

Solving (3.3) and using the boundary condition $I(t_1) = 0$, we obtain

$$I(t) = \frac{R}{\theta}[(1-\theta)^{t-t_1} - 1], t \in [0, t_1] \ . \tag{3.7}$$

Similarly, from equation (3.6) we get

$$I(t) = R(t_1 - t)\,, t \in [t_1, T] \ . \tag{3.8}$$

At time 0, the inventory level $I(0) = S$. Substituting this into (3.7) yields

$$S = \frac{R}{\theta}[(1-\theta)^{-t_1} - 1] \ . \tag{3.9}$$

Then from equation (3.1), we get

$$q = R(T - t_1) + \frac{R}{\theta}[(1-\theta)^{-t_1} - 1] \ . \tag{3.10}$$

Since the lot-size q is depleted by both demand and deterioration, and the total demand during one cycle is RT, then the total number of units, $D(t_1)$, that deteriorate during a cycle T will be

$$D(t_1) = q - RT = \frac{R}{\theta}[(1-\theta)^{-t_1} - 1] - Rt_1 \ . \tag{3.11}$$

Given unit cost c, the average deterioration cost per unit time is

$$\frac{cD(t_1)}{T} \ . \tag{3.12}$$

When t_1 is restricted to an integer, the average number of units in inventory during T can be computed as follows.

$$\bar{I}(t_1) = \frac{1}{2T}[I(0) + 2\sum_{j=1}^{j=t_1-1} I(j) + I(t_1)] \tag{3.13}$$

Note that (3.13) differs from Dave's in that the mean inventory in a period is set here as the average of the beginning and ending inventories for the period. However, when t_1 can be any real value, to compute $\bar{I}(t_1)$ becomes much more complicated if not impossible. This dilemma can be resolved by using the Trapezoid method. Since $I(t)$ is a polynomial function of t, it is a smooth function. Then (3.13) can be approximated by integration. In particular, the following is valid (Bronshtein, et al., 1997)

$$\bar{I}(t_1) = \frac{1}{T}\left(\int_0^{t_1} I(t)dt - \varepsilon\right), \tag{3.14}$$

where ε is the approximation error and can be computed by $\varepsilon = t_1 I''(\eta)/12$, $0 \le \eta \le t_1$. $I''(t) = \frac{R}{\theta}(1-\theta)^{t-t_1}\ln^2(1-\theta)$, $0 \le t \le t_1$. Therefore, $I''(\eta) \le \max\{I''(t)$: $0 \le t \le t_1\} = \frac{R}{\theta}(1-\theta)^{-t_1}\ln^2(1-\theta)$, and the error term is bounded by

$$\varepsilon \le \frac{t_1 R}{12\theta}(1-\theta)^{-t_1}\ln^2(1-\theta).$$

The integration is calculated as $\frac{1}{T}\int_0^{t_1} I(t)dt = -\frac{R[(1-\theta)^{-t_1} - 1 + t_1\ln(1-\theta)]}{T\theta\ln(1-\theta)}$. Then the relative error, $RE(\theta)$, of approximating $\bar{I}(t_1)$ by $\frac{1}{T}\int_0^{t_1} I(t)dt$ is bounded by

$$RE(\theta) = \frac{\varepsilon / T}{\frac{1}{T}\int_0^{t_1} I(t)dt} \le \frac{t_1(1-\theta)^{-t_1}\ln^3(1-\theta)}{12[1-(1-\theta)^{-t_1} - t_1\ln(1-\theta)]}. \tag{3.15}$$

When this error bound is small, the approximation is good. In doing so, the average inventory is computed by

$$\bar{I}(t_1) = -\frac{R[(1-\theta)^{-t_1} - 1 + t_1\ln(1-\theta)]}{T\theta\ln(1-\theta)}. \tag{3.16}$$

The use of this approximation will greatly simplify the computation of $\bar{I}(t_1)$. More importantly, this approximation will lead to a closed-form solution for the optimal t_1, which will be given at the end of this section.

From equation (3.7), the average number of units in inventory during T is calculated as follows:

$$\bar{I}(t_1) = \frac{1}{T}\int_0^{t_1} I(t)dt = -\frac{R[(1-\theta)^{-t_1} - 1 + t_1 \ln(1-\theta)]}{T\theta \ln(1-\theta)} \tag{3.17}$$

The average inventory holding cost per unit time is

$$h\bar{I}(t_1) \ . \tag{3.18}$$

From equation (3.8), the average shortage during T is

$$\bar{S}(t_1) = \frac{1}{T}\int_{t_1}^{T} [-I(t)]dt = \frac{R}{2T}(T - t_1)^2 \ . \tag{3.19}$$

Hence, the average shortage cost per unit time is

$$b\bar{S}(t_1) \ . \tag{3.20}$$

Adding up costs given by (3.12), (3.18) and (3.20) leads to the following total average cost of the system per unit time during one cycle

$$\begin{aligned} C(t_1) &= \frac{cD(t_1)}{T} + h\bar{I}(t_1) + b\bar{S}(t_1) \\ &= \frac{cR}{T\theta}[(1-\theta)^{-t_1} - 1] - \frac{cRt_1}{T} \\ &- \frac{hR[(1-\theta)^{-t_1} - 1 + t_1 \ln(1-\theta)]}{T\theta \ln(1-\theta)} + \frac{bR}{2T}(T - t_1)^2 \end{aligned} \tag{3.21}$$

The first-order derivative of $C(t_1)$ is given by

$$C'(t_1) = -\frac{cR}{T\theta}(1-\theta)^{-t_1} \ln(1-\theta) + \frac{hR[(1-\theta)^{-t_1} - 1]}{T\theta} + \frac{bR(t_1 - T)}{T} - \frac{cR}{T} \ . \tag{3.22}$$

Similarly, the second-order derivative of $C(t_1)$ is given by

$$C''(t_1) = \frac{cR}{T\theta}(1-\theta)^{-t_1}[\ln(1-\theta)]^2 - \frac{hR(1-\theta)^{-t_1}\ln(1-\theta)}{T\theta} + \frac{bR}{T} \ . \tag{3.23}$$

Since all the parameters are positive and $0 < \theta < 1$, it is obvious to see from equation (3.23) that $C^{''}(t_1) > 0$, for all $t \in [0, T]$. Therefore, $C(t_1)$ is a strictly convex function. This means that the solution to the equation $C^{'}(t_1) = 0$ is optimal to $\min_{t_1} \{C(t_1) | t_1 \in [0, T]\}$. Solving equation $C^{'}(t_1) = 0$, we get the optimal t_1, denoted as t_1^*, as follows:

$$t_1^* = \frac{c}{b} + \frac{W}{\ln(1-\theta)} + \frac{h}{b\theta} + T \ . \tag{3.24}$$

where W is

$$W = LambertW[\frac{c\ln(1-\theta) - h}{b\theta} \ln(1-\theta) e^{-\frac{\ln(1-\theta)(c\theta + h + b\theta T)}{b\theta}}] \ . \tag{3.25}$$

The Lambert W function is defined as

$$LambertW(x)e^{[LambertW(x)]} = x \text{, for } x > -\frac{1}{e}.$$

The result obtained here shows that the optimal value of t_1 is not affected by the demand rate. Substituting t_1^* into equations (3.9) and (3.10) yields the optimal lot-size and order-level.

3.3 A Numerical Example

The same example from Dave's paper is used to illustrate our methodology. The values of the parameters are given as follows.

(1) $R = 200$ units/month;

(2) $c = \$80.00$/unit;

(3) $h = \$1.00$/unit/month;

(4) $b = \$9.00$/unit/month;

(5) $T = 12$ months;

(6) $\theta = 0.05$

Substituting these numbers into equation (3.24) and solving by Maple 10.0, we obtain the following optimal value of t_1 as

$$t_1^* = 6.93 \ months.$$

From (3.9) and (3.10), the optimal order-level and lot-size are

$$S^* = 1708 \text{ and } q^* = 2722.$$

Also, substituting t_1^* into equation (3.21), the minimum average cost per month obtained by this method is

$$C(t_1^*) = \$4534.13 / month.$$

By Dave's method, t_1^* is computed as 7 months. Substituting this t_1^* value into equation (3.21), one gets the corresponding total monthly cost at \$4534.82, which is almost the same as the optimal total monthly cost given above. This should be expected, as the optimal t_1^* of 6.93 months is so close to 7 months. As a matter of fact, the total monthly cost as a function of t_1 is pretty flat around the optimal t_1^* as shown in Figure 3.2. This result echoes the behavior of the well-known EOQ model.

By Equation (3.15), the relative error, RE(θ), is computed as RE(0.05) < 0.17%. Since the inventory holding cost is only one of 3 cost components, the actual impact of the approximation by Equation (3.16) on the optimal total cost is far less than 0.17%. Therefore, the use of approximation (3.16) is appropriate.

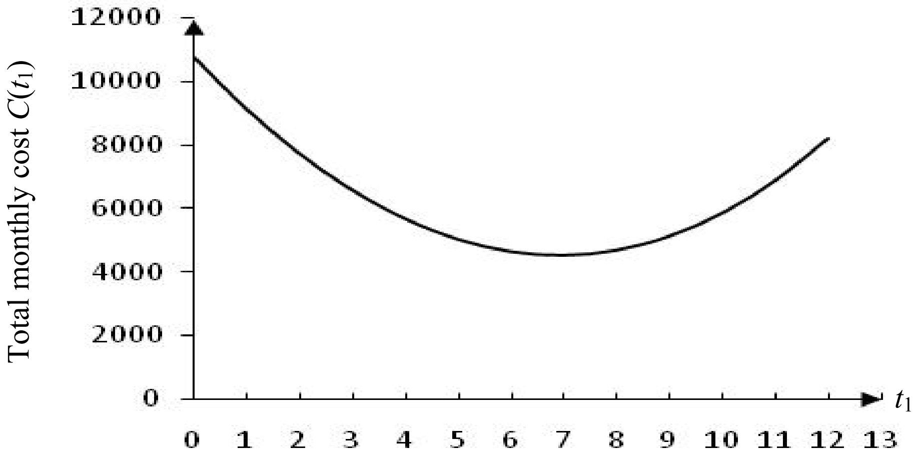


Figure 3.2 Total Monthly Cost $C(t_1)$ as a Function of t_1

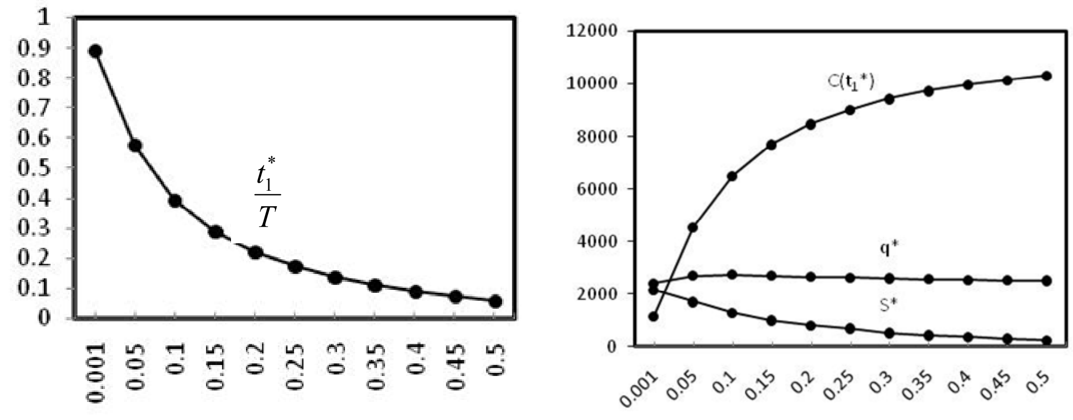
3.4 Sensitivity Analysis

The sensitivity of t_1^*, $C(t_1^*)$, q^* and s^* with respect to parameters θ, c, h, b and T is studied numerically in this section. The results are illustrated in Figure 3.3 to Figure 3.7. We use the values given in Section 4 as the base values for the various parameters. It is worth mentioning that the time unit considered in this paper is month. Moreover, to put the values of t_1^* in a perspective relative to cycle time T, we present the computational results using the ratio $\frac{t_1^*}{T}$ instead of the absolute value of t_1^*. The ratio $\frac{t_1^*}{T} = 0.5$ means that the inventory goes to zero in the middle of a cycle T. A ratio below or above 0.5 indicates the inventory goes to zero early or late in a cycle, respectively. Alternatively, a small ratio $\frac{t_1^*}{T}$ means a large backlog, while a large ratio indicates less backlogging.

The sensitivity study is conducted as follows. Only one parameter value is altered at a time, while holding all the other parameters constant. The range of θ is chosen from 0.001 to 0.5. This range represents a very little to significant deterioration. The ranges of c and b considered are, respectively, 40 ~ 360 ($/unit) and 3 ~ 24 ($/unit/month). The range of T is set to be from 3 to 24 (months). In practice, the annual inventory holding cost per unit can usually be expressed as the product of the unit purchasing cost c and an annual interest rate I (i.e., $h=Ic$), where the annual interest rate is an aggregated term comprised of cost of capital, taxes and insurance, cost of storage, and breakage and spoilage, etc. In the sensitivity study, we consider the range of the annual interest rate from 1% to 75%. This leads to the range of h values from 0.067 to 5 ($/month). The computational results and analyses are represented next.

Table 3.1 Different Optimal Values w.r.t. θ

θ	0.001	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
t_1*	0.891	0.5775	0.395	0.2875	0.2191	0.171	0.1366	0.11	0.089	0.07	0.058
S*	2152	1708	1295.5	1002.5	798.3	694.9	529.93	437.63	363.6	303	249
q*	2412	2721.4	2747.5	2712.5	2672.3	2634	2601.9	2573.6	2549	2529	2509
$C(t_1*)$	1168	4534.1	6504.3	7696.7	8479.3	9026	9428.6	9734.0	9972	10162	10315



(a) (b)

Figure 3.3 Sensitivity Analysis w.r.t. θ

(1) From Table 3.1 and Fig. 3.3(a), as θ increases from 0, the optimal ratio $\frac{t_1^*}{T}$ decreases rapidly from almost 1 until θ reaches about 0.2, and then the pace of decrease levels off and approaches close to 0. That is, for a rapidly deteriorating product (i.e., a large θ), it is better to have substantial backlogging to keep costs low, as expected. From Fig. 3.3(b), interestingly, as θ increases, the optimal lot-size q^* increases at first, peaks at around $\theta = 0.08$, and then decreases slowly. In general, q^* stays pretty steady, for all θ. However the optimal order-level s^* keeps decreasing at a much faster pace as θ increases. Moreover, the total optimal cost $C(t_1^*)$ increases very dramatically as θ increases from 0, and levels off after around $\theta = 0.3$. This indicates that at small values of θ, a little reduction of θ can lead to significant improvement of cost reduction.

Table 3.2 Different Optimal Values w.r.t. c

c	40	80	120	160	200	240	280	320	360
t_1*	0.6908	0.5775	0.496667	0.436667	0.388333	0.35	0.318333	0.290833	0.2675
S*	2119.7	1708	1430.3	1233.45	1080.05	961.59	865.82	784.15	715.93
q*	2861.7	2721.4	2638.3	2585.45	2548.05	2521.59	2501.82	2486.15	2473.93
$C(t_1*)$	3251.8	4534.1	5456.69	6157.28	6709.53	7157.04	7527.48	7839.42	8105.79

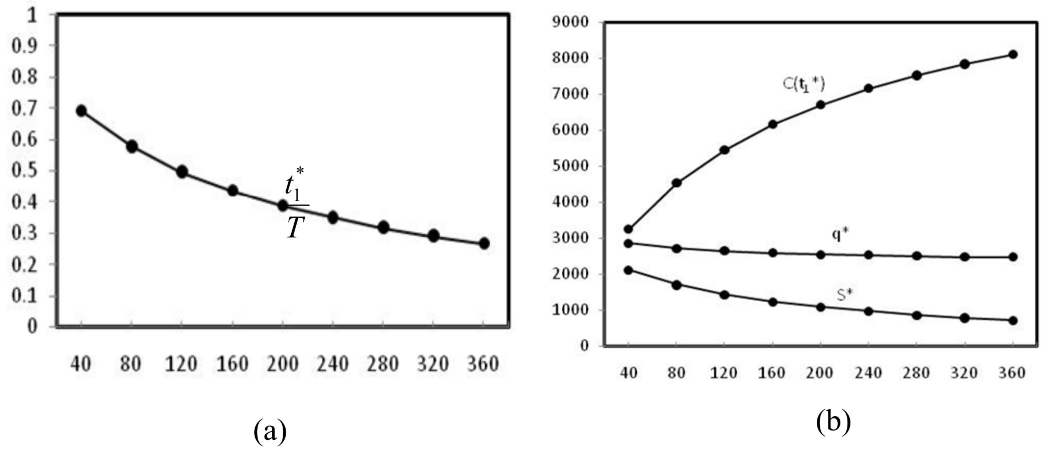
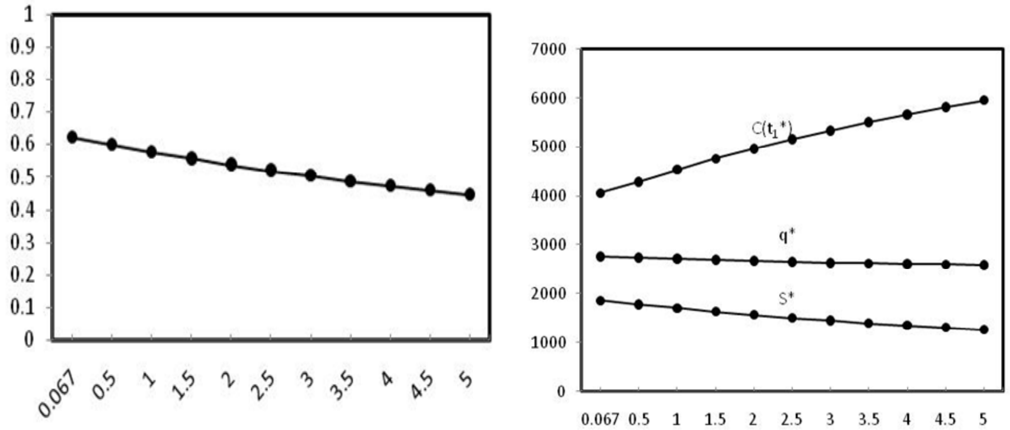


Figure 3.4 Sensitivity Analysis w.r.t. c

(2) As shown in Table 3.2 and Figure 3.4, there is a decrease in the optimal ratio $\frac{t_1^*}{T}$ with an increase in the unit price c. The optimal lot-size q^* and the optimal order-level s^* decrease as the unit price c increases. It should be noted that s^* is more sensitive than q^* with respect to the parameter c. In other words, the change of q^* is smaller than that of s^* with the increase of c. In fact, q^* is almost flat, as c changes. The optimal total monthly cost $C(t_1^*)$ increases more significantly as c increases.

Table 3.3 Different Optimal Values w.r.t. *h*

h	0.067	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
t_1*	0.62	0.6	0.5775	0.55666	0.5375	0.52	0.5033	0.4875	0.473	0.46	0.44
S*	1861	1786.9	1708	1634.64	1568.5	1508.9	1452.6	1399.7	1352	1309	1265
q*	2771	2746.9	2721.4	2698.64	2678.5	2660.9	2644.6	2629.7	2616	2605	2593
$C(t_1*)$	4066	4292.6	4534.1	4757.1	4963.9	5156.19	5335.5	5503.1	5660	5808	5947



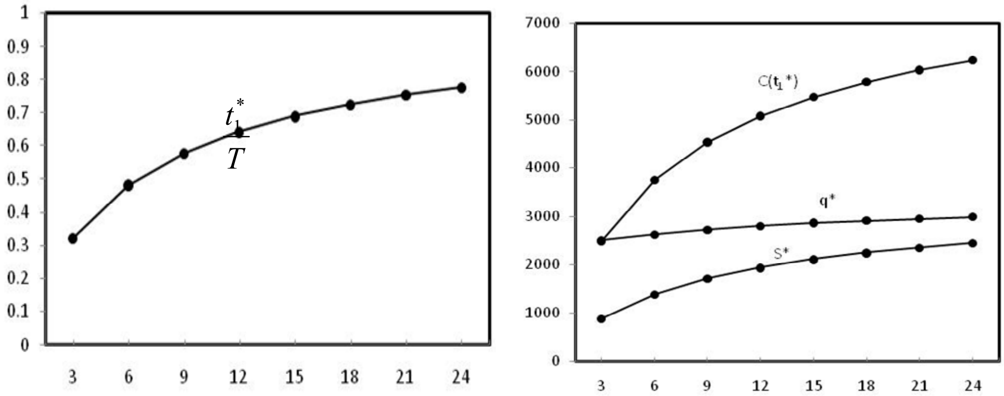
(a) (b)

Figure 3.5 Sensitivity Analysis w.r.t. *h*

(3) From Table 3.3 and Fig. 3.5, when *h* increases from 0.067 to 5, the optimal ratio $\frac{t_1^*}{T}$ decreases almost linearly within a relatively small range from 0.62 to 0.45. The trends of changes in $C(t_1^*)$, q^* and s^* with respect to inventory holding cost *h* are similar to the trends with respect to *c* but in a more or less linear fashion. Again, s^* is a little more sensitive to the changes in *h* than in q^*.

Table 3.4 Different Optimal Values w.r.t. b

b	3	6	9	12	15	18	21	24
t_1*	0.321667	0.481667	0.5775	0.641667	0.689167	0.725	0.753333	0.775833
S*	875.81	1380.44	1708	1937.29	2113.44	2249.78	2359.73	2448.42
q*	2503.81	2624.44	2721.4	2797.29	2859.44	2909.78	2951.73	2986.42
$C(t_1*)$	2484.78	3746.72	4534.13	5078.34	5478.98	5787.11	6031.86	6231.15



(a) (b)

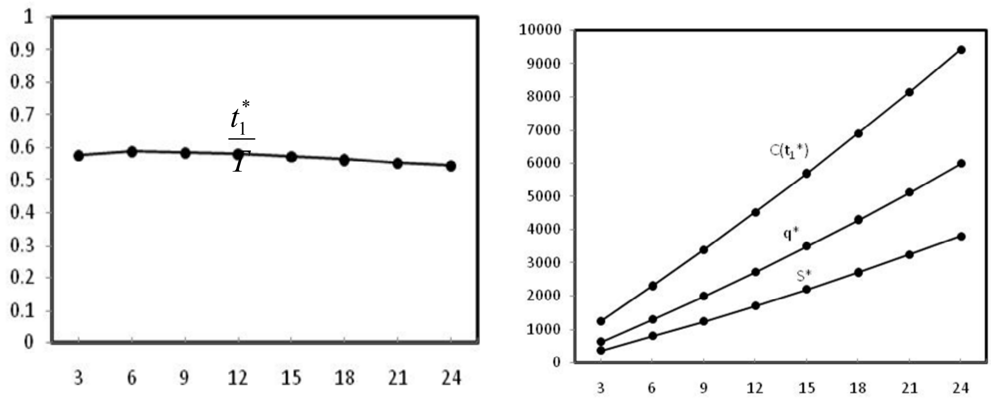
Figure 3.6 Sensitivity Analysis w.r.t. b

(4) As shown in Table 3.4 and Fig. 3.6, the optimal ratio $\frac{t_1^*}{T}$ increases pretty dramatically as the backlogging cost b increases from 3 to about 12, and then the slope of the increase becomes steady as b further increases. The total monthly cost $C(t_1^*)$ and the optimal order-level s^* behave in a very similar manner with respect to b. However, the lot-size q^* slightly increases in an almost linear fashion as b increases. This phenomenon is reasonable, since the higher the backlogging cost is, the less backlogging should be expected, which translates into larger ratio

$\frac{t_1^*}{T}$, and larger q^*, s^* and $C(t_1^*)$. Again, s^* is more sensitive to b than q^*, especially when b is small.

Table 3.5 Different Optimal Values w.r.t. *T*

T	3	6	9	12	15	18	21	24
t_1*	0.573333	0.586667	0.584444	0.5775	0.569333	0.560556	0.551429	0.542083
S*	368.93	791.52	1238.82	1708	2189.7	2711.64	3244.7	3796.07
q*	624.93	1287.52	1986.82	2721.4	3490.7	4293.64	5128.7	5994.07
C(t_1*)	1260.61	2314.63	3405.17	4534.13	5701.29	6906	8147.47	9424.8



(a) (b)

Figure 3.7 Sensitivity Analysis w.r.t. *T*

(5) From Table 3.5 and Figure 3.7, as *T* increases, the optimal ratio $\frac{t_1^*}{T}$ almost remains constant around 0.56. The optimal $C(t_1^*)$, q^* and s^* all increase dramatically in more or less linear fashions, with respect to *T*. This is somewhat expected, as larger *T* leads to higher inventory and more backlogging, since only one replenishment is considered in a cycle of *T* time units.

In conclusion, the optimal cost $C(t_1^*)$ always increases as the value of any parameter increases. The parameter θ plays a very important role in total cost, which explains why it is essential to consider the effect of deterioration for products that deteriorate. The parameter T is also very important too because all the optimal values are very sensitive to it as mentioned in part (5).

In real life, the inventory carrying cost and the shortage backlogging cost may be assessed as a given proportion to the unit price of goods. To examine the system behavior for such cases, this paper considers fixing the ratio $c{:}h{:}b$ while altering all three parameter values simultaneously. In particular, consider $h=\varepsilon_1 c$ and $b=\varepsilon_2 c$, where ε_1 and ε_2 are constants. Substituting these into equations (3.24) and (3.25) leads to the following.

$$t_1^* = \frac{1}{\varepsilon_2} + \frac{LambertW[\frac{\ln(1-\theta)-\varepsilon_1}{\varepsilon_2\theta}\cdot\ln(1-\theta)e^{-\frac{\ln(1-\theta)(\theta+\varepsilon_1+\varepsilon_2\theta T)}{\varepsilon_2\theta}}]}{\ln(1-\theta)} + \frac{\varepsilon_1}{\varepsilon_2\theta} + T\,.$$

As one can see clearly from this equation, the optimal t_1^* is now independent of all three parameters c, h, and b, provided that ε_1 and ε_2 are constants. Consequently, the optimal s^* and q^* are also independent of all c, h, and b, by equations (3.9) and (3.10). Similarly, the following equation can be obtained by using equation (3.21).

$$C(t_1^*)=c\left[\frac{R}{T\theta}((1-\theta)^{-t_1^*}-1)-\frac{Rt_1^*}{T}-\frac{\varepsilon_1 R((1-\theta)^{-t_1^*}-1+t_1^*\ln(1-\theta))}{T\theta\ln(1-\theta)}+\frac{\varepsilon_2 R}{2T}(T-t_1^*)^2\right]. \qquad (3.26)$$

The term in the brackets of equation (3.26) is independent of all c, h, and b. Therefore, the optimal cost $C(t_1^*)$ is a linear function of c (h or b), provided that ε_1 and ε_2 are constants. This insight is very significant in the sense that for companies with a fixed ratio $c:h:b$, the total cost depends strictly linearly on the unit price of a product and the optimal lot-size stays constant.

3.5 Conclusion

In this chapter, we generalized an existing inventory model by allowing the time at which inventory level reaches zero to be non-integer. Explicit formulas for order-level, lot-size, and total cost are derived. As a result, the computation is much simplified. Furthermore, the derived explicit total cost equation makes sensitivity analysis possible.

From the sensitivity analysis, it is found that the cost $C(t_1^*)$ always increases as any parameter value increases. The deterioration rate θ contributes pretty significantly to total cost, which explains why it is imperative to address the effect of deterioration for products that do deteriorate. The cycle time T also plays a rather important role as all the optimal solutions are very sensitive to it. This should be expected, though, as a single replenishment is restricted for each cycle. Another insight is that the optimal cost $C(t_1^*)$ is a linear function of c, provided that $h = \varepsilon_1 c$ and $b = \varepsilon_2 c$, where ε_1 and ε_2 are constants. This finding is pretty significant in the sense that for companies with a fixed ratio $c:h:b$, the total cost depends strictly linearly on the unit price of a product.

Chapter 4

A Discrete-in-Time Deteriorating Inventory Model with Time-Varying Demand, Variable Deterioration Rate and Waiting-Time-Dependent Partial Backlogging

A new inventory system is considered for a single deteriorating item in which the demand is varying over time, unsatisfied demands are partially backlogged depending on the waiting time up to the next replenishment, and deterioration is assumed to be a variable fraction of the on hand inventory at the beginning of each period. Time is treated as a discrete variable because in real life we always consider time on a discrete scale, i.e. in terms of complete units of days, weeks, months, etc. (Dave, 1979). Under this consideration, we are able to derive explicit solutions based on the sufficient optimality condition. The necessary optimality condition is easily proved to be true, while it is not provable in continuous time case. The first example demonstrates that our model can reduce to Dave's (1979) by considering constant demand, constant deterioration rate and complete backlogging. The second example considers a more general case with variable deterioration rate and waiting-time-dependent partial backlogging.

4.1 Assumptions and Notations

The model under consideration is developed with the following assumptions.

(1) The cycle time of T periods is known and constant.

(2) The demand of R_i units in period i is given and occurs at the end of the period.

(3) S_i denotes the inventory level at time point $i\,(i = 0,1,2,\ldots,T)$. The inventory level becomes zero at the end of time period *k*, i.e. $S_k = 0$.

(4) The replenishment rate is infinite and lead time is zero. At the beginning of a cycle, a fixed lot-size *Q* raises the inventory level to S_0.

(5) A constant fraction θ_i of the beginning on- hand inventory in period $i\,(i = 1,2,\ldots,T)$ deteriorates during that period.

(6) The fraction of shortages backlogged for period *i* is denoted as $B(\tau_i)$, for *i* = 1, 2, …, *T*, where τ_i is the waiting time until the next replenishment. It is assumed $B(\tau_i)$ is a decreasing function of τ_i. One possible form of $B(\tau_i)$ could be $\dfrac{1}{1+\alpha\tau_i}$, where the backlogging parameter α is a positive constant (Chang and Dye, 1999). This function guarantees that $0 \le B(\tau_i) \le 1$, for all *i*, and when the waiting time τ_i is zero for *i* = *T*, $B(\tau_T) = 1$, which is complete backlogging.

(7) There is neither repair nor replacement for the deteriorated items.

(8) The unit purchasing cost *c*, inventory holding cost *h* per unit per unit time, backlogging cost *b* per unit per unit time, and the penalty cost *p* per unit lost sale are all known and constant.

4.2 Model

As illustrated in Figure 4.8, the fixed lot-size *Q* raises the inventory at the beginning of the cycle to stock level S_0. This inventory of S_0 is gradually reduced due to both demand and deterioration. By assumption, $S_k = 0$ for some *k*. Demands occurring after time *k* are partially backlogged.

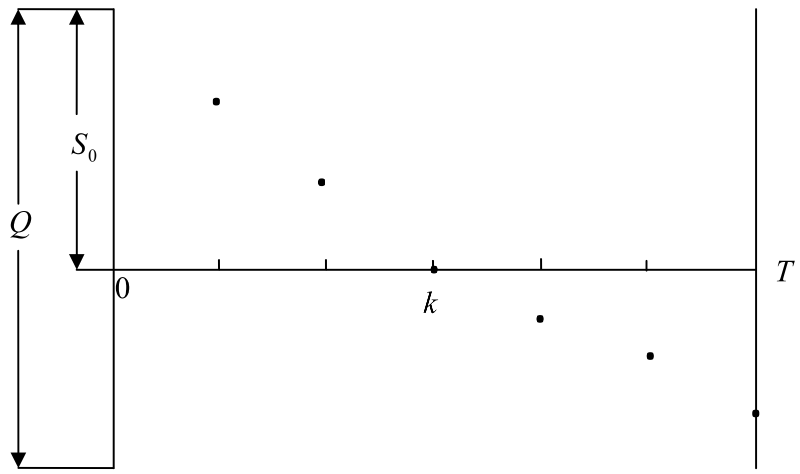


Figure 4.1 Inventory Level for the Model over the Cycle

With the depletion of inventory by both demand and deterioration, the difference equation describing the inventory level for period $i\,(i=k,k-1,\ldots 0)$ is

$$S_i = S_{i-1} - \theta_i S_{i-1} - R_i \ , \ i=k,k-1,\ldots 0,$$

i.e. $$S_{i-1} = \frac{R_i}{1-\theta_i} + \frac{S_i}{1-\theta_i} \ , \ i=k,k-1,\ldots 0.$$

From this recursive relation, we can obtain

$$S_{k-1} = \frac{R_k}{1-\theta_k} \text{ , since } S_k = 0 .$$

Similarly we can get

$$S_{k-2} = \frac{R_k}{(1-\theta_k)(1-\theta_{k-1})} + \frac{R_{k-1}}{1-\theta_{k-1}}$$

$$S_{k-3} = \frac{R_k}{(1-\theta_k)(1-\theta_{k-1})(1-\theta_{k-2})} + \frac{R_{k-1}}{(1-\theta_{k-1})(1-\theta_{k-2})}$$

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$$S_0 = \frac{R_k}{(1-\theta_k)(1-\theta_{k-1})\dots(1-\theta_1)} + \frac{R_{k-1}}{(1-\theta_{k-1})(1-\theta_{k-2})\dots(1-\theta_1)} + \dots + \frac{R_1}{1-\theta_1}$$

Therefore, the inventory levels from time 0 to time k can be rewritten as

$$S_i = \sum_{j=i+1}^{k} R_j \prod_{n=i+1}^{j} (1-\theta_n)^{-1}, \quad i = 0,1,\dots,k. \tag{4.1}$$

The order-level of the inventory system is

$$S_0 = \sum_{j=1}^{k} R_j \prod_{n=1}^{j} (1-\theta_n)^{-1}. \tag{4.2}$$

Since there are no units held in inventory from time k to time T, the deterioration will not exist during this time span. By assumptions, the shortages are partially backlogged and the partial backlogging ratio is determined by the waiting time until the next replenishment. Hence, the difference equation describing the inventory level for period $i\,(i=k,k+1,\dots T)$ is

$$S_{i+1} = S_i - R_{i+1}B(\tau_{i+1}), \quad i=k,k+1,\dots T$$

From this recursive relation, we can obtain

$S_{k+1} = -R_{k+1}B(\tau_{k+1})$, since $S_k = 0$.

Similarly, we can get

$$S_{k+2} = -R_{k+1}B(\tau_{k+1}) - R_{k+2}B(\tau_{k+2})$$

$$S_{k+3} = -R_{k+1}B(\tau_{k+1}) - R_{k+2}B(\tau_{k+2}) - R_{k+3}B(\tau_{k+3})$$

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$$S_T = -R_{k+1}B(\tau_{k+1}) - R_{k+2}B(\tau_{k+2}) - \dots - R_T B(\tau_T)$$

So the general expression of inventory levels from time *k+1* to time *T* is

$$S_i = -\sum_{j=k+1}^{i} R_j B(\tau_j)\,,\ i = k+1, k+2, \ldots, T \ . \tag{4.3}$$

The lot-size Q is the summation of the order-level and total backlogging amount, i.e.

$$Q = S_0 + \sum_{j=k+1}^{T} R_j B(\tau_j) \ . \tag{4.4}$$

From Equation (4.1), the average number of units in inventory per time unit during a cycle is

$$I_1(k) = \frac{1}{T}\sum_{i=0}^{k-1} S_i = \frac{1}{T}\sum_{i=0}^{k-1}\sum_{j=i+1}^{k} R_j \prod_{n=i+1}^{j}(1-\theta_n)^{-1} \ . \tag{4.5}$$

The average inventory holding cost per unit time is

$$h\bar{I}(t_1) \ . \tag{4.6}$$

Since deterioration rate varies each period, according to Equation (4.1), the number of units that deteriorate during a cycle is

$$D(k) = \sum_{i=1}^{k} \theta_i S_{i-1} = \sum_{i=1}^{k} \theta_i \sum_{j=i}^{k} R_j \prod_{n=i}^{j}(1-\theta_n)^{-1} \ . \tag{4.7}$$

Given unit cost c, the average deterioration cost per unit time is

$$\frac{cD(t_1)}{T} \ . \tag{4.8}$$

From Equation (4.2), the average backlogging per unit time during a cycle is

$$I_2(k) = \frac{1}{T}\sum_{i=k+1}^{T}(-S_i) = \frac{1}{T}\sum_{i=k+1}^{T}\sum_{j=k+1}^{i} R_j B(\tau_j) . \tag{4.9}$$

Hence, the average backlogging cost per unit time is

$$bI_2(k). \tag{4.10}$$

Since the partial backlogging ration is $B(\tau_i)$, the number of lost sales per cycle will be

$$LS(k) = \sum_{i=k+1}^{T} R_i \left[1 - B(\tau_i)\right]. \tag{4.11}$$

Therefore, the average penalty cost for lost sales per unit time is

$$\frac{pLS(k)}{T}. \tag{4.12}$$

Adding up costs given by (4.6), (4.8), (4.10) and (4.12) leads to the following total average cost of the system per unit time during one cycle

$$\begin{aligned} C(k) &= \frac{cD(k)}{T} + hI_1(k) + bI_2(k) + \frac{pLS(k)}{T} \\ &= \frac{c}{T}\sum_{i=1}^{k}\theta_i \sum_{j=i}^{k} R_j \prod_{n=i}^{j}(1-\theta_n)^{-1} + \frac{h}{T}\sum_{i=0}^{k-1}\sum_{j=i+1}^{k} R_j \prod_{n=i+1}^{j}(1-\theta_n)^{-1} \\ &+ \frac{b}{T}\sum_{i=k+1}^{T}\sum_{j=k+1}^{i} R_j B(\tau_j) + \frac{p}{T}\sum_{i=k+1}^{T} R_i\left[1 - B(\tau_i)\right] \end{aligned} \tag{4.13}$$

Since *k* is a non-negative integer, the following two conditions should be satisfied by the optimal value of *k*, denoted by k^*, that minimizes $C(k)$ (Sasieni et al., 1959):

$$\Delta C(k^* - 1) \le 0 \le \Delta C(k^*), \text{ for } k = 0,1,\ldots,T, \text{ and} \tag{4.14}$$

$$\Delta^2 C(k) \ge 0, \text{ for } k = 0,1,\ldots,T. \tag{4.15}$$

Using Equation (4.13), we can obtain $\Delta C(k)$ and $\Delta^2 C(k)$ as

$$\begin{aligned} \Delta C(k) &= C(k+1) - C(k) \\ &= \frac{R_{k+1}}{T}\left[c\sum_{i=1}^{k+1}\theta_i \prod_{n=i}^{k+1}(1-\theta_n)^{-1} + h\sum_{i=1}^{k+1}\prod_{n=i}^{k+1}(1-\theta_n)^{-1} - bB(\tau_{k+1})(T-k) - p\left[1 - B(\tau_{k+1})\right]\right] \end{aligned} \tag{4.16}$$

and $\Delta^2 C(k) = \Delta C(k+1) - \Delta C(k)$

$$
\begin{aligned}
&= \sum_{i=1}^{k+1} (c\theta_i + h) \prod_{n=i}^{k+1} (1-\theta_n)^{-1} \left[(1-\theta_{k+2})^{-1} - 1 \right] + (c\theta_{k+2} + h)(1-\theta_{k+2})^{-1} \\
&+ B(\tau_{k+2}) \left[b(k+1) + p - bT \right] - B(\tau_{k+1})(bk + p - bT)
\end{aligned}
\tag{4.17}
$$

Since all the parameters are positive, $0 < \theta < 1$ and $B(\tau_{k+2}) > B(\tau_{k+1})$, it is obvious to see from Equation (4.17) that $\Delta^2 C(k) > 0$, for all $k = 0,1,\ldots,T$. Therefore, condition (4.15) is satisfied. If condition (4.14) is also satisfied, then $C(k)$ will be minimized at k^*. Using Equations (4.13) and (4.16), the condition (4.14) simplifies to

$$M(k^* - 1) \le p \le M(k^*), \tag{4.18}$$

where $M(k) = \sum_{i=1}^{k+1} (c\theta_i + h) \prod_{n=i}^{k+1} (1-\theta_n)^{-1} + B(\tau_{k+1})(bk + p - bT)$. (4.19)

The result obtained here shows that the optimal value of k is not affected by the demand rate and pattern. Substituting k^* into Equations (4.2) and (4.4) yields the optimal order-level and lot-size.

4.3 Numerical Examples

This section presents several numerical examples to illustrate the developed method.

The first example uses the same basic data from Dave's 1979 paper. However, additional data, regarding lost sales penalty cost p and partial backlogging ratio $B(\tau_i)$, must be added. The data are given below.

(1) R = 200 units/month;

(2) c = \$80.00/unit;

(3) h = \$1.00/unit/month;

(4) b = \$9.00/unit/month;

(5) p = \$50/unit;

(6) $B(\tau_i)=1$;

(7) T = 12 months;

(8) $\theta=0.05$.

Such a data setting represents a constant demand rate, a constant deterioration rate, and complete backlogging. In doing so, our results can be compared to Dave's. Note that complete backlogging means no lost sales. Therefore, the results for this case are independent of p values.

The $M(k)$ values are computed by using Equation (4.19) and presented in Table 4.1 below (for k = 1 to 12).

Table 4.1 First Computational Results for $M(k)$ Values

k	1	2	3	4	5	6	7	8	9	10	11	12
$M(k)$	-38	-23.3	-8.2	7.24	23.04	39.2	55.73	72.67	90.02	107	126	144.8

Since the value of p (=50) is between $M(6)$ and $M(7)$, the optimal k^* = 7 months, by condition (18). Substituting this optimal value into Equations (4.2) and (4.4), one can obtain the optimal order-level $S_0^*=1728$ units, and the optimal lot-size $Q^*=2728$ units. These results conform to Dave's. As mentioned earlier, these results are independent of lost sales penalty cost p, since $B(\tau_i)=1$, for all i. In fact, various values of p were tried, and the same k* (= 7 months) was obtained.

The rest of this section extends to a case with variable deterioration rates and waiting-time-dependent partial backlogging ratios. Since the optimal value of k is independent of demand rate and pattern, the constant demand rate is considered in the illustration of computation. But demand patterns do have an effect on optimal order-level, lot-size, lost sales, etc. Therefore, at the end of the section, two other different demand patterns are included to illustrate such effects.

The new data are given below.

$\theta_i = \beta\gamma\, i^{\gamma-1}$, $i = 1,2,\ldots,12$ (K. Skouri, 2009);

$B(\tau_i) = \dfrac{1}{1+\alpha(T-i)}$, $i = 1,2,\ldots,12$ (Chang and Dye, 1999).

$\beta = 0.003$ and $\gamma = 2$ are used in our computational experiments. These values lead to deterioration rates slightly smaller than 5%, on the average. In addition, $\alpha = 0.5$ is considered. This α value results in backlogging ratios $B(\tau_i)$ gradually increase to 1 as i increases to T. That is, the percentage of backlogging gradually decreases as waiting time increases. Such considerations are reasonable in the practical sense, as one gets less patient when waiting time becomes longer.

The $M(k)$ values are computed and presented in Table 4.2 below (for $k = 1$ to 12).

Table 4.2 Second Computational Results for $M(k)$ Values

k	1	2	3	4	5	6	7	8	9	10	11	12
$M(k)$	-7.4	-4.6	-1.2	2.87	7.84	13.77	20.85	29.38	39.85	53.18	71.46	100.68

By condition (4.18), $p = 50$ leads to $k^* = 10$ months. This k^* is bigger than that of the previous case, because the average deterioration rate is small (< 0.05), and there is a large penalty cost for lost sales. So it is economical to keep more inventories and to have less lost sales. The corresponding optimal order-level S_0^*, the optimal lot-size Q^*, the total lost sales $LS(k^*)$ and the total amount of deterioration $D(k^*)$ during one cycle are presented in Table 4.3. This table also presents all the optimal values under two other demand patterns: linearly increasing and linearly decreasing demands, while keeping the total demands the same at 2400 units/cycle. As shown from the table, the optimal k^* stays the same (10 months) for all demand patterns. But the rest are all different. The optimal lot-sizes are almost the same for all three demand patterns, since the total demands are the same and the differences between deterioration and lost sales for all three cases are very close (See the last column of Table 4.3). However, the optimal order-levels differ dramatically. In particular, the decreasing demand case has less lost sales, while the increasing demand case has much more lost sales. The lost sales and deterioration quantities increase as the demand pattern changes from decreasing to uniform, or from uniform to increasing, as expected.

Table 4.3 The Optimal Solutions for 3 Demand Patterns

Demand Pattern	k^*	S_0^*	Q^*	$D(k^*)$	$LS(k^*)$	$D(k^*)-LS(k^*)$
$R_i = 200$, for all i	10	2300.9	2634.3	301	66.7	234.3
$R_i = \frac{400}{13} i$, for all i	10	2056.2	2651.1	363.9	112.8	251.1
$R_i = \frac{400}{13}(13-i)$, for all i	10	2545.7	2617.5	238	20.5	217.5

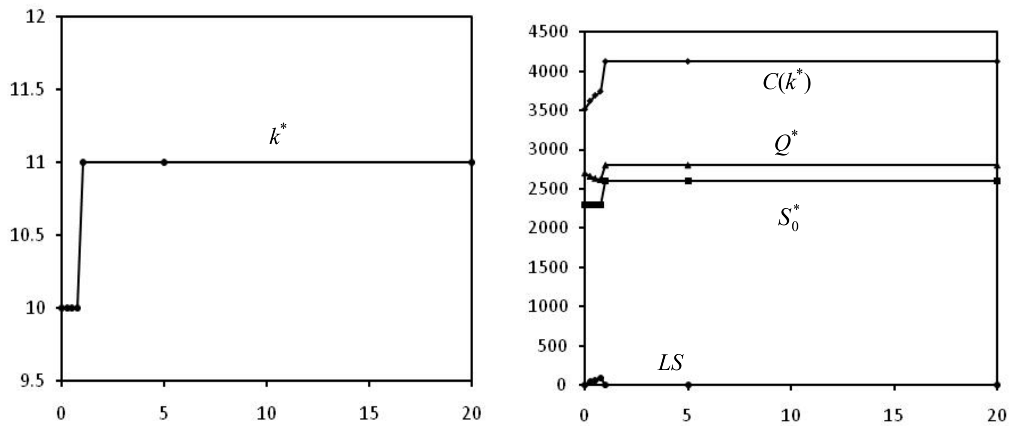
4.4 Sensitivity Analysis

This section first presents the sensitivity of k^*, $C(k^*)$, Q^*, S_0^* and LS with respect to parameters α, β, γ and T through numerical experiments. The sensitivity of the quantities with respect to parameters c, h, b and p is analytically studied and presented at the end of this section.

The parameter values of the second case in Section 4 are used as the base values. It is worth mentioning that the time unit considered in this paper is month. The sensitivity study is conducted as follows. Only one parameter value is altered at a time, while holding all the other parameters constant. The values of parameters we select are mostly the critical values at which the optimal k^* changes. The computational results and analyses are presented next.

Table 4.4 Different Optimal Values w.r.t. α

α	0	0.25	0.5	0.75	1	5	20
k*	10	10	10	10	11	11	11
S*	2300.9	2300.9	2300.9	2300.9	2600.9	2600.9	2600.9
q*	2700.9	2660.9	2634.3	2615.2	2800.9	2800.9	2800.9
LS	0	40	66.7	85.7	0	0	0
$C(k$*)	3509.6	3616.3	3687.4	3738.2	4121.1	4121.1	4121.1



(a)

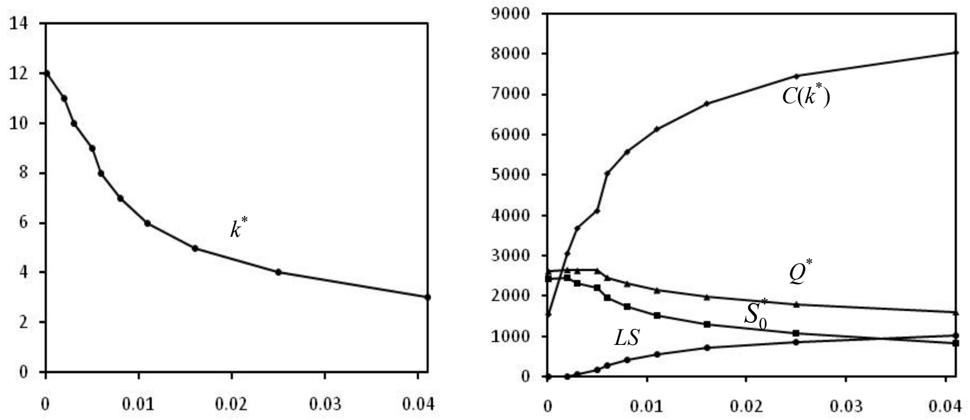
(b)

Figure 4.2 Sensitivity Analysis w.r.t. α

The range of α is chosen from 0 to 20. While $\alpha = 0$ means a complete backlogging, $\alpha > 5$ results in very little backlogging. Consequently, as α increases from 0 (i.e., the backlogging ratio decreases from 1), the optimal k^* jumps from 10 to 11 at α=1, and then stays constant at 11, afterwards. This is somewhat expected because when $\alpha \geq 1$, the partial backlogging ratio is small, which means a lot of lost sales. With lost sales penalty cost of $50/unit which is high compared to inventory and backlogging costs, there should be little or no lost sales. When k^* = 11, there are no lost sales. From Figure 4.2(b), the optimal order-level s^* has exactly the same trend as k^* because α affects s^* directly through k^*. The lot-size Q^* decreases at the beginning as α increases from 0 to 1, and then jumps to a constant after α reaches 1. This is because for k^* = 10, the optimal lot-size decreases as partial backlogging amount decreases, and k^*=11 represents complete backlogging. The amount of lost sales *LS*, on the other hand, increases rapidly at the beginning when k^*=10, and becomes zero after α reaches 1 (i.e. k^*=11). The total average cost per month $C(k^*)$ increases quickly as α changes from 0 to 1, and then remains constant.

Table 4.5 Different Optimal Values w.r.t. β

β	0.0001	0.002	0.003	0.005	0.006	0.008	0.011	0.016	0.025	0.041
k^*	12	11	10	9	8	7	6	5	4	3
S^*	2414.6	2453.3	2300.9	2197.7	1946.2	1727.9	1505.8	1284.9	1063.6	824.1
q^*	2614.6	2653.3	2634.3	2631	2459.5	2307.9	2142.9	1972.1	1795.2	1595.7
LS	0	0	66.7	166.7	286.7	420	562.9	712.9	868.4	1028.4
$C(k^*)$	1555.6	3064.3	3687.4	4117.1	5045.8	5576	6144.3	6774.8	7448.4	8037.5



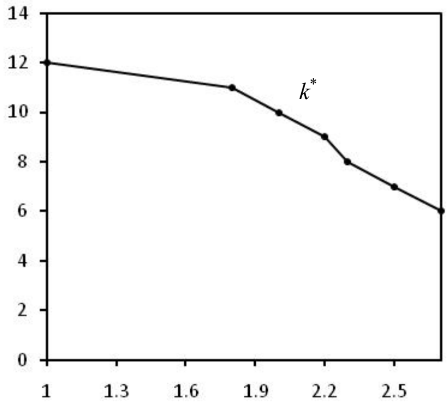
(a) (b)

Figure 4.3 Sensitivity Analysis w.r.t. β

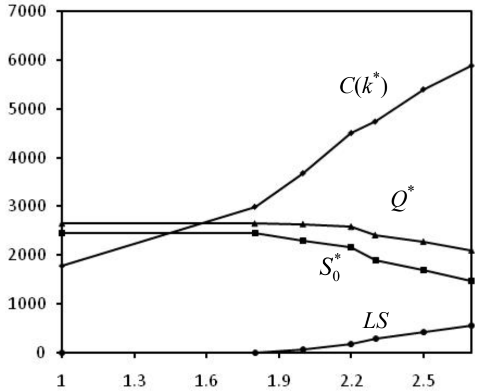
The range of β considered is from 0.0001 to 0.041. This range represents very little to significant deterioration. From Figure 4.3(a), the optimal k^* decreases dramatically as β increases. That is, for a rapidly deteriorating product, it is better to have higher backlogging amount and even more lost sales to keep costs low, as expected. From Fig. 4.4.2(b), when β increases, the optimal order-level S_0^* and lot-size Q^* decrease, while the lost sales LS increase. The total average monthly cost $C(k^*)$ increases rapidly as β increases. This indicates that a little reduction of deterioration can lead to significant improvement of cost reduction. This is especially true when the deterioration rate is small.

Table 4.6 Different Optimal Values w.r.t. γ

γ	1	1.8	2	2.2	2.3	2.5	2.7
k*	12	11	10	9	8	7	6
S*	2447.5	2444.9	2300.9	2151.3	1902.6	1702.5	1468.5
q*	2647.5	2644.9	2634.3	2584.7	2415.9	2282.5	2105.6
LS	0	0	66.7	166.7	286.7	420	562.9
$C(k$*)	1784.8	2997.8	3687.4	4513.2	4746.7	5409.1	5896



(a)



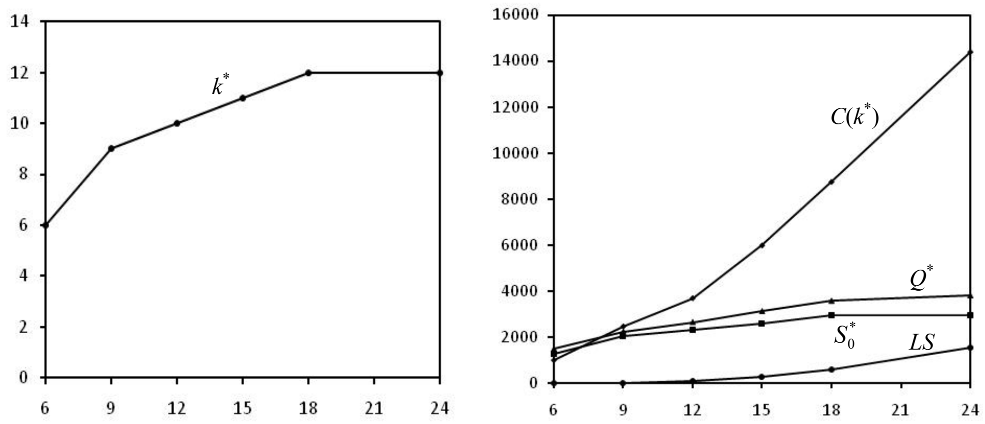
(b)

Figure 4.4 Sensitivity Analysis w.r.t. γ

Figure 4.4 depicts the sensitivity with respect to γ. The range of γ considered is from 1 to 2.7. For $\gamma = 1$, $\theta = \beta = 0.003$ is a very small constant, for all $i = 1, 2, \ldots, 12$. On the other extreme, when γ = 2.7, $\theta = 0.081 i^{1.7}$ ($i = 1,2,\ldots,12$) which indicates very significant deterioration. As one can see from Fig. 4, the changes in all quantities become significant only when $\gamma > 1.9$. This is because the deterioration rates are kind of small when $\gamma \leq 1.9$.

Table 4.7 Different Optimal Values w.r.t. Cycle Time T

T	6	9	12	15	18	24
k^*	6	9	10	11	12	12
S^*	1271	2020.8	2300.9	2600.9	2924.2	2924.2
q^*	1471	2220.8	2634.3	3114.3	3561.3	3796.3
LS	0	0	66.7	286.7	562.9	1527.9
$C(k^*)$	994.1	2463	3687.4	5980.5	8742	14387.1



(a) (b)

Figure 4.5 Sensitivity Analysis w.r.t. Cycle Time T

The range of T considered is set to be from 6 to 24 (months). From Fig. 4.4(a), the optimal k^* increases as T increases. This should be expected, as only one replenishment is considered in one cycle. The optimal order-level S_0^* and lot-size Q^* increase in the same fashion with respect to T. Again, this is somewhat expected, as larger T leads to higher inventory and more backlogging and lost sales, since only one replenishment is allowed in a cycle of T time units. Similarly, the amount of lost sales LS increases as T increases and the total average monthly cost $C(k^*)$ increases rapidly. All these point to one conclusion: it is better to keep cycle time small.

The rest of this section will discuss the sensitivity with respect to parameters c, h, b and p. In real life, the inventory carrying cost, the shortage backlogging cost and the penalty cost for lost sale may be assessed as some given proportions to the unit price of goods. To examine the system behavior for such cases, this paper considers fixing the ratio c:h:b:p while altering all four parameter values simultaneously. In particular, consider $h=\varepsilon_1 c$, $b=\varepsilon_2 c$ and $p=\varepsilon_3 c$, where ε_1, ε_2 and ε_3 are constants. Substituting these into Equations (4.18) and (4.19) leads to the following.

$$\sum_{i=1}^{k^*}(\theta_i+\varepsilon_1)\prod_{n=i}^{k^*}(1-\theta_n)^{-1}+B(\tau_{k+1})(\varepsilon_2 k^*-\varepsilon_2+\varepsilon_3-\varepsilon_2 T)\le\varepsilon_3$$
$$\le\sum_{i=1}^{k^*+1}(\theta_i+\varepsilon_1)\prod_{n=i}^{k^*+1}(1-\theta_n)^{-1}+B(\tau_{k+1})(\varepsilon_2 k^*+\varepsilon_3-\varepsilon_2 T)$$

As one can see clearly from the above, the optimal k^* is now independent of all four parameters *c*, *h*, *b*, and *p*, provided that ε_1, ε_2 and ε_3 are constants. Consequently, the optimal S_0^*, Q^* and *LS* are also independent of all *c*, *h*, *b* and *p*, by Equations (4.2), (4.4) and (4.11), respectively. Similarly, the following equation can be derived from Equation (4.13).

$$C(k)=c\left[\begin{array}{l}\dfrac{1}{T}\displaystyle\sum_{i=1}^{k}\theta_i\sum_{j=i}^{k}R_j\prod_{n=i}^{j}(1-\theta_n)^{-1}+\frac{\varepsilon_1}{T}\sum_{i=0}^{k-1}\sum_{j=i+1}^{k}R_j\prod_{n=i+1}^{j}(1-\theta_n)^{-1}\\ +\dfrac{\varepsilon_2}{T}\displaystyle\sum_{i=k+1}^{T}\sum_{j=k+1}^{i}R_jB(\tau_j)+\frac{\varepsilon_3}{T}\sum_{i=k+1}^{T}R_i\left[1-B(\tau_i)\right]\end{array}\right]$$

The terms in the brackets of the above equation are independent of all *c*, *h*, *b* and *p*. Therefore, the optimal cost $C(k^*)$ is a linear function of *c* (*h*, *b*, or *p*), provided that ε_1, ε_2 and ε_3 are constants. This insight is very significant in the sense that for companies with a fixed ratio $c:h:b:p$, the total cost depends strictly linearly on the unit price of a product and the optimal lot-size is independent of the product price when the cycle time *T* is given.

4.5 Conclusions

A discrete-in-time deteriorating inventory model with time-varying demands, variable deterioration rates and waiting-time-dependent partial backlogging ratios is addressed in this paper. This model deals with a very general case and solutions can be computed easily. This model extends a model in the literature where there is a constant deterioration rate and complete backlogging. Through extensive numerical experiments, a sensitivity study is conducted to illustrate the robustness of the proposed model.

From the sensitivity analysis, it is found that the optimal total cost always increases as any parameter value increases. The deterioration rate (through parameters β and γ) has a significant impact on total cost. This explains why it is imperative to address the effect of deterioration for products that do deteriorate. The cycle time T also plays a rather important role as all the optimal solutions are very sensitive to it. In addition, T should be set as small as possible. All these should be expected, though, as a single replenishment is restricted for each cycle. Another rather interesting insight is that the optimal total cost is a linear function of unit product price c and the optimal k^* is independent of c, h, b and p, provided that inventory holding cost $h = \varepsilon_1 c$, backlog cost $b = \varepsilon_2 c$ and lost sales penalty cost $p = \varepsilon_3 c$, where ε_1, ε_2 and ε_3 are constants. This finding is pretty significant in the sense that for companies with a fixed ratio $c : h : b : p$, the total cost depends strictly linearly on the unit price of a product and the optimal lot-size is independent of the product price when the cycle time T is fixed.

Chapter 5

Periodic-Review Stochastic Inventory Control Problem

A finite horizon inventory model for a single product is considered. The system is under periodic review and there is no fixed order cost associated with any placed order. The demand in successive periods is independent and identically distributed. A constant fraction of any positive leftover stock is deteriorated at the end of each period. Any unsatisfied demand is partially backlogged and fulfilled immediately as a new order arrives. It was proved (Porteus, 2002) that a base stock policy is optimal under complete backlogging and non-deterioration. Then this chapter can be treated as a generalization of Porteus' model by considering deterioration and partial backlogging. It is shown that the base stock policy is still optimal as long as the terminal value function is convex and second-order differentiable. The explicit base stock level is derived for a special case.

5.1 Problem Description

The problem studied in this chapter is as follows. There is a single product. The product has a random life and will deteriorate over time. The system will be run for N periods. The demand D_i in period i (= 1, 2, …, N) is stochastic. At the beginning of each period, one needs to decide if it is necessary to place an order, and if so, how much to order. When an order is placed, there is no fixed cost. There is a per-unit cost, though. Any on-hand inventory at the end of a period can be used in the next period. Any unsatisfied demand can be partially backlogged until fulfilled, or lost. The order decisions are made such that the total expected long-run cost is minimized.

It is assumed that

(1) All demands are independent and identically distributed.

(2) A constant fraction of the positive leftover stock will deteriorate.

(3) The excess demand will be partially backlogged at the end of the period

A penalty cost will be incurred for any backlogging and lost sale amount. The system is under periodic review, i.e. the inventory level is checked at the beginning of each period and a decision is made on how many to order.

Porteus (2002) has shown that a base stock level policy is optimal under complete backlogging and non-deterioration. We will follow the same logic of Porteus' and some of his proof. Our objective is to identify under what conditions the base stock level policy still holds when deterioration and partial backlogging are taken into account.

5.2 Notations

(1) c – unit purchasing cost ($/unit)

(2) h – unit holding cost, charged against positive ending inventory ($/unit)

(3) b – unit backlogging cost, charged against shortages backlogged at the end of a period ($/unit)

(4) p – penalty cost of a lost sale including lost profit ($/unit)

(5) D – generic random variable representing demand, which is i.i.d over every period

(6) Φ – one-period demand distribution

(7) ϕ – demand density distribution

(8) x – inventory level before ordering (the state of the system)

(9) y – inventory level after ordering (the decision variable)

(10) θ – contant fraction of positive leftover stock at the end of the period that is deteriorated

(11) β – constant fraction of unsatisfied demand during a period that is backlogged

(12) N – the length of the planning horizon

(13) $x^+ = \max(x,0)$

(14) $x^- = \min(x,0)$

5.3 Model

To build the model, we first study the one-period problem. Expected one-period holding, backlogging, shortage and deteriorating cost function of level y of inventory after ordering is as follows

$$\begin{aligned} L(y) &= E\left[h(y-D)^+ + c\theta(y-D)^+ + b\beta(D-y)^+ + p(1-\beta)(D-y)^+\right] \\ &= E\left[(h+c\theta)(y-D)^+ + (b\beta + p - p\beta)(D-y)^+\right] \\ &= El(y-D) \end{aligned}$$

where $l(x) = (h+c\theta)x^+ + (b\beta + p - p\beta)(-x)^+$.

If the inventory level at the end of period N is x, then the terminal cost $v(x)$ is incurred. We assume the terminal cost function is convex and second-order differentiable, one example can be

$$v(x) = -cx\,.$$

This case happens if we can obtain reimbursement of the unit cost for each leftover unit and must incur the unit cost for each unit backlogged.

The optimality equations for this model, for $1 \le t \le N$, consist of purchasing cost from inventory x to inventory y, one-period expected cost and minimum expected cost in future:

$$f_t(x) = \min_{y \ge x} \left\{ c(y-x) + L(y) + \partial \int_0^\infty f_{t+1} \left[y - D - \theta(y-D)^+ - (1-\beta)(y-D)^- \right] \phi(D) dD \right\}$$

where $f_{N+1}(x) = v(x)$.

Let $G_t(y) = cy + L(y) + \partial \int_0^\infty f_{t+1} \left[y - D - \theta(y-D)^+ - (1-\beta)(y-D)^- \right] \phi(D) dD$,

then

$$f_t(x) = \min_{y \ge x} \{ G_t(y) - cx \}.$$

Thus, the optimal decision starting with inventory level *x* in period *t* can be determined by minimizing $G_t(y)$ over $\{ y | y \ge x \}$.

In Porteus' book (2002), $G_t(y) = cy + L(y) + \alpha \int_0^\infty f_{t+1}(y-D)\phi(D)dD$. As one can see, there are two additional terms $\theta(y-D)^+$ and $(1-\beta)(y-D)^-$ in the optimality equations if we consider the effects of deterioration and partial backlogging. These two terms complicates the proof.

5.4 Base Stock Policy

As illustrated by Figure 5.1, if G_t is given as the above figure, then, if the inventory level is lower than S_t, then we should order up to it. If the inventory level is higher than S_t, we should order nothing. In other words, the decision rule is a base stock policy, which can be represented as follows (Porteus, 2002):

$$\delta(x) = \begin{cases} S_t & \text{if } x \le S_t \\ x & \text{otherwise} \end{cases}$$

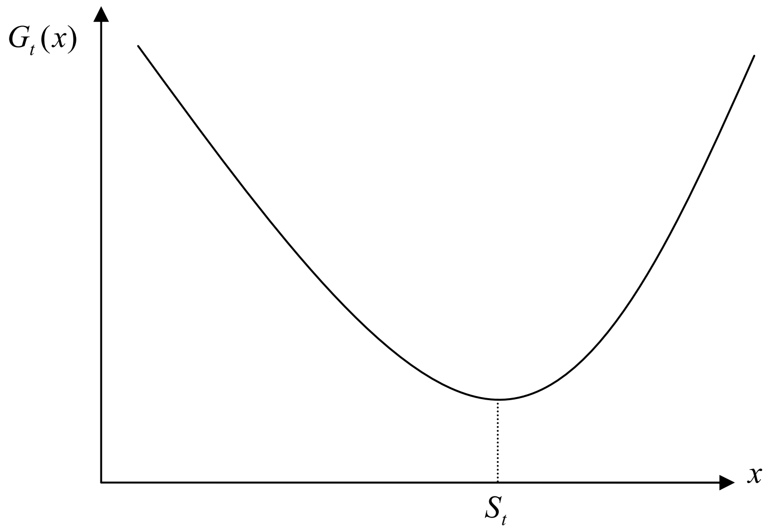


Figure 5.1 Base Stock Policy

5.5 Optimality of Base Stock Policies

Lemma 1 If f_{t+1} is convex and second-order differentiable, then the following holds.

(a) G_t is convex.

(b) A base stock policy is optimal in period t. Indeed, any minimize of G_t is an optimal base stock level.

(c) f_t is convex and second-order differentiable.

Proof (a) G_t can be written as:

$G_t(y) = cy + El(y-D) + \alpha Ef_{t+1}[y - D - \theta(y-D)^+ - (1-\beta)(y-D)^-]$.

We know l is convex, then according to the following Theorem 1, $El(y-D)$ is convex.

Theorem 1 Suppose that l is a convex function defined on **R** and the real valued function H is defined on **R** by $H(R) = E_D l(y-D)$, where D is a random variable with a density ϕ. Then H is convex on R without assuming that l is differentiable everywhere.

Proof Let y_1 and y_2 be arbitrary elements of y. Let $0 \le p \le 1$ and let $q = 1-p$. Then

$$
\begin{aligned}
& pE_D l(y_1 - D) + qE_D l(y_2 - D) \\
& = p\int_0^\infty l(y_1 - D)\phi(D)dD + q\int_0^\infty l(y_2 - D)\phi(D)dD \\
& = \int_0^\infty [pl(y_1 - D) + ql(y_2 - D)]\phi(D)dD \\
& \ge \int_0^\infty l(py_1 + qy_2 - D)\phi(D)dD \qquad \{\text{convexity of } l\} \\
& = E_D l(py_1 + qy_2 - D)
\end{aligned}
$$

which implies $E_D l(y-D)$ is convex. Proof of Thm 1 is completed.

To show $Ef_{t+1}[y - D - \theta(y-D)^+ - (1-\beta)(y-D)^-]$ is convex, since f_{t+1} is convex and second-order differentiable, we have

$$
\begin{aligned}
& \left(Ef_{t+1}[y - D - \theta(y-D)^+ - (1-\beta)(y-D)^-]\right)'' \\
& = \left(\int_0^y f_{t+1}[(1-\theta)(y-D)]\phi(D)dD + \int_y^\infty f_{t+1}[\beta(y-D)]\phi(D)dD\right)''
\end{aligned}
$$

$$
=\begin{pmatrix} \int_0^y f'_{t+1}[(1-\theta)(y-D)]\phi(D)dD + f_{t+1}[(1-\theta)(y-y)] + \\ \int_y^\infty f'_{t+1}[\beta(y-D)]\phi(D)dD - f_{t+1}[\beta(y-y)] \end{pmatrix}
$$

$$
= \int_0^y f''_{t+1}[(1-\theta)(y-D)]\phi(D)dD + f'_{t+1}[(1-\theta)(y-y)] +
$$

$$
\int_y^\infty f''_{t+1}[\beta(y-D)]\phi(D)dD - f'_{t+1}[\beta(y-y)]
$$

$$
= \int_0^y f''_{t+1}[(1-\theta)(y-D)]\phi(D)dD + \int_y^\infty f''_{t+1}[\beta(y-D)]\phi(D)dD \geq 0 \qquad \{\text{convexity of } f_{t+1}\}
$$

So $Ef_{t+1}[y-D-\theta(y-D)^+ - (1-\beta)(y-D)^-]$ is convex. Then G_t is the sum of three convex functions and therefore convex itself.

Proof of (a) is completed.

(b) (Porteus, 2002) We know $f_t(x) = \min_{y \geq x}\{G_t(y) - cx\}$. Let S_t denote a minimizer of $G_t(y)$ over all real *y*. If $x < S_t$, then the minimizing $y \geq x$ is at $y = S_t$, whereas, if $x \geq S_t$, then the minimizing *y* is at $y = x$. That is, a base stock policy with base stock level S_t is optimal for period *t*.

Proof of (b) is completed.

(c) According to the following Theorem 2, f_t is convex.

Theorem 2 (Heyman and Sobel, 1984) If X is a convex set, Y(x) is nonempty set for every $x \in X$, the set $C = \{(x,y) | x \in X, y \in Y(x)\}$ is a convex set, $g(x,y)$ is a convex function on C, $f(x) = \inf_{y \in Y(x)} g(x,y)$ and $f(x) > -\infty$ for every $x \in X$, then f is a convex function on X.

Proof Let x and $\bar{x}$ be arbitrary elements of X. Let $0 \le \theta \le 1$, and let $\overline{\theta} = 1 - \theta$. Select arbitrary $\delta > 0$. By the definition of f, there must exist $y \in Y(\bar{x})$ such that $g(x, y) \le f(x) + \delta$ and $g(\bar{x}, \bar{y}) \le f(\bar{x}) + \delta$. Then,

$$\begin{aligned}
\theta f(x) + \overline{\theta} f(\bar{x}) &\ge \theta g(x, y) + \overline{\theta} g(\bar{x}, \bar{y}) - \delta & &[\text{properties of } y \text{ and } \bar{y}] \\
&\ge g(\theta x + \overline{\theta}\bar{x}, \theta y + \overline{\theta}\bar{y}) - \delta & &[\text{convexity of } \mathrm{g}(\cdot,\cdot) \text{ on C}] \\
&\ge \mathrm{f}(\theta \mathrm{x} + \overline{\theta}\overline{\mathrm{x}}) - \delta & &[(\theta \mathrm{x} + \overline{\theta}\overline{\mathrm{x}}, \theta \mathrm{y} + \overline{\theta}\overline{\mathrm{y}}) \in \mathrm{C}]
\end{aligned}$$

Assume the inequality does not hold for $\delta = 0$, that is

$$\begin{aligned}
&\theta f(x) + \overline{\theta} f(\bar{x}) < f(\theta x + \overline{\theta}\bar{x}) \\
&\Rightarrow \theta f(x) + \overline{\theta} f(\bar{x}) = f(\theta x + \overline{\theta}\bar{x}) - z^* \quad (z^* > 0) \\
&\Rightarrow f(\theta x + \overline{\theta}\bar{x}) - z^* \ge f(\theta x + \overline{\theta}\bar{x}) - \delta
\end{aligned}$$

Since δ can be arbitrarily small, a contradiction is reached. So the inequality must hold for $\delta = 0$, which means f is convex. Proof of Thm 2 is completed.

Since all three terms of f_t are second-order differentiable, then f_t is second-order differentiable itself.

Proof of (c) is completed.

Lemma 2 (Porteus, 2002) A base stock policy is optimal in each period of a finite-horizon problem.

Proof By assumption, the terminal value function is convex and second-order differentiable. Thus, by Lemma 1, G_N is convex and a base stock policy is optimal for period N. By Lemma 1 (c), f_N is convex as well. Thus, the argument iterates backward through the periods in the sequence $t = N, N-1, \ldots 1$.

5.6 Explicit Optimal Base Stock Level

Although we have already illustrated that a base stock policy is optimal for this periodic-review stochastic inventory control problem, it is very difficult to derive the explicit form of base stock level. However, the explicit solution can be obtained for a very special case. As we mentioned before, if we assume the terminal value function is $v(x) = -cx$, then the explicit optimal base stock level can be derived as following.

Let us examine the one-period problem at the end of the time horizon. The expected ordering, holding, backlogging, shortage and deterioration cost, less any expected salvage value, in that period, starting with zero inventory and ordering y units can be written as

$$G_N(y) = cy + L(y) + \partial \int_0^{\infty} (-c)[y - D - \theta(y-D)^+ + (1-\beta)(y-D)^-]\phi(D)$$
$$= cy + \int_0^{y} (h + c\theta - \partial c + \partial\theta c)(y-D)\phi(D)dD + \int_y^{\infty} (b\beta + p - p\beta + \partial c\beta)(D-y)\phi(D)dD \ .$$

Let S denote a solution to $G_N'(S) = 0$, then S is a minimize of G_N and can be found as

$$\Phi(S) = \frac{b\beta - c(1-\partial\beta) + p(1-\beta)}{h + b\beta + c(\theta - \partial + \partial\theta + \partial\beta) + p(1-\beta)} \tag{5.1}$$

Examine f_N by plugging in the optimal decision for each state (Porteus, 2002)

$$f_N(x) = \begin{cases} G_N(S) - cx & \text{if } x \le S \\ G_N(x) - cx & \text{otherwise} \end{cases}$$

Therefore, $f_N'(x) = \begin{cases} -c & \text{if } x \le S \\ G_N'(x) - c & \text{otherwise} \end{cases}$. (5.2)

That is, $f_N(x)$ has a slope of $-c$ for $x \le S$.

Lemma 3 If f_{t+1} is convex and $f'_{t+1}(x) = -c$ for $x < S$, where S is as defined in (5.1), then the following hold.

(a) S minimizes $G_t(y)$ over all real y.

(b) The optimal base stock level in period t is also S.

(c) f_t is convex and $f'(x) = -c$ for $x < S$.

Proof (a) As in Lemma 1, G_t is convex. To see that S is a minimize of G_t,

$$\begin{aligned} G_t'(S) &= c + L'(S) + \partial\left[\int_0^S f'_{t+1}[(1-\theta)(S-D)]\phi(D)dD + \int_S^\infty f'_{t+1}[\beta(S-D)]\phi(D)dD\right] \\ &= c + (h + c\theta)\Phi(S) - (b\beta + p - p\beta)[1-\Phi(S)] - \partial c(1-\theta)\Phi(S) - \partial c\beta[1-\Phi(S)] \\ &= 0 \end{aligned}$$

Hence, *S* must be a minimize of G_t, and, therefore, by Lemma 1, part (b) must also hold.

(c) Lemma 1 ensures that f_t is convex. By calculating the consequences of using the optimal base stock level in period *t*, as was done in (5.2) for period *N*, we get

$$f_t'(x) = \begin{cases} -c & \text{if } x \le S \\ G_t'(x) - c & \text{otherwise} \end{cases}$$

Proof of Lemma 3 is completed.

Theorem 3 (Porteus, 2002) If the terminal value function v has a slope of $-c$, then a base stock policy with base stock level S defined by (2.1) is optimal for every *t*.

Proof By assumption, the terminal value function is convex and has a slope of $-c$. Thus, by Lemma 3, the optimal base stock level in period N is S. By Lemma 3 (c), f_N is convex and has a slope of $-c$ (if x<S) as well. Thus, the argument iterates backward through the periods in the sequence $t = N, N-1, \ldots 1$.

5.7 Summary

A finite horizon inventory model for a single product is considered. The system is under periodic review and there is no fixed order cost associated with any placed order. The demand in successive periods is independent and identically distributed. A constant fraction of any positive leftover stock is deteriorated at the end of each period. Any unsatisfied demand is partially backlogged and fulfilled immediately as a new order arrives. It was proved (Porteus, 2002) that a base stock policy is optimal under complete backlogging and non-deterioration. Then this chapter can be treated as a generalization of Porteus' model by considering deterioration and partial backlogging. It was shown that the base stock policy is still optimal as long as the terminal value function is convex and second-order differentiable. The explicit base stock level was derived for a special case.

Chapter 6

Stochastic Lot-Sizing Problem

The problem concerned here can be described as follows. It is a single product, single location problem. The system will be run for N periods. The product has a random life and will deteriorate over time. The customer demand in each period is stochastic. At the beginning of each period, one needs to decide if it is necessary to place an order, and if so, how much to order. A fixed order cost is incurred whenever an order is placed. There is a per-unit cost associated with each order too. Any on-hand inventory at the end of a period can be used in the next period. Any unsatisfied demand can be partially backlogged until fulfilled, or lost. The order decisions are made such that the total expected long-run cost is minimized.

It is assumed that

(1) All demands are independent and identically distributed.

(2) A constant fraction of the positive leftover stock will deteriorate.

(3) The excess demand will be partially backlogged at the end of the period

A penalty cost will be incurred for any backlogging and lost sale amount. The system is under periodic review, i.e. the inventory level is checked at the beginning of each period and a decision is made on how many to order. As one can see, the only difference between Chapter 5 and Chapter 6 is that the fixed order cost will be considered explicitly in this chapter.

Porteus (2002) has shown that a (s, S) ordering policy (see Chapter 2) is optimal under complete backlogging and non-deterioration. We will follow the same logic of Porteus' and some of his proof. Our objective is to identify under what conditions the (s, S) policy still holds when deterioration and partial backlogging are taken into account.

6.1 Notations

(1) c – unit purchasing cost ($/unit)

(2) h – unit holding cost, charged against positive ending inventory ($/unit)

(3) b – unit backlogging cost, charged against shortages backlogged at the end of a period ($/unit)

(4) p – penalty cost of a lost sale including lost profit ($/unit)

(5) K – fixed ordering cost

(6) α–one period discount factor

(7) D – generic random variable representing demand, which is i.i.d over each period

(8) Φ – one-period demand distribution

(9) ϕ – demand density distribution

(10) x – inventory level before ordering (the state of the system)

(11) y – inventory level after ordering (the decision variable)

(12) θ – constant fraction of positive leftover stock at the end of the period that is deteriorated

(13) β – constant fraction of unsatisfied demand during a period that is backlogged

(14) N – the length of the planning horizon

(15) $x^+ = \max(x,0)$

(16) $x^- = \min(x,0)$

6.2 Model

6.2.1 Formulation

Let us first examine the one-period problem. Expected one-period holding, backlogging, shortage and deteriorating cost function of level y of inventory after ordering is

$$\begin{aligned} L(y) &= E\left[h(y-D)^+ + c\theta(y-D)^+ + b\beta(D-y)^+ + p(1-\beta)(D-y)^+\right] \\ &= E\left[(h+c\theta)(y-D)^+ + (b\beta + p - p\beta)(D-y)^+\right] \\ &= El(y-D) \end{aligned}$$

where $l(x) = (h + c\theta)x^+ + (b\beta + p - p\beta)(-x)^+$.

Let $G_t(y) = cy + L(y) + \alpha\int_0^\infty f_{t+1}\left[y - D - \theta(y-D)^+ - (1-\beta)(y-D)^-\right]\phi(D)dD$, then

the optimality equations (OE) will be

$$f_t(x) = -cx + \min\left\{G_t(x), \min_{y\geq x}\left[K + G_t(y)\right]\right\}.$$

That is, there is no fixed order cost associated with zero order and a fixed order cost K will be incurred if we order something. We have to make a choice on placing an order or not. In Porteus' book, $G_t(y) = cy + L(y) + \alpha\int_0^\infty f_{t+1}(y-D)\phi(D)dD$. As one can see, if we consider the effects of deterioration and partial backlogging, there will be two additional terms $\theta(y-D)^{+}$ and $(1-\beta)(y-D)^{-}$ in the optimality equations. These two terms make the proof more challenging.

Letting $G_t^*(x)=\min\left\{G_t(x), \min_{y\geq x}\left[K+G_t(y)\right]\right\}$, then the OE can be rewritten as

$$f_t(x)=-cx+G_t^*(x).$$

6.2.2 (*s*, *S*) Policy

Since there is a fixed ordering cost incurred for any non-zero order, then the ordering cost function is concave, which is different from Periodic-Review Stochastic Lot-Sizing model. If there is no deterioration and backlogging is complete, a (*s*, *S*) policy will be optimal in each period. This policy means whenever the inventory is below some amount *s*, we will place an order to bring the inventory level up to *S* (where $s \leq S$). The order quantity is greater than or equal to *S*-*s*. If the inventory level is above *s*, we will not place an order. This ensures that the fixed ordering cost only occurs for a certain amount (i.e. ≥*S*-*s*). The order will not be placed if the amount is too small.

6.2.3 *K*-Convex Functions

A function *f*: **R→R** (a real valued function of a single real variable) is *K*-convex if *K*≥0, and for each $x \leq y$, $0 \leq \theta \leq 1$, and $\bar{\theta} = 1-\theta$ (Porteus, 2002):

$$f(\theta x+\bar{\theta}y) \leq \theta f(x)+\bar{\theta}\left[K+f(y)\right].$$

The following Lemma (Porteus, 2002) provides some important properties of *K*-convex function.

Lemma 6.1

(a) If *f* is *K*-convex and *α* is a positive scalar, then αf is *k*-convex for all $k \geq \alpha K$.

(b) The sum of a *K*-convex function and a *k*-convex function is (*K*+*k*)-convex.

(c) If f is K-convex, $x < y$, and $f(x) = K + f(y)$, then $f(z) \le K + f(y)$ for all $z \in [x, y]$.

6.2.4 Optimality of (*s*, *S*) Policy

In this section, the proof is presented about under what conditions the (*s*, *S*) policy is still optimal in each period.

Lemma 6.2 If f_{t+1} is a continuous K-convex function and $\beta = 1 - \theta$, then the following hold.

(a) G_t is a continuous K-convex function.

(b) A (*s*, *S*) policy is optimal in period *t*.

(c) G_t^* is a continuous K-convex function.

(d) f_t is a continuous K-convex function.

Proof (a) From section 6.2.1, we know that

$$G_t(y) = cy + L(y) + \alpha \int_0^\infty f_{t+1}\left[y - D - \theta(y-D)^+ - (1-\beta)(y-D)^-\right]\phi(D)dD.$$

Define $g(y) = \alpha \int_0^\infty f_{t+1}\left[y - D - \theta(y-D)^+ - (1-\beta)(y-D)^-\right]\phi(D)dD$, then

$$G_t(y) = cy + L(y) + g(y).$$

If we can show that for each $y_1 \le y_2$, $0 \le \lambda \le 1$, and $\bar{\lambda} = 1 - \lambda$, the following holds:

$$g(\lambda y_1 + \bar{\lambda} y_2) \le \lambda g(y_1) + \bar{\lambda}\left[K + g(y_2)\right].$$

Then according to Lemma 6.1 (a), $g(y)$ is a αK-convex function. Since cy and $L(y)$ are convex, then according to Lemma 6.1 (b), it can be shown that $G_t(y)$ is *k*-convex. The following is to show that $g(\lambda y_1 + \bar{\lambda} y_2) \le \lambda g(y_1) + \bar{\lambda}\left[K + g(y_2)\right]$.

$$g(\lambda y_1 + \bar{\lambda} y_2)$$

$$= \int_0^\infty f_{t+1}\left[\lambda y_1 + \bar{\lambda} y_2 - D - \theta(\lambda y_1 + \bar{\lambda} y_2 - D)^+ - (1-\beta)(\lambda y_1 + \bar{\lambda} y_2 - D)^-\right]\phi(D)dD$$

$$= \int_0^{y_1} f_{t+1}\left[(1-\theta)(\lambda y_1 + \bar{\lambda} y_2 - D)\right]\phi(D)dD$$

$$+ \int_{y_1}^{y_2} f_{t+1}\left[\lambda y_1 + \bar{\lambda} y_2 - D - \theta(\lambda y_1 + \bar{\lambda} y_2 - D)^+ - (1-\beta)(\lambda y_1 + \bar{\lambda} y_2 - D)^-\right]\phi(D)dD$$

$$+ \int_{y_2}^\infty f_{t+1}\left[\beta(\lambda y_1 + \bar{\lambda} y_2 - D)^-\right]\phi(D)dD$$

$$= \int_0^{y_1} f_{t+1}\left[(1-\theta)(\lambda y_1 + \bar{\lambda} y_2 - D)\right]\phi(D)dD$$

$$+ \int_{y_1}^{\lambda y_1 + \bar{\lambda} y_2} f_{t+1}\left[(1-\theta)(\lambda y_1 + \bar{\lambda} y_2 - D)\right]\phi(D)dD + \int_{\lambda y_1 + \bar{\lambda} y_2}^{y_2} f_{t+1}\left[\beta(\lambda y_1 + \bar{\lambda} y_2 - D)\right]\phi(D)dD$$

$$+ \int_{y_2}^\infty f_{t+1}\left[\beta(\lambda y_1 + \bar{\lambda} y_2 - D)\right]\phi(D)dD$$

$$\le \int_0^{y_1} f_{t+1}\left[(1-\theta)(\lambda y_1 + \bar{\lambda} y_2 - D)\right]\phi(D)dD$$

$$+ \int_{y_1}^{\lambda y_1 + \bar{\lambda} y_2} \left[\lambda f_{t+1}\left[(1-\theta)(y_1 - D)\right] + \bar{\lambda} f_{t+1}\left[(1-\theta)(y_2 - D)\right] + \bar{\lambda} k\right]\phi(D)dD$$

$$+\int_{\lambda y_1+\bar{\lambda}y_2}^{y_2}\left[\lambda f_{t+1}[\beta(y_1-D)]+\bar{\lambda}f_{t+1}[\beta(y_2-D)]+\bar{\lambda}k\right]\phi(D)dD$$

$$+\int_{y_2}^{\infty} f_{t+1}\left[\beta(\lambda y_1+\bar{\lambda}y_2-D)\right]\phi(D)dD$$

$$=\int_{0}^{y_1} f_{t+1}\left[(1-\theta)(\lambda y_1+\bar{\lambda}y_2-D)\right]\phi(D)dD$$

$$+\int_{y_1}^{\lambda y_1+\bar{\lambda}y_2}\left[\lambda f_{t+1}[\beta(y_1-D)]+\bar{\lambda}f_{t+1}[(1-\theta)(y_2-D)]+\bar{\lambda}k\right]\phi(D)dD$$

$$+\int_{\lambda y_1+\bar{\lambda}y_2}^{y_2}\left[\lambda f_{t+1}[\beta(y_1-D)]+\bar{\lambda}f_{t+1}[(1-\theta)(y_2-D)]+\bar{\lambda}k\right]\phi(D)dD$$

$$+\int_{y_2}^{\infty} f_{t+1}\left[\beta(\lambda y_1+\bar{\lambda}y_2-D)\right]\phi(D)dD$$

$$\le \lambda g(y_1)+\bar{\lambda}\left[K+g(y_2)\right]$$

Proof of (a) is completed.

(b) Please refer to (Porteus, 2002).

(c) Please refer to (Porteus, 2002).

(d) Please refer to (Porteus, 2002).

The proof is completed.

Lemma 6.3 If f_{t+1} is a continuous decreasing *K*-convex function, $\beta > 1-\theta$ and $\Phi(x_i) < \dfrac{b\beta + p - c - p\beta}{h + c\theta + b\beta + p - p\beta}$ ($i = 1,2,\ldots N$) where x_i is the inventory level before ordering for each period, then the following hold.

(a) G_t is a continuous *K*-convex function.

(b) A (*s*, *S*) policy is optimal in period *t*.

(c) G_t^* is a continuous *K*-convex function.

(d) f_t is a continuous decreasing *K*-convex function.

Proof (a) As of Lemma 6.2, the objective is to show that for each $y_1 \le y_2$, $0 \le \lambda \le 1$, and $\bar{\lambda} = 1 - \lambda$, $g(\lambda y_1 + \bar{\lambda} y_2) \le \lambda g(y_1) + \bar{\lambda}[K + g(y_2)]$.

$$g(\lambda y_1 + \bar{\lambda} y_2)$$

$$= \int_0^\infty f_{t+1}\left[\lambda y_1 + \bar{\lambda} y_2 - D - \theta(\lambda y_1 + \bar{\lambda} y_2 - D)^+ - (1-\beta)(\lambda y_1 + \bar{\lambda} y_2 - D)^-\right]\phi(D)dD$$

$$= \int_0^{y_1} f_{t+1}\left[(1-\theta)(\lambda y_1 + \bar{\lambda} y_2 - D)\right]\phi(D)dD$$

$$+ \int_{y_1}^{y_2} f_{t+1}\left[\lambda y_1 + \bar{\lambda} y_2 - D - \theta(\lambda y_1 + \bar{\lambda} y_2 - D)^+ - (1-\beta)(\lambda y_1 + \bar{\lambda} y_2 - D)^-\right]\phi(D)dD$$

$$+ \int_{y_2}^\infty f_{t+1}\left[\beta(\lambda y_1 + \bar{\lambda} y_2 - D)\right]\phi(D)dD$$

$$= \int_0^{y_1} f_{t+1}\left[(1-\theta)(\lambda y_1 + \bar{\lambda} y_2 - D)\right]\phi(D)dD$$

$$+ \int_{y_1}^{\lambda y_1 + \bar{\lambda} y_2} f_{t+1}\left[(1-\theta)(\lambda y_1 + \bar{\lambda} y_2 - D)\right]\phi(D)dD + \int_{\lambda y_1 + \bar{\lambda} y_2}^{y_2} f_{t+1}\left[\beta(\lambda y_1 + \bar{\lambda} y_2 - D)\right]\phi(D)dD$$

$$+ \int_{y_2}^\infty f_{t+1}\left[\beta(\lambda y_1 + \bar{\lambda} y_2 - D)\right]\phi(D)dD$$

$$\le \int_0^{y_1} f_{t+1}\left[(1-\theta)(\lambda y_1 + \bar{\lambda} y_2 - D)\right]\phi(D)dD$$

$$+ \int_{y_1}^{\lambda y_1+\bar{\lambda} y_2} \left[\lambda f_{t+1}\left[(1-\theta)(y_1-D)\right]+\bar{\lambda} f_{t+1}\left[(1-\theta)(y_2-D)\right]+\bar{\lambda} k\right]\phi(D)dD$$

$$+ \int_{\lambda y_1+\bar{\lambda} y_2}^{y_2} \left[\lambda f_{t+1}\left[\beta(y_1-D)\right]+\bar{\lambda} f_{t+1}\left[\beta(y_2-D)\right]+\bar{\lambda} k\right]\phi(D)dD$$

$$+ \int_{y_2}^{\infty} f_{t+1}\left[\beta(\lambda y_1+\bar{\lambda} y_2-D)\right]\phi(D)dD$$

$$\lambda g(y_1)+\bar{\lambda}\left[K+g(y_2)\right]$$

$$= \lambda \int_{0}^{y_1} f_{t+1}\left[(1-\theta)(y_1-D)\right]\phi(D)dD + \lambda \int_{y_1}^{\lambda y_1+\bar{\lambda} y_2} f_{t+1}\left[\beta(y_1-D)\right]\phi(D)dD$$

$$+ \lambda \int_{\lambda y_1+\bar{\lambda} y_2}^{y_2} f_{t+1}\left[\beta(y_1-D)\right]\phi(D)dD + \lambda \int_{y_2}^{\infty} f_{t+1}\left[\beta(y_1-D)\right]\phi(D)dD$$

$$+ \bar{\lambda} \int_{0}^{y_1} f_{t+1}\left[(1-\theta)(y_2-D)\right]\phi(D)dD + \bar{\lambda} \int_{y_1}^{\lambda y_1+\bar{\lambda} y_2} f_{t+1}\left[(1-\theta)(y_2-D)\right]\phi(D)dD$$

$$+ \bar{\lambda} \int_{\lambda y_1+\bar{\lambda} y_2}^{y_2} f_{t+1}\left[(1-\theta)(y_2-D)\right]\phi(D)dD + \bar{\lambda} \int_{y_2}^{\infty} f_{t+1}\left[\beta(y_2-D)\right]\phi(D)dD$$

$$+ \int_{0}^{y_1} \bar{\lambda} k\phi(D)dD + \int_{y_1}^{\lambda y_1+\bar{\lambda} y_2} \bar{\lambda} k\phi(D)dD + \int_{\lambda y_1+\bar{\lambda} y_2}^{y_2} \bar{\lambda} k\phi(D)dD + \int_{y_2}^{\infty} \bar{\lambda} k\phi(D)dD$$

$$= \int_{0}^{y_1} \left[\lambda f_{t+1}\left[(1-\theta)(y_1-D)\right]+\bar{\lambda} f_{t+1}\left[(1-\theta)(y_2-D)\right]+\bar{\lambda} k\right]\phi(D)dD$$

$$+ \int_{y_1}^{\lambda y_1+\bar{\lambda} y_2} \left[\lambda f_{t+1}\left[\beta(y_1-D)\right]+\bar{\lambda} f_{t+1}\left[(1-\theta)(y_2-D)\right]+\bar{\lambda} k\right]\phi(D)dD$$

$$+ \int_{\lambda y_1+\bar{\lambda} y_2}^{y_2} \left[\lambda f_{t+1}\left[\beta(y_1-D)\right]+\bar{\lambda} f_{t+1}\left[(1-\theta)(y_2-D)\right]+\bar{\lambda} k\right]\phi(D)dD$$

$$+ \int_{y_2}^{\infty} \left[\lambda f_{t+1}\left[\beta(y_1-D)\right]+\bar{\lambda} f_{t+1}\left[\beta(y_2-D)\right]+\bar{\lambda} k\right]\phi(D)dD$$

$$\geq \int_{0}^{y_1} f_{t+1}\left[(1-\theta)(\lambda y_1+\bar{\lambda} y_2-D)\right]\phi(D)dD$$

$$+ \int_{y_1}^{\lambda y_1+\bar{\lambda} y_2} \left[\lambda f_{t+1}\left[\beta(y_1-D)\right]+\bar{\lambda} f_{t+1}\left[(1-\theta)(y_2-D)\right]+\bar{\lambda} k\right]\phi(D)dD$$

$$+\int_{\lambda y_1+\bar{\lambda}y_2}^{y_2}\left[\lambda f_{t+1}\left[\beta(y_1-D)\right]+\bar{\lambda}f_{t+1}\left[(1-\theta)(y_2-D)\right]+\bar{\lambda}k\right]\phi(D)dD$$

$$+\int_{y_2}^{\infty}f_{t+1}\left[\beta(\lambda y_1+\bar{\lambda}y_2-D)\right]\phi(D)dD$$

Since $\beta > 1-\theta$ and f_{t+1} is a decreasing function according to the assumption, the following must hold:

$$\int_{y_1}^{\lambda y_1+\bar{\lambda}y_2} f_{t+1}\left[\beta(y_1-D)\right]\phi(D)dD \geq \int_{y_1}^{\lambda y_1+\bar{\lambda}y_2} f_{t+1}\left[(1-\theta)(y_1-D)\right]\phi(D)dD$$

and $\displaystyle\int_{\lambda y_1+\bar{\lambda}y_2}^{y_2} f_{t+1}\left[(1-\theta)(y_2-D)\right]\phi(D)dD \geq \int_{\lambda y_1+\bar{\lambda}y_2}^{y_2} f_{t+1}\left[\beta(y_2-D)\right]\phi(D)dD$.

Then $g(\lambda y_1+\bar{\lambda}y_2)\leq \lambda g(y_1)+\bar{\lambda}\left[K+g(y_2)\right]$, i.e. $g(y)$ is a K-convex function.

The proof of (a) is completed.

(b) Please refer to (Porteus, 2002).

(c) Please refer to (Porteus, 2002).

(d) According to section 6.2.1, we have:

$$f_t(x)=-cx+\min\left\{G_t(x),\min_{y\geq x}\left[K+G_t(y)\right]\right\},$$

where $G_t(x)=cx+L(x)+\partial\int_0^{\infty} f_{t+1}\left[y-D-\theta(y-D)^+-(1-\beta)(y-D)^-\right]\phi(D)dD$ and

$$\begin{aligned}L(x)&=E\left[h(x-D)^+ + c\theta(x-D)^+ + b\beta(D-x)^+ + p(1-\beta)(D-x)^+\right]\\&=\int_0^x\left[(h+c\theta)(x-D)\right]\Phi(D)dD+\int_x^{\infty}\left[(b\beta+p-p\beta)(D-x)\right]\Phi(D)dD\end{aligned}.$$

To prove $G_t(x)$ is a decreasing function, we only need to prove $\left[cx+L(x)\right]$ is decreasing because $f_{t+1}(x)$ is given to be a decreasing function.

The first derivative of $\left[cx + L(x)\right]$ is given by:

$$\left[cx + L(x)\right]' = c + h\Phi(x) + c\theta\Phi(x) - b\beta + b\beta\Phi(x) - p + p\Phi(x) + p\beta - p\beta\Phi(x)$$

$$= c - b\beta - p + p\beta + (h + c\theta + b\beta + p - p\beta)\Phi(x) \text{ , since } \Phi(y_1) < \frac{b\beta + p - c - p\beta}{h + c\theta + b\beta + p - p\beta} \text{ ,}$$

then $\left[cx + L(x)\right]' < 0$. So $\left[cx + L(x)\right]$ is a decreasing function, and therefore, G_t is a decreasing function.

Since $\min\limits_{y \geq x}\left[K + G_t(y)\right] = K + G_t(\bar{y})$, which is a constant, we obtain that

$f_t(x) = -cx + \min\left\{G_t(x), \min\limits_{y \geq x}\left[K + G_t(y)\right]\right\}$ is a decreasing function.

From part (3) we know that G_t^* is k-convex, then $f_t(x)$ is the summation of a convex function and a *k*-convex function, therefore, *k*-convex itself.

The proof is completed.

6.3 Summary

A finite horizon inventory model for a single product is considered in this chapter. The system is under periodic review and there is a fixed order cost associated with any non-zero order. The demand in successive periods is independent and identically distributed. A constant fraction of any positive leftover stock is deteriorated at the end of each period. Any unsatisfied demand is partially backlogged and fulfilled immediately as a new order arrives. Porteus (2002) proved that a (*s*, *S*) policy is optimal under complete backlogging and non-deterioration. This chapter generalized Porteus' model by considering deterioration and partial backlogging. It was shown that the (*s*, *S*) policy is still optimal for the following two cases:

(1) v_T is a continuous K-convex function and $\beta = 1 - \theta$.

(2) v_T is a continuous decreasing K-convex function, $\beta > 1 - \theta$ and $\Phi(x_i) < \dfrac{b\beta + p - c - p\beta}{h + c\theta + b\beta + p - p\beta}$ ($i = 1,2,\dots N$) where x_i is the inventory level before ordering for each period.

One drawback is that the two conditions are both rigid to some level. The explicit form of (s, S) is very difficult to obtain. We will discuss how to overcome this in Chapter 8.

Chapter 7

The Stochastic Lot-Sizing Problem with Deterioration and Service-Level Constraints

The problem considered in this chapter can be described as follows. It is a single product, single location problem. The product has a random life and will deteriorate over time. The system will be run for N periods. The customer demand in each period is stochastic. At the beginning of each period, one needs to decide if it is necessary to place an order, and if so, how much to order. A fixed order cost is incurred whenever an order is placed. There is a per-unit cost associated with each order too. Any on-hand inventory at the end of a period can be used for the next period. The net inventory at the end of each period not being negative is set to be a probability of at least α (service level). It is assumed that the value α is quite high, so that this service level incorporates the perception of the cost of backorders already and shortage cost can be neglected. The system is under periodic review, i.e. the inventory level is checked at the beginning of each period and a decision is made on how many to order.

Bookbinder and Tan (1988) first studied this problem without considering the effect of deterioration. They developed a strategy called "static-dynamic uncertainty" strategy, in which they determine the number of replenishments at the beginning of the planning horizon, and then obtain the order quantity based on the realized demand. Tarim and Kingsman (2004) improved this strategy by presenting a mixed-integer programming formulation that simultaneously determines both number of replenishments and order quantity in a single step and gives the optimal solution rather than the heuristic results given by Bookbinder and Tan.

In this chapter, we follow Tarim and Kingsman's method but take into account of the effect of deterioration. The computational results were obtained and 100 random cases were simulated for the optimal ordering policy.

7.1 Assumptions and Notations

(1) The demand d_t in period t is a random variable with known probability density $g_t(d_t)$. The demands are independent and the distribution of demand may vary from period to period.

(2) Lead time is zero, i.e. the replenishment order X_t arrives immediately at the beginning of period t, before the demand in that period occurs.

(3) The service level is α, i.e. the probability that at the end of each period the net inventory will not be negative is set to be at least α. It is assumed that the value α is quite high, so that this service level incorporates the perception of the cost of backorders already and shortage cost can be neglected.

(4) A constant fraction θ of positive leftover stock is deteriorated over one period.

(5) The length of the planning horizon is N.

(6) The inventory level at the end of period t is denoted as I_t. The initial on-hand inventory is I_0.

(7) The unit purchasing cost c, unit holding cost h and fixed ordering cost a are known.

7.2 Model

The problem can be formulated as minimizing the total expected cost $E[TC]$ over the planning horizon, as following:

Min $E[TC]$

$$= \int_{d_1}\int_{d_2}\cdots\int_{d_N}\sum_{t=1}^{N}(a\delta_t + (h + c\theta)I_t + cX_t)g_1(d_1)g_2(d_2)\cdots g_N(d_N)d(d_1)d(d_2)\cdots d(d_N) \qquad (7.1)$$

subject to

$$\delta_t = \begin{cases} 1 & \textit{if } X_t > 0, \\ 0 & \textit{otherwise}, \end{cases} \qquad t = 1,\ldots,N \qquad (7.2)$$

$$I_t = (1-\theta)(I_{t-1} + X_t - d_t)\,,\ t = 1,\ldots,N \qquad (7.3)$$

$$\Pr(I_t \geq 0) \geq \alpha\,,\ t = 1,\ldots,N \qquad (7.4)$$

$$X_t, I_t \geq 0\,,\ t = 1,\ldots,N \qquad (7.5)$$

The objective function (7.1) consists of fixed order cost, inventory holding cost, deterioration cost and purchasing cost. Constraint (7.2) means that δ_t takes the value of 1 if an order is placed in period *t* and 0 otherwise. Constraint (7.3) shows the balance of the inventory flow, i.e. inventory level is determined by order quantity, demand and deterioration. Constraint (7.4) guarantees that the service level is at least α. Constraint (7.5) makes sure all the decision variables are positively defined.

From the recursive relationship of inventory levels described by Equation (7.3), we can obtain the general expression of inventory level at period t as

$$I_t = (1-\theta)^t I_0 + \sum_{k=1}^{t}(1-\theta)^{t-k+1} X_k - \sum_{k=1}^{t}(1-\theta)^{t-k+1} d_k . \tag{7.6}$$

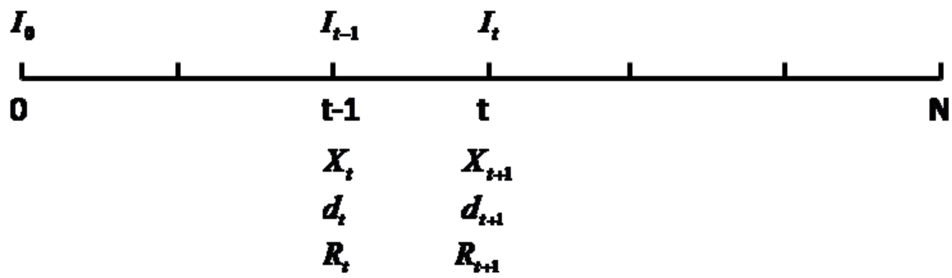


Figure 7.1 Illustration of Notation Index

We assume that there are m reviews over the N period planning horizon with orders arriving at $\{T_1, T_2, \cdots, T_m\}$, where $T_j > T_{j-1}$, $T_m \le N$. Set $T_1 = 1$ and $T_{m+1} = N+1$. Define R_{T_i} as the order-up-to level to which inventory level should be reached after replenishment at the beginning of the ith review period T_i. Then according to Fig.7.1 and equation (7.6), we have the following relationship:

$$I_t = (1-\theta)^{t-T_i+1} R_{T_i} - \sum_{k=T_i}^{t}(1-\theta)^{t-k+1} d_k \ , \ T_i \le t < T_{i+1} , \ i = 1, \ldots, m. \tag{7.7}$$

If there is no replenishment in period t, then R_t is explained as the opening stock level in period t. The relationship between R_t and I_t is as following:

$$I_t = (1-\theta)(R_t - d_t) . \tag{7.8}$$

From Figure 7.1, it is clear that if there is no replenishment in period *t*, R_t will be equal to I_{t-1}. This relationship can be defined by the following two inequalities:

$$R_t - I_{t-1} \le M\delta_t,\ t = 1,\ldots,N \tag{7.9}$$

$$R_t \ge I_{t-1},\ t = 1,\ldots,N \tag{7.10}$$

Applying Equation (7.7) to the service-level constraint $\Pr(I_t \ge 0) \ge \alpha$, we have

$$\Pr((1-\theta)^{t-T_i+1} R_{T_i} - \sum_{k=T_i}^{t} (1-\theta)^{t-k+1} d_k \ge 0) \ge \alpha$$

i.e. $\Pr(\sum_{k=T_i}^{t} (1-\theta)^{t-k+1} d_k \le (1-\theta)^{t-T_i+1} R_{T_i}) \ge \alpha$

i.e. $(1-\theta)^{t-T_i+1} R_{T_i} \ge G^{-1}_{(1-\theta)^{t-T_i+1} d_{T_i} + (1-\theta)^{t-T_i} d_{T_i+1} + \ldots + (1-\theta) d_t}(\alpha)$,

where $G_{(1-\theta)^{t-T_i+1} d_{T_i} + (1-\theta)^{t-T_i} d_{T_i+1} + \ldots + (1-\theta) d_t}(y)$ is the cumulative distribution function of $D(t) = (1-\theta)^{t-T_i+1} d_{T_i} + (1-\theta)^{t-T_i} d_{T_i+1} + \ldots + (1-\theta) d_t$.

Since $I_t = (1-\theta)^{t-T_i+1} R_{T_i} - \sum_{k=T_i}^{t} (1-\theta)^{t-k+1} d_k$, then

$$I_t \ge G^{-1}_{(1-\theta)^{t-T_i+1} d_{T_i} + (1-\theta)^{t-T_i} d_{T_i+1} + \ldots + (1-\theta) d_t}(\alpha) - \sum_{k=T_i}^{t} (1-\theta)^{t-k+1} d_k\ ,\ T_i \le t < T_{i+1},\ i = 1,\ldots,m. \tag{7.11}$$

In constraint (7.11), $G^{-1}_{(1-\theta)^{t-T_i+1} d_{T_i} + (1-\theta)^{t-T_i} d_{T_i+1} + \ldots + (1-\theta) d_t}(\alpha)$ can only be determined if the replenishment timing T_i is known. But T_i is also a decision variable in our model, so there is circularity here. Tarim and Kingsman addressed this issue by defining a new binary variable to formulate this problem as a mixed-integer programming model. Since the planning horizon is finite and consists of N periods, the following binary variable P_{tj} is defined to calculate all relevant cases of $G^{-1}_{(1-\theta)^{t-T_i+1}d_{T_i}+(1-\theta)^{t-T_i}d_{T_i+1}+\ldots+(1-\theta)d_t}(\alpha)$.

$$P_{tj} = \begin{cases} 1 & \text{if the most recent order prior to period } t \text{ was in period } t-j+1 \\ 0 & \text{otherwise} \end{cases}$$

Then $G^{-1}_{(1-\theta)^{t-T_i+1}d_{T_i}+(1-\theta)^{t-T_i}d_{T_i+1}+\ldots+(1-\theta)d_t}(\alpha) = \sum_{j=1}^{t} G^{-1}_{(1-\theta)^{j}d_{t-j+1}+(1-\theta)^{j-1}d_{t-j+2}+\ldots+(1-\theta)d_t}(\alpha)P_{tj}$,

$t = 1,\ldots,N$.

So from Equation (7.11), we have

$$I_t \geq \sum_{j=1}^{t}(G^{-1}_{(1-\theta)^{j}d_{t-j+1}+(1-\theta)^{j-1}d_{t-j+2}+\ldots+(1-\theta)d_t}(\alpha) - \sum_{k=t-j+1}^{t}(1-\theta)^{t-k+1}d_k)P_{tj},\ t = 1,\ldots,N. \qquad (7.12)$$

According to Bookbinder and Tan's (1988) argument, $G^{-1}_{(1-\theta)^{j}d_{t-j+1}+(1-\theta)^{j-1}d_{t-j+2}+\ldots+(1-\theta)d_t}(\alpha)$ can be calculated as $\sum_{k=t-j+1}^{t} E[(1-\theta)^{t-k+1}d_k] + z_{0.95}C(\sum_{k=t-j+1}^{t} E^2[(1-\theta)^{t-k+1}d_k])^{1/2}$.

The following constraint (7.13) guarantees that there can at most be only one most recent order received prior to period t and constraint (7.14) depicts the relationship between P_{tj} and δ (Tarim and Kingsman, 2004).

$$\sum_{j=1}^{t} P_{tj} = 1,\ t = 1,\ldots,N. \qquad (7.13)$$

$$P_{tj} \geq \delta_{t-j+1} - \sum_{k=t-j+2}^{t} \delta_k \text{ , } t = 1, \ldots, N \text{ , } j = 1, \ldots, t \text{ .} \tag{7.14}$$

In order to determine the timing of the replenishments before any demands become known under the "static-dynamic uncertainty" strategy, the expected values of the stochastic variables I_t, R_t and d_t must be applied.

Total cost in (7.1) can also be expressed as

$$TC = \sum_{t=1}^{N} (a\delta_t + (h + c\theta)I_t + cX_t) = \sum_{t=1}^{N} (a\delta_t + (h + c\theta)I_t + c(R_t - I_{t-1})) \text{ .}$$

Taking expected values of both sides yields the following

$$E[TC] = \sum_{t=1}^{N} \{a\delta_t + (h + c\theta)E[I_t] + c(E[R_t] - E[I_{t-1}])\} \text{ .}$$

Putting together constraints (7.8), (7.9), (7.10), (7.12), (7.13), and (7.14), the final model, denoted by M-1, is obtained as follows. Minimizing (7.1) subject to (7.2) - (7.5) is equal to solving the model M-1. In the rest of this chapter, we will use this new model M-1 to do computational tests.

Model M-1 is expressed as follows.

Min $E[TC] = \sum_{t=1}^{N} \{a\delta_t + (h + c\theta)E[I_t] + c(E[R_t] - E[I_{t-1}])\}$

Subject to $E[I_t] = (1-\theta)(E[R_t] - E[d_t])$, $t = 1, \ldots, N$.

$E[R_t] \geq E[I_{t-1}]$, $t = 1, \ldots, N$.

$$E[R_t]-E[I_{t-1}]\le M\delta_t\,,\; t=1,\ldots,N\,.$$

$$E[I_t]\ge\sum_{j=1}^{t}(G^{-1}_{(1-\theta)^j d_{t-j+1}+(1-\theta)^{j-1}d_{t-j+2}+\ldots+(1-\theta)d_t}(\alpha)-\sum_{k=t-j+1}^{t}(1-\theta)^{t-k+1}E[d_k])P_{tj}\,,\; t=1,\ldots,N\,.$$

$$\sum_{j=1}^{t}P_{tj}=1\,,\; t=1,\ldots,N\,.$$

$$P_{tj}\ge\delta_{t-j+1}-\sum_{k=t-j+2}^{t}\delta_k\,,\; t=1,\ldots,N\,,\; j=1,\ldots,t\,.$$

$$E[I_t],E[R_t]\ge 0\,,\; \delta_t,P_{tj}\in\{0,1\}\,,\; t=1,\ldots,N\,,\; j=1,\ldots,t\,.$$

7.3 A Numerical Example

To examine if our new model works, the same numerical example from Tarim and Kingsman's paper is applied except that a deterioration rate θ is added based on our extension. The purpose of this numerical example is to compare our model with the model without deterioration and show how the effect of deterioration plays a role in decision-making. Since the objective of the final model is the total expected cost, a simulation of random cases will be conducted to show how the optimal ordering policy works on each individual case. Then the comparison of each individual cost with the expected cost will reveal how the actual cost deviates from the optimal cost in practice. Also, the actual fill rate can be calculated to compare with the designated service level.

(1) $a = \$2500/order$

(2) $h = \$1/unit/period$

(3) $\theta = 0.05$

(4) $\alpha = 0.95$

(5) $z_{\alpha=0.95} = 1.645$

(6) $C = \sigma_t / \mu_t = 0.333$

Table 7.1 Forecasted Values of Demands

Period (k)	1	2	3	4	5	6	7	8	9	10
$E[d_k]$	800	850	700	200	800	700	650	600	500	200

Table 7.1 gives the forecasted values of expected demands. It is assumed that the initial inventory level is zero and the demand in each period is assumed to be normally distributed about the forecasted value under a constant coefficient of variation C=0.333. Since the service level α=0.95, then the z value for standardized normal distribution is 1.645.

Table 7.2 The Calculated Values of the Inverse Cumulative Distribution Function $G^{-1}_{(\cdot)}(\alpha)$

a.

t	*j*				
	1	2	3	4	5
1	1239	0	0	0	0
2	1316	2234.595	0	0	0
3	1084	2092.902	2935.981	0	0
4	310	1245.395	2198.945	2998.065	0
5	1239	1440.418	2189.761	3044.783	3782.77
6	1084	2025.997	2215.069	2902.639	3695.909
7	1006	1824.389	2681.901	2860.181	3501.309
8	929	1689.138	2434.231	3230.4	3399.004
9	774	1485.342	2181.765	2876.348	3623.417
10	310	957.3219	1626.003	2284.564	2942.64

b.

t	*j*				
	6	7	8	9	10
1	0	0	0	0	0
2	0	0	0	0	0
3	0	0	0	0	0
4	0	0	0	0	0
5	0	0	0	0	0
6	4386.898	0	0	0	0
7	4243.853	4893.81	0	0	0
8	4000.861	4699.103	5312.113	0	0
9	3783.181	4350.952	5010.124	5589.814	0
10	3651.025	3802.74	4341.529	4967.113	5517.409

Table 7.2 shows the calculated $G^{-1}_{(1-\theta)^j d_{t-j+1}+(1-\theta)^{j-1} d_{t-j+2}+\ldots+(1-\theta)d_t}(\alpha)$ values. For example, the element for *t*=6 and *j*=2 is calculated as 2025.997. That means the opening

inventory level in period t-j+1=5 is 2025.997 and this inventory can satisfy the demands from period 5 to period 6 with a probability of at least α=0.95.

Table 7.3 Results for $c = 0$

Period (t)	1	2	3	4	5	6	7	8	9	10
Order decision (Delta)	1	0	1	0	1	0	0	1	0	0
Closing inv. level ($E[I_t]$)	1507	624	600	380	2098	1328	644	1164	631	409
Opening inv. level ($E[R_t]$)	2386	1507	1332	600	3009	2098	1328	1825	1164	631
Order-up-to-level	2386	--	1332	--	3009	--	--	1825	--	--
Total expected cost	19390									

Table 7.4 Tarim's Results for c=0. (Tarim and Kingsman, 2004) *

Period (t)	1	2	3	4	5	6	7	8	9	10
Order decision (Delta)	1	0	1	0	1	0	0	1	0	0
Closing inv. Level ($E[I_t]$)	1490	640	599	399	2033	1333	683	1142	642	442
Opening inv. Level ($E[R_t]$)	2290	1490	1299	599	2833	2033	1333	1742	1142	642
Order-up-to-level	2290	--	1299	--	2833	--	--	1742	--	--
Total expected cost	19404									

(*) 2nd row is added.

Table 7.3 and Table 7.4 show both our results and Tarim's results when the unit purchasing cost is not considered. From the tables, it is clear that the optimal replenishment timings for both models are the same, which are in periods 1, 3, 5 and 8. The order-up-to level is higher in our model. This is because the effect of deterioration plays an important role and we have to raise the inventory level much higher since it is determined by both demand and deterioration. The total expected cost is a little bit lower in our model because the unit purchasing cost is not considered and hence the deterioration cost is not considered either.

Table 7.5 Results for $c = 4$

Period (t)	1	2	3	4	5	6	7	8	9	10
Order decision (Delta)	1	0	1	0	1	0	1	0	1	0
Closing inv. Level ($E[I_t]$)	1507	624	600	380	1295	566	1096	471	497	282
Opening inv. Level ($E[R_t]$)	2386	1507	1332	600	2164	1295	1804	1096	1023	497
Order-up-to-level	2386	--	1332	--	2164	--	1804	--	1023	--
Total expected cost	47957									

Table 7.6 Tarim's Results for $c = 4$

Period (t)	1	2	3	4	5	6	7	8	9	10
Order decision (Delta)	1	0	1	0	1	0	1	0	1	0
Closing inv. level ($E[I_t]$)	1490	640	599	399	1283	583	1085	485	495	295
Opening inv. level ($E[R_t]$)	2290	1490	1299	599	2083	1283	1735	1085	995	495
Order-up-to-level	2290	--	1299	--	2083	--	1735	--	995	--
Total expected cost	45036									

Table 7.5 and Table 7.6 show both our results and Tarim's results when the unit purchasing cost is at c=4. From both tables, we can see that the replenishments arrive in the same periods, which are periods 1, 3, 5, 7 and 9. The opening inventory level of our model is higher than that of Tarim's model, which means the order-up-to level is also higher in our model. This is because the inventory level in our model is determined by both demand and deterioration, so it needs to be raised much higher since deterioration plays a very important role. The total expected cost is much higher in our model because the effect of deterioration is taken into account explicitly and the unit purchasing cost is nonzero.

Table 7.7 Results for $c = 6$

Period (t)	1	2	3	4	5	6	7	8	9	10
Order decision (Delta)	1	0	1	0	1	0	1	0	1	0
Closing inv. Level ($E[I_t]$)	1508	624.6	600.4	380.4	1296	566	1097	471.6	497.2	282.3
Opening inv. Level ($E[R_t]$)	2387	1508	1332	600.4	2164	1296	1804	1097	1023	497.2
Order-up-to-level	2387	--	1332	--	2164	--	1804	--	1023	--
Total expected cost	62025.2									

Table 7.8 Tarim's Results for $c = 6$

Period (t)	1	2	3	4	5	6	7	8	9	10
Order decision (Delta)	1	0	1	0	1	0	1	0	1	0
Closing inv. level ($E[_{It}]$)	1490	640	599	399	1283	583	1085	485	495	295
Opening inv. level ($E[R_t]$)	2290	1490	1299	599	2083	1283	1735	1085	995	495
Order-up-to-level	2290	--	1299	--	2083	--	1735	--	995	--
Total expected cost	57624									

Table 7.7 and Table 7.8 show both our results and Tarim's results when the unit purchasing cost is set at c=6. Similarly, we can see from both tables that the replenishments still arrive in the same periods, which are periods 1, 3, 5, 7 and 9. The order-up-to level is higher in our model than that of Tarim's model. The reason is similar, i.e., the inventory level has to be raised higher since it is depleted by both demand and deterioration. The total expected cost is much higher in our model than that of Tarim's model and the difference of the total costs is higher than the case when c=4. This is because the effect of deterioration is taken into account explicitly and the unit purchasing cost in this case is higher than the previous one.

Table 7.9 Service Level and Actual Fill Rate

Service Level	95%	90%	85%
Average Fill Rate	99.70%	99.10%	98.40%

A simulation of 100 random cases is conducted for the case of c=4. As shown in Table 7.5, the optimal ordering policy for the case of c=4 is that replenishments take place in periods 1, 3, 5, 7 and 9, and the inventory levels will be raised to 2386.8, 1332, 2164, 1804.2, and 1023.4 respectively. This ordering policy will guarantee the minimum expected long-run cost.

In this simulation, we test 100 random cases each for service levels of 95%, 90% and 85%. Through the random experiment, it is found out that the actual fill rate is much higher than the service level, which one can see from Table 7.9. Actually, there is only one case for 95% service level that fill rate is lower than 95%.

Table 7.10 Worst Scenario Cost

Service Level	95%	90%	85%
Expected Cost	47957.5	46222.9	45047.4
Worst Scenario Cost	51038.4	49306.4	48129.6
Percentage Increase	6.42%	6.67%	6.84%

Table 7.10 shows that the total expected cost decreases as service level decreases. From our optimal ordering policy, one can see that it guarantees a minimum expected cost, but not necessarily minimum cost for each case. From the simulation, the worst scenario, i.e. case with maximum cost, is illustrated in Table 7.10 for each service level. As shown in Table 10, the worst case cost is about 6% higher than the total expected cost for each service level, and this percentage increases as service level decreases.

7.4 Summary

A stochastic lot-sizing problem with deterioration and service-level constraints is considered in this chapter. The system is under periodic review. The demand in successive periods is independent but not necessarily identical. A constant fraction of any on-hand inventory is deteriorated over one period. The service level is assumed to incorporate the perception of the cost of backlogging so that the shortage cost can be ignored. It was found that the order-up-to level is higher in our model than that of Tarim and Kingsman's because the inventory is depleted by both demand and deterioration. Through a simulation of 100 random cases, it was found that the actual fill rate is much higher than the service level and the total expected cost decreases as service level decreases.

Chapter 8

Conclusions and Future Work

8.1 Conclusions

This dissertation presents several discrete-in-time deteriorating inventory models and identifies optimal ordering quantities or policies for a single deteriorating product under deterministic or stochastic demand by minimizing the expected overall costs over the planning horizon. The various conditions have been considered, e.g. periodic review, time-varying deterioration rate, waiting-time-dependent partial backlogging, time-dependent demand, stochastic demand, service-level constraints etc. The computational experiments and sensitivity analysis bring a thorough and insightful understanding of the inventory control for deteriorating products. The major contributions of this dissertation are summarized as follows.

- *Deterministic inventory control for a single deteriorating product.* Under the conditions of constant demand and constant deterioration rate, a closed-form equation is derived to compute the optimal solution, while no closed-form solution was presented in literature. This greatly reduces the computational effort to identify the optimal solutions, and makes sensitivity analysis possible. Under the conditions of time-dependent demand, time-dependent deterioration rate and waiting-time-dependent partial backlogging, we are able to derive explicit solutions based on the sufficient optimality condition. The necessary optimality condition is easily proved to be true, while it is not provable in continuous time case as stated by Wu (2002).

- *Stochastic inventory control for a single deteriorating product.* To the best of our knowledge, all the deteriorating inventory control models in the existing literature are dealing with deterministic customer demand. Hence, this dissertation fills the vacancy by studying the deteriorating inventory models with stochastic customer demand and periodic review. All the findings are clearly stated in Chapter 5 and Chapter 6.
- *Stochastic inventory control for a single deteriorating product under service-level constraints.* The deteriorating inventory control model with stochastic demand and service-level constraints is first studied in this dissertation. It was found that the actual fill rate is much higher than the service level and the total expected cost decreases as service level decreases.

The 21st Century Engineering Grand Challenges, identified by National Academy of Engineering, requires innovative approaches to effectively use and manage finite resources. This will positively impact the sustainability, health, security and living of different species. Of particular importance is the reduction and management of various wastes generated in the world. Our research addresses this challenge by optimally controlling the inventory of deteriorating products and minimizing the enormous cost due to deterioration. Our efforts could be applied and yield positive results in a variety of sectors that stock deteriorating products.

8.2 Future Work

This study aims to (a) obtain optimal order quantity and useful insights for the inventory control of a single deteriorating product over a discrete time horizon with deterministic demand, variable deterioration rates and waiting-time-dependent partial backlogging ratios and (b) identify optimal ordering policy for a single deteriorating product over a finite horizon with stochastic demand and partial backlogging. For part (a), the demand patterns we consider are constant and time-varying demands. In future, more demand patterns could be taken into consideration, such as stock-dependent demand, price-dependent demand, ramp type demand, etc. The partial backlogging behaviors we consider are constant and waiting-time-dependent partial backlogging. For future research, more partial backlogging behaviors could be studied, such as exponential partial backlogging ratio, shortage-dependent partial backlogging ratio, etc. For the stochastic lot-sizing model with deterioration and backlogging in part (b), the conditions for the (s, S) policy to hold are very restrictive and the explicit optimal ordering policy is rather difficult to develop. One possible future work will be to study this problem by discretizing the customer demand into several discrete scenarios. For example, we could have three customer demand scenarios, low customer demand, high customer demand and most likely customer demand. This customer demand uncertainty could be resolved by applying multi-stage mixed-integer stochastic programming techniques, but the computation will be time-consuming since the size of the problem grows exponentially as the number of scenarios increases. A superior future research study could concentrate on the development of efficient computational algorithms for this type of problem.

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