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Inverse Kinematics

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Inverse Kinematics

Abstract

Inverse kinematics is the process of converting a Cartesian point in space into a set of joint angles to more efficiently move the end effector of a robot to a desired orientation. This project investigates the inverse kinematics of a robotic hand with fingers under various scenarios. Assuming the parameters of a provided robot, a general equation for the end effector point was calculated and used to plot the region of space that it can reach. Further, the benefits obtained from the addition of a prismatic joint versus an extra variable angle joint were considered. The results confirmed that having more movable parts, such as prismatic points and changing angles, increases the effective reach of a robotic hand.

Keywords

Kinematics, Robotics, Optimization of Robotic Movement

Erratum

This article was previously called Article 18.

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PROBLEM STATEMENT

The main objective of this project is controlling a robotic hand. We consider a four fingered robotic hand (see Figure 1). Figure 2 shows one of the robotic fingers. This finger has only one movable joint θ which means it can move only in a single direction. The links L_1, L_2 and L_3 are fixed while the interior angles of the non-movable joints are fixed at $120°$. This means each finger can only move to the front and back. Assume that the joint θ has limited freedom and can only rotate between 0° and $120°$. Let L_1 be $10"$, L_2 be 7" and L_3 be 4". Also, let the distance L_0 between the base of each finger and the center of the hand (the cross formed by the fingers) be 4". We will answer the following questions based on the information above:

1. Calculate θ if the hand has to grip a ball 3 inches in radius. Assume the center of the ball is directly above the center of the hand. Also, calculate the height (from the center of the hand) the ball should be such the fingers grip the ball in the center. (θ should be same for each finger).

2. Considering the center of the hand to be the origin, plot the end-effector point for a single finger for all values of θ. We assume a step size of 1°.

3. Redo 2, if one more joint θ_2 was added to the finger. If θ_2 could move between $60°$ and $140°$, plot the end-effector point for all values of $\theta = \theta_1$ and θ_2. We assume a step size of 1°.

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4. Redo 2, if L_1 was a prismatic joint. A prismatic joint can slide ahead or back in one direction. Assume L_1 can change from $4''$ to $12''$. Plot the end-effector point for all values of θ and L_1. We assume a step size of $1°$ and 0.1 inches.

Figure 1: A four fingered robotic hand.

MOTIVATION

Robots are programmed to be as efficient as possible in their tasks. Their efficiency is affected by their area of movement. If a robot does not have a sufficient range of motion, then its efficiency is compromised. Presently the end effectors of a robotic hand are analyzed. An effector is any device that affects the environment. In our case the end effectors are merely the tips of each finger attached to the robotic hand. End effectors generally have many uses and may have various sorts of equipment attached to it for precision, such as but not limited to drills, brushes, cameras, anti-collision sensors, and gripping tools. Engineers program robots to move their end effectors to various coordinates in 3D space. The process of converting a Cartesian point into a set of joint angles is called inverse kinematics. This project investigates a 2D version of inverse kinematics for the motion of an end effector in a robotic hand with varying angles and lengths.

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MATHEMATICAL DESCRIPTION AND SOLUTION APPROACH

I. SETUP.

The specifications of each robot vary so the location of their end effectors must be determined individually. The robotic finger, as shown in Figure 2, is divided into four line segments labeled L_0, L_1, L_2, and L_3. The joints of the robotic finger represent the end points of each line segment. Using a Cartesian plane, the location of the end effector point will be determined in regard to the base of the robotic hand. In order to find the location of this point on the Cartesian plane, mathematical calculations have to be applied to the different lengths and angles supplied.

II. Angles and Lengths

The most important angle during the duration of the project is undoubtedly θ (later to be called θ_1). The remaining angles of interest shown in Figure 2 are α, β, λ, and μ. We shall express all these other angles in terms of θ. Similarly, we will express the unknown lengths in terms of θ. For instance, the distances l_x and l_y can be determined to be

$$\begin{cases} l_x = L_1 \cos\theta \\ l_y = L_1 \sin\theta \end{cases}. \tag{1}$$

In order to find the other lengths, the unknown angles need to be expressed in terms of θ.

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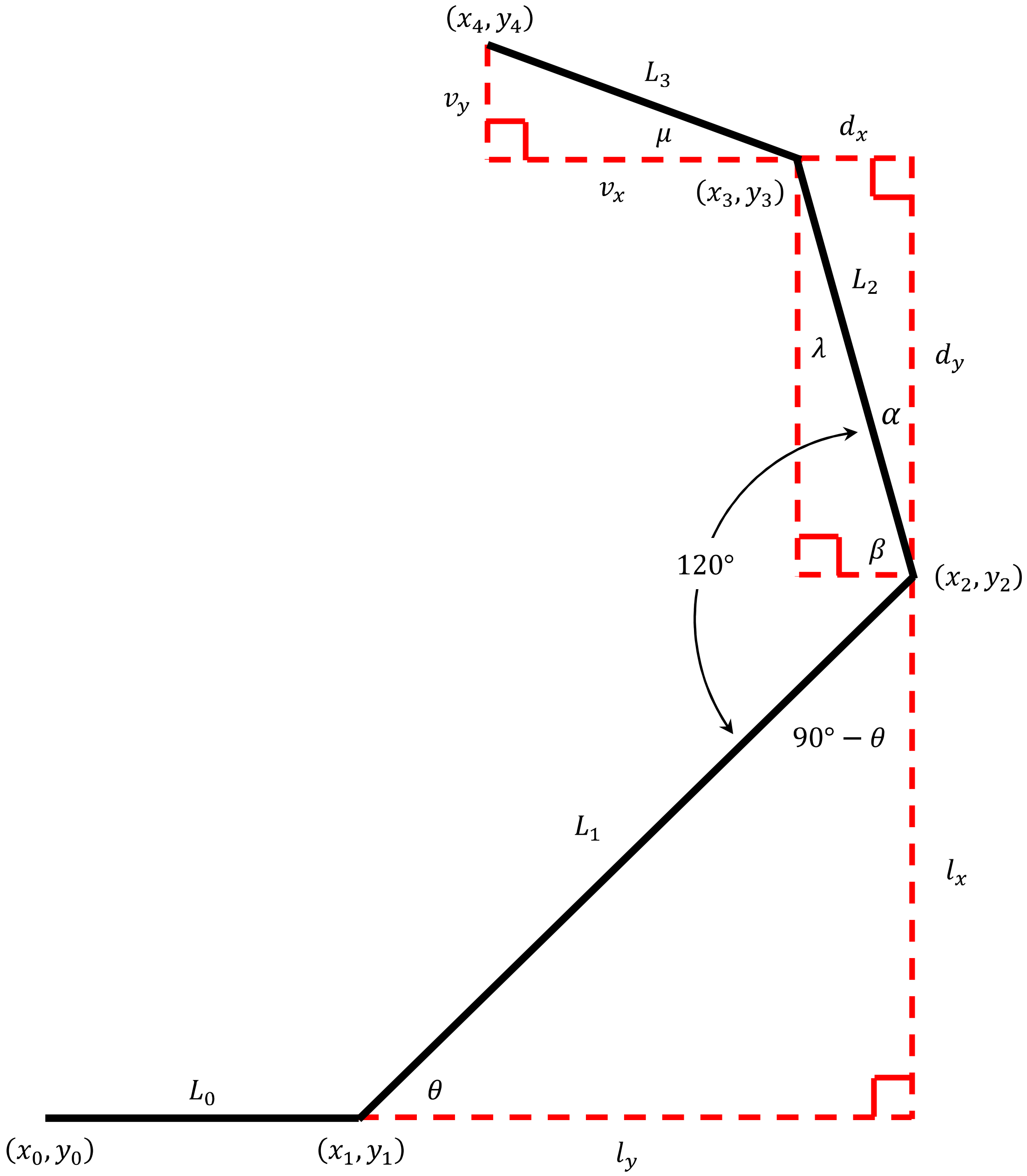

Figure 2: This figure illustrates the basic setup for the robotic finger. The finger is divided into four segments: L_0, L_1, L_2, and L_3 of lengths 4″, 10″, 7″, and 4″, respectively.

Angle $\boldsymbol{\alpha}$ The line that forms the angle $(90° - \theta)$, $120°$, and α in Figure 1 creates a total angle of $180°$. Therefore, it can be concluded that

$$\alpha = 180° - (90° - \theta) - 120° = \theta - 30°. \tag{2}$$

Keeping this in mind, the lengths of d_x and d_y can be calculated using α as follows:

$$\begin{cases} d_x = L_2 \cos\alpha = L_2 \cos(\theta - 30°) \\ d_y = L_2 \sin\alpha = L_2 \sin(\theta - 30°) \end{cases}. \tag{3}$$

Angle $\boldsymbol{\beta}$ The two angles α and β can be summed to form a right angle. Since α is expressed in terms of θ in (2), β can also be put in terms of θ through

$$\beta = 90° - \alpha = 120° - \theta. \tag{4}$$

This allows for the calculation of angle λ, which is needed for the calculation of angle μ.

Angle $\boldsymbol{\lambda}$ We know that $\lambda + \beta + 90° = 180°$ so by (4) we have that

$$\lambda = 180° - 90° - \beta = \theta - 30°. \tag{5}$$

From (2) and (5) we see that $\alpha = \lambda$. This is expected since α and λ are alternate interior angles of a transversal cutting parallel lines.

Angle $\boldsymbol{\mu}$ Since the interior joint between L_2 and L_3 is given to be $120°$, we see that

$$120° = \mu + 90° + \lambda \quad \Rightarrow \quad \mu = 30° - \lambda = 60° - \theta. \tag{6}$$

This allows for the determination of the distances

$$\begin{cases} v_x = L_3 \cos\mu = L_3 \cos(60° - \theta) \\ v_y = L_3 \sin\mu = L_3 \sin(60° - \theta) \end{cases}. \tag{7}$$

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III. POINTS

Now that the previously unknown lengths and angles are expressed in terms of θ, we calculate the position of the robotic finger joints. Each joint can be calculated in terms of the proceeding joints back to the fixed base of the hand. Thus the location of the end effector (x_4, y_4) may be calculated once the location of all the proceeding joints are calculated

Point $(\boldsymbol{x_0}, \boldsymbol{y_0})$ Naturally, we take the center of the hand to be the origin. This point coincides with the base of the robotic finger, i.e.,

$$(x_0, y_0) = (0,0). \tag{8}$$

Point $(\boldsymbol{x_1}, \boldsymbol{y_1})$ This point is the distance between the center of the robotic hand, and the first joint of the robotic finger. The distance between the center and the joint is known to be L_0, therefore the point

$$(x_1, y_1) = (L_0, 0)\ . \tag{9}$$

Point $(\boldsymbol{x_2}, \boldsymbol{y_2})$ The horizontal distance from the origin to x_2 is the sum of the distance between x_0 and x_1, and the distance between x_1 and x_2. The length from x_1 and x_2 is merely l_x which was computed in (1). This is also the first point where the y-value is not a fixed position and depends on l_y. Thus, the point

$$\begin{aligned}(x_2, y_2) &= \left(L_0 + l_x,\ l_y\right)\\ &= (L_0 + L_1 \cos\theta\,,\ L_1 \sin\theta).\end{aligned} \tag{10}$$

Point $(\boldsymbol{x_3}, \boldsymbol{y_3})$ The location of x_3 is the value of $x_2 - d_x$. Since the distance d_x has been calculated in (3), x_3 point can be easily determined in terms of x_2. The y-value at this coordinate has increased from the position of the last value, so it is the sum of the distances $y_2 = l_y$ and d_y. Thus from (10) and (3) we see that

$$\begin{aligned}(x_3, y_3) &= \left(x_2 - d_x,\ y_2 + d_y\right) \\ &= (L_0 + L_1 \cos\theta - L_2 \sin(\theta - 30°)\,,\ L_1 \sin\theta + L_2 \cos(\theta - 30°)). \end{aligned} \tag{11}$$

Point $(\boldsymbol{x_4}, \boldsymbol{y_4})$ Note from the figure that $x_4 = x_3 - v_x$ and $y_4 = y_3 + v_y$. Since (x_3, y_3) was calculated at (11) and v_x, v_y were calculated at (7), the coordinate can be described as

$$\begin{cases} x_4 = L_0 + L_1 \cos\theta - L_2 \sin(\theta - 30°) - L_3 \cos(60° - \theta) \\ y_4 = L_1 \sin\theta + L_2 \cos(\theta - 30°) + L_3 \cos(60° - \theta) \end{cases}. \tag{12}$$

Recalling that $L_0 = 4$, $L_1 = 10$, $L_2 = 7$, and $L_3 = 4$, the equations in (11) may be used as a general kinematic equation for the location of the end effector with regard to the angle θ, i.e.,

$$\begin{cases} \boldsymbol{x}(\theta) = 4 + 10\cos\theta - 7\sin(\theta - 30°) - 4\cos(60° - \theta) \\ \boldsymbol{y}(\theta) = 10\sin\theta + 7\cos(\theta - 30°) + 4\cos(60° - \theta) \end{cases}. \tag{13}$$

IV. BALL HELD BY ROBOTIC HAND.

The general formula for the tip of each robotic finger is given in (13). Problems 1 can now be approached in a simple and straightforward manner. Since the center of the ball will be placed over the center of the hand, the ball can be safely assumed to be held symmetrically by each of the robotic fingers. This being the case, it is only necessary to calculate the end effector point of one finger. It is straightforward to calculate $\boldsymbol{x}(\theta)$ from $\theta = 0°$ to $90°$ by increments of $0.1°$ using Excel. From this we find numerically that $\boldsymbol{x}(54.2°) \approx 3$. Therefore the robotic hand must angle

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θ by at least $54.2°$ or the ball will drop. At $\theta = 54.2°$, we find the vertical component to be $\boldsymbol{y}(54.2°) \approx 18.5"$. This means that the center of the ball is held approximately $18.5"$ from the base of the hand.

V. GRAPHING THE RANGE OF THE END EFFECTOR.

Assuming a step size of $1°$, the end effector point is graphed in Figure 6 in the Appendix. The graph as a whole represents the range of the end effector by a single finger. Each point represents the location of the end effector point as θ increases by $1°$.

The next situation involves the introduction of a new angle θ_2 between L_1 and L_2 (see Figure 4). To avoid confusion, the original θ is renamed as θ_1. We allow θ_2 to move from $60°$ to $140°$. In order to determine the range of the end effector point, Figure 7 plots each $1°$ movement of θ_1 against each $1°$ movement of θ_2. Each line of the same color represents θ_1 at specified angle in its overall range.

The Problem 4 introduces a prismatic joint applied to L_1. A prismatic joint is a joint that can slide ahead or back in one direction. The $120°$ between L_1 and L_2 remains constant, however the length L_1 becomes a variable between 4" and 12" as shown in Figure 5. The step size assumed for this scenario was 0.1". In this case, each line represents θ_1 plotted against varying size of L_1. Hence, the first line represents θ_1 at 0° with a length L_1 ranging from 4" to $12"$.

The final situation involves a combination of the previous two end effector graphs (see Figure 9). This graph represents θ_1 from 0° to $120°$ plotted against both a varying θ_2 and prismatic L_1. As can be anticipated, the graph allows for a wide range of movement. For clarity θ_1, θ_2, and L_1 were plotted with step sizes of $15°$, $1°$, and 2" respectively.

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DISCUSSION

In the first problem, a ball is held at the center of a robotic hand and the fingers are able to hold it bending at the same angles, allowing for symmetrical end effector points. The symmetry between two end effector points can be seen in Figure 3. The inclination of the robotic fingers can be seen via the Figure 6, which plots the location of the end effector points from 0° to 120°.

The inclusion of an extra variable angle joint added a new depth to the effectiveness of the end effector point. When θ_2 was added, the reach of the robotic finger was extend in both the x and y directions (Figure 7). Similarly the prismatic joint extended the reach of the end effector (Figure 8), but the extended range appears to be more restrictive than adding θ_2. However the graph combining both the angle θ_2 and the prismatic joint (Figure 9) shows a remarkable range for the robotic finger when compared to the original range of motion (Figure 6).

These computations emphasize how important choosing the right type of joint can affect the overall effectiveness of a robot. An engineer provided the task of designing such a robot would prefer maximized the range of the end effector while minimizing the overall cost of the robot. The graph of the combination of both the angle and the prismatic joint depict a more optimized orientation of the end effector, but the cost was not considered.

CONCLUSION AND RECOMMENDATIONS

This project assessed the efficiency of end effector points based on different functionalities of a robotic finger. A situation was provided where a robotic hand must hold a ball of radius 3″. Although symmetrical, each finger must be angled precisely to produce the

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desired result. Using the dimensions of the robotic finger, an inverse kinematic equation was derived which related the Cartesian coordinate of the end effector with the joint angle which realizes it. This equation is crucial to understanding the end effector efficiency. It was shown that the effect range of the end effector increased with the addition of a prismatic joint but most increased by the addition of a second variable joint angle.

The calculations presented here are common to most robots and engineers use these inverse kinematic equations to program robots to either move itself or its manipulators to various orientations in space. Inverse kinematic equations create a more sophisticated yet efficient method of movement for robotic constructs. It is recommended that future projects explore the efficiency of variable angled joints versus prismatic joints on end effectors in 3D space.

NOMENCLATURE

Symbols	Description	Units
$\boldsymbol{\theta}$	Angle of the movable joint exterior to L_0 and L_1	Degrees (°)
$\boldsymbol{\theta_1}$	When θ_2 is added, θ is relabeled θ_1	Degrees (°)
$\boldsymbol{\theta_2}$	An additional movable joint between L_1 and L_2	Degrees (°)
$\boldsymbol{L_0}$	Segment connecting the base of the robotic hand to L_1	Inches (“)
$\boldsymbol{L_1}$	Segment connecting L_0 with L_2	Inches (“)
$\boldsymbol{L_2}$	Segment connecting L_1 with L_3	Inches (“)
$\boldsymbol{L_3}$	Segment connecting L_2 with the end effector	Inches (“)

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APPENDIX - FIGURES

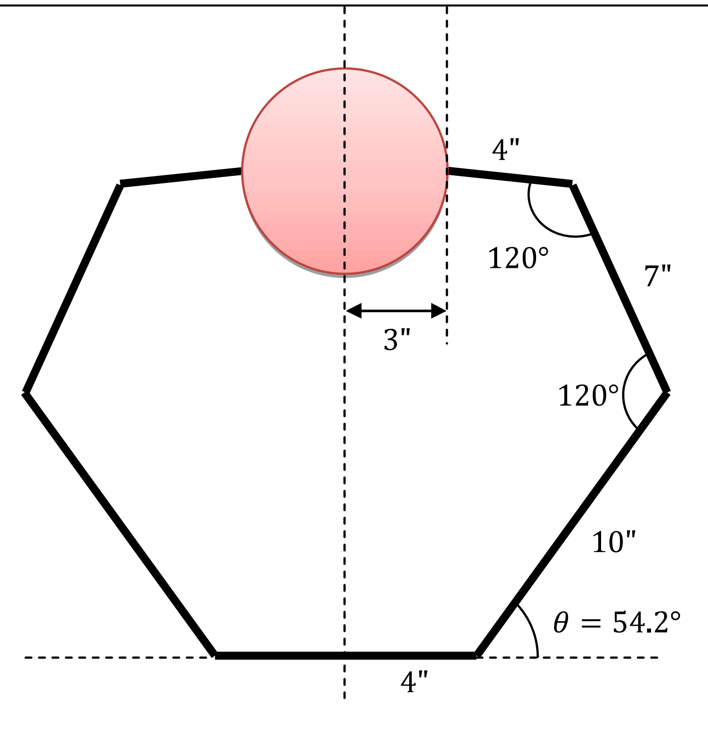

Figure 3: In the initial setup, the robotic hand holds a ball of radius 3” in space. The angle θ which achieves this position is 54.2°. The distance from the center of the ball to the base of the hand is calculated to be 18.5”.

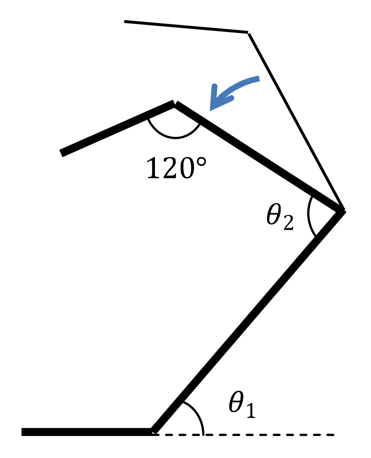

Figure 4: Robotic finger with second adjustable angle θ_2.

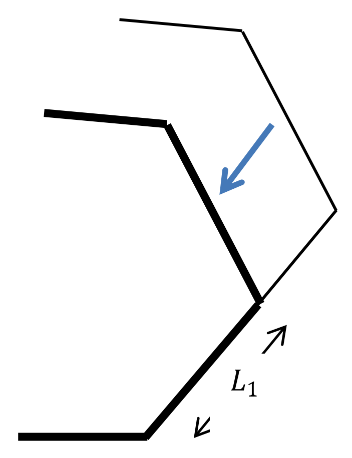

Figure 5: Robotic finger with prismatic joint located at L_1.

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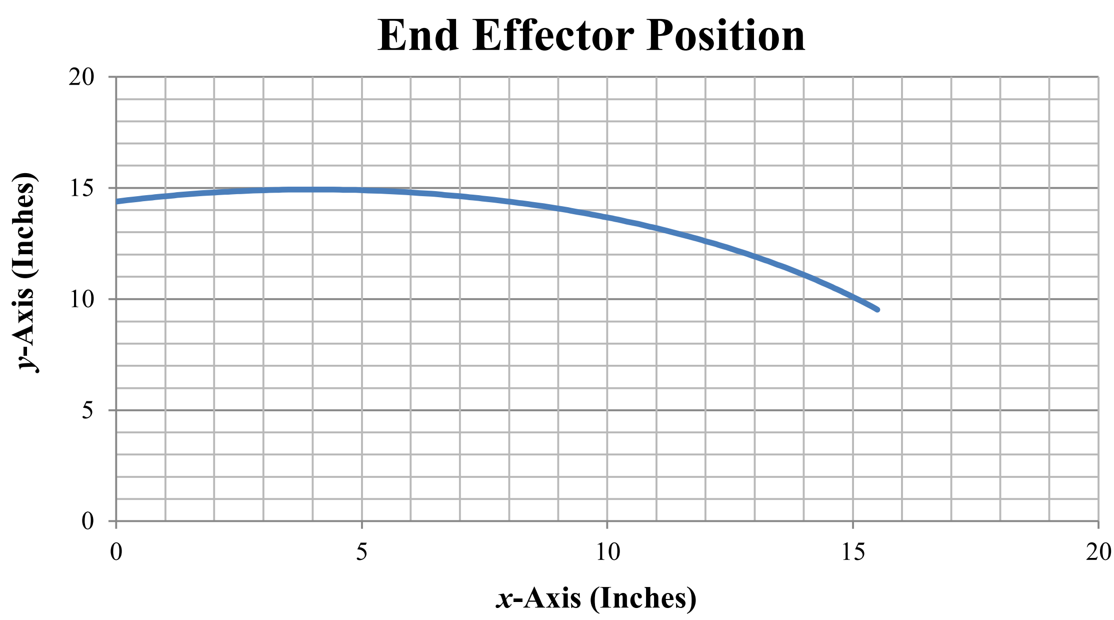

Figure 6: This graph displays the position of the end effector as θ increases from 0° to 120°.

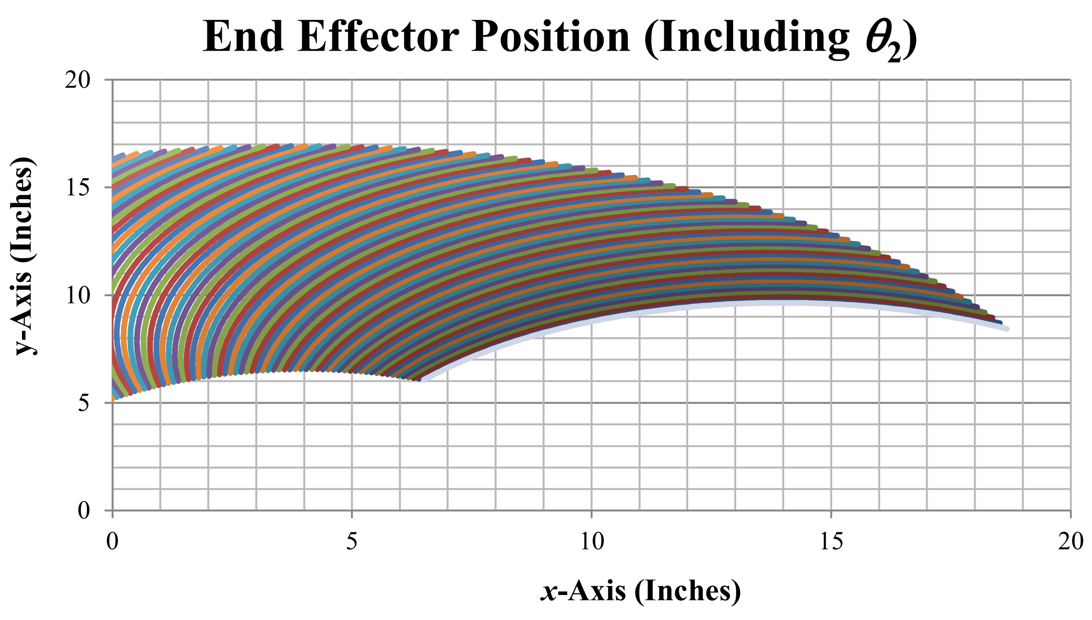

Figure 7: Each line or series in the chart represents an increment of 1° for θ_1 from 60° to 120° plotted against all the values of θ_2 between 60° to 140°.

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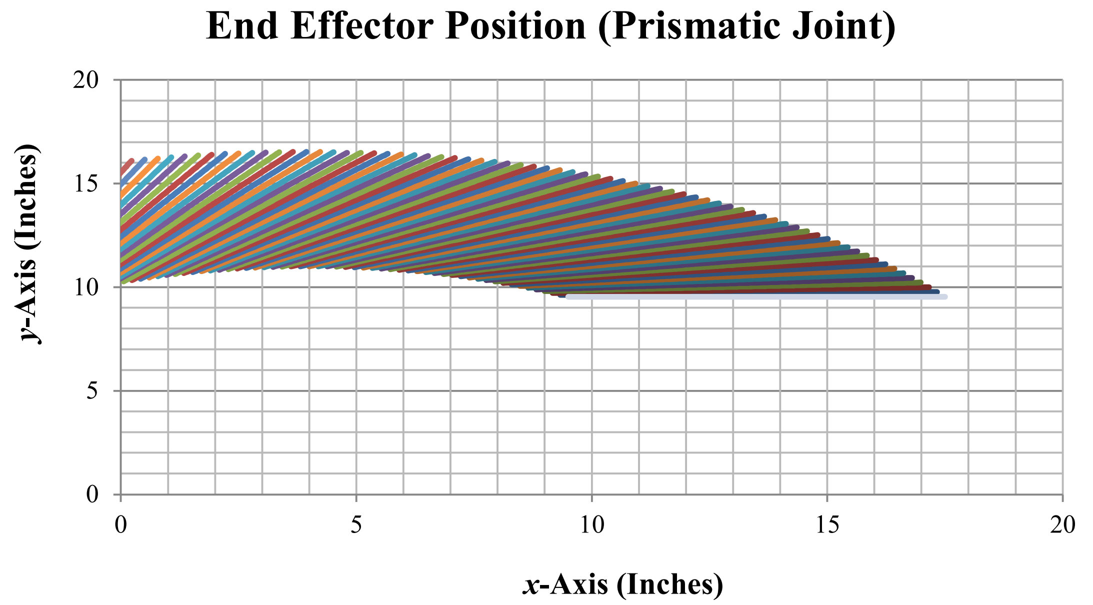

Figure 8: Each series in the graphs represents a 1° increment of θ_1 plotted against a change in the length of L_1 ranging from 4″ to 12″.

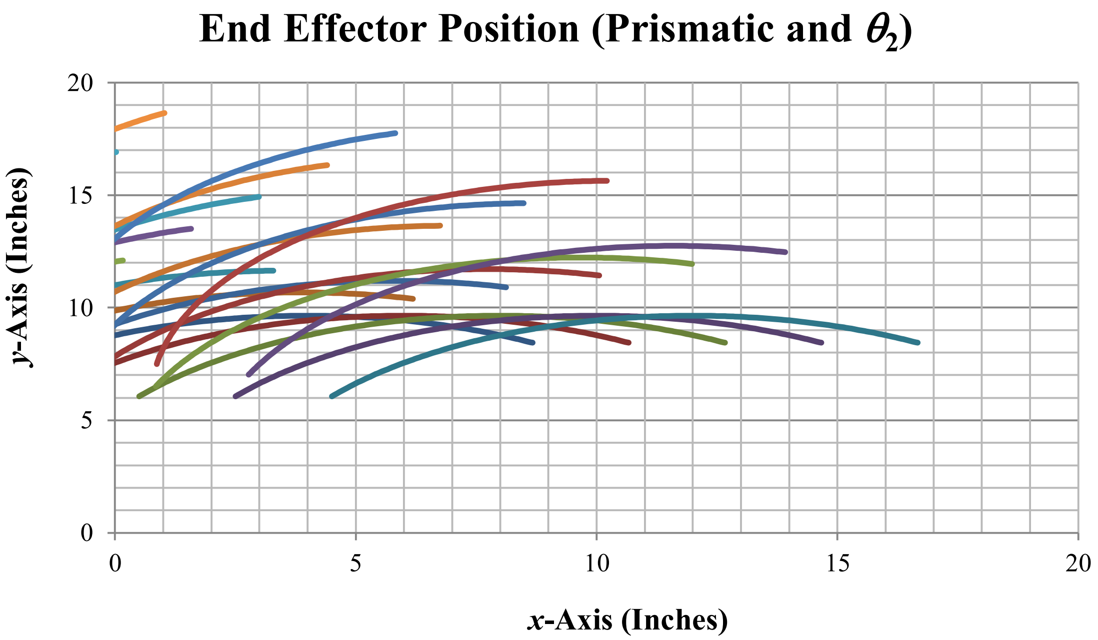

Figure 9: This graph displays the position of the end effector point while including the two variables θ_2 and the length of the prismatic joint applied to L_1. The plotted angles for θ_1 range from 0° to 120° in increments of 15°. Included angles for θ_2 are from 80° to 120° by 1° increments. The plotted prismatic joint values are 4", 6", 8", 10", and 12".