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# Parametric and Bayesian Modeling of Reliability and Survival Analysis

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# Parametric and Bayesian Modeling of Reliability and Survival Analysis

by

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## DEDICATION

Primarily to God, from Whom all true discoveries and all true knowledge in this world come from. To my beloved wife Marie and my smart and curious little girl Kailyn. To my mother Escilda, my sister Nicolasa, my mother-in-law Lydia, and my father Antonio.

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## ABSTRACT

The objective of this study is to compare Bayesian and parametric approaches to determine the best for estimating reliability in complex systems. Determining reliability is particularly important in business and medical contexts. As expected, the Bayesian method showed the best results in assessing the reliability of systems.

In the first study, the Bayesian reliability function under the Higgins-Tsokos loss function using Jeffreys as its prior performs similarly as when the Bayesian reliability function is based on the squared-error loss. In addition, the Higgins-Tsokos loss function was found to be as robust as the squared-error loss function and slightly more efficient.

In the second study, we illustrated that—through the power law intensity function—Bayesian analysis is applicable in the power law process. The power law intensity function is the key entity of the power law process (also called the Weibull process or the non-homogeneous Poisson process). It gives the rate of change of a system's reliability as a function of time. First, using real data, we demonstrated that one of our two parameters behaves as a random variable. With the generated estimates, we obtained a probability density function that characterizes the behavior of this random variable. Using this information, under the commonly used squared-error loss function and with a proposed adjusted estimate for the second parameter, we obtained a Bayesian reliability estimate of the failure probability distribution that is characterized by the power law process. Then, using a Monte Carlo simulation, we showed the superiority of the Bayesian estimate compared with the maximum likelihood estimate and also the

better performance of the proposed estimate compared with its maximum likelihood counterpart.

In the next study, a Bayesian sensitivity analysis was performed via Monte Carlo simulation, using the same parameter as in the previous study and under the commonly used squared-error loss function, using mean square error comparison. The analysis was extended to the second parameter as a function of the first, based on the relationship between their maximum likelihood estimates. The simulation procedure demonstrated that the Bayesian estimates are superior to the maximum likelihood estimates and that the selection of the prior distribution was sensitive. Secondly, we found that the proposed adjusted estimate for the second parameter has better performance under a noninformative prior.

In the fourth study, a Bayesian approach was applied to real data from breast cancer research. The purpose of the study was to investigate the applicability of a Bayesian analysis to survival time of breast cancer data and to justify the applicability of the Bayesian approach to this domain. The estimation of one parameter, the survival function, and hazard function were analyzed. The simulation analysis showed that the Bayesian estimate of the parameter performed better compared with the estimated value under the Wheeler procedure. The excellent performance of the Bayesian estimate is reflected even for small sample sizes. The Bayesian survival function was also found to be more efficient than its parametric counterpart.

In the last study, a Bayesian analysis was carried out to investigate the sensitivity to the choice of the loss function. One of the parameters of the distribution that characterized the survival times for breast cancer data was estimated applying a Bayesian

approach and under two different loss functions. Also, the estimates of the survival function were determined under the same setting. The simulation analysis showed that the choice of the squared-error loss function is robust in estimating the parameter and the survival function.

# CHAPTER 1 REVIEW OF LITERATURE AND THE PRESENT STUDIES

This chapter presents a review of the body of literature related to reliability analysis of complex systems that are relevant to the present studies. In particular, an overview of reliability and survival theory is presented, along with ordinary and empirical Bayesian methods, Bayesian point estimation, and the power law process. Finally, we introduce the structure of the problems that we study in the thesis.

## 1.1 Introduction

Failure in complex systems can have far-reaching negative effects. For instance, failure of mechanical equipment can lead to significant repairs, technical support, and loss of employee time, all of which can have a direct impact on productivity and costs (Crow, 1974; Tsokos & Shimi, 1977; Singpurwalla, 2006). Even in the field of medicine, treatment regiments can be viewed as complex systems, and knowledge of systems and their failure behavior can save lives (Tsokos & Shimi, 1977; Singpurwalla, 2006). Reliability analysis can aid in the more effective use of resources in the longevity of equipment. Its statistical equivalent survival analysis can help clinicians to decide which treatments are better for patients in terms of survival time (Crow, 1974). Klein and Moeschberger (1997) provide two relevant examples: 1) in bone marrow transplantation, survival function can be used to compare the efficacy of autologous transplant methods compared with allogenic methods, and 2) in early-stage breast cancer treatment for women, the effectiveness of radiotherapy alone can be compared with that of radiotherapy with adjuvant chemotherapy.

Reliability can be estimated in a number of ways. Bayesian and parametric approaches of estimation are some common methods of estimation. In order to obtain the more favorable of the two approaches, we conducted several studies of reliability, using both Bayesian and parametric methods in each study to determine which method shown to be more efficient in obtaining estimates. In each instance, simulated data was used to illustrate the evaluation process. However, in two of the four studies, real data were also used to demonstrate the practical implications of reliability and survival analysis.

### 1.2 Reliability and Survival Theory

*Reliability* of a process, product, or system is the probability that it will perform as specified, under the specified conditions, for the specified period of time (Blank, 2004). The purpose of reliability analysis is to evaluate the performance of an item, to predict its time to failure (TTF), and to find its failure pattern.

A reliability analysis must be based on precisely defined concepts in order to make comparisons between systems and to provide logical bases for improvement. In a reliability analysis, some commonly used statistical concepts to investigate for the subject data include TTF, reliability function, hazard rate, and reliable life. The collected data, obtained for example from a reliability test of an object or from observations of its use, are realizations of random variables.

TTF, also called *failure time*, is a random period of operation, after which any object or device of interest fails under stated environmental conditions. TTF can be denoted by the random variable X where $f(x)$ is its probability density function (pdf). The probability of failure as a function of time can be defined as

$$F(x) = P(X \leq x) = \int_0^x f(u)du, \qquad x \geq 0$$

where $F(x)$ is the probability that the device will fail by time $x$. Sometimes, the cumulative distribution function (CDF), $F(x)$, is referred to as the *unreliability function* (Tobias & Trindade, 1986).

If reliability is defined as the probability of success—that is, the probability that the device will perform its intended function for at least a period of time $x$—then we can write

$$R(x) = P(X > x) = \int_x^\infty f(u)du = 1 - F(x)$$

where $R(x)$ is the *reliability function* or the *survival function* commonly used in the life sciences and sometimes denoted by $S(x)$. The mathematical foundations of reliability and survival analyses are the same. However, the methodologies may sometimes be different (Singpurwalla, 2006).

Several concepts are relevant to the determination of reliability. *Failure rate*, the rate at which failures occur in a certain time interval $[x_1, x_2]$, can be used to help determine failure pattern. It is defined as the probability that a failure occurs in a time interval, given that a failure has not occurred prior to the beginning of the interval $x_1$. In addition, the *hazard rate* (also referred to as hazard rate function or hazard function) is relevant to reliability. It is defined by the limit of the failure rate as the length of the interval $[x_1, x_2]$ approaches zero. Thus, it is the instantaneous failure rate. The hazard rate $h(x)$ is defined as

$$h(x) = \lim_{\Delta x \to 0} \frac{R(x) - R(x + \Delta x)}{\Delta x R(x)} = \frac{1}{R(x)}\left[-\frac{dR(x)}{dx}\right]$$
$$= -\frac{dLnR(x)}{dx} = \frac{f(x)}{R(x)}$$

since $-\frac{dR(x)}{dx} = f(x)$, the TTF pdf.

The term, $h(x)dx$, represents the probability that a device that has survived to time *x* will fail in the small interval of time from *x* to $x + dx$; it also can represent the probability that a patient who has survived to time *x* will die in the small interval of time represented by $[x, \; x + dx]$. Thus, $h(x)$ is the rate of change of the conditional probability of failure given survival time *x*. The importance of the hazard rate is that it can indicate the change in the failure rate over the lifetime of a population of devices; it can also indicate the change in the death rate in the survival time of patients. In addition, it is important to note that $f(x)$ is the rate of change of the ordinary (unconditional) probability of failure. If $h(x)$ is increasing in $x \geq 0$, $f(x)$ is said to be a decreasing failure rate distribution.

*Reliable life* is yet another facet of reliability. It is represented by R and is a measure of the reliability of a device or survival of a patient at a given time $x_R$. The reliable life may be thought of as the time $x_R$ for which 100R% of the population will survive.

### 1.2.1 Ordinary Bayesian Methods in Reliability and Survival Analysis

When used to determine reliability, Bayesian methods allow the combination of operation data with any other relevant information available for reliability studies (Martz & Waller, 1982). Some possible sources of supplemental information are engineering design and test data, operating data in different environments, engineering judgments and personal experience, operating experiences with similar equipment, or efficiency data on a given treatment for a patient.

A Bayesian reliability analysis consists of the use of statistical methods in reliability problems that involve parameter estimation. In the parameter estimation, one or more of the parameters are considered to be a random variable with a nondegenerate prior probability distribution, which expresses the analyst's prior degree of belief about the parameters. Several elements are present in a good Bayesian reliability analysis; namely, a detailed justification and analysis of the prior distribution selected, with a clear understanding of the mathematical implications of this prior and thorough documentation of the data sources used in identifying and selecting the prior (Martz & Waller, 1982).

In the analysis, the selection of the prior must be considered satisfactory. Secondly, using the amount of sample test data ultimately expected, the analyst should consider a group of simulated sample test results as data. Third, for the tentative prior distribution and each of the simulated test results, the analyst should compute the resultant posterior distribution via the Bayes Theorem. Fourth, in the posterior analysis, the analyst should study the set of resulting posterior distributions to determine whether they seem reasonable in light of the simulated data. If they are reasonable, the prior distribution becomes a strong candidate for use. In addition, a Bayesian reliability analysis has two more elements: a clearly defined posterior distribution of the parameter(s) of interest and an analysis of the sensitivity of the Bayesian inferences to the prior model selected.

Sample data may be expensive or difficult to obtain in areas of application such as reliability. A Bayesian method usually requires less sample data to achieve the same quality of inferences than the method based on sampling theory. In many cases, this is the practical motivation for using a Bayesian method and represents the practical advantage in the use of prior information.

A Bayesian analysis has additional practical and important benefits. One is the increased quality of the inferences, provided the prior information accurately reflects the time variation in the parameter(s). Another benefit is the reduction in testing requirements (i.e., test time or sample size) that often occurs in Bayesian reliability demonstration test programs. Both of these are the result of formally including additional information in the analysis in the form of the prior distribution.

It is important to recognize that all statistical inferential theories—whether sampling theory, Bayesian, likelihood, or otherwise—require some degree of subjectivity in their use. Sampling theory requires assumptions about a sampling model, confidence coefficients, which estimator to use, and so on. For example, a sampling analysis of

$$f(\vec{x} \mid \theta) = \frac{1}{\theta} \exp\left\{ -\frac{1}{\theta} \sum_{i=1}^{n} x_i \right\}, \quad 0 < x_i < \infty$$

proceeds under *a priori* belief that the data were exactly exponentially distributed, that each observation had exactly the same mean life $\theta$, and that each observation was distributed exactly independently of every other sample observation.

The Bayesian method provides a satisfactory way of explicitly introducing and organizing assumptions regarding prior knowledge or ignorance. In the Bayes Theorem, these assumptions lead to posterior inferences—that is, inferences obtained once the data have been incorporated into the analysis of the reliability parameter(s) of interest.

Bayesian analysis is associated with yet another important advantage—inferences that are unacceptable must come from incorrect assumptions and not from inadequacies of the method used to provide the inferences. In this regard, the Bayesian procedure rectifies many shortcomings of the sampling theory method. Inferences based on the deductive arguments inherent in the Bayesian approach are more direct than those based on the inductive arguments of sampling theory (Martz & Waller, 1982).

The philosophical bases of the Bayesian paradigm are founded on the calculus of probabilities. However, in reality, with unique situations, the notion of frequency is not always relevant. The Bayesian paradigm allows for these kinds of situations and for situations in which no previous data exist. In such cases, the study of the uncertainty can only be based on background information. In the consideration of prior probabilities, the Bayesian paradigm enables the formal incorporation of information from the experts into the analysis (Singpurwalla, 2006). Reliability analysis is most credible when subject matter experts play a key role throughout the analysis.

### 1.2.2 Empirical Bayesian Methods in Reliability and Survival Analysis

As explained by Martz and Waller (1982), the empirical Bayes approach to reliability analysis is a class of decision theoretical procedures that uses past data as a measure for bypassing the necessity of identifying a completely unknown and unspecified prior probability distribution that has a frequency interpretation. A main difference between ordinary Bayes and empirical Bayes methods is that, unlike in empirical Bayes, in ordinary Bayes, an underlying prior probability distribution is assumed to exist with a degree of belief or a frequency interpretation. Also with the ordinary Bayes method, the parametric form of the prior probability distribution is either assumed to be completely known and specified or known except for the values of certain parameters that had to be estimated from sampled data. Whereas, in the empirical Bayes approach, the distributional form of the prior probability remains unknown. In this case, Bayes estimation methods cannot be employed apart from a hit-or-miss assumption about the unknown prior. With such an assumption and, further, that the parameter to be estimated does indeed follow an unknown prior probability distribution having a frequency interpretation, the Bayes estimates based on this assumed prior may or may not accurately approximate the true Bayes estimate that could be obtained if the true prior probability distribution were known. In such cases, the accuracy of the approximation is never really known. One can only demonstrate how well the assumed prior probability distribution performs when the true probability distribution departs from the assumption.

In addition, using the empirical Bayes method is desirable in order to avoid the need to identify a prior probability distribution. The assignment of a prior probability distribution often represents a practical difficulty in the application of Bayesian methods.

When empirical Bayes procedures can be used, it is often desirable to do so, due to their greater dependency on empirical data and fewer assumptions than strict Bayesian methods. Also, in empirical Bayes procedures, the postulated prior probability distribution must have a frequency interpretation, and certain “past” data suitable for estimating this probability distribution are assumed to be available (Martz & Waller, 1982).

In essence, the difference between empirical Bayes and ordinary Bayes is that empirical Bayes does not make explicit the form of the prior information in order to make possible a Bayes solution. Instead, the empirical Bayes method depends on the existence of prior information in the form of past estimates of either the parameter in question or some close variation of it.

In the case when estimation of reliability is done under data accumulation conditions, the analysis can be based on the empirical Bayes approach in the form of reliability estimates of all preceding types of the devices and does not require the determination of a prior probability distribution in a unique way. In the case that the prior probability distribution is known and the availability of data is not met, then the ordinary Bayesian approach is the appropriate choice. However, in the case that the prior probability distribution is known and the reliability analysis is done under data accumulation conditions, either ordinary Bayes or empirical Bayes analysis may be employed. In this case, we would use model selection criteria to select the best choice.

## 1.3 Bayesian Point Estimation

From a Bayesian point of view, a decision function $\psi(\vec{x})$ is considered to search for an estimator $\hat{\zeta}$ to approximate the unknown random parameter $\zeta$ from the observed realizations $\vec{x} = (x_1, x_2, x_3, ..., x_n)$ of independent random variables $X_1, X_2, X_3, ...X_n$ with a common pdf conditional on $\zeta$. A loss function $L(\psi(\vec{x}), \zeta) \geq 0$, represents the error of choosing $\psi(\vec{x})$ as the decision function for $\zeta$. The conditional expectation of the loss for any $\psi(\vec{x})$ when $\zeta$ is the realization of the random variable Z is called the risk and is defined by the relation

$$R[\psi(\vec{x}), \zeta] = E[L(\psi(\vec{x}), \zeta) \mid \zeta]$$

where $\zeta \in \mathrm{Z} \subseteq \mathbb{R}$ is assumed.

The expected risk, over the entire parameter space $\mathrm{Z}$ when the estimator $\psi(\vec{x})$ is used, is given by the expectation with respect to the prior probability distribution $p(\zeta)$ of $\zeta$, that is,

$$\begin{aligned}
R_{\zeta}[\psi(\vec{x})] &= E(R[\psi(\vec{x}), \zeta]) \\
&= \int_{\mathrm{Z}} R[\psi(\vec{x}), \zeta] p(\zeta) d\zeta \\
&= \int_{\mathrm{Z}} \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} L(\psi(\vec{x}), \zeta) L(\vec{x}; \zeta) d\vec{x} \right] p(\zeta) d\zeta
\end{aligned}$$

where $L(\vec{x}; \zeta)$ is the likelihood function and $p(\zeta)$ is the prior density function of Z. Since the integrand is nonnegative, interchanging the order of integration of $\vec{x}$ and $\zeta$, we obtain

$$\begin{aligned}
R_{\zeta}[\psi(\vec{x})] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left[ \int_{\mathrm{Z}} L(\psi(\vec{x}), \zeta) L(\vec{x}; \zeta) p(\zeta) d\zeta \right] d\vec{x} \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} m(\vec{x}) \left[ \int_{\mathrm{Z}} L(\psi(\vec{x}), \zeta) h(\zeta; \vec{x}) d\zeta \right] d\vec{x}
\end{aligned}$$

where

$$m(\vec{x}) = \int_{Z} L(\vec{x};\zeta)\, p(\zeta)\, d\zeta$$

is the marginal probability density function of $\vec{X}$, in the case that $Z$ is a continuous random variable. To minimize the risk $R_{\zeta}[\psi(\vec{x})]$, the decision function $\psi(\vec{x})$ is chosen so that the quantity

$$\int_{Z} L(\psi(\vec{x}),\zeta)\, h(\zeta;\vec{x})\, d\zeta$$

is a minimum. Therefore, the Bayes decision function, or Bayes estimator, for the realization $\zeta$ is the decision function $\psi$ which minimizes the expected loss

$$E\left[L(\psi(\vec{x}),\zeta) \mid \vec{x}\right] = \int_{Z} L(\psi(\vec{x}),\zeta)\, h(\zeta;\vec{x})\, d\zeta$$

with respect to the prior distribution of $Z$, $p(\zeta)$. Moreover, the Bayes solution $\psi$ minimizes the expected risk

$$R[\psi] = \min_{\psi}\left\{R_{\zeta}\left[\psi(\vec{x})\right]\right\}$$

called the Bayes risk. Clearly, the determination of the Bayes solution and risk depends on the form of the prior probability distribution $p(\zeta)$.

## 1.4 Power Law Process

A *repairable system* is one that can be restored to an operating condition by some repair process instead of replacing the entire system. We assume that, in such a system, we observe a number of failures. Let $0 < T_1 < T_2 < \ldots$ denote the TTFs of the system measured in *global time*—that is, the times are recorded from the initial start-up of the system onward.

Let $X_1, X_2, \ldots$ denote the times between failures such that $X_i = T_i - T_{i-1}$, *i=1,2,…* . Consider a complex repairable system that is tested until it fails, and then corrective action is undertaken to identify and remove the cause. The system is tested again until the next failure occurs. This process continues toward achieving a desired reliability level. This testing procedure is known as *reliability growth*.

Duane (1964) proposed the concept of the "learning curve approach" to monitor the progress of reliability improvement programs. According to Duane, this learning curve is useful in predicting the duration and the end result of such programs. This graphical method for displaying data from repairable systems can be used to gain insight into the data. It can be used to determine whether there is a trend in the time between failures. It consists of plotting the global time $t_i$ along the horizontal axis, and on the other axis the ratio of the cumulative number of failures through time $t_i$, that is, $N(t_i)$, and $t_i$, *i=1,2,…,n*. This ratio is often called the *cumulative failure rate*.

Let $N(a,b]$ denote the number of failures in the interval $(a,b]$. A counting process $N(t)$ is said to be a *Poisson process* if the following conditions exist:

1. $N(0) = 0$.
2. The independent increment property holds; i.e., for any $a < b \leq c < d$, the random variables $N(a,b]$ and $N(c,d]$ are independent. That is, counts in nonoverlapping intervals are independent.
3. There is a function $V$, called the *intensity function* of the Poisson process, such that

$$V(t) = \lim_{\Delta t \to 0} \frac{P(N(t, t+\Delta t] = 1)}{\Delta t}.$$

4. $\lim_{\Delta t \to 0} \frac{P(N(t,t+\Delta t]=2)}{\Delta t} = 0$; i.e., there are not simultaneous failures.

A consequence of these four conditions presented in the Poisson process definition is that

$$P(N(t)=n) = \frac{[\int_0^t V(x)dx]^n \exp\left\{-\int_0^t V(x)dx\right\}}{n!}, n = 0,1,2,...$$

which implies that $N(a,b]$ for any $a<b$ has a Poisson probability distribution with parameter $\int_a^b V(x)dx$.

The *non-homogeneous Poisson process* (NHPP) is a Poisson process whose intensity function is nonconstant and is an effective approach to analyzing reliability growth. Since for some repairable systems the plots of the cumulative failure rate to time were approximately linear on log-log paper, Crow (1974, 1975) proposed a NHPP with intensity function given by

$$V(t) = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1}, \beta>0, \theta>0, t>0. \tag{1.2.1}$$

This type of Poisson process is usually called the *power law process* (PLP), and its intensity function is called the *power law intensity function*. The PLP is also referred to as the *Weibull process* since the power law intensity function has the same form as the hazard function of a Weibull distribution with pdf given by

$$f(x)=\begin{cases}\frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1}\exp\left\{-\left(\frac{t}{\theta}\right)^{\beta}\right\}, \beta>0, \theta>0, t>0\\ 0 \qquad\qquad\qquad\qquad\qquad\qquad , \text{otherwise}\end{cases}$$

even when the TTF does not have a Weibull probability distribution (except for the first failure) and neither do the times between failures.

The PLP reduces to a *homogeneous Poisson process* (HPP) if $\beta$=1. In the case of $\beta$>1 the intensity function increases, which implies the reliability decreases. For $\beta$<1 the power law intensity function decreases, implying reliability growth.

The NHPP is an effective approach to analyze the reliability growth and predict the failure behavior of a given system. The following researchers reported on the fundamental aspects of reliability growth of repairable systems: Bassin (1969), Higgins and Tsokos (1981), Ascher and Feingold (1984), Engelhardt and Bain (1978, 1987), Rigdon and Basu (1990), and Ascher, Lin, and Siewiorek (1992) among others.

### 1.4.1 Review of the Analytical Power Law Process

The probability of achieving $n$ failures in a given system in the time interval (0, t] can be written as

$$P(x=n;t)=\frac{\exp\left\{-\int_0^t V(x)dx\right\}\left[\int_0^t V(x)dx\right]^n}{n!},\ t>0 \qquad (1.2.1.1)$$

where $V(t)$ is the intensity function given by (1.2.1). The reduced expression

$$P(x=n;t)=\frac{1}{n!}\exp\left\{-\left(\frac{t}{\theta}\right)^{\beta}\right\}\left(\frac{t}{\theta}\right)^{n\beta} \qquad (1.2.1.2)$$

represents the NHPP or Weibull process.

If the PLP is the underlying failure model of the TTF's $t_1, t_2, t_3, ..., t_{n-1}$, and $t_n$, the conditional reliability function of $t_n$ given $t_1, t_2, t_3, ..., t_{n-1}$ can be written as

$$R(t_n \mid t_1, t_2, t_3, ..., t_{n-1}) = \exp\left\{-\int_{t_{n-1}}^{t_n} V(x)dx\right\}, \quad t_n > t_{n-1} > 0 \qquad (1.2.1.3)$$

since it is independent of $t_1, t_2, t_3, ..., t_{n-2}$. The equation (1.2.1.3) shows the reliability as a function of the intensity function. An estimate of the reliability function can be obtained using an estimate of the intensity function, where the key entity is the parameter $\beta$. Therefore, $\beta$ affects the reliability function through the intensity function.

The maximum likelihood estimate (MLE) of $\beta$ is a function of the largest TTF, and the MLE of $\theta$ is also a function of the MLE of $\beta$ as we will show below. Let $T_1, T_2, \dots, T_n$ denote the first $n$ TTF's of the NHPP, where $T_1 < T_2 < \dots < T_n$ are measured in global time, that is, the times are recorded from the initial start-up of the system onward. Thus, the truncated conditional probability distribution function, $f_i(t \mid t_1, ..., t_{i-1})$, in the Weibull process and is given by

$$f_i(t \mid t_1, ..., t_{i-1}) = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1} \exp\left\{-\left(\frac{t}{\theta}\right)^{\beta} + \left(\frac{t_{i-1}}{\theta}\right)^{\beta}\right\}, \; t_{i-1} < t \qquad (1.2.1.4)$$

With $\vec{t} = (t_1, t_2, ..., t_n)$, the likelihood function for the first $n$ failures for the times of the NHPP $T_1 = t_1, T_2 = t_2, ..., T_n = t_n$, can be written as

$$L(\vec{t} \mid \beta) = exp\left\{-\left(\frac{t_n}{\theta}\right)^{n}\right\}\left(\frac{\beta}{\theta}\right)^{n} \prod_{i=1}^{n}\left(\frac{t_i}{\theta}\right)^{\beta-1}. \qquad (1.2.1.5)$$

The MLE for the shape parameter is given by

$$\hat{\beta}_n = \frac{n}{\sum_{i=1}^{n} \log\left(\frac{t_n}{t_i}\right)} \tag{1.2.1.6}$$

and, for the scale parameter is

$$\hat{\theta}_n = \frac{t_n}{n^{1/\hat{\beta}_n}} . \tag{1.2.1.7}$$

The theoretical background presented in the previous sections will be applied to the different problems that we will study.

### 1.5 Overview of the Studies

The present study is comprised of the investigation of five different problems. In the first study, detailed in Chapter 2, a Bayesian sensitivity analysis was performed to examine the Bayesian reliability function under the Higgins-Tsokos loss function (Higgins & Tsokos, 1980) using several probability priors. In addition, a comparison was made between the best Bayesian estimate obtained from the analysis and the Bayesian Reliability function based on the squared-error loss function. Robustness of the loss function and efficiency will be examined.

The second study is detailed in Chapter 3. The objective of the study was to illustrate the applicability of a Bayesian analysis in the NHPP through the two parameter intensity function. We performed a numerical simulation to compare the Bayesian estimates of one of the parameters and the Bayesian estimate of the intensity function under the assumption of a squared-error loss function with the maximum likelihood estimates. Moreover, we proposed an adjusted maximum likelihood estimate for the second parameter and obtained a Bayesian reliability estimate of the PLP.

The next study is a logical continuation of the previous and is detailed in Chapter 4. A Bayesian sensitivity analysis— of the same parameter as in the previous study— based on the prior selection was performed via Monte Carlo simulation. The analysis was carried out under the assumption of the squared-error loss function using mean square error comparison. The study was extended to the second parameter as a function of the first, based on the relationship between their maximum likelihood estimates.

In Chapter 5, we studied Bayesian and parametric survival analysis of real breast cancer data. The purpose of the study was twofold: to justify the applicability of the Bayesian approach to this domain and to compare the Bayesian and parametric estimates. The Bayesian estimation of one parameter, the survival function, and hazard function were analyzed and are presented in detail in the present study.

Chapter 6 is a logical extension of the previous study. A Bayesian sensitivity analysis was performed to examine the Bayesian survival function under the squared-error and the Higgins-Tsokos loss functions. The objective was to find out how robust is the selection of the squared-error loss function. Chapter 7 presents future research directions in this area of studies.

# CHAPTER 2 BAYESIAN RELIABILITY ANALYSIS OF THE WEIBULL DISTRIBUTION SUBJECT TO SEVERAL PRIORS AND THE HIGGINS-TSOKOS LOSS FUNCTION

The objective of the present study is to perform a Bayesian sensitivity analysis of the choice of the prior for estimating the reliability function associated with the 3-parameter Weibull model, where one of the parameters behaves as a random variable. In this study, first we calculated the Bayesian estimate of the parameter under the Higgins-Tsokos loss function for each of the selected priors. Then, we compared the closer estimate obtained from the analysis with the Bayesian estimate of the Weibull reliability function based on the best choice of the prior now under the commonly used squared-error loss function.

The present study is divided into four sections. In the first section, we present the background theory to develop a Bayesian analysis for the reliability function of the 3-parameter Weibull probability distribution as the underlying failure model. We present several priors as the different choices for the probabilistic behavior of the parameter assumed as a random variable. We proceeded to develop the general form of the Bayesian estimate of the parameter and the Weibull reliability function assuming the Higgins-Tsokos loss function. In the second section, we introduce the analytical form of the reliability function for each of the priors and under the assumption of the Higgins-Tsokos loss function. We also present the Bayesian estimate of the reliability function under the Jeffreys' prior and squared-error loss function. A numerical simulation of the analytical results is given in the third section. Finally, we summarize the findings in the last section.

## 2.1 Introduction

In the present study, we consider a Bayesian analysis of the three parameter Weibull life testing model whose probability density function is given by

$$w(x \mid \theta, \xi, \tau) = \frac{\xi}{\theta}(x-\tau)^{\xi-1} \exp\left\{-\frac{1}{\theta}(x-\tau)^{\xi}\right\}, \quad \xi > 0, \theta > 0, x \geq \tau \tag{2.1}$$

under the assumption that the guarantee time $\tau$ and the shape parameter $\xi$ are known (can be estimated) and $\theta$ behaves as a random variable.

A Bayesian analysis implies the use of suitable prior information in association with Bayes' Theorem and rests on the exploitation of such information as well as the belief that a parameter is not merely an unknown fixed quantity but rather a random variable with some prior probability distribution.

In life testing, as Barlow and Proschan (1965) pointed out, the exponential family has been the best known and most thoroughly explored probability distributions. However, it suffers somewhat because its constant failure rate makes it inadequate for describing the life-times of various components which wear out through normal use. As a result, the Weibull probability distribution, although somewhat more complex, has also been used as a failure probability distribution especially if the structure is suspected of having increasing (or decreasing) failure rate. In fact, the Weibull family of distributions offers more flexibility than the exponential family for the latter is but a special case of the former. Therefore, we shall be concerned with the Bayesian estimation of the associated reliability function of (2.1), that is,

$$R(t) = P(X > t) = \exp\left\{-\frac{1}{\theta}(t-\tau)^{\xi}\right\}, \quad \xi>0,\ \theta>0,\ t \geq \tau \tag{2.2}$$

by considering $\theta$ as a random variable and $\xi$ and $\tau$ are known or can be estimated.

When $\theta$ is assumed to be a random variable, we shall examine the problem for each of the following four prior probability densities of $\theta$:

(i) a general uniform probability density given by

$$p(\theta) = \frac{(a-1)(\alpha\beta)^{a-1}}{(\beta^{a-1} - \alpha^{a-1})\theta^{a}}, \quad 0 < \alpha < \theta \leq \beta\,, \tag{2.3}$$

which for $a = 0$ reduces to the uniform density on $[\alpha,\beta]$,

(ii) the exponential probability density

$$p(\theta) = \frac{1}{\lambda}\exp\left\{-\frac{1}{\lambda}\theta\right\}, \quad \lambda > 0, 0 < \theta < \infty, \tag{2.4}$$

(iii) the inverted gamma probability density

$$p(\theta) = \left(\frac{\mu}{\theta}\right)^{\nu+1}\frac{1}{\mu\Gamma(\nu)}\exp\left\{-\frac{\mu}{\theta}\right\}, \quad \mu > 0, \nu > 0, \theta > 0\,, \tag{2.5}$$

and

(iv) the Jeffreys' prior

$$p(\theta) \propto \frac{1}{\theta}, \quad \theta > 0. \tag{2.6}$$

The uniform prior probability density of $\theta$ is surely a realistic choice if one considers the possibility of some prior information concerning the range of the parameter. The inverted gamma prior will give rise to a posterior density that belongs to the same family; thus the property of closure under sampling is realized.

Bhattacharya (1967) considered a Bayesian analysis of the exponential distribution with probability density function given by

$$f(x \mid \theta) = \frac{1}{\theta} e^{-\frac{1}{\theta}x}, \quad \theta > 0, 0 < x < \infty, \tag{2.7}$$

when the parameter $\theta$ is treated as a random variable and obtained Bayesian estimates of the reliability function $R(t \mid \theta) = e^{-\frac{1}{\theta}t}$ for the three prior densities (2.3)-(2.5).

The Higgins-Tsokos loss function (1976) is given by

$$L(\hat{\zeta}, \zeta) = \frac{f_1 \exp\{f_2(\hat{\zeta} - \zeta)\} + f_2 \exp\{-f_1(\hat{\zeta} - \zeta)\}}{f_1 + f_2} - 1, \qquad f_1 > 0, f_2 > 0 \tag{2.8}$$

where $\hat{\zeta}$ represents the estimate for $\zeta$. We use this loss function since it places a heavier penalty at the extremes (over and underestimation) than in the middle compared to the squared-error loss function, which is traditionally used because of its analytical tractability (Camara & Tsokos, 2001).

The risk using the H-T loss function, with $\zeta = \theta$ and $\hat{\zeta} = \hat{\theta}$, is given by

$$E\left[L\left(\hat{\theta},\theta\right)\right] = \int_{-\infty}^{\infty}\left[\frac{f_1 \exp\left\{f_2\left(\hat{\theta}-\theta\right)\right\} + f_2 \exp\left\{-f_1\left(\hat{\theta}-\theta\right)\right\}}{f_1+f_2} - 1\right] h(\theta \mid \vec{x})d\theta$$

$$= \int_{-\infty}^{\infty}\left[\frac{f_1 \exp\left\{f_2\left(\hat{\theta}-\theta\right)\right\} + f_2 \exp\left\{-f_1\left(\hat{\theta}-\theta\right)\right\}}{f_1+f_2}\right] h(\theta \mid \vec{x})d\theta - 1$$

$$= \frac{f_1}{f_1+f_2}\exp\left\{f_2\hat{\theta}\right\}\int_{-\infty}^{\infty}\exp\left\{-f_2\hat{\theta}\right\}h(\theta \mid \vec{x})d\theta + \frac{f_2}{f_1+f_2}\exp\left\{-f_1\hat{\theta}\right\}\int_{-\infty}^{\infty}\exp\left\{f_1\hat{\theta}\right\}h(\theta \mid \vec{x})d\theta - 1$$

Then,

$$\frac{\partial E[L(\hat{\theta},\theta)]}{\partial\hat{\theta}} = \frac{f_1 f_2}{f_1+f_2}\exp\left\{f_2\hat{\theta}\right\}\int_{-\infty}^{\infty}\exp\left\{-f_2\hat{\theta}\right\}h(\theta \mid \vec{x})d\theta - \frac{f_1 f_2}{f_1+f_2}\exp\left\{-f_1\hat{\theta}\right\}\int_{-\infty}^{\infty}\exp\left\{f_1\hat{\theta}\right\}h(\theta \mid \vec{x})d\theta$$

and the minimum satisfies $\frac{\partial E[L(\hat{\theta},\theta)]}{\partial\hat{\theta}} = 0$, that is,

$$0 = \frac{f_1 f_2}{f_1+f_2}\exp\left\{f_2\hat{\theta}\right\}\int_{-\infty}^{\infty}\exp\left\{-f_2\hat{\theta}\right\}h(\theta \mid \vec{x})d\theta - \frac{f_1 f_2}{f_1+f_2}\exp\left\{-f_1\hat{\theta}\right\}\int_{-\infty}^{\infty}\exp\left\{f_1\hat{\theta}\right\}h(\theta \mid \vec{x})d\theta .$$

Then,

$$\exp\left\{\hat{\theta}(f_1+f_2)\right\} = \frac{\int_{-\infty}^{\infty}\exp\left\{f_1\theta\right\}h(\theta \mid \vec{x})d\theta}{\int_{-\infty}^{\infty}\exp\left\{-f_2\theta\right\}h(\theta \mid \vec{x})d\theta} .$$

Therefore, the Bayesian estimates of $\theta$ with respect to the Higgins-Tsokos loss function is given by

$$\hat{\theta} = \frac{1}{f_1 + f_2} \ln \left( \frac{\int_{-\infty}^{\infty} \exp\{f_1 \theta\} h(\theta \mid \vec{x}) d\theta}{\int_{-\infty}^{\infty} \exp\{-f_2 \theta\} h(\theta \mid \vec{x}) d\theta} \right),$$

and the Bayes estimate of the reliability function given by (2.2) with respect to the Higgins-Tsokos loss is obtained by evaluating

$$\hat{R}_B(t) = \frac{1}{f_1 + f_2} \ln \left( \frac{\int_{-\infty}^{\infty} \exp\{f_1 R(t)\} h(\theta \mid \vec{x}) d\theta}{\int_{-\infty}^{\infty} \exp\{-f_2 R(t)\} h(\theta \mid \vec{x}) d\theta} \right), \quad f_1 > 0,\ f_2 > 0 \tag{2.9}$$

where $h(\theta \mid \vec{x})$ is the corresponding posterior probability density under the prior $p(\theta)$. Since, $\exp\{u\} = \sum_{k=0}^{\infty} \frac{u^k}{k!}$, then

$$\exp\{f_1 R(t)\} = \sum_{k=0}^{\infty} \frac{[f_1 R(t)]^k}{k!} = \sum_{k=0}^{\infty} \frac{f_1^{\,k} [R(t)]^k}{k!}$$

and similarly

$$\exp\{-f_2 R(t)\} = \sum_{k=0}^{\infty} \frac{(-f_2)^k [R(t)]^k}{k!}.$$

Therefore, (2.9) can be written as

$$\hat{R}_B(t) = \frac{1}{f_1 + f_2} \ln \left\{ \frac{\sum_{k=0}^{\infty} \frac{(f_1)^k \int_{-\infty}^{\infty} \{R(t)\}^k h(\theta \mid \vec{x}) d\theta}{k!}}{\sum_{k=0}^{\infty} \frac{(-f_2)^k \int_{-\infty}^{\infty} \{R(t)\}^k h(\theta \mid \vec{x}) d\theta}{k!}} \right\}, \quad f_1 > 0, f_2 > 0 \tag{2.10}$$

To measure the robustness of the $\hat{R}_B(t)_1$ with respect to $\hat{R}_B(t)_2$ we use the relative efficiency (RE) of the estimate $\hat{R}_B(t)_1$ compared to the estimate $\hat{R}_B(t)_2$, defined as

$$RE = \frac{IMSE[\hat{R}_B(t)_1]}{IMSE[\hat{R}_B(t)_2]},$$

where $IMSE[\hat{R}_B(t)] = \int_0^\infty [\hat{R}_B(t) - R(t)]^2\, dt$. In the case $RE = 1$, $\hat{R}_B(t)_1$ and $\hat{R}_B(t)_2$ will be interpreted as equally effective. If $RE < 1$, $\hat{R}_B(t)_2$ is less efficient than $\hat{R}_B(t)_1$, contrary to when $RE > 1$, in which case $\hat{R}_B(t)_2$ is more efficient than $\hat{R}_B(t)_1$.

## 2.2 Stochastic Scale parameter: Development of Bayesian Reliability Model

We assume $\theta$ is a random variable and consider a random sample of *n* items whose life-times are described by (2.1). The *n* items are placed on a life test which is terminated after observing a predetermined $r \leq n$ number of failures. Let ($x_1, x_2, \ldots, x_r$)= $\bar{x}$ denote the observed ordered life times of the test items. The probability of observing *r* failures at times $x_1, x_2, \ldots, x_r$ and ( $n - r$ ) items having survived time $x_r$ is given by the likelihood of the sample

$$L(\vec{x} \mid \theta) = \frac{n!}{(n-r)!}\left(\frac{\xi}{\theta}\right)^r \prod_{i=1}^{r} (x_i - \tau)^{\xi-1} \exp\left\{-\frac{1}{\theta} S_r\right\} \qquad (2.1.1)$$

where the accumulated observed life is

$$S_r = (n-r)(x_r - \tau)^{\xi} + \sum_{i=1}^{r} (x_i - \tau)^{\xi}.$$

The likelihood of the complete sample is realized for $r = n$.

### 2.2.1 General Uniform Probability Density and Higgins-Tsokos Loss Function

Assuming the general uniform density (2.3) as the prior of $\theta$, by invoking Bayes' Theorem, we obtain the posterior density of $\theta$, that is,

$$h(\theta \mid \vec{x}) = \frac{p(\theta)L(\vec{x} \mid \theta)}{\int_\alpha^\beta p(\theta)L(\vec{x} \mid \theta)d\theta}, \quad \alpha \le \theta \le \beta .$$

Now,

$$p(\theta)L(\vec{x} \mid \theta) = C_U \frac{1}{\theta^{a+r}} \exp\left\{-\frac{1}{\theta} S_r\right\}$$

where

$$C_U = \frac{(a-1)(\alpha\beta)^{a-1}}{\beta^{a-1} - \alpha^{a-1}} \cdot \frac{n!}{(n-r)!} \xi^r \prod_{i=1}^{r} (x_i - \tau)^{\xi-1}$$

and $S_r$ as previously defined.

Then, the posterior density for the general uniform prior is

$$h_U(\theta \mid \vec{x}) = \frac{C_U \frac{1}{\theta^{a+r}} exp\left\{-\frac{1}{\theta} S_r\right\}}{\int_\alpha^\beta C_U \frac{1}{\theta^{a+r}} exp\left\{-\frac{1}{\theta} S_r\right\} d\theta}$$

$$= \frac{\frac{1}{\theta^{a+r}} \exp\left\{-\frac{1}{\theta} S_r\right\}}{\int_\alpha^\beta \frac{1}{\theta^{a+r}} \exp\left\{-\frac{1}{\theta} S_r\right\} d\theta}, \qquad \alpha \le \theta \le \beta. \qquad (2.2.1.1)$$

Using the incomplete gamma function

$$\gamma(n, z) = \int_0^z t^{n-1} \exp\{-t\} dt$$

we have

$$\gamma\left(r+a-1,\frac{S_r}{\alpha}\right)=\int_0^{S_r/\alpha} t^{(r+a-1)-1}\exp\{-t\}dt$$

$$=\int_0^{S_r/\alpha} t^{r+a-2}\exp\{-t\}dt$$

and similarly

$$\gamma\left(r+a-1,\frac{S_r}{\beta}\right)=\int_0^{S_r/\beta} t^{r+a-2}\exp\{-t\}dt.$$

Letting $t=\frac{S_r}{\theta}$, $\left(\frac{1}{\theta}\right)^{r+a}=\left(\frac{t}{S_r}\right)^{r+a}$, and $d\theta=-\frac{S_r}{t^2}dt$, then we have

$$\int_\alpha^\beta \frac{1}{\theta^{r+a}}\exp\left\{-\frac{1}{\theta}S_r\right\}d\theta=-\int_{S_r/\alpha}^{S_r/\beta}\frac{t^{r+a-2}}{(S_r)^{r+a-1}}\exp\{-t\}dt$$

$$=\int_0^{S_r/\alpha}\frac{t^{r+a-2}}{(S_r)^{r+a-1}}\exp\{-t\}dt-\int_0^{S_r/\beta}\frac{t^{r+a-2}}{(S_r)^{r+a-1}}\exp\{-t\}dt.$$

Since $x_i \geq \tau$ and $r \leq n$, we have $x_i-\tau \geq 0$ and $n-r \geq 0$ which implies $S_r \geq 0$ and $0<\alpha \leq \beta$. Therefore, $\frac{S_r}{\beta} \leq \frac{S_r}{\alpha}$, then

$$\int_\alpha^\beta \frac{1}{\theta^{r+a}}\exp\left\{-\frac{1}{\theta}S_r\right\}d\theta=\frac{1}{(S_r)^{r+a-1}}\left[\int_0^{S_r/\alpha} t^{r+a-2}\exp\{-t\}dt-\int_0^{S_r/\beta} t^{r+a-2}\exp\{-t\}dt\right].$$

Therefore, the denominator of (2.2.1.1) reduces to

$$\int_{\alpha}^{\beta}\frac{1}{\theta^{r+a}}\exp\left\{-\frac{1}{\theta}S_r\right\}d\theta=\left\{\gamma\left(r+a-1,\frac{S_r}{\alpha}\right)-\gamma\left(r+a-1,\frac{S_r}{\beta}\right)\right\}\cdot\frac{1}{(S_r)^{r+a-1}}$$

where for brevity

$$\gamma^*(w,y)=\gamma\left(w,\frac{y}{\alpha}\right)-\gamma\left(w,\frac{y}{\beta}\right)$$

then

$$\int_{\alpha}^{\beta}\frac{1}{\theta^{r+a}}\exp\left\{-\frac{1}{\theta}S_r\right\}d\theta=\gamma^*(r+a-1,S_r)\cdot\frac{1}{(S_r)^{r+a-1}}.$$

Hence, the posterior density of $\theta$ for the uniform prior (2.3) is given by

$$h_U(\theta\,|\,\vec{x})=\frac{(S_r)^{r+a-1}\exp\left\{-\frac{1}{\theta}S_r\right\}}{\gamma^*(r+a-1,S_r)},\qquad \alpha\le\theta\le\beta. \tag{2.2.1.2}$$

Now, using (2.2) and (2.2.1.2), we have

$$\begin{aligned}[R(t)]^k h_U(\theta\,|\,\vec{x})&=\exp\left\{-\frac{k}{\theta}(t-\tau)^{\xi}\right\}\cdot\left(\frac{1}{\theta}\right)^{r+a}\frac{\exp\left\{-\frac{1}{\theta}S_r\right\}(S_r)^{r+a-1}}{\gamma^*(r+a-1,S_r)}\\&=\frac{\exp\left\{-\frac{1}{\theta}\left[k(t-\tau)^{\xi}+S_r\right]\right\}(S_r)^{r+a-1}}{\gamma^*(r+a-1,S_r)}\cdot\left(\frac{1}{\theta}\right)^{r+a}.\end{aligned}$$

Then,

$$\int_{-\infty}^{\infty} \exp\{f_1 R(t)\} h_U(\theta \mid \vec{x}) d\theta = \int_{\alpha}^{\beta} \sum_{k=0}^{\infty} \frac{f_1^{\,k}}{k!} [R(t)]^k h_U(\theta \mid \vec{x}) d\theta$$

$$= \sum_{k=0}^{\infty} \frac{f_1^{\,k}}{k!} \int_{\alpha}^{\beta} [R(t)]^k h_U(\theta \mid \vec{x}) d\theta$$

$$= \sum_{k=0}^{\infty} \frac{f_1^{\,k}}{k!} \int_{\alpha}^{\beta} \frac{\exp\left\{-\frac{1}{\theta}\left[k(t-\tau)^{\xi} + S_r\right]\right\}(S_r)^{r+a-1}}{\gamma^*(r+a-1, S_r)} \cdot \left(\frac{1}{\theta}\right)^{r+a} d\theta$$

$$= \sum_{k=0}^{\infty} \frac{f_1^{\,k}}{k!} \cdot \frac{(S_r)^{r+a-1}}{\gamma^*(r+a-1, S_r)} \int_{\alpha}^{\beta} \left(\frac{1}{\theta}\right)^{r+a} \exp\left\{-\frac{1}{\theta}\left[k(t-\tau)^{\xi} + S_r\right]\right\} d\theta$$

$$= \sum_{k=0}^{\infty} \frac{f_1^{\,k}}{k!} \frac{(S_r)^{r+a-1}}{\gamma^*(r+a-1, S_r)} \cdot \gamma^*(r+a-1, k(t-\tau)^{\xi} + S_r).$$

Similarly,

$$\int_{-\infty}^{\infty} \exp\{-f_2 R(t)\} h_U(\theta \mid \vec{x}) d\theta$$

$$= \sum_{k=0}^{\infty} \frac{(-f_2)^k}{k!} \cdot \frac{(S_r)^{r+a-1}}{\gamma^*(r+a-1, S_r)} \cdot \gamma^*(r+a-1, k(t-\tau)^{\xi} + S_r).$$

Thus, using (2.10), the Bayesian reliability estimate with the general uniform as a prior pdf is

$$\hat{R}_B(t)_U = \frac{1}{f_1 + f_2} \ln\left( \frac{\sum_{k=0}^{\infty} \frac{f_1^{\,k}}{k!} \cdot \frac{(S_r)^{r+a-1}}{\gamma^*(r+a-1, S_r)} \cdot \gamma^*(r+a-1, k(t-\tau)^{\xi} + S_r)}{\sum_{k=0}^{\infty} \frac{(-f_2)^k}{k!} \cdot \frac{(S_r)^{r+a-1}}{\gamma^*(r+a-1, S_r)} \cdot \gamma^*(r+a-1, k(t-\tau)^{\xi} + S_r)} \right)$$

$$= \frac{1}{f_1 + f_2} \ln\left( \frac{\sum_{k=0}^{\infty} \frac{f_1^{\,k}}{k!} \cdot \gamma^*(r+a-1, k(t-\tau)^{\xi} + S_r)}{\sum_{k=0}^{\infty} \frac{(-f_2)^k}{k!} \cdot \gamma^*(r+a-1, k(t-\tau)^{\xi} + S_r)} \right).$$

The series in the previous expression converges for some large $k = m$. Therefore , for a large $m$, $\hat{R}_B(t)_U$ can be approximated to

$$\hat{R}_B(t)_U \approx \frac{1}{f_1 + f_2} \ln\left( \frac{\sum_{k=0}^{m} \frac{f_1^{\,k}}{k!} \cdot \gamma^*(r+a-1, k(t-\tau)^{\xi} + S_r)}{\sum_{k=0}^{m} \frac{(-f_2)^k}{k!} \cdot \gamma^*(r+a-1, k(t-\tau)^{\xi} + S_r)} \right), \qquad t \geq \tau. \qquad (2.2.1.3)$$

### 2.2.2 Exponential Probability Density and Higgins-Tsokos Loss Function

We now examine the problem when the prior density of $\theta$ is given by the exponential pdf (2.4). Using the likelihood function (2.1.1) in conjunction with Bayes' Theorem, we obtain the posterior probability density of $\theta$

$$h_E(\theta \mid \vec{x}) = \frac{\frac{1}{\lambda}\exp\left\{-\frac{1}{\lambda}\theta\right\} L(\vec{x} \mid \theta)}{\int_0^{\infty} \frac{1}{\lambda}\exp\left\{-\frac{1}{\lambda}\theta\right\} L(\vec{x} \mid \theta)\, d\theta}$$

$$= \frac{C_E \cdot \left(\frac{1}{\theta}\right)^r \exp\left\{-\frac{1}{\theta} S_r - \frac{\theta}{\lambda}\right\}}{\int_0^{\infty} C_E \cdot \left(\frac{1}{\theta}\right)^r \exp\left\{-\frac{1}{\theta} S_r - \frac{\theta}{\lambda}\right\} d\theta}$$

where

$$C_E = \frac{1}{\lambda} \cdot \frac{n!}{(n-r)!} \prod_{i=1}^{r} (x_i - \tau)^{\xi - 1} \xi^r .$$

Thus, the posterior density of $\theta$ is given by

$$h_E(\theta \mid \vec{x}) = \frac{\frac{1}{\theta^r}\exp\left\{-\left(\frac{S_r}{\theta}+\frac{\theta}{\lambda}\right)\right\}}{\int_0^\infty \frac{1}{\theta^r}\exp\left\{-\left(\frac{S_r}{\theta}+\frac{\theta}{\lambda}\right)\right\}d\theta}, \qquad 0 < \theta < \infty . \qquad (2.2.2.1)$$

The denominator can be evaluated by using the relation

$$K_\nu(az) = \frac{1}{2}a^\nu \int_0^\infty \exp\left\{-\left(\frac{1}{2}zt+\frac{a^2 z}{2t}\right)\right\}\cdot\frac{1}{t^{\nu+1}}dt \qquad (2.2.2.2)$$

where $K_\nu(az)$ is the modified Bessel function of the third kind of order $\nu$ as given by Erdélyi, et al. (1953). Hence, in (2.2.2.1), letting $\frac{a^2 z}{2} = S_r$, $\frac{z}{2} = \frac{1}{\lambda}$, and $\nu + 1 = r$, the posterior density of $\theta$ for the exponential prior is given by

$$h_E(\theta \mid \vec{x}) = \frac{(\lambda S_r)^{\frac{r-1}{2}}\left(\frac{1}{\theta}\right)^r exp\left\{-\left(\frac{S_r}{\theta}+\frac{\theta}{\lambda}\right)\right\}}{2K_{r-1}\left(2\sqrt{\frac{S_r}{\lambda}}\right)}.$$

Similarly, to develop the Bayesian estimate of the reliability function given by (2.2) with respect to the Higgins-Tsokos loss (2.10) we have

$$\exp\{f_1 R(t)\} = \sum_{k=0}^\infty \frac{f_1^k}{k!}\exp\left\{-\frac{k}{\theta}(t-\tau)^\xi\right\}.$$

Then

$$\exp\{f_1 R(t)\} h_E(\theta \mid \vec{x}) = \sum_{k=0}^{\infty} \frac{f_1^k}{k!} \exp\left\{-\frac{k}{\theta}(t-\tau)^{\xi}\right\} \frac{(\lambda S_r)^{\frac{r-1}{2}} \left(\frac{1}{\theta}\right)^r \exp\left\{-\left(\frac{S_r}{\theta}+\frac{\theta}{\lambda}\right)\right\}}{2K_{r-1}\left(2\sqrt{\frac{S_r}{\lambda}}\right)}.$$

Integrating both sides, we have

$$\int_0^{\infty} \exp\{f_1 R(t)\} h_E(\theta \mid \vec{x}) d\theta = \sum_{k=0}^{\infty} \frac{(f_1)^k}{k!} \frac{(\lambda S_r)^{\frac{r-1}{2}}}{2K_{r-1}\left(2\sqrt{\frac{S_r}{\lambda}}\right)} \int_0^{\infty} \left(\frac{1}{\theta}\right)^r \exp\left\{-\left[\frac{k}{\theta}(t-\tau)^{\xi} + \frac{S_r}{\theta} + \frac{\theta}{\lambda}\right]\right\} d\theta$$

(2.2.2.3)

Then, using the relation (2.2.2.2) with $\frac{a^2 z}{2} = k(t-\tau)^{\xi} + S_r$, $\frac{z}{2} = \frac{1}{\lambda}$, and $\nu + 1 = r$, the expression (2.2.2.3) reduces to

$$\int_0^{\infty} \exp\{f_1 R(t)\} h_E(\theta \mid \vec{x}) d\theta = \sum_{k=0}^{\infty} \frac{f_1^k}{k!} \frac{(\lambda S_r)^{\frac{r-1}{2}}}{2K_{r-1}\left(2\sqrt{\frac{S_r}{\lambda}}\right)} \cdot \frac{2K_{r-1}\left(2\sqrt{\frac{k(t-\tau)^{\xi}+S_r}{\lambda}}\right)}{\left(\lambda\left[k(t-\tau)^{\xi}+S_r\right]\right)^{\frac{r-1}{2}}}$$

$$= \sum_{k=0}^{\infty} \frac{f_1^k}{k!} \frac{(S_r)^{\frac{r-1}{2}}}{K_{r-1}\left(2\sqrt{\frac{S_r}{\lambda}}\right)} \cdot \frac{K_{r-1}\left(2\sqrt{\frac{k(t-\tau)^{\xi}+S_r}{\lambda}}\right)}{\left[k(t-\tau)^{\xi}+S_r\right]^{\frac{r-1}{2}}}.$$

where $S_r = (n-r)(x_r-\tau)^{\xi} + \sum_{i=1}^{r}(x_i-\tau)$, as previously defined.

Now, similarly we can write

$$\exp\{-f_2 R(t)\} h_E(\theta \mid \vec{x}) = \sum_{k=0}^{\infty} \frac{(-f_2)^k}{k!} \exp\left\{-\frac{k}{\theta}(t-\tau)^{\xi}\right\} \frac{(\lambda S_r)^{\frac{r-1}{2}} \left(\frac{1}{\theta}\right)^r \exp\left\{-\left(\frac{S_r}{\theta}+\frac{\theta}{\lambda}\right)\right\}}{2K_{r-1}\left(2\sqrt{\frac{S_r}{\lambda}}\right)}.$$

Then, integrating both sides, we have

$$\int_0^\infty \exp\{-f_2 R(t)\} h_E(\theta \mid \vec{x}) d\theta = \sum_{k=0}^{\infty} \frac{(-f_2)^k}{k!} \left( \frac{S_r}{k(t-\tau)^\xi + S_r} \right)^{\frac{r-1}{2}} \cdot \frac{K_{r-1}\left( 2\sqrt{\frac{k(t-\tau)^\xi + S_r}{\lambda}} \right)}{K_{r-1}\left( 2\sqrt{\frac{S_r}{\lambda}} \right)}.$$

Therefore, using (2.10), the Bayesian reliability estimate, with the exponential density as the prior probability distribution and the Higgins-Tsokos loss function is given by

$$\hat{R}_B(t)_E = \frac{1}{f_1 + f_2} \ln \left( \frac{\sum_{k=0}^{\infty} \frac{(f_1)^k}{k!} \frac{K_{r-1}\left( 2\sqrt{\frac{k(t-\tau)^\xi + S_r}{\lambda}} \right)}{[k(t-\tau)^\xi + S_r]^{\frac{r-1}{2}}}}{\sum_{k=0}^{\infty} \frac{(-f_2)^k}{k!} \frac{K_{r-1}\left( 2\sqrt{\frac{k(t-\tau)^\xi + S_r}{\lambda}} \right)}{[k(t-\tau)^\xi + S_r]^{\frac{r-1}{2}}}} \right), \quad t \geq \tau$$

The series in the previous expression converges for some large $k = m$. Therefore, for a large $m$, $\hat{R}_B(t)_E$ can be approximated to

$$\hat{R}_B(t)_E \approx \frac{1}{f_1 + f_2} \ln \left( \frac{\sum_{k=0}^{m} \frac{(f_1)^k}{k!} \frac{K_{r-1}\left( 2\sqrt{\frac{k(t-\tau)^\xi + S_r}{\lambda}} \right)}{[k(t-\tau)^\xi + S_r]^{\frac{r-1}{2}}}}{\sum_{k=0}^{m} \frac{(-f_2)^k}{k!} \frac{K_{r-1}\left( 2\sqrt{\frac{k(t-\tau)^\xi + S_r}{\lambda}} \right)}{[k(t-\tau)^\xi + S_r]^{\frac{r-1}{2}}}} \right), \quad t \geq \tau . \qquad (2.2.2.4)$$

### 2.2.3 Inverted Gamma Probability Density and Higgins-Tsokos Loss Function

Now, we proceed to obtain Bayesian Reliability estimate under the inverted gamma prior and Higgins-Tsokos loss function. Recall that the prior density of $\theta$ is given by the inverted gamma pdf (2.5), then the posterior density of $\theta$ is

$$h_{IG}(\theta \mid \vec{x}) = \frac{\left(\dfrac{\mu}{\theta}\right)^{\nu+1} \cdot \dfrac{1}{\mu\Gamma(\nu)} \exp\left\{-\dfrac{\mu}{\theta}\right\} \cdot L(\vec{x} \mid \theta)}{\int_0^\infty \left(\dfrac{\mu}{\theta}\right)^{\nu+1} \cdot \dfrac{1}{\mu\Gamma(\nu)} \exp\left\{-\dfrac{\mu}{\theta}\right\} \cdot L(\vec{x} \mid \theta) d\theta}$$

$$= \frac{C_{IG} \cdot \left(\dfrac{1}{\theta}\right)^{\nu+r+1} \exp\left\{-\dfrac{1}{\theta}(\mu + S_r)\right\}}{\int_0^\infty C_{IG} \cdot \left(\dfrac{1}{\theta}\right)^{\nu+r+1} \exp\left\{-\dfrac{1}{\theta}(\mu + S_r)\right\} d\theta}$$

where

$$C_{IG} = \frac{\mu^{\nu+1}}{\mu\Gamma(\nu)} \frac{n!}{(n-r)!} \xi^r \prod_{i=0}^{r} (x_i - \tau)^{\xi-1},$$

then

$$h_{IG}(\theta \mid \vec{x}) = \frac{\left(\dfrac{1}{\theta}\right)^{\nu+r+1} \exp\left\{-\dfrac{1}{\theta}(\mu + S_r)\right\}}{\int_0^\infty \left(\dfrac{1}{\theta}\right)^{\nu+r+1} \exp\left\{-\dfrac{1}{\theta}(\mu + S_r)\right\} d\theta}, \quad \mu > 0, \nu > 0, 0 < \theta < \infty. \qquad (2.2.3.1)$$

Letting

$$y = \frac{(S_r + \mu)}{\theta}$$

the denominator in (2.2.3.1) can be evaluated as

$$\int_0^\infty \left(\frac{1}{\theta}\right)^{\nu+r+1} \exp\left\{-\frac{1}{\theta}(\mu+S_r)\right\} d\theta = -\int_\infty^0 \left(\frac{y}{\mu+S_r}\right)^{\nu+r+1} \exp\{-y\}(\mu+S_r)y^{-2}dy$$

$$= \left(\frac{1}{\mu+S_r}\right)^{\nu+r} \Gamma(\nu+r).$$

Thus, the posterior density of $\theta$ when the prior density is the inverted gamma is given by

$$h_{IG}(\theta \mid \vec{x}) = \frac{(\mu+S_r)^{\nu+r}}{\Gamma(\nu+r)} \left(\frac{1}{\theta}\right)^{\nu+r+1} \exp\left\{-\frac{1}{\theta}(\mu+S_r)\right\}, \quad \mu>0,\ \nu>0,\ 0<\theta<\infty$$

which is also an inverted gamma pdf. Therefore, the inverted gamma prior probability density is the natural conjugate family of prior densities for the scale parameter $\theta$ of the Weibull distribution (Raiffa and Schaifer, 1961).

Now, to develop the Bayesian estimate of the reliability function given by (2.2) with respect to Higgins-Tsokos loss function and the inverted gamma prior probability density, we have

$$\exp\{f_1 R(t)\} h_{IG}(\theta \mid \vec{x}) = \sum_{k=0}^{\infty} \frac{f_1^{\,k}}{k!} \exp\left\{-\frac{k}{\theta}(t-\tau)^{\xi}\right\} \frac{(\mu+S_r)^{\nu+r}}{\Gamma(\nu+r)} \left(\frac{1}{\theta}\right)^{\nu+r+1} \exp\left\{-\frac{1}{\theta}(\mu+S_r)\right\}.$$

Then, integrating both sides, we have

$$\int_0^\infty \exp\{f_1 R(t)\} h_{IG}(\theta \mid \vec{x}) d\theta = \sum_{k=0}^{\infty} \frac{f_1^{\,k}}{k!} \frac{(\mu+S_r)^{\nu+r}}{\Gamma(\nu+r)} \int_0^\infty \left(\frac{1}{\theta}\right)^{\nu+r+1} \exp\left\{-\frac{1}{\theta}\left[k(t-\tau)^{\xi} + (\mu+S_r)\right]\right\} d\theta.$$

Substituting $y = \frac{1}{\theta}\left[k(t-\tau)^{\xi} + \left(\mu + S_r\right)\right]$in the right hand integral, we can reduce it to

$$\int_0^{\infty}\left(\frac{1}{\theta}\right)^{\nu+r+1}\exp\left\{-\frac{1}{\theta}\left[k(t-\tau)^{\xi}+\left(\mu+S_r\right)\right]\right\}d\theta = \int_0^{\infty}\left(\frac{1}{k(t-\tau)^{\xi}+\left(\mu+S_r\right)}\right)^{\nu+r}\cdot y^{\nu+r+1}\exp\{-y\}dy$$

$$= \left(\frac{1}{k(t-\tau)^{\xi}+\left(\mu+S_r\right)}\right)^{\nu+r}\Gamma(\nu+r)$$

where $S_r = (n-r)(x_r-\tau)^{\xi} + \sum_{i=1}^{r}(x_i-\tau)$ and $\Gamma(\nu+r)$is the gamma function evaluated at $\nu + r$. Thus, we have

$$\int_0^{\infty}\exp\{f_1 R(t)\}h_{IG}(\theta \mid \vec{x})d\theta = \sum_{k=0}^{\infty}\frac{f_1^{\,k}}{k!}\frac{(\mu+S_r)^{\nu+r}}{\Gamma(\nu+r)}\left(\frac{1}{k(t-\tau)^{\xi}+\left(\mu+S_r\right)}\right)^{\nu+r}\Gamma(\nu+r)$$

$$= (\mu+S_r)^{\nu+r}\sum_{k=0}^{\infty}\frac{f_1^{\,k}}{k!}\left(\frac{1}{k(t-\tau)^{\xi}+\left(\mu+S_r\right)}\right)^{\nu+r}.$$

Similarly, we can write

$$\int_0^{\infty}\exp\{-f_2 R(t)\}h_{IG}(\theta \mid \vec{x})d\theta = (\mu+S_r)^{\nu+r}\sum_{k=0}^{\infty}\frac{\left(-f_2\right)^k}{k!}\left(\frac{1}{k(t-\tau)^{\xi}+\left(\mu+S_r\right)}\right)^{\nu+r}.$$

Therefore, using expression (2.10), the Bayesian estimate of the reliability function (2.2) given by (2.10) with respect to the Higgings-Tsokos loss function and inverted gamma prior is given by

$$\hat{R}_B(t)_{IG} = \frac{1}{f_1+f_2}\ln\left(\frac{(\mu+S_r)^{\nu+r}\sum_{k=0}^{\infty}\frac{f_1^{\,k}}{k!}\cdot\left[k(t-\tau)^{\xi}+(\mu+S_r)\right]^{-\nu-r}}{(\mu+S_r)^{\nu+r}\sum_{k=0}^{\infty}\frac{(-f_2)^k}{k!}\cdot\left[k(t-\tau)^{\xi}+(\mu+S_r)\right]^{-\nu-r}}\right)$$

$$= \frac{1}{f_1+f_2}\ln\left(\frac{\sum_{k=0}^{\infty}\frac{f_1^{\,k}}{k!}\cdot\left[k(t-\tau)^{\xi}+(\mu+S_r)\right]^{-\nu-r}}{\sum_{k=0}^{\infty}\frac{(-f_2)^k}{k!}\cdot\left[k(t-\tau)^{\xi}+(\mu+S_r)\right]^{-\nu-r}}\right), \quad t \geq \tau. \tag{2.2.3.2}$$

The series in the previous expression converges for some large $k = m$. Therefore, for a large $m$, $\hat{R}_B(t)_{IG}$ can be approximated to

$$\hat{R}_B(t)_{IG} \approx \frac{1}{f_1+f_2}\ln\left(\frac{\sum_{k=0}^{m}\frac{f_1^{\,k}}{k!}\cdot\left[k(t-\tau)^{\xi}+(\mu+S_r)\right]^{-\nu-r}}{\sum_{k=0}^{m}\frac{(-f_2)^k}{k!}\cdot\left[k(t-\tau)^{\xi}+(\mu+S_r)\right]^{-\nu-r}}\right), \quad t \geq \tau. \tag{2.2.3.3}$$

### 2.2.4 Jeffreys' Prior and Higgins-Tsokos Loss Function

Finally, if the prior density of $\theta$ is the Jeffreys' prior given by (2.6), the posterior density of $\theta$ can be written as

$$h_J(\theta \mid \vec{x}) = \frac{\frac{c}{\theta}L(\vec{x} \mid \theta)}{\int_0^{\infty}\frac{c}{\theta}L(\vec{x} \mid \theta)d\theta} \tag{2.2.4.1}$$

for some constant $c$. Then, using (2.1.1), we can write (2.2.4.1 ) as

$$h_J(\theta \mid \vec{x}) = \frac{c_J \dfrac{1}{\theta^{r+1}} exp\left\{-\dfrac{1}{\theta} S_r\right\}}{\int_0^\infty c_J \dfrac{1}{\theta^{r+1}} exp\left\{-\dfrac{1}{\theta} S_r\right\} d\theta},$$

where

$$c_J = c\frac{n!}{(n-r)!}\xi^r \prod_{i=1}^{r} (x_i - \tau)^{\xi-1} .$$

Then, the posterior density of $\theta$ is given by

$$h_J(\theta \mid \vec{x}) = \frac{\left(\dfrac{1}{\theta}\right)^{r+1} \exp\left\{-\dfrac{1}{\theta} S_r\right\}}{\int_0^\infty \left(\dfrac{1}{\theta}\right)^{r+1} \exp\left\{-\dfrac{1}{\theta} S_r\right\} d\theta}, \; 0 < \theta < \infty .$$

(2.2.4.2)

By letting $y = \dfrac{S_r}{\theta}$, the denominator in (2.2.4.2) can be expressed as

$$\begin{aligned} \int_0^\infty \left(\frac{1}{\theta}\right)^{r+1} \exp\left\{-\frac{1}{\theta} S_r\right\} d\theta &= -\int_\infty^0 \left(\frac{1}{S_r}\right)^r y^{r-1} \exp\{-y\} dy \\ &= \left(\frac{1}{S_r}\right)^r \Gamma(r), \end{aligned}$$

where $S_r = (n-r)(x_r-\tau)^{\xi} + \sum_{i=1}^{r}(x_i-\tau)$ and $\Gamma(r)$ is the gamma function evaluated at $r$. Thus, the posterior density of $\theta$ when the prior density is the Jeffreys' prior is given by

$$h_J(\theta \mid \vec{x}) = \frac{(S_r)^r}{\Gamma(r)} \frac{1}{\theta^{r+1}} \exp\left\{-\frac{1}{\theta} S_r\right\}, \;\; 0 < \theta < \infty. \qquad (2.2.4.3)$$

Now, we can write

$$\exp\{f_1 R(t)\} h_J(\theta \mid \vec{x}) = \sum_{k=0}^{\infty} \frac{f_1^{\,k}}{k!} \exp\left\{-\frac{k}{\theta}(t-\tau)^{\xi}\right\} \frac{(S_r)^r}{\Gamma(r)} \left(\frac{1}{\theta}\right)^{r+1} \exp\left\{-\frac{1}{\theta} S_r\right\}, \quad \theta > 0 \;.$$

Then, integrating both sides, we have

$$\int_0^{\infty} \exp\{f_1 R(t)\} h_J(\theta \mid \vec{x}) d\theta = \sum_{k=0}^{\infty} \frac{f_1^{\,k}}{k!} \frac{(S_r)^r}{\Gamma(r)} \int_0^{\infty} \left(\frac{1}{\theta}\right)^{r+1} \exp\left\{-\frac{1}{\theta}\left[k(t-\tau)^{\xi} + S_r\right]\right\} d\theta. \qquad (2.2.4.4)$$

Substituing $y = \frac{1}{\theta}\left[k(t-\tau)^{\xi} + S_r\right]$ in the right hand integral, we can write

$$\begin{aligned} \int_0^{\infty} \left(\frac{1}{\theta}\right)^{r+1} \exp\left\{-\frac{1}{\theta}\left[k(t-\tau)^{\xi} + S_r\right]\right\} d\theta &= \left[k(t-\tau)^{\xi} + S_r\right]^{-r} \int_0^{\infty} y^{r-1} \exp\{-y\} dy \\ &= \left[k(t-\tau)^{\xi} + S_r\right]^{-r} \Gamma(r). \end{aligned}$$

Thus, we can write (2.2.4.4 ) as

$$\int_0^\infty \exp\{f_1 R(t)\} h_J(\theta \mid \vec{x}) d\theta = \sum_{k=0}^{\infty} \frac{(f_1)^k}{k!} \frac{(S_r)^r}{\Gamma(r)} \left[k(t-\tau)^{\xi} + S_r\right]^{-r} \Gamma(r)$$
$$= (S_r)^r \sum_{k=0}^{\infty} \frac{(f_1)^k}{k!} \left[k(t-\tau)^{\xi} + S_r\right]^{-r}.$$

Similarly, for the denominator in the Bayesian estimate of the reliability function (2.9) with the Jeffreys' posterior density (2.2.4.3) we can write

$$\int_0^\infty \exp\{-f_2 R(t)\} h_J(\theta \mid \vec{x}) d\theta = (S_r)^r \sum_{k=0}^{\infty} \frac{(-f_2)^k}{k!} \left[k(t-\tau)^{\xi} + S_r\right]^{-r}.$$

Therefore, using (2.10), the Bayesian reliability estimate with Jeffreys' prior and Higgins-Tsokos loss function is given by

$$\hat{R}_B(t)_{JHT} = \frac{1}{f_1+f_2} \ln\left( \frac{(S_r)^r \sum_{k=0}^{\infty} \frac{(f_1)^k}{k!} \left[k(t-\tau)^{\xi} + S_r\right]^{-r}}{(S_r)^r \sum_{k=0}^{\infty} \frac{(-f_2)^k}{k!} \left[k(t-\tau)^{\xi} + S_r\right]^{-r}} \right)$$
$$= \frac{1}{f_1+f_2} \ln\left( \frac{\sum_{k=0}^{\infty} \frac{(f_1)^k}{k!} \left[k(t-\tau)^{\xi} + S_r\right]^{-r}}{\sum_{k=0}^{\infty} \frac{(-f_2)^k}{k!} \left[k(t-\tau)^{\xi} + S_r\right]^{-r}} \right), \quad t \geq \tau.$$

The series in the previous expression converges for some large $k = m$. Therefore, for a large $m$, $\hat{R}_B(t)_{JHT}$ can be approximated by

$$\hat{R}_B(t)_{JHT} \approx \frac{1}{f_1+f_2} \ln\left( \frac{\sum_{k=0}^{m} \frac{(f_1)^k}{k!} \left[k(t-\tau)^{\xi} + S_r\right]^{-r}}{\sum_{k=0}^{m} \frac{(-f_2)^k}{k!} \left[k(t-\tau)^{\xi} + S_r\right]^{-r}} \right), \quad t \geq \tau. \qquad (2.2.4.5)$$

### 2.2.5 Jeffreys' Prior and Squared-error Loss Function

The Bayesian estimate of the reliability function given by (2.2) with respect to the squared-error loss function and using (2.2.4.3) is obtained by evaluating

$$E\left[R(t)\mid\vec{x}\right]=\int_0^\infty R(t)h_J(\theta\mid\vec{x})d\theta$$

and is given by

$$\hat{R}_B(t)_{JSQ}=\int_0^\infty\frac{(S_r)^r}{\Gamma(r)}\frac{1}{\theta^{r+1}}\exp\left\{-\frac{1}{\theta}\left[(t-\tau)^\xi+S_r\right]\right\}d\theta. \qquad (2.2.5.1)$$

Let $y=\frac{1}{\theta}\left[(t-\tau)^\xi+S_r\right]$, then (2.2.5.1) reduces to

$$\hat{R}_B(t)_{JSQ}=(S_r)^r\left[(t+\tau)^\xi+S_r\right]^{-r},\quad t\geq\tau. \qquad (2.2.5.2)$$

Each Bayesian estimate of the reliability is a decreasing function of time and is defined for all $t > 0$ regardless of the prior density of $\theta$. In addition, the Bayesian estimate using inverted gamma prior with respect to Higgins-Tsokos loss function (2.2.3.2) is reduced to the Jeffreys' prior when $v \to 0$ and $\mu \to 0$.

## 2.3 Numerical Simulation

### 2.3.1 Comparison of Bayesian Estimates of the Reliability function under the Higgins-Tsokos Loss Function

Because of the absence of "live" life-times, it is felt that an indication of the properties of the Bayesian estimates developed in the previous sections can be best determined through a Monte Carlo simulation. At the same time, a comparison is made between the Bayesian estimate and the true reliability function. In the implementation of the simulation procedure, a complete sample of 10, 50 and 100 life-times are generated where the guarantee time is taken to be zero. The following schematic diagram displays the process of the simulation.

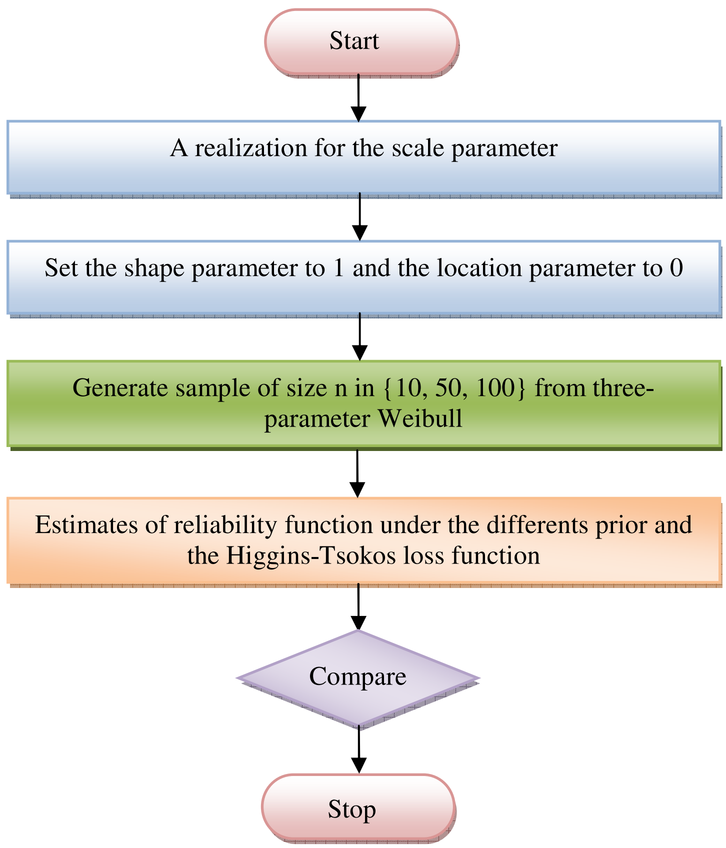



**Figure 2.1 Numerical Simulation: Comparison of the Bayesian Reliability Functions**

For a realization of the stochastic scale parameter $\theta$, random life-times distributed according to the three parameter Weibull law were simulated for each of the four prior densities discussed, and four distinct values of the shape parameter. In computing the

Bayesian estimates of the reliability according to the equations (2.2.1.3), (2.2.2.4), (2.2.3.3), and (2.2.4.5) the numerical answer is allowed to converge by varying the value $m$ until no change in the result is noted. The parameters values used for the general uniform probability density were$(\alpha,\beta) = (5,50)$ and $a = 2$, $\lambda = 30$ for the exponential pdf, and $(\mu,v) = (10,3)$ for the inverted gamma pdf. We also considered $\xi = 1$, $\tau = 0$, $f_1 = f_2 = 1$.

For a complete sample ($r = n$) with $n = 10$, we obtained the results summarized in Table 2.2 where the subscripts *U, E, IG, JHT* stand for uniform, exponential, inverted gamma, and Jeffreys' respectively under the Higgins-Tsokos loss function.

**Table 2.1 Bayesian Reliability estimates under the Higgins-Tsokos loss function for a complete sample $r = n = 10$**

| $t$ | $R(t)$ | $\hat{R}_B(t)_U$ | $\hat{R}_B(t)_E$ | $\hat{R}_B(t)_{IG}$ | $\hat{R}_B(t)_{JHT}$ |
|---|---|---|---|---|---|
| 0.4 | 0.9853 | 0.99987 | 0.99098 | 0.979225 | 0.983312 |
| 1.2 | 0.9566 | 0.99966 | 0.97323 | 0.939055 | 0.950847 |
| 2.0 | 0.9287 | 0.99942 | 0.95584 | 0.900653 | 0.919556 |
| 2.8 | 0.9016 | 0.99919 | 0.93881 | 0.863936 | 0.880394 |
| 3.6 | 0.8775 | 0.99894 | 0.92214 | 0.828826 | 0.860316 |

We can observe that these estimates are sensitive to the choice of the prior distribution, and that the reliability estimate with respect to the Higgins-Tsokos loss function, using Jeffreys' prior, is closer to the true value. Incrementing the complete sample to $r = n = 50$ we obtained the results in Table 2.3.

**Table 2.2 Bayesian Reliability estimates under the Higgins-Tsokos loss function for a complete sample $r = n = 50$**

| $t$ | $R(t)$ | $\hat{R}_B(t)_{JHT}$ |
|---|---|---|
| 0.4 | 0.9853 | 0.9882 |
| 1.2 | 0.9566 | 0.9652 |
| 2.0 | 0.9287 | 0.9426 |
| 2.8 | 0.9016 | 0.9261 |
| 3.6 | 0.8775 | 0.8991 |

and for $r = n = 100$ we obtained results in Table 2.4.

**Table 2.3 Bayesian Reliability estimates under the Higgins-Tsokos loss function for a complete sample $r = n = 100$**

| $t$ | $R(t)$ | $\hat{R}_B(t)_{JHT}$ |
|---|---|---|
| 0.4 | 0.9853 | 0.9862 |
| 1.2 | 0.9566 | 0.9693 |
| 2.0 | 0.9287 | 0.9331 |
| 2.8 | 0.9016 | 0.9077 |
| 3.6 | 0.8775 | 0.8829 |

Therefore, for a large complete sample size, the Bayesian reliability function corresponding to Jeffreys' prior is a good approximation to the true reliability function $R(t)$.

### 2.3.2 Comparison of the Best Bayesian Estimates of the Reliability Function under the Higgins-Tsokos Loss Function with respect to the Bayesian Estimate of the Reliability Function under Squared-error Loss Function

We proceed to study if any difference exists in the estimation of the reliability function when we keep the best prior obtained under the assumption of the Higgins-Tsokos loss function but we choose the squared-error loss function instead. For this new setting, we calculated the Bayesian reliability model varying the sample size with the three parameter Weibull as the underlying failure distribution. The reliability estimates with respect to the squared-error loss function and Jeffreys' prior, $\hat{R}_B(t)_{JSQ}$, are summarized in the following tables. For a complete sample of $r = n = 10$ we obtained the results given in Table 2.5.

**Table 2.4 Bayesian Reliability estimates under the Squared-error loss function for a complete sample $r = n = 10$**

| $t$ | $R(t)$ | $\hat{R}_B(t)_{JSQ}$ |
|---|---|---|
| 0.4 | 0.9853 | 0.9833 |
| 1.2 | 0.9566 | 0.9508 |
| 2.0 | 0.9287 | 0.9195 |
| 2.8 | 0.9016 | 0.8894 |
| 3.6 | 0.8775 | 0.8603 |

We proceed to increment the sample size to obtain the estimate $\hat{R}_B(t)_{JSQ}$. For a complete sample to $r = n = 50$ we obtained the results detailed in Table 2.6.

**Table 2.5 Bayesian Reliability estimates under the squared-error loss function for a complete sample *r* = *n* = 50**

| $t$ | $R(t)$ | $\hat{R}_B(t)_{JSQ}$ |
|---|---|---|
| 0.4 | 0.9853 | 0.9882 |
| 1.2 | 0.9566 | 0.9652 |
| 2.0 | 0.9287 | 0.9426 |
| 2.8 | 0.9016 | 0.9206 |
| 3.6 | 0.8775 | 0.8992 |

and, for a complete sample to $r = n = 100$ we obtained the results detailed in Table 2.6.

**Table 2.6 Bayesian Reliability estimates under the Squared-error loss function for a complete sample *r* = *n* = 100**

| $t$ | $R(t)$ | $\hat{R}_B(t)_{JSQ}$ |
|---|---|---|
| 0.4 | 0.9853 | 0.9862 |
| 1.2 | 0.9566 | 0.9593 |
| 2.0 | 0.9287 | 0.9331 |
| 2.8 | 0.9016 | 0.9077 |
| 3.6 | 0.8775 | 0.8829 |

Under the squared-error loss, in the case of the small sample, we acquired a minor underestimate for the reliability function. For a large sample it was found a very small overestimate. The estimates under Higgins-Tsokos loss are very similar to the estimates using squared-error loss function.

Almost negligible differences can be seen when the time are 2.8 and 3.6. In such cases, the squared-error loss has less error in the estimation under the assumption $\xi = 1$. For the same samples sizes but assuming $\xi = 2$, 4, and 6 we obtained the approximated values for the reliability with respect to the Higgins-Tsokos and squared-error loss functions when Jeffreys' prior was considered. We followed the numerical simulation as displayed in Figure 2.1 with the new assumed values for the shape parameter $\xi$. Tables 2.7 to 2.15 shows the comparison of the reliability estimates with respect to the true reliability function for different samples sizes.

**Table 2.7 Bayesian Reliability Estimate Values for $r = n = 10$, $\xi = 2$**

| $t$ | $R(t)$ | $\hat{R}_B(t)_{JHT}$ | $\hat{R}_B(t)_{JSQ}$ |
|---|---|---|---|
| 0.4 | 0.9941 | 0.9998 | 0.9998 |
| 1.2 | 0.9491 | 0.9985 | 0.9985 |
| 2.0 | 0.8624 | 0.9957 | 0.9957 |
| 2.8 | 0.7482 | 0.9916 | 0.9916 |
| 3.6 | 0.6191 | 0.9862 | 0.9862 |

**Table 2.8 Bayesian Reliability Estimate Values for $r = n = 50$, $\xi = 2$**

| $t$ | $R(t)$ | $\hat{R}_B(t)_{JHT}$ | $\hat{R}_B(t)_{JSQ}$ |
|---|---|---|---|
| 0.4 | 0.9941 | 0.9999 | 0.9999 |
| 1.2 | 0.9491 | 0.9994 | 0.9994 |
| 2.0 | 0.8624 | 0.9983 | 0.9983 |
| 2.8 | 0.7482 | 0.9967 | 0.9967 |
| 3.6 | 0.6191 | 0.9946 | 0.9946 |

**Table 2.9 Bayesian Reliability Estimate Values for $r = n = 100$, $\xi = 2$**

| $t$ | $R(t)$ | $\hat{R}_B(t)_{JHT}$ | $\hat{R}_B(t)_{JSQ}$ |
|---|---|---|---|
| 0.4 | 0.9941 | 0.9999 | 0.9999 |
| 1.2 | 0.9491 | 0.9990 | 0.9990 |
| 2.0 | 0.8624 | 0.9972 | 0.9972 |
| 2.8 | 0.7482 | 0.9944 | 0.9944 |
| 3.6 | 0.6191 | 0.9908 | 0.9908 |

**Table 2.10 Bayesian Reliability Estimate Values for $r = n = 10$, $\xi = 4$**

| $t$ | $R(t)$ | $\hat{R}_B(t)_{JHT}$ | $\hat{R}_B(t)_{JSQ}$ |
|---|---|---|---|
| 0.4 | 0.9905 | 1 | 1 |
| 1.2 | 0.9261 | 0.9999 | 0.9999 |
| 2.0 | 0.5532 | 0.9999 | 0.9999 |
| 2.8 | 0.1029 | 0.9999 | 0.9999 |
| 3.6 | 0.0020 | 0.9999 | 0.9999 |

**Table 2.11 Bayesian Reliability Estimate Values for $r = n = 50$, $\xi = 4$**

| $t$ | $R(t)$ | $\hat{R}_B(t)_{JHT}$ | $\hat{R}_B(t)_{JSQ}$ |
|---|---|---|---|
| 0.4 | 0.9905 | 1 | 1 |
| 1.2 | 0.9261 | 1 | 1 |
| 2.0 | 0.5532 | 0.9999 | 0.9999 |
| 2.8 | 0.1029 | 0.9999 | 0.9999 |
| 3.6 | 0.0020 | 0.9999 | 0.9999 |

**Table 2.12 Bayesian Reliability Estimate Values for *r* = *n* = 100, *ξ* = 4**

| $t$ | $R(t)$ | $\hat{R}_B(t)_{JHT}$ | $\hat{R}_B(t)_{JSQ}$ |
|---|---|---|---|
| 0.4 | 0.9905 | 0.9999 | 0.9999 |
| 1.2 | 0.9261 | 0.9999 | 0.9999 |
| 2.0 | 0.5532 | 0.9999 | 0.9999 |
| 2.8 | 0.1029 | 0.9999 | 0.9999 |
| 3.6 | 0.0020 | 0.9999 | 0.9999 |

**Table 2.13 Bayesian Reliability Estimate Values for *r* = *n* = 10, *ξ* = 6**

| $t$ | $R(t)$ | $\hat{R}_B(t)_{JHT}$ | $\hat{R}_B(t)_{JSQ}$ |
|---|---|---|---|
| 0.4 | 0.9998 | 1 | 1 |
| 1.2 | 0.8954 | 1 | 1 |
| 2.0 | 0.0937 | 1 | 1 |
| 2.8 | $1.8088 \cdot 10^{-8}$ | 1 | 1 |
| 3.6 | $1.0601 \cdot 10^{-35}$ | 1 | 1 |

**Table 2.14 Bayesian Reliability Estimate Values for *r* = *n* = 50, *ξ* = 6**

| $t$ | $R(t)$ | $\hat{R}_B(t)_{JHT}$ | $\hat{R}_B(t)_{JSQ}$ |
|---|---|---|---|
| 0.4 | 0.9998 | 1 | 1 |
| 1.2 | 0.8954 | 1 | 1 |
| 2.0 | 0.0937 | 1 | 1 |
| 2.8 | $1.8088 \cdot 10^{-8}$ | 1 | 1 |
| 3.6 | $1.0601 \cdot 10^{-35}$ | 1 | 1 |

**Table 2.15  Bayesian Reliability Estimate Values for $r = n = 100$, $\xi = 6$**

| $t$ | $R(t)$ | $\hat{R}_B(t)_{JHT}$ | $\hat{R}_B(t)_{JSQ}$ |
|---|---|---|---|
| 0.4 | 0.9998 | 1 | 1 |
| 1.2 | 0.8954 | 1 | 1 |
| 2.0 | 0.0937 | 1 | 1 |
| 2.8 | $1.8088 \cdot 10^{-8}$ | 1 | 1 |
| 3.6 | $1.0601 \cdot 10^{-35}$ | 1 | 1 |

Even when we did not obtain good results when $\xi$ = 2, 4, and 6 at different sample sizes, we observed that the estimates produced the same approximation. They suggest us that $\hat{R}_B(t)_{JHT}$ using Higgins-Tsokos loss function is robust with respect to when it is used the squared-error loss function. In the next graphs it can be observed the good approximation for $R(t)$ made by $\hat{R}_B(t)_{JHT}$.

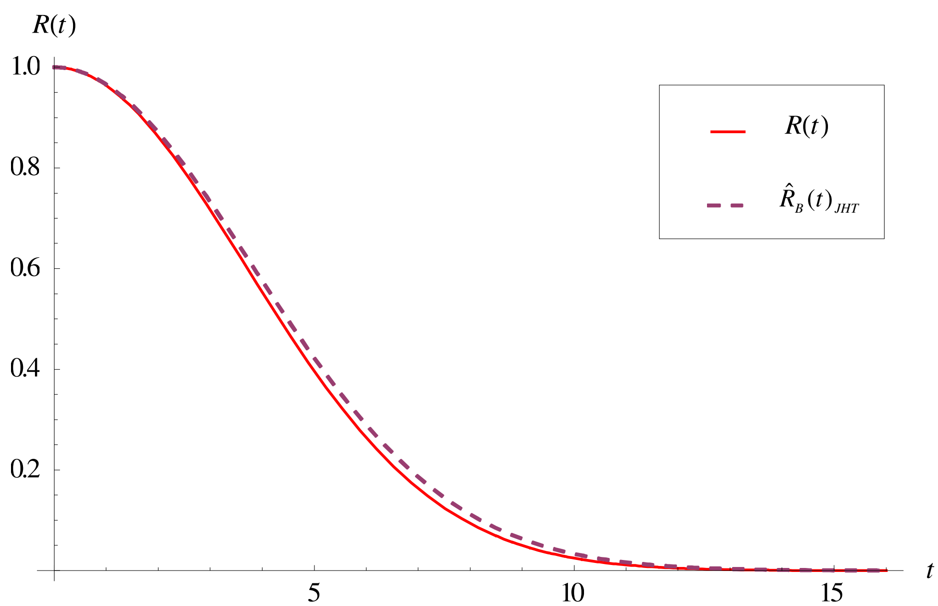


**Figure 2.2 Comparison of $\hat{R}_B(t)_{JHT}$ with respect to $R(t)$**

It can be seen that $\hat{R}_B(t)_{JHT}$ and $\hat{R}_B(t)_{JSQ}$ gave nearly the same approximation for $R(t)$.

Therefore $\hat{R}_B(t)_{JHT}$ behaves as $\hat{R}_B(t)_{JSQ}$.

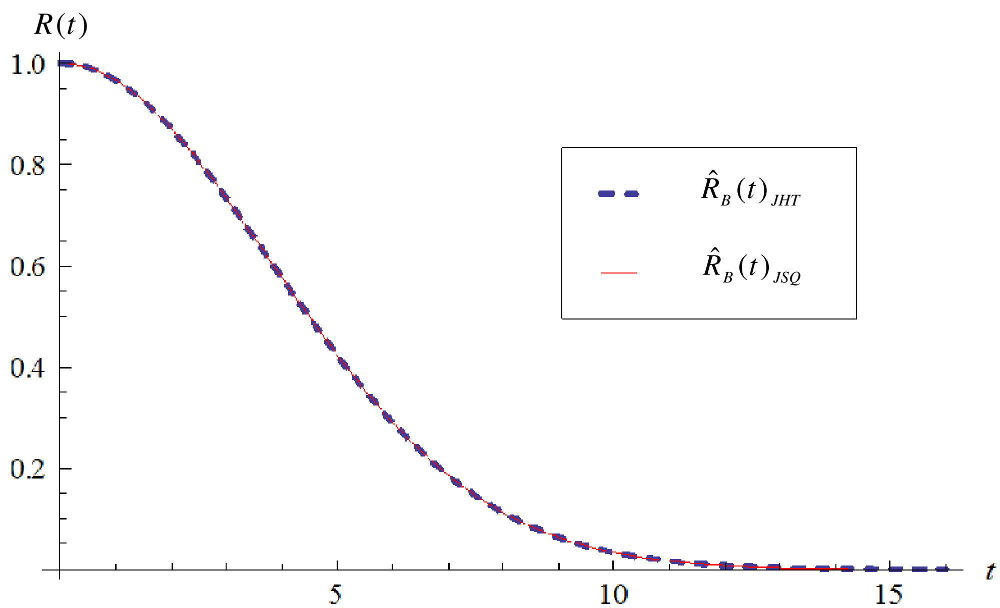



**Figure 2.3 Comparison of $\hat{R}_B(t)_{JHT}$ with respect to $\hat{R}_B(t)_{JSQ}$**

Figure 2.4 displays the differences —as a function of time— of the Bayesian estimate of the reliability function under the Jeffreys' prior and the Higgins-Tsokos loss function, and the true Weibull reliability function. It shows that there is no differences as the time increases. In addition, the differences appear to be at a very small time interval.

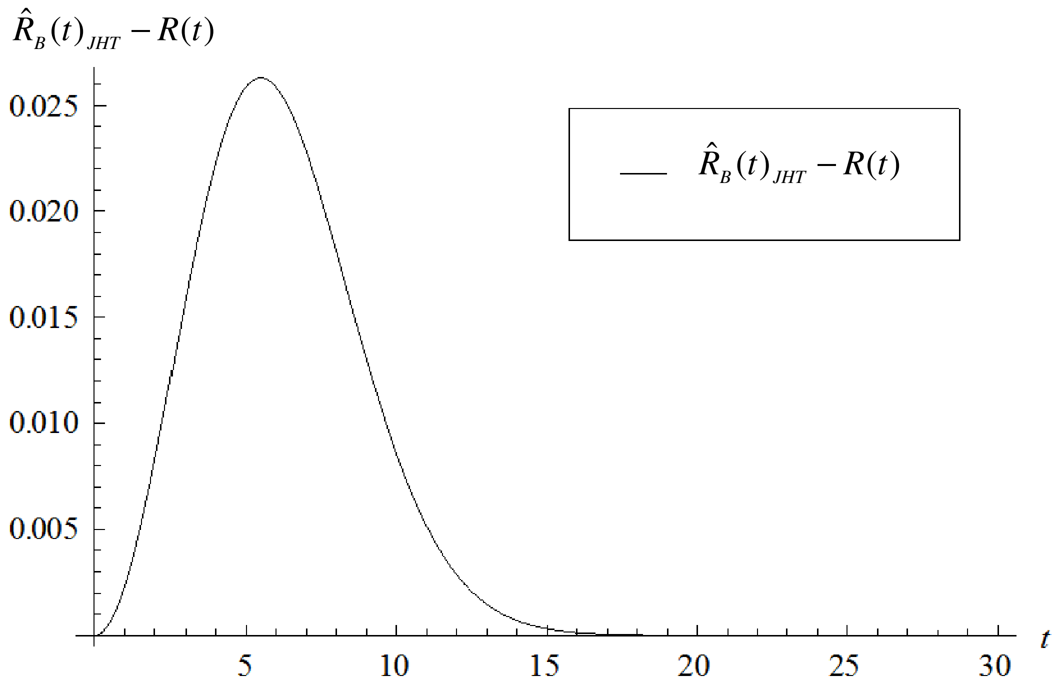



**Figure 2.4 Behavior of $\hat{R}_B(t)_{JHT} - R(t)$**

### 2.3.3 Relative Efficiency of the Reliability Estimates under Higgins-Tsokos Loss and Squared-error Loss Function

Recall that the IMSE of an estimate $\hat{R}_B(t)$ of the reliability function is defined as $\int_0^\infty [\hat{R}_B(t) - R(t)]^2 dt$, and the RE of the estimate $\hat{R}_B(t)_1$ compared to the estimate $\hat{R}_B(t)_2$ is defined as the ratio of $IMSE[\hat{R}_B(t)_1]$ and $IMSE[\hat{R}_B(t)_2]$. For a complete sample size $n = 100$, the RE of the Bayesian Reliability estimates under the Higgins-Tsokos and squared-error loss function is presented in Table 2.16.

**Table 2.16 Relative Efficiency for the Reliability estimates under the Higgins-Tsokos and squared-error loss functions**

| $\xi$ | $IMSE[\hat{R}_B(t)_{JHT}]$ | $IMSE[\hat{R}_B(t)_{JSQ}]$ | $RE$ |
|---|---|---|---|
| 1 | 0.00328848 | 0.00328827 | 1.00006 |

The calculation of the RE (Table 2.16) reveals that the reliability function under the Higgins-Tsokos loss function is as robust as the squared-error loss function and slightly more efficient.

For 500 simulations a random value of the parameter $\theta$ was generated to obtain random samples of size $n$ = 10, 50 and 100 for $\xi = 1$. The average REs were calculated and compared pairwise among them for the different choices of the priors and under the Higgins-Tsokos loss function. Computations revealed the estimate under Higgins-Tsokos loss with Jeffreys' prior has better performance. The following schematic diagram displays the process of the simulation.

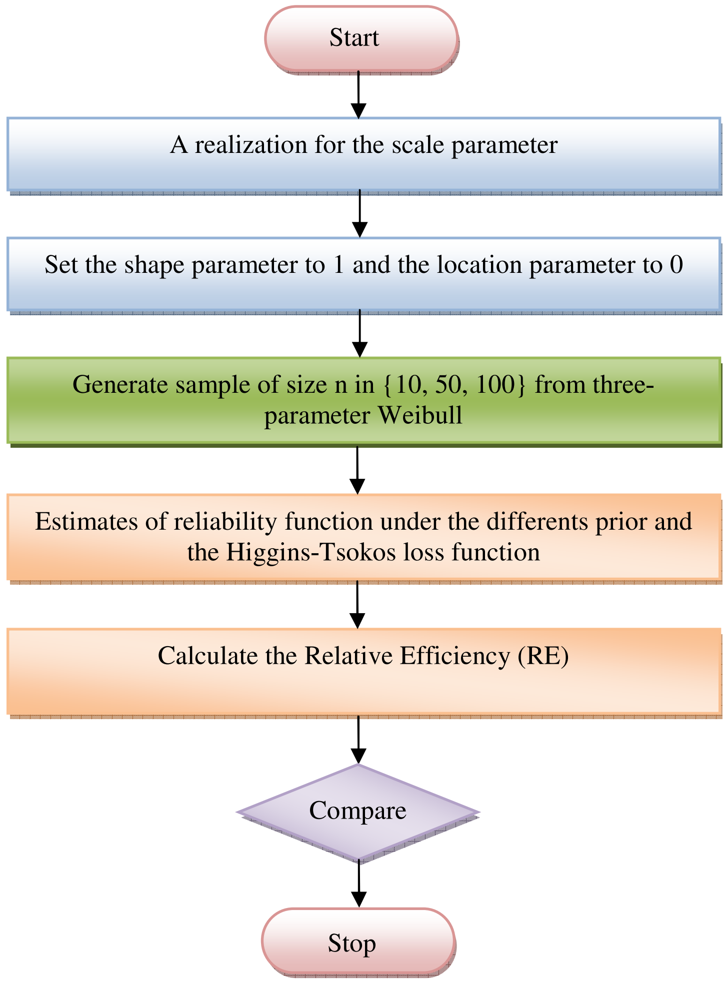



**Figure 2.5 Simulation Process to Compare the Relative Efficiency of the Bayesian Reliability Functions**

## 2.4 Conclusions

We developed the analytical Bayesian form of the reliability function where the underlying failure model is the three parameter Weibull probability distribution with the scale parameter considered to behave as a random variable and its behavior is being characterized by the general uniform, exponential, inverted gamma, and Jeffreys prior, under both the Higgins-Tsokos and the squared-error loss functions. The table below gives a summary of the analytical results.

**Table 2.17 Bayesian estimates of the Reliability Function of the Weibull probability distribution with stochastic scale parameter**

| Prior density | Reliability Function Bayesian estimate |
|---|---|
| | Respect to Higgins-Tsokos loss function |
| General Uniform pdf<br>$p(\theta)=\frac{(a-1)(\alpha\beta)^{a-1}}{(\beta^{a-1}-\alpha^{a-1})\theta^{a}},$<br>$0<\alpha<\theta\leq\beta$ | $\hat{R}_B(t)_U=\frac{1}{f_1+f_2}\ln\left(\frac{\sum_{k=0}^{\infty}\frac{f_1^{\,k}}{k!}\cdot\gamma^*(r+a-1,k(t-\tau)^{\xi}+S_r)}{\sum_{k=0}^{\infty}\frac{(-f_2)^k}{k!}\cdot\gamma^*(r+a-1,k(t-\tau)^{\xi}+S_r)}\right),t\geq\tau$ |
| Exponential pdf<br>$p(\theta)=\frac{1}{\lambda}\exp\left\{-\frac{1}{\lambda}\theta\right\},$<br>$0<\theta<\infty,\lambda>0$ | $\hat{R}_B(t)_E=\frac{1}{f_1+f_2}\ln\left(\frac{\sum_{k=0}^{\infty}\frac{(f_1)^k}{k!}\frac{K_{r-1}\left(2\sqrt{\frac{k(t-\tau)^{\xi}+S_r}{\lambda}}\right)}{[k(t-\tau)^{\xi}+S_r]^{\frac{r-1}{2}}}}{\sum_{k=0}^{\infty}\frac{(-f_2)^k}{k!}\frac{K_{r-1}\left(2\sqrt{\frac{k(t-\tau)^{\xi}+S_r}{\lambda}}\right)}{[k(t-\tau)^{\xi}+S_r]^{\frac{r-1}{2}}}}\right),t\geq\tau$ |
| Inverted gamma pdf<br>$p(\theta)=\left(\frac{\mu}{\theta}\right)^{\nu-1}\frac{1}{\mu\Gamma(\nu)}\exp\left\{-\frac{\mu}{\theta}\right\},$<br>$\theta>0,\ \mu>0,\ \nu>0$ | $\hat{R}_B(t)_{IG}=\frac{1}{f_1+f_2}\ln\left(\frac{\sum_{k=0}^{\infty}\frac{f_1^{\,k}}{k!}\cdot[k(t-\tau)^{\xi}+(\mu+S_r)]^{-\nu-r}}{\sum_{k=0}^{\infty}\frac{(-f_2)^k}{k!}\cdot[k(t-\tau)^{\xi}+(\mu+S_r)]^{-\nu-r}}\right),t\geq\tau$ |
| Jeffreys' prior<br>$p(\theta)\propto\frac{1}{\theta},$<br>$\theta>0$ | $\hat{R}_B(t)_{JHT}=\frac{1}{f_1+f_2}\ln\left(\frac{\sum_{k=0}^{\infty}\frac{(f_1)^k}{k!}[k(t-\tau)^{\xi}+S_r]^{-r}}{\sum_{k=0}^{\infty}\frac{(-f_2)^k}{k!}[k(t-\tau)^{\xi}+S_r]^{-r}}\right),t\geq\tau$ |
| | Respect to Squared error loss function |
| | $\hat{R}_B(t)_{JSQ}=(S_r)^r\left[(t+\tau)^{\xi}+S_r\right]^{-r},t\geq\tau$ |

We have identified the best prior —Jeffreys— using the IMSE of the reliability function estimate with respect to the Higgins-Tsokos loss function. For a large complete sample size, the Bayesian reliability function corresponding to Jeffreys' prior and Higgins-Tsokos loss function is a good approximation to the true reliability function $R(t)$. Having identified the best prior, we test for differences using the Higgins-Tsokos and the squared-error loss functions for the same prior. We obtained the IMSE of the reliability function estimate for the Higgins-Tsokos subject to Jeffreys prior and found approximately the same IMSE for the squared-error loss function. This implies the robustness with respect to the choice of the loss function. Moreover, it was found that the Bayesian estimate of the reliability function under the Higgins-Tsokos loss function and Jeffreys'prior is slightly more efficient than under the squared-error loss function.

# CHAPTER 3 BAYESIAN RELIABILITY APPROACH TO THE POWER LAW PROCESS

In this chapter, we illustrate the applicability of a Bayesian analysis for the Power Law Process, PLP, through the intensity function. First, we show using real data that one of the two parameters in the intensity function behaves as a random variable. We proceed to identify a prior probability distribution that characterizes its probabilistic behavior. Under the assumption of the squared-error loss function, we obtained the Bayesian estimate of the parameter and the intensity function. We compared the estimates with their MLE counterpart. In addition, we obtained a better Bayesian estimate of the intensity function proposing an adjusted MLE for the second parameter.

The first section of the chapter gives a brief review of the general concepts concerning the subject area. In addition, it points out the importance of one of the two parameters in the intensity function in the PLP. The second section shows the applicability of Bayesian analysis for the PLP using real data by demonstrating the random behavior of the parameter. Identifying its probability distribution as the prior, we proceeded to obtain the analytical form of the Bayesian estimates of the parameters, the intensity and the reliability functions. In the third section, we compared the Bayesian estimates of the parameters and the intensity function, as well as proposed MLE for the second parameter, with their MLE counterparts. In the fourth section, we show the applicability of the analytical results to real data. The fifth section summarizes the findings of the study.

## 3.1 Introduction

The reliability of a repairable system will improve with time as component defects and flaws are detected, repaired, or removed. This is the essential pattern of reliability growth. After testing several engineering systems, Duane (1964) first reported consistency in growth patterns.

As noted in Chapter 1, Crow (1974, 1975) proposed the Non-homogeneous Poisson Process, NHPP, with a failure intensity function given by

$$V(t) = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1}, \quad \beta > 0, \theta > 0, t > 0 \tag{3.1}$$

where $\beta$ is the shape parameter and $\theta$ the scale parameter, as an effective approach to analyzing the reliability growth. This failure intensity function corresponds to the hazard rate function of the Weibull distribution of the Weibull process.

In a test procedure, two types of truncation exist. *Time truncation* is applied if the test is ended at a prespecified time. *Failure truncation* describes a predetermined number of failures. If we assume failure truncation data, the conditional reliability function of the time to failure, TTF, $T_n$ given $T_1 = t_1, T_2 = t_2, T_3 = t_3, \ldots, T_{n-2} = t_{n-2}, T_{n-1} = t_{n-1}$ is a component of the intensity function.

The intensity function is a key entity in the PLP, and $\beta$, its key parameter, affects how the system improves or deteriorates over time. The PLP inference in terms of the Bayesian perspective has received the interest of various researchers, such as Kyparisis and Singpurwalla (1985); Bar-Lev, Lavi, and Reiser (1992); Lingham and Sivaganesan (1997); Kim and Sun (2000); Kim, Kim, and Kim (2003); and Kim, Choi, and Kim (2005). If $\beta = 1$, the PLP reduces to the homogeneous Poisson process with intensity $\frac{1}{\theta}$. When $\beta > 1$ the intensity function is increasing, and the failures tend to occur more frequently, which implies that the reliability of the system decreases. For $\beta < 1$ the power law intensity function decreases, implying the system is improving (i.e., the reliability of the system grows). Thus, having a good estimate of $\beta$ gives us good information about the quality of a product or system with respect to its reliability behavior.

In addition to tracking the reliability growth of a system, such modeling can be used for predictions. For safety and financial interests, it is quite important to be able to determine the next TTF after the system has experienced some failures during the developmental process. Recently, Xu and Tsokos (2011), has successfully shown that the PLP can be used to evaluate the effectiveness of drug treatment in breast cancer. In the current study, we use the Bayesian approach with simulated and available historical data to estimate the key parameter $\beta$, which has an important role in the analysis of the reliability growth for repairable systems. The MLE of $\theta$ depends on the MLE of $\beta$. Our concern is with respect to the sensitivity of $\beta$ based on the largest TTF given that the MLE of $\beta$ depends on it. To address this issue, we used real data from Crow (1974, 1975). Furthermore, in the study, we pursue the answers to the following questions:

1. Is the Bayesian analysis applicable to the PLP?
2. If yes, do the Bayesian estimates under the commonly used squared-error loss function perform better than those obtained under the parametric approach?

## 3.2 Development of the Bayesian Reliability Model

To illustrate the random behavior of the parameter $\beta$, we use Crow (1974, 1975) failure data from a system undergoing developmental testing. The forty successive failures of the system under development are given in Table 3.1:

**Table 3.1 Sample of failure times of a system under development**

| | | | | | | |
|---|---|---|---|---|---|---|
| 0.7 | 3.7 | 13.2 | 17.6 | 54.5 | 99.2 | 112.2 |
| 120.9 | 151 | 163 | 174.5 | 191.6 | 282.8 | 355.2 |
| 486.3 | 490.5 | 513.3 | 558.4 | 678.1 | 688 | 785.9 |
| 887 | 1010.7 | 1029.1 | 1034.4 | 1136.1 | 1178.9 | 1259.7 |
| 1297.9 | 1419.7 | 1571.7 | 1629.8 | 1702.4 | 1928.9 | 2072.3 |
| 2525.2 | 2928.5 | 3016.4 | 3181 | 3256.3 | | |

According to the reliability growth failure data given in Table 3.1, the system failed for the first time at 0.7 units of time, $t_1$=0.7, and it failed after the fortieth time at 3256.3 units of time, $t_{40}$= 3256.3. The MLE of the parameter $\beta$ for $n$ = 40 is

$$\hat{\beta}_{40} = \frac{40}{\sum_{i=1}^{n} \log\left(\frac{3256.3}{t_i}\right)} \approx 0.49. \tag{3.1.1}$$

If $\beta$ were treated in a non-Bayesian setting, its MLE would be given by equation (3.1.1).

In an experimental process, the largest TTF could occur at any point in the series of failures in a given system. Therefore, consider the case where the largest failure is $t_{39}$=3181. In such a case, the estimate is

$$\hat{\beta}_{39} = \frac{39}{\sum_{i=1}^{n} \log\left(\frac{3181}{t_i}\right)} \approx 0.48 \ . \qquad (3.1.2)$$

Consequently, the value of the largest TTF affects the MLE of $\beta$. In order to study the sensitivity of the MLE of $\beta$ based on the largest TTF, we continue this approach using the reliability growth data provided in Table 3.1. The sequence of the MLE of $\beta$ that we obtained is recorded in Table 3.2.

**Table 3.2 MLE of the key parameter $\beta$, in the intensity function of a PLP, based on the reliability growth failure data given in Table 3.1**

| | | | | | | |
|---|---|---|---|---|---|---|
| 0.49 | 0.48 | 0.48 | 0.48 | 0.50 | 0.53 | 0.54 |
| 0.56 | 0.56 | 0.55 | 0.56 | 0.57 | 0.56 | 0.56 |
| 0.55 | 0.56 | 0.54 | 0.52 | 0.53 | 0.54 | 0.55 |
| 0.53 | 0.56 | 0.55 | 0.53 | 0.50 | 0.55 | 0.57 |
| 0.66 | 0.65 | 0.61 | 0.58 | 0.58 | 0.52 | 0.48 |
| 0.52 | 0.79 | 0.71 | 1.20 | | | |

Since differences are observed in the MLEs, we do not consider the parameter $\beta$ as a constant, but as a random variable. This consideration provides the opportunity to apply Bayesian analysis in the PLP.

An application of a goodness-of-fit test (GOF) to the MLEs of $\beta$ showed that they follow the four-parameter Burr probability distribution $g(\beta;\alpha,\gamma,\delta,\kappa)$, known as 4-parameter Burr type XII distribution, with pdf given by

$$g(\beta) = g(\beta;\alpha,\gamma,\delta,\kappa) = \begin{cases} \dfrac{\alpha\kappa\left(\dfrac{\beta-\gamma}{\delta}\right)^{\alpha-1}}{\delta\left(1+\left(\dfrac{\beta-\gamma}{\delta}\right)^{\alpha}\right)^{\kappa+1}}, & \gamma \le \beta < \infty \\ 0 & \text{, otherwise} \end{cases} \tag{3.1.3}$$

where the hyperparameters κ, α, δ, and γ are being estimated in the GOF test applied to the $\beta$ estimates.

Some basic characteristics of the identified prior are the expected value of the variable $\beta$ is given by

$$E[\beta] = \delta \cdot \frac{\Gamma\left(\kappa - \frac{1}{\alpha}\right)\Gamma\left(1+\frac{1}{\alpha}\right)}{\Gamma(\kappa+1)} + \gamma, \tag{3.1.4}$$

the $(1-\alpha)100\%$ lower confidence limit, LCL, for the parameter $\beta$ is given by

$$\text{LCL}= \chi^2_{1-\alpha/2} \cdot \frac{\hat{\beta}_n}{2n}, \tag{3.1.5}$$

and the $(1-\alpha)100\%$ upper confidence limit, UCL, for the parameter $\beta$ is given by

$$\text{UCL}= \chi^2_{\alpha/2} \cdot \frac{\hat{\beta}_n}{2n} \tag{3.1.6}$$

where $\chi^2_{1-\alpha}$ is the $\alpha$ quantile corresponding to a Chi-square distribution with $2(n-1)$ degrees of freedom. Thus, $P\left[\chi^2_{1-\alpha/2}\cdot\frac{\hat{\beta}_n}{2n}\leq\beta\leq\chi^2_{\alpha/2}\cdot\frac{\hat{\beta}_n}{2n}\right]\geq(1-\alpha)100\%.$

Using the real failure data in Table 3.1, according the equations (3.1.4) to (3.1.6), the expected value of the parameter $\beta$ is approximately 0.5684 with 95% confidence limits given by 0.3395 and 0.6386. That is, $P[0.3395\leq\beta\leq 0.6386]\geq 95\%$.

A Bayesian analysis implies the use of suitable prior information in association with the Bayes Theorem and rests on the exploitation of such information, as well as the belief that a parameter is not merely an unknown fixed quantity but rather a random variable with some prior probability distribution. Therefore, to follow a Bayesian analysis, since the parameter $\beta$ behaves as a random variable, we consider the density (3.1.3) as its prior, along with the squared-error loss function.

The Bayesian estimate of $\beta$ with respect to the squared-error loss function is given by

$$\hat{\beta}_B=\int_{-\infty}^{\infty}\beta\cdot h(\beta\mid\vec{t})d\beta \tag{3.1.7}$$

where the posterior probability density $h$ of $\beta$, using the Bayes Theorem, is given by

$$h(\beta\mid\vec{t})=\frac{L(\vec{t}\mid\beta)g(\beta)}{\int_{-\infty}^{\infty}L(\vec{t}\mid\beta)g(\beta)d\beta}. \tag{3.1.8}$$

Then, the Bayesian estimate of $\beta$, under the squared-error loss function, is

$$\hat{\beta}_B = \int_{-\infty}^{\infty} \beta \cdot h(\beta \mid \vec{t}) d\beta = \frac{\displaystyle\int_{\gamma}^{\infty} \frac{\beta^{n+1}}{\theta^n} \exp\left\{-\left(\frac{t_n}{\theta}\right)^{\beta}\right\} \prod_{i=1}^{n} \left(\frac{t_i}{\theta}\right)^{\beta-1} \frac{\left(\frac{\beta-\gamma}{\delta}\right)^{\alpha-1}}{\left(1+\left(\frac{\beta-\gamma}{\delta}\right)^{\alpha}\right)^{\kappa+1}} d\beta}{\displaystyle\int_{\gamma}^{\infty} \frac{\beta^{n}}{\theta^n} \exp\left\{-\left(\frac{t_n}{\theta}\right)^{\beta}\right\} \prod_{i=1}^{n} \left(\frac{t_i}{\theta}\right)^{\beta-1} \frac{\left(\frac{\beta-\gamma}{\delta}\right)^{\alpha-1}}{\left(1+\left(\frac{\beta-\gamma}{\delta}\right)^{\alpha}\right)^{\kappa+1}} d\beta}. \qquad (3.1.9)$$

With the use of equation (1.2.1.3), the conditional reliability of $t_i$, the analytical structure of the conditional Bayesian reliability estimate for the PLP that is subject to the above information is given by

$$\hat{R}_B(t_i \mid t_1, t_2, \ldots, t_{i-1}) = \exp\left\{-\int_{t_{i-1}}^{t_i} \hat{V}_B^{'}(x) dx\right\}, \quad t_i > t_{i-1} > 0$$

where

$$\hat{V}_B^{'}(t) = \frac{\hat{\beta}_B}{\theta}\left(\frac{t}{\theta}\right)^{\hat{\beta}_B - 1}, \quad \theta > 0, t > 0.$$

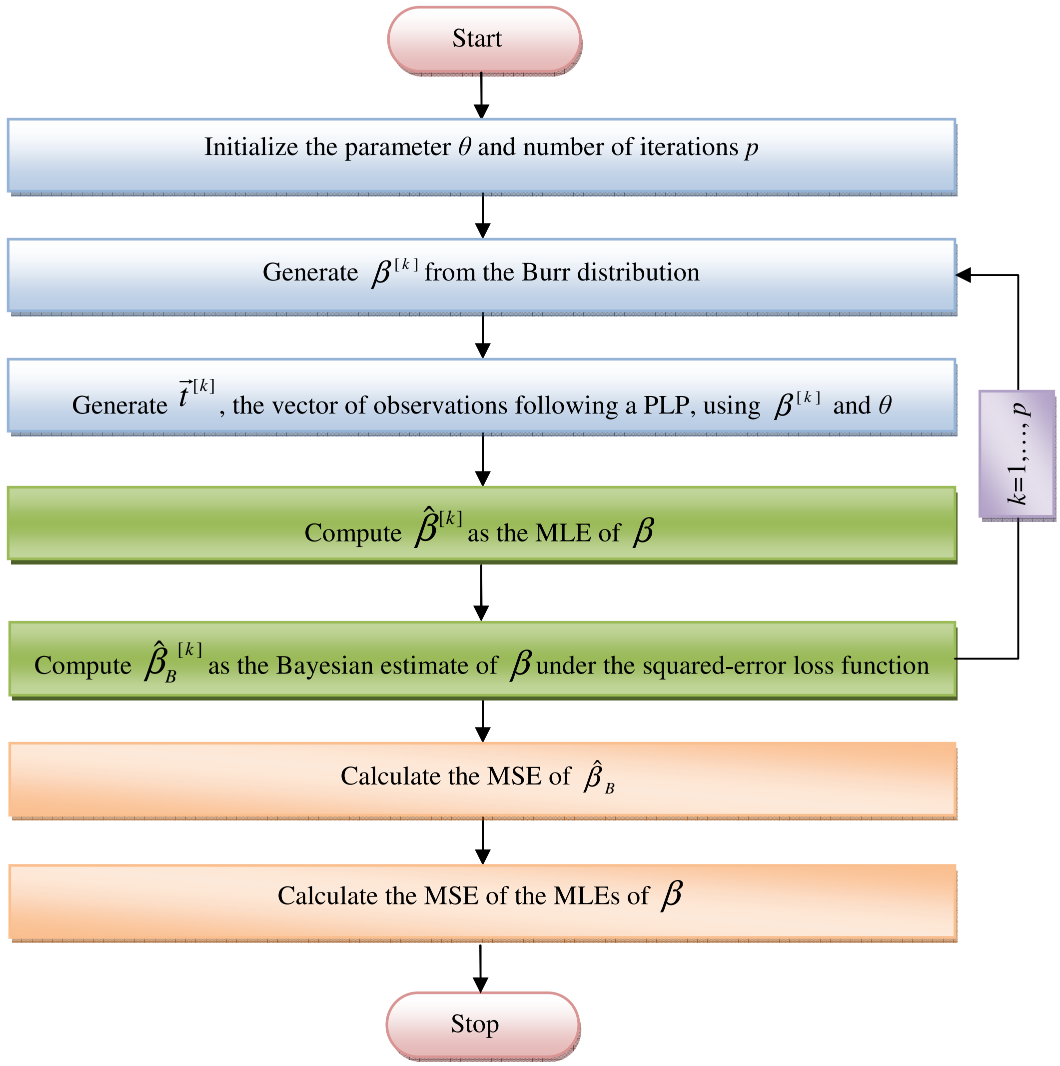
In order to continue our analysis, we proceed to simulate data with the PLP as the underlying failure distribution.

### 3.3 Numerical Simulation

A Monte Carlo simulation was used to compare the Bayesian and the MLE approaches. The parameter $\beta$ of the intensity function for the PLP was calculated using numerical integration techniques in conjunction with a Monte Carlo simulation to obtain its Bayesian estimate. Substituting this estimate in the intensity function, we obtained the estimated Bayesian intensity function.

For a given value of the parameter $\theta$, a stochastic value for the parameter $\beta$ was generated from the identified prior probability density. For a pair of values of $\theta$ and $\beta$, we generated 500 samples of 40 TTFs that follow a NHPP. This procedure was repeated 1,000 times and for three distinct values of $\theta$. The procedure that we followed is summarized in Algorithm 1, below:

*Algorithm 1. Simulation to Analyze the Bayesian Estimate of β for a Given Value of θ*




For each sample of size 40, the Bayesian estimates and MLEs of the parameter were calculated when $\theta \in \{0.5, 1.7441, 4\}$. The comparison is based on the mean squared error (MSE) averaged over the 500,000 repetitions. The results are given in Table 3.3. It is observed that $\hat{\beta}_B$ is superior to $\hat{\beta}$ in estimating $\beta$.

**Table 3.3 MSEs for Bayesian estimates and MLEs of $\beta$ for $n$ = 40 over 500,000 repetitions**

| θ | MSE of $\hat{\beta}_B$ | MSE of $\hat{\beta}$ |
|---|---|---|
| 0.5 | 0.00072 | 0.013492 |
| 1.7441 | 0.00077 | 0.013581 |
| 4 | 0.00078 | 0.013712 |

For different samples sizes, the Bayesian estimates and the MLEs of the parameter $\beta$ were calculated averaging over 500 repetitions. Table 3.4 displays the simulated result of comparing a true value of $\beta$ with respect to its MLE and Bayesian estimate for $n$ = 20, 40, …, 200. The Bayesian estimate of $\beta$ has smaller error than the MLE of $\beta$. This is reflected even with the small sample size of 40 (Figure 3.1).

**Table 3.4 Bayesian estimates and MLEs for the parameter $\beta$ = 0.7054 averaged over 500 repetitions**

| $n$ | $\hat{\beta}_B$ | $\hat{\beta}$ |
|---|---|---|
| 20 | 0.6982 | 0.7834 |
| 40 | 0.7004 | 0.7472 |
| 60 | 0.7056 | 0.7343 |
| 80 | 0.7054 | 0.7241 |
| 100 | 0.7044 | 0.7220 |
| 120 | 0.7054 | 0.7201 |
| 140 | 0.7053 | 0.7158 |
| 160 | 0.7049 | 0.7142 |
| 180 | 0.7047 | 0.7120 |
| 200 | 0.7056 | 0.7114 |

Again, the Bayesian estimate is uniformally closer to the true value of $\beta$ than its MLE, even for a very small sample size of $n$ = 20. A graphical comparison of the true estimate of $\beta$ along with the Bayesian and MLE as a function of sample size is given below by Figure 3.1.

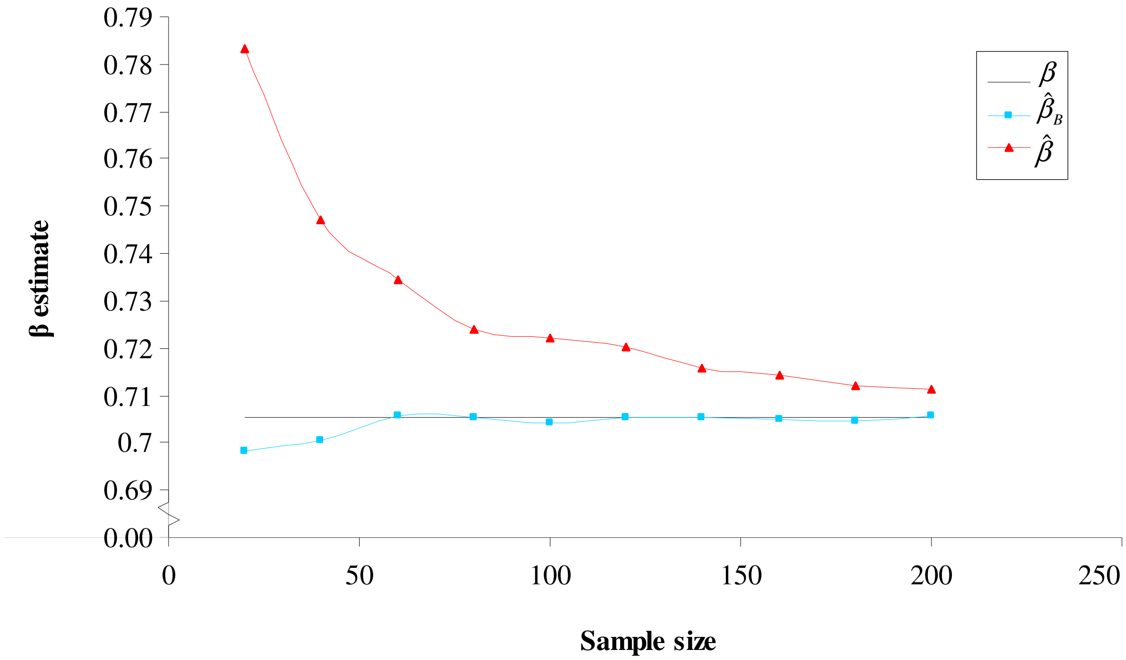



**Figure 3.1 $\beta$ Estimates vs. Sample Size**

For different sample sizes and the same $\beta$, the MLE of the parameter $\theta$ and the corresponding MSE were computed, averaging over the 500 repetitions. Table 3.5 and Figure 3.2 show the results for $\beta$ = 0.7054 and $\theta$ = 1.7441 in addition to the inferior performance for the MLE of $\theta$ and the slow convergence of its MSE values.

**Table 3.5 Averaged $\theta$ MLE and its MSE over 500 repetitions**

| $n$ | $\theta$ | $\hat{\theta}$ | MSE of $\hat{\theta}$ |
|---|---|---|---|
| 40 | 1.7441 | 2.8740 | 7.3411 |
| 80 | 1.7441 | 2.3715 | 3.6187 |
| 160 | 1.7441 | 2.1502 | 1.9106 |
| 200 | 1.7441 | 2.0598 | 1.4721 |

Since the Bayesian estimate for $\beta$ is superior to its MLE, we propose to adjust the MLE of the parameter $\theta$ using equation (1.2.1.6) with $\hat{\beta}_B$ instead of $\hat{\beta}_n$. This proposed adjusted estimate, $\hat{\theta}^*$, was averaged over the 500 repetitions. The results for $\theta$ = 1.7441 are shown in Table 3.6. It can be appreciated that, based on the Bayesian influence on $\beta$, $\hat{\theta}^*$ is a better estimate than the MLE, as expected. This can be seen on Figure 3.3, which shows the excellent performance of $\hat{\theta}^*$.

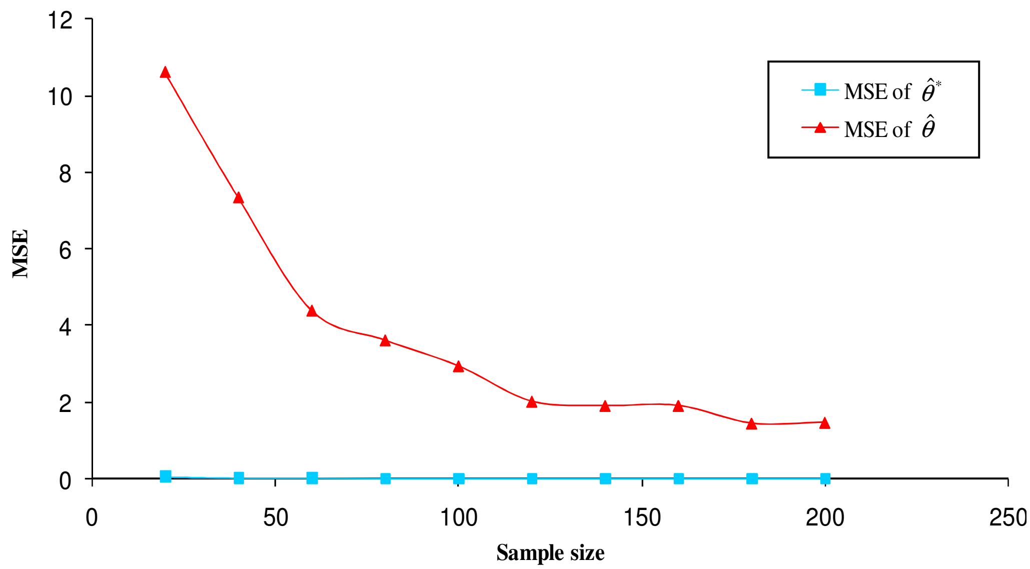



**Figure 3.2 MSE of $\theta$ Estimates vs. Sample Size**

**Table 3.6 Comparison of the adjusted estimate and MLE of $\theta$ with respect to the sample size, for $\theta = 1.7441$**

| $N$ | $\hat{\theta}^*$ | $\hat{\theta}$ | MSE of $\hat{\theta}^*$ | MSE of $\hat{\theta}$ |
|---|---|---|---|---|
| 20 | 1.5898 | 3.1491 | 0.0501 | 10.6103 |
| 40 | 1.6802 | 2.8740 | 0.0140 | 7.34106 |
| 60 | 1.7009 | 2.5525 | 0.0077 | 4.39211 |
| 80 | 1.7108 | 2.3715 | 0.0049 | 3.61871 |
| 100 | 1.7207 | 2.3286 | 0.0030 | 2.94527 |
| 120 | 1.7252 | 2.2361 | 0.0022 | 2.02210 |
| 140 | 1.7266 | 2.1569 | 0.0019 | 1.91071 |
| 160 | 1.7286 | 2.1502 | 0.0017 | 1.91061 |
| 180 | 1.7301 | 2.0751 | 0.0013 | 1.44870 |
| 200 | 1.7306 | 2.0598 | 0.0014 | 1.47206 |

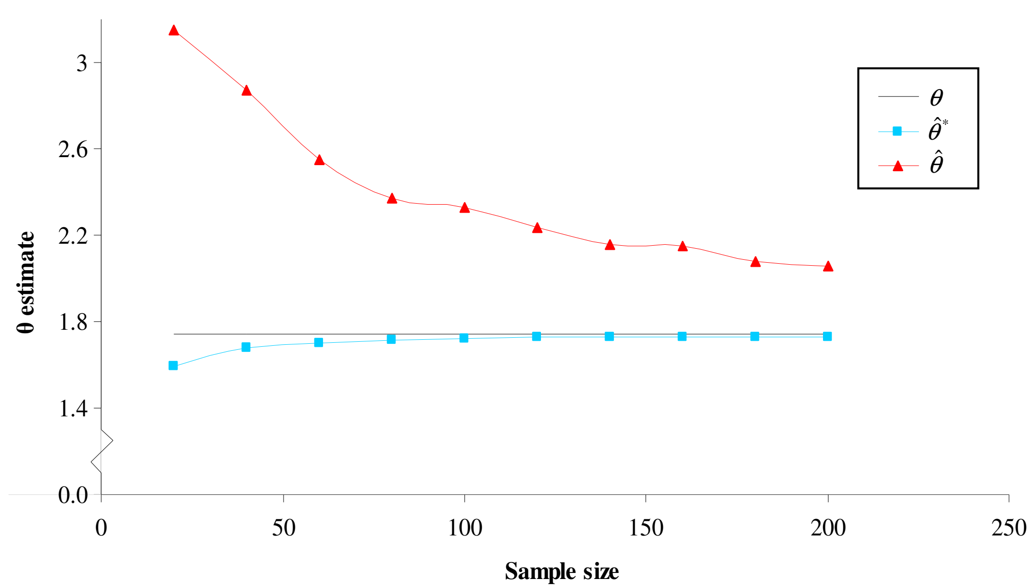



**Figure 3.3 $\theta$ Estimates vs. Sample Size**

We computed our proposed estimate for the parameter $\theta$ and its MSE over 500 repetitions for different values of $\theta$ and sample size $n$ = 160. The results are given in Table 3.7.

**Table 3.7 MSE of the proposed estimate for different values of $\theta$ with $n$ = 160.**

| $\theta$ | $\hat{\theta}^*$ | MSE of $\hat{\theta}^*$ |
|---|---|---|
| 0.5 | 0.4955 | 0.00013 |
| 1.7441 | 1.7286 | 0.00172 |
| 4 | 3.9685 | 0.00899 |

For a fixed value of $\theta$ = 1.7441 and a sample size similar to the size of the collected data, $n$=40, the estimates of the intensity function $\hat{V}'(t)$ and $\hat{V}_B'(t)$ were obtained when we used $\hat{\beta}$ and $\hat{\beta}_B$, respectively. That is,

$$\hat{V}'(t) = \frac{\hat{\beta}}{\theta}\left(\frac{t}{\theta}\right)^{\hat{\beta}-1}, \theta > 0, t > 0$$

and

$$\hat{V}_B'(t) = \frac{\hat{\beta}_B}{\theta}\left(\frac{t}{\theta}\right)^{\hat{\beta}_B-1}, \theta > 0, t > 0 .$$

Their graphs (Figure 3.4) show the superior performance of $\hat{V}_B'(t)$.

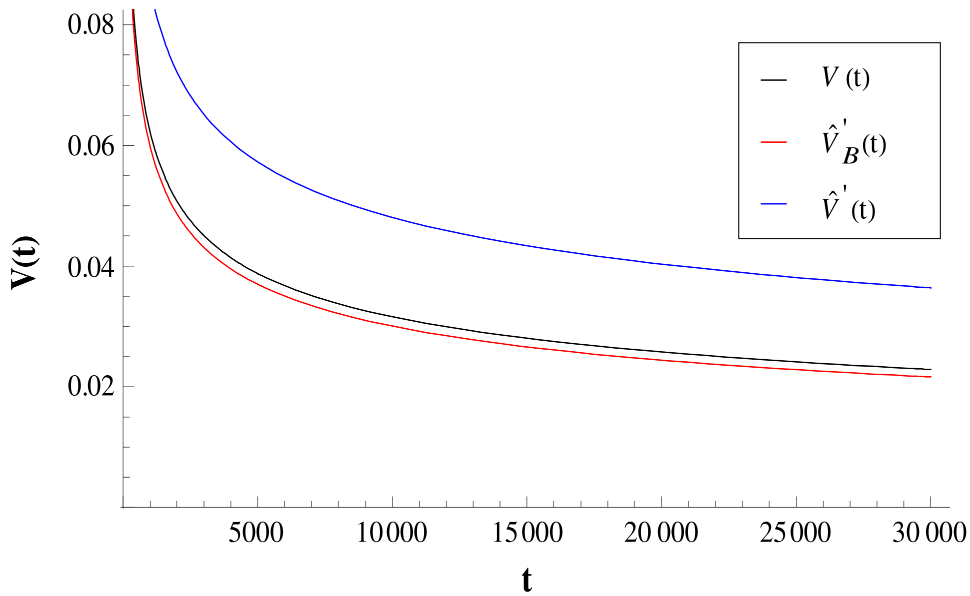



**Figure 3.4 Graph for $\theta$ = 1.7441 and the corresponding $\beta$ Bayesian estimates and MLEs used in $\hat{V}_B^{'}(t)$ and $\hat{V}^{'}(t)$, $n$ = 40**

In order to obtain a Bayesian estimate of the intensity function $\hat{V}_B$, we substituted the Bayesian estimate of $\beta$ and its corresponding MLE of $\theta$:

$$\hat{V}_B(t) = \frac{\hat{\beta}_B}{\hat{\theta}}\left(\frac{t}{\hat{\theta}}\right)^{\hat{\beta}_B - 1}, t > 0$$

The MLE of the intensity function, $\hat{V}$, is obtained using the MLEs of $\beta$ and $\theta$. That is,

$$\hat{V}(t) = \frac{\hat{\beta}}{\hat{\theta}}\left(\frac{t}{\hat{\theta}}\right)^{\hat{\beta} - 1}, t > 0$$

The Bayesian MLE of the intensity function under the influence of the Bayesian estimate of $\beta$, denoted by $\hat{V}_B^*$, is obtained by substituting $\hat{\beta}_B$ and $\hat{\theta}^*$:

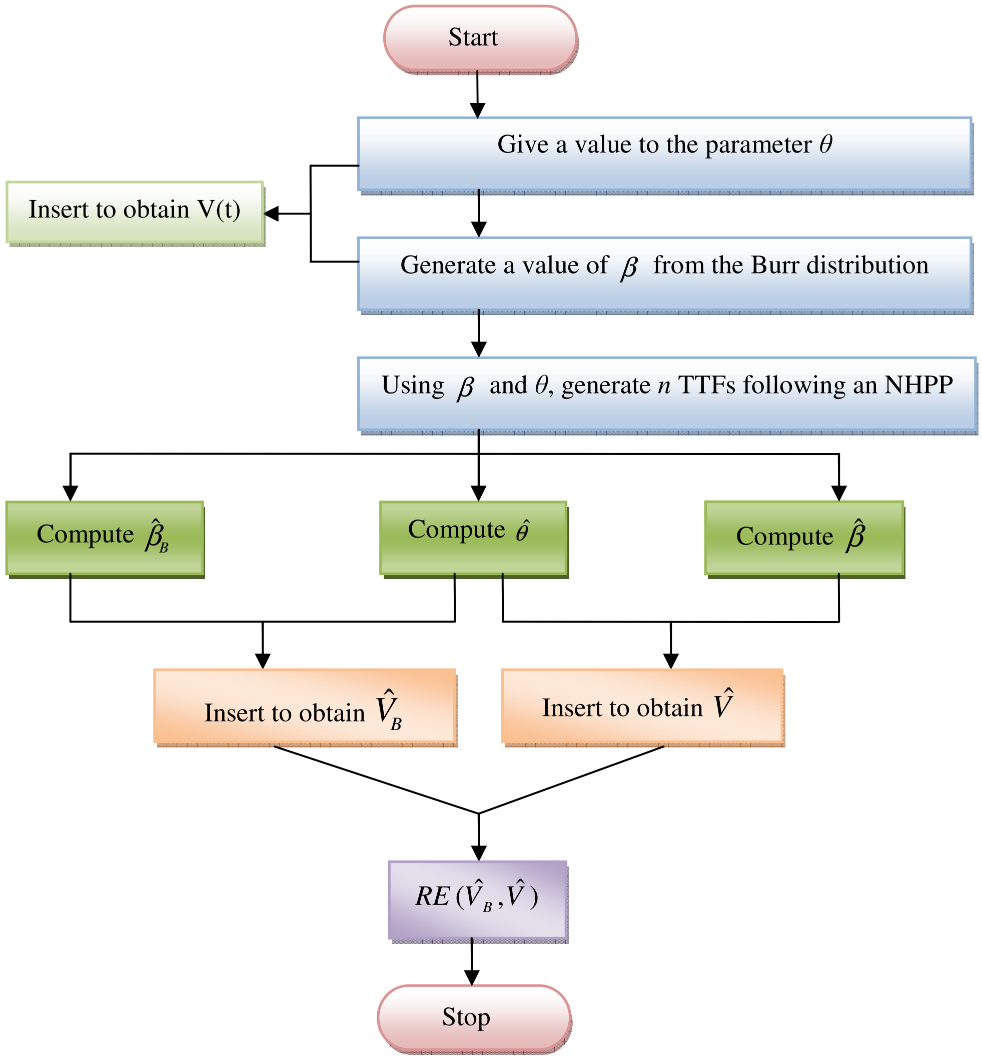
$$\hat{V}_B^*(t) = \frac{\hat{\beta}_B}{\hat{\theta}^*}\left(\frac{t}{\hat{\theta}^*}\right)^{\hat{\beta}_B - 1}, t > 0.$$

To measure the robustness of $\hat{V}_B$ with respect to $\hat{V}$, we calculated the relative efficiency (RE) of the estimate $\hat{V}_B$ compared with the estimate $\hat{V}$ defined as

$$RE(\hat{V}_B, \hat{V}) = \frac{\int_{-\infty}^{\infty} [\hat{V}_B(t) - V(t)]^2 \, dt}{\int_{-\infty}^{\infty} [\hat{V}(t) - V(t)]^2 \, dt} \qquad (3.2.1)$$

If RE = 1, $\hat{V}_B$ and $\hat{V}$ will be interpreted as equally effective. If RE < 1, $\hat{V}_B$ is more efficient than $\hat{V}$, contrary to RE > 1, in which case $\hat{V}_B$ is less efficient than $\hat{V}$. This procedure follows Algorithm 2, given below. Similarly, we compared $\hat{V}_B^*$ and $\hat{V}_B$.

*Algorithm 2. Simulation to Analyze the Bayesian Estimate of the Intensity Function*




Using the values from Tables 3.4 through 3.6, for $n$ = 40, we compared $\hat{V}_B^*, \hat{V}_B$, and $\hat{V}$ using equation (3.2.1). The results are given in Table 3.8.

**Table 3.8 Relative Efficiency of $\hat{V}_B$ with respect to $\hat{V}$ when $\beta$ = 0.7054, $\hat{\beta}_B = 0.7054$ , $\hat{\beta} = 0.7472$, $\hat{\theta}^* = 1.6802$ , $\hat{\theta} = 2.8740$ , $\theta = 1.7441$ , $n$ = 40**

| $V(t),\ t>0$ | $\hat{V}_B(t),\ t>0$ | $\hat{V}(t),\ t>0$ | $\hat{V}_B^*(t),\ t>0$ | $RE(\hat{V}_B^*, \hat{V})$ | $RE(\hat{V}_B^*, \hat{V}_B)$ |
|---|---|---|---|---|---|
| $0.4765t^{-0.2946}$ | $0.3344t^{-0.2996}$ | $0.3395t^{-0.2528}$ | $0.4869t^{-0.2996}$ | $<1$ | $<1$ |

For the comparison of $\hat{V}_B^*$ and $\hat{V}$, the denominator in equation (3.2.1) dominates the numerator. This implies that the intensity function using $\hat{\beta}_B$ and $\hat{\theta}^*$ is more efficient than the intensity function under $\hat{\beta}$ and $\hat{\theta}$. Comparing $\hat{V}_B^*$ and $\hat{V}_B$, we obtained a similar result, establishing the superior relative efficiency of $\hat{V}_B^*$. The corresponding graphs for the intensity functions are given by Figure 3.5.

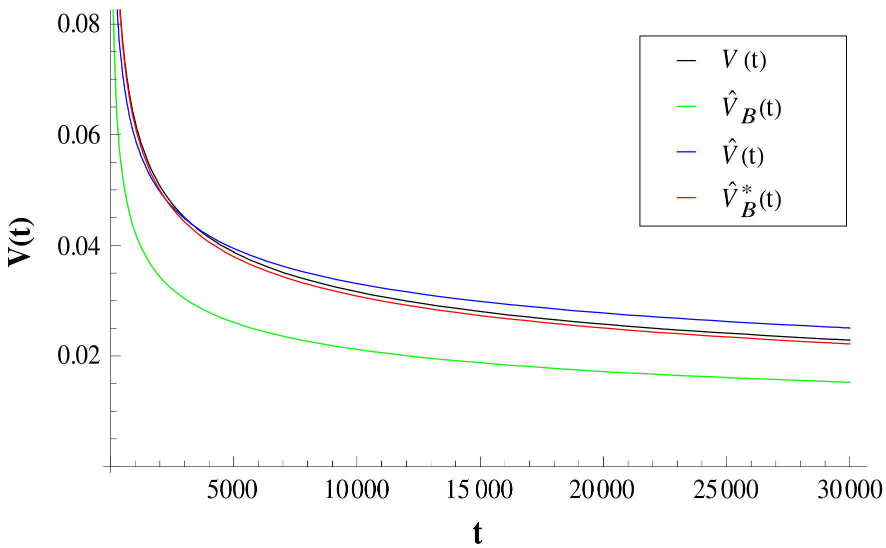
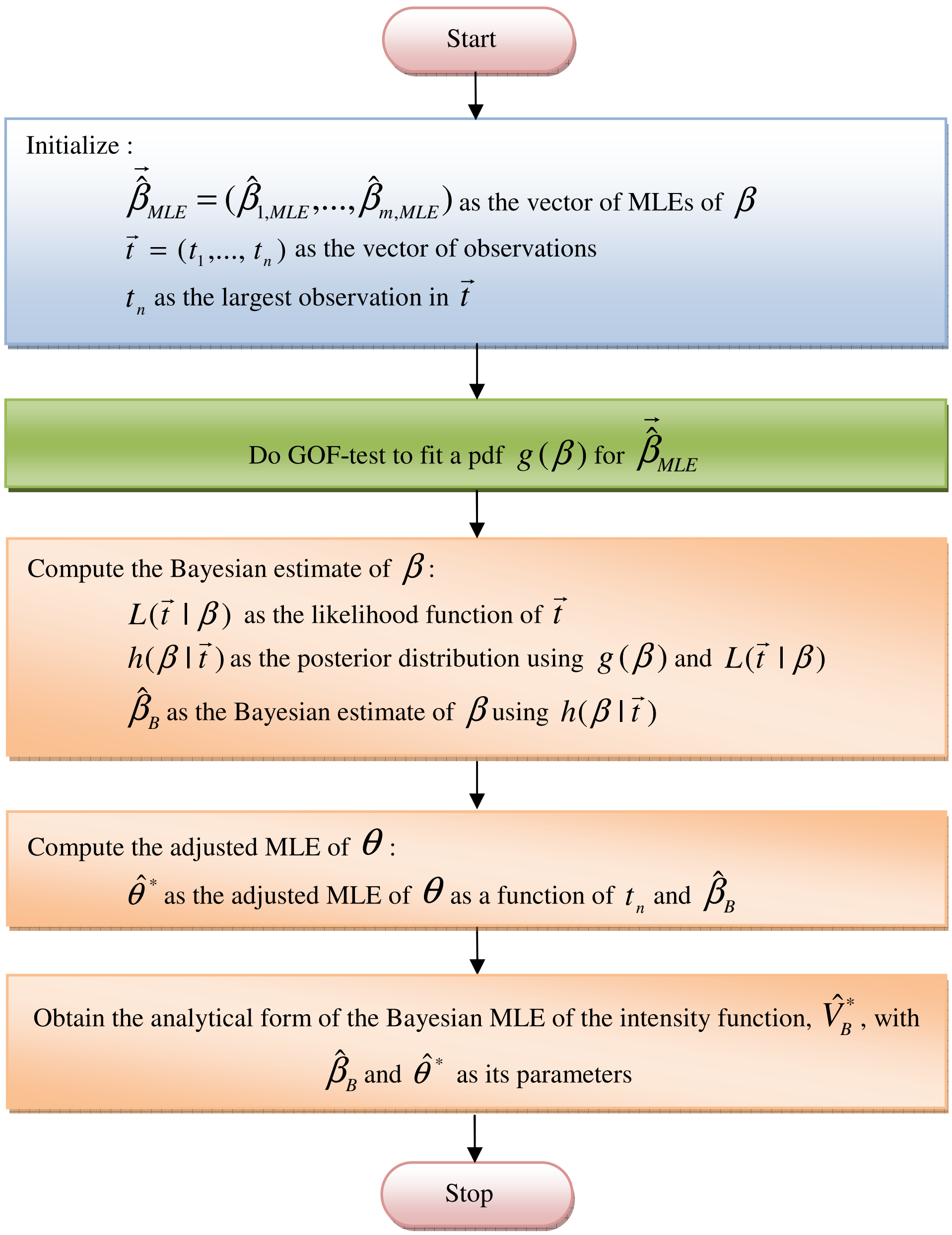



**Figure 3.5 Estimates of the intensity function for $\beta = 0.7054$, $\hat{\beta}_B = 0.7004$, $\hat{\beta} = 0.7472$, $\theta = 1.7441$, $\hat{\theta}^* = 1.6802$, $\hat{\theta} = 2.8740$, $n = 40$**

## 3.4 Using real data

Using the reliability growth data from Table 3.1, we computed $\hat{\beta}_B$ and the better estimate $\hat{\theta}^*$ in order to obtain the Bayesian intensity function. We follow the algorithm that is given below to obtain the Bayesian intensity function for the given real data.

*Algorithm 3. Estimate of the Intensity Function Using the Real Data*



For the failure data of Crow provided by Tsokos (1995), $\hat{\beta}_B$ is approximately 0.4851, and $\hat{\theta}^*$ is approximately 1.6234. Therefore, with the use of $\hat{\theta}^*$, the Bayesian MLE of the intensity function for the data is approximately

$$\hat{V}_B^*(t) = 0.3835 \cdot t^{-0.5149}, \quad t > 0 .$$

A graphical display of $\hat{V}_B^*(t)$ is given below by Figure 3.6.

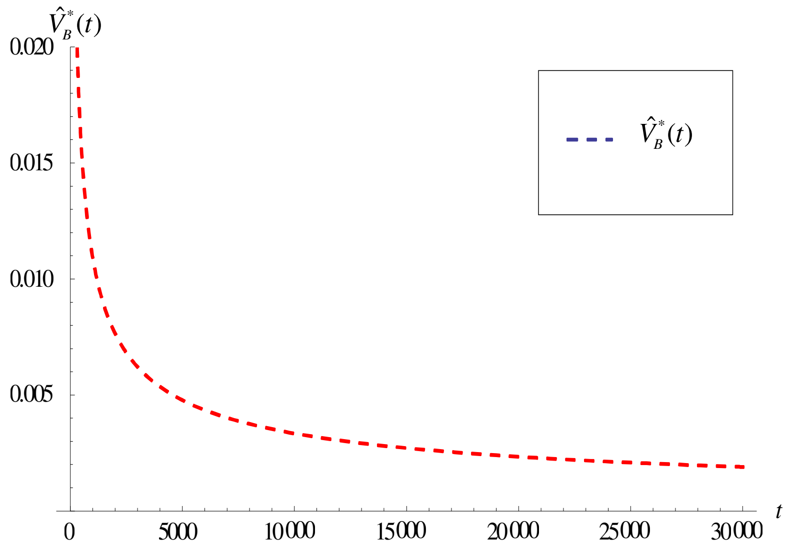



**Figure 3.6 Bayesian MLE of the intensity function, $\hat{V}_B^*(t)$, with $\hat{\beta}_B = 0.4851$ and $\hat{\theta}^* = 1.6234$.**

To obtain a Bayesian MLE for the reliability function, we use this Bayesian estimate for the intensity function. The analytical form for the corresponding Bayesian reliability estimate, based on the data, is given by

$$\hat{R}_B(t_i \mid t_1, t_2, ..., t_{i-1}) = \exp\left\{-0.3835 \int_{t_{i-1}}^{t_i} x^{-0.5149} dx\right\}, \qquad t_i > t_{i-1} > 0$$

## 3.5 Conclusions

In the present study, we considered that the key parameter $\beta$ in the intensity function in the NHPP could behave as a random variable, and our analysis showed that its prior probabilistic characterization is the Burr type XII probability distribution. We developed the analytical structure of the Bayesian reliability estimate of the PLP subject to the mentioned prior along with the squared-error loss function.

We used real data in addition to numerical simulation to illustrate the usefulness of having developed the Bayesian analytical procedure. Based on the Monte Carlo simulation, the Bayesian estimate is superior to the MLE of $\beta$. This is reflected even with a small sample size for which the MLE of $\theta$ had inferior performance. Moreover, the MSE of the MLE for this parameter shows slow convergence. The inferior results in estimating the MLE of $\theta$ lie in the fact that this estimate depends on the MLE of $\beta$.

Because of the superior performance of $\hat{\beta}_B$, the MLE of $\theta$ was adjusted, thereby producing a better estimate for $\theta$ under the mentioned Bayesian influence. In addition, for a particular value of $\theta$, the $V(t)$ estimate with $\beta = \hat{\beta}_B$ is better when compared with $\hat{V}'(t)$ using the MLE of $\beta$, $\hat{\beta}$. Moreover, the computation of RE implies that $\hat{V}_B^*(t)$ is more efficient when compared with $\hat{V}(t)$ and $\hat{V}_B(t)$.

The main contributions of this study, that are expected to have a direct impact in future research on Bayesian and parametric approaches to reliability analysis in complex systems problems, are:

- An innovative way to investigate if a Bayesian analysis is applicable to estimate the key parameter of the intensity function in a PLP, taking advantage of the dependency of the MLE for this parameter on the last TTF which could be less or greater than the largest TTF provided in the available data.

- The derivation of the analytical form of the Bayesian estimate of the key parameter in the power law intensity function as a function of the Bayesian estimate of the key parameter $\beta$ and the MLE of the parameter $\theta$.
- The development of the analytical form of the Bayesian estimate for the intensity function as a function of the estimates of the key parameter, $\hat{\beta}_B$, and $\theta$ adjusted, $\hat{\theta}^*$.

All the analytical findings are given in the following table:

**Table 3.9 Analytical Form of the Estimates**

| Estimate | Analytical Form |
|---|---|
| Bayesian estimate of the key parameter $\beta$ | $\hat{\beta}_B = \dfrac{\int_{\gamma}^{\infty} \frac{\beta^{n+1}}{\theta^n} \exp\left\{-\left(\frac{t_n}{\theta}\right)^{\beta}\right\} \prod_{i=1}^{n} \left(\frac{t_i}{\theta}\right)^{\beta-1} \frac{\left(\frac{\beta-\gamma}{\delta}\right)^{\alpha-1}}{\left(1+\left(\frac{\beta-\gamma}{\delta}\right)^{\alpha}\right)^{\kappa+1}} d\beta}{\int_{\gamma}^{\infty} \frac{\beta^{n}}{\theta^n} \exp\left\{-\left(\frac{t_n}{\theta}\right)^{\beta}\right\} \prod_{i=1}^{n} \left(\frac{t_i}{\theta}\right)^{\beta-1} \frac{\left(\frac{\beta-\gamma}{\delta}\right)^{\alpha-1}}{\left(1+\left(\frac{\beta-\gamma}{\delta}\right)^{\alpha}\right)^{\kappa+1}} d\beta}$ |
| Reliability Function Estimate | $\hat{R}_B(t_i \mid t_1, t_2, \ldots, t_{i-1}) = \exp\left\{-\int_{t_{i-1}}^{t_i} \hat{V}_B'(x)dx\right\}, \qquad t_i > t_{i-1} > 0$ |
| Intensity Function Estimate under the Bayesian influence | $\hat{V}_B'(t) = \frac{\hat{\beta}_B}{\theta}\left(\frac{t}{\theta}\right)^{\hat{\beta}_B-1}, \theta > 0, t > 0$ |
| Intensity Function Estimate under the parametric influence | $\hat{V}'(t) = \frac{\hat{\beta}}{\theta}\left(\frac{t}{\theta}\right)^{\hat{\beta}-1}, \theta > 0, t > 0$ |
| Intensity Function Estimate under the Bayesian and parametric influence | $\hat{V}_B(t) = \frac{\hat{\beta}_B}{\hat{\theta}}\left(\frac{t}{\hat{\theta}}\right)^{\hat{\beta}_B-1}, t > 0$ |
| Parametric Intensity Function Estimate | $\hat{V}(t) = \frac{\hat{\beta}}{\hat{\theta}}\left(\frac{t}{\hat{\theta}}\right)^{\hat{\beta}-1}, t > 0$ |
| Intensity Function Estimate under the influence of the Bayesian estimates | $\hat{V}_B^*(t) = \frac{\hat{\beta}_B}{\hat{\theta}^*}\left(\frac{t}{\hat{\theta}^*}\right)^{\hat{\beta}_B-1}, t > 0$ |
| Bayesian Estimate of the Intensity Function for the data | $\hat{V}_B^*(t) = 0.3835 \cdot t^{-0.5149}, t > 0$ |
| Reliability Function Estimate for the data | $\hat{R}_B(t_i \mid t_1, t_2, \ldots, t_{i-1}) = \exp\left\{-0.3835\int_{t_{i-1}}^{t_i} x^{-0.5149}dx\right\}, \qquad t_i > t_{i-1} > 0$ |

# CHAPTER 4 BAYESIAN ROBUSTNESS ANALYSIS FOR THE POWER LAW PROCESS BASED ON THE PRIOR SELECTION

The objective of the present study is to perform sensitivity analysis in the selection of the prior in the PLP in a Bayesian setting. We compared the Bayesian estimates of one of the two parameters that are inherent in the intensity function with its MLE. In addition, we compared the corresponding adjusted MLEs of the second parameter applying the proposed adjusted MLE that we studied in the previous chapter.

In the first section of this chapter we present an overview of the PLP. In the next section we define the priors as the probability characterization of one of the parameters in the intensity function and using squared-error loss function, we develop the analytical form of the Bayesian estimates of the parameter. In the third section, we compared the Bayesian estimates of the parameter and with its MLE counterpart. We also compared the adjusted MLE of the second parameter defined in the previous chapter. The last section presents the conclusions of the study.

## 4.1 Introduction

As we mentioned in Chapter 3, reliability growth $\beta$ is the key parameter in the PLP intensity function. As noted previously, a study of the growth of the reliability of systems is usually centered on the evaluation of the probability of a system failure as a function of the age of the system. When the failure intensity of a system changes with time, the NHPP with the failure intensity function given by

$$V(t) = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1}, \; \mathrm{t} > 0, \beta > 0, \theta > 0 \tag{4.1}$$

where $\beta$ and θ are the shape and scale parameters, respectively, has usually been used as the underlying failure distribution of repairable systems. Here, we seek the answer to the following question: Is the Bayesian estimate of the key parameter, $\beta$, in the PLP sensitive to the selection of the prior? In the present study, we assume that the parameter $\beta$ behaves as a random variable and using simulated data governed by a PLP, we proceed to perform Bayesian sensitivity analysis subject to prior selection for $\beta$ and under the commonly used squared-error loss function.

To measure the robustness of the $\hat{\beta}_1$ with respect to $\hat{\beta}_2$ we compare their MSE. If $\mathrm{MSE}(\hat{\beta}_1) < \mathrm{MSE}(\hat{\beta}_2)$, $\hat{\beta}_1$ is more efficient than $\hat{\beta}_2$. For the case when $\mathrm{MSE}(\hat{\beta}_1) > \mathrm{MSE}(\hat{\beta}_2)$, then $\hat{\beta}_2$ is more efficient than $\hat{\beta}_1$. When $\mathrm{MSE}(\hat{\beta}_1) = \mathrm{MSE}(\hat{\beta}_2)$, we conclude that $\hat{\beta}_1$ is equally as efficient as $\hat{\beta}_2$.

## 4.2 Maximum Likelihood Estimates in the Power Law Process

The probability of achieving $n$ failures of a given system in the time interval (0, t] can be written as

$$P(x=n;t)=\frac{\exp\left\{-\int_0^t V(x)dx\right\}\left[\int_0^t V(x)dx\right]^n}{n!},\quad t>0 \tag{4.1.1}$$

where $V(t)$ is the intensity function given by equation (4.1).

Let $T_1, T_2, T_3, ..., T_n$ be the first $n$ TTF of the NHPP, where $T_1 < T_2 < T_3 < ... < T_n$ are recorded from the initial start-up of the system onward. Thus, the truncated conditional probability distribution function, $f_i(t \mid t_1, ..., t_{i-1})$, in the Weibull process is given by

$$f_i(t \mid t_1,...,t_{i-1})=\frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1}\exp\left\{-\left(\frac{t}{\theta}\right)^{\beta}+\left(\frac{t_{i-1}}{\theta}\right)^{\beta}\right\},\quad \mathrm{t_{i-1}}<t. \tag{4.1.2}$$

The likelihood function for $T_1 = t_1, T_2 = t_2, ..., T_n = t_n$, with $\vec{t} = (t_1, t_2, t_3, ..., t_n)$, is given by

$$L(\vec{t} \mid \beta)=\left(\frac{\beta}{\theta}\right)^n\cdot\exp\left(-\left(\frac{t_n}{\theta}\right)^{\beta}\right)\cdot\prod_{i=1}^{n}\left(\frac{t_i}{\theta}\right)^{\beta-1} \tag{4.1.3}$$

The MLE for the shape parameter is

$$\hat{\beta}_{MLE} = \frac{n}{\sum_{i=1}^{n} \log\left(\frac{t_n}{t_i}\right)} \tag{4.1.4}$$

and, for the scale parameter is

$$\hat{\theta}_{MLE} = \frac{t_n}{n^{1/\hat{\beta}}}. \tag{4.1.5}$$

Our first interest is to compare the Bayesian estimates for $\beta$ for each of two assumed priors, and with respect to its MLE given by equation (4.1.4), assuming $\beta$ behaves as a random variable and $\theta$ as known. Secondly, we compare equation (4.1.5) with an adjusted MLE considered as a function of $\beta$.

## 4.3 Bayesian Analytical Form of the Stochastic Parameter $\beta$

Let $\beta$ be a random variable and the 4-parameter Burr type XII distribution given by Burr (1942),

$$g(\beta) = g(\beta;\alpha,\gamma,\delta,\kappa) = \begin{cases} \dfrac{\alpha\kappa\left(\dfrac{\beta-\gamma}{\delta}\right)^{\alpha-1}}{\delta\left(1+\left(\dfrac{\beta-\gamma}{\delta}\right)^{\alpha}\right)^{\kappa+1}}, & \gamma \le \beta < \infty \\ 0 & \text{, otherwise} \end{cases} \tag{4.2.1}$$

as the true probability distribution of $\beta$. We shall examine the problem for each of the following prior densities of $\beta$.

i) Jeffreys prior

$$g(\beta) = \frac{1}{\beta}, \quad \beta > 0 \tag{4.2.2}$$

ii) the Inverted Gamma

$$g(\beta) = \left(\frac{\mu}{\beta}\right)^{\nu+1} \frac{1}{\mu\Gamma(\nu)} \exp\left\{-\frac{\mu}{\beta}\right\}, \quad \beta > 0, \mu > 0, \nu > 0 \tag{4.2.3}$$

and we consider the well-known squared-error loss function, given by

$$L(\beta, \hat{\beta}) = (\beta - \hat{\beta})^2 \tag{4.2.4}$$

where $\hat{\beta}$ is the estimate of $\beta$.

Using the Bayes Theorem, the posterior probability distribution of $\beta$ is given by

$$h(\beta \mid \vec{t}\,) = \frac{L(\vec{t} \mid \beta) g(\beta)}{\int_{-\infty}^{\infty} L(\vec{t} \mid \beta) g(\beta) d\beta}. \tag{4.2.5}$$

The Bayesian estimate of the key parameter $\beta$, with respect to the squared-error loss function (4.2.4), is obtained by evaluating

$$\hat{\beta} = \int_{-\infty}^{\infty} \beta \cdot h(\beta \mid \vec{t}\,) d\beta. \tag{4.2.6}$$

### 4.3.1 The Jeffreys' Prior

Assuming Jeffreys prior (4.2.2) as the prior of $\beta$ and using the likelihood (4.1.3) and (4.2.5), the posterior density of $\beta$ is given by

$$h_J(\vec{t} \mid \beta) = \frac{\exp\left\{-\left(\frac{t_n}{\theta}\right)^{\beta}\right\}\frac{\beta^{n-1}}{\theta^{n\beta}}\prod_{i=1}^{n}(t_i)^{\beta-1}}{\int_0^{\infty}\exp\left\{-\left(\frac{t_n}{\theta}\right)^{\beta}\right\}\frac{\beta^{n-1}}{\theta^{n\beta}}\prod_{i=1}^{n}(t_i)^{\beta-1}d\beta}. \tag{4.3.1.1}$$

Thus, the Jeffreys Bayesian estimate of the key parameter $\beta$ under the squared-error loss function, using equations (4.2.6) and (4.3.1.1), we have

$$\hat{\beta}_B^J = \frac{\int_0^{\infty}\exp\left\{-\left(\frac{t_n}{\theta}\right)^{\beta}\right\}\left(\frac{\beta}{\theta}\right)^{n}\prod_{i=1}^{n}\left(\frac{t_i}{\theta}\right)^{\beta-1}d\beta}{\int_0^{\infty}\exp\left\{-\left(\frac{t_n}{\theta}\right)^{\beta}\right\}\frac{\beta^{n-1}}{\theta^{n\beta}}\prod_{i=1}^{n}(t_i)^{\beta-1}d\beta}. \tag{4.3.1.2}$$

We can not obtain a close solution for $\hat{\beta}_B^J$ and we must rely on a numerical estimate. Also note that it depends on knowing or being able to estimate the scale parameter $\theta$.

### 4.3.2 The Inverted Gamma Prior

The following is an examination of the problem when the prior density of $\beta$ is given by the inverted gamma (4.2.3). Using the likelihood function (4.1.3) and (4.2.5), the posterior density of $\beta$ is given by

$$h_{IG}(\vec{t} \mid \beta) = \frac{\frac{\beta^{n-\nu-1}}{\theta^{n\beta}} \exp\left\{-\left(\frac{t_n}{\theta}\right)^{\beta} - \frac{\mu}{\beta}\right\} \prod_{i=1}^{n} (t_i)^{\beta-1}}{\int_0^{\infty} \frac{\beta^{n-\nu-1}}{\theta^{n\beta}} \exp\left\{-\left(\frac{t_n}{\theta}\right)^{\beta} - \frac{\mu}{\beta}\right\} \prod_{i=1}^{n} (t_i)^{\beta-1} d\beta}. \qquad (4.3.2.1)$$

Thus, the Bayesian estimate of $\beta$ under the inverted gamma prior and squared-error loss function, using (4.2.6) and (4.3.2.1), is given by

$$\hat{\beta}_B^{IG} = \frac{\int_0^{\infty} \frac{\beta^{n-\nu}}{\theta^{n\beta}} \exp\left\{-\left(\frac{t_n}{\theta}\right)^{\beta} - \frac{\mu}{\beta}\right\} \prod_{i=1}^{n} (t_i)^{\beta-1}}{\int_0^{\infty} \frac{\beta^{n-\nu-1}}{\theta^{n\beta}} \exp\left\{-\left(\frac{t_n}{\theta}\right)^{\beta} - \frac{\mu}{\beta}\right\} \prod_{i=1}^{n} (t_i)^{\beta-1} d\beta}. \qquad (4.3.2.2)$$

Here as well, we can not get a close form solution of $\hat{\beta}_B^{IG}$ and we will obtain numerical estimates.

### 4.3.3 The Burr Probability Distribution as Prior

In order to assess the computational procedure, we also consider the Bayesian estimate of $\beta$ assuming the 4-parameter Burr type XII probability distribution as a prior of $\beta$. Under this consideration, using (4.1.3), (4.2.1), and (4.2.5), the posterior distribution of $\beta$ is given by

$$h_B(\beta \mid \vec{t}) = \frac{\left(\frac{\beta}{\theta}\right)^n \exp\left\{-\left(\frac{t_n}{\theta}\right)^\beta\right\} \prod_{i=1}^{n} \left(\frac{t_i}{\theta}\right)^{\beta-1} \frac{\left(\frac{\beta-\gamma}{\delta}\right)^{\alpha-1}}{\left(1+\left(\frac{\beta-\gamma}{\delta}\right)^{\alpha}\right)^{\kappa+1}}}{\int_{\gamma}^{\infty} \left(\frac{\beta}{\theta}\right)^n \exp\left\{-\left(\frac{t_n}{\theta}\right)^\beta\right\} \prod_{i=1}^{n} \left(\frac{t_i}{\theta}\right)^{\beta-1} \frac{\left(\frac{\beta-\gamma}{\delta}\right)^{\alpha-1}}{\left(1+\left(\frac{\beta-\gamma}{\delta}\right)^{\alpha}\right)^{\kappa+1}} d\beta}. \tag{4.3.3.1}$$

Therefore, the Burr Bayesian estimate of the parameter $\beta$ can be written as

$$\hat{\beta}_B = \int_{-\infty}^{\infty} \beta \cdot h(\beta \mid \vec{t}) d\beta = \frac{\int_{\gamma}^{\infty} \frac{\beta^{n+1}}{\theta^n} \exp\left\{-\left(\frac{t_n}{\theta}\right)^\beta\right\} \prod_{i=1}^{n} \left(\frac{t_i}{\theta}\right)^{\beta-1} \frac{\left(\frac{\beta-\gamma}{\delta}\right)^{\alpha-1}}{\left(1+\left(\frac{\beta-\gamma}{\delta}\right)^{\alpha}\right)^{\kappa+1}} d\beta}{\int_{\gamma}^{\infty} \frac{\beta^{n}}{\theta^n} \exp\left\{-\left(\frac{t_n}{\theta}\right)^\beta\right\} \prod_{i=1}^{n} \left(\frac{t_i}{\theta}\right)^{\beta-1} \frac{\left(\frac{\beta-\gamma}{\delta}\right)^{\alpha-1}}{\left(1+\left(\frac{\beta-\gamma}{\delta}\right)^{\alpha}\right)^{\kappa+1}} d\beta}. \tag{4.3.3.2}$$

Table 4.1 below provides a summary of the Bayesian estimates of the key parameter $\beta$ in the PLP.

**Table 4.1 Bayesian Estimates for the Key Parameter $\beta$ in a PLP**

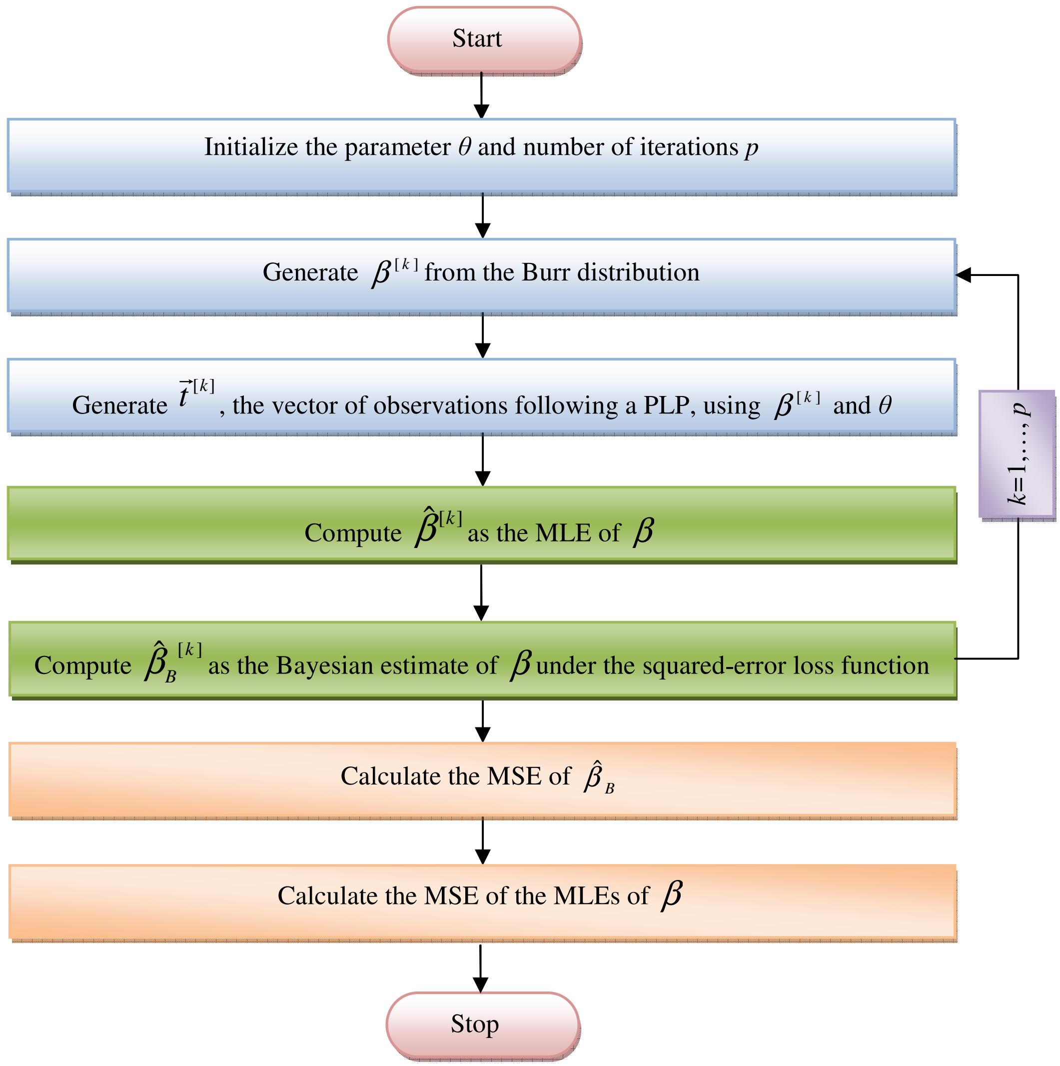
| *Prior Density* | $\beta$ *Bayesian Estimate Under the Squared-Error Loss Function* |
|---|---|
| The Burr pdf $g(\beta)=g(\beta;\alpha,\gamma,\delta,\kappa)=\begin{cases}\dfrac{\alpha\kappa\left(\frac{\beta-\gamma}{\delta}\right)^{\alpha-1}}{\delta\left(1+\left(\frac{\beta-\gamma}{\delta}\right)^{\alpha}\right)^{\kappa+1}}, & \gamma\le\beta<\infty\\ 0 & \text{, otherwise}\end{cases}$ | $\hat{\beta}_B=\dfrac{\int_{\gamma}^{\infty}\frac{\beta^{n+1}}{\theta^{n}}\exp\left\{-\left(\frac{t_n}{\theta}\right)^{\beta}\right\}\prod_{i=1}^{n}\left(\frac{t_i}{\theta}\right)^{\beta-1}\frac{\left(\frac{\beta-\gamma}{\delta}\right)^{\alpha-1}}{\left(1+\left(\frac{\beta-\gamma}{\delta}\right)^{\alpha}\right)^{\kappa+1}}d\beta}{\int_{\gamma}^{\infty}\frac{\beta^{n}}{\theta^{n}}\exp\left\{-\left(\frac{t_n}{\theta}\right)^{\beta}\right\}\prod_{i=1}^{n}\left(\frac{t_i}{\theta}\right)^{\beta-1}\frac{\left(\frac{\beta-\gamma}{\delta}\right)^{\alpha-1}}{\left(1+\left(\frac{\beta-\gamma}{\delta}\right)^{\alpha}\right)^{\kappa+1}}d\beta}$ |
| Jeffreys' Prior $g(\beta)=\frac{1}{\beta},\quad \beta>0$ | $\hat{\beta}_B^{J}=\dfrac{\int_0^{\infty}\exp\left\{-\left(\frac{t_n}{\theta}\right)^{\beta}\right\}\left(\frac{\beta}{\theta}\right)^{n}\prod_{i=1}^{n}\left(\frac{t_i}{\theta}\right)^{\beta-1}d\beta}{\int_0^{\infty}\exp\left\{-\left(\frac{t_n}{\theta}\right)^{\beta}\right\}\frac{\beta^{n-1}}{\theta^{n\beta}}\prod_{i=1}^{n}(t_i)^{\beta-1}d\beta}$ |
| The Inverted Gamma pdf $g(\beta)=\left(\frac{\mu}{\beta}\right)^{\nu+1}\frac{1}{\mu\Gamma(\nu)}\exp\left\{-\frac{\mu}{\beta}\right\},\quad \beta>0,\mu>0,\nu>0$ | $\hat{\beta}_B^{IG}=\dfrac{\int_0^{\infty}\frac{\beta^{n-\nu}}{\theta^{n\beta}}\exp\left\{-\left(\frac{t_n}{\theta}\right)^{\beta}-\frac{\mu}{\beta}\right\}\prod_{i=1}^{n}(t_i)^{\beta-1}}{\int_0^{\infty}\frac{\beta^{n-\nu-1}}{\theta^{n\beta}}\exp\left\{-\left(\frac{t_n}{\theta}\right)^{\beta}-\frac{\mu}{\beta}\right\}\prod_{i=1}^{n}(t_i)^{\beta-1}d\beta}$ |

Below are details of the analysis we conducted using Monte Carlo simulation to generate data governed by a PLP.

## 4.4 Numerical Simulation

In the implementation of the simulation procedure, we followed the Algorithm 1 given in Chapter 3 and reproduced here for the reader's convenience.

*Algorithm 1. Simulation to Analyze the Bayesian Estimate of $\beta$ for a Given Value of $\theta$*




Random TTF's distributed according to the PLP are simulated for a realization of the stochastic scale parameter $\beta$, which follows a Burr type XII probability distribution. Numerical integration techniques were used to compute the Bayesian estimates of the key parameter $\beta$ according to the equations (4.3.1.2), (4.3.2.2), and (4.3.3.2) for each of the three prior densities presented in section 4.2 and three distinct values of $\theta$. Samples of size 40, 50, 60, 70, 80, 100, 120, 140, 160, and 180 were generated where the parameter $\theta$ was assumed to be 1.7441. The results, for 500 repetitions, are shown in Table 4.2. It can be observed in Table 4.2 that the Bayesian estimate of the key parameter $\beta$ under the Jeffreys' prior and squared-error loss function produces a small error.

**Table 4.2 MLE and Bayesian estimates, with Burr, Jeffreys, and Inverted Gamma as priors under squared-error loss function, for the parameter $\beta$ in a PLP with $\theta$ = 1.7441 and 1,000 samples with different sizes.**

| $n$ | $\beta$ | $\hat{\beta}_B$ | $\hat{\beta}_B^J$ | $\hat{\beta}_{MLE}$ | $\hat{\beta}_B^{IG}$ |
|---|---|---|---|---|---|
| 40 | 0.7054 | 0.7037 | 0.7072 | 0.7378 | 0.6957 |
| 50 | 0.7054 | 0.7041 | 0.7066 | 0.7378 | 0.6984 |
| 60 | 0.7054 | 0.7040 | 0.7058 | 0.7336 | 0.6995 |
| 70 | 0.7054 | 0.7041 | 0.7056 | 0.7214 | 0.7005 |
| 80 | 0.7054 | 0.7046 | 0.7058 | 0.7257 | 0.7017 |
| 100 | 0.7054 | 0.7053 | 0.7062 | 0.7210 | 0.7031 |
| 120 | 0.7054 | 0.7051 | 0.7058 | 0.7173 | 0.7034 |
| 140 | 0.7054 | 0.7052 | 0.7058 | 0.7152 | 0.7039 |
| 160 | 0.7054 | 0.7052 | 0.7057 | 0.7153 | 0.7041 |
| 180 | 0.7054 | 0.7050 | 0.7054 | 0.7153 | 0.7040 |

The MSE of the Bayesian $\beta$ estimates with respect to the sample size, shown in Figure 4.1 below, indicated the poor performance of the MLE of $\beta$.

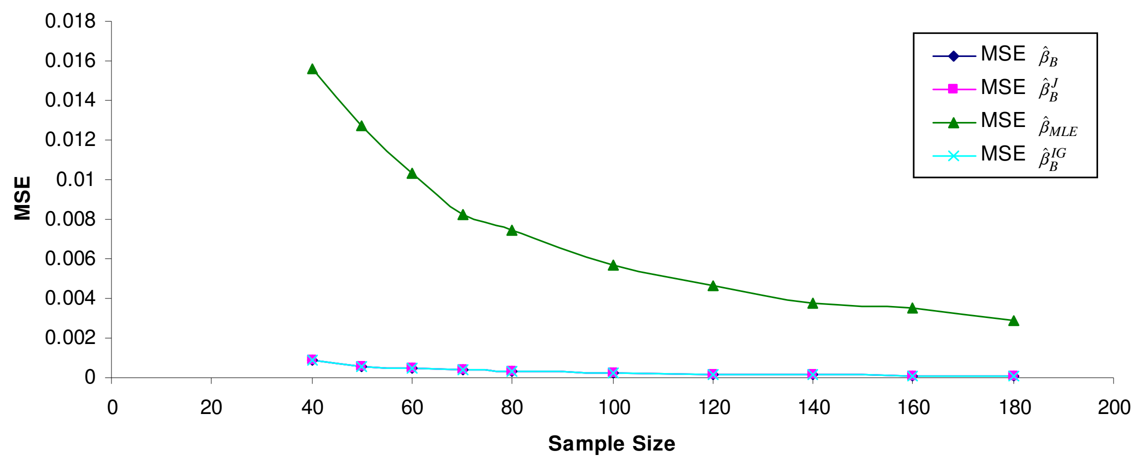



**Figure 4.1 MSE of the MLE and Bayesian estimates of the parameter $\beta$ for different sample sizes, with $\theta$ = 1.7441**

Eliminating the MSE of the $\hat{\beta}_{MLE}$ to more closely observe the MSE of the Bayesian estimates of the key parameter $\beta$, we determined that the Bayesian estimates under Burr, Jeffreys, and inverted gamma priors have good performance, and they tend to converge to the true value beyond the sample size $n$ = 180, as can be observed in Figure 4.2, given below.

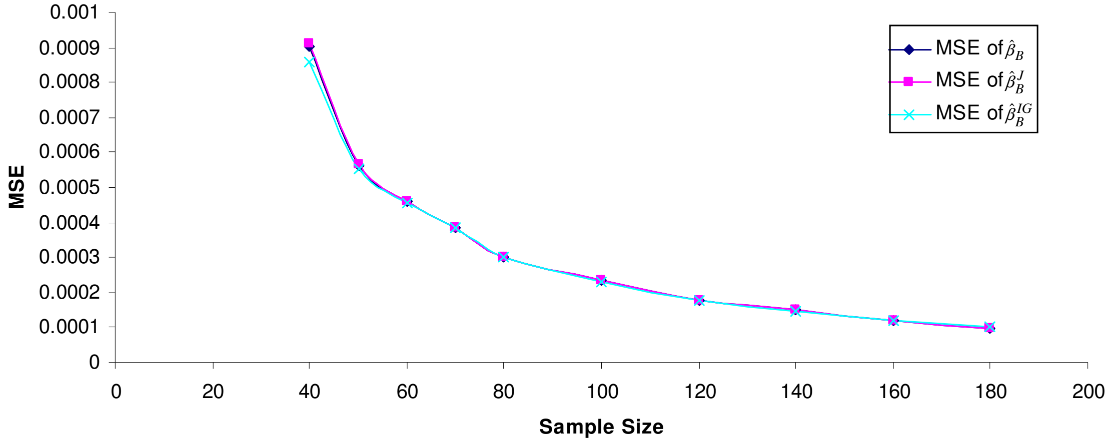



**Figure 4.2 MSE of the Bayesian estimates of the parameter $\beta$ with respect to the sample size, with $\theta = 1.7441$.**

Even when the Bayesian estimates for the parameter $\beta$ are more efficient than those for their counterpart MLEs, the Jeffreys and Burr Bayesian estimates are closer than the inverted gamma Bayesian estimate, which converges more slowly among them. The Bayesian estimate of $\beta$ under the Burr probability distribution as its prior tends to underestimate while the Jeffreys tends to overestimate. This behavior is shown in Figure 4.3. The conditions of the problem where the estimate is involved may influence the selection or preference for one or the other. For example, knowing that a device is needed for a patient's life may influence the selection of the prior when estimating the value of the parameter $\beta$.

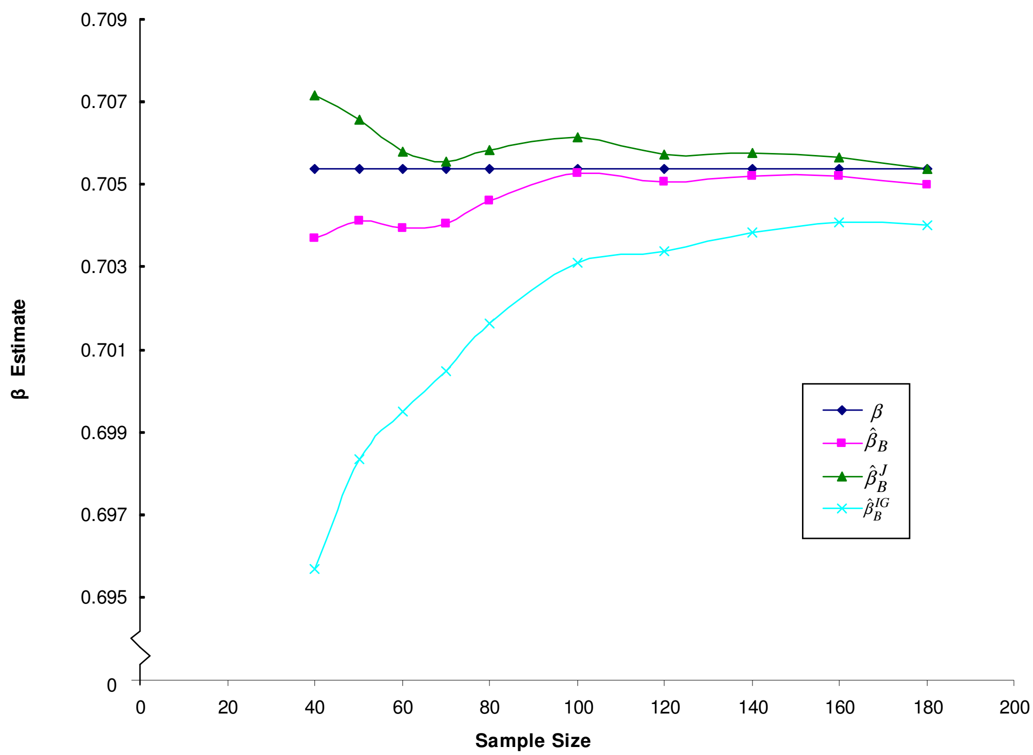



**Figure 4.3 Bayesian estimates for the key parameter *β* with respect to sample size.**

For each sample of size 40, the Bayesian estimates and MLEs of the parameter were calculated when $\theta \in \{0.5, 1.7441, 4\}$. The comparison is based on the MSE averaged over the 500,000 simulated samples. The results are given in Table 4.3. It can be observed that $\hat{\beta}_B$ is superior to $\hat{\beta}_{MLE}$ in estimating $\beta$, with sample size $n$ = 40, while maintaining a consistent behavior for the different values of $\theta$. For the case in which we misleadingly assumed the true probability distribution of the key parameter $\beta$, we obtained that the Jeffreys Bayesian estimate of $\beta$ has the best performance when compared with the inverted gamma Bayesian estimate of $\beta$, indicating that the Bayesian estimate of $\beta$ is sensitive to the choice of its prior.

**Table 4.3 MSE of $\beta$ Bayesian estimates with Burr, Jeffreys, and Inverted Gamma as priors under squared-error loss function, and MSE of MLE of the parameter $\beta$ in a NHPP for 500,000 samples with $n$ = 40 and different values of the parameter $\theta$.**

| $\theta$ | MSE of $\hat{\beta}_B$ | MSE of $\hat{\beta}_B^J$ | MSE of | MSE of $\hat{\beta}_B^{IG}$ |
|---|---|---|---|---|
| 0.5 | 0.001283 | 0.001292 | 0.02536 | 0.007002 |
| 1.7441 | 0.001323 | 0.001335 | 0.02408 | 0.006991 |
| 4 | 0.001356 | 0.001377 | 0.02364 | 0.006941 |

Molinares and Tsokos (2010) proposed an adjusted estimate for the parameter $\theta$, given by

$$\hat{\theta}^* = \frac{t_n}{n^{1/\hat{\beta}_B}} \tag{4.3.1}$$

where $\hat{\beta}_B$ is the Bayesian estimate of the key parameter.

Using the different Bayesian estimates obtained in the computation, we used equation (4.3.1) to calculate the adjusted value of the parameter $\theta$. The results are shown in Table 4.4, where $\hat{\theta}^*$, $\hat{\theta}_J^*$ ,and $\hat{\theta}_{IG}^*$ are the adjusted $\theta$ estimates and $\hat{\theta}^*$ is used with Burr, $\hat{\theta}_J^*$ is used with Jeffreys, and $\hat{\theta}_{IG}^*$ is used with inverted gamma Bayesian $\beta$ estimates**.** We observed the inferior performance of the MLE approach compared with the Bayesian approach. Among the adjusted estimates of $\theta$, those corresponding with the $\beta$ Jeffreys estimate outperformed the other adjusted $\theta$ estimates, being followed very closely by the adjusted $\theta$ using Burr Bayesian $\beta$ estimates.

**Table 4.4 MSE for the MLE and adjusted estimate for the parameter $\theta$ in a NHPP with $\beta$ = 0.7054 for 1,000 samples with different sizes.**

| $n$ | $\beta$ | $\theta$ | $\hat{\theta}^*$ | $\hat{\theta}_J^*$ | $\hat{\theta}_{MLE}$ | $\hat{\theta}_{IG}^*$ | MSEof $\hat{\theta}^*$ | MSE of $\hat{\theta}_J^*$ | MSE of $\hat{\theta}_{MLE}$ | MSE of $\hat{\theta}_{IG}^*$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 40 | 0.7054 | 1.7441 | 1.6686 | 1.7119 | 2.6353 | 1.5728 | 0.0160 | 0.0113 | 6.1623 | 0.0399 |
| 50 | 0.7054 | 1.7441 | 1.6911 | 1.7240 | 2.6344 | 1.6160 | 0.0097 | 0.0073 | 5.8616 | 0.0232 |
| 60 | 0.7054 | 1.7441 | 1.7023 | 1.7285 | 2.5796 | 1.6412 | 0.0073 | 0.0058 | 4.8583 | 0.0161 |
| 70 | 0.7054 | 1.7441 | 1.7023 | 1.7240 | 2.3650 | 1.6511 | 0.0066 | 0.0053 | 3.7872 | 0.0135 |
| 80 | 0.7054 | 1.7441 | 1.7128 | 1.7313 | 2.4245 | 1.6688 | 0.0047 | 0.0039 | 3.8154 | 0.0094 |
| 100 | 0.7054 | 1.7441 | 1.7189 | 1.7330 | 2.2942 | 1.6848 | 0.0034 | 0.0029 | 2.5909 | 0.0063 |
| 120 | 0.7054 | 1.7441 | 1.7228 | 1.7342 | 2.2201 | 1.6953 | 0.0026 | 0.0022 | 2.3545 | 0.0045 |
| 140 | 0.7054 | 1.7441 | 1.7260 | 1.7355 | 2.1518 | 1.7030 | 0.0020 | 0.0017 | 1.8922 | 0.0033 |
| 160 | 0.7054 | 1.7441 | 1.7292 | 1.7373 | 2.1624 | 1.7095 | 0.0016 | 0.0015 | 2.0208 | 0.0026 |
| 180 | 0.7054 | 1.7441 | 1.7321 | 1.7391 | 2.1398 | 1.7146 | 0.0013 | 0.0012 | 1.6152 | 0.0021 |

The $\theta$ MLE behaves poorly in comparison with the adjusted estimates. Its MSE converges slowly. Even for a small sample size ($n$ = 40), the adjusted estimates of $\theta$ tend to converge rapidly. This can be observed in Figures 4.4 and 4.5.

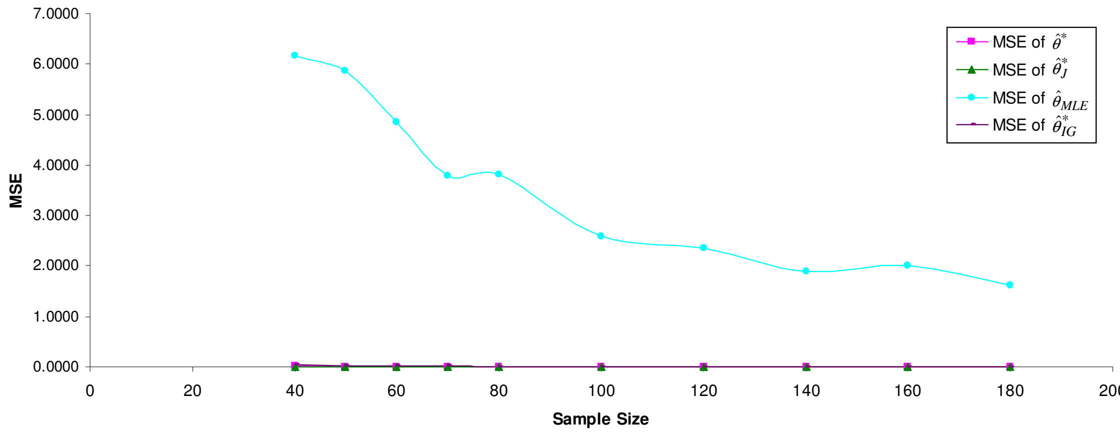



**Figure 4.4 MSE of $\theta$ adjusted estimates with respect to sample size**

Closely examining the behavior of MSE of the adjusted estimate of $\theta$ with respect to the sample size (Figure 4.5), we can observe that superior performance is achieved with Jeffreys prior, which is the most efficient of the priors regardless of sample size.

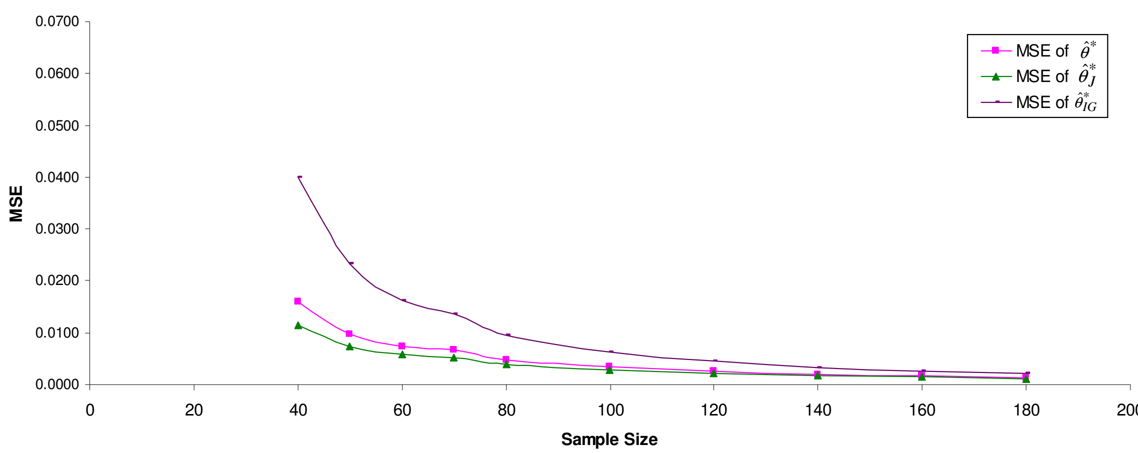



**Figure 4.5 MSE of $\theta$ Estimate versus Sample Size**

The adjusted estimate of $\theta$ with the $\beta$ Jeffreys Bayesian estimate tends to be closer to the true value, and almost as efficient as the adjusted estimates when the Burr Bayesian $\beta$ estimate is used. In addition, it converges rapidly with sample size as small as $n = 40$ (Figure 4.5).

## 4.5 Conclusions

In the present study, we considered that the key parameter $\beta$ in the intensity function in the NHPP could behave as a random variable, and we assumed its prior probabilistic characterizations as the Burr type XII probability distribution, Jeffreys, and the inverted gamma along with the squared-error loss function. We developed the analytical structure of the Bayesian $\beta$ estimate of the PLP subject to the above assumptions.

We used numerical simulation to illustrate the sensitivity to the selection of the prior. On the basis of the Monte Carlo simulation, if the true prior distribution was misleadingly chosen among the studied priors, the better selection would be Jeffreys, indicating that the Bayesian estimate of $\beta$ is sensitive to the choice of the prior. However, for over 500,000 samples with a small sample size, a lower MSE would be found if the assumed prior was a Burr probability density when the key parameter $\beta$ actually follows a Burr distribution, as expected, although the difference between selecting Burr or Jeffreys could be considered negligible.

In our study, it was shown that the Bayesian estimates are superior to the MLEs of $\beta$. This is reflected even with a small sample size. The MSE of the Bayesian $\beta$ estimates with respect to the sample size shows the poor performance of the MLE. The Burr Bayesian estimate of $\beta$ tends to underestimate while the Jeffreys tends to overestimate. The Burr, Jeffreys, and inverted gamma Bayesian $\beta$ estimates gave a good performance, and they tend to converge to the true value beyond the sample size $n$ = 180. Even when the Bayesian estimates for the key parameter $\beta$ are more efficient than their counterpart MLEs, those corresponding with the Jeffreys and Burr priors are closer than that the corresponding with the inverted gamma prior, which converges more slowly among them.

The adjusted MLE of $\theta$ produced a better estimate under the mentioned Bayesian influence. The MLE of $\theta$ had inferior performance in the case of using the MLE formula to estimate the parameter $\theta$. Moreover, the MSE of the parameter $\theta$ shows slow convergence. In the case of assuming Burr or Jeffreys priors, we can see that both tend to converge to the true value for small sample sizes.

Among the adjusted estimates of $\theta$, those corresponding to the $\beta$ Jeffreys Bayesian estimate outperform the other adjusted $\theta$ estimates, being followed very closely by the adjusted $\theta$ using $\beta$ Burr Bayesian estimates. The adjusted $\theta$ estimate when the $\beta$ Jeffreys Bayesian estimate is used tends to be closer to the true value, and almost equally efficient as the adjusted estimates of $\theta$ when the $\beta$ Burr Bayesian estimate is used. In addition, it converges rapidly for a small sample size. Under close examination of the behavior of the MSE of the adjusted estimate of $\theta$ with respect to the sample size, we observed that superior performance is achieved with Jeffreys prior, which is the most efficient regardless the sample size.

The main contribution of this study that is expected to have a direct impact in future research on Bayesian and parametric approaches to reliability analysis in complex systems problems is the proposed estimate for one of the parameters in the PLP with better performance under the Bayesian influence than its maximum likelihood counterpart given by

$$\hat{\theta}^* = \frac{t_n}{n^{1/\hat{\beta}_B}}$$

where $\hat{\beta}_B$ is the Bayesian estimate of the key parameter in the intensity function, $t_n$ is the largest TTF, and $n$ is the sample size.

# CHAPTER 5 PARAMETRIC AND BAYESIAN SURVIVAL ANALYSIS FOR BREAST CANCER

In the present study, we investigated the applicability of performing Bayesian analysis for survival times of breast cancer patients assuming an informative prior based on the variability exhibited by one parameter of the Johnson $S_B$ distribution. In addition, we compared the Bayesian estimates of the survival and hazard functions with respect to their parametric counterparts.

The chapter is divided into five main sections. In the first section, we present a brief theoretical and literature review for the four parameter Johnson $S_B$ model. The next section describes a parametric procedure to obtain the approximated estimates of the parameter inherited within the subject model. The third section justifies the applicability of a Bayesian analysis to the survival time for breast cancer data. A sequence of 40 samples were extracted from a large database and obtained the four parameter estimates for each sequence. A comparison of the approximated estimates of the parameters behaves as a random variable rather than a being a fixed value. Thus, we proceeded with a Bayesian analysis. We utilized the 40 estimates of the subject parameter to identify their probability distribution as the prior. Then, we obtained the analytical form of the Bayesian estimate of the parameter, the survival and hazard functions.

In the fourth section, we proceeded with the analysis through Monte Carlo simulation. Random samples from the Johnson $S_B$ model were generated. We compared the estimates of the parameter obtained applying the parametric and Bayesian approaches using the MSE as the criteria. In addition, we compared the estimates of the survival and hazard functions using the relative efficiency as the measure of robustness. The fifth section summarizes the findings of the study.

We performed our Bayesian analysis and compared results with the parametric approach, assuming the data are independent and identically distributed. We used the survival time of breast cancer patients provided by the SEER database and performed the Kolmogorov-Smirnov (K-S) GOF test. The aforementioned methods were used to answer the following questions:

1. Is the Bayesian analysis applicable to the survival time of breast cancer data?
2. Is the Bayesian approach applicable to this subject area?
3. Do the Bayesian estimates of the survival and hazard function perform better than their parametric counterparts?

## 5.1 Parametric Survival Analysis

Cancer of the breast is ranked as the second highest cause of cancer death among women, without considering nonmelanoma skin cancer. According to the Surveillance, Epidemiology, and End Results (SEER) database of the US National Cancer Institute from 2003 through 2007, the median age at diagnosis for women with breast cancer was 61 years. On the basis of diagnosis rates from 2005 through 2007, an estimated 12.15% of women born today will be diagnosed with breast cancer at some point during their lifetime.

The American Cancer Society estimated that 209,060 new cases of invasive breast cancer were diagnosed and that 40,230 women died of breast cancer in the United States in 2010. In addition, approximately 54,010 women were diagnosed with carcinoma *in situ* of the breast during the same year. The incidence of breast cancer has increased steadily in the United States over the past few decades, but breast cancer mortality seems to be declining, suggesting a benefit from early detection and more effective treatment.

Most breast cancers occur in women over the age of 50, and the risk is especially high for women over age 60. Detection of breast cancer at an early stage, when the disease is less severe, provides a greater chance of survival. The overall 5-year relative survival is a measure of net survival that is calculated by comparing the observed overall survival with the expected survival from a comparable set of people who do not have cancer to measure the excess of mortality that is associated with a cancer diagnosis. According to data from 1999 through 2006 obtained from 17 SEER geographic areas, the overall 5-year relative survival was 89.0%. In addition to serving as a predictor for the probability of survival, disease severity is also of critical importance in determining an individual's breast cancer treatment (The North Carolina Comprehensive Breast Cancer Control Coalition, 1995). Studying the survival rate helps, for instance, in indicating the efficacy of new treatments. Therefore, this study is applicable to the medical profession and breast cancer patients.

The present study is based on the survival time of breast cancer extracted from the SEER database. The fit of the survival times correspond to a four parameter Johnson $S_B$ distribution. The four parameter Johnson $S_B$ distribution is one of the three types of transformations to normally distributed variables with a range of variation bounded at both extremities with pdf defined as

$$f(x;\delta,\gamma,\xi,\lambda)=\frac{\delta}{\lambda y(1-y)\sqrt{2\pi}}e^{-\frac{1}{2}\left[\gamma+\delta\log\left(\frac{y}{1-y}\right)\right]^2}, \quad \lambda>0, \delta>0, \xi<x<\xi+\lambda \tag{5.1.1}$$

where $y=\dfrac{x-\xi}{\lambda}$, and the transformation $z=\gamma+\delta\log\left(\dfrac{y}{1-y}\right)$ is a standard normal variable (Johnson, 1949).

The corresponding CDF for the 4-parameter Johnson $S_B$ distribution is given by

$$F(x)=\Phi\left(\gamma+\delta\log\left(\frac{x-\xi}{\xi+\lambda-x}\right)\right), \tag{5.1.2}$$

where $\Phi(\cdot)$ is the CDF of a standard normal distribution. The survival function and the hazard function are defined by

$$\begin{aligned} S(x)&=1-F(x)\\ &=1-\Phi\left(\gamma+\delta\log\left(\frac{x-\xi}{\xi+\lambda-x}\right)\right), \end{aligned} \tag{5.1.3}$$

and

$$h(x) = \frac{f(x)}{S(x)}$$

$$= \frac{\delta \cdot \lambda \cdot e^{\frac{-1}{2}\left[\gamma + \delta \log\left(\frac{x-\xi}{\lambda+\xi-x}\right)\right]^2}}{(x-\xi)(\lambda+\xi-x)\sqrt{2\pi}\left[1-\Phi\left(\gamma+\delta\log\left(\frac{x-\xi}{\xi+\lambda-x}\right)\right)\right]}, \quad \xi < x < \xi + \lambda, \tag{5.1.4}$$

respectively.

Because of its flexibility, the Johnson $S_B$ distribution has been used to model in areas such as forestry (Amaro, Reed, & Soares, 2003; Jerez, Dean, Cao, and Roberts, 2005; Fonseca, Marques, and Parresol, 2009), airspace simulation (McGovern & Kalish , 2009), reliability (Takaragi, Sasaki, & Shingai, 1982; Takaragi, Sasaki, & Shingai, 1985), epidemiology (Flynn, 2005), quality control (Castagliola, Celano, & Fichera, 2010), agriculture (Zhang, & Wang, 2010), and medical science (Ness, Holmes, Klein, & Dittus, 2000; Roberts, Wang, Klein, Ness, & Dittus, 2007; Mage & Donner, 2009), among others. To the best of the authors' knowledge, this kind of distribution has not been applied to model breast cancer data.

## 5.2 Parametric Estimation

Even when the Johnson $S_B$ is known by its flexibility, the estimation process for the system becomes difficult without considering the $\xi$ and $\lambda$ as known, especially when the four parameters have to be estimated. In the case when the parameters $\xi$ and $\lambda$ are known, the maximum likelihood approach leads to the following estimates, for $\gamma$ and $\delta$ respectively,

$$\hat{\gamma}_{MLE} = -\frac{\bar{f}}{s_f}, \ \hat{\delta}_{MLE} = \frac{1}{s_f} \tag{5.2.1}$$

where $\bar{f}$ is the sample mean of the transformations $f(x) = \dfrac{x - \xi}{\xi + \lambda - x}$ of the realizations of *X*, and $s_f^{\ 2}$ is the second central sample moment of these transformed values of *X* (Johnson, 1949).

Several methods have been developed to focus on the estimation of these parameters, i.e. the algorithm to estimate $\delta$ and $\gamma$ presented by Hill, Hill, and Holder (1976). The variation on this algorithm for estimating the four parameters of the $S_B$ distribution is based upon the method-of-moments outlined by Johnson and Kitchen (1971) and presented by Flynn (2006), and also on the exploration of the $S_B$ distribution (Flynn, 2004), the Bayesian estimation of the four parameters assuming non-informative priors (Tsionas, 2001), the estimation on sample percentiles reported by Slifker and Shapiro (1980) and Mage (1980), and the estimation of δ and γ using a similar procedure of the percentile methods presented by Wheeler (1980). The method of maximum likelihood, in general, has not being useful when all four parameters have to be estimated (Lambert, 1970) and may produce preposterous values for the estimates if the sample is small or if the skewness of the distribution is considerable compared with the method based on percentiles of a sample (Siekerski, 1992 & Vroon, 1981). The reason for this might be extremely fat tails of the likelihood (Tsionas, 2001).

Flynn (2006) pointed out that the Wheeler quantile procedure may provide performance superior to that of the percentile method. However, Wheeler (1980, p.727) indicated that the estimates of $\delta$ and $\gamma$ in practice "should provide good starting values for accurate iterative schemes". To estimate the parameters $\delta$ and $\gamma$, Wheeler used the relationships

$$\hat{\delta} = \frac{z_n}{2\log(b)} \text{ and } \hat{\gamma} = -\hat{\delta}\log(a) \tag{5.2.2}$$

where

$$a = \frac{t-b^2}{1-t\cdot b^2}, \qquad b = \frac{1}{2}t_b + \left[\left(\frac{1}{2}t_b\right)^2 - 1\right]^2, \quad t = \frac{x_n - x_0}{x_0 - x_p},$$

and

$$t_b = \frac{(x_m - x_0)(x_n - x_p)}{(x_n - x_m)(x_0 - x_p)},$$

with $x_p, x_0, x_m$, and $x_n$ substituted by the sample quantiles at $-z_n, 0, \frac{z_n}{2}$, and $z_n$, respectively. $z_n$ is a z normal standard value determined by $\Phi(z_n) = \frac{n-\frac{1}{2}}{N}$ for an integer $n$ at most, equal to the sample size $n$.

Since the $t_b$ relationship is symmetric with respect to the tails in that if $x_n$ and $x_m$ are interchanged with $x_p$ and $x_k$ in the expression for $t_b$, where $x_k$ is the sample quantile at $-\frac{z_n}{2}$, the result $t_b = \frac{1+b^2}{b^2}$ is still obtained. Wheeler indicated

$$\frac{1}{2}\left(\frac{(x_m - x_0)(x_n - x_p)}{(x_n - x_m)(x_0 - x_p)} + \frac{(x_k - x_0)(x_p - x_n)}{(x_p - x_k)(x_0 - x_n)}\right)$$

as the value for $t_b$ may in practice be better in estimating $\delta$. This value for $t_b$ was used in our computations.

Wheeler pointed out that, once the parameters $\gamma$ and $\delta$ are estimated, the parameters $\xi$ and $\lambda$ can be determined by the usually adequate simple linear regression. In this approach, the linear regression is given by $x = \xi + \lambda \cdot \dfrac{w}{1+w}$, where $w = \exp\left(\dfrac{z-\hat{\gamma}}{\hat{\delta}}\right)$ and $z$ corresponds to the sample quantile of $x$. Once we obtained the estimate for the parameter, we obtained the analytical form of the estimates of the survival and hazard functions by substituting the parameter estimate.

With the estimate $\hat{\lambda}$ of $\lambda$ under the Wheeler approach, the analytical structures of the parametric estimates of the survival and hazard functions are analyzed substituting $\hat{\lambda}$ in equations (5.1.3) and (5.1.4):

$$\hat{S}(x) = 1 - \Phi\left(\gamma + \delta \log\left(\frac{x-\xi}{\xi+\hat{\lambda}-x}\right)\right), \tag{5.2.3}$$

and

$$\hat{h}(x) = \frac{\delta \cdot \hat{\lambda} \cdot e^{\frac{-1}{2}\left[\gamma+\delta\log\left(\frac{x-\xi}{\hat{\lambda}+\xi-x}\right)\right]^2}}{(x-\xi)(\hat{\lambda}+\xi-x)\sqrt{2\pi}\left[1-\Phi\left(\gamma+\delta\log\left(\frac{x-\xi}{\xi+\hat{\lambda}-x}\right)\right)\right]}, \quad \xi < x < \xi + \hat{\lambda}, \tag{5.2.4}$$

respectively.

## 5.3 Justification for Bayesian Analysis

We used the Wheeler procedure in order to get an idea of the variability in the estimation of the four parameters. We took 40 sub-samples of size *n*=5,000 of survival times for breast cancer patients provided by the SEER database of the US National Cancer Institute. For each sub-sample, we performed a GOF test, which indicated that they followed the Johnson $S_B$ distribution. The basic statistics of the estimates for the parameters, analyzed with the Wheeler procedure, are in Table 5.1.

**Table 5.1 Variance for the estimates of the four parameters of the Johnson $S_B$ for 40 sub-samples with 5,000 survival times for breast cancer patients**

| | $\hat{\gamma}$ | $\hat{\delta}$ | $\hat{\xi}$ | $\hat{\lambda}$ |
|---|---|---|---|---|
| Variance | 0.00068 | 0.00065 | 0.87746 | 3.10592 |

From this table, we observed that the parameter $\lambda$ exhibits variability, implying that it is no longer a fixed value but behaves as a random variable. Considering the parameter $\lambda$ as a random variable, we proceeded to perform a GOF test. The result of the test showed a Nakagami distribution. The Nakagami distribution has pdf defined as

$$p(\lambda) = \frac{2}{\Gamma(m)}\left(\frac{m}{\omega}\right)^{m} \lambda^{2m-1} \exp\left\{-\frac{m}{\omega}\lambda^2\right\}, \qquad m \geq 0.5,\ \omega > 0,\ \lambda \geq 0 . \qquad (5.3.1)$$

Thus, for the 40 estimates of $\lambda$ we have the following analytical form of the estimate of $p(\lambda)$ using the estimates of the parameter inherent in (5.3.1)

$$\hat{p}(\lambda) = \frac{2}{\Gamma(\hat{m})}\left(\frac{\hat{m}}{\hat{\omega}}\right)^{\hat{m}} \lambda^{2\hat{m}-1} \exp\left\{-\frac{\hat{m}}{\hat{\omega}}\lambda^2\right\}.$$

The CDF of the probability density (5.3.1) is given by

$$F(\lambda) = \frac{1}{\Gamma(m)} \gamma\left(m, \frac{m}{\omega}\lambda^2\right), \quad m \geq 0.5, \omega > 0, \lambda \geq 0,$$

in terms of the incomplete gamma function, which is defined by

$$\gamma(n, z) = \int_0^z t^{n-1} \exp\{-t\} dt.$$

The expected value of the pdf (5.3.1) is given by

$$\frac{\Gamma\left(m + \frac{1}{2}\right)}{\Gamma(m)} \left(\frac{\omega}{m}\right)^{\frac{1}{2}}.$$

## 5.4 Bayesian Survival Analysis

We have identified the failure probability distribution to be the 4-parameter Johnson $S_B$ probability distribution and identified the prior probability distribution of $\lambda$ to be the 2-paramenter Nakagami probability distribution. Thus, we proceed to develop the Bayesian survival analysis.

With the assumption that the survival times $X_1, X_2, X_3, ..., X_n$ are independent and identically distributed following the Johnson $S_B$ probability distribution (5.1.1), the likelihood function is given by

$$L(\vec{x}; \lambda) = L(\vec{x}; \lambda, \delta, \gamma, \xi) = \left(\frac{\delta}{\sqrt{2\pi}}\right)^n e^{\frac{-1}{2}\sum_{i=1}^{n}\left[\gamma + \delta \log\left(\frac{x_i - \xi}{\lambda + \xi - x_i}\right)\right]^2} \prod_{i=1}^{n} \frac{\lambda}{(x_i - \xi)(\lambda + \xi - x_i)} \qquad (5.4.1)$$

where $\vec{x} = (x_1, x_2, x_3, ..., x_n)$ represents the realizations of $X_1, X_2, X_3, ..., X_n$. By invoking the Bayes Theorem and using the pdf (5.3.1) as the prior for $\lambda$, the posterior density $h(\lambda \mid \vec{x})$ of $\lambda$ is given by

$$h(\lambda;\vec{x}) = \frac{L(\vec{x};\lambda)\,p(\lambda)}{\int_\Lambda L(\vec{x};\lambda)\,p(\lambda)\,d\lambda}$$

$$= \frac{\lambda^{n+2m-1}\left(\frac{\delta}{\sqrt{2\pi}}\right)^n \frac{2}{\Gamma(m)}\left(\frac{m}{\omega}\right)^m e^{\frac{-m}{\omega}\lambda^2-\frac{1}{2}\sum_{i=1}^n\left[\gamma+\delta\log\left(\frac{x_i-\xi}{\lambda+\xi-x_i}\right)\right]^2}\prod_{i=1}^n\frac{1}{(x_i-\xi)(\lambda+\xi-x_i)}}{\int_\Lambda \lambda^{n+2m-1}\left(\frac{\delta}{\sqrt{2\pi}}\right)^n \frac{2}{\Gamma(m)}\left(\frac{m}{\omega}\right)^m e^{\frac{-m}{\omega}\lambda^2-\frac{1}{2}\sum_{i=1}^n\left[\gamma+\delta\log\left(\frac{x_i-\xi}{\lambda+\xi-x_i}\right)\right]^2}\prod_{i=1}^n\frac{1}{(x_i-\xi)(\lambda+\xi-x_i)}d\lambda} \quad (5.4.2)$$

where $\Lambda$ is the parameter space for $\lambda$. Then, the Bayesian estimate of $\lambda$, under the squared-error loss function is given by

$$\hat{\lambda}_B = \frac{\int_\Lambda \lambda L(\vec{x};\lambda)\,p(\lambda)\,d\lambda}{\int_\Lambda L(\vec{x};\lambda)\,p(\lambda)\,d\lambda}$$

$$= \frac{\int_0^\infty \lambda^{n+2m}\left(\frac{\delta}{\sqrt{2\pi}}\right)^n \frac{2}{\Gamma(m)}\left(\frac{m}{\omega}\right)^m e^{\frac{-m}{\omega}\lambda^2-\frac{1}{2}\sum_{i=1}^n\left[\gamma+\delta\log\left(\frac{x_i-\xi}{\lambda+\xi-x_i}\right)\right]^2}\prod_{i=1}^n\frac{1}{(x_i-\xi)(\lambda+\xi-x_i)}d\lambda}{\int_0^\infty \lambda^{n+2m-1}\left(\frac{\delta}{\sqrt{2\pi}}\right)^n \frac{2}{\Gamma(m)}\left(\frac{m}{\omega}\right)^m e^{\frac{-m}{\omega}\lambda^2-\frac{1}{2}\sum_{i=1}^n\left[\gamma+\delta\log\left(\frac{x_i-\xi}{\lambda+\xi-x_i}\right)\right]^2}\prod_{i=1}^n\frac{1}{(x_i-\xi)(\lambda+\xi-x_i)}d\lambda} \quad (5.4.3)$$

Therefore, the Bayesian estimates for the survival and hazard functions, substituting $\hat{\lambda}_B$ in equations (5.1.3) and (5.1.4), are given by

$$\hat{S}_B(x) = 1-\Phi\left(\gamma+\delta\log\left(\frac{x-\xi}{\xi+\hat{\lambda}_B-x}\right)\right) \quad (5.4.4)$$

and

$$\hat{h}_B(x) = \frac{\delta \cdot \hat{\lambda}_B \cdot e^{-\frac{1}{2}\left[\gamma + \delta \log\left(\frac{x-\xi}{\hat{\lambda}_B + \xi - x}\right)\right]^2}}{(x-\xi)(\hat{\lambda}_B + \xi - x)\sqrt{2\pi}\left[1 - \Phi\left(\gamma + \delta \log\left(\frac{x-\xi}{\hat{\lambda}_B + \xi - x}\right)\right)\right]}, \qquad (5.4.5)$$

respectively, where $\hat{\lambda}_B$ is given by equation (5.4.3).

## 5.5 Numerical Comparison

### 5.5.1 Comparison of the Bayesian and Parametric Estimates of the Parameter $\lambda$

We performed a simulation study, for samples of size varying from 30 to 300, of the estimates of $\delta$ and $\gamma$ using the Wheeler procedure to compare $z_n$ corresponding to the 95th and 99th percentiles in order to obtain more accurate estimates. The best estimates were obtained when the 99th percentile was used.

In order to develop an analysis of the estimates of $\lambda$ under the parametric and Bayesian approaches, we simulated samples from the Johnson $S_B$ distribution with $\xi$=0 since the survival times are nonnegatives. The values used for $\gamma$ and $\delta$ were the average of the estimates of these parameters obtained from the 40 SEER samples with the use of the Wheeler procedure. For a true value for $\lambda$, different samples of a given sample size $n$ were simulated. Table 5.2 shows the results averaging over 1,000 simulations.

**Table 5.2 Estimates of the parameter $\lambda$ and their MSE $\xi$ = 0, $\gamma$ = 0.048, $\delta$ = 0.76 based on 1,000 simulated samples of size *n***

| $n$ | $\lambda$ | $\hat{\lambda}$ | $\hat{\lambda}_B$ | MSE( $\hat{\lambda}$ ) | MSE( $\hat{\lambda}_B$ ) |
|---|---|---|---|---|---|
| 10 | 86.0315 | 80.4642 | 85.6026 | 294.3800 | 0.1950 |
| 20 | 86.0315 | 82.4348 | 85.6051 | 100.0006 | 0.2026 |
| 30 | 86.0315 | 83.9228 | 85.6165 | 115.9455 | 0.2042 |
| 40 | 86.0315 | 84.3001 | 85.6067 | 52.1192 | 0.2163 |
| 50 | 86.0315 | 84.0397 | 85.6133 | 34.1327 | 0.2180 |
| 60 | 86.0315 | 84.6173 | 85.6250 | 26.5543 | 0.2177 |
| 70 | 86.0315 | 84.6157 | 85.6226 | 24.6979 | 0.2279 |
| 100 | 86.0315 | 85.0068 | 85.6674 | 16.2845 | 0.2156 |
| 200 | 86.0315 | 85.4841 | 85.7156 | 7.6730 | 0.2135 |
| 300 | 86.0315 | 85.7720 | 85.7523 | 5.2270 | 0.2028 |
| 500 | 86.0315 | 85.8189995 | 85.83374 | 3.0925138 | 0.1812803 |

Table 5.2 shows that, even with a small sample size, the Bayesian estimate of $\lambda$ performs better than the parametric estimate of $\lambda$. Table 5.2 and Figure 5.1 show that the convergence occurs for a sample size beyond $n$ = 200 for the parametric estimate of $\lambda$.

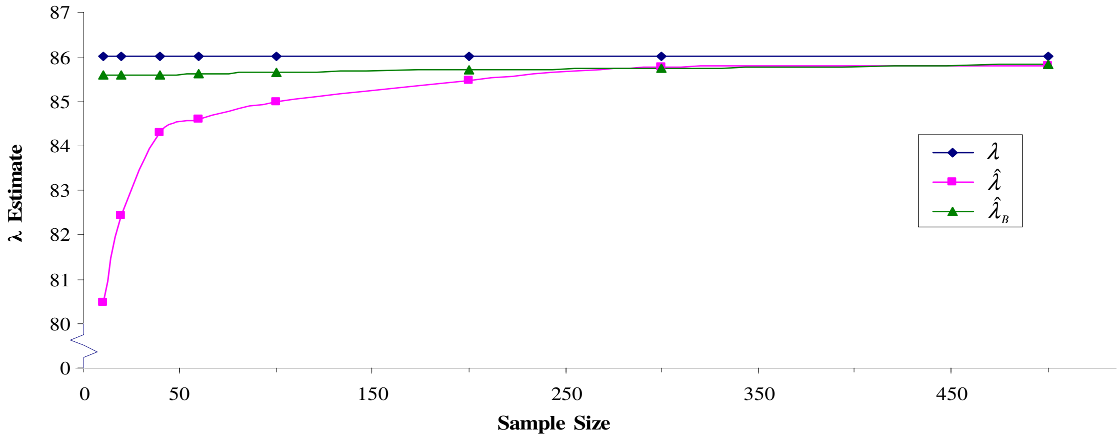



**Figure 5.1 Comparison of the true value of $\lambda$ and its estimates based on 1,000 simulated samples, with $\xi$ = 0, $\gamma$ = 0.048, $\delta$ = 0.76, with respect to different sample sizes**

Figure 5.2 displays the comparison of the behavior for the MSE for both estimates of $\lambda$. The figure shows that a convergence is reached beyond a sample size $n$=200, as suggested in Figure 5.1.

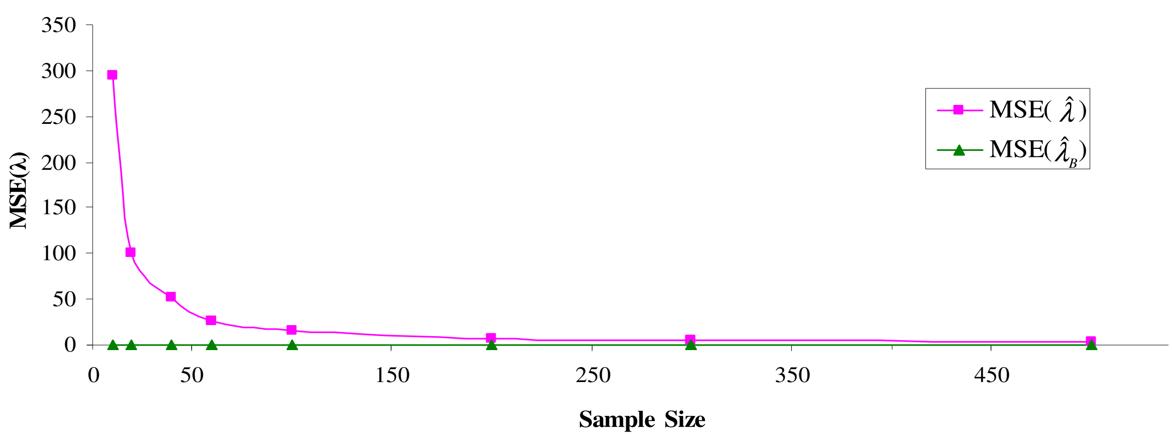



**Figure 5.2 Comparison of MSE of the estimates of $\lambda$, with $\xi$ = 0, $\gamma$ = 0.048, $\delta$=0.76, with respect to different sample sizes**

Figure 5.3 illustrates a closer view of the behavior of the MSE of the estimates of $\lambda$ for small values of the MSE, showing that the convergence for the parametric estimate of $\lambda$ has not been reached at sample size $n$ = 500.

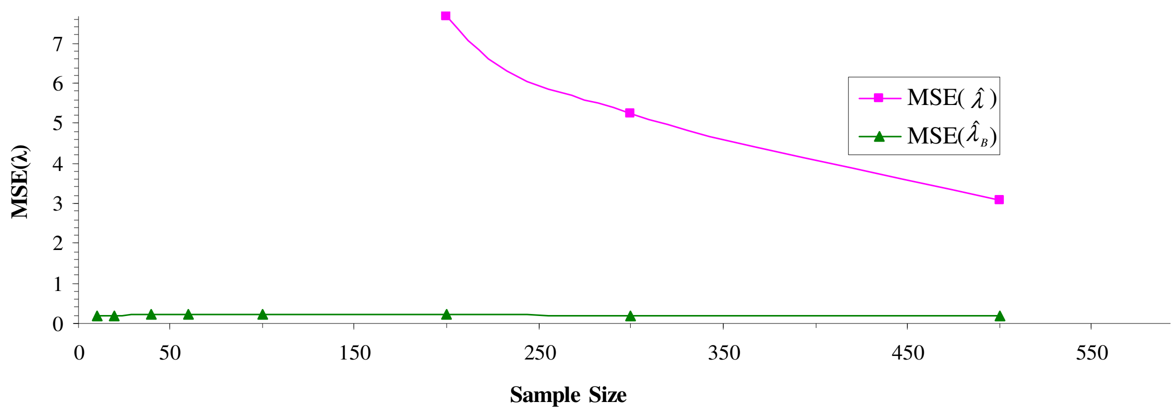



**Figure 5.3 Comparison of MSE of the estimates of $\lambda$, with $\xi$ = 0, $\gamma$ = 0.048, $\delta$ = 0.76, with respect to different sample sizes for small values of the MSE**

To analyze the overall behavior of the estimation of the parameter $\lambda$, we took a random value for $\lambda$ generated from its informative prior, and different samples of a given sample size $n$ were simulated from the Johnson $S_B$ distribution. Table 5.3 shows the results of the MSE of the parameter $\lambda$, averaging over 5,000 simulations for $n$ = 40.

**Table 5.3 MSE of the parametric and Bayesian estimates of $\lambda$ over 5,000 simulations of samples with $n$ = 40, $\gamma$ = 0.048, $\delta$ = 0.76, and $\xi$ = 0**

| MSE( $\hat{\lambda}$ ) | MSE( $\hat{\lambda}_B$ ) |
|---|---|
| 48.7780 | 0.4217 |

When comparing the values of the MSEs with the Bayesian estimate of λ and the parametric estimate $\hat{\lambda}$ , we observed that the former has better performance than the latter.

### 5.5.2 Comparison of the Bayesian and Parametric Estimate of the Survival and Hazard Functions

In order to compare the parametric and Bayesian estimates of the survival and hazard functions, we used the same values for the parameter $\delta$, $\gamma$ and $\xi$ considered in the estimation of the parameter $\lambda$. The value for $\lambda$ used to obtain the Bayesian estimate of hazard and survival function is the Bayesian estimate of $\lambda$, $\hat{\lambda}_B$. The value of $\lambda$ used to obtain the parametric estimate of hazard and survival function is the parametric estimate of $\lambda$, $\hat{\lambda}$ , which is obtained applying the Wheeler procedure.

To measure the robustness of $\hat{S}(x)$ with respect to $\hat{S}_B(x)$, we calculated the RE of the estimate $\hat{S}(x)$ compared with the estimate $\hat{S}_B(x)$ defined as

$$RE(\hat{S},\hat{S}_B) = \frac{\int_{-\infty}^{\infty}\left[S(x)-\hat{S}(x)\right]^2 dx}{\int_{-\infty}^{\infty}\left[S(x)-\hat{S}_B(x)\right]^2 dx}$$

If RE = 1, $\hat{S}(x)$ and $\hat{S}_B(x)$ will be interpreted as equally effective. If RE < 1, $\hat{S}(x)$ is more efficient than $\hat{S}_B(x)$, contrary to the case in which RE > 1 and, thus, $\hat{S}_B(x)$ is more efficient than $\hat{S}(x)$. For the different samples size, the RE was greater than 1. In particular, for $n$ = 40, RE of $\hat{S}(x)$ compared with $\hat{S}_B(x)$ was equal to 16.7271. This result means that $\hat{S}_B(x)$ is more efficient than $\hat{S}(x)$ in accordance with the superior behavior of the Bayesian estimate of the parameter $\lambda$.

Figure 5.4 shows the comparison of the estimates of the survival function with respect to the true survival function for $n$ = 40 and $\lambda$ = 86.0315. In this case, both of the estimates are close to true survival function.

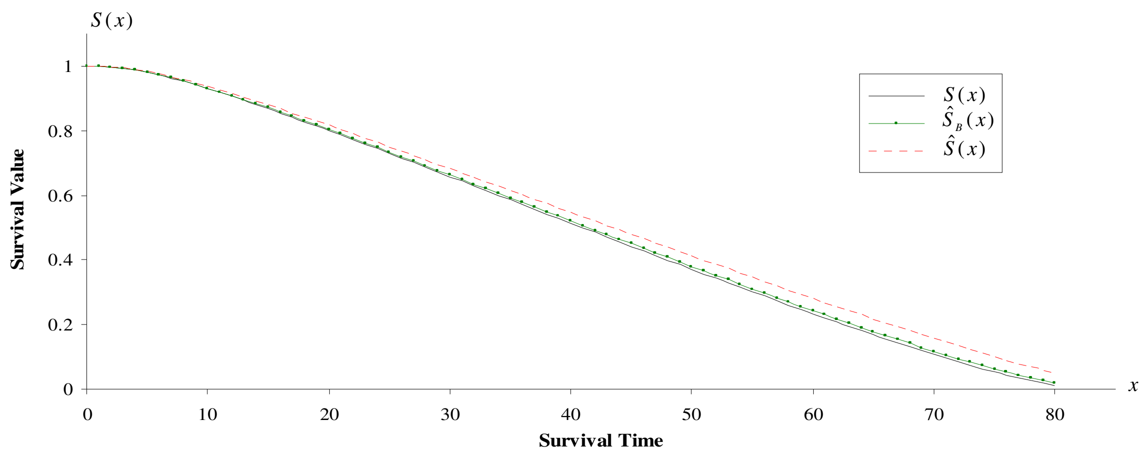



**Figure 5.4 Comparison of the survival function estimates for $n$ = 40, with $\lambda$ = 86.0315, $\xi$ = 0, $\gamma$ = 0.048, and $\delta$ = 0.76, with respect to the true survival function.**

Even when the estimates of the survival functions are close to the true survival function, Figures 5.5 to 5.7 show that the Bayesian survival function is closer than the parametric survival function estimate.

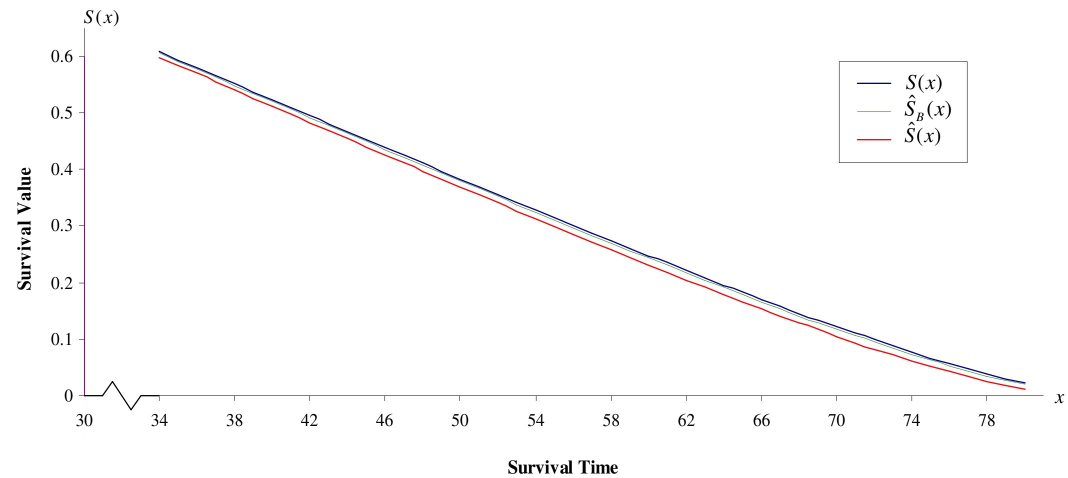



**Figure 5.5 Comparison of the survival function estimates with respect to the true survival function for $n$ = 40 and $\lambda$ = 86.0315 in the survival time interval [34, 80]**

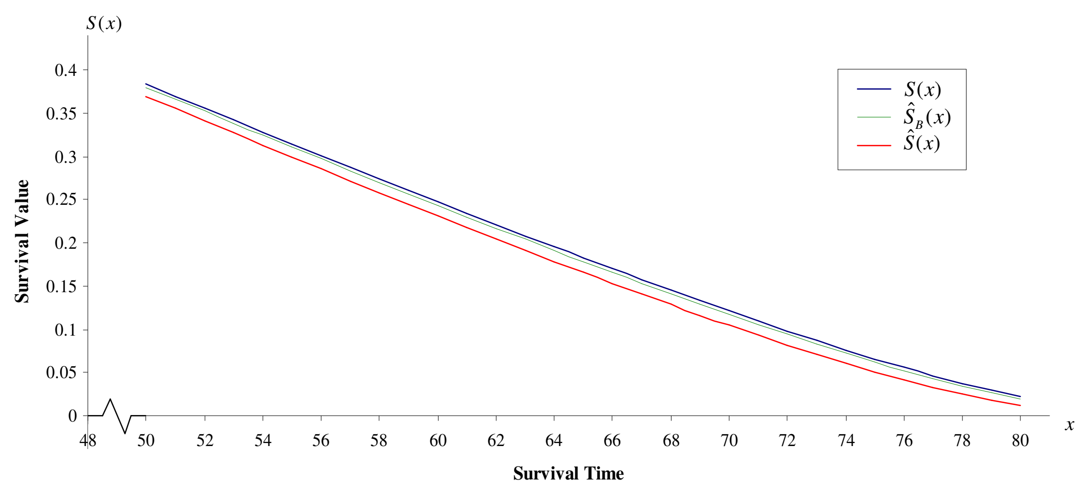



**Figure 5.6 Comparison of the survival function estimates with respect to the true survival function for $n$ = 40 and $\lambda$ = 86.0315 in the survival time interval [50, 80]**

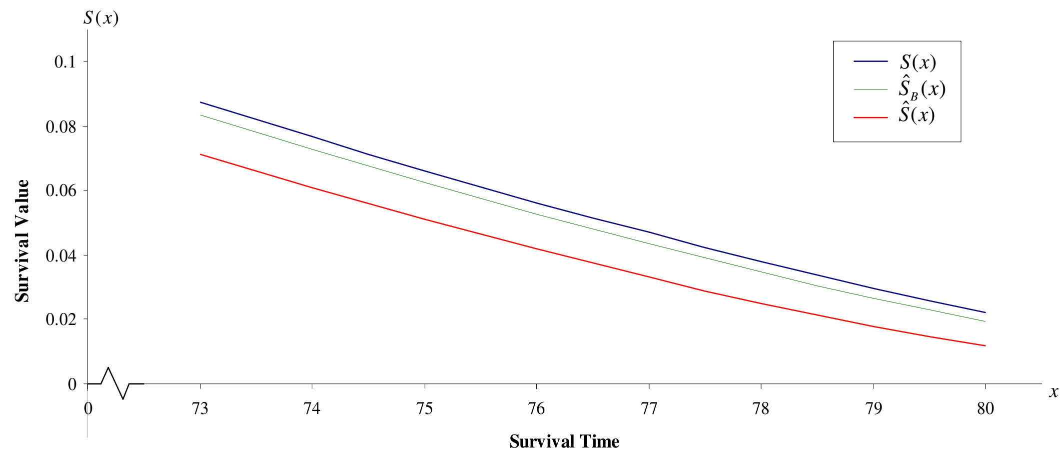



**Figure 5.7 Comparison of the survival function estimates with respect to the true survival function for $n$ = 40 and $\lambda$ = 86.0315 in the survival time range [73, 80]**

We compared the hazard function estimates with respect to the true hazard function for $n$ = 40 and $\lambda$ = 86.0315. Figure 5.8 shows that for a survival time greater than 60 the Bayesian hazard function becomes closer to the true hazard function.

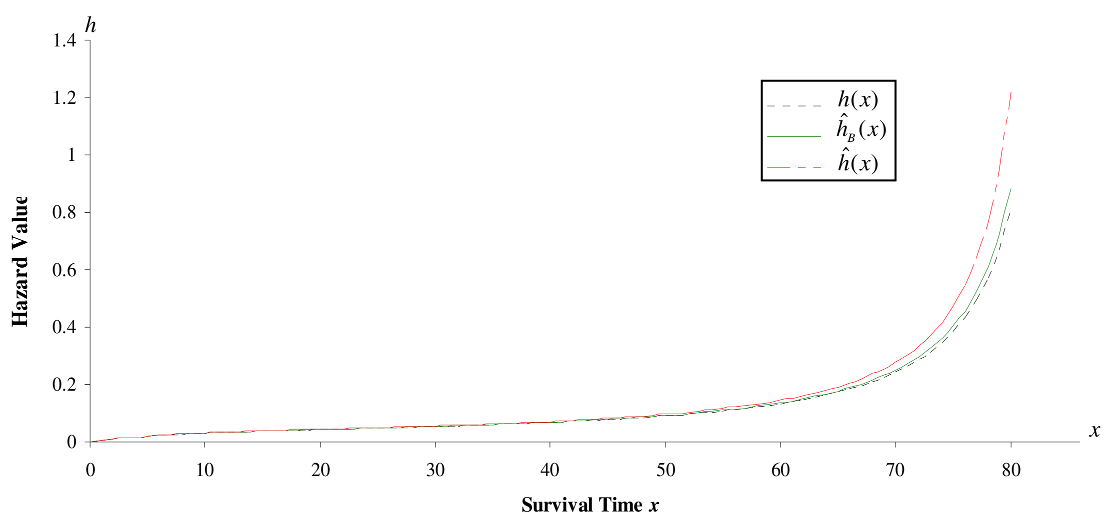



**Figure 5.8 Comparison of the estimates of $h(x)$, for $n$ = 40 and with $\xi$ = 0, $\gamma$ = 0.048, $\delta$ = 0.76.**

Figure 5.9 shows a closer view of the behavior of the survival estimates, for $n$=40 and $\lambda$ = 86.0315, in the survival interval [73, 80]. It can be seen that the Bayesian hazard function is closer than the parametric hazard estimate. In addition, it shows that the true hazard function is overestimated by both approaches.

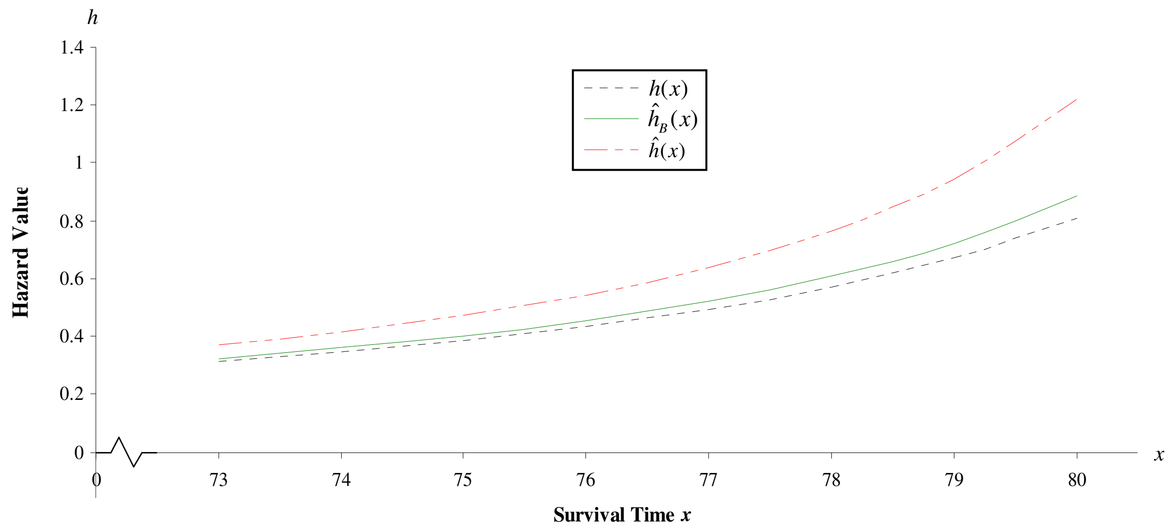



**Figure 5.9 Comparison of the hazard function estimates with respect to the true hazard function for $n$ = 40 and $\lambda$ = 86.0315 in the survival time range [73, 80]**

We present comparisons of the survival and hazard functions for two other values of the parameter $\lambda$ randomly generated from the Nakagami distribution**.** Figure 5.10 shows that the Bayesian survival function is closer than the parametric survival estimate. In addition, both estimates of the survival function underestimate the true survival function for this particular value of the parameter $\lambda$.

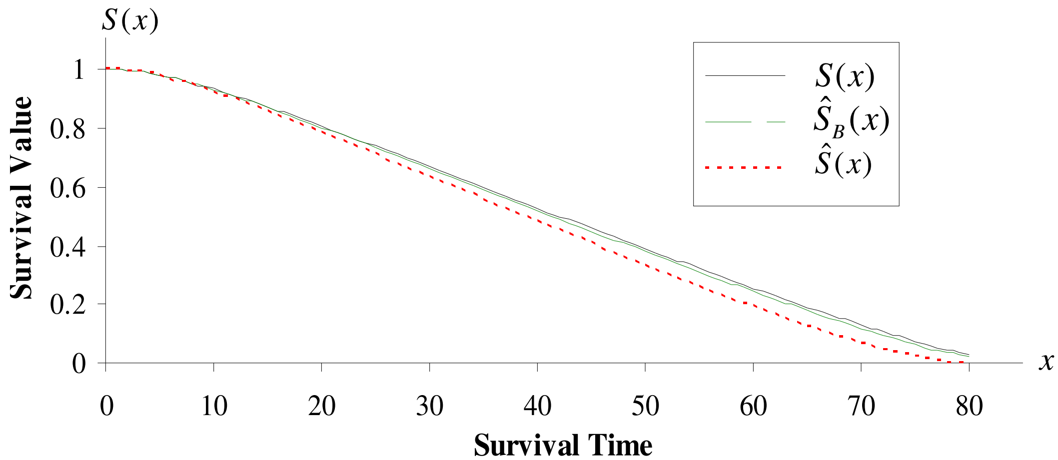



**Figure 5.10 Comparison of the survival function estimates with respect to the true hazard function for $n$ = 40, with $\xi$ = 0, $\gamma$ = 0.048, $\delta$ = 0.76, and $\lambda$ = 86.8738**

Figure 5.11 shows the comparison of the estimates of the hazard function for $\lambda$=86.8738. It shows the same behavior as in Figure 5.8.

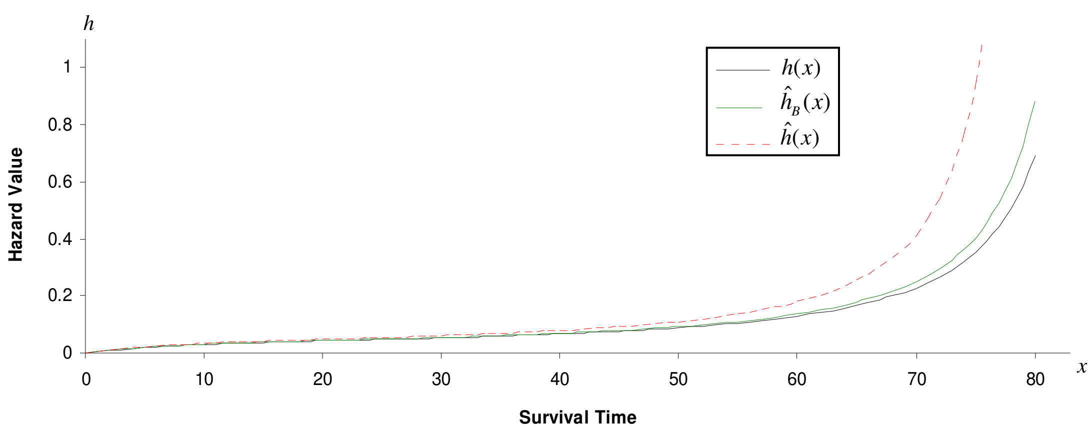



**Figure 5.11 Comparison of the hazard function estimates with respect to the true hazard function for $n$ = 40 and $\lambda$ = 86.8738**

For the particular value of the parameter λ = 84.5404, Figure 5.12 shows the comparison of the survival function estimates. It can be seen that both approaches overestimated the true survival values.

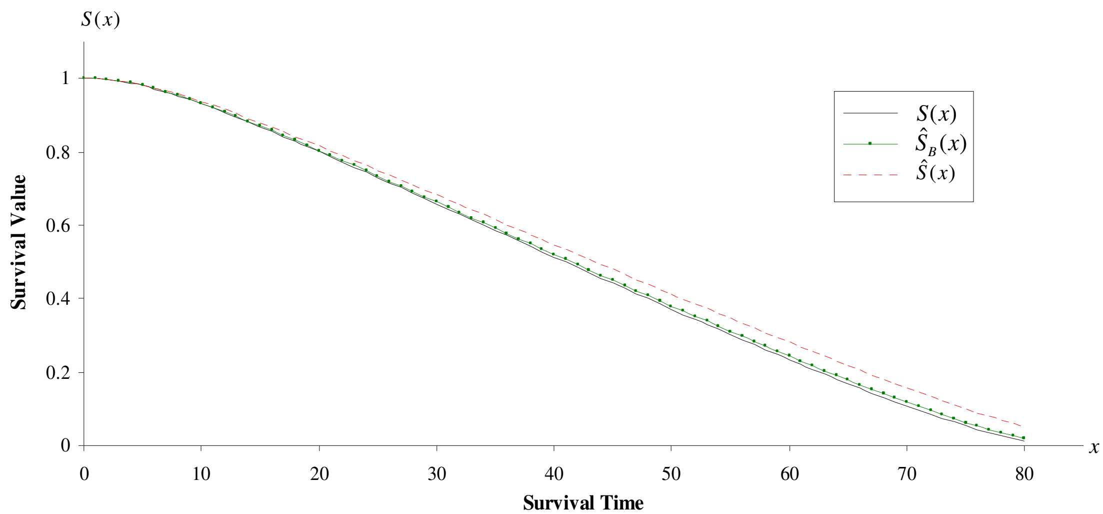



**Figure 5.12 Comparison of the survival function estimates with respect to the true hazard function for *n* = 40, with $\xi = 0$, $\gamma = 0.048$, $\delta = 0.76$, and $\lambda = 84.5404$**

Figure 5.13 shows the comparison of the hazard function estimates for the value of $\lambda$ = 84.5404 and the true hazard function. Both of the estimates of the hazard function underestimated the hazard values. In addition, the Bayesian hazard function is closer than the parametric estimate of the hazard function.

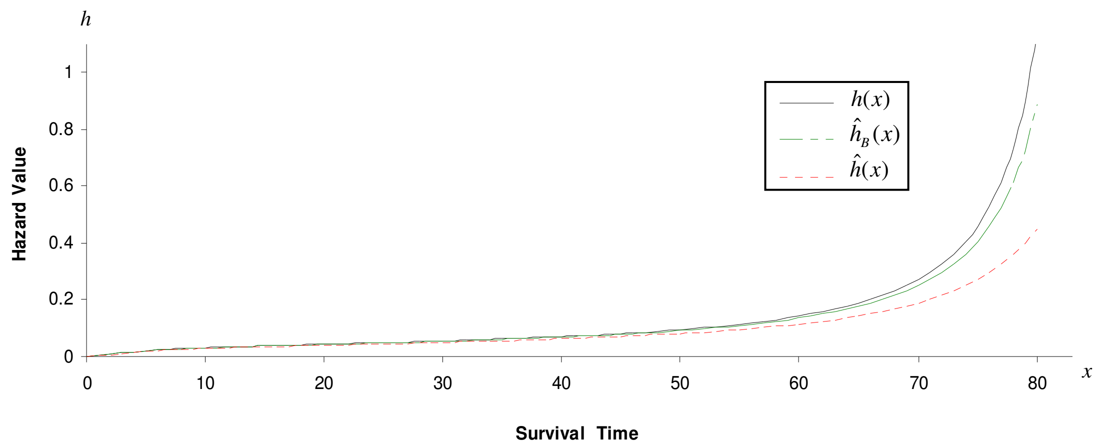



**Figure 5.13 Comparison of the hazard function estimates with respect to the true hazard function for $n$ = 40 and $\lambda$ = 84.5404**

### 5.5.3 Bayesian and Parametric Estimates of the Survival and Hazard Functions for Real Breast Cancer Data

For a random sample of size $n$ = 40 extracted from the breast cancer data provided by the SEER database, we obtained the following behavior for the estimates of the survival and hazard functions, Figure 5.14 and 5.15, respectively.

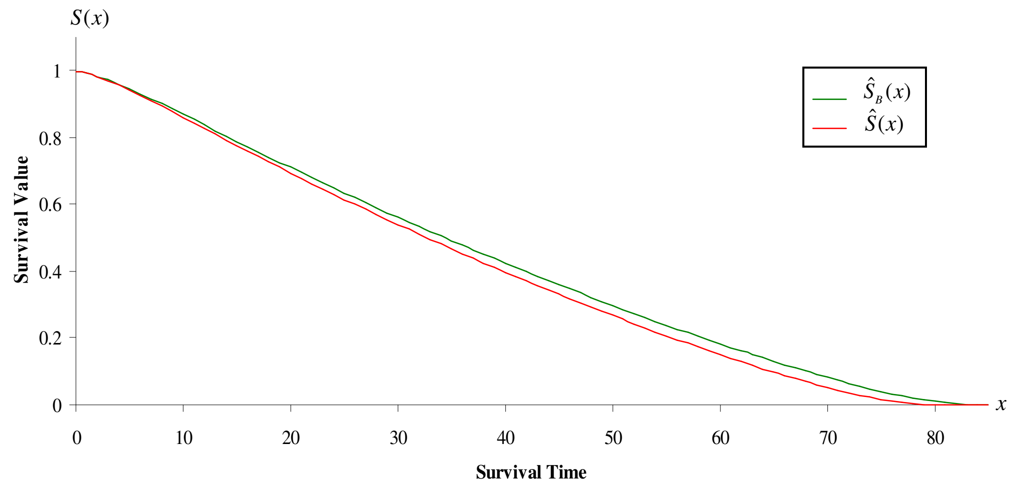



**Figure 5.14 Survival function estimates for a random sample of size $n$ = 40 extracted from the breast cancer data provided by the SEER database.**

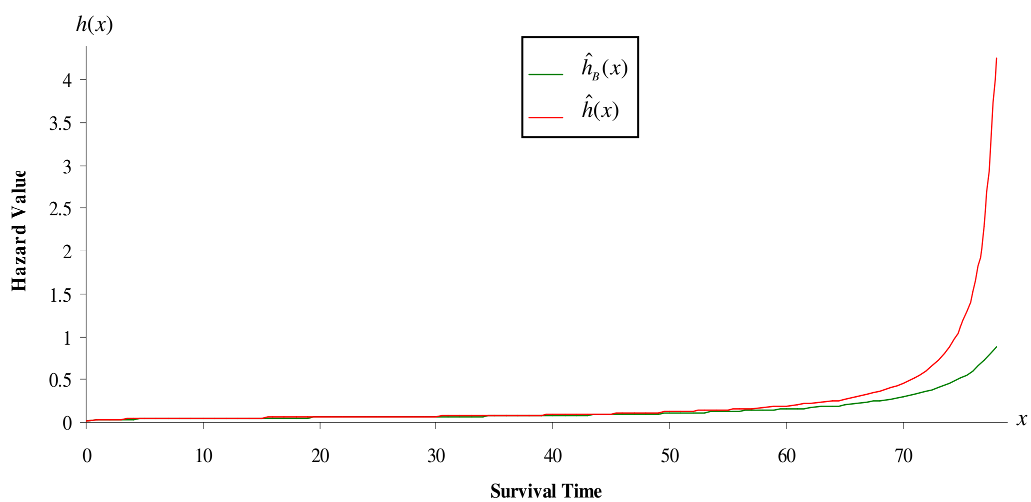



**Figure 5.15 Hazard function estimates for a random sample of size $n$ = 40 extracted from the breast cancer data provided by the SEER database.**

## 5.6 Conclusions

We have shown that the parameter $\lambda$ of the Johnson $S_B$ distribution exhibits variability in its estimated values, allowing us to consider it as a random variable with a pdf fitted with the K-S test. Therefore, we developed a Bayesian analysis assuming this pdf as its prior information and applied the Bayes Theorem in conjunction with the squared-error loss function to obtain its Bayesian estimate.

The simulation analysis showed that the Bayesian estimate of the parameter $\lambda$ performed better than the estimate value under the Wheeler procedure. The excellent behavior of the Bayesian estimate is reflected even for small sample sizes for $\lambda$= 85.0315. Small values of the MSE of the Bayesian estimates for sample size as small as $n$ = 10 reflected this finding.

We compared the estimates of the survival function with those of the true survival function when $n$ = 40 and for three values of $\lambda$: 86.0315, 86.8738 and 84.5404. We noticed than under these values for $\lambda$, the estimates of the survival function are close to the true survival function, but the Bayesian survival function estimate is closer than the parametric survival estimate. Among these three values of $\lambda$, the survival estimates underestimated the survival values when $\lambda$ = 84.5404. The RE for each sample size was greater than 1, implying the Bayesian survival function is more efficient.

In addition, we compared the estimates of the hazard function with those of the true hazard function when $n$ = 40 and for the same values of $\lambda$. The hazard values were underestimated when $\lambda$ = 84.5404.

The overall analysis reflected that Bayesian estimates for the parameter $\lambda$ of the Johnson $S_B$ distribution produced better estimates than those of the Wheeler procedure. The survival times documented in breast cancer data used in this analysis followed this distribution and, on the basis of our results, applying the Bayesian approach is a good choice to obtain estimates of the survival and hazard functions.

The main contributions of this study are:

- The demonstration of the applicability of the Bayesian approach to survival analysis of breast cancer patient data with survival times following the Johnson $S_B$ distribution.
- We developed the analytical Bayesian estimate for one parameter of the underlying model.
- We obtained the analytical form of the Bayesian estimate of the Johnson $S_B$ survival function which performed better than its parametric counterpart.
- We obtained the Bayesian estimate of the hazard function of the Johnson $S_B$ model which performed better than its parametric counterpart.

# CHAPTER 6 SENSITIVITY OF THE CHOICE OF THE LOSS FUNCTION FOR A BAYESIAN SURVIVAL ANALYSIS

The purpose of the present study was to perform a Bayesian sensitivity analysis to the selection of the loss function. We compared the Bayesian estimates of one of the four parameters —considered as a random variable— of the Johnson $S_B$ distribution. In addition, we compared the Bayesian estimates of the survival functions under the assumption of the selected loss functions.

The chapter is divided into five sections. In the first two sections, we present a brief review of the Higgins-Tsokos loss function and the Johnson $S_B$ model. The third section is a deduction of the Bayesian estimate of the parameter of the underlying model under the assumed loss functions. Moreover, we developed the analytical form of the Bayesian estimates of the survival function. The fourth section corresponds to the comparison of these estimates. The last section summarizes the findings of the study.

## 6.1 Introduction

This chapter serves as a continuation of the problem presented in the previous chapter, in that it examines the sensitivity of the choice of the loss function. We assumed that the 4-parameter Johnson $S_B$ distribution characterizes the behavior of the survival times and that one of the parameters behaves as a random variable. Assuming a

Nakagami probability distribution as the parameter prior, the primary objective is to answer the following questions within a Bayesian framework:

1. How robust is the assumption of the squared-error loss function being challenged by the assumption of the Higgins-Tsokos loss functions in estimating the parameter?
2. How robust is the assumption of the squared-error loss function being challenged by the assumption of the Higgins-Tsokos loss functions in estimating the Johnson $S_B$ survival function?

To answer these questions, we performed a Bayesian analysis through simulation. To measure the robustness of the choice of the loss function, we computed the RE of the Bayesian survival function under the squared-error loss function with respect to the one under the Higgins-Tsokos loss function and compared the Bayesian estimates of the parameter $\lambda$ by computing their MSE.

## 6.2 Higgins-Tsokos Loss Function

In a Bayesian decision-theoretic framework, a loss function $L(\cdot,\cdot)$ is a nonnegative function of the unknown random parameter $\zeta$ and a decision function $\psi$ that minimizes the conditional expected loss incurring in choosing the estimates of $\zeta$. The minimum $\hat{\zeta}$ is reached when the quantity $\int_Z L(\psi(\vec{x}),\zeta)h(\zeta;\vec{x})d\zeta$ is a minimum. In this expression, $h(\zeta;\vec{x})$ is the posterior density function of $\zeta$.

In our analysis, we are interested in the Higgins-Tsokos loss function and the well known squared-error loss function. The latter has been used since, as long as the error is reasonable, the loss is of the same magnitude for both high and low estimates. In addition, the loss becomes substantial only when the estimate is grossly off the true value.

The Higgins-Tsokos loss function (Higgins & Tsokos, 1976) is given by

$$L(\hat{\zeta},\zeta) = \frac{f_1 \exp\{f_2(\hat{\zeta}-\zeta)\} + f_2 \exp\{-f_1(\hat{\zeta}-\zeta)\}}{f_1+f_2} - 1, \quad f_1 > 0, f_2 > 0 \qquad (6.1.1)$$

where $\hat{\zeta}$ is the estimate for $\zeta$. The Higgins-Tsokos loss function is useful because it places a heavier penalty at the extremes —over and underestimation— than in the middle compared to the squared-error loss function (Camara and Tsokos, 2001). The Higgins-Tsokos loss function has been of interest to develop sensitivity analysis (Camara and Tsokos, 1999), to introduce Monte Carlo integration (Camara and Tsokos, 2005), and to derive approximate confidence interval for the mean of a normal population (Camara, 2009).

## 6.3 Revisiting the 4-parameter Johnson $S_B$ Distribution

The pdf of the 4-parameter Johnson $S_B$ distribution is defined as

$$f(x;\delta,\gamma,\xi,\lambda) = \frac{\delta}{\lambda y(1-y)\sqrt{2\pi}} e^{-\frac{1}{2}\left[\gamma+\delta\log\left(\frac{y}{1-y}\right)\right]^2}, \quad \lambda > 0, \delta > 0, \xi < x < \xi+\lambda \qquad (6.2.1)$$

where $y = \dfrac{x-\xi}{\lambda}$, and the transformation $z = \gamma + \delta \log\left(\dfrac{y}{1-y}\right)$ is a standard normal variable (Johnson, 1949). The corresponding survival function is defined by

$$S(x) = 1 - \Phi\left(\gamma + \delta \log\left[\frac{x-\xi}{\xi+\lambda-x}\right]\right), \tag{6.2.2}$$

where $\Phi(\cdot)$ is the CDF of a standard normal distribution.

With the assumption that the survival times $X_1, X_2, X_3, ..., X_n$ are independent and identically distributed following the Johnson $S_B$ distribution (6.2.1), the likelihood function is given by

$$L(\vec{x};\lambda) = \left(\frac{\delta}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum_{i=1}^{n}\left[\gamma+\delta\log\left(\frac{x_i-\xi}{\lambda+\xi-x_i}\right)\right]^2} \prod_{i=1}^{n} \frac{\lambda}{(x_i-\xi)(\lambda+\xi-x_i)} \tag{6.2.3}$$

where $\vec{x} = (x_1, x_2, x_3, ..., x_n)$ represents the realizations of $X_1, X_2, X_3, ..., X_n$.

## 6.4 Bayesian Estimates of the Parameter λ and the Corresponding Survival function

Consider the parameter $\lambda$ as a random variable and the Nakagami distribution as its informative pdf defined as

$$p(\lambda) = \frac{2}{\Gamma(m)}\left(\frac{m}{\omega}\right)^m \lambda^{2m-1} exp\left\{-\frac{m}{\omega}\lambda^2\right\}, \quad m \geq 0.5, \omega > 0, \lambda \geq 0, \tag{6.3.1}$$

By invoking the Bayes Theorem, the posterior $h(\lambda;\vec{x})$ of $\lambda$ is given by

$$h(\lambda;\vec{x}) = \frac{\lambda^{n+2m-1}\left(\frac{\delta}{\sqrt{2\pi}}\right)^n \frac{2}{\Gamma(m)}\left(\frac{m}{\omega}\right)^m e^{-\frac{m}{\omega}\lambda^2-\frac{1}{2}\sum_{i=1}^{n}\left[\gamma+\delta\log\left(\frac{x_i-\xi}{\lambda+\xi-x_i}\right)\right]^2} \prod_{i=1}^{n}\frac{1}{(x_i-\xi)(\lambda+\xi-x_i)}}{\int_\Lambda \lambda^{n+2m-1}\left(\frac{\delta}{\sqrt{2\pi}}\right)^n \frac{2}{\Gamma(m)}\left(\frac{m}{\omega}\right)^m e^{-\frac{m}{\omega}\lambda^2-\frac{1}{2}\sum_{i=1}^{n}\left[\gamma+\delta\log\left(\frac{x_i-\xi}{\lambda+\xi-x_i}\right)\right]^2} \prod_{i=1}^{n}\frac{1}{(x_i-\xi)(\lambda+\xi-x_i)}} \tag{6.3.2}$$

where $\Lambda$ is the parameter space for $\lambda$. Then, the Bayesian estimate of $\lambda$, under the squared-error loss function is given by

$$\hat{\lambda}_{SQ} = \frac{\int_0^\infty \lambda^{n+2m} \left(\frac{\delta}{\sqrt{2\pi}}\right)^n \frac{2}{\Gamma(m)} \left(\frac{m}{\omega}\right)^m e^{-\frac{m}{\omega}\lambda^2 - \frac{1}{2}\sum_{i=1}^n \left[\gamma + \delta \log\left(\frac{x_i - \xi}{\lambda + \xi - x_i}\right)\right]^2} \prod_{i=1}^n \frac{1}{(x_i - \xi)(\lambda + \xi - x_i)} d\lambda}{\int_0^\infty \lambda^{n+2m-1} \left(\frac{\delta}{\sqrt{2\pi}}\right)^n \frac{2}{\Gamma(m)} \left(\frac{m}{\omega}\right)^m e^{-\frac{m}{\omega}\lambda^2 - \frac{1}{2}\sum_{i=1}^n \left[\gamma + \delta \log\left(\frac{x_i - \xi}{\lambda + \xi - x_i}\right)\right]^2} \prod_{i=1}^n \frac{1}{(x_i - \xi)(\lambda + \xi - x_i)} d\lambda}, \quad (6.3.3)$$

and the Bayesian estimate of $\lambda$ with respect to the Higgins-Tsokos loss function is expressed as

$$\begin{aligned} \hat{\lambda}_{HT} &= \frac{1}{f_1 + f_2} ln\left\{ \frac{\int_{-\infty}^\infty e^{f_1\lambda} h(\lambda; \vec{x}) d\lambda}{\int_{-\infty}^\infty e^{-f_2\lambda} h(\lambda; \vec{x}) d\lambda} \right\} \\ &= \frac{1}{f_1 + f_2} ln\left\{ \frac{\int_0^\infty \lambda^{n+2m-1} e^{f_1\lambda - \frac{m}{\omega}\lambda^2 - \frac{1}{2}\sum_{i=1}^n \left[\gamma + \delta \log\left(\frac{x_i - \xi}{\xi + \lambda - x_i}\right)\right]^2} \prod_{i=1}^n \frac{1}{(x_i - \xi)(\xi + \lambda - x_i)} d\lambda}{\int_0^\infty \lambda^{n+2m-1} e^{-f_2\lambda - \frac{m}{\omega}\lambda^2 - \frac{1}{2}\sum_{i=1}^n \left[\gamma + \delta \log\left(\frac{x_i - \xi}{\xi + \lambda - x_i}\right)\right]^2} \prod_{i=1}^n \frac{1}{(x_i - \xi)(\xi + \lambda - x_i)} d\lambda} \right\} \end{aligned} \quad (6.3.4)$$

Therefore, the Bayesian estimate of the survival function under the squared-error loss function is obtained by substituting $\hat{\lambda}_{SQ}$ in equation (6.2.2) and is given by

$$\hat{S}_{SQ}(x) = 1 - \Phi\left(\gamma + \delta \log\left[\frac{x - \xi}{\xi + \hat{\lambda}_{SQ} - x}\right]\right) \quad (6.3.5)$$

The Bayesian estimate of the survival function under the Higgins-Tsokos loss function is

$$\hat{S}_{HT}(x) = 1 - \Phi\left(\gamma + \delta \log\left[\frac{x-\xi}{\xi + \hat{\lambda}_{HT} - x}\right]\right) \tag{6.3.6}$$

obtained by substituting $\hat{\lambda}_{HT}$ in equation (6.2.2).

From a decision-theoretic framework and applying a Bayesian approach, our aim is to analyze the differences in the estimation of the parameter $\lambda$ and the estimates of the survival function incurred by applying the Higgins-Tsokos loss function instead of the squared-error loss function. We proceed with our analysis through Monte Carlo simulation.

## 6.5 Numerical Comparison

We simulated 1,000 samples of size in {10, 40, 100} from the Johnson $S_B$ distribution with the parameter $\lambda$ generated from the Nakagami distribution and taking $\xi$=0, $\gamma$ = 0.048, and $\delta$ = 0.76. We proceeded to calculate the MSE of the Bayesian estimate of the parameter under the squared-error loss function $\hat{\lambda}_{SQ}$ and the MSE of the Bayesian estimate of the parameter under the Higgins-Tsokos loss function denoted by $\hat{\lambda}_{HT}$. The results are given in Table 6.1. Their MSE are approximately equal.

**Table 6.1 MES of the Bayesian Estimates for the Parameter $\lambda$ of Johnson $S_B$ Distribution under the Squared-Error and the Higgins-Tsokos Loss Functions Based on 1,000 Simulated Samples of Sizes 10, 40, and 100**

| $n$ | $MSE(\hat{\lambda}_{SQ})$ | $MSE(\hat{\lambda}_{HT})$ |
|---|---|---|
| 10 | 0.48417 | 0.48422 |
| 40 | 0.46485 | 0.46481 |
| 100 | 0.39966 | 0.39971 |

To compare the corresponding estimates of the survival functions $\hat{S}_{QT}(t)$ and $\hat{S}_{HT}(t)$, their RE were calculated and averaged. The results are in Table 6.2. The Bayesian estimates of the survival function under the squared-error and the Higgins-Tsokos loss functions are approximately equally efficient. However, the Higgins-Tsokos loss function is slightly more efficient than the squared-error loss function.

**Table 6.2 Average of the RE of the Bayesian Estimate for the Survival Function under the Squared-Error Loss Function with respect to the Bayesian Estimate under the Higgins-Tsokos Loss Function based on 1,000 Simulated Samples of Sizes 10, 40 and 100**

| $n$ | $RE(\hat{S}_{SQ}(t), \hat{S}_{HT}(t))$ |
|---|---|
| 10 | 1.001783246 |
| 40 | 1.015216086 |
| 100 | 1.011343741 |

1,000 samples of sizes 10, 40 and 100 were generated from the Johnson $S_B$ distribution with $\lambda$ = 85.6, $\xi$ = 0, $\gamma$ = 0.048, and $\delta$ = 0.76. The MSE of the Bayesian estimate $\hat{\lambda}_{SQ}$ and the MSE of the Bayesian estimate of $\hat{\lambda}_{HT}$ are given in Table 6.3.Their MSE are approximately equal.

**Table 6.3 MSE of the Bayesian Estimates for the Parameter $\lambda$ of Johnson $S_B$ Distribution under the Squared-Error and the Higgins-Tsokos Loss Functions Based on 1,000 Simulated Samples of Sizes 10, 40, and 100 with $\lambda$ = 85.6**

| $n$ | $\hat{\lambda}_{SQ}$ | $\hat{\lambda}_{HT}$ | $MSE(\hat{\lambda}_{SQ})$ | $MSE(\hat{\lambda}_{HT})$ |
|---|---|---|---|---|
| 10 | 85.58512 | 85.58588 | 0.00596 | 0.00609 |
| 40 | 85.58670 | 85.58851 | 0.02678 | 0.02730 |
| 100 | 85.57344 | 85.57661 | 0.05385 | 0.05452 |

Using these estimates, the Bayesian survival functions were developed under the squared-error and the Higgins-Tsokos loss functions. The RE of the Bayesian survival function under the squared-error loss function with respect to the one under the Higgins-Tsokos loss function were calculated for $\lambda$ = 85.6. The Res are approximately equal (Table 6.4). Nevertheless, for this particular value of $\lambda$, the Higgins-Tsokos loss function is slightly more efficient than the squared-error loss function.

**Table 6.4 RE of the Bayesian Estimate for the Survival Function under the Squared-Error Loss Function with respect to the Bayesian Estimate under the Higgins-Tsokos Loss Function for $\lambda$ = 85.6 based on 1,000 Simulated Samples of Sizes 10, 40, and 100**

| $n$ | $\lambda$ | $RE(\hat{S}_{SQ}(t), \hat{S}_{HT}(t))$ |
|---|---|---|
| 10 | 85.6 | 1.11055 |
| 40 | 85.6 | 1.33989 |
| 100 | 85.6 | 1.28945 |

## 6.6 Conclusions

In the present study, we assumed the parameter $\lambda$ in the underlying Johnson $S_B$ distribution for survival times could behave as a random variable, and we considered its prior probabilistic characterization as the Nakagami probability density function along with the squared-error and Higgins-Tsokos loss functions. We developed the Bayesian estimates of the parameter $\lambda$ and the analytical structure of the corresponding survival function estimates subject to the above. In addition, we compared the estimates of the parameter $\lambda$ using their MSE as the criteria and the Bayesian estimates of the survival functions were compared calculating their RE.

We used numerical simulation to illustrate the sensitivity to the selection of the loss function. On the basis of the Monte Carlo simulation, the Bayesian approach applied under either loss functions produced approximately the same estimates for the parameter $\lambda$. For over 1,000 simulated samples of different sizes, with the parameter $\lambda$ generated from the Nakagami probability density function —in particular for $\lambda = 85.6$— the Bayesian estimates of the parameter had approximately equal MSE.

In addition, for each of the considered sample sizes, the averaged RE of the survival functions estimates were approximately equal to 1; implying the robustness of the squared-error loss function. The behavior of the RE of these estimates was illustrated for the realization $\lambda = 85.6$.

The main contributions of this study can be summarized as:

- The development of the analytical structure of the Bayesian estimate of one of the parameters in the Johnson $S_B$ model under the squared-error loss function.
- Development of the analytical structure of the Bayesian estimate of one of the parameter in the Johnson $S_B$ model under the Higgins-Tsokos loss function.
- Development of the analytical structure of the Bayesian estimate of the survival function of the Johnson $S_B$ model under the squared-error loss function.
- Development of the analytical structure of the Bayesian estimate of the survival function of the Johnson $S_B$ model under the Higgins-Tsokos loss function.

# CHAPTER 7 FUTURE RESEARCH

In chapter 3 we showed that one of the parameters in the intensity function of the PLP behaves as a random variable and developed a Bayesian estimate for it. The MLE analytical form of the subject parameter depends on the last ordered failure time. This dependency produces a sensitivity behavior in the MLE of the parameter. As a future study, we are interested in developing an analytical form that is maximum ordered statistic free.

In chapter 5, for the four parameters Johnson SB probability distribution, we considered that one of the parameters behaves as a random variable. Although there is another parameter that behaves as a random variable, we only developed a Bayesian estimate for the parameter with the largest variance. In a future research, we are interested in considering a bivariate probability distribution that involves two of the parameters that behave as a random variable and to develop Bayesian estimates for them.

Further research efforts should also focus on the use of the kernel density estimation method. Suppose we do not have enough estimates to fit the prior probability distribution of the parameter, or parameters, which behave as a random variable. For this case, we proceed to investigate the applicability of the kernel density estimation method to obtain the pdf of the parameter(s) and use it to develop the analytical form of the Bayesian estimate for the subject parameter(s).

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