2011

Generalizations of a Laplacian-Type Equation in the Heisenberg Group and a Class of Grushin-Type Spaces

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Generalizations of a Laplacian-Type Equation in the Heisenberg Group and a Class of Grushin-Type Spaces

by

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A thesis submitted in partial fulfillment
of the requirements for the degree of
Master of Arts
Department of Mathematics & Statistics
College of Arts and Sciences
University of South Florida

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Date of Approval:
June 27, 2011

Keywords: Fundamental Solution, Nonlinear Potential Theory, p-Laplace operator, Partial Differential Equations, Sub-Riemannian Geometry

Acknowledgments

The author would like to thank the following people for their support in the writing of this thesis. First, I would like to thank my husband Ross for his constant encouragement and support, without which I would not be here today. I would also like to thank both my mother and father for their love and support. I would like to thank my mother-in-law and father-in-law for believing in me. I would also like to thank my fellow graduate students who provided both encouragement and comedic relief during difficult times. I would like to thank the Department of Mathematics and Statistics at the University of South Florida for giving me the opportunity to continue my mathematics education. Finally, I would like to thank my advisor, Thomas Bieske, for his support throughout the writing of this thesis and for making me a better mathematician.

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Abstract

In [2], Beals, Gaveau and Greiner find the fundamental solution to a 2-Laplace-type equation in a class of sub-Riemannian spaces. This fundamental solution is based on the well-known fundamental solution to the p-Laplace equation in Grushin-type spaces [4] and the Heisenberg group [6]. In this thesis, we look to generalize the work in [2] for a p-Laplace-type equation. After discovering that the "natural" generalization fails, we find two generalizations whose solutions are based on the fundamental solution to the p-Laplace equation.

Chapter 1

Introduction and Motivation

In recent years, there has been mathematical interest in sub-Riemannian spaces. In calculus and classical physics, one models natural phenomena using the Euclidean space $\mathbb{R}^n$. In Euclidean spaces, a key geometric property is that every direction is mathematically identical. Though this is appropriate for many models, there are some phenomena that is not suited for modeling in this environment, for instance, parallel parking a car. The sideways direction is clearly more difficult to travel in than the forward or reverse direction. The proper spaces for models of such phenomena are sub-Riemannian spaces, which generalize Euclidean spaces by restricting the direction of the tangent vectors, reflecting the ease or difficulty in traveling in that direction.

Sub-Riemannian spaces are generalizations of $\mathbb{R}^n$ and Riemannian spaces. Recall that the tangent space to $\mathbb{R}^n$ is n-dimensional and that the dimension of a tangent space to a Riemannian space equals the dimension of the space itself. Curves in both Riemannian spaces and $\mathbb{R}^n$ can have tangent vectors in any direction. However, this is not the case with sub-Riemannian spaces. Curves in sub-Riemannian spaces are allowed tangent vectors in only certain restricted directions. In particular, the dimension of the tangent space is less than the dimension of the space. Because of this, one application of sub-Riemannian spaces is control theory, in which one tries to travel from one point to another only along certain routes [1].

Sub-Riemannian spaces have a variety of desirable mathematical properties. As suggested by the above, these spaces are metric spaces, based on the lengths of restricted curves. In addition, some sub-Riemannian spaces also have a (possibly non-abelian) algebraic group law. However, not every sub-Riemannian space has an algebraic group law. See [1] for a further discussion.

Most importantly for our purposes, sub-Riemannian spaces have a differentiable structure. Vector fields can be defined on the tangent space, and so there is a calculus on sub-Riemannian spaces [8]. Using the calculus, one can define partial differential equations and begin to study their solutions and properties.

One key partial differential equation is the p-Laplace equation, a prototype for the study of partial differential equations. The study of partial differential equations in sub-Riemannian spaces is a relatively new field, and there are many unanswered questions concerning the behavior of solutions in these spaces. The p-Laplace equation forms the basis of nonlinear potential theory [10]. Thus, there is an interest in finding solutions to this equation in sub-Riemannian spaces. This is not an easy task, and solutions to the p-Laplace equation have been found in only a few sub-Riemannian spaces, namely in the so-called groups of Heisenberg-type [6, 9] and in certain classes of the so-called Grushin-type spaces [1, 4]. Arguments in [1, 4, 6, 9] exploit the underlying geometry of the particular space and the results of [6, 9] require an algebraic group law.

In [2], solutions to a generalization of the 2-Laplace equation were found in a wide class of sub-Riemannian spaces. This class includes some of the spaces in [6, 9, 4, 1]. The methodology of [2] mixes the geometric properties of the space with the linearity of the 2-Laplace operator. In this thesis, we study the generalization of [2] and look to extend it to an equation based on the p-Laplace equation. Because the p-Laplace equation is nonlinear, we face many technical issues. The first of which is the proper way to generalize the original equation. In Section 3 we discuss the original equations of [2] and in Section 4, we find that the seemingly "natural" generalization is not the correct one. In Section 5, we will find 2 generalizations for which the equations are indeed solutions. This exercise helps us to gain some understanding as to the behavior of solutions to partial differential equations and in particular, to solutions of the p-Laplace equation.

Since the p-Laplace equation is not linear, we are limited to two specific sub-Riemannian spaces. We concern ourself with Grushin-type planes, which are two-dimensional sub-Riemannian spaces lacking a group law, and the (first) Heisenberg group, a three-dimensional sub-Riemannian space possessing a group law. We explore these environments in Section 2.

Chapter 2
The Environments

As mentioned above, we are going to examine partial differential equations in Grushin-type planes, which are 2-dimensional Grushin-type spaces, and the Heisenberg group, which is a model for the groups of Heisenberg-type. We will first recall the construction of these spaces and then highlight the main properties, including how calculus in these spaces is different from Euclidean calculus.

2.1 Grushin-type planes

We begin with $\mathbb{R}^2$, possessing coordinates (y_1, y_2), a real number a, a non-zero real number c and a positive integer n. We use them to make the vector fields:

$$Y_1 = \frac{\partial}{\partial y_1} \text{ and } Y_2 = c(y_1 - a)^n \frac{\partial}{\partial y_2}.$$

Recall that the Lie bracket of two vectors fields, $\mathfrak{X}$ and $\mathfrak{Z}$, is given by

$$[\mathfrak{X}, \mathfrak{Z}] = \mathfrak{X}\mathfrak{Z} - \mathfrak{Z}\mathfrak{X}.$$

Note that if $\mathfrak{X}$ is a r^{th} order derivative and $\mathfrak{Z}$ is a s^{th} order derivative, then the Lie bracket yields a $(r+s)^{th}$ order derivative. For our vector fields, the only (possibly) nonzero Lie bracket is

$$[Y_1, Y_2] = Y_1 Y_2 - Y_2 Y_1 = cn(y_1 - a)^{n-1} \frac{\partial}{\partial y_2}.$$

Note this is different from Euclidean $\mathbb{R}^n$, in which every Lie bracket is zero. The brackets in $\mathbb{R}^n$ are zero because mixed partial derivatives of smooth functions are equal.

Because n is a positive integer, we see that applying the Lie bracket n number of times gives us a nonzero vector at $y_1 = a$, namely,

$$Z \equiv [Y_1, [Y_1, \cdots [Y_1, Y_2]] \cdots] = cn! \frac{\partial}{\partial y_2}.$$

The vector fields $\mathfrak{X}_1, \ldots, \mathfrak{X}_n$ are said to satisfy Hörmander's condition if $\mathfrak{X}_1, \ldots, \mathfrak{X}_n$, together with their iterated Lie brackets,

$$[\mathfrak{X}_i, \mathfrak{X}_j], \ldots, [[\mathfrak{X}_i, \mathfrak{X}_j], \mathfrak{X}_k], \ldots$$

span the tangent space at every point p. Since Y_1 and Y_2, as defined above, span $\mathbb{R}^2$ when $y_1 \neq a$, and Y_1 and Z span $\mathbb{R}^2$ when $y_1 = a$, it follows that Hörmander's condition is satisfied by these vector fields.

We will put an inner product on $\mathbb{R}^2$, denoted $\langle \cdot, \cdot \rangle_{\mathbb{G}}$, with related norm $\| \cdot \|_{\mathbb{G}}$, so that the collection $\{Y_1, Y_2\}$ forms an orthonormal basis. This inner product is singular when Y_2 vanishes. In Euclidean coordinates, the inner product is given by

$$\begin{aligned}\langle be_1 + de_2, se_1 + te_2 \rangle_{\mathbb{G}} &= (b, d) \begin{pmatrix} 1 & 0 \\ 0 & (c^2(y_1 - a)^{2n})^{-1} \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} \\ &= bs + dt(c^2(y_1 - a)^{2n})^{-1}\end{aligned}$$

where e_1 and e_2 are the standard basis vectors for $\mathbb{R}^2$.

We then have a sub-Riemannian space that we will call g_n, which is also the tangent space to a generalized Grushin-type plane $\mathbb{G}_n$. Points in $\mathbb{G}_n$ will also be denoted by $p = (y_1, y_2)$.

We consider a distance on $\mathbb{G}_n$ which is defined for points p and q as follows:

$$d_{\mathbb{G}}(p, q) = \inf_{\Gamma} \int \sqrt{\langle \gamma'(t), \gamma'(t) \rangle_{\mathbb{G}}}\, dt.$$

Here Γ is the set of all curves γ such that $\gamma(0) = p$, $\gamma(1) = q$ and

$$\gamma'(t) \in \operatorname{span}\{Y_1(\gamma(t)), Y_2(\gamma(t))\}.$$

This distance is called the Carnot-Carathéodory distance.

Chow's theorem asserts that if M is a connected space, and if vector fields $\{\mathfrak{X}_i\}_{i=1}^k$ satisfy Hörmander's condition, then any two points of M can be connected by curves whose tangents lie in $\operatorname{span}\{\mathfrak{X}_1, \mathfrak{X}_2, \ldots, \mathfrak{X}_k\}$. Since $\mathbb{R}^2$ is connected, and as stated above, Hörmander's condition is satisfied, we have that Γ, as defined, is non-empty and so $d_{\mathbb{G}}$ is indeed a metric.

The Carnot-Carathéodory distance between $p = (y_1, y_2)$ and $q = (\widehat{y_1}, \widehat{y_2})$, can be estimated via Theorem 7.34 in [1] by

$$d_{\mathbb{G}}(p, q) \approx |\widehat{y_1} - y_1| + |\widehat{y_2} - y_2|^{\frac{1}{n+1}}.$$

The explicit computation of the corresponding geodesics is outside the scope of this thesis. When $n = 1$, the distances and geodesics are explicitly computed in [7].

We shall now discuss calculus on the Grushin-type planes. Given a smooth function f on $\mathbb{G}_n$, we define the horizontal gradient of f as

$$\nabla_0 f(p) = \big(Y_1 f(p), Y_2 f(p)\big).$$

This definition of the horizontal gradient differs from that of the standard gradient of f on $\mathbb{R}^2$. If $y_1 = a$, the second coordinate of the horizontal gradient, as defined above, will *always* be zero. In the sense of the standard gradient on $\mathbb{R}^2$, we will always have a non-zero second coordinate when $\frac{\partial f}{\partial y_2}$ is non-zero.

In $\mathbb{R}^2$, the second-order derivative matrix has entries given by:

$$(D^2 f(p))_{ij} = \frac{\partial^2 f}{\partial y_i \partial y_j}(p).$$

This matrix is symmetric, since $\frac{\partial^2 f}{\partial y_i \partial y_j} = \frac{\partial^2 f}{\partial y_j \partial y_i}$ for all smooth functions.

For $\mathbb{G}_n$, the case is different. The second-order horizontal derivative matrix has entries given by:

$$(D^2 f(p))_{ij} = Y_i Y_j f(p).$$

Because $[Y_i, Y_j]$ need not be zero, we have $Y_i Y_j \neq Y_j Y_i$. So, this matrix is not necessarily symmetric. We shall symmetrize this matrix. In doing so, we see that the symmetrized second-order horizontal derivative matrix is given by

$$\begin{aligned} ((D^2 f(p))^\star)_{11} &= Y_1 Y_1 f(p), \\ ((D^2 f(p))^\star)_{12} = ((D^2 f(p))^\star)_{21} &= \frac{1}{2}\big(Y_1 Y_2 f(p) + Y_2 Y_1 f(p)\big), \\ \text{and } ((D^2 f(p))^\star)_{22} &= Y_2 Y_2 f(p). \end{aligned}$$

Using these derivatives, we have the following natural definitions:

DEFINITION 2.1.1

- *The function $f : \mathbb{G}_n \to \mathbb{R}$ is said to be $C^1_{\mathbb{G}}$ at the point $p = (y_1, y_2)$ with $y_1 \neq a$ if $Y_i f$ is continuous at p for $i = 1, 2$. Similarly, the function f is $C^2_{\mathbb{G}}$ at p if $Y_i Y_j f$ is continuous at p for $i, j = 1, 2$.*

- *The function $f : \mathbb{G}_n \to \mathbb{R}$ is said to be $C^1_{\mathbb{G}}$ at the point $p = (a, y_2)$ if $Y_1 f$ is continuous at p. Similarly, the function f is $C^2_{\mathbb{G}}$ at p if $Y_1 Y_1 f$ is continuous at p and, if $n = 1$, $Y_1 Y_2 f$ is continuous at p.*

Claim 2.1 *A function that is $C^1_{\mathbb{G}}$ at p does not have to be Euclidean C^1 at p. For example, let $n = 1$, $a = 0$ and $c = 1$, consider $f(y_1, y_2) = \sqrt{y_2}$.*

Proof. This function is not Euclidean C^1 at $p = (0, 0)$ because $\frac{\partial f}{\partial y_2}$ does not exist. To show that $f(y_1, y_2)$ is $C^1_{\mathbb{G}}$ we need only that $Y_1 f$ is continuous. Since for all p, $Y_1 f = \frac{\partial f}{\partial y_1} = 0$, we see that this is the case. □

Using these derivatives, we consider a key operator on $C^2_{\mathbb{G}}$ functions called the $\mathtt{p}$-Laplacian for $1 < \mathtt{p} < \infty$, given by

$$\begin{aligned}
\Delta_{\mathtt{p}} f &= \operatorname{div}(\|\nabla_0 f\|_{\mathbb{G}}^{\mathtt{p}-2} \nabla_0 f) && (2.1)\\
&= Y_1\big(\|\nabla_0 f\|_{\mathbb{G}}^{\mathtt{p}-2} Y_1 f\big) + Y_2\big(\|\nabla_0 f\|_{\mathbb{G}}^{\mathtt{p}-2} Y_2 f\big)\\
&= \frac{1}{2}(\mathtt{p}-2)\|\nabla_0 f\|_{\mathbb{G}}^{\mathtt{p}-4} Y_1 \|\nabla_0 f\|^2 Y_1 f + \|\nabla_0 f\|_{\mathbb{G}}^{p-2} Y_1 Y_1 f\\
&\quad + \frac{1}{2}(\mathtt{p}-2)\|\nabla_0 f\|_{\mathbb{G}}^{\mathtt{p}-4} Y_2 \|\nabla_0 f\|_{\mathbb{G}}^2 Y_2 f + \|\nabla_0 f\|^{p-2} Y_2 Y_2 f\\
&= \frac{1}{2}(\mathtt{p}-2)\|\nabla_0 f\|_{\mathbb{G}}^{\mathtt{p}-4}\big(Y_1 \|\nabla_0 f\|_{\mathbb{G}}^2 Y_1 f + Y_2 \|\nabla_0 f\|^2 Y_2 f\big) && (2.2)\\
&\quad + \|\nabla_0 f\|_{\mathbb{G}}^{\mathtt{p}-2}\big(Y_1 Y_1 f + Y_2 Y_2 f\big)\\
&= \|\nabla_0 f\|_{\mathbb{G}}^{\mathtt{p}-4}\bigg((\mathtt{p}-2)\frac{1}{2}\langle \nabla_0 \|\nabla_0 f\|_{\mathbb{G}}^2, \nabla_0 f\rangle_{\mathbb{G}}\\
&\quad + \|\nabla_0 f\|_{\mathbb{G}}^2\big(Y_1 Y_1 f + Y_2 Y_2 f\big)\bigg).
\end{aligned}$$

2.2 The Heisenberg group

We begin with $\mathbb{R}^3$ using the coordinates (x_1, x_2, x_3) and consider the vector fields $\{X_1, X_2, X_3\}$, defined by

$$X_1 = \frac{\partial}{\partial x_1} - \frac{x_2}{2}\frac{\partial}{\partial x_3}, \; X_2 = \frac{\partial}{\partial x_2} + \frac{x_1}{2}\frac{\partial}{\partial x_3} \text{ and } X_3 = \frac{\partial}{\partial x_3}.$$

These vector fields obey the relation

$$[X_1, X_2] = X_3,$$

with all other Lie brackets equal to zero.

Claim 2.2 *The vector fields X_1, X_2 and $X_3 \equiv [X_1, X_2]$ are linearly independent and so, Hörmander's condition is satisfied.*

Proof. We have

$$\begin{aligned} 0 &= c_1X_2 + c_2X_2 + c_3X_3 \\ &= c_1e_1 + \frac{-c_1x_2}{2}e_3 + c_2e_2 + \frac{c_2x_1}{2}e_3 + c_3e_3 \\ &= c_1e_1 + c_2e_2 + \left(\frac{-c_1x_2}{2} + \frac{c_2x_1}{2} + c_3\right)e_3. \end{aligned}$$

This happens only when $c_1 = 0, c_2 = 0$ and $c_3 = 0$. Since we are in $\mathbb{R}^3$, these 3 vectors form a basis. □

Because we have one nonzero Lie bracket, we get a Lie Algebra denoted h_1 that decomposes as a direct sum

$$h_1 = V_1 \oplus V_2$$

where $V_1 = \text{span}\{X_1, X_2\}$ and $V_2 = \text{span}\{X_3\}$. The algebra h_1 is stratified, that is $[V_1, V_1] = V_2$ and $[V_1, V_2] = 0$.

We may put an inner product on $\mathbb{R}^3$, denoted $\langle \cdot, \cdot \rangle_{\mathbb{H}}$, with related norm $\| \cdot \|_{\mathbb{H}}$ so that the collection $\{X_1, X_2, X_3\}$ forms an orthonormal basis. In Euclidean coordinates, the inner product is given by

$$\begin{aligned} &\langle ae_1 + be_2 + ce_3, re_1 + se_2 + te_3\rangle_{\mathbb{H}} = \\ &(a,b,c)\begin{pmatrix} 1+\frac{x_2^2}{4} & \frac{-x_1x_2}{4} & \frac{x_2}{2} \\ \frac{-x_1x_2}{4} & 1+\frac{x_1^2}{4} & \frac{-x_1}{2} \\ \frac{x_2}{2} & \frac{-x_1}{2} & 1 \end{pmatrix}\begin{pmatrix} r \\ s \\ t \end{pmatrix} \\ &= ar + bs + ct - (cs+bt)\frac{x_1}{2} + (at+cr)\frac{x_2}{2} \\ &\quad + bs\frac{x_1^2}{4} + ad\frac{x_2^2}{4} - (br+as)\frac{x_1x_2}{4} \end{aligned}$$

where e_1, e_2 and e_3 are the standard basis vectors for $\mathbb{R}^2$.

We will now compute the exponential map. Recall that the exponential map identifies elements of a Lie Algebra, g, with elements of a Lie Group, G. Let $\mathbb{X}$ be a left-invariant vector field and define $\gamma_{\mathbb{X}}$ as the unique integral curve satisfying

$$\begin{cases} \gamma'_{\mathbb{X}}(t)\big|_{t=0} = \mathbb{X} \\ \gamma_{\mathbb{X}}(0) = 0. \end{cases}$$

We define the map $\exp : g \to G$ by $\exp(\mathbb{X}) = \gamma_{\mathbb{X}}(1)$. The Lie Group corresponding to h_1 is denoted by $\mathbb{H}^1$ and defined by $\mathbb{H}^1 = \exp(h_1)$. This Lie Group is called the (first) Heisenberg group.

PROPOSITION 1 The choice of vector fields and their Lie bracket relations forces the exponential map to be the identity and so elements of $\mathbb{H}^1$ and the corresponding Lie Algebra h_1 can be identified with each other. Namely,

$$x_1X_1 + x_2X_2 + x_3X_3 \in h_1 \leftrightarrow (x_1, x_2, x_3) \in \mathbb{H}^1.$$

Proof.

$$\begin{aligned} & x_1X_1 + x_2X_2 + x_3X_3 = \\ & \quad x_1\left(\frac{\partial}{\partial x_1} - \frac{x_2}{2}\frac{\partial}{\partial x_3}\right) + x_2\left(\frac{\partial}{\partial x_2} + \frac{x_1}{2}\frac{\partial}{\partial x_3}\right) + x_3\left(\frac{\partial}{\partial x_3}\right) \\ &= x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + \left(\frac{-x_1x_2}{2} + \frac{x_1x_2}{2} + x_3\right)\frac{\partial}{\partial x_3} \\ &= x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3} \end{aligned}$$

We require a curve γ that satisfies the initial value problem

$$\begin{cases} \gamma'(t) = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3} \\ \gamma(0) = 0. \end{cases}$$

Using elementary calculus, we find that

$$\gamma(t) = (x_1t, x_2t, x_3t)$$

and this gives that

$$\gamma(1) = (x_1, x_2, x_3).$$

Thus

$$\exp(x_1X_1 + x_2X_2 + x_3X_3) = (x_1, x_2, x_3).$$

□

PROPOSITION 2 The Heisenberg group $\mathbb{H}^1$ is an algebraic group. For any p, q in $\mathbb{H}^1$, written as $p = (x_1, x_2, x_3)$ and $q = (\widehat{x_1}, \widehat{x_2}, \widehat{x_3})$ the group multiplication law is given by

$$p \cdot q = \big(x_1 + \widehat{x_1}, x_2 + \widehat{x_2}, x_3 + \widehat{x_3} + \frac{1}{2}(x_1\widehat{x_2} - x_2\widehat{x_1})\big).$$

Before proving the proposition, we must first discuss the Baker-Campbell-Hausdorff theorem. This theorem states that if $\mathbb{G}$ is a simply-connected Lie group with an associated Lie algebra g and $\exp : g \to \mathbb{G}$ is the exponential map, then

$$\exp X \exp Z = \exp(X + Z + \frac{1}{2}[X, Z] + \frac{1}{12}[X, [X, Z]] - \frac{1}{12}[Z, [X, Z]] + \dots)$$

[11, p.173-174]. Since $[X_1, X_2] \equiv X_3$ is the only non-zero Lie Bracket, we shall require only the first few terms of this formula, namely

$$\exp X \exp Z = \exp(X + Z + \frac{1}{2}[X, Z]).$$

Proof. The Baker-Campbell-Hausdorff Theorem, together with Proposition 3, gives that the group law can be determined by

$$\begin{aligned}
p * q &= (x_1, x_2, x_3) * (\widehat{x_1}, \widehat{x_2}, \widehat{x_3}) \\
&= \exp(x_1X_1 + x_2X_2 + x_3X_3) \exp(\widehat{x_1}X_1 + \widehat{x_2}X_2 + \widehat{x_3}X_3) \\
&= \exp\big(x_1X_1 + x_2X_2 + x_3X_3 + \widehat{x_1}X_1 + \widehat{x_2}X_2 + \widehat{x_3}X_3 \\
&\quad + \frac{1}{2}[x_1X_1 + x_2X_2 + x_3X_3,\ \widehat{x_1}X_1 + \widehat{x_2}X_2 + \widehat{x_3}X_3]\big) \\
&= \exp\big((x_1 + \widehat{x_1})X_1 + (x_2 + \widehat{x_2})X_2 + (x_3 + \widehat{x_3})X_3 \\
&\quad + \frac{1}{2}(x_1\widehat{x_2}[X_1, X_2] + x_2\widehat{x_1}[X_2, X_1])\big) \\
&= \exp\big((x_1 + \widehat{x_1})X_1 + (x_2 + \widehat{x_2})X_2 + (x_3 + \widehat{x_3})X_3 \\
&\quad + \frac{1}{2}(x_1\widehat{x_2} - x_2\widehat{x_1})X_3\big) \\
&= \big(x_1 + \widehat{x_1}, x_2 + \widehat{x_2}, x_3 + \widehat{x_3} + \frac{1}{2}(x_1\widehat{x_2} - x_2\widehat{x_1})\big)
\end{aligned}$$

□

This multiplication law is not abelian and so we consider left multiplication. We choose left multiplication because the vector fields $\{X_1, X_2, X_3\}$ are left-invariant vector fields. Using the formula for left multiplication by a point p from above, we may compute the differential matrix of left multiplication as

$$DL_p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{x_2}{2} & \frac{x_1}{2} & 1 \end{pmatrix}.$$

The vector field X_i is then $DL_p e_i$ where e_i is the usual standard Euclidean vector at the origin.

As in the Grushin-type planes, the natural metric on $\mathbb{H}^1$ is the Carnot-Carathéodory metric given by

$$d_{\mathbb{H}}(p,q) = \inf_{\Gamma} \int_0^1 \sqrt{\langle \gamma'(t), \gamma'(t) \rangle_{\mathbb{H}}}\, dt$$

where the set Γ is the set of all curves γ such that

$$\gamma(0) = p, \gamma(1) = q \text{ and } \gamma'(t) \in V_1.$$

Since Hörmander's condition is satisfied, and since $\{X_1, X_2, X_3\}$ span $\mathbb{R}^3$, we have that by Chow's theorem, Γ, as defined above, is non-empty and $d_{\mathbb{H}}$ is indeed a metric on $\mathbb{H}^1$. Because the vectors X_1 and X_2 are left-invariant, the distance is invariant under left multiplication, that is, for points $p, q, r \in \mathbb{H}^1$,

$$d_{\mathbb{H}}(r \cdot p, r \cdot q) = d_{\mathbb{H}}(p,q).$$

Thus, we may let q be the origin.

We now discuss calculus in the Heisenberg Group.

Given a smooth function $f : \mathbb{H}^1 \to \mathbb{R}$, we define the horizontal gradient by

$$\nabla_0 f = (X_1 f, X_2 f),$$

and the full gradient by

$$\nabla_1 f = (X_1 f, X_2 f, X_3 f).$$

Much like in the Grushin case, the second-order horizontal derivative matrix is not necessarily symmetric, since $X_1 X_2 f(p) \neq X_2 X_1 f(p)$. We will symmetrize this matrix, and we see that the symmetrized second-order horizontal derivative matrix $(D^2 f(p))^\star$ is given by

$$\begin{aligned} ((D^2 f(p))^\star)_{11} &= X_1 X_1 f(p), \\ ((D^2 f(p))^\star)_{12} = ((D^2 f(p))^\star)_{21} &= \frac{1}{2}\big(X_1 X_2 f(p) + X_2 X_1 f(p)\big), \\ \text{and } ((D^2 f(p))^\star)_{22} &= X_2 X_2 f(p). \end{aligned}$$

DEFINITION 2.2.1 *A function $f : \mathbb{H}^1 \to \mathbb{R}^3$ is $C^1_{\mathbb{H}}$ at the point p if $X_i f(p)$ is continuous at p for all $i = 1, 2$ and f is $C^2_{\mathbb{H}}$ at p if $X_i X_j f(p)$ is continuous at p for all $i, j = 1, 2$.*

Claim 2.3 *A function that is $C^1_{\mathbb{H}}$ at p does not have to be Euclidean C^1 at p. For example, let $p = (0, 0, 0)$ and let $u(x_1, x_2, x_3) = \sqrt{x_3}$.*

Proof. This function is not Euclidean C^1 at $(0, 0, 0)$ because $\frac{\partial u}{\partial x_3}$ does not exist. To show that $u(x_1, x_2, x_3)$ is $C^1_{\mathbb{H}}$ at $(0, 0, 0)$, we consider the following:

$$\begin{aligned} X_1 \sqrt{x_3}\big|_{(0,0,0)} &= \frac{\partial}{\partial x_1}\left(\sqrt{x_3}\right) = 0 \text{ and} \\ X_2 \sqrt{x_3}\big|_{(0,0,0)} &= \frac{\partial}{\partial x_2}\left(\sqrt{x_3}\right) = 0. \end{aligned}$$

Thus, $u(x_1, x_2, x_3)$ is indeed $C^1_{\mathbb{H}}$ at $(0, 0, 0)$. □

As in the Grushin case, we use these derivatives to consider a key operator on $C^2_{\mathbb{H}}$ functions called the $\mathtt{p}$-Laplacian for $1 < \mathtt{p} < \infty$, given by

$$\begin{aligned} \Delta_{\mathtt{p}} u &= \operatorname{div}(\|\nabla_0 u\|_{\mathbb{H}}^{\mathtt{p}-2} \nabla_0 u) \qquad (2.3) \\ &= X_1\big(\|\nabla_0 u\|_{\mathbb{H}}^{\mathtt{p}-2} X_1 u\big) + X_2\big(\|\nabla_0 u\|_{\mathbb{H}}^{\mathtt{p}-2} X_2 u\big) \\ &= \frac{1}{2}(\mathtt{p}-2)\|\nabla_0 u\|_{\mathbb{H}}^{\mathtt{p}-4} X_1 \|\nabla_0 u\|^2 X_1 u + \|\nabla_0 u\|_{\mathbb{H}}^{p-2} X_1 X_1 u \\ &\quad + \frac{1}{2}(\mathtt{p}-2)\|\nabla_0 u\|_{\mathbb{H}}^{\mathtt{p}-4} X_2 \|\nabla_0 u\|^2 X_2 u + \|\nabla_0 u\|^{p-2} X_2 X_2 u \\ &= \frac{1}{2}(\mathtt{p}-2)\|\nabla_0 u\|_{\mathbb{H}}^{\mathtt{p}-4}\big(X_1 \|\nabla_0 u\|_{\mathbb{H}}^2 X_1 u + X_2 \|\nabla_0 u\|_{\mathbb{H}}^2 X_2 u\big) \qquad (2.4) \\ &\quad + \|\nabla_0 u\|_{\mathbb{H}}^{\mathtt{p}-2}\big(X_1 X_1 u + X_2 X_2 u\big) \\ &= \|\nabla_0 u\|_{\mathbb{H}}^{\mathtt{p}-4}\Bigg((\mathtt{p}-2)\frac{1}{2}\langle \nabla_0 \|\nabla_0 u\|_{\mathbb{H}}^2, \nabla_0 u\rangle_{\mathbb{H}} \\ &\quad + \|\nabla_0 u\|_{\mathbb{H}}^2\big(X_1 X_1 u + X_2 X_2 u\big)\Bigg). \end{aligned}$$

Chapter 3

Motivating Results

3.1 Grushin-type Planes

In their paper, Bieske and Gong [4] found the following in the Grushin-type planes.

THEOREM 3.1 ([4]) *Let* $1 < p < \infty$. *Let*

$$f(y_1, y_2) = c^2(y_1 - a)^{(2n+2)} + (n+1)^2(y_2 - b)^2.$$

For $p \neq n+2$, *let*

$$\tau_p = \frac{n+2-p}{(2n+2)(1-p)}$$

and let

$$\psi_p = \begin{cases} f(y_1, y_2)^{\tau_p} & p \neq n+2 \\ \log f(y_1, y_2) & p = n+2. \end{cases}$$

Then on $\mathbb{G} \setminus \{(a,b)\}$, *we have*

$$\Delta_p \psi_p = 0.$$

In the Grushin-type planes, Beals, Gaveau and Greiner [2] extend this equation as shown in the following theorem. The proof proved by [2] is done abstractly. In this thesis, we shall prove the theorem directly.

THEOREM 3.2 (BGG,C) *Let* $L \in \mathbb{R}$. *Following BGG [2], we shall consider the following quantities,*

$$\begin{aligned} \alpha &= \frac{-n}{(2n+2)}(1+L) \\ \beta &= \frac{-n}{(2n+2)}(1-L) \end{aligned}$$

where $L \in \mathbb{R}$. We use these constants with the functions

$$\begin{aligned} g(y_1, y_2) &= c(y_1 - a)^{n+1} + i(n+1)(y_2 - b) \\ h(y_1, y_2) &= c(y_1 - a)^{n+1} - i(n+1)(y_2 - b) \end{aligned}$$

to define our main function $f(y_1, y_2)$, given by

$$f(y_1, y_2) = g(y_1, y_2)^{\alpha} h(y_1, y_2)^{\beta}$$

Then, $\Delta_2 f + iL[Y_1, Y_2]f = 0$.

Proof. Suppressing the variables (y_1, y_2) and recalling that

$$[Y_1, Y_2]f = cn(y_1 - a)^{n-1} \frac{\partial f}{\partial y_2},$$

we compute:

$$\begin{aligned} Y_1 f &= g^{\alpha-1} h^{\beta-1} (n+1) c (y_1 - a)^n (\alpha h + \beta g) \\ Y_1 Y_1 f &= g^{\alpha-2} h^{\beta-2} c (n+1) (y_1 - a)^{n-1} \times \\ &\quad \Big(ngh(\alpha h + \beta g) + c(n+1)(y_1 - a)^{n+1} \times \\ &\quad \Big(gh(\alpha + \beta) + (\alpha h + \beta g)\big((\alpha - 1)h + (\beta - 1)g\big)\Big)\Big) \\ Y_2 f &= i(n+1) c (y_1 - a)^n g^{\alpha-1} h^{\beta-1} (\alpha h - \beta g) \\ Y_2 Y_2 f &= -(n+1)^2 c^2 (y_1 - a)^{2n} g^{\alpha-2} h^{\beta-2} \times \\ &\quad \big(gh(-\alpha - \beta) + (\alpha h - \beta g)\big((\alpha - 1)h - (\beta - 1)g\big)\big) \\ [Y_1, Y_2] f &= i(n+1) cn (y_1 - a)^{n-1} g^{\alpha-1} h^{\beta-1} (\alpha h - \beta g) \end{aligned}$$

Thus,

$$
\begin{aligned}
&Y_1Y_1f + Y_2Y_2f + iL[Y_1,Y_2]f = \\
&\quad g^{\alpha-2}h^{\beta-2}c(n+1)(y_1-a)^{n-1}\Big(ngh(\alpha h+\beta g) \\
&\quad + c(n+1)(y_1-a)^{n+1} \times \\
&\quad \Big(gh(\alpha+\beta) + (\alpha h+\beta g)\big((\alpha-1)h+(\beta-1)g\big) \\
&\quad - gh(-\alpha-\beta) - (\alpha h-\beta g)\big((\alpha-1)h-(\beta-1)g\big)\Big) \\
&\quad + iL(ingh(\alpha h-\beta g))\Big) \\
&= g^{\alpha-1}h^{\beta-1}c(n+1)(y_1-a)^{n-1}\Big(n(\alpha h+\beta g) \\
&\quad + c(n+1)(y_1-a)^{n+1}\big(2(\alpha+\beta) + \\
&\quad (2\alpha(\beta-1)+2\beta)(\alpha-1)\big) - Ln(\alpha h-\beta g)\Big) \\
&= g^{\alpha-1}h^{\beta-1}c(n+1)(y_1-a)^{n-1}\Big(n(\alpha h+\beta g) \\
&\quad + c(n+1)(y_1-a)^{n+1}\big(2(\alpha+\beta) \\
&\quad + 2\alpha(\beta-1)+2\beta(\alpha-1)\big) - Ln(\alpha h-\beta g)\Big) \\
&= g^{\alpha-1}h^{\beta-1}c(n+1)(y_1-a)^{n-1}\Big(n(\alpha h+\beta g) \\
&\quad + c(n+1)(y_1-a)^{n+1}4\alpha\beta\ - Ln(\alpha h-\beta g)\Big) \\
&= g^{\alpha-1}h^{\beta-1}c(n+1)(y_1-a)^{n-1} \times \\
&\quad \Big(n\beta(1+L) + n\alpha(1-L) + 4\alpha\beta c(n+1)(y_1-a)^{n+1}\Big) \\
&= g^{\alpha-1}h^{\beta-1}c(n+1)(y_1-a)^{n-1} \times \\
&\quad \Big(2\frac{(1+L)(1-L)n^2}{-2n-2}c(y_1-1)^{n+1} \\
&\quad + 4\frac{(1+L)(1-L)n^2}{(-2n-2)^2}c(n+1)(y_1-1)^{n+1}\Big) = 0.
\end{aligned}
$$

□

3.2 The Heisenberg Group

In their paper, Capogna, Danielli, and Garofalo [6] proved the following theorem.

THEOREM 3.3 ([6]) *Let* $1 < p < \infty$. *In* $\mathbb{H}^1 \setminus \{0\}$, *let*

$$u(x_1, x_2, x_3) = ((x_1^2 + x_2^2)^2 + 16x_3^2).$$

For $p \neq 4$, *let*

$$\eta_p = \frac{4-p}{4(1-p)},$$

and let

$$\zeta_p = \begin{cases} u(x_1, x_2, x_3)^{\eta_p} & p \neq 4 \\ \log u(x_1, x_2, x_3) & p = 4. \end{cases}$$

Then we have

$$\Delta_p \zeta_p = 0.$$

In the Heisenberg Group, [2] extend this equation as shown in the following theorem. As in the Grushin case, they have computed this abstractly, we will prove the theorem directly.

THEOREM 3.4 (BGG) *Let* $L \in \mathbb{R}$. *Following [2], we shall consider the following constants,*

$$\begin{aligned} \eta &= \frac{L-1}{2} \\ \tau &= \frac{-(L+1)}{2} \end{aligned}$$

together with the functions,

$$\begin{aligned} v(x_1, x_2, x_3) &= (x_1^2 + x_2^2) - 4ix_3 \\ w(x_1, x_2, x_3) &= (x_1^2 + x_2^2) + 4ix_3 \end{aligned}$$

to define our main function, $u(x_1, x_2, x_3)$

$$u(x_1, x_2, x_3) = v(x_1, x_2, x_3)^{\eta} w(x_1, x_2, x_3)^{\tau}.$$

Then, $\Delta_2 u + iL[X_1, X_2]u = 0$.

Proof. Suppressing the variables (x_1, x_2, x_3), we compute

$$\begin{aligned}
X_1 u &= 2v^{\eta-1}w^{\tau-1}\big((\eta w+\tau v)x_1+(\eta w-\tau v)ix_2\big)\\
X_1X_1 u &= 2v^{\eta-2}w^{\tau-2}\Big(2\big((\eta w+\tau v)x_1^2\\
&\quad + (\eta w-\tau v)ix_1x_2\big)\big((\eta-1)w+(\tau-1)v\big)\\
&\quad - 2i\big((\eta w+\tau v)x_1x_2+(\eta w-\tau v)ix_2^2\big)\big(-(\eta-1)w+(\tau-1)v\big)\\
&\quad + vw\big(2(x_1^2+x_2^2)(\tau+\eta)+vw(\eta w+\tau v)\big)\Big)\\
X_2 u &= 2v^{\eta-1}w^{\tau-1}\big((\eta w+\tau v)x_2+(-\eta w+\tau v)ix_1\big)\\
X_2X_2 u &= 2v^{\eta-2}w^{\tau-2}\Big(2\big((\eta w+\tau v)x_2^2\\
&\quad + (-\eta w+\tau v)ix_1x_2\big)\big((\eta-1)w+(\tau-1)v\big)\\
&\quad + 2i\big((\eta w+\tau v)x_1x_2+(-\eta w+\tau v)ix_1^2\big)\big(-(\eta-1)w+(\tau-1)v\big)\\
&\quad + vw\big(2(x_1^2+x_2^2)(\eta+\tau)+(\eta w+\tau v)\big)\Big)\\
X_3 u &= 4iv^{\eta-1}w^{\tau-1}(-\eta w+\tau v).
\end{aligned}$$

Thus,

$$\begin{aligned}
&X_1X_1u+X_2X_2u+iLX_3u=\\
&\qquad 2v^{\eta-2}w^{\tau-2}\times\big(2\big((\eta w+\tau v)x_1^2+(\eta w-\tau v)ix_1x_2\big)\big((\eta-1)w+(\tau-1)v\big)\\
&\qquad -2i\big((\eta w+\tau v)x_1x_2+(\eta w-\tau v)ix_2^2\big)\big(-(\eta-1)w+(\tau-1)v\big)\\
&\qquad +vw\big(2(x_1^2+x_2^2)(\tau+\eta)+vw(\eta w+\tau v)\big)+2\big((\eta w+\tau v)x_2^2\\
&\qquad +(-\eta w+\tau v)ix_1x_2\big)\big((\eta-1)w+(\tau-1)v\big)\\
&\qquad +2i\big((\eta w+\tau v)x_1x_2+(-\eta w+\tau v)ix_1^2\big)\big(-(\eta-1)w+(\tau-1)v\big)\\
&\qquad +vw\big(2(x_1^2+x_2^2)(\eta+\tau)+(\eta w+\tau v)\big)+iLvw\big(4ivw(-\eta w+\tau v)\big)\big)\\
&= 2v^{\eta-1}w^{\tau-1}\times\big(2(\eta w+\tau v)+4(\eta+\tau)(x_1^2+x_2^2)\\
&\qquad +4(x_1^2+x_2^2)(2\eta\tau-\eta-\tau)-4L(-\eta w+\tau v)\big)\\
&= 2^2v^{\eta-1}w^{\tau-1}\times\big((\eta w+\tau v)+(x_1^2+x_2^2)(1-L^2)-L(-\eta w+\tau v)\big)\\
&= 2^2v^{\eta-1}w^{\tau-1}\left((x_1^2+x_2^2)(1-L^2)+\left(\frac{-1+L^2}{2}\right)w+\left(\frac{-1+L^2}{2}\right)v\right)\\
&= 2^2v^{\eta-1}w^{\tau-1}\big((x_1^2+x_2^2)(1-L^2)+(x_1^2+x_2^2)(-1+L^2)\big)=0.
\end{aligned}$$

□

Observation 1 *In $\mathbb{G} \setminus \{(a,b)\}$, we have when $p = 2$,*

$$f_2(y_1, y_2) = \left(c^2(y_1 - a)^{2n+2} + (n+1)^2(y_2 - b)^2\right)^{-\dfrac{n}{2n+2}}$$

solves

$$\Delta_2 f_2 = 0.$$

Also,

$$\widehat{f_L}(y_1, y_2) = g(y_1, y_2)^{-\frac{n}{2n+2}(1+L)} h(y_1, y_2)^{-\frac{n}{2n+2}(1-L)}$$

where

$$g(y_1, y_2) = c(y_1 - a)^{n+1} + i(n+1)(y_2 - b)$$

and

$$h(y_1, y_2) = c(y_1 - a)^{n+1} - i(n+1)(y_2 - b),$$

solves

$$\Delta_2 \widehat{f_L} + iL[Y_1, Y_2]\widehat{f_L} = 0.$$

Notice that the equations and solutions coincide when $L = 0$. That is,

$$\widehat{f_0} = f_2.$$

Similarly, in $\mathbb{H}^1 \setminus \{0\}$,

$$u_2(x_1, x_2, x_3) = \left((x_1^2 + x_2^2)^2 + 16x_3^2\right)^{-\frac{1}{2}}$$

solves

$$\Delta_2 u_2 = 0.$$

Also,

$$\widehat{u_L}(x_1, x_2, x_3) = v(x_1, x_2, x_3)^{\frac{L-1}{2}} w(x_1, x_2, x_3)^{-\frac{(L+1)}{2}}$$

where

$$v(x_1, x_2, x_3) = (x_1^2 + x_2^2) - 4ix_3$$

and

$$w(x_1, x_2, x_3) = (x_1^2 + x_2^2) + 4ix_3$$

solves

$$\Delta_2 \widehat{u_L} + iL[X_1, X_2]\widehat{u_L} = 0.$$

Again, the equations and solutions coincide when $L = 0$. So again,

$$\widehat{u_0} = u_2.$$

These proofs rely on the linearity of Δ_2 and lead us to ask:

Main Question 1 *Can we extend this relationship in both $\mathbb{G} \setminus \{(a,b)\}$ and in $\mathbb{H}^1 \setminus \{0\}$ for all $p,\ 1 < p < \infty$? That is, can we find an equation $\Phi(p, L, \phi)$ so that $\phi_{p,L}$ solves $\Phi(p, L, \phi_{p,L}) = 0$, for all $p,\ 1 < p < \infty$ and for all $L \in \mathbb{R}$. In addition, we should have*

$$\Phi(2, L, \phi_{2,L}) = \Delta_2 \phi_{2,L} + iL[Z_1, Z_2]\phi_{2,L} = 0 \quad \textit{and} \tag{3.1}$$

$$\Phi(p, 0, \phi_{p,0}) = \Delta_p \phi_{p,0} = 0, \tag{3.2}$$

Where $Z_i = Y_i$ in the Grushin case and $Z_i = X_i$ in the Heisenberg case.

In order to answer this question, we first look at a good candidate for what the solution should be.

3.3 The Core Functions

3.3.1 The Core Grushin Function

For the Grushin-type planes, we consider the following for $\mathtt{p} \neq n + 2$:

$$\alpha = \frac{n + 2 - \mathtt{p}}{(1 - \mathtt{p})(2n + 2)}(1 + L)$$

$$\beta = \frac{n + 2 - \mathtt{p}}{(1 - \mathtt{p})(2n + 2)}(1 - L)$$

where $L \in \mathbb{R}$. We use these constants with the functions

$$g(y_1, y_2) = c(y_1 - a)^{n+1} + i(n + 1)(y_2 - b)$$

$$h(y_1, y_2) = c(y_1 - a)^{n+1} - i(n + 1)(y_2 - b)$$

to define our main function $f_{\mathtt{p},L}(y_1, y_2)$, given by

$$f_{\mathtt{p},L}(y_1, y_2) = \begin{cases} g(y_1, y_2)^{\alpha} h(y_1, y_2)^{\beta} & \mathtt{p} \neq n + 2 \\ \log\left(g^{1+L} h^{1-L}\right) & \mathtt{p} = n + 2. \end{cases}$$

From Theorems 3.1 and 3.2, we have $f_{\mathtt{p},L}(y_1, y_2)$ solves

$$\Delta_{\mathtt{p}} f_{\mathtt{p},L} + iL[Y_1, Y_2] f_{\mathtt{p},L} = 0$$

when $\mathtt{p}$ is arbitrary and $L = 0$ or when $\mathtt{p} = 2$ and for all L.

3.3.2 The Core Heisenberg Function

In the Heisenberg group, for $\mathtt{p} \neq 4$, we consider the following quantities:

$$\begin{aligned} \eta &= \frac{4-\mathtt{p}}{4(1-\mathtt{p})}(1-L) \\ \tau &= \frac{4-\mathtt{p}}{4(1-\mathtt{p})}(1+L) \end{aligned}$$

where $L \in \mathbb{R}$. We use these constants with the functions

$$\begin{aligned} v(x_1, x_2, x_3) &= (x_1^2 + x_2^2) - 4ix_3 \\ w(x_1, x_2, x_3) &= (x_1^2 + x_2^2) + 4ix_3 \end{aligned}$$

to define our main function $u_{\mathtt{p},L}(x_1, x_2, x_3)$, given by

$$u_{\mathtt{p},L}(x_1, x_2, x_3) = \begin{cases} v(x_1, x_2, x_3)^{\eta} w(x_1, x_2, x_3)^{\tau} & \mathtt{p} \neq 4 \\ \log\left(v^{1-L} w^{1+L}\right) & \mathtt{p} = 4. \end{cases}$$

From Theorems 3.3 and 3.4, we have $u_{\mathtt{p},L}(x_1, x_2, x_3)$ solves

$$\Delta_{\mathtt{p}} u_{\mathtt{p},L} + iL[X_1, X_2] u_{\mathtt{p},L} = 0$$

when $\mathtt{p}$ is arbitrary and $L = 0$ or when $\mathtt{p} = 2$ and for all L.

Chapter 4

A Negative Result

The "natural" generalization of the equation $\Delta_2\phi + iL[Z_1,Z_2]\phi$ is $\Delta_{\mathtt{p}}\phi + iL[Z_1,Z_2]\phi$ where $Z_i = Y_i$ in the Grushin-type planes and $Z_i = X_i$ in the Heisenberg group. We now consider this equation in each of our environments.

4.1 Grushin-Type Planes

Using the previous section, we consider the following functions in Grushin-type planes.

Due to our previous observation, we hypothesize that the same equation and solution should work for $\mathtt{p}$ arbitrary and for all $L \in \mathbb{R}$.

We will suppress the subscripts on the function f and on $\|\cdot\|$ for the upcoming computations.

THEOREM 4.1 *Let $f_{p,L}, \alpha, \beta$ be as in the previous section. Let $\mathtt{p} \neq n+2$ and let $L \in \mathbb{R}$ with $L \neq \pm 1$. Then in $\mathbb{G} \setminus \{(a,b)\}$*

$$\begin{aligned}&\Delta_p f_{p,L} + iL(p-1)\|\nabla_0 f_{p,L}\|^{p-2}[Y_1,Y_2]f_{p,L}\\ &-\|\nabla_0 f_{p,L}\|^{p-2}\frac{L^2}{L^2-1}(-4)\left(\frac{(p-2)(1+np)}{2+n-p}\right)(Y_2 g^\alpha)(Y_2 h^\beta) = 0.\end{aligned}$$

In particular, $\Delta_p f_{p,L} + iL[Y_1,Y_2]f_{p,L}$ need not be zero.

Proof. Recall from equation (2.2),

$$\begin{aligned}\Delta_{\mathtt{p}} f &= \|\nabla_0 f\|^{\mathtt{p}-4}\Big((\mathtt{p}-2)\frac{1}{2}\Big(Y_1\|\nabla_0 f\|^2(Y_1 f) + Y_2\|\nabla_0 f\|^2(Y_2 f)\Big)\\ &\quad + \|\nabla_0 f\|^2\big(Y_1Y_1 f + Y_2Y_2 f\big)\Big).\end{aligned}$$

Thus to show

$$\begin{aligned}&\Delta_{\mathtt{p}} f + iL(\mathtt{p}-1)\|\nabla_0 f\|^{\mathtt{p}-2}[Y_1,Y_2]f\\ &-\|\nabla_0 f\|^{\mathtt{p}-2}\frac{L^2}{L^2-1}(-4)\left(\frac{(\mathtt{p}-2)(1+n\mathtt{p})}{2+n-\mathtt{p}}\right)(X_2 g^\alpha)(X_2 h^\beta) = 0,\end{aligned}$$

we will consider

$$\begin{aligned}
&\|\nabla_0 f\|^{\mathtt{p}-4}\Big((\mathtt{p}-2)\frac{1}{2}\Big(Y_1\|\nabla_0 f\|^2(Y_1 f)+Y_2\|\nabla_0 f\|^2(Y_2 f)\Big)\\
&+\|\nabla_0 f\|^2\big(Y_1Y_1 f+Y_2Y_2 f\big)\Big)+iL(\mathtt{p}-1)\|\nabla_0 f\|^{\mathtt{p}-2}[Y_1,Y_2]f\\
&-\|\nabla_0 f\|^{\mathtt{p}-2}\frac{L^2}{L^2-1}(-4)\left(\frac{(\mathtt{p}-2)(1+n\mathtt{p})}{2+n-\mathtt{p}}\right)(X_2 g^{\alpha})(X_2 h^{\beta})\\
=\ &\|\nabla_0 f\|^{\mathtt{p}-4}\Bigg(\Big((\mathtt{p}-2)\frac{1}{2}\Big(Y_1\|\nabla_0 f\|^2(Y_1 f)+Y_2\|\nabla_0 f\|^2(Y_2 f)\Big)\\
&+\|\nabla_0 f\|^2\big(Y_1Y_1 f+Y_2Y_2 f\big)\Big)+iL(\mathtt{p}-1)\|\nabla_0 f\|^2[Y_1,Y_2]f\\
&-\|\nabla_0 f\|^2\frac{L^2}{L^2-1}(-4)\left(\frac{(\mathtt{p}-2)(1+n\mathtt{p})}{2+n-\mathtt{p}}\right)(X_2 g^{\alpha})(X_2 h^{\beta})\Bigg).
\end{aligned}$$

We need only show that

$$\begin{aligned}
&\Big((\mathtt{p}-2)\frac{1}{2}\Big(Y_1\|\nabla_0 f\|^2(Y_1 f)+Y_2\|\nabla_0 f\|^2(Y_2 f)\Big)\\
&+\|\nabla_0 f\|^2\big(Y_1Y_1 f+Y_2Y_2 f\big)\Big)+iL(\mathtt{p}-1)\|\nabla_0 f\|^2[Y_1,Y_2]f\\
&-\|\nabla_0 f\|^2\frac{L^2}{L^2-1}(-4)\left(\frac{(\mathtt{p}-2)(1+n\mathtt{p})}{2+n-\mathtt{p}}\right)(X_2 g^{\alpha})(X_2 h^{\beta})=0.
\end{aligned}$$

To show this, we will require the following quantities. We compute for $\mathtt{p}\neq n+2$,

$$\begin{aligned}
Y_1 f &= c(n+1)(y_1-a)^n g^{\alpha-1}h^{\beta-1}(\alpha h+\beta g)\\
\overline{Y_1 f} &= c(n+1)(y_1-a)^n g^{\beta-1}h^{\alpha-1}(\alpha g+\beta h)\\
Y_2 f &= ic(n+1)(y_1-a)^n g^{\alpha-1}h^{\beta-1}(\alpha h-\beta g)\\
\overline{Y_2 f} &= ic(n+1)(y_1-a)^n g^{\beta-1}h^{\alpha-1}(-\alpha g+\beta h)\\
\|\nabla_0 f\|^2 &= 4c^2(n+1)^2(y_1-a)^{2n}g^{\alpha+\beta-1}h^{\alpha+\beta-1}(\alpha^2+\beta^2)\\
Y_1(Y_1 f) &= c(n+1)(y_1-a)^{n-1}g^{\alpha-2}h^{\beta-2}\Big(ngh(\alpha h+\beta g)+\\
&\quad c(n+1)(y_1-a)^{n+1}\Big((\alpha h+\beta g)\big((\alpha-1)h-(\beta-1)g\big)\\
&\quad +gh(\alpha+\beta)\Big)\Big)\\
Y_2(Y_2 f) &= -c^2(n+1)^2(y_1-a)^{2n}g^{\alpha-2}h^{\beta-2}\times\\
&\quad\Big((\alpha h+\beta g)\big((\alpha-1)h-(\beta-1)g\big)-gh(\alpha+\beta)\Big).
\end{aligned}$$

To proceed, we shall require the following,

$$\begin{aligned} Y_1\|\nabla_0 f\|^2 &= 2^2c^2(n+1)^2(\alpha^2+\beta^2)(y_1-a)^{2n-1}g^{\alpha+\beta-2}h^{\alpha+\beta-2} \\ &\quad \times \big(ngh + c^2(n+1)(\alpha+\beta-1)(y_1-a)^{2n+2}\big) \end{aligned}$$

and

$$\begin{aligned} Y_2\|\nabla_0 f\|^2 &= 2^2c^3(n+1)^4(\alpha^2+\beta^2)(y_1-a)^{3n}(y_2-b) \\ &\quad \times (\alpha+\beta-1)g^{\alpha+\beta-2}h^{\alpha+\beta-2}. \end{aligned}$$

Using the above quantities, we now compute

$$\begin{aligned} & Y_1\|\nabla_0 f\|^2(Y_1 f) + Y_2\|\nabla_0 f\|^2(Y_2 f) = \\ &\quad 2^2c^3(n+1)^3(\alpha^2+\beta^2)(y_1-a)^{3n-1}g^{2\alpha+\beta-3}h^{\alpha+2\beta-3} \\ &\quad \times \Big((\alpha h+\beta g)\big(ngh + c^2(n+1)(\alpha+\beta-1)(y_1-a)^{2n+2}\big) \\ &\quad + ic(n+1)^2(y_1-a)^{n+1}(y_2-b)(\alpha+\beta-1)(\alpha h-\beta g)\Big) \end{aligned}$$

and

$$\begin{aligned} & \|\nabla_0 f\|^2(Y_1Y_1 f + Y_2Y_2 f) = \\ &\quad 2c^3(n+1)^3(\alpha^2+\beta^2)(y_1-a)^{3n-1}g^{2\alpha+\beta-3}h^{\alpha+2\beta-3} \\ &\quad \times \Big(ngh(\alpha h+\beta g) + 4c(n+1)(y_1-a)^{n+1}gh(\alpha\beta)\Big). \end{aligned}$$

We can then calculate

$$\begin{aligned} & (\mathtt{p}-2)\frac{1}{2}\Big(Y_1\|\nabla_0 f\|^2(Y_1 f) + Y_2\|\nabla_0 f\|^2(Y_2 f)\Big) \\ & + \|\nabla_0 f\|^2\big(Y_1Y_1 f + Y_2Y_2 f\big) \\ = \; & 2c^3(n+1)^3(\alpha^2+\beta^2)(y_1-a)^{3n-1}g^{2\alpha+\beta-3}h^{\alpha+2\beta-3} \\ & \times \Big((p-1)(\alpha h+\beta g)ngh \\ & + (p-2)c^2(n+1)(y_1-a)^{2n+2}(\alpha+\beta-1)(\alpha h+\beta g) \\ & + ic(p-2)(n+1)^2(y_1-a)^{n+1}(y_2-b)(\alpha+\beta-1)(\alpha h-\beta g) \\ & + 2^2c(n+1)(y_1-a)^{n+1}\alpha\beta gh\Big). \end{aligned}$$

We will need the above quantity with

$$iL(\mathtt{p}-1)\|\nabla_0 f\|^2[Y_1,Y_2]f \quad = \quad -2L(p-1)c^3 n(n+1)^3(\alpha^2+\beta^2)$$

$$\times (y_1-a)^{3n-1}(\alpha h-\beta g)g^{2\alpha+\beta-2}h^{\alpha+2\beta-2}$$

and

$$\|\nabla_0 f\|^2 \frac{L^2}{L^2-1}(-4)\left(\frac{(\mathtt{p}-2)(1+n\mathtt{p})}{2+n-\mathtt{p}}\right)(X_2 g^\alpha)(X_2 h^\beta) =$$

$$-2^3c^4(n+1)^4(y_1-a)^{4n}(\alpha^2+\beta^2)\alpha\beta g^{2\alpha+\beta-2}h^{\alpha+2\beta-2}$$

$$\times \frac{(L^2)(\mathtt{p}-2)(1-n\mathtt{p})}{(L^2-1)(2+n-\mathtt{p})}.$$

We let Λ be defined as

$$\Lambda \quad = \quad \Big((\mathtt{p}-2)\frac{1}{2}\Big(Y_1\|\nabla_0 f\|^2(Y_1 f)+Y_2\|\nabla_0 f\|^2(Y_2 f)\Big)$$

$$+\|\nabla_0 f\|^2\big(Y_1Y_1 f+Y_2Y_2 f\big)\Big)+iL(\mathtt{p}-1)\|\nabla_0 f\|^2[Y_1,Y_2]f$$

$$-\|\nabla_0 f\|^2\frac{L^2}{L^2-1}(-4)\left(\frac{(\mathtt{p}-2)(1+n\mathtt{p})}{2+n-\mathtt{p}}\right)(X_2 g^\alpha)(X_2 h^\beta).$$

We then compute

$$\begin{aligned}
\Lambda \;=\;& 2c^3(n+1)^3(\alpha^2+\beta^2)(y_1-a)^{3n-1}g^{2\alpha+\beta-3}h^{\alpha+2\beta-3}\\
&\times\Big((p-1)(\alpha h+\beta g)ngh\\
&+(p-2)c^2(n+1)(y_1-a)^{2n+2}(\alpha+\beta-1)(\alpha h+\beta g)\\
&+ic(p-2)(n+1)^2(y_1-a)^{n+1}(y_2-b)(\alpha+\beta-1)(\alpha h-\beta g)\\
&+2^2c(n+1)(y_1-a)^{n+1}\alpha\beta gh\Big)\\
&+\Big(-2L(p-1)c^3n(n+1)^3(\alpha^2+\beta^2)\\
&\times(y_1-a)^{3n-1}(\alpha h-\beta g)g^{2\alpha+\beta-2}h^{\alpha+2\beta-2}\Big)\\
&-\Big(-2^3c^4(n+1)^4(y_1-a)^{4n}(\alpha^2+\beta^2)\alpha\beta g^{2\alpha+\beta-2}h^{\alpha+2\beta-2}\\
&\times\frac{(L^2)(\mathtt{p}-2)(1-n\mathtt{p})}{(L^2-1)(2+n-\mathtt{p})}\Big)\\
=\;& 2c^3(n+1)^3(\alpha^2+\beta^2)(y_1-a)^{3n-1}g^{2\alpha+\beta-3}h^{\alpha+2\beta-3}\\
&\Bigg(ngh(p-1)\big((\alpha h+\beta g)-L(\alpha h-\beta g)\big)\\
&2^2c(n+1)(y_1-a)^{n+1}\alpha\beta gh\left(1+\frac{(L^2)(\mathtt{p}-2)(1-n\mathtt{p})}{(L^2-1)(2+n-\mathtt{p})}\right)\\
&c(p-2)(n+1)(y_1-a)^{n+1}(\alpha+\beta-1)gh(\alpha+\beta)\Bigg)\\
=\;& 2c^3(n+1)^3(\alpha^2+\beta^2)(y_1-a)^{3n-1}g^{2\alpha+\beta-3}h^{\alpha+2\beta-3}\\
&\times\left(n(p-1)(h+g)\left(\frac{(-1+L^2)(2+n-\mathtt{p})}{2(1+n)(\mathtt{p}-1)}\right)\right.\\
&\left.+c(n+1)(y_1-a)^{n+1}\left(-\frac{n(-1+L^2)(2+n-\mathtt{p})}{(n+1)^2}\right)\right)\\
=\;& 2c^3(n+1)^3(\alpha^2+\beta^2)(y_1-a)^{3n-1}g^{2\alpha+\beta-3}h^{\alpha+2\beta-3}\\
&\times\left(c(y_1-a)^{n+1}\left(\frac{n(-1+L^2)(2+n-\mathtt{p})}{(n+1)}\right)\right.\\
&\left.+c(y_1-a)^{n+1}\left(-\frac{n(-1+L^2)(2+n-\mathtt{p})}{(n+1)}\right)\right)=0.
\end{aligned}$$

□

4.2 The Heisenberg group

Similar to the Grushin case in Theorem 4.1, in the Heisenberg group, Theorems 3.3 and 3.4 lead us to hypothesize that $u_{\mathrm{p},L}(x_1, x_2, x_3)$ should solve

$$\Delta_{\mathrm{p}} u_{\mathrm{p},L} + iL[X_1, X_2]u_{\mathrm{p},L} = 0$$

for p arbitrary and $L \in \mathbb{R}$

Unfortunately, we discover this is not the case. Again, we will suppress the subscripts on the function u and on $\|\cdot\|$ throughout our calculations.

THEOREM 4.2 *Let* $\mathrm{p} \neq 4$. *Then in* $\mathbb{H}^1 \setminus \{0\}$,

$$\Delta_p u + iL[X_1, X_2]u$$

need not be zero.

Proof. Recall from equation (2.4),

$$\begin{aligned}\Delta_{\mathrm{p}} u &= \|\nabla_0 u\|^{\mathrm{p}-4}\Big((\mathrm{p}-2)\frac{1}{2}\Big(X_1\|\nabla_0 u\|^2(X_1 u) + X_2\|\nabla_0 u\|^2(X_2 u)\Big)\Big) \\ &\quad + \|\nabla_0 u\|^2\big(X_1X_1u + X_2X_2u\big)\Big).\end{aligned}$$

Thus to compute

$$\Delta_{\mathrm{p}} u + iL[X_1, X_2]u,$$

we will consider

$$\begin{aligned}&\|\nabla_0 u\|^{\mathrm{p}-4}\Big((\mathrm{p}-2)\frac{1}{2}\Big(X_1\|\nabla_0 u\|^2(X_1 u) \\ &+ X_2\|\nabla_0 u\|^2(X_2 u)\Big) + \|\nabla_0 u\|^2\big(X_1X_1u + X_2X_2u\big)\Big) + iLX_3 u\end{aligned}$$

We compute:

$$\begin{aligned}
X_1u &= 2v^{\eta-1}w^{\tau-1}\big((\eta w+\tau v)x_1+(\eta w-\tau v)ix_2\big)\\
\overline{X_1u} &= 2w^{\eta-1}v^{\tau-1}\big((\eta v+\tau w)x_1-(\eta v-\tau w)ix_2\big)\\
X_2u &= 2v^{\eta-1}w^{\tau-1}\big((\eta w+\tau v)x_2-(\eta w-\tau v)ix_1\big)\\
\overline{X_2u} &= 2w^{\eta-1}v^{\tau-1}\big((\eta v+\tau w)x_2+(\eta v-\tau w)ix_1\big)\\
\|\nabla_0 u\|^2 &= 2^3(\eta^2+\tau^2)v^{\eta+\tau-1}w^{\eta+\tau-1}(x_1^2+x_2^2)\\
X_1X_1u &= 2v^{\eta-2}w^{\tau-2}\Big(2\big((\eta w+\tau v)x_1^2\\
&\quad +(\eta w-\tau v)ix_1x_2\big)\big((\eta-1)w+(\tau-1)v\big)\\
&\quad -2i\big((\eta w+\tau v)x_1x_2+(\eta w-\tau v)ix_2^2\big)\big(-(\eta-1)w+(\tau-1)v\big)\\
&\quad +vw\big(2(x_1^2+x_2^2)(\tau+\eta)+vw(\eta w+\tau v)\big)\Big)\\
X_2X_2u &= 2v^{\eta-2}w^{\tau-2}\Big(2\big((\eta w+\tau v)x_2^2\\
&\quad +(-\eta w+\tau v)ix_1x_2\big)\big((\eta-1)w+(\tau-1)v\big)\\
&\quad +2i\big((\eta w+\tau v)x_1x_2+(-\eta w+\tau v)ix_1^2\big)\big(-(\eta-1)w+(\tau-1)v\big)\\
&\quad +vw\big(2(x_1^2+x_2^2)(\eta+\tau)+(\eta w+\tau v)\big)\Big)\\
iLX_3u &= -4Lv^{\eta-1}w^{\tau-1}(-\eta w+\tau v).
\end{aligned}$$

To proceed, we shall require the following,

$$\begin{aligned}
X_1\|\nabla_0 u\|^2 &= 2^4(\eta^2+\tau^2)v^{\eta+\tau-2}w^{\eta+\tau-2}\\
&\quad\times\Bigg(x_1vw+2(\eta+\tau-1)x_1(x_1^2+x_2^2)^2\\
&\quad -8(\eta+\tau-1)x_2x_3(x_1^2+x_2^2)\Bigg)
\end{aligned}$$

and

$$\begin{aligned}
X_2\|\nabla_0 u\|^2 &= 2^4(\eta^2+\tau^2)v^{\eta+\tau-2}w^{\eta+\tau-2}\\
&\quad\times\Bigg(x_2vw+2(\eta+\tau-1)x_2(x_1^2+x_2^2)^2\\
&\quad +8(\eta+\tau-1)x_1x_3(x_1^2+x_2^2)\Bigg).
\end{aligned}$$

Using the above, we can now compute

$$
\begin{aligned}
&X_1\|\nabla_0 u\|^2(X_1u) + X_2\|\nabla_0 u\|^2(X_2u) = \\
&\quad 2^5(\eta^2+\tau^2)v^{2\eta+\tau-3}w^{\eta+2\tau-3} \\
&\quad \times \Big((\eta w+\tau v)vw(x_1^2+x_2^2) + \big(2(\eta w+\tau v)(\eta+\tau-1)(x_1^2+x_2^2)^3\big) \\
&\quad - 8(\eta w-\tau v)(\eta+\tau-1)ix_3(x_1^2+x_2^2)^2\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
&\|\nabla_0 u\|^2\big(X_1X_1u + X_2X_2u\big) = \\
&\quad 2^4(\eta^2+\tau^2)v^{2\eta+\tau-3}w^{\eta+2\tau-3}(x_1^2+x_2^2) \\
&\quad \Big(2vw(\eta w+\tau v) + 4vw(\eta+\tau)(x_1^2+x_2^2) \\
&\quad + 2\big((\eta-1)w+(\tau-1)v\big)(\eta w+\tau v)(x_1^2+x_2^2) \\
&\quad + 2\big(-(\eta-1)w+(\tau-1)v\big)(\eta w-\tau v)(x_1^2+x_2^2)\Big).
\end{aligned}
$$

We can now compute

$$
\begin{aligned}
&(\mathtt{p}-2)\frac{1}{2}\Big(X_1\|\nabla_0 u\|^2(X_1u) + X_2\|\nabla_0 u\|^2(X_2u)\Big) + \|\nabla_0 u\|^2\big(X_1X_1u + X_2X_2u\big) = \\
&-\left(\frac{1}{(-1+\mathtt{p})^4}\right) L(1+L^2)(-4+\mathtt{p})^3(x_1^2+x_2^2)g^{\frac{4+L(-4+\mathtt{p})+5\mathtt{p}}{4-4\mathtt{p}}}h^{\frac{4+4L+5\mathtt{p}-L\mathtt{p}}{4-4\mathtt{p}}} \\
&\big(L(-4+\mathtt{p})(x_1^2+x_2^2) + 4i(-1+\mathtt{p})\mathtt{p}x_3\big).
\end{aligned}
$$

Using this, we calculate:

$$
\Delta_{\mathtt{p}}u + iLX_3u = -8Lv^{\frac{1}{2}(-3+L)}\big(L(x_1^2+x_2^2) - 4ix_3\big)w^{\frac{1}{2}(-3-L)}.
$$

□

Chapter 5

Some Positive Results

5.1 A Traditional Divergence Form

The equation given in Theorem 4.1 is not in divergence form and the extra term is worrysome. So we shall try a divergence form. We first put the original 2-Laplace equation in divergence form to get

$$\Delta_2\phi + iL[Z_1, Z_2]\phi = \operatorname{div}\begin{pmatrix} Z_1\phi + iLZ_2\phi \\ Z_2\phi - iLZ_1\phi \end{pmatrix} \tag{5.1}$$

where Z_i is Y_i in the Grushin-type planes and Z_i is X_i in the Heisenberg group.

Inspired by the definition of Δ_{p} in equations (2.1) and (2.3), we consider

$$\overline{\Delta_{\mathrm{p}}}\phi = \operatorname{div}\left(\left\| \begin{matrix} Z_1\phi + iLZ_2\phi \\ Z_2\phi - iLZ_1\phi \end{matrix} \right\|^{p-2} \begin{pmatrix} Z_1\phi + iLZ_2\phi \\ Z_2\phi - iLZ_1\phi \end{pmatrix}\right), \tag{5.2}$$

where Z_i is Y_i in the Grushin case and Z_i is X_i in the Heisenberg case.

5.1.1 Grushin-type planes

Using equation (5.2), we have the following theorem. We will again suppress the subscripts on the function f and on $\|\cdot\|$.

THEOREM 5.1

$$\overline{\Delta_p}f = \operatorname{div}\left(\left\| \begin{matrix} Y_1f + iLY_2f \\ Y_2f - iLY_1f \end{matrix} \right\|^{p-2} \begin{pmatrix} Y_1f + iLY_2f \\ Y_2f - iLY_1f \end{pmatrix}\right) = 0.$$

Proof. First, we let

$$\Upsilon \;=\; \begin{pmatrix} \Upsilon_1 \\ \Upsilon_2 \end{pmatrix} = \begin{pmatrix} Y_1f + iLY_2f \\ Y_2f - iLY_1f \end{pmatrix},$$

and then we consider the following reduction:

$$
\begin{aligned}
\overline{\Delta_{\mathrm{p}}} f &= \operatorname{div}(\|\Upsilon\|^{\mathrm{p}-2}\Upsilon) \\
&= Y_1\big(\|\Upsilon\|^{\mathrm{p}-2}\Upsilon_1\big) + Y_2\big(\|\Upsilon\|^{\mathrm{p}-2}\Upsilon_2\big) \\
&= \frac{1}{2}(\mathrm{p}-2)\|\Upsilon\|^{\mathrm{p}-4} Y_1\|\Upsilon\|^2\Upsilon_1 + \|\Upsilon\|^{p-2} Y_1\Upsilon_1 \\
&\quad + \frac{1}{2}(\mathrm{p}-2)\|\Upsilon\|^{\mathrm{p}-4} Y_2\|\Upsilon\|^2\Upsilon_2 + \|\Upsilon\|^{p-2} Y_2\Upsilon_2 \\
&= \frac{1}{2}(\mathrm{p}-2)\|\Upsilon\|^{\mathrm{p}-4}\big(Y_1\|\Upsilon\|^2\Upsilon_1 + Y_2\|\Upsilon\|^2\Upsilon_2\big) \\
&\quad + \|\Upsilon\|^{\mathrm{p}-2}\big(Y_1\Upsilon_1 + Y_2\Upsilon_2\big) \\
&= \|\Upsilon\|^{\mathrm{p}-4}\Bigg(\frac{1}{2}(\mathrm{p}-2)\big(Y_1\|\Upsilon\|^2\Upsilon_1 + Y_2\|\Upsilon\|^2\Upsilon_2\big) \\
&\quad + \|\Upsilon\|^2\big(Y_1\Upsilon_1 + Y_2\Upsilon_2\big)\Bigg).
\end{aligned}
$$

Thus to show $\overline{\Delta_{\mathrm{p}}} f = 0$, we need only show that

$$
\Lambda = \frac{1}{2}(\mathrm{p}-2)\big(Y_1\|\Upsilon\|^2\Upsilon_1 + Y_2\|\Upsilon\|^2\Upsilon_2\big) + \|\Upsilon\|^2\big(Y_1\Upsilon_1 + Y_2\Upsilon_2\big) = 0.
$$

Case 1: We compute for $\mathrm{p} \neq n+2$,

$$
\begin{aligned}
Y_1 f &= c(n+1)(y_1-a)^n g^{\alpha-1} h^{\beta-1}(\alpha h + \beta g) \\
Y_2 f &= ic(n+1)(y_1-a)^n g^{\alpha-1} h^{\beta-1}(\alpha h - \beta g) \\
Y_1 f + iLY_2 f &= c(n+1)(y_1-a)^n g^{\alpha-1} h^{\beta-1}\big(\alpha h(1-L) + \beta g(1+L)\big) \\
\overline{Y_1 f + iLY_2 f} &= c(n+1)(y_1-a)^n h^{\alpha-1} g^{\beta-1}\big(\alpha g(1-L) + \beta h(1+L)\big) \\
Y_2 f - iLY_1 f &= ic(n+1)(y_1-a)^n g^{\alpha-1} h^{\beta-1}\big(\alpha h(1-L) - \beta g(1+L)\big) \\
\overline{Y_2 f - iLY_1 f} &= -ic(n+1)(y_1-a)^n h^{\alpha-1} g^{\beta-1}\big(\alpha g(1-L) - \beta h(1+L)\big) \\
\left\| \begin{matrix} Y_1 f + iLY_2 f \\ Y_2 f - iLY_1 f \end{matrix} \right\|^2 &= 2c^2(n+1)^2(y_1-a)^{2n} g^{\alpha+\beta-1} h^{\alpha+\beta-1} \\
&\quad \times \big(\alpha^2(1-L)^2 + \beta^2(1+L)^2\big).
\end{aligned}
$$

We then calculate:

$$Y_1(Y_1 f + iLY_2 f) + Y_2(Y_2 f - iLY_1 f) = \left(\frac{1}{(-1+\mathtt{p})^2 gh}\right)\Big(c^2(-1+L^2)(1+n)(2+n-\mathtt{p})(-2+\mathtt{p}) \times (y_1-a)^{2n} h^{\frac{(-1+L)(2+n-\mathtt{p})}{2(1+n)(-1+\mathtt{p})}} g^{-\frac{(1+L)(2+n-\mathtt{p})}{2(1+n)(-1+\mathtt{p})}}\Big).$$

We can then calculate

$$Y_1\left(\left\| \begin{array}{c} Y_1 f + iLY_2 f \\ Y_2 f - iLY_1 f \end{array} \right\|^2\right) = -\left(\frac{1}{(-1+\mathtt{p})^3 gh}\right)\Big(2c^2(-1+L^2)^2 \times (1+n)(2+n-\mathtt{p})^2 (y_1-a)^{-1+2n} h^{\frac{1}{1+n}+\frac{1}{1-\mathtt{p}}} \times \left(c^2(y_1-a)^{2+2n} + n(1+n)(-1+\mathtt{p})(y-b)^2\right) g^{\frac{1}{1+n}+\frac{1}{1-\mathtt{p}}}\Big)$$

and

$$Y_2\left(\left\| \begin{array}{c} Y_1 f + iLY_2 f \\ Y_2 f - iLY_1 f \end{array} \right\|^2\right) = \left(\frac{1}{(-1+\mathtt{p})^3 gh}\right)\Big(2c^3(-1+L^2)^2 \times (1+n)(2+n-\mathtt{p})^2(1+n\mathtt{p})(y_1-a)^{3n} h^{\frac{1}{1+n}+\frac{1}{1-\mathtt{p}}} \times (b-y_2) g^{\frac{1}{1+n}+\frac{1}{1-\mathtt{p}}}\Big).$$

Using the above quantities, we compute

$$Y_1\left(\left\| \begin{array}{c} Y_1 f + iLY_2 f \\ Y_2 f - iLY_1 f \end{array} \right\|^2\right)\left(Y_1 f + iLY_2 f\right) + Y_2\left(\left\| \begin{array}{c} Y_1 f + iLY_2 f \\ Y_2 f - iLY_1 f \end{array} \right\|^2\right)\left(Y_2 f - iLY_1 f\right) = -\left(\frac{1}{(-1+\mathtt{p})^4 (gh)^2}\right)\Big(2c^4(-1+L^2)^3(1+n)(2+n-\mathtt{p})^3(y_1-a)^{4n} \times h^{\frac{(-3+L)(2+n-\mathtt{p})}{2(1+n)(-1+\mathtt{p})}} g^{-\frac{(3+L)(2+n-\mathtt{p})}{2(1+n)(-1+\mathtt{p})}}\Big).$$

and

$$\left\| \begin{array}{c} Y_1 f + iLY_2 f \\ Y_2 f - iLY_1 f \end{array} \right\|^2 \big(Y_1(Y_1 f + iLY_2 f) + Y_2(Y_2 f - iLY_1 f)\big) = \left(\frac{1}{(-1+\mathtt{p})^4 (gh)^2}\right)\Big(c^4(-1+L^2)^3(1+n)(2+n-\mathtt{p})^3(-2+\mathtt{p})(y_1-a)^{4n} \times h^{\frac{(-3+L)(2+n-\mathtt{p})}{2(1+n)(-1+\mathtt{p})}} g^{-\frac{(3+L)(2+n-\mathtt{p})}{2(1+n)(-1+\mathtt{p})}}\Big).$$

We can then calculate

$$\begin{aligned}\Lambda &= \frac{1}{2}(\mathtt{p}-2)\Bigg(-\left(\frac{1}{(-1+\mathtt{p})^4(gh)^2}\right)\Big(2c^4(-1+L^2)^3(1+n)(2+n-\mathtt{p})^3(y_1-a)^{4n} \\ &\quad \times h^{\frac{(-3+L)(2+n-\mathtt{p})}{2(1+n)(-1+\mathtt{p})}} g^{-\frac{(3+L)(2+n-\mathtt{p})}{2(1+n)(-1+\mathtt{p})}}\Big)\Bigg) \\ &\quad + \Bigg(\left(\frac{1}{(-1+\mathtt{p})^4(gh)^2}\right)\Big(c^4(-1+L^2)^3(1+n)(2+n-\mathtt{p})^3(-2+\mathtt{p})(y_1-a)^{4n} \\ &\quad \times h^{\frac{(-3+L)(2+n-\mathtt{p})}{2(1+n)(-1+\mathtt{p})}} g^{-\frac{(3+L)(2+n-\mathtt{p})}{2(1+n)(-1+\mathtt{p})}}\Big)\Bigg) = 0.\end{aligned}$$

So we have $\overline{\Delta_{\mathtt{p}}}f = 0$ when $\mathtt{p} \neq n+2$.

Case 2: For $\mathtt{p} = n+2$, we compute:

$$\begin{aligned}Y_1 f &= c(n+1)(y_1-a)^n\left(\frac{1+L}{g}+\frac{1-L}{h}\right) \\ Y_2 f &= ic(n+1)(y_1-a)^n\left(\frac{1+L}{g}-\frac{1-L}{h}\right) \\ Y_1 f + iLY_2 f &= -\left(\frac{1}{gh}\right)\left(2c^2(L^2-1)(n+1)(y_1-a)^{2n+1}\right) \\ \overline{Y_1 f + iLY_2 f} &= Y_1 f + iLY_2 f && (5.3) \\ Y_2 f - iLY_1 f &= \left(\frac{1}{gh}\right)\left(2c(L^2-1)(n+1)^2(y_1-a)^n(y_2-b)\right) \\ \overline{Y_2 f - iLY_1 f} &= Y_2 f - iLY_1 f && (5.4) \\ \left\| \begin{matrix} Y_1 f + iLY_2 f \\ Y_2 f - iLY_1 f \end{matrix} \right\|^2 &= \left(\frac{1}{gh}\right)\left(4c^2(L^2-1)^2(n+1)^2(y_1-a)^{2n}\right).\end{aligned}$$

We then calculate:

$$\begin{aligned}&Y_1(Y_1 f + iLY_2 f) + Y_2(Y_2 f - iLY_1 f) = \\ &\qquad -\left(\frac{1}{gh}\right)\left(2c^2(L^2-1)n(n+1)(y_1-a)^{2n}\right).\end{aligned}$$

We can then calculate

$$Y_1\left(\left\| \begin{array}{c} Y_1 f + iLY_2 f \\ Y_2 f - iLY_1 f \end{array} \right\|^2\right) = -\left(\frac{1}{(gh)^2}\right)\Big(8c^2(L^2-1)^2(n+1)^2$$

$$\times (y_1-a)^{2n-1}\big(c^2(y_1-a)^{2n+2} + n(n+1)^2(y_2-b)^2\big)\Big)$$

$$Y_2\left(\left\| \begin{array}{c} Y_1 f + iLY_2 f \\ Y_2 f - iLY_1 f \end{array} \right\|^2\right) = -\left(\frac{1}{(gh)^2}\right)\big(8c^3(L^2-1)^2(n+1)^4$$

$$\times (y_1-a)^{3n}(y_2-b)\big).$$

Using the above quantities, we compute

$$Y_1\left(\left\| \begin{array}{c} Y_1 f + iLY_2 f \\ Y_2 f - iLY_1 f \end{array} \right\|^2\right)\big(Y_1 f + iLY_2 f\big)$$

$$+Y_2\left(\left\| \begin{array}{c} Y_1 f + iLY_2 f \\ Y_2 f - iLY_1 f \end{array} \right\|^2\right)\big(Y_2 f - iLY_1 f\big) =$$

$$\left(\frac{1}{(gh)^2}\right)\big(16c^4(L^2-1)^3(n+1)^3(y_1-a)^{4n}\big)$$

and

$$\left\| \begin{array}{c} Y_1 f + iLY_2 f \\ Y_2 f - iLY_1 f \end{array} \right\|^2 \big(Y_1(Y_1 f + iLY_2 f) + Y_2(Y_2 f - iLY_1 f)\big) =$$

$$-\left(\frac{1}{(gh)^2}\right)\big(8c^4(L^2-1)^3 n(n+1)^3(y_1-a)^{4n}\big).$$

We can then calculate

$$\begin{aligned}\Lambda &= \frac{1}{2}(\mathtt{p}-2)\Big(\left(\frac{1}{(gh)^2}\right)\big(16c^4(L^2-1)^3(n+1)^3(y_1-a)^{4n}\big)\Big)\\ &\quad + \Big(-\left(\frac{1}{(gh)^2}\right)\big(8c^4(L^2-1)^3 n(n+1)^3(y_1-a)^{4n}\big)\Big)\\ &= \left(\frac{1}{(gh)^2}\right)\big(8c^4(L^2-1)^3 n(n+1)^3(y_1-a)^{4n}\big)\\ &\quad - \left(\frac{1}{(gh)^2}\right)\big(8c^4(L^2-1)^3 n(n+1)^3(y_1-a)^{4n}\big) = 0.\end{aligned}$$

Thus $\overline{\Delta_{\mathtt{p}}} f = 0$ for $1 < \mathtt{p} < \infty$ and for all $L \in \mathbb{R}$. □

5.1.2 The Heisenberg Group

Using equation (5.2), we have the following theorem. We will again suppress the subscripts on the function u and on $\|\cdot\|$.

THEOREM 5.2

$$\overline{\Delta_p}u = \operatorname{div}\left(\left\| \begin{array}{c} X_1u + iLX_2u \\ X_2u - iLX_1u \end{array} \right\|^{p-2} \left(\begin{array}{c} X_1u + iLX_2u \\ X_2u - iLX_1u \end{array} \right)\right) = 0.$$

Proof. First, we let

$$\Upsilon \;=\; \left(\begin{array}{c} \Upsilon_1 \\ \Upsilon_2 \end{array} \right) = \left(\begin{array}{c} X_1u + iLX_2u \\ X_2u - iLX_1u \end{array} \right),$$

and then we consider the following reduction:

$$\begin{aligned}
\overline{\Delta_{\mathrm{p}}}u &= \operatorname{div}(\|\Upsilon\|^{\mathrm{p}-2}\Upsilon) \\
&= X_1\big(\|\Upsilon\|^{\mathrm{p}-2}\Upsilon_1\big) + X_2\big(\|\Upsilon\|^{\mathrm{p}-2}\Upsilon_2\big) \\
&= \frac{1}{2}(\mathrm{p}-2)\|\Upsilon\|^{\mathrm{p}-4}X_1\|\Upsilon\|^2\Upsilon_1 + \|\Upsilon\|^{\mathrm{p}-2}X_1\Upsilon_1 \\
&\quad + \frac{1}{2}(\mathrm{p}-2)\|\Upsilon\|^{\mathrm{p}-4}X_2\|\Upsilon\|^2\Upsilon_2 + \|\Upsilon\|^{\mathrm{p}-2}X_2\Upsilon_2 \\
&= \frac{1}{2}(\mathrm{p}-2)\|\Upsilon\|^{\mathrm{p}-4}\big(X_1\|\Upsilon\|^2\Upsilon_1 + X_2\|\Upsilon\|^2\Upsilon_2\big) \\
&\quad + \|\Upsilon\|^{\mathrm{p}-2}\big(X_1\Upsilon_1 + X_2\Upsilon_2\big) \\
&= \|\Upsilon\|^{\mathrm{p}-4}\Bigg(\frac{1}{2}(\mathrm{p}-2)\big(X_1\|\Upsilon\|^2\Upsilon_1 + X_2\|\Upsilon\|^2\Upsilon_2\big) \\
&\quad + \|\Upsilon\|^2\big(X_1\Upsilon_1 + X_2\Upsilon_2\big)\Bigg).
\end{aligned}$$

Thus to show $\overline{\Delta_{\mathrm{p}}}u = 0$, we need only show that

$$\Lambda = \frac{1}{2}(\mathrm{p}-2)\big(X_1\|\Upsilon\|^2\Upsilon_1 + X_2\|\Upsilon\|^2\Upsilon_2\big) + \|\Upsilon\|^2\big(X_1\Upsilon_1 + X_2\Upsilon_2\big) = 0.$$

Case 1: We compute for $\mathtt{p} \neq 4$,

$$
\begin{aligned}
X_1 u &= 2v^{\eta-1}w^{\tau-1}\big((\eta w+\tau v)x_1-(-\eta w+\tau v)ix_2\big)\\
X_2 u &= 2v^{\eta-1}w^{\tau-1}\big((\eta w+\tau v)x_2+(-\eta w+\tau v)ix_1\big)\\
X_1u+iLX_2u &= 2v^{\eta-1}w^{\tau-1}\Bigg((\eta w+\tau v)(x_1+iLx_2)\\
&\quad +(\eta w-\tau v)(Lx_1+ix_2)\Bigg)\\
\overline{X_1u+iLX_2u} &= 2w^{\eta-1}v^{\tau-1}\Bigg((\eta v+\tau w)(x_1-iLx_2)\\
&\quad +(\eta v-\tau w)(Lx_1-ix_2)\Bigg)\\
X_2u-iLX_1u &= 2v^{\eta-1}w^{\tau-1}\Bigg((\eta w+\tau v)(x_2-iLx_1)\\
&\quad +(-\eta w+\tau v)(ix_1-Lx_2)\Bigg)\\
\overline{X_2u-iLX_1u} &= 2w^{\eta-1}v^{\tau-1}\Bigg((\eta v+\tau w)(x_2+iLx_1)\\
&\quad +(-\eta v+\tau w)(ix_1+Lx_2)\Bigg)\\
\left\| \begin{matrix} X_1u+iLX_2u \\ X_2u-iLX_1u \end{matrix} \right\|^2 &= -\left(\frac{1}{(-1+\mathtt{p})^2}\right)(-1+L^2)^2(-4+\mathtt{p})^2(x_1^2+x_2^2)v^{\frac{2+\mathtt{p}}{2-2\mathtt{p}}}w^{\frac{2+\mathtt{p}}{2-2\mathtt{p}}}.
\end{aligned}
$$

We then calculate

$$
\begin{aligned}
&X_1(X_1u+iLX_2u)+X_2(X_2u-iLX_1u)=\\
&\quad -\left(\frac{1}{(-1+\mathtt{p})^2}\right)\left(3(-1+L^2)(-4+\mathtt{p})(-2+\mathtt{p})(x_1^2+x_2^2)v^{-\frac{L(-4+\mathtt{p})+3\mathtt{p}}{4(-1+\mathtt{p})}}w^{\frac{4L+3\mathtt{p}-L\mathtt{p}}{4-4\mathtt{p}}}\right).
\end{aligned}
$$

We shall also require the following:

$$X_1\left(\left\| \begin{array}{c} X_1u+iLX_2u \\ X_2u-iLX_1u \end{array} \right\|^2\right) = \left(\frac{1}{(-1+\mathtt{p})^3(vw)^2}\right)\Big(2(-1+L^2)^2(-4+\mathtt{p})^2$$
$$\times\, v^{\frac{-4+\mathtt{p}}{2(-1+\mathtt{p})}} w^{\frac{-4+\mathtt{p}}{2(-1+\mathtt{p})}} \big(-3x_1(x_1^2+x_2^2)^2$$
$$+4(2+\mathtt{p})x_2(x_1^2+x_2^2)x_3+16(-1+\mathtt{p})x_1x_3^2\big)\Big)$$

$$X_2\left(\left\| \begin{array}{c} X_1u+iLX_2u \\ X_2u-iLX_1u \end{array} \right\|^2\right) = -\left(\frac{1}{(-1+\mathtt{p})^3(vw)^2}\right)\Big(2(-1+L^2)^2(-4+\mathtt{p})^2$$
$$\times\, v^{\frac{-4+\mathtt{p}}{2(-1+\mathtt{p})}} w^{\frac{-4+\mathtt{p}}{2(-1+\mathtt{p})}} \big(3x_2(x_1^2+x_2^2)^2$$
$$+4(2+\mathtt{p})x_1(x_1^2+x_2^2)x_3-16(-1+\mathtt{p})x_2x_3^2\big)\Big).$$

Using the above quantities, we compute

$$X_1\left(\left\| \begin{array}{c} X_1u+iLX_2u \\ X_2u-iLX_1u \end{array} \right\|^2\right)\big(X_1u+iLX_2u\big)+X_2\left(\left\| \begin{array}{c} X_1u+iLX_2u \\ X_2u-iLX_1u \end{array} \right\|^2\right)\big(X_2u-iLX_1u\big) =$$
$$\left(\frac{1}{(-1+\mathtt{p})^3(vw)^2}\right)\Big(6(-1+L^2)^3(-4+\mathtt{p})^3(x_1^2+x_2^2)^2 v^{-\frac{(-3+L)(-4+\mathtt{p})}{4(-1+\mathtt{p})}} w^{\frac{(3+L)(-4+\mathtt{p})}{4(-1+\mathtt{p})}}\Big)$$

and

$$\left\| \begin{array}{c} X_1u+iLX_2u \\ X_2u-iLX_1u \end{array} \right\|^2\big(X_1(X_1u+iLX_2u)+X_2(X_2u-iLX_1u)\big) =$$
$$-\left(\frac{1}{(-1+\mathtt{p})^4}\right)\Big(3(-1+L^2)^3(-4+\mathtt{p})^3(-2+\mathtt{p})(x_1^2+x_2^2)^2 v^{\frac{4+L(-4+\mathtt{p})+5\mathtt{p}}{4-4\mathtt{p}}} w^{\frac{4+4L+5\mathtt{p}-L\mathtt{p}}{4-4\mathtt{p}}}\Big).$$

We can then calculate

$$\begin{aligned}
\Lambda &= \frac{1}{2}(\mathtt{p}-2)\Bigg(\left(\left(\frac{1}{(-1+\mathtt{p})^3(vw)^2}\right)\Big(6(-1+L^2)^3(-4+\mathtt{p})^3(x_1^2+x_2^2)^2\right. \\
&\quad \left.\times v^{-\frac{(-3+L)(-4+\mathtt{p})}{4(-1+\mathtt{p})}} w^{\frac{(3+L)(-4+\mathtt{p})}{4(-1+\mathtt{p})}}\Big)\right) \\
&\quad + \left(-\left(\frac{1}{(-1+\mathtt{p})^4}\right)\Big(3(-1+L^2)^3(-4+\mathtt{p})^3(-2+\mathtt{p})(x_1^2+x_2^2)^2\right. \\
&\quad \left.\times v^{\frac{4+L(-4+\mathtt{p})+5\mathtt{p}}{4-4\mathtt{p}}} w^{\frac{4+4L+5\mathtt{p}-L\mathtt{p}}{4-4\mathtt{p}}}\Big)\right) \\
&= \left((\mathtt{p}-2)\left(\frac{1}{(-1+\mathtt{p})^3}\right)\Big(3(-1+L^2)^3(-4+\mathtt{p})^3(x_1^2+x_2^2)^2\right. \\
&\quad \left.\times v^{\frac{4+L(-4+\mathtt{p})+5\mathtt{p}}{4-4\mathtt{p}}} w^{\frac{4+4L+5\mathtt{p}-L\mathtt{p}}{4-4\mathtt{p}}}\Big)\right) \\
&\quad + \left(-\left(\frac{1}{(-1+\mathtt{p})^4}\right)\Big(3(-1+L^2)^3(-4+\mathtt{p})^3(-2+\mathtt{p})(x_1^2+x_2^2)^2\right. \\
&\quad \left.\times v^{\frac{4+L(-4+\mathtt{p})+5\mathtt{p}}{4-4\mathtt{p}}} w^{\frac{4+4L+5\mathtt{p}-L\mathtt{p}}{4-4\mathtt{p}}}\Big)\right) = 0.
\end{aligned}$$

So we have $\overline{\Delta_{\mathtt{p}}}u = 0$ when $\mathtt{p} \neq 4$.

Case 2: For $\mathtt{p} = 4$ we compute

$$\begin{aligned}
X_1u &= \left(\frac{1}{vw}\right)\Big(4\Big((x_1-iLx_2)(x_1^2+x_2^2) - 4(iLx_1-x_2)x_3\Big)\Big) \\
X_2u &= \left(\frac{1}{vw}\right)\Big(4\Big((iLx_1+x_2)(x_1^2+x_2^2) + 4(x_1-iLx_2)x_3\Big)\Big) \\
X_1u+iLX_2u &= -\left(\frac{1}{vw}\right)\Big(4(L^2-1)(x_1^3+x_1x_2^2-4x_2x_3)\Big) \\
\overline{X_1u+iLX_2u} &= X_1u+iLX_2u \qquad (5.5) \\
X_2u-iLX_1u &= -\left(\frac{1}{vw}\right)\Big(4(L^2-1)(x_2^3+x_1^2x_2+4x_1x_3)\Big) \\
\overline{X_2u-iLX_1u} &= X_2u-iLX_1u \qquad (5.6) \\
\left\|\begin{matrix} X_1u+iLX_2u \\ X_2u-iLX_1u \end{matrix}\right\|^2 &= \left(\frac{1}{vw}\right)\Big(16(L^2-1)^2(x_1^2+x_2^2)\Big).
\end{aligned}$$

We then calculate

$$X_1(X_1u + iLX_2u) + X_2(X_2u - iLX_1u) = \\ -\left(\frac{1}{vw}\right)\Big(8(L^2-1)^2(x_1^2+x_2^2)\Big).$$

We shall also require the following:

$$\begin{aligned} X_1\left(\left\| \begin{matrix} X_1u + iLX_2u \\ X_2u - iLX_1u \end{matrix} \right\|^2\right) &= -\left(\frac{1}{(vw)^2}\right)\Big(32(L^2-1)^2\big(x_1(x_1^2+x_2^2)^2 \\ &\quad - 8x_2(x_1^2+x_2^2)x_3 - 16x_1x_3^2\big)\Big) \\ X_2\left(\left\| \begin{matrix} X_1u + iLX_2u \\ X_2u - iLX_1u \end{matrix} \right\|^2\right) &= -\left(\frac{1}{(vw)^2}\right)\Big(32(L^2-1)^2\big(x_2(x_1^2+x_2^2)^2 \\ &\quad + 8x_1(x_1^2+x_2^2)x_3 - 16x_2x_3^2\big)\Big). \end{aligned}$$

Using the above quantities, we compute

$$X_1\left(\left\| \begin{matrix} X_1u + iLX_2u \\ X_2u - iLX_1u \end{matrix} \right\|^2\right)\big(X_1u + iLX_2u\big) \\ + X_2\left(\left\| \begin{matrix} X_1u + iLX_2u \\ X_2u - iLX_1u \end{matrix} \right\|^2\right)\big(X_2u - iLX_1u\big) =$$

$$\left(\frac{1}{(vw)^2}\right)\Big(128(L^2-1)^3(x_1^2+x_2^2)^2\Big)$$

and

$$\left\| \begin{matrix} X_1u + iLX_2u \\ X_2u - iLX_1u \end{matrix} \right\|^2\big(X_1(X_1u + iLX_2u) + X_2(X_2u - iLX_1u)\big) = \\ -\left(\frac{1}{(vw)^2}\right)\Big(128(L^2-1)^3(x_1^2+x_2^2)^2\Big).$$

We can then calculate

$$\begin{aligned}\Lambda &= \frac{1}{2}(\mathrm{p}-2)\Bigg(\left(\frac{1}{(vw)^2}\right)\left(128(L^2-1)^3(x_1^2+x_2^2)^2\right)\Bigg)\\ &\quad+\Bigg(-\left(\frac{1}{(vw)^2}\right)\left(128(L^2-1)^3(x_1^2+x_2^2)^2\right)\Bigg)\\ &= \left(\frac{1}{(vw)^2}\right)\left(128(L^2-1)^3(x_1^2+x_2^2)^2\right)\\ &\quad-\left(\frac{1}{(vw)^2}\right)\left(128(L^2-1)^3(x_1^2+x_2^2)^2\right)=0.\end{aligned}$$

Thus $\overline{\Delta_{\mathrm{p}}}u=0$ when $1<\mathrm{p}<\infty$ and for all $L\in\mathbb{R}$. □

5.2 An Unusual Divergence form

When working on the previous equation, we discovered a second equation that also meets our conditions. We consider:

$$\operatorname{div}\left((\nabla_0\phi\cdot\overline{\nabla_0\phi})^{\frac{p-2}{2}}\begin{pmatrix} Z_1\phi+iLZ_2\phi\\ Z_2\phi-iLZ_1\phi\end{pmatrix}\right)=0, \tag{5.7}$$

where Z_i is Y_i in the Grushin case and Z_i is X_i in the Heisenberg case. We note that when $\mathrm{p}=2$, this equation becomes

$$\Delta_2\phi+iL[Z_1,Z_2]\phi=0.$$

Also, when p is arbitrary and $L=0$, this equation becomes

$$\Delta_{\mathrm{p}}\phi=0,$$

because the exponents are equal in this case, allowing us to multiply the complex parts together, producing a real-valued function.

5.2.1 Grushin-type planes

We again suppress the subscripts on f and on $\|\cdot\|$.

THEOREM 5.3 *For all $1<p<\infty$ with and all $L\in\mathbb{R}$, we have:*

$$\operatorname{div}\left(\left(\nabla_0 f\cdot\overline{\nabla_0 f}\right)^{\frac{p-2}{2}}\begin{pmatrix} Y_1+iLY_2\\ Y_2-iLY_1\end{pmatrix}f\right)=0$$

in $\mathbb{G}\setminus\{(a,b)\}$.

Proof. Let

$$\Upsilon = \begin{pmatrix} \Upsilon_1 \\ \Upsilon_2 \end{pmatrix} = \begin{pmatrix} Y_1 f + iLY_2 f \\ Y_2 f - iLY_1 f \end{pmatrix}$$

and consider the following:

$$\begin{aligned} \widetilde{\Delta_{\mathtt{p}}} f &= \operatorname{div}\left(\big(\nabla_0 f \cdot \overline{\nabla_0 f}\big)^{\frac{\mathtt{p}-2}{2}} \Upsilon \right) \\ &= \operatorname{div}\left(\big((Y_1 f)^2 + (Y_2 f)^2\big)^{\frac{\mathtt{p}-2}{2}} \Upsilon \right) \end{aligned}$$

with the reduction

$$\begin{aligned} \widetilde{\Delta_{\mathtt{p}}} f &= \operatorname{div}\left(\big((Y_1 f)^2 + (Y_2 f)^2\big)^{\frac{\mathtt{p}-2}{2}} \Upsilon \right), \\ &= Y_1\left(\big((Y_1 f)^2 + (Y_2 f)^2\big)^{\frac{\mathtt{p}-2}{2}} \Upsilon_1 \right) + Y_2\left(\big((Y_1 f)^2 + (Y_2 f)^2\big)^{\frac{\mathtt{p}-2}{2}} \Upsilon_2 \right) \\ &= \frac{1}{2}(\mathtt{p}-2)\big((Y_1 f)^2 + (Y_2 f)^2\big)^{\frac{\mathtt{p}-2}{2}-1} Y_1\big((Y_1 f)^2 + (Y_2 f)^2\big)\Upsilon_1 \\ &\quad + \frac{1}{2}(\mathtt{p}-2)\big((Y_1 f)^2 + (Y_2 f)^2\big)^{\frac{\mathtt{p}-2}{2}-1} Y_2\big((Y_1 f)^2 + (Y_2 f)^2\big)\Upsilon_2 \\ &\quad + \big((Y_1 f)^2 + (Y_2 f)^2\big)^{\frac{\mathtt{p}-2}{2}} Y_1 \Upsilon_1 + \big((Y_1 f)^2 + (Y_2 f)^2\big)^{\frac{\mathtt{p}-2}{2}} Y_2 \Upsilon_2 \\ &= \frac{1}{2}(\mathtt{p}-2)\big((Y_1 f)^2 + (Y_2 f)^2\big)^{\frac{\mathtt{p}-2}{2}-1} \Big(Y_1\big((Y_1 f)^2 + (Y_2 f)^2\big)\Upsilon_1 \\ &\quad + Y_2\big((Y_1 f)^2 + (Y_2 f)^2\big)\Upsilon_2\Big) + \big((Y_1 f)^2 + (Y_2 f)^2\big)^{\frac{\mathtt{p}-2}{2}} \big(Y_1 \Upsilon_1 + Y_2 \Upsilon_2\big) \\ &= \big((Y_1 f)^2 + (Y_2 f)^2\big)^{\frac{\mathtt{p}-2}{2}-1} \left(\frac{1}{2}(\mathtt{p}-2)\Big(Y_1\big((Y_1 f)^2 + (Y_2 f)^2\big)\Upsilon_1 \right. \\ &\quad \left. + Y_2\big((Y_1 f)^2 + (Y_2 f)^2\big)\Upsilon_2\Big) + \big((Y_1 f)^2 + (Y_2 f)^2\big)\big(Y_1 \Upsilon_1 + Y_2 \Upsilon_2\big) \right). \end{aligned}$$

Thus to show

$$\widetilde{\Delta_{\mathtt{p}}} f = 0,$$

we need only show that

$$\begin{aligned} \Lambda &= \frac{1}{2}(\mathtt{p}-2)\Big(Y_1\big((Y_1 f)^2 + (Y_2 f)^2\big)\Upsilon_1 + Y_2\big((Y_1 f)^2 + (Y_2 f)^2\big)\Upsilon_2\Big) \\ &\quad + \big((Y_1 f)^2 + (Y_2 f)^2\big)\big(Y_1 \Upsilon_1 + Y_2 \Upsilon_2\big) = 0. \end{aligned}$$

Case 1: Let $\mathtt{p} \neq n+2$. We then compute

$$\begin{aligned} Y_1 f + iLY_2 f &= c(n+1)(y_1-a)^n g^{\alpha-1} h^{\beta-1}\big(\alpha h(1-L)+\beta g(1+L)\big) \\ Y_2 f - iLY_1 f &= ic(n+1)(y_1-a)^n g^{\alpha-1} h^{\beta-1}\big(\alpha h(1-L)-\beta g(1+L)\big) \\ \big((Y_1 f)^2+(Y_2 f)^2\big) &= \big(4c^2(n+1)^2(y_1-a)^{2n} g^{2\alpha-1} h^{2\beta-1}\alpha\beta\big). \end{aligned}$$

We then calculate:

$$\begin{aligned} &Y_1(Y_1 f + iLY_2 f) + Y_2(Y_2 f - iLY_1 f) = \\ &\quad \left(\frac{1}{(-1+\mathtt{p})^2 gh}\right)\Bigg(c^2(-1+L^2)(1+n)(2+n-\mathtt{p})(-2+\mathtt{p}) \\ &\quad \times (y_1-a)^{2n} h^{\frac{(-1+L)(2+n-\mathtt{p})}{2(1+n)(-1+\mathtt{p})}} g^{-\frac{(1+L)(2+n-\mathtt{p})}{2(1+n)(-1+\mathtt{p})}}\Bigg). \end{aligned}$$

We also require

$$\begin{aligned} &\Big((Y_1 f)^2+(Y_2 f)^2\Big)\Big(Y_1(Y_1 f + iLY_2 f) + Y_2(Y_2 f - iLY_1 f)\Big) = \\ &\quad \left(\frac{1}{(-1+p)^4 (gh)^2}\right)\Bigg(c^4(-1+L^2)^3(1+n)(2+n-p)^3(-2+p) \\ &\quad \times (y_1-a)^{4n} h^{\frac{3(-1+L)(2+n-p)}{2(1+n)(-1+p)}} g^{-\frac{3(1+L)(2+n-p)}{2(1+n)(-1+p)}}\Bigg). \end{aligned}$$

We can then calculate

$$\begin{aligned} &Y_1\Big((Y_1 f)^2+(Y_2 f)^2\Big)(Y_1 f + iLY_2 f) = \\ &\quad 4c^3(n+1)^3(y_1-a)^{3n-1} g^{3\alpha-3} h^{3\beta-3}\alpha\beta \\ &\quad \times \Big(2ngh\big(\alpha h(1-L)+\beta g(1+L)\big) \\ &\quad + c(n+1)(y_1-a)^{n+1}\big(\alpha h(1-L) \\ &\quad + \beta g(1+L)\big)\big((2\alpha-1)h+(2\beta-1)g\big)\Big) \end{aligned}$$

and

$$\begin{aligned} &Y_2\Big((Y_1 f)^2+(Y_2 f)^2\Big)(Y_2 f - iLY_1 f) = \\ &\quad 4c^4(n+1)^4(y_1-a)^{4n} g^{3\alpha-3} h^{3\beta-3}\alpha\beta \\ &\quad \times \Bigg(\Big(g(2\beta-1)-h(2\alpha-1)\Big)\Big(\alpha h(1-L)-\beta g(1+L)\Big)\Bigg). \end{aligned}$$

We then sum the two quantities, which gives

$$
\begin{aligned}
&Y_1\Big((Y_1)^2+(Y_2^2)\Big)(Y_1f+iLY_2f)+Y_2\Big((Y_1)^2+(Y_2^2)\Big)(Y_2f-iLY_1f)=\\
&\quad 2^3c^3(n+1)^3(y_1-a)^{3n-1}g^{3\alpha-2}h^{3\beta-2}\alpha\beta\\
&\quad\times\Big(n\big(\alpha h(1-L)+\beta g(1+L)\big)\\
&\quad+c(n+1)(y_1-a)^{n+1}\big(\alpha(2\beta-1)(1-L)+\beta(2\alpha-1)(1+L)\big)\Big).
\end{aligned}
$$

We then compute:

$$
\begin{aligned}
\Lambda &= \frac{1}{2}(\mathtt{p}-2)\Big(Y_1\big((Y_1f)^2+(Y_2f)^2\big)(Y_1f+iLY_2f)\\
&\quad +Y_2\big((Y_1f)^2+(Y_2f)^2\big)(Y_2f-iLY_1f)\Big)\\
&\quad +\big((Y_1f)^2+(Y_2f)^2\big)\big(Y_1(Y_1f+iLY_2f)+Y_2(Y_2f-iLY_1f)\big)\\
&= 2^2c^3(n+1)^3\alpha\beta(y_1-a)^{3n-1}g^{3\alpha-2}h^{3\beta-2}\times\\
&\quad\Big(n\Big(\alpha h(1-L)+\beta g(1+L)\Big)(p-1)\\
&\quad +c(n+1)(y_1-a)^{n+1}\Big((p-2)\big(\alpha(2\beta-1)(1-L)\\
&\quad +\beta(2\alpha-1)(1+L)\big)+4\alpha\beta\Big)\\
&= 2^2c^3(n+1)^3\alpha\beta(y_1-a)^{3n-1}g^{3\alpha-2}h^{3\beta-2}\times\\
&\quad\Bigg(\frac{c(-1+L^2)n(2+n-p)(y_1-a)^{1+n}}{1+n}\\
&\quad -\frac{c(-1+L^2)n(2+n-p)(y_1-a)^{1+n}}{1+n}\Bigg)=0.
\end{aligned}
$$

Thus $\widetilde{\Delta_{\mathtt{p}}}f=0$ for $\mathtt{p}\neq n+2$.

Case 2: For $\mathtt{p}=n+2$ We notice in equations (5.3) and (5.4) that

$$Y_1f+iLY_2f=\overline{Y_1f+iLY_2f}$$

and

$$Y_2f-iLY_1f=\overline{Y_2f-iLY_1f}.$$

This gives that

$$\big((Y_1f)^2+(Y_2f)^2\big)=\left\|\begin{matrix}Y_1f+iLY_2f\\ Y_2f-iLY_1f\end{matrix}\right\|^2$$

and so the proof of this is the same as 5.1.1. Thus $\widetilde{\Delta_{\mathtt{p}}}f=0$ for $1<\mathtt{p}<\infty$ and for all $L\in\mathbb{R}$. □

5.2.2 The Heisenberg Group

We suppress the subscripts on the function u and on $\|\cdot\|$.

THEOREM 5.4 *For all* $1 < p < \infty$ *and all* $L \in \mathbb{R}$, *we have:*

$$\operatorname{div}\left(|\nabla_0 u \cdot \overline{\nabla_0 u}|^{p-2}\begin{pmatrix} X_1 + iLX_2 \\ X_2 - iLX_1 \end{pmatrix} u_{p,L}\right) = 0$$

in $\mathbb{H} \setminus \{(0,0,0)\}$.

Proof. Let

$$\Upsilon = \begin{pmatrix} \Upsilon_1 \\ \Upsilon_2 \end{pmatrix} = \begin{pmatrix} X_1 u + iLX_2 u \\ X_2 u - iLX_1 u \end{pmatrix},$$

and we consider the following:

$$\begin{aligned} \widetilde{\Delta_{\mathrm{p}}} u &= \operatorname{div}\left(\left(\nabla_0 u \cdot \overline{\nabla_0 u}\right)^{\frac{\mathrm{p}-2}{2}} \Upsilon\right) \\ &= \operatorname{div}\left(\left((X_1 u)^2 + (X_2 u)^2\right)^{\frac{\mathrm{p}-2}{2}} \Upsilon\right), \end{aligned}$$

and the reduction:

$$\begin{aligned} \widetilde{\Delta_{\mathrm{p}}} u &= \operatorname{div}\left(\left((X_1 u)^2 + (X_2 u)^2\right)^{\frac{\mathrm{p}-2}{2}} \Upsilon\right), \\ &= X_1\left(\left((X_1 u)^2 + (X_2 u)^2\right)^{\frac{\mathrm{p}-2}{2}} \Upsilon_1\right) + X_2\left(\left((X_1 u)^2 + (X_2 u)^2\right)^{\frac{\mathrm{p}-2}{2}} \Upsilon_2\right) \\ &= \frac{1}{2}(\mathrm{p}-2)\left((X_1 u)^2 + (X_2 u)^2\right)^{\frac{\mathrm{p}-2}{2}-1} X_1\left((X_1 u)^2 + (X_2 u)^2\right)\Upsilon_1 \\ &\quad + \frac{1}{2}(\mathrm{p}-2)\left((X_1 u)^2 + (X_2 u)^2\right)^{\frac{\mathrm{p}-2}{2}-1} X_2\left((X_1 u)^2 + (X_2 u)^2\right)\Upsilon_2 \\ &\quad + \left((X_1 u)^2 + (X_2 u)^2\right)^{\frac{\mathrm{p}-2}{2}} X_1\Upsilon_1 + \left((X_1 u)^2 + (X_2 u)^2\right)^{\frac{\mathrm{p}-2}{2}} X_2\Upsilon_2 \\ &= \frac{1}{2}(\mathrm{p}-2)\left((X_1 u)^2 + (X_2 u)^2\right)^{\frac{\mathrm{p}-2}{2}-1}\Big(X_1\left((X_1 u)^2 + (X_2 u)^2\right)\Upsilon_1 \\ &\quad + X_2\left((X_1 u)^2 + (X_2 u)^2\right)\Upsilon_2\Big) + \left((X_1 u)^2 + (X_2 u)^2\right)^{\frac{\mathrm{p}-2}{2}}\left(X_1\Upsilon_1 + X_2\Upsilon_2\right) \\ &= \left((X_1 u)^2 + (X_2 u)^2\right)^{\frac{\mathrm{p}-2}{2}-1}\Bigg(\frac{1}{2}(\mathrm{p}-2)\Big(X_1\left((X_1 u)^2 + (X_2 u)^2\right)\Upsilon_1 \\ &\quad + X_2\left((X_1 u)^2 + (X_2 u)^2\right)\Upsilon_2\Big) \\ &\quad + \left((X_1 u)^2 + (X_2 u)^2\right)\left(X_1\Upsilon_1 + X_2\Upsilon_2\right)\Bigg). \end{aligned}$$

Thus to show

$$\widetilde{\Delta_{\mathrm{p}}} u = 0,$$

we need only show that

$$\begin{aligned}\Lambda &= \frac{1}{2}(\mathrm{p}-2)\big(X_1\big((X_1u)^2+(X_2u)^2\big)\Upsilon_1 + X_2\big((X_1u)^2+(X_2u)^2\big)\Upsilon_2\big)\\ &\quad + \big((X_1u)^2+(X_2u)^2\big)\big(X_1\Upsilon_1 + X_2\Upsilon_2\big) = 0.\end{aligned}$$

Case 1: Let $\mathrm{p} \neq 4$. We then calculate

$$\begin{aligned}X_1u + iLX_2u &= 2v^{\eta-1}w^{\tau-1}\Big((\eta w+\tau v)(x_1+iLx_2)\\ &\quad + (\eta w - \tau v)(Lx_1 + ix_2)\Big)\\ X_2u - iLX_1u &= 2v^{\eta-1}w^{\tau-1}\Big((\eta w+\tau v)(x_2-iLx_1)\\ &\quad + (-\eta w + \tau v)(ix_1 - Lx_2)\Big)\\ \big((X_1u)^2+(X_2u)^2\big) &= 2^2v^{2(\eta-1)}w^{2(\tau-1)}\Big((\eta w+\tau v)^2(x_1^2+x_2^2)\\ &\quad - (\eta w-\tau v)^2x_2^2 - (-\eta w+\tau v)^2x_1^2\Big).\end{aligned}$$

We can then compute:

$$\begin{aligned}&X_1(X_1u+iLX_2u)+X_2(X_2u-iLX_1u) =\\ &\quad -\left(\frac{1}{(-1+p)^2}\right)\left(3(-1+L^2)(-4+p)(-2+p)(x1^2+x2^2)v^{-\frac{L(-4+p)+3p}{4(-1+p)}}w^{\frac{4L+3p-Lp}{4-4p}}\right).\end{aligned}$$

We also require

$$\begin{aligned}&\Big((X_1u)^2+(X_2u)^2\Big)\Big(X_1(X_1u+iLX_2u)+X_2(X_2u-iLX_1u)\Big) =\\ &\quad -\left(\frac{1}{(-1+\mathrm{p})^4}\right)\Big(3(-1+L^2)^3(-4+\mathrm{p})^3(-2+\mathrm{p})(x_1^2+x_2^2)^2\\ &\quad \times v^{\frac{4+3L(-4+\mathrm{p})+5\mathrm{p}}{4-4\mathrm{p}}}w^{\frac{4-3L(-4+\mathrm{p})+5\mathrm{p}}{4-4\mathrm{p}}}\Big).\end{aligned}$$

We then calculate

$$X_1\Big((X_1u)^2+(X_2u)^2\Big)(X_1u+iLX_2u)+X_2\Big((X_1)^2+(X_2^2)\Big)(X_2u-iLX_1u)=$$
$$\left(\frac{1}{(-1+\mathtt{p})^4(vw)^2}\right)\Bigg(6(-1+L^2)^3(-4+\mathtt{p})^3(x_1^2+x_2^2)^2$$
$$\times\, v^{-\frac{3(-1+L)(-4+\mathtt{p})}{4(-1+\mathtt{p})}}w^{\frac{3(1+L)(-4+\mathtt{p})}{4(-1+\mathtt{p})}}\Bigg).$$

We then compute:

$$\begin{aligned}
\Lambda &= \frac{1}{2}(\mathtt{p}-2)\Bigg(\left(\frac{1}{(-1+\mathtt{p})^4(vw)^2}\right)\Bigg(6(-1+L^2)^3(-4+\mathtt{p})^3(x_1^2+x_2^2)^2\\
&\quad\times v^{-\frac{3(-1+L)(-4+\mathtt{p})}{4(-1+\mathtt{p})}}w^{\frac{3(1+L)(-4+\mathtt{p})}{4(-1+\mathtt{p})}}\Bigg)\Bigg)\\
&\quad\Bigg(-\left(\frac{1}{(-1+\mathtt{p})^4}\right)\Bigg(3(-1+L^2)^3(-4+\mathtt{p})^3(-2+\mathtt{p})(x_1^2+x_2^2)^2\\
&\quad\times v^{\frac{4+3L(-4+\mathtt{p})+5\mathtt{p}}{4-4\mathtt{p}}}w^{\frac{4-3L(-4+\mathtt{p})+5\mathtt{p}}{4-4\mathtt{p}}}\Bigg)\Bigg)\\
&= \Bigg(\left(\frac{1}{(-1+\mathtt{p})^4}\right)\Bigg(3(-1+L^2)^3(-4+\mathtt{p})^3(-2+\mathtt{p})(x_1^2+x_2^2)^2\\
&\quad\times v^{\frac{4+3L(-4+\mathtt{p})+5\mathtt{p}}{4-4\mathtt{p}}}w^{\frac{4-3L(-4+\mathtt{p})+5\mathtt{p}}{4-4\mathtt{p}}}\Bigg)\Bigg)\\
&\quad+\Bigg(-\left(\frac{1}{(-1+\mathtt{p})^4}\right)\Bigg(3(-1+L^2)^3(-4+\mathtt{p})^3(-2+\mathtt{p})(x_1^2+x_2^2)^2\\
&\quad\times v^{\frac{4+3L(-4+\mathtt{p})+5\mathtt{p}}{4-4\mathtt{p}}}w^{\frac{4-3L(-4+\mathtt{p})+5\mathtt{p}}{4-4\mathtt{p}}}\Bigg)\Bigg)=0.
\end{aligned}$$

Thus $\widetilde{\Delta_{\mathtt{p}}}u=0$ for $\mathtt{p}\neq 4$.

Case 2: Similar to the Grushin Case, for $\mathtt{p}=4$ we notice in equations (5.5) and (5.6) that

$$X_1u+iLX_2u=\overline{X_1u+iLX_2u}$$

and

$$X_2u-iLX_1u=\overline{X_2u-iLX_1u}.$$

This gives that

$$\big((X_1u)^2+(X_2u)^2\big)=\left\|\begin{array}{c}X_1u+iLX_2u\\ X_2u-iLX_1u\end{array}\right\|^2,$$

and so the proof of this is the same as the proof for 5.1.2. Thus $\widetilde{\Delta_{\mathtt{p}}} u = 0$ for all $1 < \mathtt{p} < \infty$ and for all $L \in \mathbb{R}$. $\square$

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