Thermo-Mechanical Beam Element for Analyzing Stresses in Functionally Graded Materials

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Thermo-Mechanical Beam Element for Analyzing

Stresses in Functionally Graded Materials

by

Simón A. Caraballo

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
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Dedication

First, I give praise and honor to my Lord and savior Jesus Christ for allowing me to accomplish this goal.

I dedicate this dissertation to my beloved wife Marisolin and my daughter Marielsy; without their love and emotional support this work would have not been possible. I would like to thank them for their sacrifice, patience and love, and for all that they have given to me in my life.

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Nomenclature

The following symbols are used in this dissertation:

\( A_{11}, A_{55}, B_{11}, D_{11} \)  \( \) stiffness coefficients

\( AT_{11}, BT_{11} \)  \( \) thermal stiffness coefficients

\( BL \)  \( \) beam length

\( E(z) \)  \( \) Young’s modulus

\( G(z) \)  \( \) shear modulus

\( L \)  \( \) element length

\( S \)  \( \) strain energy

\( t \)  \( \) beam thickness

\( u \)  \( \) axial displacement

\( w \)  \( \) transverse displacement

\( \varepsilon_{xx} \)  \( \) axial strain

\( \alpha(z) \)  \( \) coefficient of thermal expansion

\( \phi^0 \)  \( \) rotation of reference axis about y-axis

\( \gamma_{xz} \)  \( \) shear strain

\( \sigma_{xx} \)  \( \) axial stress
\( \tau_{xz} \) shear stress

\( \Delta T(x, z) \) temperature change from reference state

\( \{ F \} \) mechanical force vector

\( \{ R1 \} \) first thermal force vector

\( \{ R2 \} \) second thermal force vector

\( \{ u \} \) element displacement vector

\( \{ \hat{u} \} \) nodal displacement vector

\( \{ \theta \} \) element thermal vector

\( \{ \hat{\theta} \} \) nodal thermal vector

\( [K] \) element stiffness matrix

\( [B] \) strain-displacement matrix

\( [\mathbf{N}(x)] \) shape function

**Subscripts**

\( b \) value at bottom of beam

\( t \) value at top of beam

**Superscripts**

\( ^0 \) with respect to reference plane
Abstract

Modeling at the structural scale most often requires the use of beam and shell elements. This simplification reduces modeling complexity and computation requirements but sacrifices the accuracy of through-the-thickness information. Several studies have reported various design approaches for analyzing functionally graded material structures. One of these studies proposed a two-node beam element for functionally graded materials (FGMs) based on first order shear deformable (FOSD) theory. The derivation of governing equations included spatial temperature variation. However, only the constant temperature case was carried through in the element formulation. This investigation explore the effects of spatial temperature variation in the axial and through-the-thickness direction of this proposed element and present a new standard three-node beam finite element modified for structure constructed of FGMs. Also, the influence of the temperature dependency of the thermo-elastic material properties on the thermal stresses distribution was studied. In addition, variations in the layer thicknesses within multilayer beam models were studied to determine the effect on stresses and factor of safety. Finally, based on the specific factor of safety, which combines together the strength and mass of the beam, the best layer thicknesses for the beam models were established.
The key contributions expected from this research are:

1. development and implementation of a three-node beam element as a finite element code into the commercial computational tool MATLAB® to analyze thermo-mechanical stresses in structures constructed of functionally graded materials;

2. a strategy to simulate different load cases in structures constructed of functionally graded materials;

3. an analysis of the influence of the FGM interlayer thickness on the factor of safety especific gravity ratio in structures constructed of functionally graded materials under thermo-mechanical loads;

4. and an analysis/comparison of the advantages/benefits of using structures constructed of functionally graded materials with respect to those constructed with homogenous materials.
Chapter 1 Introduction

Motivation

The main benefit of using functionally graded materials (FGMs) instead of traditional materials is that the internal composition of their component materials can be tailored to satisfy the requirements of a given structure. Although much of this technology has not been fully implemented, the internal structure of the material could be prepared to manufacture hybrid high temperature pressure vessels or other thermal structures. Before attempting to fabricate complicated applications out of FGMs, it is very important that the tools for structural analysis are developed. This work is an important step in being able to properly design mechanical structures using a functionally graded material system.

On a grander scale, the ultimate goal of this research is to help determine if the structures constructed of functionally graded materials can be used instead of traditional materials within the context of the needed applications. One of these applications is a space shuttle, where the aluminum substructure is shielded by a thermal protection system (TPS) barrier consisting of several layers of primers, tile, adhesives, fibers, and coatings. The core metallic of structures made of FGMs could resist higher temperatures and the structure size requirements can be reduced. Additionally, mass could be
minimized by tailoring the ingredient of each component based upon the load and stress interactions present in different areas of the mechanical structure.

Functionally graded materials (FGMs) are of increasing importance as designers seek a way to address structures under combined thermal and mechanical loads. The finite element method is commonly employed to analyze structures where beam, plate/shell, and solid elements are used. The question arises as to how to implement element formulations for structures composed of FGMs. As an important step to achieve this goal, a first order shear deformable (FOSD) beam model is investigated and applied to beams subjected to spatial variations in temperature.

**Research Goals**

The goal of this research is to determine if the structures constructed of functionally graded materials can be used instead of traditional materials. The study will focus on the modeling and simulation of:

1. Functionally graded beam structures with material properties varying throughout the thickness of the beam.
2. Functionally graded beam structures with temperature-dependent thermo-elastic material properties.
3. Elastic thermo-mechanical stresses in FGM structures.
The major outcomes of the present research are the following:

1. Develop a finite element program to analyze thermo-mechanical stresses in structures constructed of functionally graded materials.
2. The performance of the proposed element formulation is presented throughout comparisons with FGMs model available in related literature.
3. Methodology to conduct analytical and numerical simulations of thermal loading studies conducted on the FGMs beam structures in one and two dimensions.
4. Simulate structures constructed of functionally graded materials with and without temperature dependence of the material properties.
5. Analyze the influence of layer thicknesses within multilayer beam models on the factor of safety/specific gravity ratio in structures constructed of functionally graded materials under thermo-mechanical loads.
6. Analyze and compare the advantages and benefits of using structures constructed of functionally graded materials with respect to those constructed with homogenous materials.
Dissertation Organization

This research work is organized where Chapter 1 presents an introduction, motivation, the research goals and major outcomes of this work. Then Chapter 2 discusses previous research work on the analysis and design of FGM structures. A review of current research in this area is also introduced.

Chapter 3 presents fundamental theoretical aspects of FGMs and their applications. It also introduces the conceptual idea of FGMs and their distinct features in comparison with other engineering materials. Approaches for modeling and calculating the effective properties of FGMs are discussed. Considerations of the temperature dependence of material properties for FGMs are presented. Typical engineering applications of FGMs are also provided.

Chapter 4 presents a detailed formulation of the governing equations for analyzing functionally graded material models. Formulations of the equations of motion are developed for a first order shear deformable beam. Two finite element formulations are presented. The first formulation is a two-node formulation based on Chakraborty et al. [1] where a beam element is developed to study the thermo-elastic behavior of functionally graded beam structures. This is followed by a new three-node element formulation.

Chapter 5 discusses the results of the analyses performed in this research. Numerical simulations of thermal loading studies conducted on the FGMs beam structures are presented. Also, this chapter describes a study to determine the influence of
manipulating the FGM layer thickness on the factor of safety and the specific factor of safety in these structures under thermal loads.

Finally, Chapter 6 presents a summary of the work conducted and a discussion of the main conclusions drawn. The chapter also offers recommendations that emerge from this work for future research in the field.
Chapter 2 Review of Relevant Literature

Introduction

This chapter summarizes previous research work in the analysis and design of FGM structures. The extensive research in this field, which started with the pioneering work of Suresh and Mortensen [2], Reddy [3], and Sankar [4] has led to the development of several design approaches for analyzing FGM structures that are currently used in many applications throughout the world. A brief description of the most applicable works in this area is presented as follows.

Reddy [5] worked on characterizing the theoretical formulation of FGMs to include the derivations of equations used to calculate material properties throughout the thickness of the material based on the through-the-thickness distribution of materials. Na and Kim [6] studied the thermo-mechanical buckling of FGMs using a finite element discretization method. Cooley [7] researched FGM shell panels under thermal loading also using the finite element method. Hill and Lin [8] concentrated their research of FGMs in the field of residual stress measurement in a ceramic-metallic graded material using experimental procedures that released residual stresses by making incisions into the material and measuring the resulting change in stress with strain gages. Hill et al. [9] participated in studying the fracture testing of layered (as opposed to a continuous function) FGMs.
Some work in the area of FGM aluminum matrix composites include the study of Kang and Rohatgi [10] who performed a transient thermal analysis of solidification in a centrifugal casting for composite materials containing particle segregation.

Another contributor to the field, Sankar [4] showed that a functionally graded Euler-Bernoulli beam is subject to the same limitations normally associated with beam theory under mechanical loading. For comparison between a beam theory and the elasticity solution, a simply supported beam with a sinusoidal distributed load was solved. Later, Sankar and Tzeng [11] expanded upon Sankar’s [4] earlier work by investigating beams with through-the-thickness temperature gradients.

Additionally, Chakraborty et al. [1] proposed a two-node beam element for FGMs based on FOSD theory and applied it to static, thermal, free vibration and wave propagation problems. The assumed displacement field of the element satisfies the general solution to the static part of the governing equations. Static condensation (Cook et al. [12], Wilson [13]) is used to reduce the number of unknowns in the elements displacement field to the number of degrees of freedom within the element. The derivation of governing equations included spatial temperature variation.

Even though Chakraborty et al. [1] work constitutes an important contribution to the FGMs field, it presents some limitations. For example, only the one-dimensional constant temperature case was carried through in the finite element formulation. Another limitation is that the temperature dependency of the material properties was not considered for the analyzed models.

To address these limitations, this work will investigate the effects of spatial two-dimensional temperature variation in the axial and through-the-thickness direction of the
element proposed by Chakraborty et al. [1]. Additionally, a more accurate three-node beam element will be formulated for analyzing the FGMs structures. Much more important, since FGMs structures are usually subjected to high ranges of temperatures (20°C-800°C), temperature dependency of the material properties is considered in this investigation, which will produce more realistic simulations and analyses of the structures being studied.
Chapter 3 Theoretical Background

Introduction

This chapter presents the fundamental theoretical aspects of FGMs and their applications. First, the conceptual idea of FGMs and their distinct features in comparison with other engineering materials is introduced. Also, approaches for modeling and calculating the effective properties of FGMs are discussed. In addition, important considerations of the temperature dependence for FGM material properties are presented. Finally, some typical engineering applications of FGMs are reviewed.

FGM Theoretical Fundamentals

Conceptual Idea of FGMs

The term functionally graded materials (FGMs) refers to solid objects or parts that usually consist of multiple materials or embedded components, that is, they are materially heterogeneous. The term “heterogeneous object” is defined for those objects with and/or multiple material objects with clear material domains [14].

A FGM consists of a material whose properties change from one surface to another according to a smooth continuous function based on the position throughout the
thickness of the material. Most often, this material consists of ceramic and metallic constituents. One surface is generally a pure metal while the opposite surface is usually pure ceramic or a majority ceramic. The metal portion of the material acts in the role of a structural support while the ceramic provides thermal protection when subjected to harsh temperatures. The function describing the material variation throughout the material and more importantly the material property variation makes it possible to tailor the function to suit the needs of various applications. Examples of different types of material grading in functionally graded materials are shown in Figure 1 as presented by Refs. [15-17].

Figure 1. Examples of material grading in functionally graded materials. Reprinted from Refs. [15-17] with permission from Elsevier.
The continuous change in the microstructure of FGMs distinguishes them from the fiber-reinforced laminated composite materials, which have a mismatch of mechanical properties across an interface due to two discrete materials bonded together. As a result the constituents of the fiber-matrix composites are prone to debonding at extremely high thermal loading. Also, the anisotropic constitution of laminated composite structures often results in stress concentrations near material and geometric interfaces that can lead to damage in the form of delamination, matrix cracking, and adhesive bond separation [5]. Continuous or nearly continuous gradual change in material properties of FGMs reduces significantly these problems, making them a desirable choice for adverse thermal gradient applications.

FGMs alleviate these problems because they consist of a continuous variation of material properties from one surface to the other. The continuous nature of the variation lessens the stress concentrations which become troublesome in a laminated composite material. Also the smooth transition through the various material properties reduces both thermal and residual stresses [18]. In most cases the material progresses from a metal on one surface to a ceramic or mostly ceramic on the opposite surface, with a smooth transition throughout the center of the material. Also the material properties can change in any orientation across a material, but the majority of applications to date deal with a material in which the properties change through the thickness of the material.

The material transitions from a metal to a ceramic by increasing the percentage of ceramic material present in the metal until the appropriate percentage is reached or a pure ceramic is achieved (See Figure 2).
Since the material does not have a dramatic change in material properties at any one point through the thickness, it would not cause a large stress concentration. This material usually exists where there is an extreme temperature gradient which is designated by $T_{\text{hot}}$ and $T_{\text{cold}}$ in Figure 2. The ceramic face of the material is generally exposed to a high temperature, while the metallic face is usually subjected to a relatively cooler temperature. The smooth transition of material properties allows for a material whose properties provide thermal protection as well as structural integrity reducing the possibilities of failure within the structure. This reduction of failure is of critical importance in space programs where thermal protection tiles are laminated to the metallic structure of the space shuttle to handle the extreme temperatures during re-entry into the earth’s atmosphere. These tiles are susceptible to cracking and debonding at the superstructure/tile interface due to abrupt transition between thermal expansion coefficients. The smooth transitions between material properties reduces the potential cracking and debonding of thermal protection tiles laminated to structural members.

Figure 2. Graphical FGM representation of gradual transition in the direction of the temperature gradient. Reprinted from Ref. [19] with permission from Elsevier.
The capabilities of the FGMs are quite flexible as one can vary the materials used as well as the function of composition throughout the material at which they transition from surface to surface. A specific metal and ceramic can be chosen for the particular application to capitalize on the positive characteristics of each of the materials. Also, the function between the two outside materials can be mathematically maximized and tailored specifically to meet the needs of the desired application as shown by [20].

Functionally graded materials (FGMs) are new advanced multifunctional composites where the volume fractions of the reinforcements phase(s) (or dispersoids) vary smoothly. This is achieved by using reinforcements with different properties, sizes, and shapes, as well as by interchanging the functions of the reinforcement and matrix phases in a continuous manner. The result is a microstructure bearing continuous changes in thermal and mechanical properties at the macroscopic or continuum scale [21]. In other words, FGM is usually a combination of two materials or phases that show a gradual transition of properties from one side of sample to the other. This gradual transition allows the creation of superior and multiple properties without any mechanically weak interface. Moreover, the gradual change of properties can be tailored to different applications and service environments.

This new concept of materials engineering hinges on materials science and mechanics due to the integration of the material and structural considerations into the final design of structural components. Because of the many variables that control the design of functionally graded microstructures, full utilization of the FGMs potential requires the development of appropriate modeling strategies for their response to combined thermo-mechanical loads.
FGMs are ideal candidates for applications involving severe thermal gradients, varying from thermal structures in advanced aircraft and aerospace engines to computer circuit boards. These materials were introduced to take advantage of ideal behavior of its constituents, for example, heat and corrosion resistance of ceramics together with mechanical strength and toughness of metals [22].

Effective Properties of FGMs

To study FGMs, a model must be created that describes the function of composition throughout the material. In Figure 3, the volume fraction, $V_c$, describes the volume of ceramic at any point $z$ throughout the thickness $t$ according to a parameter $n$ which controls the shape of the function (as seen in Figure 4). $V_c$ is given by

$$V_c(z) = \left( \frac{z}{t} + \frac{1}{2} \right)^n$$

(1)
It follows that the volume fraction of metal, \( V_m(z) \), in the FGM is \( 1 - V_c(z) \). A graphical representation of various values of the parameter \( n \) can be seen in Figure 4.

![Graph showing the effect of grading parameter \( n \) on the volume fraction \( V_c \)](image)

**Figure 4.** Effect of the grading parameter \( n \) on the volume fraction \( V_c \)

The area to the right of each line represents the amount of metal, and the area to the left represents the amount of ceramic in the material. It should be noted that, as \( n \to 0 \), the material approaches to a homogeneous ceramic, while as \( n \to \infty \), the material becomes entirely metal. For \( 0 < n < \infty \), the material will contain both metal and ceramic. When \( n = 1 \), the distribution is linear containing equal portions of ceramic and metal. According to Nakamura and Sampath [23], the values of \( n \) should be taken in the range of \([1/3, 3]\), as values outside this range will produce an FGM having too much of one phase.
One of the most common methods to determine the effective properties of FGMs is the rule of mixtures, where the material properties through the thickness vary as a function of the volume fraction and are given by

\[ P(z) = (P_t - P_b)V_c(z) + P_b \]  

(2)

As Figure 5 shows, the variables \( P_t \) and \( P_b \) are the material property at the top and bottom, respectively. \( P_t \) corresponds to the material property of the pure ceramic and \( P_b \) corresponds to the material property of the pure metal. \( V_c(z) \) is given by Eq. (1).

![Graded Layer](image)

Figure 5. Material properties throughout the FGM layer.

Even though the rule of mixtures given by Eq. (2) is widely used for most researchers in the FGM field, this rule is very general and it does not always give a realistic value of the properties in question. In fact, more appropriate formulas have been found by Nemat-Alla [24] to address the limitations of the rule of mixtures.
For the analysis of the FGM in this research, formulas shown in Table 1 will be used for estimating the effective values of the thermo-mechanical properties. It should be noted that these formulas are particular cases of zero material porosity [24].

At this point, it is important to mention that in the formulas given in Table 1, the thermo-mechanical properties of each material in the composite beam are also a function of the temperature. The influence of the temperature on the material properties will be discussed in detail in “The Temperature Dependence of Material Properties for FGM” section in this chapter.
Table 1. Effective property formulas of FGMs [24].

<table>
<thead>
<tr>
<th>Material property</th>
<th>Effective property formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thermal conductivity ( k )</td>
<td>[ k(z) = k_t \left( 1 + \frac{3(k_b - k_t)V_m(z)}{3k_tV_m(z) + (k_b + 2k_t)V_c(z)} \right) ]</td>
</tr>
<tr>
<td>Modulus of elasticity ( E )</td>
<td>[ E(z) = E_t \left( \frac{E_t + (E_b - E_t)(V_c(z))^{2/3}}{E_t + (E_b - E_t)[(V_c(z))^{2/3} - V_c(z)]} \right) ]</td>
</tr>
<tr>
<td>Coefficient of thermal expansion ( \alpha )</td>
<td>[ \alpha(z) = (\alpha_t - \alpha_c)V_c(z) + \alpha_b + \frac{V_m(z)V_c(z)(\alpha_t - \alpha_b)(K_t - K_c)}{(K_t - K_b)V_c(z) + K_b + (3K_tK_c/4G_m)} ]</td>
</tr>
<tr>
<td>Poisson’s ratio ( \nu )</td>
<td>[ \nu(z) = (\nu_t - \nu_b)V_c(z) + \nu_b ]</td>
</tr>
<tr>
<td>Density ( \rho )</td>
<td>[ \rho(z) = (\rho_t - \rho_b)V_c(z) + \rho_b ]</td>
</tr>
<tr>
<td>Yield strength ( \sigma_y )</td>
<td>[ \sigma_y(z) = (\sigma_{ty} - \sigma_{yb})V_c(z) + \sigma_{yb} ]</td>
</tr>
</tbody>
</table>

In Table 1, \( K \) and \( G \) are the bulk's modulus and modulus of rigidity, respectively. Also, the undefined parameters are given by

\[
K_t = \frac{E_t}{3(1-2\nu_t)}, \quad K_b = \frac{E_b}{3(1-2\nu_b)}, \\
G_t = \frac{E_t}{2(1+\nu_t)}, \quad G_b = \frac{E_b}{2(1+\nu_b)}.
\]

The subscripts \( t \) and \( b \) stand for the material property at the top and bottom, respectively for the corresponding property. \( t \) corresponds to the material property of the pure ceramic, and \( b \) corresponds to the material property of the pure metal.
Consideration of Temperature Dependence of Material Properties for FGMs

FGMs are generally used in applications where high temperature environments/fields are involved. In these high temperature environments, some material properties (thermal conductivity \( k \), coefficient of thermal expansion \( \alpha \), modulus of elasticity \( E \), and yield strength \( \sigma_y \) are of particular pertinence to this work) become temperature-dependent [25]. In fact, the composite beam model structures that will be analyzed in Chapter 5 are subject to high levels of temperature. Therefore, this section reviews important aspects of the influence of temperature in the thermo-mechanical properties of the materials used in the composite models to be studied in this work.

The influence of the temperature on the material properties have been reported by various researchers and in handbooks of engineering materials. For example, Chen and Awaji [26] studied the temperature dependence of the mechanical properties of aluminum titanate (AT), and found that both the fracture strength and fracture toughness increased considerably with increase in temperature. They also found the temperature dependence of elastic modulus and thermal conductivity of AT ceramics as shown in Figure 6. Also, Yang et al. [27] presented thermo-mechanical post-buckling analysis of cylindrical panels that are made of FGMs with temperature-dependent thermo-elastic properties. They found that the temperature-independent solutions are about 9\%-18\% higher than the temperature-dependent solutions, that is, the buckling temperature is considerably overestimated when the temperature-dependence of the material properties is not taken into consideration. Finally, Richerson [28] and Murray [29] present several engineering materials frequently used in high temperature applications and how their thermo-elastic properties vary with temperature (see Figure 7-10).
Figure 6. Temperature dependence of elastic modulus and thermal conductivity for aluminum titanate ceramics. Reprinted from Ref. [26] with permission from Elsevier.

Figure 6 shows the thermal conductivity temperature dependence for several engineering materials. Clearly, we can see that the temperature has a strong effect on the thermal conductivity of ceramics materials. While in most ceramic materials, $k$ decreases as $T$ increases, in other materials $k$ increases with $T$. Platinum has high thermal conductivity that increases with temperature up to at least 1200 °C. It can be observed that the materials with complex crystal structures have lower thermal conductivities. Also, the presence of foreign atoms decreases the thermal conductivity [30]. For example, zirconia stabilized with MgO or CaO has low thermal conductivity and is very useful as a high-temperature refractory material. The highest conductivities are achieved in the least complex structures, that is, structures consisting of a single element or similar atomic weight or with no extraneous atoms in solid solution. When comparing to metals, in general ceramics (nonconductor materials) have lower thermal conductivities than metals (conductor materials). However, nonconductor materials such as beryllium oxide,
diamond, and silicon carbide are exceptions to this rule [31]. A detailed discussion on thermal conductivity is beyond the scope of this work; readers interested in a more detailed treatment of thermal conductivity are referred to the literature in Ref. [32].

Figure 7. Temperature dependence of thermal conductivity for several engineering materials. Reprinted from Ref. [32] with permission from Taylor and Francis Group, LLC.

Figure 8 shows the linear thermal expansion as a function of temperature for several materials. Between room temperature and 400°C, thermal expansion is relatively small for mullite and alumina compared to polyethylene, nylon, and aluminum alloys. Above 400°C this trend is reverted. In general, for temperatures above 400°C, zircon,
mullite, alumina, ZrO₂, and Ni-base superalloy exhibit the greatest expansion, while fused SiO₂ and aluminum silicate (LiAlSi₂O₆) have the least thermal expansion. It is convenient to mention that the last two materials (SiO₂, and LiAlSi₂O₆) have very little dimensional change as a function of temperature and can therefore withstand extreme thermal cycling or thermal shock without fracturing [33]. Low-thermal-expansion ceramic materials have broad potential for both domestic and industrial applications. Fused silica is one of the best thermal-shock-resistant materials available. One of the best application example of this material is put in practice when it is fabricated in a porous foam, which is used for lining critical surface of the space shuttle that are exposed to high temperature during ascent and reentry to the atmosphere. This material combines the low thermal expansion to prevent thermal shock damage and the very low thermal conductivity to protect the underlying structures which are less thermal resistant [33].
Figure 8. Temperature dependence of the linear thermal expansion for several engineering materials.
Reprinted from Ref. [32] with permission from Taylor and Francis Group, LLC.
Figure 9. Temperature dependence of the Young's modulus for several ceramic materials. Reprinted from Ref. [32] with permission from Taylor and Francis Group, LLC.

Figure 9 depicts the effect of the temperature on the elastic modulus of typical ceramics. As we can observe, for each material $E$ decreases slightly as the temperature increases. SiC and TiC have the highest moduli, followed by $\text{Al}_2\text{O}_3$, $\text{Si}_3\text{N}_4$, $\text{MgO}$, and $\text{ThO}_2$. $\text{ZrO}_2$ and $\text{MgAl}_2\text{O}_4$ have relatively low moduli, and $\text{LiAlSi}_2\text{O}_6$ has the lowest modulus of this group.
The strength of nearly all ceramic materials decreases as the temperature increases as shown in Figure 10. It can be seen that the strength of ceramics changes only slightly for several hundred degrees up to a temperature where the strength decreases significantly. This appears to occur for most ceramics materials at intermediate temperatures. At higher temperatures, the rate of strength decrease is more rapid, generally attributed to non-elastic effects [33]. For a wide range of temperature, CaO exhibit only small temperature-dependent changes in the yield strength up to 1100 °C.
In the FGM beam models to be analyzed in Chapter 5, basically two materials will be used; these are steel and alumina (Al\(_2\)O\(_3\)). The thermal properties for the materials are shown in Table 2 and Table 3. These material property data were collected from engineering manuals ([34], [35]), material handbooks ([36, 37]), and an online database of material properties [38].

Table 2. Thermal properties of steel [31, 35, 39].

<table>
<thead>
<tr>
<th>(T(\degree C))</th>
<th>(k) (W/m K)</th>
<th>(E) (GPa)</th>
<th>(\alpha \times 10^6) (1/K)</th>
<th>(\sigma_y) (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>61.8</td>
<td>194</td>
<td>11.4</td>
<td>420</td>
</tr>
<tr>
<td>27</td>
<td>60.7</td>
<td>204</td>
<td>11.6</td>
<td>397</td>
</tr>
<tr>
<td>100</td>
<td>57.8</td>
<td>195</td>
<td>12.1</td>
<td>381</td>
</tr>
<tr>
<td>200</td>
<td>53.5</td>
<td>204</td>
<td>12.7</td>
<td>362</td>
</tr>
<tr>
<td>300</td>
<td>49.0</td>
<td>193</td>
<td>13.3</td>
<td>380</td>
</tr>
<tr>
<td>400</td>
<td>44.5</td>
<td>188</td>
<td>13.9</td>
<td>359</td>
</tr>
<tr>
<td>500</td>
<td>40.2</td>
<td>183</td>
<td>14.4</td>
<td>313</td>
</tr>
<tr>
<td>600</td>
<td>35.7</td>
<td>167</td>
<td>14.8</td>
<td>284</td>
</tr>
<tr>
<td>700</td>
<td>31.2</td>
<td>141</td>
<td>15.0</td>
<td>167</td>
</tr>
<tr>
<td>800</td>
<td>27.3</td>
<td>106</td>
<td>14.8</td>
<td>72</td>
</tr>
<tr>
<td>900</td>
<td>26.0</td>
<td>74</td>
<td>12.6</td>
<td>44</td>
</tr>
</tbody>
</table>

Table 3. Thermal properties of alumina [31, 38].

<table>
<thead>
<tr>
<th>(T(\degree C))</th>
<th>(k) (W/m K)</th>
<th>(E) (GPa)</th>
<th>(\alpha \times 10^6) (1/K)</th>
<th>(\sigma_y) (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>50.45</td>
<td>415</td>
<td>4.75</td>
<td>459</td>
</tr>
<tr>
<td>27</td>
<td>42.00</td>
<td>408</td>
<td>5.55</td>
<td>455</td>
</tr>
<tr>
<td>100</td>
<td>29.51</td>
<td>393</td>
<td>6.86</td>
<td>442</td>
</tr>
<tr>
<td>200</td>
<td>21.56</td>
<td>380</td>
<td>7.42</td>
<td>424</td>
</tr>
<tr>
<td>300</td>
<td>16.92</td>
<td>373</td>
<td>7.79</td>
<td>407</td>
</tr>
<tr>
<td>400</td>
<td>13.54</td>
<td>371</td>
<td>8.15</td>
<td>390</td>
</tr>
<tr>
<td>500</td>
<td>10.62</td>
<td>370</td>
<td>8.43</td>
<td>375</td>
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<td>8.77</td>
<td>368</td>
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<td>700</td>
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</tr>
<tr>
<td>800</td>
<td>7.08</td>
<td>353</td>
<td>9.29</td>
<td>350</td>
</tr>
<tr>
<td>900</td>
<td>6.45</td>
<td>336</td>
<td>9.53</td>
<td>349</td>
</tr>
</tbody>
</table>
Figure 11 and Figure 12 plot the temperature dependence of these two materials. We can observe that for both materials, the thermoelastic properties vary significantly with the temperature for a wide range of temperature (0°C-900°C), which confirms the behavior discussed previously for several engineering materials in Figure 7-10. Also, from these figures it can be seen that the properties are not linear functions of the temperature and in general have large variations with temperature. Therefore, a cubic-splines interpolation is used to fit a model for the temperature-dependent material property data of these two materials. During the solution of the FGM beam model problems to be analyzed in Chapter 5, these fitted models are incorporated into the numerical procedure for solving these problems. The solution details of these models are treated in the two-dimensional numerical solution for a 3-layer FG Beam section in Chapter 4.
Figure 11. Temperature dependence of the thermoelastic properties of steel.
Figure 12. Temperature dependence of the thermoelastic properties of alumina.
FGMs Applications

As technology progresses at an ever increasing rate, the need for advanced capability materials becomes a priority in the engineering of more complex and higher performance systems. This need can be seen in many fields in which engineers are exploring the applications of these new engineered materials. Aerospace engineers trying to incorporate new and improved capabilities into air and space systems are pushing the envelope for what current materials can physically handle. FGMs are a relatively new technology and are being studied for the use in components exposed to harsh temperature gradients.

There is an extensive variety of applications in engineering practice which requires materials performance to vary with locations within the component. One of these applications is shown in Figure 13, where FGM is used to improve the thermo-mechanical performance of a turbine blade design. Another application, is shown in Figure 14, where graded region (FGM) is inserted between a metal and a ceramic tip for relaxation of stress concentration in lathe bits.
Figure 13. FGM application for a turbine blade design. Reprinted from Ref. [40] with permission from Elsevier.

Figure 14. FGM application for relaxation of stress concentration in lathe bits. Reprinted from Ref. [41] with permission from Elsevier.

There are many more current and future applications for FGMs. Most of them include space shuttles and aeronautical applications. One of these is the aircraft exhaust wash structures which separate exhaust gas from aircraft structure for vehicles which have internally exhausted engines, that is, stealth aircraft and UAVs (Unmanned Aerial Vehicles) with engines that do not exhaust directly to the atmosphere. FGMs are also being used in the thermoelectric devices for energy conversion and the semiconductor industry. FGMs are also being used as thermal barrier coating in gas turbine engines [42]. As research into this material progresses and the cost for manufacturing decreases, it is inevitable that many other industries which deal with severe thermal gradients will begin investigating the usefulness of FGMs.
Chapter 4 Formulation of Governing Equations

Introduction

This chapter presents the fundamental aspects of the beam theory for analyzing FGM structures that serves as a basis for developing this research. Detailed formulations of the governing equations for analyzing functionally graded material models are presented. Formulations of the equations of motion are developed for a first order shear deformable beam. It includes two finite element formulations. The first element formulation is a two-node formulation based on the work of Chakraborty et al. [1]. The second formulation is a new three-node beam element formulation to study the thermo-elastic behavior of functionally graded beam structures.

Beam Theory for FGM Structures

The axial and transverse displacements using first-order shear deformation theory for a beam are given by

\[ u = u^0(x) - z \phi^0(x) \]  

(3)
\[ w = w^0(x) \]  

(4)

where \( u^0 \) and \( w^0 \) are the mid-plane axial and transverse displacements, \( \phi^0 \) is the rotation about \( y \) axis, and \( z \) is measured from the reference plane as shown in Figure 15. Using Eqs. (3) and (4), and adding the strain due to temperature, the linear strains displacement relations are:

\[
\begin{align*}
\varepsilon_{xx} &= u^0_{,x} - z\phi^0_{,x} - \alpha(z,T)\Delta T(x,z) \\
\gamma_{xz} &= -\phi^0 + w^0_{,x}
\end{align*}
\]  

(5)

where \( (\cdot)_{,x} \) represents differentiation with respect to \( x \), \( \alpha(z) \) is the coefficient of thermal expansion, \( \Delta T \) is the temperature change from a stress-free state. In general, the temperature can vary along the length and through the thickness.

The constitutive relations are given by

\[
\begin{align*}
\sigma_{xx} &= E(z)\varepsilon_{xx}, \\
\tau_{xz} &= G(z)\gamma_{xz},
\end{align*}
\]  

(6)

where \( E(z) \) and \( G(z) \) are Young’s modulus and shear modulus, respectively, and are allowed to vary through the thickness.

The strain energy, \( S \) is given by

\[
S = \frac{1}{2} \int_0^L \int_A (\sigma_{xx} \varepsilon_{xx} + \tau_{xz} \gamma_{xz}) dAdx,
\]  

(7)
where $L$ and $A$ are the length and the area of cross-section of the beam, respectively.

Eq. (7) can also be written as

$$S = \frac{1}{2} \int_0^L \int_A \left( E(z)\varepsilon_{xx}^2 + G(z)\gamma_{xz}^2 \right) dAdx,$$  

(8)

Using Eq. (5), the strain energy, $S$ is expressed in terms of displacement field as

$$S = \frac{1}{2} \int_0^L \int_A \left[ E(z)((u^0_{\varepsilon_x})^2 - 2u^0_{\varepsilon_x} z\phi^0_{\varepsilon_x} - 2u^0_{\varepsilon_x} \alpha(z)\Delta T(x,z) 
+ (z\phi^0_{\varepsilon_x})^2 + 2z\phi^0_{\varepsilon_x} \alpha(z)\Delta T(x,z) + (\alpha(z)\Delta T(x,z))^2 \right] 
+ G(z)[(-\phi^0)^2 - 2\phi^0 w^0_{\varepsilon_x} + (w^0_{\varepsilon_x})^2] dAdx.$$

(9)

Integrating over the area and, from composite laminate theory, the following stiffness and thermal stiffness coefficients are
\[
[A_{11} \quad B_{11} \quad D_{11}] = \int_A E(z)[1 \ z \ z^2]dA, \\
A_{55} = \int_A G(z)dA,
\]

(10)

\[
[AT_{11} \quad BT_{11}] = \int_A E(z)\alpha(z)\Delta T(x, z)[1 \ z]dA.
\]

(11)

Using the relations in Eqs. (10) and (11), Eq. (9) can be rewritten as

\[
S = \frac{1}{2} \int_0^L \left[ A_{11}(u_0', x)^2 - 2B_{11}u_0'^2 + D_{11}(\phi_0', x)^2 + 2\Delta T_{11}u_0'^2 + A_{55}(\phi_0, x)^2 - 2A_{55}\phi_0 w_0, x + A_{55}(w_0, x)^2 \right] + \int_A E(z)(\alpha(z))^2 (\Delta T(x, z))^2 dA]dx.
\]

(12)

Taking the first variation with respect to the nodal degrees of freedom and applying Green’s theorem in conjunction with integration by parts transforms Eq. (12) into the equations of motion in terms of \( u_0, \ w_0, \) and \( \phi_0 \) as

\[
\delta u_0 : -A_{11}u_0', xx + B_{11}\phi_0', xx + AT_{11}, x = 0,
\]

(13)

\[
\delta w_0 : A_{55}(\phi_0', x - w_0', xx) = 0,
\]

(14)

\[
\delta \phi_0 : B_{11}u_0', xx - D_{11}\phi_0', xx - BT_{11}, x - A_{55}(w_0', x - \phi_0) = 0.
\]

(15)
The force boundary conditions obtained from the line integral are

\[ A_{i1}u_{ix}^{0} - B_{i1}\phi_{ix}^{0} - AT_{11} = N_{x}, \]  \hspace{1cm} (16)

\[ A_{55}(w_{ix}^{0} - \phi^{0}) = V_{x}, \]  \hspace{1cm} (17)

\[ (-B_{i1}u_{ix}^{0} + D_{i1}\phi_{ix}^{0} + BT_{11}) = M_{x}. \]  \hspace{1cm} (18)

where \( N_{x}, V_{x} \) and \( M_{x} \) denote axial force, shear force and bending moment acting at the boundary. Note that when the temperature is constant along the axis of the beam, temperature does not enter into the governing equations but enters as a force term in the boundary conditions.

**Finite Element Formulations**

Now that the equations of motion are developed for a first order shear deformable beam, the following two sections will develop the element formulation. The first formulation is based on Chakraborty et al. [1] followed by the three node element formulation.

**Two-node Element Formulation**

Chakraborty et al. [1] obtained the same governing equations expressed in Eqs. (13)-(15). However, they developed the element for a constant temperature case only.
Herein, the derived element formulation contains provisions for axial and through the thickness temperature gradients. For clarity, the variables in Chakraborty et al. [1] have been retained where possible.

The element formulation starts by developing the interpolation functions based on the displacement field. The exact form for the degrees of freedom used in the general solution of Eqs. (13)-(15) are

\[ u^0 = c_1 + c_2 x + c_3 x^2, \]  
(19)

\[ w^0 = c_4 + c_6 x + c_8 x^2 + c_7 x^3, \]  
(20)

\[ \phi^0 = c_8 + c_9 x + c_{10} x^2. \]  
(21)

Each node in an element has three degrees of freedom and there are two nodes per element giving a total of six unknowns per element. A review of Eqs. (19)-(21) gives ten unknown coefficients, \( c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, \) and \( c_{10}. \) Static condensation [12]-[13] is employed to reduce the number of unknowns to six. Substituting Eqs. (19)-(21) into Eqs. (13)-(15) yields the following system of equations

\[-2A_1 c_3 + 2B_{11} c_{10} + AT_{11,1} = 0, \]  
(22)

\[-A_{55} (2c_6 + 6c_7 x - c_9 - 2c_{10} x) = 0, \]  
(23)

\[-A_{55} (c_5 + 2c_6 x + 3c_7 x^2 - c_8 - c_9 x - c_{10} x^2) + 2B_{11} c_5 - 2D_{11} c_{10} - BT_{11,1} = 0. \]  
(24)
Solving the system of equations (22)-(24), the following relationships are established

\[
c_3 = \frac{1}{2} (\alpha(c_8 - c_5) + \alpha),
\]
\[
c_7 = \frac{1}{6} (\beta(c_8 - c_5) + \beta),
\]
\[
c_8 = c_9 / 2,
\]
\[
c_{10} = \frac{1}{2} (\beta(c_8 - c_5) + \beta),
\]

where

\[
\alpha = B_{11}A_{55} / (A_{11}D_{11} - B_{11}^2),
\]
\[
\beta = A_{11}A_{55} / (A_{11}D_{11} - B_{11}^2),
\]
\[
\alpha = (D_{11}AT_{11,\nu} - B_{11}BT_{11,\nu}) / (A_{11}D_{11} - B_{11}^2),
\]
\[
\beta = (A_{11}BT_{11,\nu} - B_{11}AT_{11,\nu}) / (A_{11}D_{11} - B_{11}^2).
\]

In Eq. (26), \(\alpha\) and \(\beta\) relate to coupling between the stiffness coefficients, while \(\alpha\) and \(\beta\) give the coupling between stiffness coefficients and the axial gradient of the thermal stiffness coefficients. If the axial gradient is zero, the terms \(\alpha\) and \(\beta\) are both equal to zero.

With the aid of the relationships established from static condensation, Eqs. (19)-(21) are rewritten in the form
\[ u^0 = c_1 + c_2 x + \frac{1}{2} \alpha (c_8 - c_5) x^2 + \frac{1}{2} \alpha_\phi x^2, \]  
(27)

\[ w^0 = c_4 + c_3 x + \frac{1}{2} c_\phi x^2 + \frac{1}{6} \beta (c_8 - c_5) x^3 + \frac{1}{6} \beta_\phi x^3, \]  
(28)

\[ \phi^0 = c_8 + c_9 x + \frac{1}{2} \beta (c_8 - c_5) x^2 + \frac{1}{2} \beta_\phi x^2. \]  
(29)

Note that the end result of static condensation is a coupling between the mid-plane displacements and mid-plane rotations through the stiffness coefficients and the gradient in the axial direction of the thermal stiffness coefficients. Eqs. (27)-(29) can be rewritten in matrix form

\[
\begin{bmatrix}
\{u(x)\} \\
\{w(x)\} \\
\{\phi(x)\}
\end{bmatrix} = [N(x)] \{a\} + \{\theta(x)\},
\]  
(30)

where

\[
\{a\} = \{c_1, c_2, c_4, c_5, c_8, c_9\}^T \text{ and } \{\theta(x)\} = \{\frac{1}{2} \alpha_\phi x^2, \frac{1}{6} \beta_\phi x^3, \frac{1}{2} \beta_\phi x^2\}^T.
\]

Solving for the unknown constants, \{a\}, in terms of nodal variables requires evaluating Eq. (30) at the nodes \(x = 0\) and \(x = L\) as (see Figure 16)
Figure 16. Nodes and degrees of freedom for the 2-node element

\[
\begin{pmatrix}
\{u(0)\} \\
\{u(L)\}
\end{pmatrix}
= 
\begin{bmatrix}
N(0) \\
N(L)
\end{bmatrix}
\{a\}
+ 
\begin{bmatrix}
\{\theta(0)\} \\
\{\theta(L)\}
\end{bmatrix}.
\]

(31)

Rearranging and using a more compact notation, Eq. (31) is written as

\[
\{\hat{u} - \hat{\theta}\} = [G]^{-1}\{a\},
\]

(32)

where

\[
[G]^{-1}
= 
\begin{bmatrix}
N(0) \\
N(L)
\end{bmatrix}.
\]

Solving for the unknown constants, \{a\}, yields

\[
\{a\} = [G]\{\hat{u} - \hat{\theta}\}.
\]

(33)
The term \( \{ \hat{u} \} = \{ u_1, w_1, \phi_1, u_2, w_2, \phi_2 \}^T \) is the nodal displacement vector for the element and \( \{ \hat{\theta} \} = \{ \theta(0), \theta(L) \}^T \) is the thermal gradient contribution vector at nodes. Now the displacements at any point in the element can be expressed in terms of nodal displacements by substituting Eq. (33) into Eq. (30)

\[
\{ u(x) \} = [N(x)] \{ \hat{u} - \hat{\theta} \} + \{ \theta(x) \} \tag{34}
\]

Recognizing that \([N(x)][G]\) is equal to the shape function for the element, \([N(x)]\). The full expression for \([N(x)]\) is given in Appendix A of Ref. [1]. Temperature gradient in the axial direction results in \( \{ \hat{\theta} \} \) and \( \{ \theta(x) \} \) terms. The terms \( \{ \hat{\theta} \} \) and \( \{ \theta(x) \} \) are zero when the beam does not contain a thermal gradient in the axial direction.

The element stiffness matrix is determined by

\[
[K] = \int_{V} [B]^T \begin{bmatrix} E(z) & 0 \\ 0 & G(z) \end{bmatrix} [B] dV , \tag{35}
\]

where \([B]\) is the strain-displacement matrix. Performing the matrix multiplication, integrating over the volume, and using the definitions in Eq. (10) gives the terms of the element stiffness matrix in closed form. The element stiffness matrix is given in Appendix A of Ref. [1].
The final step is formulating the thermal load vectors. The first thermal load vector is from the force boundary conditions in Eqs. (16)-(18)

\[ \{ R1 \} = \{ -AT_{11}, 0, BT_{11}, AT_{11}, 0, -BT_{11} \}^T. \]  

The second thermal load vector comes from Eqs. (13)-(15). Specifically the terms \( AT_{11,x} \) and \( BT_{11,x} \) can be treated as distributed loads and applied at the nodes as

\[ \{ R2 \} = \int_0^L \begin{bmatrix} AT_{11,x} \\ 0 \\ -BT_{11,x} \end{bmatrix} \begin{bmatrix} N_1^T \\ N_2^T \end{bmatrix} dx. \]  

Eq. (37) can be numerically integrated for a general temperature distribution. If the temperature distribution is linear along the length of the beam, \( AT_{11,x} \) and \( BT_{11,x} \) are constants, and the integration of Eq. (37) results in

\[ \{ R2 \} = \begin{bmatrix} L AT_{11,x}/2 \\ L^2 A_5 \left( AT_{11,x} B_{11} - BT_{11,x} A_{11} \right) / \xi \\ L \xi /2 \zeta \\ L AT_{11,x}/2 \\ -L^2 A_5 \left( AT_{11,x} B_{11} - BT_{11,x} A_{11} \right) / \xi \\ L \xi /2 \zeta \end{bmatrix}, \]  

where

\[ \xi = AT_{11,x} A_5 B_{11} L^2 + 12 BT_{11,x} D_{11} A_{11} - 12 BT_{11,x} B_{11}^2, \]

\[ \zeta = 12 B_{11}^2 - A_5 A_{11} L^2 - 12 D_{11} A_{11}. \]
For this special case, the thermal load vector \( \{R2\} \) is constant along the beam length and we have the closed form of \( \{R2\} \) given by Eq. (38). However, for a more general form of the temperature distribution, the terms \( AT_{11,x} \) and \( BT_{11,x} \) are no longer constants and it is necessary to numerically integrate Eq. (37) for each element of the beam model.

When \( \{R2\} \) is assembled globally for the model, the second and fifth terms cancel for all but the end elements. Finally, the full expression for the element with a general thermal load can be expressed as

\[
\{ F \} = [K] \{ \dot{u} \} + \{ R1 \} + \{ R2 \}
\]

**Three-node FOSD Element**

In developing the three-node element, the same displacement field given in Eqs. (19)-(21) is used. By choosing an interpolation function for \( w^0 \) one order in \( x \) higher than \( \phi^0 \) meets one of the requirements to prevent shear locking of the element [3]. Static condensation is again used to reduce the number of unknown coefficients from ten to nine. The relationship is found to be

\[
c_{10} = 3c_7,
\]

(40)
coupling the transverse displacement and the rotation. However, unlike in the two-node element formulation, the stiffness coefficients and the gradient of the thermal stiffness coefficients do not make an appearance in the interpolation functions, and therefore in the shape functions. Proceeding as in the two-node element case to solve for the unknown coefficients \( \{a\} \) in terms of nodal variables, we evaluate Eq. (29) at the nodes \( x = 0 \), \( x = L/2 \) and \( x = L \) (see Figure 17), the following matrix is written

\[
\begin{bmatrix}
\{u(0)\} \\
\{u(L/2)\} \\
\{u(L)\}
\end{bmatrix} =
\begin{bmatrix}
N(0) \\
N(L/2) \\
N(L)
\end{bmatrix}
\begin{bmatrix}
a \\
\end{bmatrix},
\]

or in similar notation as Eq. (32)

\[
\{\hat{a}\} = [G]^{-1}\{a\}.
\]

Figure 17. Nodes and degrees of freedom for the 3-node element
The shape function for the three-node element becomes

\[ \{u(x)\} = [N(x)] \{\hat{u}\}, \quad (43) \]

where the non-zero elements of the shape function \([N(x)]\) for the three-node element are given by

\[
\begin{align*}
N_{11} &= N_{22} = N_{33} = \frac{L^2 - 3xL + 2x^2}{L^2} \\
N_{14} &= N_{25} = N_{36} = -4x \left( \frac{x - L}{L^2} \right) \\
N_{17} &= N_{28} = N_{39} = \frac{x(2x - L)}{L^2} \\
N_{23} &= N_{29} = \frac{x(L^2 - 3xL + 2x^2)}{3L^2} \\
N_{26} &= \frac{-2x(L^2 - 3xL + 2x^2)}{3L^2}
\end{align*}
\]

(44)

The thermal load vectors for the three-node element are arrived at through a similar process as in the two-node element formulation case finding that

\[
\{R_1\} = \{-AT_{11}, 0, BT_{11}, 0, 0, 0, AT_{11}, 0, -BT_{11}\}^T,
\quad (45)
\]
and

\[
\{R2\} = \begin{bmatrix}
L AT_{11,x} / 6 \\
0 \\
-L BT_{11,x} / 6 \\
2L AT_{11,x} / 3 \\
0 \\
-L BT_{11,x} / 3 \\
L AT_{11,x} / 6 \\
0 \\
-L BT_{11,x} / 6
\end{bmatrix}
\] (46)

Again, as in the 2-node element case, this value of \( \{R2\} \) in Eq. (46) is valid only for the particular case when the temperature distribution thermal load is linear. For a general form of the temperature distribution, the terms \( AT_{11,x} \) and \( BT_{11,x} \) are no longer constants and it is necessary to numerically integrate Eq. (37) for each element of the beam model.

Next we find the element stiffness matrix for the three-node element. Proceeding in similar way as in the two-node element case, the nonzero elements in the upper diagonal of the stiffness matrix are found to be:

\[
K_{11} = K_{77} = \frac{7A_{11}}{3L} ; \quad K_{13} = K_{79} = \frac{-7B_{11}}{3L}
\]

\[
K_{14} = K_{47} = \frac{-8A_{11}}{3L} ; \quad K_{16} = K_{67} = \frac{8B_{11}}{3L}
\]
\[ K_{17} = \frac{A_{11}}{3L}; K_{19} = K_{37} = -\frac{B_{11}}{3L} \]

\[ K_{22} = K_{88} = \frac{7A_{55}}{3L}; K_{23} = -K_{89} = \frac{A_{55}}{2} \]

\[ K_{25} = K_{28} = -\frac{8A_{55}}{3L}; K_{29} = -K_{38} = \frac{-A_{55}}{6} \]

\[ K_{26} = K_{39} = -K_{35} = -\frac{2A_{55}}{3}; K_{28} = \frac{A_{55}}{3L} \]

\[ K_{33} = K_{99} = \frac{21D_{11} + A_{55}L^2}{9L}; K_{34} = K_{49} = \frac{8B_{11}}{3L} \]

\[ K_{36} = K_{69} = \frac{-24D_{11} + A_{55}L^2}{9L}; K_{39} = \frac{6D_{11} - A_{55}L^2}{18L} \]

\[ K_{44} = 16A_{11} \frac{1}{3L}; K_{44} = -16B_{11} \frac{1}{3L} \]

\[ K_{55} = -16A_{55} \frac{1}{3L}; K_{66} = \frac{4 \left(12D_{11} + A_{55}L^2 \right)}{9L} \]
Temperature Profile Modeling

This section presents the mathematical formulation and solution of the heat conduction steady-state problem for composite FGMs beam models under thermal loading. The solution serves as a foundation to conduct the analytical and numerical simulations in this research. Two formulations are presented. The first is a formulation to find the one-dimensional temperature distribution for a 3-layer beam with a middle FGM layer. This is followed by a more realistic numerical formulation for finding the two-dimensional temperature distribution for the same 3-layer beam.

One-dimensional Heat Conduction Steady-State Exact Solution for a 3-layer FG Beam

In this part, we consider the solution of the heat conduction steady-state problem in a composite beam consisting of 3 layers, which are assumed to be in perfect thermal contact. Figure 18 shows the geometry coordinates and the boundary conditions for this problem.
Figure 18. Three-layer beam with perfect thermal contact at the interface.

The mathematical formulation of this problem is given as

$$\frac{d}{dz} \left[ k_1 \frac{dT_1(z)}{dz} \right] = 0 , \quad -(h_1 + a) < z < -a \quad (48)$$

$$\frac{d}{dz} \left[ k_2(z) \frac{dT_2(z)}{dz} \right] = 0 , \quad -a < z < a \quad (49)$$

$$\frac{d}{dz} \left[ k_3(z) \frac{dT_3(z)}{dz} \right] = 0 , \quad a < z < (a + h_2) \quad (50)$$

subject to the boundary and interface conditions

$$T_i = T_b \quad \text{at} \quad z = -(h_1 + a) \quad (51)$$
\[
\begin{align*}
  k_1 \frac{dT_1(z)}{dz} &= k_2(z) \frac{dT_2(z)}{dz} \quad \text{at } z = -a \\
  k_2(z) \frac{dT_2(z)}{dz} &= k_3 \frac{dT_3(z)}{dz} \quad \text{at } z = a \\
  T_3 &= T_f \quad \text{at } z = (a + h_2)
\end{align*}
\]

where \( k_1, k_2(z), \) and \( k_3 \) are the thermal conductivities coefficients for steel, graded layer, and alumina, respectively (see Figure 19). The solution to the equations \((48)-(50)\) subject to the boundary and interface conditions given by Eqs. \((51)-(56)\) can be found numerically. Several special cases can result in exact solutions such as when \( k_1 = k_b \) and \( k_3 = k_i \) are constant throughout layers 1 and 3, while \( k_2(z) \) is assumed to vary only in the direction of the beam thickness according to

\[
k_2(z) = k_i e^{-0.5\ln\left(\frac{k_i}{k_b}\right)(1 - \frac{z}{a})}
\]

The solution of the ordinary differential equations \((48)-(50)\), for each layer is given in the form

\[
T_1(z) = C_1 z + C_2,
\]

\[
T_2(z) = C_3 \left( \frac{k_i}{k_b} \right)^{-0.5z} + C_4,
\]

\[
T_3(z) = C_5 z + C_6.
\]
The solution involves two unknown constants for each layer; then, for a 3-layer problem, 6 unknown constants are to be determined. Substituting the solution given by Eqs. (58)-(60) into the boundary and interface conditions (51)-(56), one obtains 6 equations for the determination of the 6 unknown constants. The final solution for each layer is then given by

\[
T_1(z) = \frac{k_1 \ln(k_i / k_b)(T_i - T_b)(z + a) + \ln(k_i / k_b)(k_i h_1 T_i + k_b h_2 T_b) + 2aT_i(k_i - k_b)}{(k_i h_1 + k_2 h_b) \ln(k_i / k_b) + 2a(k_i - k_b)}
\]  
(61)

\[
T_2(z) = \frac{2k_1 a \left( \frac{k t}{k b} \right)^{\frac{1}{2} - \frac{a}{z}} (T_i - T_b) + \ln(k_i / k_b)(k_i h_1 T_i + k_b h_2 T_b) + 2a(T_i - k_b T_b)}{(k_i h_1 + k_2 h_2) \ln(k_i / k_b) + 2a(k_i - k_b)}
\]  
(62)

\[
T_3(z) = \frac{k_1 \ln(k_i / k_b)(T_i - T_b)(z + a) + \ln(k_i / k_b)(k_i h_1 T_i + k_b h_2 T_b) + 2aT_i(k_i - k_b)}{(k_i h_1 + k_2 h_b) \ln(k_i / k_b) + 2a(k_i - k_b)}
\]  
(63)

Figure 19 shows the depth-wise exact temperature distribution for sample materials and geometrical parameters given in Table 4.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>h_1</td>
<td>0.025 m</td>
</tr>
<tr>
<td>h_2</td>
<td>0.025 m</td>
</tr>
<tr>
<td>a</td>
<td>0.025 m</td>
</tr>
<tr>
<td>T_b</td>
<td>20 °C</td>
</tr>
<tr>
<td>T_i</td>
<td>400 °C</td>
</tr>
<tr>
<td>k_b</td>
<td>51.9 W/m°C (Steel)</td>
</tr>
<tr>
<td>k_l</td>
<td>13.75 W/m°C (Al₂O₃)</td>
</tr>
</tbody>
</table>
Figure 19. Depth-wise exact temperature distribution obtained from the solution of the heat conduction differential equation.
Two-dimensional Heat Conduction Steady-State Numerical Solution for a 3-Layer FG Beam with Temperature Dependency of the Material Properties

Now, we present the mathematical formulation of the two-dimensional heat conduction steady-state problem of a composite beam consisting of three parallel layers, which are assumed to be in perfect thermal contact. Figure 20 shows the geometry coordinates and the boundary conditions for this problem. Different from the one-dimensional profile modeling case, in this problem the material parameter thermal conductivity $k$ depends on the temperature $T$ itself, which is the dependent variable of this problem.

Figure 20. Three-layer beam geometry and boundary conditions
The two-dimensional mathematical formulation of this problem is given as

\[
\frac{\partial}{\partial z} \left[ k_1(T) \frac{\partial T_1(x, z)}{\partial z} \right] + \frac{\partial}{\partial x} \left[ k_1(T) \frac{\partial T_1(x, z)}{\partial x} \right] = 0 \quad \text{in } z_1 < z < z_2, \ 0 < x < L \quad (64)
\]

\[
\frac{\partial}{\partial z} \left[ k_2(z, T) \frac{\partial T_2(x, z)}{\partial z} \right] + \frac{\partial}{\partial x} \left[ k_2(z, T) \frac{\partial T_2(x, z)}{\partial x} \right] = 0 \quad \text{in } z_2 < z < z_3, \ 0 < x < L \quad (65)
\]

\[
\frac{\partial}{\partial z} \left[ k_3(T) \frac{\partial T_3(x, z)}{\partial z} \right] + \frac{\partial}{\partial x} \left[ k_3(T) \frac{\partial T_3(x, z)}{\partial x} \right] = 0 \quad \text{in } z_3 < z < z_4, \ 0 < x < L \quad (66)
\]

subject to the boundary and interface conditions

\[
T_1(x, z) = T_b \quad \text{at } z = z_1 \quad \text{(67)}
\]

\[
k_1(T) \frac{\partial T_1(x, z)}{\partial z} = k_2(z, T) \frac{\partial T_2(x, z)}{\partial z} \quad \text{at } z = z_2 \quad \text{(68)}
\]

\[
T_2(x, z) = T_2 \quad \text{at } z = z_2 \quad \text{(69)}
\]

\[
k_2(z, T) \frac{\partial T_2(x, z)}{\partial z} = k_3(T) \frac{\partial T_3(x, z)}{\partial z} \quad \text{at } z = z_3 \quad \text{(70)}
\]

\[
T_3(x, z) = T_3 \quad \text{at } z = z_3 \quad \text{(71)}
\]

\[
T_4(x, z) = T_r \quad \text{at } z = z_4 \quad \text{(72)}
\]

\[
T(x, z) = T_L \quad \text{at } x = 0 \quad \text{(73)}
\]

\[
T(x, z) = T_R \quad \text{at } x = L \quad \text{(74)}
\]
where \( k_1(T) \), \( k_2(z,T) \), and \( k_3(T) \) are the thermal conductivities coefficients for steel, graded layer, and alumina, respectively. Since layers 1 and 3 are homogenous materials, their thermal conductivities \( k_1 \) and \( k_3 \) are considered independent of the position \( z \) throughout layers 1 and 3; however they still depend on the temperature \( T \). The thermal conductivity of the graded layer \( k_2(z,T) \) is assumed to vary in the direction of the beam thickness and with the temperature according to

\[
k(z,T) = k_i(T) \left( 1 + \frac{3[k_2(T) - k_i(T)]V_m(z)}{3k_i(T)V_m(z) + [(k_b(T) + 2k_i(T))]V_c(z)} \right)
\]  

(75)

The partial differential equations (64)-(66) are classified as elliptic type [43]. This kind of equation is also well-known as the homogeneous Laplace equation. It is important to realize that this problem becomes nonlinear due to the nonlinearity introduced to the governing differential equation by the variation of the thermal conductivity \( k(z,T) \) with the temperature \( T \), which is the dependent variable itself of this problem. This nonlinear heat conduction steady-state problem was solved iteratively using a finite element partial differential equation solver using the computational tool MATLAB®. During the solution, the temperature dependency of the thermo-mechanical material properties is considered. That is, during the solution process of finding the temperature distribution in the different layers, these material properties are updated iteratively according to the actual temperature at the particular geometrical position. The material property data was fitted using cubic-spline interpolation, as discussed in the temperature dependence of material properties section in Chapter 3, and incorporated into the
numerical procedure so that the solver can interpolate to determine the thermal conductivity of a material at any temperature. The resulting temperature distribution will be used as a thermal load into a finite element code for analyzing stresses in FGM beam models in Chapter 5.
Chapter 5 Analyses and Results

Introduction

This chapter discusses the results of the analyses performed in this research. The analyses are performed using the computational tool MATLAB®. Comparisons with FGMs models available in related literature are made. Also, analytical and numerical simulations of thermal loading studies conducted on the FGMs beam structures are presented. The beam models are studied to show the performance of the element formulations presented in Chapter 4. The mathematical formulation and solution details of these problems are included in Chapter 4 as well. Additionally, this chapter introduces a study to determine the influence of manipulating the FGM layer thicknesses on the factor of safety in structures constructed of functionally graded materials under thermal loads.

Comparisons of the Element Formulation Simulations with Related Literature

This section will present simulations of FGMs model results available in the related literature for comparison purposes. These comparisons will reveal the performance of the element formulations presented in Chapter 4. Two groups of
comparisons will be presented. The first group involves an example of thermal stress distribution in a tri-layered FGM model analyzed by Suresh and Mortensen [2]. The second group of comparison involves the analysis of FGM beams in stress smoothening when more than one type of material is present in the structures, as presented by Chakraborty et al. [1]. For both groups, problems will be revisited and their results will be compared with the results of this research [1, 2].

**Comparison with Suresh and Mortensen's Model**

The formulated element in this work is used to compare with an example of thermal stress distribution in a tri-layered FGM model analyzed by Suresh and Mortensen [2]. The model considered is a system of Ni-graded layer (GL)-Al₂O₃ tri-layered composed beam as shown in Figure 21. In this literature, the thermoelastic properties within the graded layer vary linearly with \( z \), according to

\[
E = E_0 + \Delta E \frac{z}{a} \quad \text{for} \quad -a \leq z \leq a, \tag{76}
\]

\[
\alpha = \alpha_0 + \Delta \alpha \frac{z}{a} \quad \text{for} \quad -a \leq z \leq a, \tag{77}
\]

where \( \Delta \) preceding a property refers to the change in that property for a change in temperature \( \Delta T \), and the subscript \( _0 \) on a property refers to the value of that property at the initial reference temperature. At this point, it important to mention that even though
this literature uses a different approach for calculating the material properties and does not give details about the actual properties values used in its model, our intention here is to make a qualitative comparison rather than a quantitative one. With this in mind, we proceed to compare the results for the axial thermal stress distribution throughout the thickness found in the referenced literature and the present work.

Figure 21. Geometry and nomenclature for a tri-layered composed beam model from literature reference [1].

Figure 22 shows the spatial variation of the thermal axial stress $\sigma_{xx}$ (in plane stress) throughout the thickness of the Ni-GL-Al$_2$O$_3$ tri-layered beam subject to a temperature drop of 100 °C (from an initial stress-free reference temperature) for the particular geometrical condition that $h_1 = h_2 = h$ and that $a/h = 0.6$. The constituent materials of this model and their properties are given in Table 5.
Table 5. Thermo-elastic properties of nickel and alumina at 300 K

<table>
<thead>
<tr>
<th>Property</th>
<th>Parameter value</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Nickel (GPa)</td>
</tr>
<tr>
<td>Young's modulus of elasticity</td>
<td>207</td>
</tr>
<tr>
<td>Shear modulus</td>
<td>76</td>
</tr>
<tr>
<td>Coefficient of thermal expansion</td>
<td>13.1×10⁻⁶ °C⁻¹</td>
</tr>
</tbody>
</table>

When comparing the axial stress distribution found in the present work shown in Figure 22 (a) with that in the reference literature (Figure 22 (b)), the following tendencies are revealed:

1. when there is no graded layer between the ceramic and metal layer, large values of stresses are developed at the interface;

2. the near-interface region of the metallic layer is in tension, while the corresponding region for the ceramic layer is in compression (there is considerable abrupt change in magnitude and sign of the stress at the interface);

3. when a graded interlayer is introduced, the magnitude of the stress at the interface can be significantly reduced and the abrupt change in the stress sign is eliminated;

4. the stresses vary linearly with \( z \) within the metallic and ceramic layer, and approximately parabolically within the functionally graded layer.

From these trends, we can conclude that the qualitative results obtained in this work are very similar to the referenced literature, which demonstrates a suitable performance of the element formulations presented in this work.
Figure 22. Axial thermal stress distribution in a Ni-Graded Layer-Al₂O₃ trilayer beam subject to a ΔT = -100 °C.

Figure 22 (b) reprinted from Suresh and Mortensen [2] with permission from Maney Publishing.
Comparison with Chakraborty et al. Models

Next, we compare the axial and shear stress through the thickness results obtained with formulated element in this work with the results obtained in Ref. [1] for a bi-material beam model where the transition is made smooth by inserting a thin FGM layer. The materials considered and their properties are given in Table 6. Using these materials a functionally graded cantilever composite beam of 0.5 m length and unit width subjected to three different loads are considered as illustrated in Figure 23. The topmost material is steel and bottom layer is alumina. An FGM interlayer is placed in between these layers. Material properties vary according to the exponential law given by

\[ P(z) = P_t e^{-0.5 \ln \left( \frac{P_t}{P_b} \right) (1 - \frac{z}{t})} \]  

(78)

where \( P(z) \) describes a typical material property (\( E, \alpha, G \), etc.) at any point \( z \) throughout the thickness \( t \). The variables \( P_t \) and \( P_b \) are the material property at the top and bottom, respectively.

Table 7 specify the loading cases applied to the analyzed models. Each of these loads is applied to three different geometrical configurations of the two material constituent of this model. The first is a bi-material beam contains two layers; the second is a partial functional graded composite beam (PFGM) consisting of 3 layers where the middle region is a FGM that transitions the material properties from the bottom layer to
the top layer; the last a 1-layer beam composed of a functional graded material (FGM) through the entire thickness.

Figure 23. Geometry and loading cases for models from literature paper [1].

Table 6. Thermo-elastic properties of steel and alumina at 300 K

<table>
<thead>
<tr>
<th>Property</th>
<th>Parameter value</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Steel</td>
</tr>
<tr>
<td>Modulus of elasticity</td>
<td>210 GPa</td>
</tr>
<tr>
<td>Shear modulus</td>
<td>80 GPa</td>
</tr>
<tr>
<td>Coefficient of thermal expansion</td>
<td>14.0×10⁻⁶ °C⁻¹</td>
</tr>
</tbody>
</table>

Table 7. Loading cases for models from literature paper [1].

<table>
<thead>
<tr>
<th>Case</th>
<th>Load type</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Unit transverse load applied at the tip (1 N)</td>
</tr>
<tr>
<td>2</td>
<td>Unit axial load applied at the tip (1 N)</td>
</tr>
<tr>
<td>3</td>
<td>Thermal load ( \Delta T = 5 ) °C</td>
</tr>
</tbody>
</table>
The results of this comparison are summarized in Figure 24–28.

From the Figures 24–28, we can observe the following similarities and discrepancies when comparing this work and the referenced literature results. Both results agree as follows:

1. in the absence of FGM layer between the ceramic and metal layer, the stress distributions are discontinuous at the interface;
2. the introduction of a small FGM layer smoothens the stresses to the tune of about 300 N/m² and 10 N/m² stress jump of the axial and shear stress, respectively (the abrupt value change in the stress is eliminated);
3. for load cases 1 and 2, the axial stresses vary linearly with $z$ within the metallic and ceramic layer, and approximately parabolically within the functionally graded layer;
4. for load cases 1 and 2, the shear stresses are constant throughout the metallic and ceramic layer, and approximately parabolically within the functionally graded layer.

The results disagree in the axial stress for the load case 3 (thermal load). In the present work, the axial stress vary linearly with $z$ within the metallic and ceramic layer, while for the referenced paper the axial stress is constant throughout these layers. From these observations, we can conclude that, except for thermal load case, the qualitative and quantitative results obtained in this work are very similar to the referenced paper, which demonstrates a proper performance of the element formulations presented in this work.
a) Present work’s results

Figure 24. Axial stress through the thickness for case 1. Figure 24 (b) reprinted from Ref. [1] with permission from Elsevier.

b) Reference paper

a) Present work’s results

Figure 25. Transverse shear stress through the thickness for case 1. Figure 25 (b) reprinted from Ref. [1] with permission from Elsevier.
a) Present work’s results

Figure 26. Axial stress through the thickness for case 2. Figure 26 (b) reprinted from Ref. [1] with permission from Elsevier.

a) Present work’s results

Figure 27. Axial stress through the thickness for case 3. Figure 27 (b) reprinted from Ref. [1] with permission from Elsevier.
a) Present work’s results

Figure 28. Transverse shear stress through the thickness for case 3.  
Figure 28 (b) reprinted from Ref. [1] with permission from Elsevier.
Simulations with Generic Temperature Distributions and Temperature Independence of the Material Properties

This section analyzes the beam configurations shown in Figure 29. The models are composed of a cantilever beam with the support at the origin. The beam is 100 mm long and 10 mm thick. The beam width is not important because in the first-order shear deformation theory for a beam this is classified as cylindrical bending.

![Beam configurations](image)

a) FGM beam  
b) Bi-material beam  
c) Bi-material with average interlayer beam  
d) PFGM beam

Figure 29. Beam configurations.

The beam contains two materials arranged in four different configurations. The first, Figure 29 (a) is a 1-layer beam composed of a functional graded material (FGM) through the entire thickness. The second, Figure 29 (b) is a bi-material beam contains two
layers without FGM region similar to a traditional layered composite. The third one, Figure 29 (c) is a composite beam consisting of 3 parallel layers where the middle region is a homogeneous layer whose properties values are the average of the material properties of the bottom layer (homogeneous material) and the top layer (homogeneous material). The last, Figure 29 (d) is a partial functional graded composite beam (PFGM) consisting of 3 parallel layers where the middle region is a FGM that transitions the material properties from the bottom layer (homogeneous material) to the top layer (homogeneous material). The beam models are subject to different thermal loads using generic temperature distributions (some of them from the related literature); its mechanicals and thermal properties are independent of temperature.

The FGM beam has a modulus ratio, $E_t/E_b$, of 5 with $E_b$ equal to 1GPa. The Poisson’s ratio for both materials is fixed at 0.3125. The coefficient of thermal expansion has a ratio of $\alpha_t/\alpha_b$ of 1/5 with $\alpha_t$ equal to $10^{-4}$. Within the FGM region, the thermo-mechanical material properties, the modulus, thermal conductivity and coefficient of thermal expansion vary through the thickness following the corresponding formula presented in Table 1 of Chapter 3.

Figure 30 shows details about the beam geometry and boundary conditions of the different models to be analyzed. The PFGM beam contains three sections where the middle region is a FGM that transitions the material properties from section 1 to section 3. Sections 1 and 3 are a quarter of the beam thickness with a constant modulus and coefficient of thermal expansion equal to the bottom and top of the FGM example, respectively. The bi-material beam contains two sections of equal thickness with no FGM region similar to a traditional layered composite.
a) FGM beam

b) Bi-material beam

c) Bi-material with average interlayer beam

d) PFGM beam

Figure 30. Beam geometry and boundary conditions.

The temperature distributions applied to the models are summarized in Table 8. The temperature distributions are with reference to stress free configuration. Case 1 represents a constant temperature. Case 2 contains a thermal gradient in the through the thickness only. Case 3 contains an axial thermal gradient. Case 4 combines cases 2 and 3 to give a distribution with an axial and through the thickness temperature gradient.
Table 8. Temperature distributions.

<table>
<thead>
<tr>
<th>Case</th>
<th>( \Delta T(x,z) ) - °C</th>
<th>( \Delta T_r / \Delta T_b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>( 100 \exp \left( 230 \left( z + \frac{h}{2} \right) \right) )</td>
<td>10</td>
</tr>
<tr>
<td>3</td>
<td>( x / BL )</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>( 100 \exp \left( 230 \left( z + \frac{h}{2} \right) \right) x / BL )</td>
<td>10</td>
</tr>
</tbody>
</table>

In the first example, a constant temperature distribution is applied to the beam and the normalized axial stress through the normalized thickness is shown in Figure 31. Both element formulations yield nearly identical results for all four beam material combinations. The combination of boundary conditions and thermal loading produces no gradients in the axial direction so Figure 31 applies along the length of the beam. The transverse shearing stresses are equal to zero for this load case. The bi-material beam contains the highest peak stress followed by the PFGM and FGM as expected showing the advantage of smooth as opposed to discontinuous transitions in material properties. Only at the bottom surface is the stress in FGM greater than either the PFGM or bi-material beam.
Figure 31. Normalized axial stress through the thickness for case 1, $\Delta T=100$.

The stress is normalized by $(E_b \alpha_b \Delta T)$ and transverse coordinate ($z$) is normalized by the beam thickness.

The second example contains an exponential through the thickness variation in temperature. The temperature change at the top of the beam is $1000^\circ C$ and the bottom is $100^\circ C$ from a stress free state temperature. The normalized axial stress in the transverse direction is shown in Figure 32. The combination of boundary conditions and thermal loading produce a beam whose stress is invariant to the axial direction. The transverse shear stress is equal to zero as well. In terms of comparing the two formulations, both give identical results.
Figure 32. Normalized axial stress through the thickness for case 2, $\Delta T(z) = 100\exp\left(230(z + h/2)\right)$.

The stress is normalized by $(E_b\alpha_b\Delta T_b)$ and transverse coordinate $(z)$ is normalized by the beam thickness.

The next case contains a linear axial temperature distribution from $0^\circ C$ at the cantilevered end to $100^\circ C$ at the free end. Figure 33 displays the normalized axial stress through the thickness of the beam. The normalized stress does not vary along the length of the beam. Case 1 and 3 are very similar and only show minor differences in the normalized axial stress. However, transverse shear stress is present as shown in Figure 34 when using the equilibrium equation but zero when calculated from the constitutive relations. The 2-node beam formulation gives poor results while the three-node beam formulation gives acceptable results. This is due to the differences in the second derivative of the shape function which is used to calculate the shearing stress from the equilibrium equation.
Figure 33. Normalized axial stress through the thickness for case 3, $\Delta T(x) = x / BL$.

The axial stress, $\sigma_{xx}(x)$, is normalized by $(E_0 \alpha_0 \Delta T_0(x))$ and transverse coordinate $(z)$ is normalized by the beam thickness.

Figure 34. Normalized transverse shear stress through the thickness for case 3, $\Delta T(x) = x / BL$.

The shear stress, $\tau_{xz}(x)$, is normalized by $(E_0 \alpha_0 \Delta T_0(x))$ and transverse coordinate $(z)$ is normalized by the beam thickness.
For a traditional two-node beam element the first derivative with respect to $x$ of the shape function is constant and the second derivative is a zero matrix. Because of the coupling terms from static condensation, the second derivative of the shape function for the two-node beam element formulated in the preceding chapter is not the zero matrix. However, the resulting shear stress when using this matrix is undesirable and gives a maximum shearing stress at the top of the beam which should be zero. In contrast, the three node element formulation gives a reasonable result. It should be emphasized that the magnitude of the transverse shear stress is small and is given on the $y$-axis on the right hand side of the graph for the three-node element. The absolute value obtained from the difference between the normalized axial stress of case 1 and case 3 is of the same order as the absolute value of the transverse shear stress.

The final case is a combination of cases 2 and 3 which gives a temperature distribution with a gradient in the axial and transverse thermal direction. The normalized axial stress in the through the thickness direction is shown in Figure 35 and the corresponding transverse shear stress using the equilibrium equation in Figure 36. The transverse shear stress from the constitutive relations is zero. The normalization procedure produces a graph independent of the axial location. The axial stress in this case does not vary significantly from case 2 with the same transverse variation in temperature. Again, both elements produce nearly identical results for the axial stress but vary greatly when considering the transverse shearing stress.
Figure 35. Normalized axial stress through the thickness for case 4, \( \Delta T(x,z) = 100\exp\left(230(z + h/2)\right) x / BL \).

The axial stress, \( \sigma_{xx}(x) \), is normalized by \( E_o\alpha_p\Delta T_p(x) \) and transverse coordinate \( (z) \) is normalized by the beam thickness.

Figure 36. Normalized transverse shear stress through the thickness for case 4, \( \Delta T(x,z) = 100\exp\left(230(z + h/2)\right) x / BL \).

The shear stress, \( \tau_{xz}(x) \), is normalized by \( E_o\alpha_p\Delta T_p(x) \) and transverse coordinate \( (z) \) is normalized by the beam thickness.
Simulations with Actual Temperature Distributions with and without Temperature Dependence of the Material Properties

This section analyzes the examples considered in the preceding section using the actual temperature distribution found by solving the heat conduction steady-state problem for the different composite beam models under thermal loading. The material properties law, assumptions, geometry and layers configuration are as per the preceding section. The temperature distribution was found as described in the “Two-dimensional Heat Conduction Steady-State Numerical Solution for a 3-Layer FG Beam” section in Chapter 4. Also, the examples are analyzed considering the temperature dependency of the thermo-elastic material properties. The temperature-dependent material property data was collected from engineering manuals, material handbooks, and database of material properties web sites [31, 35, 39]. The material property data was fitted using cubic-spline interpolation and incorporated into the numerical procedure. The nonlinear heat conduction steady-state problem was solved iteratively using a finite element solver. The solutions details of this problem are given in the two-dimensional heat conduction steady-state problem section in Chapter 4.

The following figures show boundary conditions, thermal conductivity and temperature profiles for the analyzed models.

The first analyzed model is a two-layer beam composited of steel and alumina as shown in Figure 37. This model will serve as a baseline reference to compare how the thermal conductivity temperature-dependence affects the temperature distribution and thermal stresses. It also will reveal how the temperature distribution and thermal stresses behaves when varying interlayers are introduced.
Figure 37. Beam geometry and boundary conditions (Bimaterial)

Figure 38 shows thermal conductivity $k$ distribution with and without temperature dependence. Figure 38 (a) shows that the thermal conductivity is constant throughout the entire layer for each material (51.26 W/m.K for steel and 18.41 W/m.K for alumina, both at 235 °C) when temperature influence is not considered. However, Figure 38 (b) reveals the actual thermal conductivity distribution when temperature dependence is taken into account. We can observe that when temperature dependence is considered, the previously assumed constant thermal conductivities values for steel and alumina vary from about 45 to 62 W/m.K for steel and from about 12 to 40 W/m.K for alumina. The significance or effect of this observation can be seen in the temperature profile distribution shown in Figure 39. It can be seen that at a particular position $z$ other than a boundary, the temperatures are higher in Figure 39 (a). In other words, the heat insulation effect of alumina is higher when the temperature dependence is considered. This can be explained by the fact that as the temperature increases, the thermal
conductivity decreases as studied previously in the “Consideration of Temperature Dependence of Material Properties” section in Chapter 3 (see Figure 11 and Figure 12).

Figure 38. Thermal conductivity $k$ distribution with and without temperature dependence (Bimaterial case).

Figure 39. Temperature profile with and without temperature dependence (Bimaterial case).
The next analyzed model is a three-layer beam composit of a steel bottom layer, an alumina top layer, and a homogeneous material interlayer, as shown in Figure 40. The material properties of this interlayer are taken as the average values of the steel and alumina. Figure 41 shows thermal conductivity $k$ distribution with and without temperature dependence for this model. Again, we can see from Figure 41 (a) that the thermal conductivity is constant throughout the entire layer for each material (51.26 W/m.K for steel, 18.41 W/m.K for alumina, and 34.83 W/m.K for the average interlayer; properties are taken at the average temperature 235 $^\circ$C) when temperature influence is not considered. As in the preceding bi-material model, the actual thermal conductivity distribution for this model is very different when temperature dependence is taken into account (Figure 41 (b)). Here we can observe that when temperature dependence is considered, the actual thermal conductivities values for steel, alumina, and the average interlayer vary from 55 to 62 W/m.K, 12 to 40 W/m.K, and 30 to 50 W/m.K, respectively. Again, as in the bi-material model, this difference on the thermal conductivity distribution affects the temperature profile distribution as shown in Figure 42. Similar behavior in comparison with the bi-material model can be seen here. That is, for a particular position $z$ other than a boundary, the temperatures are higher in Figure 42 (a). Also in this model, it is found that the heat insulation effect of alumina in Figure 42 (b) is higher than in Figure 42 (a). Now, comparing this model with the bi-material model, we see that in this model the heat insulation effect of alumina is higher.
Figure 40. Beam geometry and boundary conditions (Average interlayer)
Figure 41. Thermal conductivity $k$ distribution with and without temperature dependence (Average interlayer case).

Figure 42. Temperature profile with and without temperature dependence (Average interlayer case).
Now, we study the effect of substituting the homogeneous material interlayer by a functionally graded material (FGM) interlayer, as shown in Figure 43. The material properties of this interlayer are calculated according to the formulas given in Table 1 in Chapter 3.

Figure 43. Beam geometry and boundary conditions (FGM interlayer)

Figure 44 depicts the thermal conductivity $k$ distribution with and without temperature dependence for this model. For this model, we can see from Figure 44 (a) that the thermal conductivity is constant throughout the entire layer for the homogenous layers (51.26 W/m.K for steel and 18.41 W/m.K for alumina, both at 235 °C), but it changes continuously from 51.26 to 18.41 W/m K for the FGM interlayer when temperature influence is not considered. In Figure 44 (b) we see that the actual thermal conductivity distribution is very different from the results in Figure 44 (a) where...
temperature dependence was not considered. For this model, the actual thermal conductivities values for steel, alumina, and the FGM interlayer vary from about 60 to 62 W/m.K, 12 to 40 W/m.K, and 20 to 60 W/m.K, respectively. The influence of this different behavior is manifested in the temperature profile distribution shown in Figure 45. As in the two preceding models, similar results are found for this model; that is, the temperatures are higher in Figure 45 (a) than in Figure 45 (b) for a particular position $z$ other than a boundary. Again, in this model the heat insulation effect of alumina in Figure 45 (b) is higher than in Figure 45 (a). When comparing this model with the bi-material and average interlayer models, we see that in this model the heat insulation effect of alumina is higher. This fact can be use in engineering applications where insulation effects need to be improved.
Figure 44. Thermal conductivity $k$ distribution with and without temperature dependence (FGM interlayer case).

Figure 45. Temperature profile with and without temperature dependence (FGM interlayer case).
To compare the thermo-elastic behavior of the three preceding models, we now analyze the thermal stresses and factor of safety in these models subjected to the corresponding temperature distribution found for each model. The results of this comparison are summarized in the following figures.

The normalized axial stress through the thickness is shown in Figure 46. From Figure 46 (a), we observe that when temperature dependence is considered, the absolute value of axial stress is diminished at particular position z within the steel layer. For the FGM and ceramic layers, the axial stress behavior is non-uniform. As we can see, within these two last layers, there are regions where the absolute value of the axial stress is diminished when temperature dependence is considered and regions where the behavior is opposite.

When comparing the influence of the temperature for the three analyzed models, it can be seen from Figure 46 (b) that the absolute value of axial stress is diminished in the average and FGM interlayer model within the steel layer. For the FGM and ceramic layers, the axial stress behavior is non-uniform.

From these results, apparently nothing definitive can be concluded yet regarding the influence of including the temperature dependency in the design of the beam. However, later on in this section, we will see that the inclusion of the factor of safety and the specific factor of safety as a design criteria will allow us to chose the best design.
Figure 46. Normalized axial stress through the thickness for actual temperature distribution.

The stress is normalized by \((E_b \alpha_b \Delta T)\) and transverse coordinate \((z)\) is normalized by the beam thickness.
Figure 47 displays the transverse shear stress through the thickness of the beam. From Figure 47 (a), it is found that for the FGM interlayer model, the absolute value of the shear stress diminishes throughout the entire beam when temperature dependency is considered. When comparing the influence of the temperature for the three analyzed models, it can be seen from Figure 47 (b) that apparently the bi-material model gives the lowest levels of the absolute value of shear stress compared to the average and FGM interlayer model. However, as commented for axial stresses results, we cannot make a final decision or conclusion regarding which model is better until we include the factor of safety and the specific factor of safety as design criteria.
Figure 47. Normalized transverse shear stress through the thickness for actual temperature distribution.

The shear stress, $\tau_{xz}(x)$, is normalized by $\left( E_b \alpha_b \Delta T_b(x) \right)$ and transverse coordinate $(z)$ is normalized by the beam thickness.
As discussed earlier in this section, we calculate the factor of safety of the models to have a decision criterion for finding the most convenient beam model.

Figure 48 displays the factors of safety and their corresponding position of calculation for the different analyzed models. Table 9 summarizes the numerical values of these factors of safety.

Figure 48. Factor of safety for the different models
Table 9. Factor of safety for the different models.

<table>
<thead>
<tr>
<th>Case</th>
<th>Temperature independent</th>
<th>Temperature dependent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bimaterial (No FGM interlayer)</td>
<td>1.8964</td>
<td>1.3521</td>
</tr>
<tr>
<td>Tri-layer (Average interlayer)</td>
<td>1.6756</td>
<td>1.2869</td>
</tr>
<tr>
<td>Tri-layer (FGM interlayer)</td>
<td>2.0445</td>
<td>1.1257</td>
</tr>
</tbody>
</table>

From the results in Table 9, we see that in general the factor of safety of the models decreases when temperature dependency is considered.

Although the factor of safety is shown to decrease by adding an interlayer, these results are only for the special model case shown in Figure 40 and Figure 43. In a later section, we will show how a different three-layer case gives higher factor of safety.
Influence of the Interlayer Thickness on the Factor of Safety in Composite Beams

This section describes a study to determine the influence of manipulating the FGM interlayer thickness of the beam on the factor of safety in structures constructed of functionally graded materials under thermal loads. This study will allow, among other benefits, an analysis/comparison of the advantages/benefits of using structures constructed of functionally graded materials with respect to those constructed with homogenous materials. The beam models to be used in this study are shown in Figure 49. As noted, they are essentially the same composed cantilever beams studied in previous sections. The interested outputs are the factor of safety and the maximum temperature on the beams layers constructed of FGMs under thermal loads. The finite element program developed in chapter 3 is used to automate this study.

First, the bi-material model is studied to find out the maximum thickness of the metallic layer able to meet the maximum temperature constraint in that layer. The upper layer of the beam (ceramic) was allowed to be made thinner as the lower layer (metallic) was increased in thickness. Once this maximum possible thickness was found, it served

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Figure 49. Beam models for studying the effect of the FGM interlayer thickness in the factor of safety.
as baseline thickness of the metallic layer for studying the influence of the graded interlayer thickness on the factor of safety for the 3-layer composite beam.

**Determination of the Baseline Thickness of the Metallic Layer for Studying the Influence of the FGM Interlayer in the Factor of Safety**

For different metallic layer thicknesses, the maximum temperature in the metallic layer was calculated for the bi-material beam model (steel/Al₂O₃). Without losing generality, the maximum temperature allowable within the steel layer was set to 160 °C. Although the factor of safety is also calculated for the bi-material models in this section, it was not used as a determining factor in finding the baseline thickness of the metallic layer; it was included just to have a preliminary idea of its behavior when changing the layer thicknesses of the model.

<table>
<thead>
<tr>
<th>Steel thickness (m)</th>
<th>Alumina Thickness (m)</th>
<th>Max. temp. steel (°C)</th>
<th>Factor of safety</th>
</tr>
</thead>
<tbody>
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<td>0.0005</td>
<td>0.0095</td>
<td>27.90</td>
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<td>0.0010</td>
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<td>0.0020</td>
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<td>0.0075</td>
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<td>0.0045</td>
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<td>0.00379</td>
<td>159.22</td>
<td>1.2166</td>
</tr>
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<td>0.00378</td>
<td>159.58</td>
<td>1.2161</td>
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<td>0.00377</td>
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<td>1.2157</td>
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<tr>
<td>0.00624</td>
<td>0.00376</td>
<td>160.31</td>
<td>1.2153</td>
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</tbody>
</table>
The numerical results of this study are shown in Table 10. The results reveal that as the steel thickness is increased and the ceramic layer thickness is decreased the maximum temperature in steel increases.

From the results we can establish that the baseline thickness of the metallic layer is 0.00623 m. Also, as a preliminary examination, we can see that the factor of safety of the beam tends to diminish as we reduce the ceramic material thickness from the beam. This fact gives us a criterion for choosing the placement of the FGM interlayer in next section.
Effect of Thickness of the Graded Interlayer in the Factor of Safety for the Tri-layer Model

For different graded interlayer thicknesses, the factor of safety and the specific factor of safety were calculated for the 3-layer beam model (steel/FG/Al₂O₃). The numerical results of this study are shown in Table 11.

The maximum temperature allowable within the steel layer was set to 160 °C. For the FGM interlayer the maximum temperature is constraint to satisfy the following condition, based on the rule of mixtures,

\[ T_{FGM}(z) \leq 160V_s(z) + 450V_c(z) \] (79)

where \( T_{FGM}(z) \) refers to the temperature in the FGM interlayer, and \( V_s \) and \( V_c \) the volume fraction of the steel and ceramic layer, respectively.

Regarding the placement of the FGM interlayer, we found in preliminary computations of the tri-layer model that the temperature constraints of the model do not allow the interlayer to be a replacement toward the ceramic layer. Based on this fact, we set the following conditions for this study:

1. we take the baseline thickness of the metallic layer found in the previous section (0.00623 m) as the maximum thickness of the steel to meet the maximum temperature requirement within it;
2. the baseline thickness of the ceramic layer was fixed at 0.00377 m;
3. the FGM layer thickness was increased in the direction of the steel layer, that is, toward the bottom boundary face, diminishing the amount of steel from the model.
Table 11. Layer thickness variation for the 3-layer model.

<table>
<thead>
<tr>
<th>Steel thickness (m)</th>
<th>FGM interlayer thickness (m)</th>
<th>Alumina thickness (m)</th>
<th>Max. temp. steel (°C)</th>
<th>Max. temp. FGM (°C)</th>
<th>Factor of safety</th>
<th>Specific factor of safety</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00623</td>
<td>0.0000</td>
<td>0.00377</td>
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<td>159.94</td>
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</tr>
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<td>0.00377</td>
<td>20.55</td>
<td>190.63</td>
<td>1.30</td>
<td>0.25116</td>
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</tbody>
</table>

Figure 50 shows the factor of safety as a function of FGM interlayer thickness. From this figure, we can see that the factor of safety of the beam tends to behave nonevenly as the interlayer thickness increases. Initially, for relatively low interlayer thicknesses (0 to 0.0010 m), the factor of safety decreases, then for thicknesses from 0.0010 to 0.0045 m it increases up to its maximum value of 1.38. For thicknesses between 0.0045 to 0.0062 m, the factor of safety starts decreasing its value again up to 1.30. As we can see, this value of the factor of safety is not that low compared to those found for low interlayer thicknesses. It is important to realize that for small interlayer thicknesses, the amount of metallic material in the beam is high, while for larger interlayer thicknesses the content of metal is low. From this fact, we can conclude that, in general, the factor of safety tends to improve as we increase the FGM interlayer thickness in the beam.
From Figure 50, and based in the factor of safety criterion, we could tend to decide that the best FGM interlayer thickness is 0.0045 m where its factor of safety has a maximum value of 1.38. However, as we discuss next, we will see that this is not the best decision criterion.

Figure 50. Effect of thickness of FGM interlayer in the factor of safety for the tri-layer model
To make a better decision criterion for finding the best interlayer thickness, we use the specific factor of safety of the model, which is given by

\[
SFS = \frac{FS}{SG}
\]  

(80)

where \(SFS\), \(FS\), and \(SG\) are the specific factor of safety, factor of safety, and specific gravity of the beam, respectively. The \(SFS\) ratio is a convenient decision parameter in determining the interlayer thickness since it combines together the strength and mass of the beam.

Figure 51 shows the specific factor of safety as a function of FGM interlayer thicknesses. As in the preceding factor of safety case, similar behavior can be seen here. That is, the specific factor of safety of the beam tends to behave nonevenly as the
interlayer thickness increases. Again, for low interlayer thicknesses from 0 to 0.0005 m, the $SFS$ decreases; for thicknesses from 0.0005 to 0.0050 m the $SFS$ increases up to its maximum value of 0.253. However, differently from factor of safety case, for thicknesses between 0.0050 to 0.0062 m, the $SFS$ tends to flatten out its value to 0.253. As we can see, this value of the specific factor of safety is not that low compared to those found for low interlayer thicknesses.

Finally, from Figure 51, and based in the specific factor of safety criterion, we can decide that the best FGM interlayer thickness for the given conditions is 0.0050 m where its $SFS$ has a maximum value of 0.253. Even though this FGM interlayer thickness (0.0050 m) seems to be relatively close to the one found using the $FS$ criterion (0.0045 m), for a different applications and/or conditions this small difference could be very significant, especially in engineering applications that are highly sensitive to the geometrical parameters.
Chapter 6 Conclusions and Future Work

Introduction

This chapter summarizes the findings of the analyses and the models studied in this dissertation. Also, potential practical applications and benefits of this work within industry are discussed. Finally, recommendations for future research are made to supplement the modeling and analyzing techniques for functionally graded materials structures presented in this work.

Conclusions

From the simulation results for the beam models, both elements (2-node and 3-node) perform equally in the example cases presented in terms of axial stress and transverse shear stress when calculated from the constitutive relations. However, when the shearing stress is calculated using the equilibrium equation, only the three-node element performs well. The inclusion of the axial gradient for the examples chosen does not alter the axial stresses significantly but does produce differences in the transverse shear stress as calculated from the equilibrium equation.
The 3-node beam element model was implemented into a finite element code in MATLAB and code verification was performed on a composite cantilever beam. Benchmark comparisons of finite element predictions of stress field with the analytical solutions for a composite cantilever beam resulted in a good agreement. Simulations were also successfully performed on different beam models, which demonstrate the ability of the 3-node beam element model to simulate thermo-mechanical stresses in different structures and under different mechanical and thermal loading conditions.

Comparisons of the element formulation with FGM models available in related literature are presented. In general, from the results of these comparisons, we can conclude that the qualitative and quantitative results obtained in this work are very similar to the referenced literature, which demonstrates a suitable performance of the element formulations presented in this work.

From the beam model simulations with actual temperature distributions with and without temperature dependence of the thermo-elastic material properties, it was revealed that when temperature dependence is taken into account, the temperature profile distribution within the model is very different from the results obtained when temperature dependency is not considered. The heat insulation effect of alumina is higher when the temperature dependence is considered. It was also found that introducing a FGM interlayer between the bi-material beam model produce higher heat insulation effect when comparing with the bi-material and average interlayer models. This fact can be used in engineering applications where insulation effects need to be improved.

From the study of the influence of the FGM interlayer thicknesses on the factor of safety in beam structures constructed of FGMs under thermal loads, it can be concluded
that the interlayer thickness significantly influences the stress distribution, factor of safety, and the specific factor of safety of the structure.

In answer to the question posed in the introduction about how to implement element formulations for structures composed of FGMs, it can be stated that the implementation involved several steps:

1. the ability to integrate the variation of material properties through-the-thickness needs to be added to the material library for beam elements;
2. explore the effects of spatial temperature variation in the axial and through-the-thickness direction of the finite element;
3. consider the influence of the temperature dependency of the material properties on the thermal stresses;
4. study the effect of the constituent layer thicknesses on the stresses, factor of safety, and specific factor of safety.
**Recommendations and Future Work**

Even though the results found in this work were compared with the related literature, they should be used only as approximations, as further experimental testing should be used to verify the simulations results.

The following recommendations and future work is suggested.

1. We recommend further investigation of functionally graded beam structures with material properties varying in directions other than through-the-thickness.

2. One could develop a design of experiments study on the influence of the variables affecting the factor of safety/mass ratio in structures constructed of functionally graded materials under thermo-mechanical loads. This study would allow, among other benefits, an analysis/comparison of the advantages/benefits of using structures constructed of functionally graded materials with respect to those constructed with homogenous materials.

3. A further investigation regarding the techniques for estimating effective material properties of functionally graded materials is desirable. In the graded layer of real FGMs, ceramic and metal particles of arbitrary shapes are mixed up in arbitrary dispersion structures. Hence, the prediction of the thermo-elastic properties is not a simple problem, but complicated due to the shape and orientation of particles, the dispersion structure, and the volume fraction. This situation implies that the reliability of material-property estimations becomes an important key for designing a FGM that meets the required performance.
References


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