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Contributions to the degree theory for perturbation of maximal monotone maps

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Contributions to the Degree Theory for Perturbations of Maximal Monotone Maps

by

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Dedication

To My Beloved Family and Professor F.K.A. Allotey, for without them, I wouldn't have come this far

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Joseph Quarcoo

ABSTRACT

Let X be a real reflexive separable locally uniformly convex Banach space with locally uniformly convex dual space X^*. Let $T : X \supset D(T) \to 2^{X^*}$ be maximal monotone with $0 \in T(0)$, $0 \in intD(T)$ and $C : X \supset D(C) \to X^*$. Assume that $L \subset D(C)$ is a dense linear subspace of X, C is of class $(S_+)_L$ and $\langle Cx, x\rangle \geq -\psi(\|x\|)$, $x \in D(C)$, where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing. A new topological degree is developed for the sum $T + C$ in chapter one.

This theory extends the recent degree theory for the operators C of type $(S_+)_{0,L}$ in [15]. Unlike such a recent extension to multivalued $(S_+)_{0,L}$-type operators, the current approach utilizes the approximate degree $d(T_t + C, G, 0)$, $t \downarrow 0$, where $T_t = (T^{-1} + tJ^{-1})^{-1}$ and G is an open bounded subset of X and is such that $0 \in G$, for the single-valued mapping $T_t + C$. The subdifferential $\partial\varphi$, for φ belonging to a large class of proper convex lower semicontinuous functions, gives rise to operators T to which this degree theory applies. Theoretical applications to problems of Nonlinear Analysis are included, as well as applications from the field of partial differential equations.

Let $T : X \supset D(T) \to 2^{X^*}$ be maximal monotone with compact resolvents, i.e, the operator $(T + \epsilon J)^{-1} : X^* \to X$ is compact for every $\epsilon > 0$. We present a relevant result in chapter 2 that says there exists an open ball around zero in the image of a relatively open set by a continuous and bounded perturbation of a maximal monotone operator with compact resolvents.

The generalized degree function for compact perturbations of m-accretive opera-

tors established by Y.-Z Chen in [7] is extended to the case of a multivalued compact perturbations of maximal monotone maps by appealing to the topological degree for set-valued compact fields in locally convex spaces introduced by Tsoy Wo-Ma in [25]. Such is the content of the third chapter.

A unified eigenvalue theory is developed for the pair (T, S), where $T : X \supset D(T) \to 2^{X^*}$ is a quasimonotone-type operator which belong to the so-called $A_G(QM)$ class introduced by Arto Kittila in [23] and S is a bounded demicontinuous mapping of class $(S)_+$. Conditions are given for the existence of a pair $(x, \lambda) \in (0, \infty) \times (D(T + S) \cap \partial G)$ such that $Tx + \lambda Sx \ni 0$. This is the content of Chapter 4.

1 Introduction

Functional analysis provides an abstract framework to investigate differential or integral equations. By an adequate choice of underlying function spaces, the problems for these equations can be rewritten as an operator equation $Tx = y$, where T is a mapping from a Banach space X into another Banach space Y and $y \in Y$. A useful way to obtain information about the solution set is topological degree theory.
Such a theory consists of an *algebraic count* of the number of solutions of the equation $Tx = y$. The value of the degree function for any such count is an ordinary integer. This integer may be positive, negative or zero. For finite dimensional situations, positive counts corresponds to solutions at which the mapping T is orientation-preserving, while negative count corresponds to solutions at which T is not orientation-preserving.

Although topological degree was one of the earliest tools in dealing with such operator equations, it has stood in the core of all nonlinear analysis. For a long time, the concept of degree was studied especially within algebraic topology. To enlarge its audience, significant contributions for an analytic approach of the topological degree have been revealed in the last decades.

The classical topological degree in the space $\mathbb{R}^n$ for continuous mapping $T : \mathbb{R}^n \to \mathbb{R}^n$ was introduced by L.E.J. Brouwer in 1912, and was uniquely determined by four axioms: the normalization property, existence condition, additivity with respect to the domain and invariance under homotopies. In 1934 Leray and Schauder generalized the Brouwer degree to infinite dimensional Banach spaces for mappings of the form $I + T$, where I is the identity map and T is compact. Their construction was based on the finite dimensional Brouwer degree.

Since 1934, various generalizations of the degree theory have been defined. I.V.

Skrypnik in 1973 and F.E. Browder in the early eighties in a series of articles [27] and [3]-[5] respectively, which extended the concept of a topological degree to nonlinear mappings of monotone type. Their methods were based on Galerkin-type approximations for which the finite dimensional Brouwer degree is well-defined.

The interest in the definitions of nonlinear mappings of monotone type arises from the fact that these properties can be verified under concrete hypotheses for the maps between Sobolev spaces obtained from elliptic operators in generalized divergence form.

In 1999, A.G. Kartsatos and I.V. Skrypnik in [15] introduced for the first time the topological degree theories for densely defined mappings involving operators of type $(S_+)_{0,L}$. By "densely defined", we mean the domain of such a mapping contains a linear dense subspace. Here, again, the construction of the degree was based on the classical Brouwer degree.

In 2005 [16], the same authors introduced a new topological degree for densely defined quasibounded $(\tilde{S})_+$- perturbations of multivalued maximal monotone operators in reflexive Banach spaces. This new degree theory was a substantial extension of Browder's degree theory in [4].

This work is organized in four chapters. In Chapter one we define a new topological degree for densely defined $(S_+)_L$ perturbation of multivalued maximal monotone operators. This new topological degree theory is in the spirit of [15] and [16] but we do not use the finite dimensional Brouwer degree as in [15], and the condition that the perturbation is quasibounded with respect to a maximal monotone operator is no longer assumed. The construction of this new degree theory follows that of [16]. We have applied this new degree in study of existence, surjectivity and mapping theorems.

In Chapter 2, we present a relevant results that says there exists an open ball around zero in the image of a relatively open set by a continuous and bounded perturbation of a maximal monotone operator with compact resolvents under various boundary conditions.

Recently Y.-Z. Chen in [7] established a generalized degree function for compact perturbations of m-accretive operators and showed that this degree function has the crucial properties of the topological degree.

X. Fu and S. Song in [9] extended [7] to the case where the compact perturbation is multivalued. Z. Guan and A. G. Kartsatos in [11] extended it to compact perturbations of maximal monotone maps. We extend the results in [11] to multivalued compact perturbations of maximal monotone maps by appealing to the topological degree for set-valued compact fields in locally convex spaces introduced by Tsoy Wo-Ma in [25].

The eigenvalue problem, $Tx + \lambda Sx \ni 0$ for $\lambda > 0$, had been considered by many authors including [8], [12], [14] and [17] for various classes of T and S. In Chapter four, we consider this inclusion for $T \in A_G(QM)$, a class of quasimonotone-type mapping introduced by Arto Kittila in [23] and S is a bounded demicontinuous mapping of class (S_+).

2 Degree Theory for Multivalued $(S_+)_L$-Perturbation of Maximal Monotone Operators

2.1 Preliminaries

In order to discuss the degree for the maps from X to X^*, where X is a reflexive Banach space, we need to introduce first the appropriate mappings.

Definition 2.1.1 *A map $T : X \supset D(T) \to 2^{X^*}$ is said to be "monotone" if*

$$\langle x^* - y^*, x - y \rangle \geq 0$$

for all $(x, x^*), (y, y^*) \in Gr(T)$. Here $Gr(T)$ denotes the graph of T and $\langle ., . \rangle$ the duality bracket for the pair $\langle X^*, X \rangle$, i.e, $\langle x^*, x \rangle$ is the value of the functional x^* at x. We say that T is "maximal monotone" if it is monotone and for any $(u, u^*) \in X \times X^*$ for which $\langle u^* - x^*, u - x \rangle \geq 0$, for all $(x, x^*) \in Gr(T)$, we have $(u, u^*) \in Gr(T)$.

Definition 2.1.2 *Let $B \subseteq X$ and $f : B \to X^*$. We say that f is of class (S_+) if*
(i) f is demicontinuous i.e. $x_n \to x$ in B implies $f(x_n) \rightharpoonup f(x)$ in X^* and
(ii) if $\{x_n : n \geq 1\} \subseteq B$ and $x_n \rightharpoonup x$ for some $x \in X$ and

$$\limsup_{n\to\infty} \langle f(x_n), x_n - x \rangle \leq 0,$$

implies $x_n \to x$ in X.

Definition 2.1.3 *A sequence of operators $T_n : X \to X^*$, $n \geq 1$, is said to satisfy condition (S_q) on a set $A \subset X$ if for every sequence $\{x_n\} \subset A$, such that $x_n \rightharpoonup x$, and $T_n(x_n) \to y$, we have $x_n \to x$.*

Definition 2.1.4 *The operator $T : X \supset D(T) \to 2^{X^*}$ is said to be "pseudomonotone" if $x_n \rightharpoonup x$ as $n \to \infty$ and*

$$\limsup_{n\to\infty} \langle v_n, x_n - x \rangle \leq 0, \qquad v_n \in T(x_n),$$

implies $x \in D(T)$ and

$$\langle v, x - y \rangle \leq \liminf_{n\to\infty} \langle v_n, x_n - y \rangle$$

for all $y \in X$, $v \in T(x)$.

Definition 2.1.5 *The operator $T : X \supset D(T) \to 2^{X^*}$ is said to be "generalized pseudomonotone" if $x_n \rightharpoonup x$ and $v_n \rightharpoonup v$, with $v_n \in T(x_n)$ as $n \to \infty$ and*

$$\limsup_{n\to\infty} \langle v_n, x_n - x \rangle \leq 0,$$

then $\langle v_n, x_n \rangle \to \langle v, x \rangle$, $x \in D(T)$ and $v \in T(x)$.

Definition 2.1.6 *The operator $T : X \supset D(T) \to 2^{X^*}$ is said to be quasimonotone if for any sequence (x_n) in $D(T)$ with $x_n \rightharpoonup x$ in X we have*

$$\limsup_{n\to\infty} \langle v_n, x_n - x \rangle \geq 0$$

with $v_n \in T(x_n)$.

By a well-known renorming theorem due to Troyanski [29], given a reflexive Banach space, we can always renorm it equivalently so that both X and X^* are locally uniformly convex. Thus without loss of generality we assume without further mention that both X and X^* are locally uniformly convex.

We define $J : X \to X^*$, "the duality" map of X, by

$$J(x) = \{x^* \in X^* : \langle x^*, x\rangle = \|x\|^2 = \|x^*\|_*{}^2\}$$

J is a well-defined, singled-valued map from X to X^* which is a homeomorphism, maximal monotone and of class $(S)_+$.

An operator $T : X \supset D(T) \to Y$, with Y another real Banach space, is "bounded" if it maps bounded subsets of $D(T)$ onto bounded sets. It is "compact" if it is continuous and maps bounded subsets of $D(T)$ onto relatively compact subsets of Y. It is "demicontinuous" ("completely continuous") if it is strong-weak (weak-strong) continuous on $D(T)$.

Definition 2.1.7 *$T : X \supset D(T) \to 2^{X^*}$ is "strongly quasibounded" if for each $S > 0$ there exists $K(S) > 0$ such that*

$$\|x\| \le S, \quad \langle u, x\rangle \le S, \quad \textit{for some } u \in Tx,$$

imply $\|u\| \le K(S)$.

Browder and Hess have shown in [6, Proposition 14] that, a monotone operator T is strongly quasibounded if $0 \in int D(T)$.

The following lemma can be found in Zeidler [31, p. 915].

Lemma 2.1.8 *Let $T : X \supset D(T) \to 2^{X^*}$ be maximal monotone. Then the following are true*

(i) *$\{x_n\} \subset D(T)$, $x_n \to x_0$ and $Tx_n \ni y_n \rightharpoonup y_0$ imply $x_0 \in D(T)$ and $y_0 \in Tx_0$.*

(ii) *$\{x_n\} \subset D(T)$, $x_n \rightharpoonup x_0$ and $Tx_n \ni y_n \to y_0$ imply $x_0 \in D(T)$ and $y_0 \in Tx_0$.*

From lemma 1.1.7 we see that either one of (i), (ii) implies that the graph $G(T)$ of the operator T is closed, i.e. $G(T) \equiv \{(x, u)\ ;\ x \in D(T),\ u \in Tx\}$ is a closed subset of $X \times X^*$.

2.2 Construction of the Degree

In what follows X is a separable reflexive Banach space. Let $A : X \supset D(A) \to X^*$ with $D(A)$ dense in X. We assume that there exists a subspace L of the space X such that

$$L \subset D(A), \quad \overline{L} = X. \tag{1.1}$$

Denote by $F(L)$ the set of all finite dimensional subspaces of L. We can choose a sequence $\{F_n\},\ n \in N$, such that, for each $n \in N$,

$$F_n \in F(L), \quad F_n \subset F_{n+1}, \quad \dim F_n = n, \quad \text{and} \quad \overline{\cup_n F_n} = X. \tag{1.2}$$

We let

$$L(F_n) = \bigcup_{n=1}^{\infty} F_n. \tag{1.3}$$

Definition 2.2.1 *We say that the operator $C : X \supset D(C) \to X^*$ satisfies Condition $(S_+)_{0,L}$, if for every sequence $\{F_n\}$ satisfying (1.2) and every sequence $\{x_n\} \subset L$ with*

$$x_n \rightharpoonup x_0, \quad \limsup_{n\to\infty} \langle Cx_n, x_n \rangle \leq 0, \quad \lim_{n\to\infty} \langle Cx_n, y \rangle = 0,$$

for some $x_0 \in X$ and any $y \in L(F_n)$, it follows that $x_n \to x_0, x_0 \in D(C)$ and $Cx_0 = 0$. We say that the operator $C : X \supset D(C) \to X^*$ satisfies condition $(S_+)_L$ if the operator $C_h : D(C) \to X^*$, defined by $C_h u = Cu - h$, satisfies condition $(S_+)_{0,L}$ for any $h \in X^*$.

We need the following three conditions on the operator C:

(c_1) there exists a subspace L of X which satisfies (1.1) and is such that the operator C satisfies Condition $(S_+)_L$;

(c_2) for every $F \in F(L)$, $y \in L$ the mapping $a(F,y) : F \to \mathbb{R}^+$, defined by $a(F,y)(x) = \langle Cx, y\rangle$, is continuous.

(c_3) There exists a function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ which is nondecreasing and such that

$$\langle Cx, x\rangle \geq -\psi(\|x\|), \quad x \in D(C).$$

Lemma 2.2.2 *Let $T : X \supset D(T) \to 2^{X^*}$ be maximal monotone and such that $0 \in D(T)$ and $0 \in T(0)$. Then the mapping $(t,x) \to T_t x$ is continuous on the set $(0,\infty) \times X$, where $T_t = (T^{-1} + tJ^{-1})^{-1}$.*

Proof. Fix a $\delta > 0$. Let $\{x_n\} \subset X$, $\{t_n\} \subset [\delta, \infty)$ be such that $x_n \to x_0$ and $t_n \to t_0$. Let $y_n^* = T_{t_n}(x_n)$. Then for some $z_n \in D(T)$ with $y_n^* \in T(z_n)$,

$$(T^{-1} + t_n J^{-1})y_n^* \ni x_n = z_n + t_n J^{-1} y_n^*. \tag{1.4}$$

Using the monotonicity of the operator T and the condition $0 \in T(0)$ we get

$$\langle y_n^*, x_n\rangle = \langle y_n^*, z_n\rangle + t_n\langle y_n^*, J^{-1}y_n^*\rangle \geq \delta\|y_n^*\|^2$$

which gives the bounded of the sequence $\{y_n^*\}$, and hence the boundedness of $\{z_n\}$ by 1.4. Since X and X^* are reflexive, we may assume that $y_n^* \rightharpoonup y_0$, $z_n^* \rightharpoonup z_0$ and $J^{-1}y_n^* \rightharpoonup j_0$. Using this and the monotonicity of the duality mapping $J^{-1} : X^* \to X = X^{**}$, we obtain

$$\lim_{n\to\infty} \langle y_n^* - y_0, x_n\rangle = 0, \qquad \liminf_{n\to\infty} \langle y_n^* - y_0, t_n J^{-1} y_n^*\rangle \geq 0. \tag{1.5}$$

The second of (1.5) follows from

$$\langle y_n^* - y_0, t_n J^{-1}y_n^* - t_n J^{-1}y_0\rangle \geq t_n(\|y_n^*\| - \|y_0\|)^2,$$

which implies

$$\langle y_n^* - y_0, t_n J^{-1} y_n^* \rangle \geq \langle y_n^* - y_0, t_n J^{-1} y_0 \rangle.$$

From (1.4) and (1.5) we have

$$\limsup_{n\to\infty} \langle y_n^* - y_0, z_n \rangle \leq \lim_{n\to\infty} \langle y_n^* - y_0, x_n \rangle + \limsup_{n\to\infty} -[\langle y_n^* - y_0, t_n (J^{-1} y_n^* \rangle],$$

which says

$$\limsup_{n\to\infty} \langle y_n^* - y_0, z_n \rangle \leq 0.$$

But

$$\langle y_n^*, z_n \rangle = \langle y_n^* - y_0, z_n \rangle + \langle y_0, z_n \rangle.$$

This yields

$$\limsup_{n\to\infty} \langle y_n^*, z_n \rangle \leq \langle y_0, z_0 \rangle. \tag{1.6}$$

Let $v \in D(T)$, $v_0 \in Tv$. Using the monotonicity of the operator T, we get

$$\langle y_n^* - v_0, z_n - v \rangle \geq 0,$$

or

$$\langle y_n^*, z_n \rangle \geq \langle y_n^*, v \rangle + \langle v_0, z_n \rangle - \langle v_0, v \rangle.$$

We conclude that

$$\liminf_{n\to\infty} \langle y_n^*, z_n \rangle \geq \langle y_0, v \rangle + \langle v_0, z_0 \rangle - \langle v_0, v \rangle. \tag{1.7}$$

(1.6) and (1.7) imply

$$\langle y_0 - v_0, z_0 - v \rangle \geq 0.$$

Since the point $(v, v_0) \in Gr(T)$ is arbitrary, we have $z_0 \in D(T)$ and $y_0 \in Tz_0$ by the maximal monotonicity of the operator T. Thus, we may take in (1.7) $v = z_0$ to arrive at

$$\liminf_{n\to\infty} \langle y_n^*, z_n \rangle \geq \langle y_0, z_0 \rangle. \tag{1.8}$$

From (1.4), the first equality in (1.5) and (1.8) we obtain

$$\limsup_{n\to\infty}\langle y_n^* - y_0, t_n J^{-1} y_n^*\rangle \le 0.$$

Using the (S_+)-property of the operator J^{-1} we obtain $y_n^* \to y_0$. Passing to the limit in (1.4) and taking into consideration that $y_0 \in Tz_0$, we get

$$x_0 = z_0 + t_0 J^{-1} y_0 \in (T^{-1} + t_0 J^{-1}) y_0.$$

Thus, $y_0 = (T^{-1} + t_0 J^{-1})^{-1} x_0 = T_{t_0} x_0$, and we are done.

Lemma 2.2.3 *Assume that the operators $T : X \supset D(T) \to 2^{X^*}$, $T_0 : X \supset D(T_0) \to X^*$ are maximal monotone with $0 \in D(T) \cap D(T_0)$ and $0 \in T(0) \cap T_0(0)$. Assume further that, $T + T_0$ is maximal monotone. Assume that there is a positive sequence $\{t_n\}$ such that $t_n \downarrow 0$, a sequence $\{x_n\} \subset D(T_0)$ and a sequence $w_n \in T_0 x_n$ such that $x_n \rightharpoonup x_0 \in X$ and $T_{t_n} x_n + w_n \rightharpoonup y_0^* \in X^*$. Then the following are true:*

(i) *the inequality*

$$\lim_{n\to\infty}\langle T_{t_n} x_n + w_n, x_n - x_0\rangle < 0$$

is impossible;

(ii) *if*

$$\lim_{n\to\infty}\langle T_{t_n} x_n + w_n, x_n - x_0\rangle = 0,$$

then $x_0 \in D(T + T_0)$ and $y_0^* \in (T + T_0) x_0$.

The following lemma follows from the proof of a lemma of Browder and Hess [6, Proposition 12].

Lemma 2.2.4 *Assume that $T : X \supset D(T) \to 2^{X^*}$ is maximal monotone and bounded* *(i.e. if M is a bounded set in X, then the set*

$$\bigcup_{x\in D(T)\cap M} Tx$$

is bounded. Then if M is a bounded set in X the set

$$\{T_t x : (t,x) \in (0,\infty) \times (D(T)\cap M)\}$$

is also bounded.

Proof. From the proof of a lemma of [27], we see that if $(t,x) \in (0,\infty) \times D(T)\cap M$, then $\|T_t x\| \leq \|z\|$ for any $z \in Tx$. Thus, if $K > 0$ is an upper bound for the set $T(D(T)\cap M)$, we have $\|T_t x\| \leq K,\ x \in D(T)\cap M$, as well.

The next lemma is practically contained in the proof of Theorem 7 of Browder and Hess in [6]. We give the full proof here for completeness and future reference.

Lemma 2.2.5 *Let $T : X \supset D(T) \to 2^{X^*}$ be maximal monotone and such that $0 \in$ $intD(T)$ and $0 \in T(0)$. Let $\{t_n\} \subset (0,\infty)$ and $\{u_n\} \subset X$ be such that*

$$\|u_n\| \leq S, \quad \langle T_{t_n} u_n, u_n\rangle \leq S_1,$$

where S, S_1 are positive constants. Then there exists a number $K > 0$ such that $\|T_{t_n} u_n\| \leq K$ for all $n = 1, 2, \ldots$.

Proof. Let the assumptions of the lemma be satisfied. We set

$$w_n = T_{t_n} u_n = (T^{-1} + t_n J^{-1})^{-1} u_n.$$

This implies that

$$\begin{aligned}
T^{-1}(w_n) + t_n J^{-1}(w_n) &\ni u_n \\
T^{-1}(w_n) &\ni u_n - t_n J^{-1}(w_n) \\
&= u_n - t_n J^{-1}(T_{t_n}(u_n) \\
&= J_{t_n}(u_n)
\end{aligned}$$

Then we have $w_n \in TJ_{t_n}u_n$. If we let $x_n = J_{t_n}u_n$, then

$$T^{-1}(w_n) \ni J_{t_n}(u_n) = x_n = u_n - t_n J^{-1}(w_n).$$

This implies

$$\begin{aligned}
t_n J^{-1}(w_n) &= u_n - x_n \\
J^{-1}(w_n) &= \frac{1}{t_n}(u_n - x_n) \\
w_n &= J(\frac{1}{t_n}(u_n - x_n)) \\
t_n w_n &= J(u_n - x_n).
\end{aligned}$$

Thus,

$$\begin{aligned}
\langle w_n, x_n \rangle \quad &= \langle w_n, u_n - t_n J^{-1} w_n \rangle \\
&= \langle w_n, u_n \rangle - t_n \langle w_n, J^{-1} w_n \rangle \\
&= \langle w_n, u_n \rangle - t_n \|w_n\|^2 \\
&\leq \langle T_{t_n} u_n, u_n \rangle \\
&\leq S_1.
\end{aligned} \tag{1.9}$$

From (1.9), we also obtain

$$t_n \|w_n\|^2 = \langle w_n, u_n \rangle - \langle w_n, x_n \rangle.$$

Since $0 \in T(0)$ and $w_n \in Tx_n$, we have $\langle w_n, x_n \rangle \geq 0$, which implies $t_n \|w_n\|^2 \leq S_1$. Now, if $\{w_n\}$ is unbounded, we may assume that $\|w_n\| \to \infty$ and $\|w_n\| \leq \|w_n\|^2$ for all n. Thus, $t_n\|w_n\| \leq S_1$ and

$$t_n\|w_n\| = \|J(u_n - x_n)\| = \|u_n - x_n\|$$

implies that $\{x_n\}$ is bounded. We note that since $0 \in intD(T)$, the operator T is strongly quasibounded. Thus, the boundedness of $\{x_n\}$ and $\{\langle w_n, x_n\rangle\}$ $(w_n \in Tx_n)$ imply, by the strong quasiboundedness of T, the boundedness of $\{w_n\}$, i.e., a contradiction. It follows that $\{T_{t_n}u_n\}$ is bounded.

Lemma 2.2.6 *Assume that the operator C satisfies conditions (c_1) and (c_2). Assume that the operator $T : X \supset D(T) \to 2^{X^*}$ is maximal monotone such that $0 \in D(T)$ and $0 \in T(0)$.*
Then, for each $t \in (0,\infty)$, the operator $T_t + C$ satisfies the condition $(S_+)_{0,L}$.

Proof. We know that, for each $t \in (0,\infty)$, the operator T_t is bounded, continuous and maximal monotone. To show that the operator $T_t + C$ satisfies the condition $(S_+)_{0,L}$, we assume that $\{x_n\} \subset L$ is such that

$$x_n \rightharpoonup x_0 \in X, \quad \limsup_{n\to\infty}\langle T_tx_n + Cx_n, x_n\rangle \leq 0, \quad \lim_{n\to\infty}\langle T_tx_n + Cx_n, y\rangle = 0 \qquad (1.10)$$

for every $y \in L(F_n)$. Since T_t is bounded, we may assume that $T_tx_n \rightharpoonup v^* \in X^*$. Thus, from the last equality of (1.10), along with

$$\langle Cx_n, y\rangle = \langle T_tx_n + Cx_n, y\rangle - \langle T_tx_n, y\rangle, \quad y \in L(F_n),$$

we get

$$\lim_{n\to\infty}\langle Cx_n + v^*, y\rangle = 0, \quad y \in L(F_n). \qquad (1.11)$$

Now, we observe that

$$\langle Cx_n + v^*, x_n\rangle = \langle T_tx_n + Cx_n, x_n\rangle - \langle T_tx_n, x_n - x_0\rangle - \langle T_tx_n, x_0\rangle + \langle v^*, x_n\rangle, \quad (1.12)$$

which, in view of (1.10) and the monotonicity of T_t, implies

$$\begin{aligned}\limsup_{n\to\infty}\langle Cx_n+v^*,x_n\rangle &\leq \limsup_{n\to\infty}\langle T_tx_n+Cx_n,x_n\rangle-\liminf_{n\to\infty}\langle T_tx_n,x_n-x_0\rangle\\ &\leq -\liminf_{n\to\infty}\langle T_tx_n,x_n-x_0\rangle\\ &\leq 0. \end{aligned}\tag{1.13}$$

We have used the fact that

$$\liminf_{n\to\infty}\langle T_tx_n,x_n-x_0\rangle\geq 0, \tag{1.14}$$

which is an immediate consequence of

$$\langle T_tx_n,x_n-x_0\rangle=\langle T_tx_n-T_tx_0,x_n-x_0\rangle+\langle T_tx_0,x_n-x_0\rangle\geq\langle T_tx_0,x_n-x_0\rangle.$$

Since C is of type $(S_+)_L$, (1.11) and (1.13) imply $x_n\to x_0,\ x_0\in D(C)=D(T_t+C)$ and $C(x_0)+v^*=0$, or $T_tx_0+Cx_0=0$. It follows that the operator T_t+C satisfies condition $(S_+)_{0,L}$.

We now give a theorem that will allow us to define our degree from [16] on the operator T_t+C.

Theorem 2.2.7 *Assume that the operators T, C are as in Lemma 1.2.6 with $0\in \operatorname{int}D(T)$ and C satisfying condition (c_3). Let $G\subset X$ be open and bounded and assume that*

$$Tx+Cx\neq 0,\ x\in D(T+C)\cap\partial G.$$

Then there exists $t_0>0$ such that $T_tx+Cx\neq 0$ for any $(t,x)\in(0,t_0]\times(L\cap\partial G)$.

Proof. Assume that our assertion is false. Then there exist sequences $\{t_n\} \subset (0,\infty)$, $\{x_n\} \subset L \cap \partial G$ such that $t_n \downarrow 0$, $x_n \rightharpoonup x_0 \in X$ and

$$T_{t_n} x_n + C x_n = 0.$$

Since $\{x_n\}$ is bounded, we apply condition (c_3) to get

$$\langle T_{t_n} x_n, x_n \rangle = -\langle C x_n, x_n \rangle \le \psi(\|x_n\|) \le \psi(S),$$

where S is bound for the sequence $\{x_n\}$. Thus, Lemma 1.2.5 implies $\{T_{t_n} x_n\}$ is bounded, and we may assume that $T_{t_n} x_n \rightharpoonup v_0^*$. Now, if we assume that

$$\liminf_{n\to\infty} \langle T_{t_n} x_n, x_n - x_0 \rangle < 0, \tag{1.15}$$

then there exists a subsequence of $\{x_n\}$, denoted again by $\{x_n\}$, such that

$$\lim_{n\to\infty} \langle T_{t_n} x_n, x_n - x_0 \rangle = -q < 0.$$

By (i) of Lemma 1.2.3, we have a contradiction. It follows that

$$\liminf_{n\to\infty} \langle T_{t_n} x_n, x_n - x_0 \rangle \ge 0. \tag{1.16}$$

This and

$$\langle C x_n + v^*, x_n \rangle = \langle T_{t_n} x_n + C x_n, x_n \rangle - \langle T_{t_n} x_n, x_n - x_0 \rangle - \langle T_{t_n} x_n, x_0 \rangle + \langle v^*, x_n \rangle \tag{1.17}$$

imply

$$\limsup_{n\to\infty} \langle C x_n + v^*, x_n \rangle \le -\liminf_{n\to\infty} \langle T_{t_n} x_n, x_n - x_0 \rangle \le 0. \tag{1.18}$$

Now, let $y \in L(F_n)$. Then, since $C x_n \rightharpoonup -v^*$,

$$\lim_{n\to\infty} \langle C x_n + v^*, y \rangle = 0. \tag{1.19}$$

Since the operator C is of type $(S_+)_L$, we have $x_n \to x_0 \in D(C)$ and $Cx_0 + v^* = 0$. Thus, since $T_{t_n} x_n \rightharpoonup v^*$,

$$\lim_{n\to\infty} \langle T_{t_n} x_n, x_n - x_0 \rangle = 0.$$

Invoking Lemma 1.2.3, we see now that $x_0 \in D(T)$ and $v^* = -Cx_0 \in Tx_0$, or $Tx_0 + Cx_0 \ni 0$. This, however, is a contradiction to our boundary condition because $x_0 \in D(T + C) \cap \partial G$.

In order to be able to define our new degree mapping, we need to show that the degree $d(T_t + C, G, 0)$ is constant for all sufficiently small values of t. We need the following three definitions and Theorem A from [15].

Definition 2.2.8 *Let $A : X \supset D(A) \to X^*$ satisfy conditions $(c_1), (c_2)$ and assume that $Ax \neq 0,\ x \in D(A) \cap \partial G$, where $G \subset X$ is open and bounded. Then the degree $d(A, G, 0)$ is defined by*

$$d(A, G, 0) = \lim_{n\to\infty} deg(A_n, G_n, 0) \tag{1.20}$$

where $deg(A_n, G_n, 0)$ is the Brouwer degree of the finite-dimensional mapping A_n, defined by

$$A_n x = \sum_{i=1}^{n} \langle Ax, v_i \rangle v_i, \quad x \in F_n,$$

$G_n = G \cap F_n$ and $F_n = span\{v_1, v_2, \ldots, v_n\}$.

Remark 2.2.9 *As we can easily see from the above definition, the only points in $D(A) \cap \partial G$ that matter here are the points in $F_n \cap \partial G_n$. Thus, any boundary condition of the type $x \notin D(A) \cap \partial G$ can actually be replaced by the condition $x \notin L \cap \partial G$.*

In what follows, the symbol $d(T, G, 0)$ denotes the degree function defined in [15] for operators of the type $(S_+)_{0,L}$ and open bounded sets $G \subset X$. Let $G \subset X$ be open and bounded. Let $A_t : X \supset D(A_t) \to X^*,\ t \in [0, 1]$, be a one-parameter family of nonlinear operators. We assume that there exists a subspace L of X and a sequence $\{F_n\}$ as in (1.2) and (1.3) such that

$$\overline{L} = X,\ L\{F_n\} \subset D(A_t), \quad t \in [0,1]. \tag{1.21}$$

Definition 2.2.10 *We say that the family $\{A_t\}$ satisfies Condition $(S_+)^{(t)}_{0,L}$ if whenever $\{x_n\} \subset L\{F_n\},\ \{t_n\} \subset [0,1]$ are such that $x_n \rightharpoonup x_0,\ t_n \to t_0$ and*

$$\limsup_{n\to\infty}\langle A_{t_n}x_n, x_n\rangle \leq 0, \quad \lim_{n\to\infty}\langle A_{t_n}x_n, y\rangle = 0, \tag{1.22}$$

for some $x_0 \in X$ and any $y \in L\{F_n\}$, it follows that $x_n \to x_0,\ x_0 \in D(A_{t_0})$ and $A_{t_0}x_0 = 0$.

Definition 2.2.11 *Let $A^{(i)} : X \supset D(A^{(i)}) \to X^*,\ i = 0, 1,$ satisfy the conditions $(c_1), (c_2)$ with a common space L. The operators $A^{(0)}, A^{(1)}$ are called "homotopic" with respect to the bounded open set $G \subset X$ if there exists a one-parameter family of operators $A_t : X \supset D(A_t) \to X^*,\ t \in [0,1]$, such that*

1. *$A^{(i)} = A_i,\ i = 0, 1;\ A_t(x) \neq 0,\ (t,x) \in [0,1] \times (D(A_t) \cap \partial G)$;*
2. *the family $\{A_t\}$ satisfies Condition $(S_+)^{(t)}_{0,L}$ with respect to the space L;*
3. *for every space $F \subset L\{F_n\}$ and every $y \in L\{F_n\}$ the mapping $a(\tilde{F}, y) : F \times [0,1] \to \mathbb{R}$ defined by $\tilde{a}(F,y)(x,t) = \langle A_t x, y\rangle$ is continuous.*

If A_t is replaced by $A_t - s(t)$, where $s : [0,1] \to X^*$ is a continuous curve, and it satisfies condition $(S_+)^{(t)}_{0,L}$, then we say that $\{A_t\},\ t \in [0,1]$, is a homotopy of class $(S_+)^{(t)}_L$.

Theorem 2.2.12 *Let $A^{(i)} : X \supset D(A^{(i)}) \to X^*,\ i = 0, 1,$ satisfy the conditions (c_1), (c_2) with a common space L, and assume that the operators $A^{(0)}, A^{(1)}$ are homotopic* *with respect to the open and bounded set $G \subset X$. Then*

$$d(A^{(0)}, G, 0) = d(A^{(1)}, G, 0). \tag{1.23}$$

Theorem 2.2.13 *Assume that $T : X \supset D(T) \to 2^{X^*}$ is maximal monotone with $0 \in intD(T)$ and $0 \in T(0)$. Assume that the operator $C : X \supset D(C) \to X^*$ satisfies conditions $(c_1) - (c_3)$. Let $G \subset X$ be open and bounded and such that $Tx + Cx \neq 0,\ x \in D(T + C) \cap \partial G$. Let $t_0 > 0$ be the constant of Theorem 1.2.7. Then the degree $d(T_t + C, G, 0)$ is constant for $t \in (0, t_0]$.*

Proof. Following Theorem 1.2.12, it suffices to show that for any two numbers $t_1, t_2 \in (0, t_0]$ we have

$$d(T_{t_1} + C, G, 0) = d(T_{t_2} + C, G, 0). \tag{1.24}$$

Obviously, the degree $d(T_t, G, 0)$ is well defined because, according to Theorem 1.2.7 (see also Remark 1.2.9), $T_t x + Cx \neq 0$ for every $x \in L \cap \partial G$. Now, let $t_1, t_2 \in (0, t_0]$ be given with $t_1 < t_2$ and consider the curve

$$s(t) := tt_1 + (1 - t)t_2, \quad t \in [0, 1].$$

We are going to show that the family $A_t := T_{s(t)} + C$ satisfies the condition $(S_+)^{(t)}_{0,L}$. As we already said above, $A_t(x) = T_{s(t)}x + Cx \neq 0,\ (t, x) \in [0, 1] \times (D(A_t) \cap \partial G)$, where $D(A_t) = D(C)$. This is because $s(t) \in (0, t_0], t \in [0, 1]$. Let $\{t_n\} \subset [0, 1],\ \{x_n\} \subset L(F_n),\ y \in L(F_n)$ be such that $x_n \rightharpoonup x_0,\ t_n \to \tilde{t}_0$ and

$$\limsup_{n\to\infty} \langle T_{s(t_n)}x_n + Cx_n, x_n\rangle \leq 0, \quad \lim_{n\to\infty} \langle T_{s(t_n)}x_n + Cx_n, y\rangle = 0. \tag{1.25}$$

From the first of (1.25) we obtain that the sequence $\{\langle T_{s(t_n)}x_n + Cx_n, x_n\rangle\}$ is bounded above. Letting S be such a bound, we have

$$\langle T_{s(t_n)}x_n, x_n\rangle \leq -\langle Cx_n, x_n\rangle + S \leq \psi(\|x_n\|) + S \leq \psi(S_1) + S,$$

where S_1 is bound for the sequence $\{x_n\}$. Using Lemma 1.2.5, we obtain the boundedness of the sequence $\{T_{s(t_n)}x_n\}$. We may thus assume that $T_{s(t_n)}x_n \rightharpoonup v^*$. Then, since $\langle T_{s(t_n)}x_n, y\rangle \to \langle v^*, y\rangle$, we obtain from the last of (1.25)

$$\lim_{n\to\infty} \langle Cx_n + v^*, y\rangle = 0. \tag{1.26}$$

We now observe that

$$\langle T_{s(t_n)}x_n, x_n - x_0\rangle = \langle T_{s(t_n)}x_n - T_{s(t_n)}x_0, x_n - x_0\rangle + \langle T_{s(t_n)}x_0, x_n - x_0\rangle.$$

By the monotonicity of $T_{s(t_n)}$ and Lemma 1.2.2, we obtain

$$\begin{aligned}
\liminf_{n\to\infty} \langle T_{s(t_n)}x_n, x_n - x_0\rangle &\geq \liminf_{n\to\infty} \langle T_{s(t_n)}x_0, x_n - x_0\rangle \\
&= \lim_{n\to\infty} \langle T_{s(t_n)}x_0, x_n - x_0\rangle \\
&= \langle T_{s(\tilde{t}_0)}x_0, 0\rangle \\
&= 0.
\end{aligned}$$

We next show that

$$\limsup \langle Cx_n + v^*, x_n\rangle \leq 0.$$

We have

$$\begin{aligned}
\langle Cx_n + v^*, x_n\rangle &= \langle Cx_n + T_{s(t_n)}x_n, x_n\rangle - \langle T_{s(t_n)}x_n, x_n\rangle + \langle v^*, x_n\rangle \\
&= \langle Cx_n + T_{s(t_n)}x_n, x_n\rangle - \langle T_{s(t_n)}x_n, x_n\rangle + \langle T_{s(t_n)}x_n, x_0\rangle \\
&\quad -\langle T_{s(t_n)}x_n, x_0\rangle + \langle v^*, x_n\rangle \\
&= \langle Cx_n + T_{s(t_n)}x_n, x_n\rangle - \langle T_{s(t_n)}x_n, x_n - x_0\rangle - \langle T_{s(t_n)}x_n, x_0\rangle \\
&+ \langle v^*, x_n\rangle.
\end{aligned}$$

Hence

$$\limsup \langle Cx_n + v^*, x_n\rangle \leq \limsup \langle Cx_n + T_{s(t_n)}x_n, x_n\rangle - \liminf \langle T_{s(t_n)}x_n, x_n - x_0\rangle$$

$$
\begin{aligned}
& -\langle v^*, x_0\rangle + \langle v^*, x_0\rangle \\
\leq\ & 0.
\end{aligned}
$$

Since C is of type $(S_+)_L$, we see that $x_n \to x_0 \in D(C) = D(A_{\tilde{t}_0})$ and $Cx_0 = -v^*$. Thus, by Lemma 1.2.2, $T_{s(t_n)}x_n \to T_{s(\tilde{t}_0)}x_0$ and $A_{\tilde{t}_0}x_0 = T_{s(\tilde{t}_0)}x_0 + Cx_0 = 0$. Condition 3 of Definition 1.2.11 is trivially satisfied because the operator $(t,x) \to T_t(x)$ is continuous and C satisfies condition c_2, hence (1.24) is true and the proof of the $(S_+)^{(t)}_{0,L}$-property of the operator family A_t is complete. Thus by Theorem 1.2.12, we have

$$
d(A_1, G, 0) \;=\; d(A_0, G, 0),
$$

which means that

$$
\begin{aligned}
d(T_{s(1)} + C, G, 0) &= d(T_{s(0)} + C, G, 0) \\
d(T_{t_1} + C, G, 0) &= d(T_{t_2} + C, G, 0)
\end{aligned}
$$

and this completes the proof. We are now ready for the definition of the new degree.

Definition 2.2.14 *Let the assumptions of Theorem 1.2.13 be satisfied. Then the new degree mapping, deg$(T+C, G, 0)$, is defined by*

$$
deg(T + C, G, 0) = d(T_t + C, G, 0), \quad t \in (0, t_0].
$$

where d is the degree for densely defined $(S_+)_{0,L}$ operators. If $p^* \in X^*$ is such that $p^* \notin (T+C)(D(T+C) \cap \partial G)$, then

$$
deg(T + C, G, p^*) := d(T_t + C - p^*, G, 0).
$$

It is easy to see that if the operator C satisfies the conditions $(c_1) - (c_3)$, then so does the operator $Cx - p^*$. In fact, (c_2) is trivially satisfied. The property (c_1)

follows from the fact that since $C - h$ is of type $(S_+)_{0,L}$ for all $h \in X^*$, we have that $C - p^* - h = C - (p^* + h)$ is also of type $(S_+)_{0,L}$ for all $h \in X^*$. The property (c_3) follows from

$$\langle Cx - p^*, x\rangle \geq -\psi(\|x\|) - \|p^*\|\|x\| \geq -(1 + \|p^*\|)\widetilde{\psi}(\|x\|),$$

where $\widetilde{\psi}(t) = \max\{\psi(t), t\},\ t \in R_+$.
In what follows, we will use the symbol $d(T + C, G, 0)$ instead of $deg(T + C, G, 0)$. We should note here that if $D(T) = X$ and T is bounded, then we do not need the condition involving the ψ-function on the operator C. This is because, according to Lemma 1.2.4, if $\{x_n\}$ is bounded and $\{t_n\} \subset (0, \infty)$, then $\{T_{t_n} x_n\}$ is also bounded.

2.3 Properties of the degree mapping

We need the following condition.
Condition (T1)
$T : X \supset D(T) \to 2^{X^*}$ is maximal monotone with $0 \in intD(T)$ and $0 \in T(0)$.

Theorem 2.3.1 *Assume that the operator T satisfies $(T1)$. Let $G \subset X$ be open and bounded. Let d denote the degree mapping defined in Definition 1.2.14. Then the following statements are true.*

(i) *If $0 \in G$, then, for every $\lambda > 0$,*

$$d(T + \lambda J, G, 0) = 1.$$

If $0 \notin J(\overline{G})$,

$$d(J, G, 0) = 0.$$

(ii) *if $p^* \notin (T + C)(D(T + C) \cap \partial G)$ and $d(T + C, G, p^*) \neq 0$, then there exists $x \in D(T + C) \cap G$ such that $(T + C)x \ni p^*$;*

(iii) *if* $0 \in G$*, then the degree* $d(H(t,\cdot), G, 0)$ *is well-defined and invariant, where the homotopy* H *is of the type*

$$H(t,x) \equiv t(T + C_1)x + (1-t)C_2 x, \quad t \in [0,1], \tag{1.27}$$

provided that $0 \notin H(t,\cdot)(\partial G),\ t \in [0,1]$*. Here,* C_1 *satisfies* $(c_1) - (c_3)$ *with the function* ψ_1 *and* $C_2 : X \to X^*$ *is bounded, demicontinuous, strictly monotone, of type* (S_+)*, and satisfies* $C_2(0) = 0$ *and* $\langle C_2 x, x\rangle \geq \psi_2(\|x\|),\ x \in D(C_2)$*, where* $\psi_2 :\to \mathbb{R}^+ \to \mathbb{R}^+$ *is strictly increasing, continuous and such that* $\psi_2(0) = 0$*. In particular,* $d(T + C_1, G, 0) = d(C_2, G, 0) = 1$*.*

(iv) *Under the assumptions on* $T,\ C_1,\ C_2$ *and* G *in (iii), the degree* $d(H(t,\cdot), G, 0)$ *is well-defined and invariant, where the homotopy* H *is of the type*

$$H(t,x) \equiv t(T + C_1)x + C_2 x, \quad t \in [0,1], \tag{1.28}$$

provided that $0 \notin H(t,\cdot)(\partial G),\ t \in [0,1]$*. In particular,* $d(T + C_1 + C_2, G, 0) = d(C_2, G, 0) = 1$*.*

(v) *The degree* $d(H(t,\cdot), G, 0)$ *is invariant under homotopies of the type*

$$H(t,x) \equiv (T + C)x - y^*(t), \quad t \in [0,1],$$

where $y^* : [0,1] \to X^*$ *is a continuous curve. Here,* $0 \notin H(t,\cdot)(\partial G),\ t \in [0,1]$*.*

(vi) *If* $G_1,\ G_2$ *are open and subsets of* G *such that* $G_1 \cap G_2 = \emptyset$ *and* $0 \notin (T + C)(D(T + C) \cap (\overline{G} \setminus (G_1 \cup G_2)))$*, then*

$$d(T + C, G, 0) = d(T + C, G_1, 0) + d(T + C, G_2, 0).$$

Proof. Property (i) follows from the fact that $d(T_t + \lambda J, G, 0) = 1$ for all sufficiently small $t > 0$, because our degree for $T_t + \lambda J$ is the Skrypnik degree from [28]. This follows also from the more general homotopy argument about (iii) below.

Property (ii) follows also from the fact that if $p^* \notin (T+C)(D(T+C) \cap \partial G)$ then $p^* \notin (T_t + C)(L \cap \partial G)$ for all sufficiently small $t > 0$, which implies, by our original degree theory, that $p^* \in R(T_t + C)$ for the same values of t.
To see this, let x_n be such a solution, $x_n \in G$. Then we can find $t_n \in (0, \infty)$ such that $T_{t_n}(x_n) + Cx_n = p^*$. We may assume that $x_n \rightharpoonup x_0$ and $t_n \downarrow 0$. Then

$$\begin{aligned} \langle T_{t_n}(x_n), x_n \rangle &= \langle p^*, x_n \rangle - \langle Cx_n, x_n \rangle \\ &\leq \|p^*\| \|x_n\| + \psi(\|x_n\|) \\ &\leq \|p^*\| + \psi(S) := S_1 \end{aligned}$$

By lemma 1.2.5 , $T_{t_n}(x_n)$ is bounded. Hence we assume that $T_{t_n}(x_n) \rightharpoonup v^*$, which implies $Cx_n = p^* - T_{t_n}(x_n) \rightharpoonup p^* - v^*$. We are going to show that

$$\lim_{n \to \infty} \langle T_{t_n}(x_n), x_n - x_0 \rangle = 0.$$

Let $y \in L(F_n)$. Then

$$\langle Cx_n + v^*, y \rangle = \langle Cx_n + T_{t_n}(x_n), y \rangle - \langle T_{t_n}(x_n), y \rangle + \langle v^*, y \rangle.$$

Therefore

$$\lim_{n \to \infty} \langle Cx_n + v^*, y \rangle = \langle p^*, y \rangle$$

or

$$\lim_{n \to \infty} \langle Cx_n + v^* - p^*, y \rangle = 0.$$

Now

$$\langle Cx_n + v^* - p^*, x_n \rangle = \langle Cx_n + T_{t_n}(x_n) - p^*, x_n \rangle + \langle v^*, x_n \rangle - \langle T_{t_n}(x_n), x_n - x_0 \rangle - \langle T_{t_n}(x_n), x_0 \rangle.$$

Since T_{t_n} is monotone

$$\liminf_{n \to \infty} \langle T_{t_n}(x_n), x_n - x_0 \rangle \geq 0.$$

Hence

$$\limsup_{n\to\infty}\langle Cx_n + v^* - p^*, x_n\rangle \le 0.$$

Since C is of type $(S_+)_L$, we have that $x_n \to x_0$, $x_0 \in \overline{G}$ and $Cx_0 = p^* - v^*$. By the strong convergence of x_n we have that

$$\lim_{n\to\infty}\langle T_{t_n}(x_n), x_n - x_0\rangle = 0.$$

By Lemma 1.2.3, $x_0 \in D(T)$ and $v^* \in T(x_0)$. But $p^* \notin (T+C)(D(T+C)\cap\partial G)$, hence $p^* = v^* + Cx_0 \in (T+C)x_0$ and therefore $x_0 \in D(T+C)\cap G$.
We are going to elaborate on properties (iii) and (iv). To this end, we let first the assumptions on T, C_1, C_2 and G in (iii) be satisfied. We consider the homotopy mapping

$$H_1(t,s,x) = s(T_t + C_1)x + (1-s)C_2x, \quad s \in [0,1],$$

and show that there exists $t_0 > 0$ such that the equation $H_1(t,s,x) = 0$ has no solution $x \in L\cap\partial G$ for all sufficiently small $t \in (0,t_0]$ and all $s \in [0,1]$. Let us assume that this is not true. Then there exist sequences $\{s_n\} \subset [0,1]$, $\{t_n\} \subset (0,\infty)$ and $\{x_n\} \subset L\cap\partial G$ such that $s_n \to s_0 \in [0,1]$, $t_n \downarrow 0$, $x_n \rightharpoonup x_0 \in X$, $C_2x_n \rightharpoonup h_2^*$ and

$$s_n(T_{t_n} + C_1)x_n + (1-s_n)C_2x_n = 0. \tag{1.29}$$

We notice first that $s_n \neq 0$ because $C_2x_n = 0$ implies $x_n = 0 \notin \partial G$ by the strict monotonicity of the operator C_2. We consider two cases:

(j) $s_0 = 0$;

(jj) $s_0 \in (0,1]$.

For the case (i), we divide (1.29) by s_n to get

$$\begin{aligned}\langle T_{t_n}x_n, x_n\rangle &= -\langle C_1x_n, x_n\rangle - \left(\frac{1}{s_n} - 1\right)\langle C_2x_n, x_n\rangle \\ &\le -\langle C_1x_n, x_n\rangle \le \psi_1(\|x_n\|) \le \psi_1(S),\end{aligned} \tag{1.30}$$

where ψ_1 is the function from condition (c_3) for the operator C_1, and S is an upper bound for the sequence $\{x_n\}$. Using Lemma 1.2.5 and the strong quasiboundedness of the operator T, we see that the sequence $\{T_{t_n}x_n\}$ is bounded. We may thus assume $T_{t_n}x_n \rightharpoonup v^* \in X^*$. In addition, from (1.30) we also see that

$$\begin{aligned} -\psi_1(S) \le -\psi_1(\|x_n\|) &\le \langle C_1 x_n, x_n\rangle \\ &\le -\left(\frac{1}{s_n}-1\right)\langle C_2 x_n, x_n\rangle \le -\left(\frac{1}{s_n}-1\right)\psi_2(\|x_n\|). \end{aligned} \tag{1.31}$$

Since the sequence $\{\|x_n\|\}$ is bounded, it has a subsequence, denoted again by $\{\|x_n\|\}$, which converges to a number $q \ge 0$. If $q > 0$, then $\psi_2(\|x_n\|) \to \psi_2(q) > 0$ and

$$-\lim_{n\to\infty}\left(\frac{1}{s_n}-1\right)\psi_2(\|x_n\|) = -\infty.$$

However, this contradicts with (1.31) because the right-hand side of it is bounded below. Consequently, $\|x_n\| \to 0$. This is a contradiction again because it implies $x_n \to x_0 = 0 \in G$, while $x_n \in \partial G$. For the case (jj), let $s_0 \in (0,1]$. Again, we may take $T_{t_n}x_n \rightharpoonup v^*$, which, along with (1.29), gives

$$C_1 x_n \rightharpoonup -v^* - \left(\frac{1}{s_0}-1\right)h_2^*. \tag{1.32a}$$

Fixing $y \in L(F_n)$, we see that (1.32) implies

$$\lim_{n\to\infty}\left\langle C_1 x_n + v^* + \left(\frac{1}{s_0}-1\right)h_2^*, y\right\rangle = 0. \tag{1.32b}$$

We are going to show that

$$\liminf_{n\to\infty}\langle s_n T_{t_n}x_n + (1-s_n)C_2 x_n, x_n - x_0\rangle \ge 0. \tag{1.33}$$

To this end, we rewrite (1.33) as follows

$$\liminf_{n\to\infty}(a_n + b_n) \ge 0, \tag{1.34}$$

where both sequences a_n, b_n are bounded and

$$a_n = s_n \langle T_{t_n} x_n, x_n - x_0 \rangle, \quad b_n = (1 - s_n)\langle C_2 x_n, x_n - x_0 \rangle. \tag{1.35}$$

Assume that (1.34) is not true. Then there exists a subsequence of $\{n\}$, denoted by $\{n\}$ again, such that

$$\lim_{n\to\infty} (a_n + b_n) < 0. \tag{1.36}$$

Obviously, this implies that there exists a further subsequence of $\{n\}$, denoted again by $\{n\}$, such that one of the following is true:

$$\lim_{n\to\infty} a_n = -q, \quad \lim_{n\to\infty} b_n = -q, \tag{1.37}$$

where q is a positive constant. Let us assume that the first of (1.37) is true. Then, since $s_n \to s_0$, we have

$$\lim_{n\to\infty} \langle T_{t_n} x_n, x_n - x_0 \rangle = -\frac{q}{s_0} < 0.$$

However, this is impossible by (i) of Lemma 1.2.3. Let us assume that the second of (1.37) is true. Then, again,

$$\lim_{n\to\infty} \{(1 - s_n)\langle C_2 x_n, x_n - x_0 \rangle\} < 0. \tag{1.38}$$

If $s_0 = 1$, then (1.38) is impossible, because the factor to the right of $(1 - s_n)$ in it is bounded. Thus, $s_0 \in (0, 1)$. This, along with (1.38), says

$$\lim_{n\to\infty} \langle C_2 x_n, x_n - x_0 \rangle = -\frac{q}{1 - s_0} < 0, \tag{1.39}$$

where q is a positive constant. By the (S_+)-property of C_2, we have $x_n \to x_0$, which contradicts (1.39). We have shown that (1.33) is true. Naturally, since $s_n \to s_0 \in$

$(0,1]$, we also have

$$\liminf_{n\to\infty}\langle T_{t_n}x_n+\left(\frac{1}{s_n}-1\right)C_2x_n, x_n-x_0\rangle\geq 0. \tag{1.40}$$

Now, let

$$u^*=v^*+\left(\frac{1}{s_0}-1\right)h_2^*.$$

We have

$$\begin{aligned}\langle C_1x_n+u^*,x_n\rangle &= \langle T_{t_n}x_n+C_1x_n+[(1-s_n)/s_n]C_2x_n,x_n\rangle\\ &\quad -\langle T_{t_n}x_n+[(1-s_n)/s_n]C_2x_n,x_n-x_0\rangle\\ &\quad -\langle T_{t_n}x_n+[(1-s_n)/s_n]C_2x_n,x_0\rangle+\langle u^*,x_n\rangle. \end{aligned}\tag{1.41}$$

We recall that $\{x_n\}$ satisfies the equation (1.29). Having this in mind, as well as (1.40), we arrive at

$$\begin{aligned}\limsup_{n\to\infty}\langle C_1x_n+u^*,x_n\rangle &\leq -\liminf_{n\to\infty}\langle T_{t_n}x_n+\left(\frac{1}{s_n}-1\right)C_2x_n,x_n-x_0\rangle\\ &\quad -\langle u^*,x_0\rangle+\langle u^*,x_0\rangle\\ &= \liminf_{n\to\infty}\langle T_{t_n}x_n+\left(\frac{1}{s_n}-1\right)C_2x_n,x_n-x_0\rangle\\ &\leq 0. \end{aligned}\tag{1.42}$$

Using (1.32b), (1.42) and the $(S_+)_L$-property of the operator C_1, we obtain $x_n\to x_0\in D(C_1)$ and $C_1x_0=-v^*-[(1-s_0)/s_0]h_2^*$. Since

$$\lim_{n\to\infty}\langle T_{t_n}x_n,x_n-x_0\rangle=0,$$

Lemma 1.2.3 says that $x_0\in D(T)$ and $v^*\in Tx_0$. Since $C_2x_n\to C_2x_0=h_2^*$, we have $s_0(Tx_0+Cx_0)+(1-s_0)C_2x_0\ni 0$, i.e., a contradiction with our boundary condition $0\notin H(t,\cdot)(\partial G)$, $t\in[0,1]$. It follows that there exists $t_0>0$ such that the equation $H_1(t,s,x)=0$ has no solution $x\in\partial G$ for any $(t,s)\in(0,t_0]\times[0,1]$.

We must now show that the degree $d(H_1(t,s,\cdot),G,0)$ is independent of $s\in[0,1]$

for a fixed $t \in (0, t_0]$. To this end, we fix $t \in (0, t_0]$ and show that the mapping $A_s x := H_1(t, s, x)$ satisfies the conditions $(S_+)^{(s)}_{0,L}$ and (c_3). The latter is obviously true because the operator $x \to T_t x$ is continuous, the operator C_2 is demicontinuous, and the operator C_1 satisfies (c_2). To show that $H_1(t, s, x)$ is of type $(S_+)^{(s)}_{0,L}$, let us assume that for a sequence $\{s_n\} \in [0, 1]$, a sequence $x_n \in L$, and any functional $y \in L(F_n)$, we have $s_n \to s_0 \in [0, 1]$, $x_n \rightharpoonup x_0$, $T_t x_n \rightharpoonup v^*$, $C_2 x_n \rightharpoonup h_2^*$,

$$\limsup_{n\to\infty} \langle A_{s_n} x_n, x_n \rangle = \limsup_{n\to\infty} \langle s_n(T_t x_n + C_1 x_n) + (1 - s_n) C_2 x_n, x_n \rangle \leq 0$$

and

$$\lim_{n\to\infty} \langle A_{s_n} x_n, y \rangle = \lim_{n\to\infty} \langle s_n(T_t x_n + C_1 x_n) + (1 - s_n) C_2 x_n, y \rangle = 0. \tag{1.43}$$

Let as assume that $s_0 = 0$. If there exists a subsequence $\{s_{n_k}\}$ such that $s_{n_k} = 0$, $k = 1, 2, \ldots$, then

$$0 \leq \limsup_{k\to\infty} \psi_2(\|x_{n_k}\|) \leq \limsup_{k\to\infty} \langle C_2 x_{n_k}, x_{n_k} \rangle \leq 0 \tag{1.44}$$

implies $\|x_{n_k}\| \to 0$, i.e. $x_{n_k} \to 0$. This says $x_0 = 0 \in X = D(A_{s_0})$ and $A_{s_0} x_0 = 0$. We may thus assume that $s_n > 0, n = 1, 2, \ldots$. Using the first of (1.43) we now see that

$$\begin{aligned} s_n \langle C_1 x_n, x_n \rangle &= \langle A_{s_n} x_n, x_n \rangle - s_n \langle T_t x_n, x_n \rangle - (1 - s_n) \langle C_2 x_n, x_n \rangle \\ &\leq \langle A_{s_n} x_n, x_n \rangle - (1 - s_n) \psi_2(\|x_n\|). \end{aligned} \tag{1.45}$$

Since the sequence $\{\|x_n\|\}$ is bounded, it has a subsequence that converges to a number $q \geq 0$. If $q > 0$, then denoting this subsequence by $\{\|x_n\|\}$ we obtain from (1.45)

$$\begin{aligned} 0 &= -\lim_{n\to\infty} (s_n \psi_1(S)) \leq -\lim_{n\to\infty} (s_n \psi_1(\|x_n\|)) \leq \limsup_{n\to\infty} s_n \langle C_1 x_n, x_n \rangle \\ &\leq -\lim_{n\to\infty} \psi_2(\|x_n\|) = -\psi_2(q) < 0, \end{aligned}$$

where $\|x_n\| \leq S$ for all n. Consequently, we must have $\|x_n\| \to 0$, which implies $x_n \to x_0 = 0 \in D(A_{s_0}) = X$ and $A_{s_0} x_0 = 0$.

We now have to consider the case $s_0 \in (0,1]$. We first note that (1.43) implies the following two relations

$$\limsup_{n\to\infty} \langle T_t x_n + C_1 x_n + \left(\frac{1}{s_n} - 1\right) C_2 x_n, x_n \rangle \leq 0$$

and

$$\lim_{n\to\infty} \langle T_t x_n + C_1 x_n + \left(\frac{1}{s_n} - 1\right) C_2 x_n, y \rangle = 0. \tag{1.46}$$

From the second of (1.46) we get, for all $y \in L(F_n)$,

$$\lim_{n\to\infty} \langle C_1 x_n + v^* + \tilde{s} h_2^*, y \rangle = 0, \tag{1.47}$$

where $\tilde{s} = (1 - s_0)/s_0$. We need to show that

$$\limsup_{n\to\infty} \langle C_1 x_n + v^* + \tilde{s} h_2^*, x_n \rangle \leq 0. \tag{1.48}$$

We set $\tilde{s}_n = (1 - s_n)/s_n$ and observe that

$$\liminf_{n\to\infty} \langle T_t x_n + \tilde{s}_n C_2 x_n, x_n - x_0 \rangle \geq 0. \tag{1.49}$$

In fact, this follows from

$$\begin{aligned} \langle (T_t + \tilde{s}_n C_2) x_n, x_n - x_0 \rangle &= \langle (T_t + \tilde{s}_n C_2) x_n - (T_t + \tilde{s}_n C_2) x_0, x_n - x_0 \rangle \\ &\quad + \langle (T_t + \tilde{s}_n C_2) x_0, x_n - x_0 \rangle \\ &\geq \langle (T_t + \tilde{s}_n C_2) x_0, x_n - x_0 \rangle. \end{aligned}$$

Consequently,

$$\begin{aligned} \langle C_1 x_n + v^* + \tilde{s} h_2^*, x_n \rangle &= \langle T_t x_n + C_1 x_n + \tilde{s}_n C_2 x_n, x_n \rangle - \langle T_t x_n + \tilde{s}_n C_2 x_n, x_n - x_0 \rangle \\ &\quad - \langle T_t x_n + \tilde{s}_n C_2 x_n, x_0 \rangle + \langle v^* + \tilde{s} h_2^*, x_0 \rangle. \end{aligned} \tag{1.50}$$

Again,

$$\limsup_{n\to\infty}\langle C_1x_n + v^* + \tilde{s}h_2^*, x_n\rangle \le -\liminf_{n\to\infty}\langle T_tx_n + \tilde{s}_nC_2x_n, x_n - x_0\rangle \le 0,$$

thus, (1.48) is true. From (1.47), (1.48) and the $(S_+)_L$-property of C_1, we obtain $x_n \to x_0 \in D(C_1)$, $T_tx_n \to T_tx_0$ and $C_2x_n \rightharpoonup C_2x_0$. Since $C_1x_0 + v^* + \tilde{s}h_2^* = 0$, we have

$$s_0(T_tx_0 + C_1x_0) + (1-s_0)C_2x_0 = 0,$$

which concludes the proof of the fact that the family of operators A_s, $s \in [0,1]$, satisfies the condition $(S_+)_{0,L}^{(s)}$.

Since, in our case, $A^{(0)}$ and $A^{(1)}$ may be replaced by any two operators A_{s_0}, A_{s_1} from the family $A_s, s \in [0,1]$, (1.23) says

$$d(H_1(t,s,\cdot), G, 0) = const., \quad s \in [0,1].$$

Now, let $s_1, s_2 \in [0,1]$ with $s_1 \neq s_2$ and $t_1 \in (0, t_0]$ be given. Then, from what we have shown above,

$$d(H(s_1,\cdot),G,0) = d(H_1(t_1,s_1,\cdot),G,0) = d(H_1(t_1,s_2,\cdot),G,0) = d(H(s_2,\cdot),G,0).$$

In particular,

$$d(T+C_1,G,0) = d(H(1,\cdot),G,0) = d(H(0,\cdot),G,0) = d(C_2,G,0) = 1.$$

The last equality follows from the fact that the degree $d(C_2,G,0)$ is the classical Skrypnik degree [28] which equals 1 (cf. Browder [4, Theorem 3, (iv)]).

Property (v) follows easily from the definition of the degree mapping.

To prove property (vi) we note that if $0 \notin (T+C)\big((D(T+C)\cap(\overline{G}\setminus G_1\cup G_2)\big)$ then $0 \notin (T_t+C)\big((D(T+C)\cap(\overline{G}\setminus G_1\cup G_2)\big)$. Hence for $t \in [0,t_0)$ and by definition of our degree we have that

$$\begin{aligned} d(T+C,G,0) &= d(T_t+C,G,0) \\ &= d(T_t+C,G_1,0)+d(T_t+C,G_2,0) \\ &= d(T+C,G_1,0)+d(T+C,G_2,0). \end{aligned}$$

2.4 Applications in Nonlinear Analysis

Example

As an example we consider the following operators. Let Ω be a bounded open subset of $\mathbb{R}^n$ with boundary $\partial\Omega$ belonging to $C^{2,\alpha}$, for some $\alpha > 0$. For $i = 0, 1, ..., n$, let $a_i : \Omega \times \mathbb{R} \to \mathbb{R}$, be such that, $a_i(x, u)$ is measurable w.r.t $x \in \Omega$ for all $u \in \mathbb{R}$, and continuous w.r.t $u \in \mathbb{R}$ for almost all $x \in \Omega$. We also assume that for $i = 0, 1, ..., n$, the following inequalities are satisfied.

$$\begin{aligned} |a_i(x,u)| &\leq \nu_1, \quad i=1,2,...,n \\ |a_0(x,u)| &\leq \nu_1|u|+a(x) \end{aligned}$$

where, ν_1 is a positive constant and $a \in L^2(\Omega)$. With $X = L^2(\Omega)$, we define the operators S and C as follows.

$$Su = \Delta u, \qquad D(S) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega),$$

and

$$Cu = \sum_{i=1}^{n} a_i(x,u)\frac{\partial u}{\partial x_i} + a_0(x,u), \qquad D(C) = W^{1,2}(\Omega).$$

Now consider first a closed and convex set $K \subset X$. Let $\varphi_K : X \to \mathbb{R}_+ \cup \{\infty\}$ be defined by

$$\varphi_K(x) = \begin{cases} 0, & \text{if } x \in K; \\ \infty, & \text{otherwise.} \end{cases}$$

The function φ_K is proper, convex and lower semicontinuous on X. For any $x \in K$, the subdifferential of φ with respect to the convex set K is defined as

$$\partial\varphi_K(x) = \{x^* : \langle x^*, y - x\rangle \leq 0, \quad y \in K\}.$$

Also,

$$\begin{cases} D(\partial\varphi_K) = K \text{ and } 0 \in \partial\varphi_K(x), \ x \in K \\ \partial\varphi_K(x) = \{0\}, \quad x \in intK. \end{cases}$$

The operator $\partial\varphi_K : X \to 2^{X^*}$ is maximal monotone and $0 \in intD(\partial\varphi_K)$ and $0 \in (\partial\varphi_K(0))$ with $intK \neq \emptyset$. If we let $K = \overline{B_r(0)}$ then

$$\partial\varphi_{\overline{B_r(0)}}(u) = \begin{cases} 0, & \|u\| < r; \\ \{\lambda Jx : \lambda \geq 0\}, & \|u\| = r \ ; \\ \emptyset, & \|u\| > r \ . \end{cases}$$

Then $Tu = -Su + \partial\varphi_{\overline{B_r(0)}}(u)$ is maximal monotone with $0 \in intD(T) = intD(\partial\varphi)$ which is a nontrivial example of an operator T that can be covered by our present theory. Also, by the conditions on $a_i(x,u)$, $i = 0, 1, ..., n$, defined by the above inequalities, it is easily verified that the operator C satisfies condition $c_1 - c_3$. We check c_2 and c_3 below.

By the definition of the operator C, we have

$$\langle Cu, u\rangle = \int_\Omega \sum_{i=1}^n a_i(x,u)u\frac{\partial u}{\partial x_i}dx + \int_\Omega a_0(x,u)udx.$$

Hence

$$|\langle Cu, u\rangle| \leq \nu_1 \sum_{i=1}^n \|u\|_{L^2(\Omega)}\|\frac{\partial u}{\partial x_i}\|_{L^2(\Omega)} + \nu_1\|u\|^2_{L^2(\Omega)} + \|a\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}.$$

Since

$$\begin{aligned}\|u\|_{W^{1,2}(\Omega)} &= \Big(\sum_{|\alpha|\leq 1} \|D^\alpha u\|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}} \\ &= \Big(\|u\|^2_{L^2(\Omega)} + \sum_{i=1}^n \|\frac{\partial u}{\partial x_i}\|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}},\end{aligned}$$

it follows that

$$\|u\|_{L^2(\Omega)} \leq \|u\|_{W^{1,2}(\Omega)} \qquad \text{and} \qquad \sum_{i=1}^n \|\frac{\partial u}{\partial x_i}\|_{L^2(\Omega)} \leq \|u\|_{W^{1,2}(\Omega)}.$$

Hence we have

$$|\langle Cu, u\rangle| \leq 2\nu_1 \|u\|^2_{W^{1,2}(\Omega)} + \|a\|_{L^2(\Omega)}\|u\|_{W^{1,2}(\Omega)}. \tag{$*$}$$

Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $\psi(r) = 2\nu_1 r^2 + \beta r$, where $\beta \in \mathbb{R}^+$. Then ψ is nondecreasing. It follows from $(*)$ that

$$\langle Cu, u\rangle \geq -\psi(\|u\|_{W^{1,2}(\Omega)}).$$

Next we show that c_2 is also satisfied. We choose $L = \{\phi \in C^\infty(\Omega) : \|\phi\|_{W^{1,2}(\Omega)} < \infty\}$. Then, L is a dense linear subspace of $W^{1,2}(\Omega)$. Let $y \in L$ and $F \in F(L)$. We show that the mapping $a(F, y) : F \to \mathbb{R}$ defined by $a(F, y)u = \langle Cu, y\rangle$ is continuous. Let $u_k \to u$ in $W^{1,2}(\Omega)$ as $k \to \infty$. Then

$$\begin{aligned}\|u_k - u\|_{W^{1,2}(\Omega)} &= \Big(\sum_{|\alpha|\leq 1} \|D^\alpha (u_k - u)\|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}} \to 0 \\ &= \Big(\|u_k - u\|^2_{L^2(\Omega)} + \sum_{i=1}^n \|\frac{\partial (u_k - u)}{\partial x_i}\|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}} \to 0.\end{aligned}$$

Hence $\frac{\partial u_k}{\partial x_i} \to \frac{\partial u}{\partial x_i}$ in $L^2(\Omega)$ for all $1 \leq i \leq n$ and $\|u_k - u\|_{L^2(\Omega)} \to 0$. Therefore there exists a subsequence of $\{u_k\}$, still denoted by $\{u_k\}$ and a function $v \in L^2(\Omega)$ such that $u_k(x) \to u(x)$ a.e. in Ω and $|u_k(x)| \leq |v(x)|$ a.e. in Ω for all $k \geq 1$. Since the $a_i(x,u)$ are continuous in u for all $i = 0, 1, ..., n$, we have $a_i(x, u_k(x)) \to a(x, u(x))$ a.e. in Ω for $i = 0, 1, ..., n$. Hence, for a subsequence, we have

$$\begin{aligned} |\langle Cu_k - Cu, y\rangle| &\leq \sum_{i=1}^{n} \int_{\Omega} |a_i(x, u_k(x))| |\frac{\partial u_k}{\partial x_i} - \frac{\partial u}{\partial x_i}| |y| dx \\ &+ \sum_{i=1}^{n} \int_{\Omega} |a_i(x, u_k(x)) - a_i(x, u(x))| |y| |\frac{\partial u}{\partial x_i}| dx \\ &+ \int_{\Omega} |a_0(x, u_k(x)) - a_0(x, u(x))| |y| dx. \end{aligned}$$

This implies

$$\begin{aligned} |\langle Cu_k - Cu, y\rangle| &\leq \nu_1 \sum_{i=1}^{n} \int_{\Omega} \| \frac{\partial u_k}{\partial x_i} - \frac{\partial u}{\partial x_i} \|_{L^2(\Omega)} \|y\|_{L^2(\Omega)} dx \\ &+ \sum_{i=1}^{n} \int_{\Omega} |a_i(x, u_k(x)) - a_i(x, u(x))| |y| |\frac{\partial u}{\partial x_i}| dx \\ &+ \int_{\Omega} |a_0(x, u_k(x)) - a_0(x, u(x))| |y| dx. \end{aligned}$$

For $i = 0, 1, ..., n$, we also have

$$|a_i(x, u_k) - a_i(x, u)| \leq 2\nu_1$$

and

$$|a_0(x, u_k) - a_0(x, u)| \leq 2(\nu_1 |v| + a(x)).$$

Therefore, by the dominated convergence theorem, we have

$$\int_{\Omega} |a_i(x, u_k(x)) - a_i(x, u(x))|^2 dx \to 0$$

for all $i = 0, 1, .., n$. Since, also, $\| \frac{\partial u_k}{\partial x_i} - \frac{\partial u}{\partial x_i} \|_{L^2(\Omega)} \to 0$ and $|\langle Cu_k - Cu, y\rangle| \to 0$, the results follows.

In what follows, we give first give an abstract result involving the existence of zeros.

Theorem 2.4.1 ***Existence of Zeros***

Assume that $G \subset X$ is open and bounded with $0 \in G$. Assume that $T : X \supset D(T) \to 2^{X^*}$ satisfies condition (T1). Assume that $C : X \supset D(C) \to X^*$ satisfies conditions $c_1 - c_3$. Assume that

$$Tx + Cx + \lambda Jx \not\ni 0, \quad (\lambda, x) \in (0, \Lambda) \times (D(T + C) \cap \partial G), \tag{1.51}$$

for a fixed constant Λ such that

$$\Lambda > \frac{Q_2}{Q_1^2} := Q, \tag{1.52}$$

where

$$Q_1 = \inf_{x \in \partial G} \{\|x\|\}, \quad Q_2 = \psi(\sup_{x \in \partial G} \{\|x\|\}). \tag{1.53}$$

Then the inclusion

$$Tx + Cx \ni 0 \tag{1.54}$$

has a solution $x \in D(T+C) \cap \overline{G}$. If (1.51) holds also with $\lambda = 0$, then $x \in D(T+C) \cap G$.

Proof. Suppose the conclusion of the theorem is false. That is,

$$Tx + Cx \not\ni 0 \tag{155a}$$

for $x \in D(T + C) \cap \overline{G}$. We show first that there exists $\varepsilon_0 > 0$ such that

$$Tx + Cx + \epsilon Jx \not\ni 0, \quad (\epsilon, x) \in (0, \epsilon_0] \times (D(T + C) \cap \partial G). \tag{1.55b}$$

In fact, if this is not true, then there exist sequences $\{\varepsilon_n\} \subset (0, \infty)$, $\{x_n\} \subset \partial G$, $v_n^* \in Tx_n$ such that $\epsilon_n \downarrow 0$, $x_n \rightharpoonup x_0 \in X$ and

$$v_n^* + Cx_n + \epsilon_n Jx_n = 0, \tag{1.56}$$

which implies

$$\limsup_{n\to\infty} \langle v_n^* + Cx_n + \epsilon_n Jx_n, x_n \rangle = 0.$$

Then

$$\begin{aligned} \langle v_n^*, x_n \rangle &= -\langle Cx_n, x_n \rangle - \epsilon_n \langle Jx_n, x_n \rangle \\ &\leq \psi(\|x_n\|) - \epsilon_n \|x_n\|^2 \\ &\leq \psi(S), \end{aligned}$$

where S is the bounded for the sequence $\{x_n\}$. From the quasiboundedness of T and condition (c_3) we get the boundedness of $\{v_n^*\}$. Assuming that $v_n^* \rightharpoonup v^*$, we also get $Cx_n \rightharpoonup -v^*$ and $\langle Cx_n + v^*, y \rangle \to 0$, where $y \in L(F_n)$. On the other hand, let $v_0 \in T(x_0)$. Then

$$\begin{aligned} \langle v_n^* + \epsilon_n Jx_n, x_n - x_0 \rangle &= \langle v_n^*, x_n - x_0 \rangle + \epsilon_n \langle Jx_n, x_n - x_0 \rangle \\ &= \langle v_n^* - x_0, x_n - x_0 \rangle + \langle v_0, x_n - x_0 \rangle + \epsilon_n \langle Jx_n - Jx_0, x_n - x_0 \rangle \\ &+ \epsilon_n \langle Jx_0, x_n - x_0 \rangle. \end{aligned}$$

Hence, by the monotonicity of T and J, we have

$$\liminf_{n\to\infty} \langle v_n^* + \epsilon_n Jx_n, x_n - x_0 \rangle \geq 0.$$

Therefore, we have

$$\langle Cx_n + v^*, x_n \rangle = \langle v_n^* + Cx_n + \epsilon_n Jx_n, x_n \rangle - \langle v_n^* + \epsilon_n Jx_n, x_n - x_0 \rangle - \langle v_n^* - v + \epsilon_n Jx_n^*, x_0 \rangle,$$

and the fact that $\epsilon_n \|Jx_n\| \to 0$ as $n \to \infty$, implies

$$\limsup_{n\to\infty} \langle Cx_n + v^*, x_n \rangle \leq 0.$$

Condition $(S_+)_L$ on C implies that $x_n \to x_0$ and $Cx_0 = -v^*$. From Lemma 1.1.7, we get $x_0 \in D(T)$ and $Tx_0 + Cx_0 \ni 0$, i.e., a contradiction to (1.55a) because $x_0 \in D(T+C) \cap \partial G$. Therefore (1.55b) is true. We pick ϵ_0 further so that $Q + \varepsilon_0 < \Lambda$, fix $\epsilon \in (0, \epsilon_0]$, and consider the homotopy inclusion

$$t(Tx + Cx + \epsilon Jx) + (1-t)Jx \ni 0. \tag{1.57}$$

We are going to show that (1.57) has no solution $x \in D(T+C) \cap \partial G$ for any $t \in [0,1]$. To this end, assume that the contrary is true. Then there exist sequences $\{t_n\} \subset [0,1]$, $\{x_n\} \subset D(T+C) \cap \partial G$, $v_n^* \in Tx_n$ such that $t_n \to t_0 \in [0,1]$,
$x_n \rightharpoonup x_0 \in X$, $Jx_n \rightharpoonup j^* \in X^*$ and

$$t_n(v_n^* + Cx_n + \epsilon Jx_n) + (1-t_n)Jx_n = 0. \tag{1.58}$$

By what we have just shown above, $t_n \neq 1$. Also, $t_n \neq 0$ because $0 \in G$, $J(0) = 0$ and J is injective. It follows that $t_n > 0$.
Now, for all $n \geq 1$,

$$\begin{aligned} \|x_n\| &\leq \sup \|x_n\| \\ \psi(\|x_n\|) &\leq \psi(\sup \|x_n\|) \\ &= Q_2. \end{aligned}$$

Also, from (c_3), the monotonicity of T and the fact that $0 \in T(0)$, we have

$$\begin{aligned} -\psi(\|x_n\|) &\leq \langle Cx_n, x_n \rangle \\ &\leq \langle Cx_n, x_n \rangle + \langle v_n^*, x_n \rangle \\ &= \langle v_n^* + Cx_n, x_n \rangle. \end{aligned}$$

From (1.58) and above we have

$$\begin{aligned} -Q_2 &\leq -\psi(\|x_n\|) \leq \langle v_n^* + Cx_n, x_n \rangle \\ &= -\varepsilon\|x\|^2 - [(1-t_n)/t_n]\|x_n\|^2 \end{aligned}$$

$$< \quad -[(1-t_n)/t_n]\|x_n\|^2 \le -[(1-t_n)/t_n]Q_1^2, \tag{1.59}$$

and

$$\frac{1-t_n}{t_n} < \frac{Q_2}{Q_1^2} = Q,$$

or

$$t_n > \frac{1}{Q+1}.$$

It follows that $t_0 \in (0,1]$. From (1.58), we have

$$\begin{aligned} \langle v_n^*, x_n \rangle &= -\langle Cx_n, x_n \rangle - \epsilon\|x_n\|^2 - \frac{1-t_n}{t_n}\|x_n\|^2 \\ &\le \psi(\|x_n\|) - \left(\epsilon + \frac{1-t_n}{t_n}\right)\|x_n\|^2 \\ &\le \psi(S), \end{aligned}$$

where S is the bound for $\{x_n\}$, which says that $\{v_n^*\}$ is bounded. Let us assume that $v_n^* \rightharpoonup v^*$. Then (1.58) says that

$$Cx_n \rightharpoonup -v^* - (\epsilon + \tilde{t}_0)j^*,$$

where $\tilde{t}_0 = [(1-t_0)/t_0]$. Consequently,

$$\lim_{n\to\infty} \langle Cx_n + v^* + (\epsilon + \tilde{t}_0)j^*, y \rangle = 0$$

for every $y \in L(F_n)$. It is now easy to obtain, as we did several times before, that

$$\limsup_{n\to\infty} \langle Cx_n + v^* + (\epsilon + \tilde{t}_0)j^*, x_n \rangle \le 0.$$

We only note here that we have used Lemma 1.2.3 (i), to conclude that

$$\liminf_{n\to\infty} \langle v_n^* + (\epsilon + \tilde{t}_0)j^*, x_n - x_0 \rangle \ge 0.$$

By the $(S_+)_L$-property of C, we obtain $x_n \to x_0 \in D(C) \cap \partial G$ and $Cx_0 = -v^* - (\epsilon +$

$\tilde{t}_0)Jx_0$. Lemma 1.1.7 says that $x_0 \in D(T)$ and $Tx_0 + Cx_0 + (\epsilon + \tilde{t}_0)Jx_0 \ni 0$, i.e, a contradiction to our assumed hypothesis (1.51) because

$$0 < \epsilon + \tilde{t}_0 \leq \epsilon_0 + \tilde{t}_0 \leq \epsilon_0 + Q < \Lambda.$$

It is easy to see that the operator $T + \varepsilon J$ satisfies condition (T1). It is also easy to see that the operator J satisfies the conditions on C_2 of Theorem 1.3.1, (iii). Thus, according to (iii) of Theorem 1.3.1, the mapping

$$H(t, x) := t(Tx + Cx + \varepsilon J)x + (1 - t)Jx$$

is an admissible homotopy for our degree. It follows that

$$d(H(1, \cdot), G, 0) = d(T + C + \epsilon J, G, 0) = d(H(0, \cdot), G, 0) = d(J, G, 0) = 1.$$

This says that the inclusion

$$Tx + Cx + \epsilon Jx \ni 0$$

has a solution x_ϵ for every $\epsilon > 0$. We let $\epsilon = \epsilon_n = 1/n$ and $x_{\epsilon_n} = x_n$. Then we have, for some $v_n^* \in Tx_n$,

$$v_n^* + Cx_n + (1/n)Jx_n = 0.$$

We may assume that $x_n \rightharpoonup x_0 \in X$. Let S be the bound for $\{x_n\}$, then

$$\begin{aligned} \langle v_n^*, x_n \rangle &= -\langle Cx_n, x_n \rangle - \frac{1}{n}\|x_n\|^2 \\ &\leq \psi(\|x_n\|) \leq \psi(S). \end{aligned}$$

By the quasiboundedness of T, we obtain that $\{v_n^*\}$ is bounded. Let $v_n^* \rightharpoonup v^*$. Then, $Cx_n \rightharpoonup -v^*$ and $\langle Cx_n + v^*, y \rangle \to 0$. But $\frac{1}{n}Jx_n \to 0$ as $n \to \infty$.
Therefore $\lim_{n\to\infty}(v_n^* + Cx_n) = 0$.

We are going to show that

$$\limsup_{n\to\infty}\langle Cx_n + v^*, x_n\rangle \le 0.$$

Obviously

$$\begin{aligned}\langle Cx_n + v^*, x_n\rangle &= \langle v_n^* + Cx_n, x_n\rangle - \langle v_n^*, x_n\rangle + \langle v^*, x_n\rangle - \langle v_n^*, x_0\rangle + \langle v_n^*, x_0\rangle \\ &= \langle v_n^* + Cx_n, x_n\rangle - \langle v_n^*, x_n - x_0\rangle + \langle v^*, x_n\rangle - \langle v_n^*, x_0\rangle.\end{aligned}$$

Let $u_0 \in T(x_0)$. Then since T is maximal monotone we have

$$\langle v_n^*, x_n - x_0\rangle \ge \langle u_0, x_n - x_0\rangle.$$

Therefore

$$\liminf_{n\to\infty}\langle v_n^*, x_n - x_0\rangle \ge \liminf_{n\to\infty}\langle u_0, x_n - x_0\rangle = 0.$$

Hence

$$\limsup_{n\to\infty}\langle Cx_n+v^*, x_n\rangle \le \limsup_{n\to\infty}\langle v_n^*+Cx_n, x_n\rangle - \liminf_{n\to\infty}\langle v_n^*, x_n-x_0\rangle + \langle v^*, x_0\rangle - \langle v^*, x_0\rangle \le 0$$

and, consequently, $x_n \to x_0 \in D(C)$ and $Cx_0 = -v^*$. Once again, by Lemma 1.1.7, $x_0 \in D(T)$, $Tx_0 + Cx_0 \ni 0$ and $x_0 \in \overline{G}$, and the proof is complete. If we assume that (1.51) is also true with $\lambda = 0$, then, obviously, $x_0 \in G$.

Interestingly enough, it turns out that the boundary condition (1.51), for $\Lambda = +\infty$, is the same as

$$Tx + Cx + (\partial\varphi_{\overline{B_r(0)}}(x) \setminus \{0\}) \not\ni 0, \quad x \in \partial B_r(0). \tag{1.60}$$

In fact, as Kenmochi notes in [22],

$$\partial\varphi_{\overline{B_r(0)}}(x) = \begin{cases} 0, & \|x\| < r\ ; \\ \{\lambda Jx : \lambda \geq 0\}, & \|x\| = r\ ; \\ \emptyset, & \|x\| > r\ . \end{cases}$$

The following corollary to Theorem 1.5.1 contains a surjectivity result.

Corollary 2.4.2 *Assume that the operator $T : X \supset D(T) \to 2^{X^*}$ satisfies (T1). Assume that $C : X \supset D(C) \to X^*$ satisfies conditions $(c_1) - (c_3)$. Assume that for an open bounded set $G \subset X$ containing zero and some $f^* \in X^*$ we have*

$$\langle w + Cx - f^*, x\rangle \geq 0, \quad (x, w) \in (D(T + C) \cap \partial G) \times Tx. \tag{1.61}$$

Then there exists a solution $x \in D(T + C) \cap G$ of the inclusion

$$Tx + Cx \ni f^*. \tag{1.62}$$

If (1.61) is replaced by

$$\lim_{\substack{\|x\| \to \infty \\ x \in D(T+C) \cap \partial G \\ w \in Tx}} \frac{\langle w + Cx, x\rangle}{\|x\|} = +\infty, \tag{1.63}$$

then the operator $T + C$ is surjective.

Proof. We first note that under the condition (1.61) we have

$$\langle w + Cx - f^* + \lambda Jx, x\rangle \geq \lambda\langle Jx, x\rangle = \lambda\|x\|^2 > 0$$

for any $\lambda > 0$, any $x \in D(T + C) \cap \partial G$ and any $w \in Tx$. Consequently, (1.51) is true for $\Lambda = +\infty$ and $C - f^*$ in place of the operator C. As we have noted earlier, the operator $x \to Cx - f^*$ satisfies the conditions $c_1 - c_3$ for any $f^* \in X^*$. Our conclusion follows in this case from Theorem 1.5.1.

To show that $T+C$ is surjective under (1.63), we fix $f^* \in X^*$. Then, by (1.63), there exists a ball $G = B_r(0) \subset X$ such that

$$\frac{\langle w + Cx, x\rangle}{\|x\|} > \|f^*\|, \quad (x, w) \in (D(T+C) \cap \partial G) \times Tx. \tag{1.64}$$

Consequently, for the same (x, w) as above and all $\lambda \geq 0$

$$\begin{aligned} \langle w + Cx - f^* + \lambda Jx, x\rangle &\geq \langle w + Cx, x\rangle - \|f^*\|\|x\| \\ &= \|x\| \left(\frac{\langle w + Cx, x\rangle}{\|x\|} - \|f^*\| \right) \\ &> 0. \end{aligned}$$

Thus, again, (1.51) is true for all $\lambda \geq 0$ and Theorem 1.5.1 applies to obtain (1.62) for some $x \in D(T+C) \cap B_r(0)$.

2.5 Further Applications

In what follows denote J_ψ the duality mapping with the gauge function ψ. The function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, strictly increasing and such that $\psi(0) = 0$ and $\psi(r) \to \infty$ as $r \to \infty$.

The mapping J_ψ is continuous, bounded, surjective, strictly and maximal monotone and satisfies condition $(S)_+$.

Also $\langle J_\psi(x), x\rangle = \psi(\|x\|)\|x\|$ and $\|J_\psi(x)\| = \psi(\|x\|)$.

If G is an open bounded subset in X containing zero, then $d(J_\psi, G, 0) = 1$. The following proposition shows how we can solve important approximate problem for the operator $T + C$. This approximate problem, inclusion (1.65) below can be used in a variety of problems in nonlinear analysis which includes problems of solvability, existence of eigenvalues, ranges of sums, invariance of domain and bifurcation.

In what follows, we assume that T satisfies the condition (T_1) and C satisfies conditions $c_1 - c_3$.

Proposition 2.5.1 *Assume the operator T satisfies the condition (T_1) and C satisfies* *conditions $c_1 - c_3$. Let G be an open and bounded subset of X with $0 \in G$. Assume also that $(H(t,.))(\partial G) \not\ni p^*$, $t \in [0,1]$, where*

$$H(t,x) = t(T + C - p^* + \epsilon J_\psi)(x) + (1-t)J_\psi(x),$$

with $p^* \in X^*$ fixed and ϵ a positive constant. Then the degree $d(H(t,.), G, 0)$ is well defined and constant for all $t \in [0,1]$.
In particular, the inclusion

$$Tx + Cx + \epsilon J_\psi(x) \ni p^* \tag{1.65}$$

is solvable in G.

Proof. The conclusion of this proposition follows from (i)-(iii) of Theorem 1.3.1. In fact, one may take here $C_1 = C - p^* + \epsilon J_\psi$ and $C_2 = J_\psi$. We will show in the next theorem that C_1 satisfies $c_1 - c_3$ and J_ψ satisfies the conditions of C_2 in Theorem 1.3.1. Then the homotopy invariance in (iii) in Theorem 1.3.1 says that $d(H(t,.), G, 0)$ is constant for all $t \in [0,1]$. Hence it follows that

$$d(T + C + \epsilon J_\psi, G, 0) = d(J_\psi, G, 0) = 1,$$

and by (ii) of Theorem 1.3.1 implies (1.65).

Theorem 2.5.2 ***(Surjectivity)***
Let T satisfy (T_1). Let $C : X \supset D(C) \to X^*$ satisfy $c_1 - c_3$. Assume that there is a constant $Q > 0$ and $\beta : [Q, \infty) \to \mathbb{R}^+$, with $\beta(r) \to 0$ as $r \to \infty$, such that for every $x \in D(T) \cap D(C)$ with $\|x\| \geq Q$ and every $u \in T(x)$ we have

$$\langle u + Cx, x \rangle \geq -\beta(\|x\|)\psi(\|x\|)\|x\|, \tag{1.66}$$

where ψ is a gauge function. Then, for every $\epsilon > 0$, $R(T + C + \epsilon J_\psi) = X^*$.

If, in addition,

$$\liminf_{\|x\|\to\infty}\Big(\frac{|Tx+Cx|}{\psi(\|x\|)} : x \in D(T)\cap D(C)\Big) > 0, \tag{1.67}$$

then $R(T+C) = X^*$.

Proof. We fix $p^* \in X^*$, $\epsilon > 0$, and consider the problem

$$H(t,x) = t(T + C - p^* + \epsilon J_\psi)(x) + (1-t)J_\psi(x) \ni 0, \qquad t \in [0,1], \tag{$*$}$$

and apply (iii) Theorem 1.3.1. To this end, we need to show that the operator $U = C + \epsilon J_\psi - p^*$ satisfies $c_1 - c_3$ and the operator J_ψ satisfies the conditions on C_2 as in Theorem 1.3.1.

J_ψ is continuous, bounded, surjective, strictly and maximal monotone and satisfies condition $(S)_+$.We also have $J_\psi(0) = 0$.

Now,

$$\langle J_\psi(x), x\rangle = \psi(\|x\|)\|x\|.$$

If we take $\psi_2(r) = \psi(r)r$ as in (iii) Theorem 1.3.1, then $\langle J_\psi(x), x\rangle \geq \psi_2(\|x\|)$, $\psi_2 : \mathbb{R}^+ \to \mathbb{R}^+$ is strictly increasing, continuous and such that $\psi_2(0) = 0$.

To show that U satisfies c_1, let $L \subset D(C)$, $\overline{L} = X$ and such that C satisfies $c_1 - c_3$. Let $\{F_n\}$ be a sequence satisfying (1.2), $\{x_n\} \subset L$ such that $x_n \rightharpoonup x_0$ for some $x_0 \in X$. Fix h^* in X^* and let

$$\limsup_{n\to\infty}\langle Ux_n - h, x_n\rangle \leq 0, \qquad \lim_{n\to\infty}\langle Ux_n - h, y\rangle = 0$$

for any $y \in L(F_n)$. We are going to show that $x_n \to x_0$, $x_0 \in D(U) = D(C)$ and $U(x_0) = h^*$. By definition of U, we have

$$Cx_n - p^* - h = Ux_n - h - \epsilon J_\psi(x_n).$$

Hence

$$\langle Cx_n - p^* - h, y\rangle = \langle Ux_n - h, y\rangle - \langle \epsilon J_\psi(x_n), y\rangle.$$

Since J_ψ is bounded, we assume that $J_\psi(x_n) \rightharpoonup j$. Then

$$\lim_{n\to\infty} \langle Cx_n - p^* - h, y\rangle = -\langle \epsilon j, y\rangle.$$

Hence we conclude that

$$\lim_{n\to\infty} \langle Cx_n - p^* - h + \epsilon j, y\rangle = 0.$$

Now,

$$\begin{aligned}
\langle Cx_n - p^* - h + \epsilon j, x_n\rangle &= \langle Ux_n - h, x_n\rangle - \epsilon\langle J_\psi x_n, x_n\rangle + \epsilon\langle j, x_n\rangle \\
&= \langle Ux_n - h, x_n\rangle - \epsilon\langle J_\psi x_n, x_n\rangle + \epsilon\langle j, x_n\rangle - \epsilon\langle J_\psi x_n, x_0\rangle \\
&+ \epsilon\langle J_\psi x_n, x_0\rangle \\
&= \langle Ux_n - h, x_n\rangle - \epsilon\langle J_\psi x_n, x_n - x_0\rangle - \epsilon\langle J_\psi x_n, x_0\rangle + \epsilon\langle j, x_n\rangle.
\end{aligned}$$

But

$$\langle J_\psi x_n - J_\psi x_0, x_n - x_0\rangle \geq 0$$

since J_ψ is monotone. Hence

$$\liminf_{n\to\infty} \langle J_\psi x_n, x_n - x_0\rangle \geq \liminf_{n\to\infty} \langle J_\psi x_0, x_n - x_0\rangle = 0.$$

Therefore

$$\begin{aligned}
\limsup_{n\to\infty} \langle Cx_n - p^* - h + \epsilon j, x_n\rangle &\leq \limsup_{n\to\infty} \langle Ux_n - h, x_n\rangle - \epsilon \liminf_{n\to\infty} \langle J_\psi x_n, x_n - x_0\rangle \\
&\leq 0.
\end{aligned}$$

Since C is of type $(S_+)_L$, we have $x_n \to x_0$, $x_0 \in D(C) = D(C - p^* + \epsilon J_\psi) = D(U)$, and

$$Cx_0 - p^* - h - \epsilon j = 0.$$

Since J_ψ is continuous, we have $\epsilon J_\psi(x_n) \to \epsilon J_\psi(x_0) = \epsilon j$ by the uniqueness of limits. Therefore

$$U(x_0) = Cx_0 - p^* - \epsilon J_\psi(x_0) = h^*.$$

We show that c_2 is satisfied. To this end, let $F \in F(L)$, $y \in L$. We show that the mapping $b(F, y) : F \to \mathbb{R}$ defined by

$$b(F, y)(x) = \langle Cx - p^* + \epsilon J_\psi(x), y \rangle,$$

is continuous. Let $\{x_n\} \subset F$ such that $x_n \to x_0$. Then since J_ψ is continuous we have $\epsilon J_\psi(x_n) \to \epsilon J_\psi(x_0)$.
Since $a(F, y)(x) = \langle Cx, y \rangle$ is continuous we have that $\langle Cx_n - p^*, y \rangle \to \langle Cx - p^*, y \rangle$. Therefore

$$b(F, y)(x_n) = \langle (Cx_n - p^* + \epsilon J_\psi(x_n), y \rangle \to \langle (Cx - p^* + \epsilon J_\psi x, y \rangle = b(F, y)(x).$$

Finally

$$\begin{aligned}
\langle Cx - p^* + \epsilon J_\psi(x), x \rangle &= \langle Cx, x \rangle + \epsilon \langle J_\psi(x), x \rangle - \langle p^*, x \rangle \\
&\geq -\psi_1(\|x\|) + \epsilon \psi(\|x\|)\|x\| - \|p^*\|\|x\| \\
&\geq -\psi_1(\|x\|) - \|p^*\|\|x\| \\
&= -(\psi_1(\|x\|) + \|p^*\|\|x\|).
\end{aligned}$$

Let

$$\phi(r) = \psi_1(r) + \lambda r \qquad \lambda \geq 0, r \in \mathbb{R}^+.$$

Since ψ is nondecreasing, ϕ is also nondecreasing. Hence U satisfies $c_1 - c_3$.
We next show that all the solutions of $(*)$ are bounded by a constant which is independent of $t \in [0, 1]$. To this end, assume that there is a sequence $\{t_m\} \subset [0, 1]$, and a sequence $\{x_m\} \subset D(H)$ such that $\|x_m\| \to \infty$ as $m \to \infty$.

If there exist a subsequence $\{t_{m_k}\}$ of $\{t_m\}$ such that $t_{m_k} = 0$, k =1,2,..., then $x_{m_k} = 0$ for all k, which contradicts $\|x_{m_k}\| \to \infty$ as $k \to \infty$. We may assume that $t_m > 0$, m =1,2,...,. Then $D(H) = D(T) \cap D(C)$ and

$$t_m T x_m + C x_m - p^* + \epsilon J_\psi(x_m) + (1-t) J_\psi(x_m) \ni 0,$$

or, for some $u_m \in T(x_m)$,

$$t_m u_m + (C x_m - p^*) + [1 - t_m(1-\epsilon)] J_\psi(x_m) = 0. \qquad (**)$$

By the hypothesis, and assuming that $\|x_m\| \geq Q$ for all m, we find

$$\begin{aligned} \langle u_m + C x_m - p^*, x_m \rangle &\geq -\langle p^*, x_m \rangle - \beta(\|x_m\|)\psi(\|x_m\|)\|x_m\| \\ &\geq -\|p^*\|\|x_m\| - \beta(\|x_m\|)\psi(\|x_m\|)\|x_m\| \\ &= -\Big(\frac{\|p^*\|}{\psi(\|x_m\|)} + \beta(\|x_m\|)\Big)\psi(\|x_m\|)\|x_m\| \\ &= -\tilde{\beta}(\|x_m\|)\psi(\|x_m\|)\|x_m\|, \end{aligned}$$

where $\tilde{\beta}(\|x_m\|) \to 0$ as $m \to \infty$. Using this along with $(**)$ we have

$$\begin{aligned} \epsilon\psi(\|x_m\|)\|x_m\| &\leq [1 - t_m(1-\epsilon)]\psi(\|x_m\|)\|x_m\| \\ &\leq -t_m\langle u_m + c x_m - p^*, x_m \rangle \\ &\leq t_m \tilde{\beta}(\|x_m\|)\psi(\|x_m\|)\|x_m\|. \end{aligned}$$

This implies $\epsilon \leq t_m \tilde{\beta}(\|x_m\|) \to 0$ as $m \to \infty$, i.e., a contradiction.
Thus, there exist $r > 0$ such that all possible solutions of $(*)$ lie in the ball $B_r(0)$. Consequently, no solution of $(*)$ lies in $\partial B_r(0)$, and the degree mapping $d(H(t,.), B_r(0), 0)$ is well defined. Therefore, by (iii) of Theorem 1.3.1, we have $d(H(t,.), B_r(0), 0)$ is fixed for all $t \in [0,1]$. This means

$$d(H(1,.), B_r(0), 0) = d(H(0,.), B_r(0), 0)$$

or

$$d(T + C - p^* + \epsilon J_\psi, B_r(0), 0) = d(J_\psi, B_r(0), 0) = 1.$$

By (ii) of Theorem 1.3.1, we have that the inclusion

$$T(x) + C(x) + \epsilon J_\psi(x) \ni p^*$$

is solvable for every $\epsilon > 0$.

Let x_n be a solution of

$$Tx + Cx + \frac{1}{n} J_\psi(x) \ni p^*. \qquad (**)_n$$

We assume that (1.67) holds and show that the sequence $\{x_n\}$ is bounded. To this end, assume there exist a subsequence of $\{x_n\}$, denoted again by $\{x_n\}$, such that $\|x_n\| \to \infty$. Then there exists $\alpha > 0$ such that

$$\liminf_{n\to\infty} \Big(\frac{|Tx_n + Cx_n|}{\psi(\|x_n\|)} \Big) \geq \liminf_{\|x\|\to\infty} \Big(\frac{|Tx + Cx|}{\psi(\|x\|)} : x \in D(T) \cap D(C) \Big) = \alpha.$$

However, for some $u_n \in T(x_n)$, we have from $(**)_n$

$$\|u_n + Cx_n\| = \|p^* - \frac{1}{n} J_\psi(x_n)\| \leq \frac{1}{n}\psi(\|x_n\|) + \|p^*\|$$

and

$$\begin{aligned} \alpha &= \liminf_{n\to\infty} \Big(\frac{|Tx_n + Cx_n|}{\psi(\|x_n\|)} \Big) \\ &\leq \liminf_{n\to\infty} \Big(\frac{\|Tx_n + Cx_n\|}{\psi(\|x_n\|)} \Big) \\ &\leq \liminf_{n\to\infty} \Big(\frac{1}{n} + \frac{\|p^*\|}{\psi(\|x_n\|)} \Big) = 0, \end{aligned}$$

i.e., a contradiction. Since $\{x_n\}$ is bounded, then

$$\|\frac{1}{n} J_\psi(x_n)\| = \frac{1}{n}\psi(\|x_n\|) \leq \frac{1}{n}\psi(S),$$

where $\|x_n\| \leq S$. Hence

$$\lim_{n\to\infty} \|\frac{1}{n} J_\psi(x_n)\| = 0.$$

Then from $(**)_n$ and for some sequence $u_n \in T(x_n)$, we have

$$\lim_{n\to\infty} (u_n + Cx_n) = p.$$

Again from $(**)_n$

$$\begin{aligned}
\langle u_n, x_n \rangle &= \langle p^*, x_n \rangle - \langle Cx_n, x_n \rangle - \frac{1}{n} \langle J_\psi(x_n), x_n \rangle \\
&\leq \|p^*\|S + \psi_1(\|x_n\|) - \frac{1}{n} \psi(\|x_n\|)\|x_n\| \\
&\leq \|p^*\|S + \psi_1(S).
\end{aligned}$$

By the strong quasiboundedness of T, we have that the sequence $\{u_n\}$ is bounded, so we may assume that $u_n \rightharpoonup u_0$. Hence $Cx_n \rightharpoonup p^* - u_0$.
Let $y \in L(F_n)$. Then

$$\langle Cx_n - p^*, y \rangle = \langle u_n + Cx_n - p^*, y \rangle - \langle u_n, y \rangle.$$

It follows that

$$\lim_{n\to\infty} \langle Cx_n - p^*, y \rangle = -\langle u_0, y \rangle,$$

or

$$\lim_{n\to\infty} \langle Cx_n - p^* + u_0, y \rangle = 0.$$

We next show that

$$\limsup_{n\to\infty} \langle Cx_n - p^* + u_0, x_n \rangle \leq 0.$$

Let $v \in T(x_0)$. Then by the monotonicity of T, we have

$$\langle u_n - v, x_n - x_0 \rangle \geq 0.$$

Hence

$$\liminf_{n\to\infty}\langle u_n, x_n - x_0\rangle \geq \liminf_{n\to\infty}\langle v, x_n - x_0\rangle = 0.$$

Now

$$\begin{aligned}
\langle Cx_n - p^* + u_0, x_n\rangle &= \langle u_n + Cx_n - p^*, x_n\rangle - \langle u_n, x_n\rangle + \langle u_0, x_n\rangle - \langle u_n, x_0\rangle \\
&+ \langle u_n, x_0\rangle \\
&= \langle u_n + Cx_n - p^*, x_n\rangle - \langle u_n, x_n - x_0\rangle + \langle u_0, x_n\rangle - \langle u_0, x_n\rangle.
\end{aligned}$$

Hence

$$\limsup_{n\to\infty}\langle Cx_n - p^* + u_0, x_n\rangle \leq \limsup_{n\to\infty}\langle u_n + Cx_n - p^*, x_n\rangle - \liminf_{n\to\infty}\langle u_n, x_n - x_0\rangle.$$

Therefore

$$\limsup_{n\to\infty}\langle Cx_n - p^* + u_0, x_n\rangle \leq 0.$$

Since C is of type $(S_+)_L$, we have $x_n \to x_0$, $x_0 \in D(C)$ and $Cx_0 = p^* - u_0$. By Lemma 1.1.7, we have $x_0 \in D(T)$ and $u_0 \in T(x_0)$. Therefore $x_0 \in D(T) \cap D(C) = D(T+C)$ and $Tx_0 + Cx_0 \ni p^*$. This says that $R(T+C) = X^*$.

Another corollary of Theorem 1.6.2 and its proof is the following.

Corollary 2.5.3 *Assume T satisfies (T_1), and $C : X \supset D(C) \to X^*$ satisfies $c_1 - c_3$. Assume that*

(a) *There exists a constant $k > 0$, $Q > 0$ such that*

$$\langle u + Cx, x\rangle \geq -k\|x\|, \quad x \in D(T)\cap D(C), u \in Tx, \text{and} \quad \|x\| \geq Q; \qquad (1.68)$$

(b) *$(T+C)^{-1}$ is bounded.*

Then $R(T+C) = X^*$

Proof. We first observe from (a) that

$$\langle u + Cx, x\rangle \geq \frac{-k}{\|x\|}\|x\|^2 = -\beta(\|x\|)\|x\|^2, \quad x \in D(T) \cap D(C), \|x\| \geq Q$$

where $\beta(r) = \frac{k}{r}$. Thus (1.66) is true for $\psi(r) = r$. Consequently, Theorem 1.6.2 implies $R(T + C + \epsilon J) = X^*$, i.e, given $p^* \in X^*$, the inclusion $Tx + Cx + \epsilon Jx \ni p^*$ is solvable for $\epsilon > 0$. Here $J_\psi = J$ is the normalized duality mapping.

Let us fix $p^* \in X^*$ and consider a solution x_n of the inclusion

$$Tx_n + Cx_n + \frac{1}{n}Jx_n \ni p^*. \tag{1.69}$$

To show that $\{x_n\}$ is bounded. Let assume that the contrary is true. Then, without loss of generality, we may assume that $\|x_n\| \geq Q$, $n = 1, 2, ...,$. Then by (1.69) for some $y_n \in Tx_n$,

$$y_n + Cx_n + \frac{1}{n}Jx_n = p^*. \tag{1.70}$$

Then by (1.68)

$$\langle y_n + Cx_n, x_n\rangle = \langle p^*, x_n\rangle - \frac{1}{n}\|x_n\|^2 \geq -k\|x_n\|,$$

or

$$-k\|x_n\| \leq -\frac{1}{n}\|x_n\|^2 + \langle p^*, x_n\rangle \leq -\frac{1}{n}\|x_n\|^2 + \|p^*\|\|x_n\|.$$

This implies

$$\frac{1}{n}\|x_n\| \leq \|p^*\| + k.$$

From (1.70) we have

$$\|y_n + Cx_n\| \leq \|p^*\| + \frac{1}{n}\|x_n\| \leq 2\|p^*\| + k. \tag{1.71}$$

Since $y_n + Cx_n \in (T + C)x_n$, $x_n \in (T + C)^{-1}(y_n + Cx_n)$.

From (1.71) and the boundedness of $(T + C)^{-1}$, we have $\{x_n\}$ is bounded, i.e., a

contradiction. Since
$$\frac{1}{n}\|Jx_n\| = \frac{1}{n}\|x_n\| \to 0.$$
as $n \to \infty$, we conclude from (1.70) that

$$\lim_{n\to\infty} (y_n + Cx_n) \to p^*.$$

By (1.70) again,

$$\begin{aligned}
\langle y_n, x_n\rangle &= \langle p^*, x_n\rangle - \langle Cx_n, x_n\rangle - \frac{1}{n}\langle Jx_n, x_n\rangle \\
&\leq \|p^*\|S + \psi_1(\|x_n\|) - \frac{1}{n}\|x_n\|^2 \\
&\leq \|p^*\|S + \psi_1(S),
\end{aligned}$$

where S is the bound for $\{x_n\}$. By the strong quasiboundedness of T, we have that the sequence $\{y_n\}$ is bounded, so we may assume that $y_n \rightharpoonup y_0$. Hence $Cx_n \rightharpoonup p^* - y_0$. Let $y \in L(F_n)$. Then

$$\langle Cx_n - p^*, y\rangle = \langle y_n + Cx_n - p^*, y\rangle - \langle y_n, y\rangle.$$

It follows that
$$\lim_{n\to\infty} \langle Cx_n - p^*, y\rangle = -\langle y_0, y\rangle,$$
or
$$\lim_{n\to\infty} \langle Cx_n - p^* + y_0, y\rangle = 0.$$
By the same argument as in Theorem 1.6.2 we have,

$$\limsup_{n\to\infty} \langle Cx_n - p^* + y_0, x_n\rangle \leq 0.$$

We conclude that $x_0 \in D(T + C)$ and $p^* \in (T + C)x_0$.

Theorem 2.5.4 ***Leray-Schauder Condition***

Let T satisfy (T_1), and $C : X \supset D(C) \to X^*$ satisfies $c_1 - c_3$. Assume further that there exists an open bounded convex set $G \subset X$ containing zero and such that the inclusion

$$Tx + Cx \ni \lambda Jx \tag{1.72}$$

has no solution $x \in D(T+C) \cap \partial G$ for any $\lambda \leq 0$. Then the inclusion $Tx + Cx \ni 0$ has a solution $x \in D(T+C) \cap G$.

Proof. We consider the homotopy equation

$$H(t,x) = t(Tx + Cx + \epsilon Jx) + (1-t)Jx \ni 0. \tag{1.73}$$

For $t = 1$ we have $Tx + Cx + \epsilon Jx \ni 0$, which says that $Tx + Cx \ni -\epsilon Jx$. By (1.72), this inclusion has no solution $x \in \partial G$.

Also, for $t = 0$, $Jx = 0$. Since J is injective and $J(0) = 0$, we conclude that $x = 0 \notin \partial G$. Let us now assume that for some $t \in (0,1)$ the inclusion (1.73) has a solution $x \in \partial G$. Then

$$Tx + Cx + [(\frac{1}{t} - 1) + \epsilon]Jx \ni 0,$$

which contradicts the assumption of the theorem. Thus by proposition 1.6.1, the inclusion $Tx + Cx + \epsilon Jx \ni 0$ is solvable in G for every $\epsilon > 0$.

Let $x_n \in G$ be a solution of

$$Tx_n + Cx_n + \frac{1}{n}Jx_n \ni 0, \tag{1.74}$$

Then for some $y_n \in Tx_n$, we have

$$\lim_{n\to\infty} \|y_n + Cx_n\| = \lim_{n\to\infty} \frac{1}{n}\|Jx_n\| = \lim_{n\to\infty} \frac{1}{n}\|x_n\| = 0.$$

We conclude that

$$\lim_{n\to\infty}(y_n + Cx_n) = 0.$$

From (1.74), we have for some $y_n \in Tx_n$

$$\begin{aligned}
\langle y_n, x_n\rangle &= -\langle Cx_n, x_n\rangle - \frac{1}{n}\langle Jx_n, x_n\rangle \\
&\leq \psi_1(\|x_n\|) - \frac{1}{n}\|x_n\|^2 \\
&\leq \psi_1(S),
\end{aligned}$$

where S is the bound for $\{x_n\}$. By the strong quasiboundedness of T, we have that the sequence $\{y_n\}$ is bounded, so we may assume that $y_n \rightharpoonup y_0$. Hence $Cx_n \rightharpoonup -y_0$. Let $y \in L(F_n)$. Then

$$\langle Cx_n, y\rangle = \langle y_n + Cx_n, y\rangle - \langle y_n, y\rangle.$$

It follows that

$$\lim_{n\to\infty}\langle Cx_n, y\rangle = -\langle y_0, y\rangle,$$

or

$$\lim_{n\to\infty}\langle Cx_n + y_0, y\rangle = 0.$$

We are going to show that

$$\limsup_{n\to\infty}\langle Cx_n + y_0, x_n\rangle \leq 0.$$

We observe that

$$\begin{aligned}
\langle Cx_n + y_0, x_n\rangle &= \langle y_n + Cx_n, x_n\rangle - \langle y_n, x_n\rangle + \langle y_0, x_n\rangle - \langle y_n, x_0\rangle + \langle y_n, x_0\rangle \\
&= \langle y_n + Cx_n, x_n\rangle - \langle y_n, x_n - x_0\rangle + \langle y_0, x_n\rangle - \langle y_n, x_0\rangle.
\end{aligned}$$

Let $u_0 \in T(x_0)$. Then since T is maximal monotone we have

$$\langle y_n, x_n - x_0\rangle \geq \langle u_0, x_n - x_0\rangle.$$

Therefore

$$\liminf_{n\to\infty}\langle y_n, x_n - x_0\rangle \geq \liminf_{n\to\infty}\langle u_0, x_n - x_0\rangle = 0.$$

Hence

$$\limsup_{n\to\infty}\langle Cx_n + y_0, x_n\rangle \leq \limsup_{n\to\infty}\langle y_n + Cx_n, x_n\rangle - \liminf_{n\to\infty}\langle y_n, x_n - x_0\rangle \leq 0.$$

Since C is of type $(S_+)_L$, we have $x_n \to x_0$, $x_0 \in D(C)$ and $Cx_0 + y_0 = 0$. By Lemma 1.1.7, we have $x_0 \in D(T)$ and $y_0 \in T(x_0)$. Therefore $x_0 \in D(T) \cap D(C) = D(T + C)$ and $Tx_0 + Cx_0 \ni 0$. Obviously $x_0 \in \overline{coG} = \overline{G}$. But $x_0 \notin \partial G$ by the assumption of the theorem and hence we have our conclusion.

Corollary 2.5.5 *Let T satisfy (T_1), and $C : X \supset D(C) \to X^*$ satisfies $c_1 - c_3$. Assume further that there exists an open bounded convex set $G \subset X$ containing zero and such that for every $x \in D(T+C) \cap \partial G$ and every $u \in Tx$ we have*

$$\langle u + Cx, x\rangle \geq 0.$$

Then the inclusion $Tx + Cx \ni 0$ has a solution $x \in D(T+C) \cap G$.

Proof. Suppose that the inclusion

$$Tx + Cx \ni \lambda Jx \tag{1.75}$$

has a solution $x \in D(T+C) \cap \partial G$ for $\lambda \leq 0$. Then for some $u \in Tx$, we have from (1.75) that

$$u + Cx = \lambda Jx.$$

This implies

$$\langle u + Cx, x\rangle = \lambda\langle Jx, x\rangle = \lambda\|x\|^2 \leq 0,$$

i.e., a contradiction to the hypothesis of the corollary. Hence $Tx + Cx \ni \lambda Jx$ has no solution $x \in D(T+C) \cap \partial G$. By Theorem 1.6.5, the result follows.

2.6 Further Mapping Theorems for the New Degree

Definition 2.6.1 *Let $G \subset X$ be open. An operator $T : X \supset D(T) \to 2^{X^*}$ is called "locally monotone" on G if for every $x_0 \in D(T) \cap G$, there exists a ball $\overline{B_r(x_0)} \subset G$ such that T is monotone on $D(T) \cap \overline{B_r(x_0)}$.*

Theorem 2.6.2 ***Equivalent Conditions for the Existence of zeros***
Let T satisfy (T_1), and $C : X \supset D(C) \to X^*$ satisfies $c_1 - c_3$. Assume further that for some open bounded set $G \subset X$, the operator $T + C$ is locally monotone on G. Then the following are equivalent:

(a) $0 \in (T+C)(D(T) \cap G)$;

(b) *there exists $r > 0$ and $x_0 \in D(T) \cap G$ such that $\overline{B_r(x_0)} \subset G$ and $\langle u + Cx, x - x_0 \rangle \geq 0$, for every $(x, u) \in D(T+C) \cap \partial B_r(x_0) \times Tx$;*

(c) *there exists $r > 0$ and $x_0 \in D(T) \cap G$ such that $\overline{B_r(x_0)} \subset G$ and $(T+C)x \not\ni \lambda J(x - x_0)$, for every $(\lambda, x) \in (-\infty, 0) \times D(T+C) \cap \partial B_r(x_0)$.*

Proof. Assume that $0 \in (T+C)(D(T) \cap G)$. Then there exists $x_0 \in D(T) \cap G$ such that $0 \in (T+C)x_0$. Since T is locally monotone on G, there exists a ball $\overline{B_r(x_0)} \subset G$ such that T is monotone on $D(T+C) \cap \overline{B_r(x_0)}$. Consequently we have

$$\langle u + Cx, x - x_0 \rangle \geq 0,$$

for every $(x, u) \in D(T+C) \cap \partial B_r(x_0) \times Tx$. It follows that $(a) \Rightarrow (b)$.
To show that $(b) \Rightarrow (c)$, assume that (b) holds and let $(T+C)x \ni \lambda J(x - x_0)$, for

some $(\lambda, x) \in (-\infty, 0) \times D(T+C) \cap \partial B_r(x_0)$. Then for some $u \in Tx$

$$0 \leq \langle u + Cx, x - x_0 \rangle = \lambda \langle J(x - x_0), x - x_0 \rangle = \|x - x_0\|^2 < 0.$$

This is a contradiction. Hence $(b) \Rightarrow (c)$.

Let (c) hold. We consider the approximate problem

$$Tx + Cx + \frac{1}{n} J(x - x_0) \ni 0. \tag{1.76}$$

It is actually possible to replace $J_\psi(x)$ by $J_\psi(x - x_0)$ in various homotopies provided $x_0 \in G$. Hence by Theorem 1.6.5, (1.76) is solvable in $B_r(x_0)$ for any $n = 1, 2, ...,$ where in this case $\lambda = \frac{-1}{n}$.

Let x_n be a solution of (1.76) lying in $B_r(x_0)$. We may assume that $x_n \rightharpoonup \tilde{x} \in \overline{B_r(x_0)}$. Then for some $y_n \in Tx_n$, we have

$$y_n + Cx_n + \frac{1}{n} J(x_n - x_0) = 0. \tag{1.77}$$

Hence

$$\lim_{n \to \infty} \|y_n + Cx_n\| = \lim_{n \to \infty} \frac{1}{n} \|J(x_n - x_0)\| = \lim_{n \to \infty} \frac{1}{n} \|x_n - x_0\| = 0.$$

Therefore

$$\lim_{n \to \infty} (y_n + Cx_n) = 0.$$

Let $y \in L(F_n)$. Then

$$\lim_{n \to \infty} \langle y_n + Cx_n, y \rangle = 0.$$

From (1.77), we have

$$\begin{aligned} \langle y_n, x_n \rangle &= -\langle Cx_n, x_n \rangle - \frac{1}{n} \langle J(x_n - x_0), x_n \rangle \\ &\leq \psi(\|x_n\|) + \frac{1}{n} \|J(x_n - x_0)\| \|x_n\| \leq \psi(r') + \frac{1}{n} r r' \\ &\leq \psi(r') + r r' \end{aligned}$$

where r' is the bound for the sequence $\{x_n\}$. Since T is strongly quasibounded, we have that $\{y_n\}$ is bounded. Hence we may assume that that $y_n \rightharpoonup y_0$. Therefore

$$Cx_n = -\frac{1}{n}J(x_n - x_0) - y_n \rightharpoonup -y_0.$$

Let $y \in L(F_n)$. Then

$$\langle Cx_n, y\rangle = \langle y_n + Cx_n, y\rangle - \langle y_n, y\rangle.$$

It follows that

$$\lim_{n\to\infty} \langle Cx_n, y\rangle = -\langle y_0, y\rangle,$$

or

$$\lim_{n\to\infty} \langle Cx_n + y_0, y\rangle = 0.$$

We next show that

$$\limsup_{n\to\infty} \langle Cx_n + y_0, x_n\rangle \leq 0.$$

Now

$$\begin{aligned}
\langle Cx_n + y_0, x_n\rangle &= \langle y_n + Cx_n, x_n\rangle - \langle y_n, x_n\rangle + \langle y_0, x_n\rangle - \langle y_n, \tilde{x}\rangle + \langle y_n, \tilde{x}\rangle \\
&= \langle y_n + Cx_n, x_n\rangle - \langle y_n, x_n - \tilde{x}\rangle + \langle y_0, x_n\rangle - \langle y_n, \tilde{x}\rangle.
\end{aligned}$$

Let $u_0 \in T(\tilde{x})$. Then since T is maximal monotone we have

$$\langle y_n, x_n - \tilde{x}\rangle \geq \langle u_0, x_n - \tilde{x}\rangle.$$

Therefore

$$\liminf_{n\to\infty} \langle y_n, x_n - \tilde{x}\rangle \geq \liminf_{n\to\infty} \langle u_0, x_n - \tilde{x}\rangle = 0.$$

Hence

$$\limsup_{n\to\infty}\langle Cx_n+y_0,x_n\rangle \le \limsup_{n\to\infty}\langle y_n+Cx_n,x_n\rangle - \liminf_{n\to\infty}\langle y_n, x_n-\tilde{x}\rangle + \langle y_0,\tilde{x}\rangle - \langle y_0,\tilde{x}\rangle \le 0.$$

Since C is of type $(S_+)_L$, we have $x_n \to \tilde{x}$, $\tilde{x}\in D(C)$ and $C\tilde{x}+y_0=0$.
By Lemma 1.1.7, we have $\tilde{x}\in D(T)$ and $y_0 \in T(\tilde{x})$. Therefore $\tilde{x}\in D(T)\cap D(C) = D(T+C)$ and $T\tilde{x}+C\tilde{x}\ni 0$. However $\tilde{x}\notin \partial G$ because $\overline{B_r(x_0)}\subset G$. Thus $\tilde{x}\in D(T+C)\cap G$. Therefore $0\in (T+C)(D(T+C)\cap G)$.

Theorem 2.6.3 ***Balls in the Range of T+C***
Let T satisfy (T_1), and $C: X\supset D(C)\to X^*$ satisfies $c_1 - c_3$. Assume further that there is a bounded open subset G of X with $0\in G$, and there exists a constant $r>0$ and $z_0^*\in X^*$ such that

$$\|z_0^*\| < r \le |Tx+Cx|, \quad x\in D(T+C)\cap\partial G, \tag{1.78}$$

and

$$\langle u+Cx-z_0^*, x\rangle \ge 0, \quad x\in D(T+C)\cap \partial G, \quad u\in Tx. \tag{1.79}$$

Then $B_r(0)\subset (T+C)(D(T+C)\cap\overline{coG})$. If, moreover, G is convex, then $B_r(0)\subset (T+C)(D(T+C)\cap G)$.

Proof. Fix $p^*\in B_r(0)$ and consider the approximate problem

$$Tx+Cx+\frac{1}{n}Jx_n \ni p^*. \tag{1.80}$$

We also consider the homotopy mappings

$$\begin{aligned} H_1(t,x) &= t(Tx+Cx-z_0^*)+\frac{1}{n}Jx \\ &= t(Tx+Cx-z_0^*+\frac{1}{n}Jx)+\frac{1}{n}(1-t)Jx \end{aligned} \tag{1.81}$$

and

$$H_2(t,x) = Tx + Cx + \frac{1}{n}Jx - tz_0^* - (1-t)p^*. \tag{1.82}$$

We show that (1.81)and (1.82) are admissible homotopies. That is $(H_i(t,.))(\partial G) \not\ni 0$, for $i = 1, 2$. Suppose first that $(H_i(t,.))(\partial G) \ni 0$ and that for $i = 1$, $H_1(t,x)$ has a solution $x_t \in \partial(G)$. Then for $u \in Tx_t$, we have

$$t(u + Cx_t - z_0^*) + \frac{1}{n}Jx_t = 0.$$

If $t = 0$, then $Jx_t = 0$. Since J is injective we have $x_t = 0$, i.e., a contradiction since $0 \in intG$. Consider $t \in (0,1]$. Then

$$\begin{aligned} 0 &= \langle u + Cx_t - z_0^*, x_t\rangle + \frac{1}{nt}\langle Jx_t, x_t\rangle \\ &\geq \frac{1}{nt}\langle Jx_t, x_t\rangle \\ &= \frac{1}{nt}\|x_t\|^2 \\ &> 0, \end{aligned}$$

i.e., a contradiction. Hence $d(H_1(t,.), G, 0)$ is well-defined for all $t \in [0,1]$.

We next show that the degree $d(H_2(t,.), G, 0)$ is well-defined. To this end, let $x_t \in \partial G$ be a solution of (1.82). Then for $u \in Tx_t$ we have

$$u + Cx_t + \frac{1}{n}Jx_t - tz_0^* - (1-t)p^* = 0. \tag{1.83}$$

Let $Q = \sup\{\|x\| : x \in \partial G\}$ and fix $\epsilon \in (0,r)$ and an integer $n_0 > 0$ so that

$$r - \epsilon > \frac{1}{n_0}Q + \max\{\|p^*\|, \|z_0^*\|\}. \tag{1.84}$$

Then

$$|Tx + Cx| \geq r > r - \epsilon, \quad x \in \partial G \cap D(T).$$

Since (1.84) holds for $n > n_0$, instead of n_0, we consider only values of such that $n \geq n_0$. Then from (1.83),we have

$$\begin{aligned}
0 &= \|u + Cx_t + \frac{1}{n}Jx_t - tz_0^* - (1-t)p^*\| \\
&\geq |Tx_t + Cx_t| - \frac{1}{n}\|x_t\| - (t\|z_0^*\| + (1-t)\|p^*\|) \\
&\geq |Tx_t + Cx_t| - [\frac{1}{n}\|x_t\| + \max\{\|p^*\|, \|z_0^*\|\}] \\
&\geq |Tx_t + Cx_t| - [\frac{1}{n}Q + \max\{\|p^*\|, \|z_0^*\|\}] \\
&> r - \epsilon - [\frac{1}{n}Q + \max\{\|p^*\|, \|z_0^*\|\}] > 0.
\end{aligned}$$

This is a contradiction. Hence $d(H_2(t,.), G, 0)$ is well-defined for all $t \in [0,1]$ and $n \geq n_0$. Thus we see that when n is sufficiently large, say $n \geq n_0$, both homotopies are admissible and the degrees $d(H_1(t,.), G, 0)$ and $d(H_2(t,.), G, 0)$ are well-defined and constant for $t \in [0,1]$. However

$$d(H_1(0,.), G, 0) = d(\frac{1}{n}J, G, 0) = 1.$$

It follows that

$$d(H_2(0,.), G, 0) = d(H_2(1,.), G, 0) = d(H_1(1,.), G, 0) = d(H_1(0,.), G, 0) = 1,$$

or

$$\begin{aligned}
d(T + C + \frac{1}{n}J - p^*, G, 0) &= d(T + C + \frac{1}{n}J - z_0^*, G, 0) \\
&= d(T + C + \frac{1}{n}J, G, 0) \\
&= d(\frac{1}{n}J, G, 0) = 1.
\end{aligned}$$

This implies that the inclusion (1.80) is solvable in G for all large n. Let us assume that this is true for all $n = 1, 2, ...$, and consider a solution $x_n \in G$ of (1.80). We may assume that $x_n \rightharpoonup x_0 \in \overline{coG}$. For some $y_n \in Tx_n$ and (1.80), we have

$$\lim_{n\to\infty} \|y_n + Cx_n - p^*\| = \lim_{n\to\infty} \frac{1}{n}\|Jx_n\| = \lim_{n\to\infty} \frac{1}{n}\|x_n\| = 0.$$

By (1.80), we have

$$\begin{aligned}\langle y_n, x_n\rangle &= \langle p^*, x_n\rangle - \langle Cx_n, x_n\rangle - \frac{1}{n}\langle Jx_n, x_n\rangle \\ &\le \|p^*\|S + \psi_1(\|x_n\|) - \frac{1}{n}\|x_n\|^2 \\ &\le \|p^*\|S + \psi_1(S),\end{aligned}$$

where S is an upper bound for $\{x_n\}$. Since T is strongly quasiboundedness, we have that the sequence $\{y_n\}$ is bounded, so we may assume that $y_n \rightharpoonup y_0$. Hence $Cx_n \rightharpoonup p^* - y_0$.

Let $y \in L(F_n)$. Then

$$\langle Cx_n - p^*, y\rangle = \langle y_n + Cx_n - p^*, y\rangle - \langle y_n, y\rangle.$$

It follows that

$$\lim_{n\to\infty}\langle Cx_n - p^*, y\rangle = -\langle y_0, y\rangle,$$

or

$$\lim_{n\to\infty}\langle Cx_n - p^* + y_0, y\rangle = 0.$$

We next show that

$$\limsup_{n\to\infty}\langle Cx_n - p^* + y_0, x_n\rangle \le 0.$$

Let $v \in T(x_0)$. Then by the monotonicity of T, we have

$$\langle y_n - v, x_n - x_0\rangle \ge 0.$$

Hence

$$\liminf_{n\to\infty}\langle y_n, x_n - x_0\rangle \geq \liminf_{n\to\infty}\langle v, x_n - x_0\rangle = 0.$$

Now

$$\begin{aligned}\langle Cx_n - p^* + y_0, x_n\rangle &= \langle y_n + Cx_n - p^*, x_n\rangle - \langle y_n, x_n\rangle + \langle y_0, x_n\rangle - \langle y_n, x_0\rangle + \langle y_n, x_0\rangle \\ &= \langle y_n + Cx_n - p^*, x_n\rangle - \langle y_n, x_n - x_0\rangle + \langle y_0, x_n\rangle - \langle y_0, x_n\rangle.\end{aligned}$$

Hence

$$\limsup_{n\to\infty}\langle Cx_n - p^* + y_0, x_n\rangle \leq \limsup_{n\to\infty}\langle y_n + Cx_n - p^*, x_n\rangle - \liminf_{n\to\infty}\langle y_n, x_n - x_0\rangle.$$

Therefore

$$\limsup_{n\to\infty}\langle Cx_n - p^* + y_0, x_n\rangle \leq 0.$$

Since C is of type $(S_+)_L$, we have $x_n \to x_0$, $x_0 \in D(C)$ and $Cx_0 = p^* - y_0$. By Lemma 1.1.7, we have $x_0 \in D(T)$ and $y_0 \in T(x_0)$. Therefore $x_0 \in D(T) \cap D(C) = D(T + C)$ and $Tx_0 + Cx_0 \ni p^*$.

Consequently $B_r(0) \subset (T + C)(D(T + C) \cap \overline{coG})$ which finishes the proof of the first inclusion. If, in addition G is convex, then $\overline{coG} = \overline{G}$. Hence

$$B_r(0) \subset (T + C)(D(T + C) \cap \overline{G}).$$

But the boundary of G is excluded from this inclusion because $p^* \in B_r(0)$ implies

$$\|u + Cx\| \geq |Tx + Cx| > \|p^*\|, \qquad x \in D(T + C) \cap \partial G, \quad u \in Tx$$

or

$$\|u + Cx - p^*\| \geq \|u + Cx\| - \|p^*\| > 0, \quad x \in D(T + C) \cap \partial G, \quad u \in Tx.$$

Thus, $B_r(0) \subset (T + C)(D(T + C) \cap G)$.

3 Balls in the Range of a Continuous Bounded Perturbation of a Maximal Monotone Operator with Compact Resolvents

In what follows, when we say that a maximal monotone operator $T : X \supset D(T) \to 2^{X^*}$ has "compact resolvents" ("completely continuous resolvents") we mean that for every $\varepsilon > 0$ the operator $(T + \varepsilon J)^{-1} : X^* \to X$ is compact (completely continuous). Actually, if $(T + \varepsilon J)^{-1}$ is compact for some $\varepsilon > 0$, then it is compact for all $\varepsilon > 0$ (cf. [19, Lemma 3 and the resolvent identity on page 1690]). The following relevant result says that there exists an open ball around zero in the image of a relatively open set by a continuous and bounded perturbation of a maximal monotone operator with compact resolvents. For a set $A \subset X^*$, we set $|A| = \inf\{\|y^*\| \ : \ y^* \in A\}$.
We state the following important lemma without proof.

Lemma 3.0.4 *Let X and X^* be locally uniformly convex Banach spaces and let $\{x_n\}$ be a sequence in X for which $\langle Jx_n - Jx, x_n - x\rangle \to 0$ for a given element x of X. Then $x_n \to x$ in X.*

Theorem 3.0.5 *Let $G \subset X$ be open and bounded and containing zero. Let the following assumptions be satisfied:*

(i) $T : X \supset D(T) \to 2^{X^*}$, *with* $0 \in D(T)$, *is maximal monotone and has compact resolvents;*

(ii) $C : \overline{G} \to X^*$ *is continuous and bounded*

(iii) *there exist a constant $r > 0$ and $z_0^* \in X^*$ such that*

$$\|z_0^*\| < r \leq |Tx + Cx|, \quad x \in D(T) \cap \partial G \tag{$*$}$$

and

$$\langle u + Cx - z_0^*, x\rangle \geq 0, \quad x \in D(T) \cap \partial G,\ u \in Tx. \tag{$**$}$$

Then $\overline{B_r(0)} \subset \overline{(T + C)(D(T) \cap G)}$. If, moreover, $T + C$ is of type (S) and G is convex, then $B_r(0) \subset (T + C)(D(T) \cap G)$.

Proof. Since $0 \in D(T) \cap G$ we consider the mappings $\tilde{T} : x \to T(x) - v_0,\ x \in D(T)$, and $\tilde{C} : x \to C(x) + v_0,\ x \in \overline{G}$, where v_0 is a fixed vector in $T(0)$.
By definition of $\tilde{T}$, we have $\tilde{T}(0) = T(0) - v_0$. Since $v_0 \in T(0)$, it follows that $0 \in T(0) - v_0 = \tilde{T}(0)$ and

$$\|z_0^*\| < r \leq |Tx + Cx| = |Tx - v_0 + Cx + v_0| = |\tilde{T}x + \tilde{C}x|, \quad x \in D(T) \cap \partial G.$$

Also, for every $x \in D(T) \cap \partial G$ and every $u \in \tilde{T}x$, and $u^* \in Tx$, we have

$$\begin{aligned}
\langle u + \tilde{C}x - z_0, x\rangle &= \langle u + Cx + v_0 - z_0^*, x\rangle \\
&= \langle u^* - v_0 + Cx + v_0 - z_0^*, x\rangle \\
&= \langle u^* + Cx - z_0^*, x\rangle \geq 0. \qquad (2.1)
\end{aligned}$$

Moreover, $\tilde{T}$ is maximal monotone and its resolvent $(\tilde{T} + J)^{-1}$ is compact. Actually, all the resolvents $(t\tilde{T} + \lambda J)^{-1},\ t > 0,\ \lambda > 0$, are compact (cf., for example, Kartsatos [19, p. 1684].
We are planning to solve the approximate equation

$$\begin{aligned}
\tilde{T}x + \tilde{C}x + (1/n)Jx \ni p^* &\equiv Tx - v_0 + Cx + v_0 + (1/n)Jx \ni p^* \\
&\equiv Tx + Cx + (1/n)Jx \ni p^* \qquad (*)_n
\end{aligned}$$

for $x \in D(\tilde{T})$, where $p^* \in B_r(0)$ is a fixed vector. To this end, we consider the homotopies

$$H_1(t,x) \equiv x - (t\tilde{T} + (1/n)J)^{-1}(-t(\tilde{C}x - z_0^*)), \tag{2.2}$$

and

$$H_2(t,x) \equiv x - (\tilde{T} + (1/n)J)^{-1}(-(\tilde{C}x - tz_0^*) + (1-t)p^*), \tag{2.3}$$

for a fixed positive integer n and $t \in [0,1]$. We let $Q = \sup\{\|x\| \ : \ x \in \partial G\}$ and fix $\epsilon \in (0,r)$ and the integer $n_0 > 0$ so that

$$r - \epsilon > (1/n_0)Q + \max\{\|p^*\|, \|z_0^*\|\}. \tag{2.4}$$

This is possible because we also have $\|p^*\| < r$ and $\|z_0^*\| < r$. Then

$$|\tilde{T}x + \tilde{C}x| \geq r > r - \epsilon, \ x \in \partial G \cap D(T).$$

Since (2.4) holds for any $n > n_0$ instead of n_0, we consider only values of n such that $n \geq n_0$. We now show that the mappings $H_i(t,x) \equiv x - F_i(t,x)$ are actually homotopies of compact transformations. This means that each mapping $F_i : [0,1] \times \overline{G} \to X, \ i = 1,2$, is compact. To show that

$$F_1(t,x) = (t\tilde{T} + (1/n)J)^{-1}(-t(\tilde{C}x - z_0^*))$$

is continuous, let $(t_m, x_m) \in (0,1] \times \overline{G}$ be such that $t_m \to t_0 \geq 0$ and $x_m \to x_0 \in X$. We may assume that $t_m > 0$. We set

$$y_m = (t_m\tilde{T} + (1/n)J)^{-1}(-t_m(\tilde{C}x_m - z_0^*))$$

and

$$y_0 = (t_0\tilde{T} + (1/n)J)^{-1}(-t_0(\tilde{C}x_0 - z_0^*)).$$

This implies that

$$t_m\tilde{T}y_m = -(1/n)Jy_m - t_m(Cx_m - z_0^*)$$

and

$$t_0\tilde{T}y_0 = -(1/n)Jy_0 - t_0(Cx_0 - z_0^*).$$

Hence it follows that

$$\langle t_m\tilde{T}y_m - t_0\tilde{T}y_0, y_m - y_0\rangle + (1/n)\langle Jy_m - Jy_0, y_m - y_0\rangle = \langle D_m, y_m - y_0\rangle,$$

where

$$D_m = t_0\tilde{C}x_0 - t_m\tilde{C}x_m + (t_m - t_0)z_0^* \to 0 \quad \text{as } m \to \infty.$$

Here, and in what follows, we sometimes use the symbol $\tilde{T}x,\ x \in D(\tilde{T})$, to denote a single appropriate element of the set $\tilde{T}x$. But

$$t_m\tilde{T}y_m - t_0\tilde{T}y_0 = t_m(\tilde{T}y_m - \tilde{T}y_0) + (t_m - t_0)\tilde{T}y_0.$$

Hence using the monotonicity property of $\tilde{T}$, we obtain

$$\begin{aligned}\langle t_m\tilde{T}y_m - t_0\tilde{T}y_0, y_m - y_0\rangle &= \langle t_m(\tilde{T}y_m - \tilde{T}y_0) + (t_m - t_0)\tilde{T}y_0, y_m - y_0\rangle \\ &\geq (t_m - t_0)\langle\tilde{T}y_0, y_m - y_0\rangle.\end{aligned}$$

Therefore it follows that

$$\begin{aligned}(1/n)\langle Jy_m - Jy_0, y_m - y_0\rangle &\leq -(t_m - t_0)\langle\tilde{T}y_0, y_m - y_0\rangle + \langle D_m, y_m - y_0\rangle \\ &\leq \Big(|t_m - t_0|\|\tilde{T}y_0\| + \|D_m\|\Big)\|y_m - y_0\|. \qquad (2.5)\end{aligned}$$

To find a bound for the sequence $\{\|y_m\|\}$, we evaluate

$$t_m\tilde{T}y_m + (1/n)Jy_m = -t_m(\tilde{C}x_m - z_0^*) \qquad (2.6)$$

at y_m and use the fact that $\langle\tilde{T}y_m, y_m\rangle \geq 0$ to obtain

$$(1/n)\|y_m\|^2 = (1/n)\langle Jy_m, y_m\rangle \leq -t_m\langle\tilde{C}x_m - z_0^*, y_m\rangle \leq t_m\|\tilde{C}x_m - z_0^*\|\|y_m\|, \quad (2.7)$$

from which follows the boundedness of the sequence $\{y_m\}$. Using this in (2.5), we see that

$$\lim_{m\to\infty} \langle Jy_m - Jy_0, y_m - y_0\rangle = 0.$$

Lemma 2.0.4, implies $\{y_m\}$ converges strongly to y_0 as $m \to \infty$, i.e. the continuity of $F_1(t,x)$ on the set $[0,1]\times X$. We now have to show that the mapping $F_1(t,x)$ maps $[0,1]\times G$ into a relatively compact subset of X. Let $(t_m, x_m) \in [0,1]\times G$. Since $\{t_m\}$ is bounded, we may assume that $t_m \to t_0 \in [0,1]$. If $t_0 = 0$, then the boundedness of the sequences $\{y_m\}$ and $\{\tilde{C}x_m\}$ and (2.7) imply $y_m \to 0$. Therefore, it suffices to assume that $t_0 > 0$. We observe first that (2.7) implies again the boundedness of the sequence $\{y_m\}$. We also note that (2.6) is satisfied. Thus, the boundedness of $\{y_m\}$, $\{\tilde{C}y_m\}$ and $t_0 > 0$ imply the boundedness of $\tilde{T}y_m$. We rewrite (2.6) as follows:

$$\tilde{T}y_m + (1/n)Jy_m = -t_m(\tilde{C}x_m - z_0^*) + (1-t_m)\tilde{T}y_m. \tag{2.8}$$

Then since Jy_m and $\tilde{T}y_m$ are bounded,

$$-t_m(\tilde{C}x_m - z_0^*) + (1-t_m)\tilde{T}y_m$$

is bounded. This implies

$$y_m = (\tilde{T} + (1/n)J)^{-1}\left[-t_m(\tilde{C}x_m - z_0^*) + (1-t_m)\tilde{T}y_m\right],$$

i.e., $\{y_m\}$ lies in a compact set. It therefore contains a convergent subsequence. This completes the proof of the compactness of the operator $F_1(t,x)$ on $[0,1]\times\overline{G}$. To show the compactness of the operator $F_2(t,x)$, we first observe that its continuity follows easily from the continuity of the operator $\tilde{C}$ and the continuity of the resolvent $(\tilde{T} + (1/n)J)^{-1}$. The fact that $F_2(t,x)$ maps $[0,1]\times\overline{G}$ into a relative compact set follows easily from the continuity and boundedness of $\tilde{C}$ and the compactness of the above resolvent. We are now going to show that

$$d(H_1(1,\cdot), G, 0) = 1 \tag{2.9}$$

and then

$$d(H_2(0,\cdot),G,0) = d(H_1(1,\cdot),G,0) = 1, \tag{2.10}$$

where $d = d(\cdot,\cdot,\cdot)$ denotes the Leray-Schauder degree. To show (2.9), we show first that the degree there is well-defined. To this end, let us assume that the homotopy equation

$$H_1(t,x) \equiv x - F_1(t,x) = 0$$

has a solution $x_t \in \partial G$, for some $t \in [0,1]$. Then

$$t(\tilde{T}x_t + \tilde{C}x_t - z_0^*) + (1/n)Jx_t = 0. \tag{2.11}$$

Obviously, $t = 0$ implies $x_t = 0$, i.e. a contradiction because 0 is an interior point of G. Thus, $t \in (0,1]$ and, after dividing (2.11) by t and using $(**)$, we obtain,

$$\begin{aligned} 0 &= \langle \tilde{T}x_t + \tilde{C}x_t - z_0^*, x_t\rangle + \langle [1/(nt)]Jx_t, x_t\rangle \\ &\geq [1/(nt)]\langle Jx_t, x_t\rangle \\ &= [1/(nt)](\|x_t\|^2 \\ &> 0. \end{aligned}$$

Consequently, the Leray-Schauder degree $d(H_1(t,\cdot),G,0)$ is well-defined for all $t \in [0,1]$ and equals 1 because we have $0 \in G$ and $d(H_1(0,\cdot),G,0) = d(I,G,0) = 1$. We now show that the degree $d(H_2(t,\cdot),G,0)$ is well defined. To this end, let (2.3) have a solution $x_t \in \partial G$. Then we have

$$\tilde{T}x_t + \tilde{C}x_t + (1/n)Jx_t - tz_0^* - (1-t)p^* = 0.$$

This implies

$$\begin{aligned} 0 &= \|\tilde{T}x_t + \tilde{C}x_t + (1/n)Jx_t - tz_0^* - (1-t)p^*\| \\ &\geq |\tilde{T}x_t + \tilde{C}x_t| - (1/n)\|x_t\| - (t\|z_0^*\| + (1-t)\|p^*\|) \\ &\geq |\tilde{T}x_t + \tilde{C}x_t| - [(1/n)\|x_t\| + \max\{\|p^*\|, \|z_0^*\|\}] \end{aligned}$$

$$
\begin{aligned}
&\geq \quad |\tilde{T}x_t + \tilde{C}x_t| - [(1/n)Q + \max\{\|p^*\|, \|z_0^*\|\}] \\
&> \quad r - \epsilon - [(1/n)Q + \max\{\|p^*\|, \|z_0^*\|\}] \\
&> \quad 0.
\end{aligned}
$$

This contradiction says that $d(H_2(t,\cdot), G, 0)$ is well-defined for all $t \in [0,1]$ and equals the degrees $d(H_1(1,\cdot), G, 0)$ and $d(H_2(0,\cdot), G, 0)$. We conclude that $d(H_2(0,\cdot), G, 0) = 1$, which implies

$$x - (\tilde{T} + (1/n)J)^{-1}(\tilde{C}x + p^*) = 0,$$

or

$$\tilde{T}x + \tilde{C}x + (1/n)Jx \ni p^*. \tag{2.12}$$

for some $x \in G$. Thus, we have the solvability of $(*)_n$ for each $n \geq n_0$, i.e. the solvability of the inclusion

$$Tx + Cx + (1/n)Jx \ni p^* \tag{2.13}$$

with solution $x_n \in G,\ n \geq n_0$. Since G is bounded, $\{x_n\}$ is bounded. From (2.13) we have

$$\|Tx_n + Cx_n - p^*\| = (1/n)\|Jx_n\| \to 0$$

or

$$Tx_n + Cx_n \to p^*$$

as $n \to \infty$. This implies $p^* \in \overline{(T+C)(D(T) \cap G)}$.
This says that $B_r(0) \subset \overline{(T+C)(D(T) \cap G)}$ and implies $\overline{B_r(0)} \subset \overline{(T+C)(D(T) \cap G)}$. To show the second conclusion of the theorem, assume that $T+C$ is of type (S). Let $x_n \in D(T) \cap G$ solve the inclusion (2.13). Since $\{x_n\}$ is bounded, we may assume that $x_n \rightharpoonup x \in \overline{\text{co}G} = \overline{G}$. Here, co$G$ denotes the convex hull of the set G. We have from (2.13)

$$\langle Tx_n + Cx_n, x_n - x_0\rangle = -(1/n)\langle Jx_n, x_n - x_0\rangle + \langle p^*, x_n - x_0\rangle$$

and

$$\lim_{n\to\infty} \langle Tx_n + Cx_n, x_n - x_0\rangle = 0.$$

Since $T+C$ is of type (S), we have $x_n \to x_0$. But C is continuous implies $Cx_n \to Cx_0$. From (2.13) and for some $y_n \in Tx_n$, we have

$$y_n = -Cx_n - (1/n)Jx_n + p^* \to -Cx_0 + p^*.$$

Since T is closed and $Cx_n \to Cx_0$, we have $x_0 \in D(T)$ and $Tx_0 \ni -Cx_0 + p^*$. Thus, $p^* \in (T+C)(D(T)\cap \overline{G})$. However, $x_0 \notin (T+C)(D(T)\cap \partial G)$ because $|Tx_0 + Cx_0| \geq r > \|p^*\|$. The proof is complete.

Corollary 3.0.6 *Let $T : X \supset D(T) \to 2^{X^*}$, with $D(T)$ unbounded and containing zero, be maximal monotone with compact resolvents and $C : \overline{D(T)} \to X^*$ continuous and bounded. Assume that there exist $z_0^* \in X^*$ and a constant $r > 0$ such that*

$$\|z_0^*\| < r \leq \liminf_{\substack{\|x\|\to\infty \\ x\in D(T)}} |Tx + Cx|.$$

Assume, further, that there exists a constant $r_1 > 0$ such that for every $x \in D(T)$ with $\|x\| \geq r_1$ and every $u \in Tx$ we have

$$\langle u + Cx - z_0^*, x\rangle \geq 0.$$

Then $\overline{B_r(0)} \subset \overline{R(T+C)}$. If, moreover, $T+C$ is of type (S), we have $B_r(0) \subset R(T+C)$.

Proof. The proof follows the steps of the proof of Theorem 2.0.5. The only difference is that a number $Q \geq r_1$ is now picked so that (2.4) is true and the degrees are computed on $B_Q(0)$ instead of G.

If, in addition, $0 \in T(0)$ in Theorem 2.0.5, then we choose $v_0 = 0$. A similar remark holds for Corollaries 2.0.6 and 2.0.7 . Theorem 2.0.5 is of course true if the compactness of the resolvents of T is replaced by the compactness of the operator C. However, this case is covered by the results of Yang [30] who used Browder's degree in [4]. If, in this case, we take $C = 0$, we obtain the following corollary. We would also like to mention that if the domain of a maximal monotone operator T is bounded, then T^{-1} is locally bounded, which implies that T is surjective (cf., for example, Pascali and Sburlan [27, p.147]).

Corollary 3.0.7 *Let $T : X \supset D(T) \to 2^{X^*}$, with $D(T)$ unbounded and containing 0, be maximal monotone. Assume that for some positive constants r, r_1 and some $z_0^* \in X^*$ we have*

$$\|z_0^*\| < r \leq \liminf_{\substack{\|x\|\to\infty \\ x\in D(T)}} |Tx|$$

and

$$\langle u - z_0^*, x\rangle \geq 0, \quad \text{for every } x \in D(T) \text{ with } \|x\| \geq r_1,\ u \in Tx. \tag{2.14}$$

Then $B_r(0) \subset R(T)$. If, in particular, $z_0^* \in T(0)$, then (2.14) is automatically satisfied for all $x \in D(T)$, $u \in Tx$.

Proof. Theorem 2.0.5 remains true if the compactness of the resolvent of T is replaced by the compactness of C. Assuming that this is the case, we take $C = 0$. From the proof of Theorem 2.0.5 (see also Corollary 2.0.6), we get that the equation

$$Tx + (1/n)Jx \ni p^* \tag{2.15}$$

is solvable with solution x_n lying in a fixed ball $B_Q(0)$. Passing to a subsequence if necessary, we let $x_n \rightharpoonup x$. Then (2.15) implies $y_n \to p^*$ for some sequence $y_n \in Tx_n$. Again, the demiclosedness property of T (see Lemma 1.1.7, (ii)) implies that $x \in D(T)$ and $p^* \in Tx$. Thus, $B_r(0) \subset R(T)$.
If the boundary condition (2.14) is actually true for every $x \in D(T)$ and $u \in Tx$, then the maximal monotonicity of T implies that $z_0^* \in T(0)$. If the boundary condition $(**)$

holds for every functional $z_0^* \in X^*$, then we don't need the second homotopy function $H_2(t, x)$ in the proof of Theorem 2.0.5. In fact, the first homotopy will ensure that every $z_0^* \in B_r(0)$ lies in the appropriate range set. This situation is guaranteed if we replace $(**)$ by the following:

$$\langle u + Cx, x\rangle \geq -r\|x\|, \quad \text{for every } x \in D(T) \cap \partial G,\ u \in Tx. \tag{2.16}$$

This is because we would then have

$$\langle u + Cx - z_0^*, x\rangle = \langle u + Cx, x\rangle - \langle z_0^*, x\rangle > \langle u + Cx, x\rangle - r\|x\| \geq 0. \tag{2.17}$$

4 The Generalized Topological Degree for Multivalued Compact Perturbation of Maximal Monotone Operators

Let X be a real reflexive Banach space with dual X^*. Let G is an open bounded subset on X. Let $T : X \supset D(T) \to 2^{X^*}$ be m-accretive and $C : \overline{G} \to X^*$ be a compact mapping. Recently Y.-Z Chen in [7] developed a generalized degree theory for $T - C$. His results were extended by X. Fu and S. Song in [9] to the case where the compact perturbation C was multivalued. The authors, Z. Guan and A. G. Kartsatos in [11] extended this generalized degree to the case where T is maximal monotone and C, a single-valued compact mapping.

We extend the results in [11] to the case where C is a multivalued compact mapping. Unlike [7], [9] and [11], where they appealed to the Leray-Schauder degree theory, in our case we appeal to the degree of Tsoy-Wo Ma in [25].

Definition 4.0.8 *A multifunction $G : B \subseteq X \to 2^{X^*}$ is said to belong to class (P) if it maps bounded sets to relatively compact sets, for every $x \in B$, $G(x)$ is a closed and convex subset of X^*, and $G(.)$ is u.s.c in the sense that for every closed set $C \subseteq X^*$, $G^{-1}(C) = \{x \in B : G(x) \cap C \neq \emptyset\}$ is closed in X.*

4.1 The Tsoy-Wo Ma Degree

Let A be an open subset of a locally convex Hausdorff space E, $p \in E$ and $\Gamma(E)$ the family of all nonempty compact convex subset of E. Let $f : \overline{A} \to \Gamma(E)$ be a set-valued compact field such that $p \notin f(\partial(A))$. By a compact field we mean $f = I - F$, where F

is a compact mapping. Let $g : \overline{A} \to \Gamma(E)$ be a finite dimensional set-valued compact field, i.e, $g = I - G$, where G is compact and finite dimensional. A set-valued map $h : [0,1] \times \overline{A} \to \Gamma(E)$ is called a set-valued homotopy if the map $H : [0,1] \times \overline{A} \to \Gamma(E)$ defined by $H(t,x) = x - h(t,x)$ is compact and $p \notin h([0,1] \times \overline{A})$. Two set-valued compact fields f and f' are said to be homotopic if there exists a set-valued homotopy h such that $h_0 = f$ and $h_1 = f'$, where $h_t(x) = h(t,x)$ for all $(t,x) \in [0,1] \times \overline{A}$.
Now let E_1 be any finite dimensional vector space containing $G(\overline{A})$ and p. Let g be homotopic f in the sense of the above definition and $p \notin g(\partial(A))$.
Then the Tsoy-Wo Ma degree, we denote by d_{MA}, of the set-valued compact field f, is defined as

$$d_{MA}(f, A, p) = d_1(g|_{\overline{A \cap E_1}}, A \cap E_1, p),$$

where the degree d_1 is evaluated in the finite dimensional vector space E_1.
In what follows Ω is an open bounded subset of the reflexive Banach space X.

4.2 The Generalized degree $d(T - G, D(T) \cap \Omega, y)$

Definition 4.2.1 *Let $T : X \supset D(T) \to 2^{X^*}$ be maximal monotone and $G : \overline{\Omega} \to 2^{X^*}$ a multifunction of class (P). Let $y \in X^*$ and λ a positive number with $y \notin (\lambda J + T - G)(D(T) \cap \partial\Omega)$. We define the degree*

$$d(\lambda J + T - G, D(T) \cap \Omega, y) = d_{MA}(I - (T + \lambda J)^{-1}(G + y), \Omega, 0),$$

when $D(T) \cap \Omega \neq \emptyset$,

$$d(\lambda J + T - G, D(T) \cap \Omega, y) = 0 \qquad \textit{when} \qquad D(T) \cap \Omega = \emptyset.$$

It is easy to see that $y \notin (\lambda J + T - G)(D(T) \cap \partial\Omega)$ implies that $0 \notin (I - (T + \lambda J)^{-1}(G + y))(\partial\Omega)$. Indeed, if $0 \in (I - (T + \lambda J)^{-1}(G + y))(\partial\Omega)$, then there exists $x \in \partial\Omega$ such that $x = (T + \lambda J)^{-1}(g + y)$ for some $g \in G(x)$. This implies that $x \in D(T)$ and $Tx + \lambda Jx \ni g + y$. Hence it follows that $y \in (\lambda J + T - G)(D(T) \cap \partial\Omega)$,

i.e, a contradiction. Since $(T+\lambda J)^{-1}$ is continuous and G is of class (P), hence $(T+\lambda J)^{-1}(G+y)$ is a multivalued(set-valued) compact mapping. Therefore the degree $d_{MA}(I-(T+\lambda J)^{-1}(G+y),\Omega,0)$ is well-defined.
We will show in the sequel that the degree is independent of $\lambda>0$. The following theorem is a homotopy property from the Tsoy-Wo Ma degree. We state it without proof.

Theorem 4.2.2 *Let $H:[0,1]\times\overline{\Omega}\to 2^X$ be defined by $H(t,x)=x-h(t,x)$, where $h:[0,1]\times\overline{\Omega}\to 2^X$ is compact. If $0\notin H(t,x)$ for all $t\in[0,1]$ and $x\in\partial\Omega$, then $d_{MA}(H(t,.),\Omega,0)$ is independent of $t\in[0,1]$.*

Proposition 4.2.3 *Let $T:X\supset D(T)\to 2^{X^*}$ be maximal monotone and $G:\overline{\Omega}\to 2^{X^*}$ a multivalued of class (P), $y\in X^*$ with $y\notin\overline{(T-G)(D(T)\cap\partial\Omega)}$. Then there exist $\lambda_0>0$ such that $y\notin(\lambda J+T-G)(D(T)\cap\partial\Omega)$ for $\lambda\in(0,\lambda_0]$ and the degree $d_{MA}(I-(T+\lambda J)^{-1}(G+y),\Omega,0)$ is independent of $\lambda\in(0,\lambda_0]$.*

Proof. Suppose such λ_0 does not exist, then there exist $\{x_n\}\subset D(T)\cap\partial\Omega$, $\lambda_n\to 0$ such that $y=\lambda_n Jx_n+v_n-g_n$ for some $v_n\in Tx_n$ and $g_n\in G(x_n)$. Then $v_n-g_n=y-\lambda_n Jx_n$. Passing to subsequence if necessary, we assume that $x_n\rightharpoonup x_0$ and $g_n\to g^*$. Since J is bounded we have $\lambda_n Jx_n\to 0$. Hence $v_n-g_n\to y$. Hence $y\in\overline{(T-G)(D(T)\cap\partial\Omega)}$, i.e., a contradiction. Let $H(t,x):[0,1]\times\overline{\Omega}\to 2^X$ be defined by

$$H(t,x)=x-[(t\lambda_1+(1-t)\lambda_2)J+T]^{-1}(G(x)+y),$$

for $\lambda_1,\lambda_2\in(0,\lambda_0]$ and $\lambda_1>\lambda_2$. We note that $\lambda'=t\lambda_1+(1-t)\lambda_2\in(0,\lambda_0]$. We show that $0\notin H(t,x)$ for all $t\in[0,1]$ and $x\in\partial\Omega$. Suppose this is false. Then there exists $(t,x)\in[0,1]\times\partial\Omega$ such that

$$0\in H(t,x)=x-[(t\lambda_1+(1-t)\lambda_2)J+T]^{-1}(G(x)+y),$$

or

$$0=x-[(t\lambda_1+(1-t)\lambda_2)J+T]^{-1}(g+y).$$

for some $g \in G(x)$. Thus, $x \in D(T)$ and

$$g + y = (t\lambda_1 + (1-t)\lambda_2)J(x) + v,$$

for some $v \in T(x)$. Therefore

$$y \in (\lambda^{'} J + T - G)(D(T) \cap \partial\Omega),$$

where $\lambda^{'} = t\lambda_1 + (1-t)\lambda_2 \in (0, \lambda_0]$, i.e., a contradiction. Hence by the homotopy invariance property of the Ma-degree, we have

$$d_{MA}(H(t,.), \Omega, 0) = d_{MA}(I - [(t\lambda_1 + (1-t)\lambda_2)J + T]^{-1}(G + y), \Omega, 0)$$

is fixed for all $t \in [0,1]$. Hence

$$d_{MA}(H(1,.), \Omega, 0) = d_{MA}(H(0,.), \Omega, 0),$$

meaning that

$$d_{MA}(I - (T + \lambda_1 J)^{-1}(G + y), \Omega, 0) = d_{MA}(I - (T + \lambda_2 J)^{-1}(G + y), \Omega, 0).$$

Hence the proposition.

Definition 4.2.4 *Let $T : X \supset D(T) \to 2^{X^*}$ be maximal monotone and $G : \overline{\Omega} \to 2^{X^*}$ a multifunction of class (P), $y \in X^*$ and $y \notin \overline{(T - G)(D(T) \cap \partial\Omega)}$; We define the generalized topological degree as follows.*

$$\begin{aligned} d(T - G, D(T) \cap \Omega, y) &= \lim_{\lambda \to 0} d(\lambda J + T - G, D(T) \cap \Omega, y) \\ &= \lim_{\lambda \to 0} d_{MA}(I - (T + \lambda J)^{-1}(G + y), \Omega, 0). \end{aligned}$$

It is then clear that, the above proposition justifies this definition.

Theorem 4.2.5 *Let $T : X \supset D(T) \to 2^{X^*}$ be maximal monotone and $G_i : \overline{\Omega} \to 2^{X^*}$, for $i = 1, 2$ are mappings of class (P). Let $H : [0,1] \times \overline{\Omega} \to 2^X$ be defined by $H(t,x) = x - (T + \lambda J)^{-1}(tG_1(x) + (1-t)G_2(x) + y(t))$ for all $t \in [0,1]$ and $x \in \overline{\Omega}$. Suppose that $\{y(t), 0 \le t \le 1\}$ is a continuous curve in X^* with $y(t) \notin (\lambda J + T - G_t)(D(T) \cap \partial\Omega)$, where $G_t = tG_1 + (1-t)G_2$. Then $d_{MA}(H(t,.), \Omega, y(t))$ is independent of $t \in [0,1]$.*

Proof. We first show that $h(t,x) = (T + \lambda J)^{-1}(tG_1(x) + (1-t)G_2(x) + y(t))$ is a compact map. To this end, let $(t_n, x_n) \in [0,1] \times \overline{\Omega}$ and let

$$w_n = (T + \lambda J)^{-1}(t_n G_1(x_n) + (1-t_n)G_2(x_n) + y(t_n)).$$

Let $g_n \in G_1(x_n)$ and $h_n \in G_2(x_n)$. By passing to a subsequence we assume that $t_n \to t_0 \in [0,1]$, $g_n \to g$ and $h_n \to h$. Hence

$$t_n g_n + (1-t_n)h_n + y(t_n) \to t_0 g + (1-t_0)h + y(t_0).$$

Since $(T + \lambda J)^{-1}$ is continuous, we have w_n has a convergent subsequence.
We next show that $0 \notin H(t,x)$ for all $t \in [0,1]$ and $x \in \partial\Omega$. Suppose this is false. Then there exists $(t,x) \in [0,1] \times \partial\Omega$ such that

$$0 \in H(t,x) = x - (\lambda J + T)^{-1}(tG_1(x) + (1-t)G_2(x) + y(t)),$$

or

$$0 = x - (\lambda J + T)^{-1}(tg_1(x) + (1-t)g_2(x) + y(t)).$$

for some $g_1 \in G_1(x)$ and $g_2 \in G_2(x)$. Thus, $x \in D(T)$ and

$$tg_1(x) + (1-t)g_2(x) + y(t) = \lambda J(x) + v,$$

for some $v \in T(x)$. Therefore $y(t) \in (\lambda J + T - G_t)(x)$ and $x \in D(T) \cap \partial\Omega$, i.e., a contradiction. By Theorem 3.2.2, $d_{MA}(H(t,.), \Omega, y(t))$ is independent of $t \in [0,1]$. This means by the Definition 3.2.4, we have $d(T - G_t, D(T) \cap \Omega, y(t))$ is fixed for all $t \in [0,1]$.

4.3 Properties of the Generalized Degree

Theorem 4.3.1 *Let X be a reflexive Banach space and such that X and its dual X^* are locally uniformly convex. Let $T : X \supset D(T) \to 2^{X^*}$ be maximal monotone and $G : \overline{\Omega} \to 2^{X^*}$ a multifunction of class (P). Let $y \in X^*$ with $y \notin \overline{(T - G)(D(T) \cap \partial\Omega)}$. We have*

(i) If $y \in (T + \lambda J)(D(T) \cap \Omega)$, then $d(\lambda J + T, D(T) \cap \Omega, y) = 1$

(ii) If $d(T - G, D(T) \cap \Omega, y) \neq 0$, then $y \in \overline{(T - G)(D(T) \cap \Omega)}$.

(iii) If Ω_1 and Ω_2 are two disjoint open subset of Ω such that $y \notin \overline{(T - G)(D(T) \cap (\overline{\Omega} \setminus \Omega_1 \cup \Omega_2))}$, then

$$d(T - G, D(T) \cap \Omega, y) = d(T - G, D(T) \cap \Omega_1, y) + d(T - G, D(T) \cap \Omega_2, y).$$

Proof. (i) Since $y \in (T + \lambda J)(D(T) \cap \Omega)$, then $(T + \lambda J)^{-1}(y) \in D(T) \cap \Omega \subset \Omega$. Therefore we have

$$d(\lambda J + T, D(T) \cap \Omega, y) = d_{MA}(I - (T + \lambda J)^{-1}(y), \Omega, 0) = 1.$$

(ii) By Definition 3.2.4, if $d(T - G, D(T) \cap \Omega, y) \neq 0$, then $d(\lambda J + T - G, D(T) \cap \Omega, y) \neq 0$ for $\lambda > 0$ small. This implies also by Definition 3.2.1 we have $d_{MA}(I - (T + \lambda J)^{-1}(G + y), \Omega, 0) \neq 0$. Then there exist $x_\lambda \in \Omega$ and $g_\lambda \in G(x_\lambda)$ such that

$$x_\lambda - (T + \lambda J)^{-1}(g_\lambda + y) = 0.$$

Since $(T + \lambda J)^{-1} : X^* \to D(T)$, we have $x_\lambda \in D(T)$ and $g_\lambda + y \in T(x_\lambda) + \lambda J(x_\lambda)$, so $y \in T(x_\lambda) + \lambda J(x_\lambda) - g_\lambda$. Hence $y = v_\lambda + \lambda J(x_\lambda) - g_\lambda$, $v_\lambda \in T(x_\lambda)$. As $\lambda \to 0$,

$\lambda J(x_\lambda) \to 0$ and $v_\lambda - g_\lambda \to y$. Therefore $y \in \overline{(T-G)(D(T)\cap\Omega)}$.

(iii) Since $y \notin \overline{(T-G)(D(T)\cap(\overline{\Omega}\setminus\Omega_1\cup\Omega_2))}$, we have by the first part of proposition 3.2.3

$$y \notin (\lambda J + T - G)(D(T)\cap(\overline{\Omega}\setminus\Omega_1\cup\Omega_2))$$

for $\lambda > 0$ small. Then

$$0 \notin (I - (T+\lambda J)^{-1}(G+y))(\overline{\Omega}\setminus\Omega_1\cup\Omega_2).$$

Hence

$$\begin{aligned}
d(\lambda J + T - G, D(T)\cap\Omega, y) &= d_{MA}(I-(T+\lambda J)^{-1}(G+y),\Omega,0)\\
&= d_{MA}(I-(T+\lambda J)^{-1}(G+y),\Omega_1,0)\\
&\quad +d_{MA}(I-(T+\lambda J)^{-1}(G+y),\Omega_2,0)\\
&= d(\lambda J + T - G, D(T)\cap\Omega_1, y)\\
&+ d(\lambda J + T - G, D(T)\cap\Omega_1, y).
\end{aligned}$$

Allow $\lambda \to 0$, then the result follow immediately.

5 An Eigenvalue Problem for S_+ Perturbation of Nonlinear Operators Approximated by Quasimonotone Mappings

Let X be a reflexive Banach space and G an open bounded subset of X. Let $T : X \supset D(T) \to 2^{X^*}$, $S : \overline{G} \to 2^{X^*}$ and λ, Λ be positive real numbers.
The eigenvalue problem

$$T(u) + \lambda S(u) \ni 0 \tag{E}$$

had been considered by many authors recently. Z. Guan and A. G. Kartsatos in [12] considered the cases where the operator T is maximal monotone and S is either bounded, demicontinuous and of class (S_+) or S is densely defined satisfying certain conditions or quasibounded w.r.t to T. Also in another paper by A. G. Kartsatos and I. V. Skrypnik in [17], they considered the cases where the operator T is (S_+), maximal monotone, m-accretive, maximal monotone with compact resolvents and m-accretive with compact resolvent with S either compact or continuous and bounded. They showed that there exists $(\lambda, x) \in (0, \Lambda] \times (D(T) \cap \partial G)$ such that the inclusion (E) is satisfied.
P. M. Fitzpatrick and W. V. Petryshyn in [8] considered the case where T is A-proper and S compact.
We give another result of the eigenvalue problem (E), where $T \in A_G(QM)$, a class of quasimonotone-type mappings introduced by Arto Kittila in [23] when S is bounded demicontinuous mapping of class $(S)_+$.

5.1 Classes of Multivalued Mappings of Monotone Type

We introduce in this section, the classes of mappings we shall be dealing with in the sequel. Denote by

$$\begin{aligned} F_G(S_+) &= \{T : \overline{G} \to X^* | T \text{ is bounded, demicontinuous and of type } (S_+)\}; \\ F_G(PM) &= \{T : \overline{G} \to X^* | T \text{ is bounded and pseudomonotone}\}; \\ F_G(QM) &= \{T : \overline{G} \to X^* | T \text{ is bounded, demicontinuous and quasimonotone}\}. \end{aligned}$$

Next we define new classes of maps $A_G(S_+)$, $A_G(PM)$ and $A_G(QM)$ of multivalued mappings which are approximated by single-valued mappings of class $F_G(S_+)$, $F_G(PM)$ and $F_G(QM)$, respectively. The $(S)_+$, pseudomonotonicity and quasimonotonicity conditions are applied to the approximating sequences rather the mapping itself.

Definition 5.1.1 *Let G be an open subset of X and let $T : X \supset D(T) \to 2^{X^*}$. T is of class $A_G(S_+)$ if there is a sequence (T_n)(**an approximating sequence of** T) in $F_G(S_+)$ with the following properties:*

- *(A_1) For each $C > 0$ there exists a $K \geq 0$ such that $\langle\, T_n(u), u\rangle \geq -K$ for all $u \in \overline{G}$, $\|u\| \leq C$ and for all $n \in \mathbb{N}$.*
- *(A_2) Let $(t_n) \subset [0, 1]$, $(u_n) \in \overline{G}$ and T_{m_n} be a subsequence of (T_n). If $t_n \to 0$, $u_n \to u$ in X and $t_n T_{m_n}(u_n) \rightharpoonup z$ in X^*, then $z = 0$.*
- *(A_3) Let $(u_n) \in \overline{G}$ and T_{m_n} be a subsequence of (T_n). If $u_n \rightharpoonup u$ in X and $T_{m_n}(u_n) \rightharpoonup w$ in X^* and $\limsup_{n\to\infty}\langle T_{m_n}(u_n), u_n\rangle \leq \langle w, u\rangle$, then $u_n \to u$ in X, $u \in D(T)$ and $w \in T(u)$.*

Definition 5.1.2 *T is of class $A_G(PM)$ if there is an approximating sequence (T_n) in $F_G(PM)$ satisfying (A_1), (A_2) and the following condition:*

- *(A_4) Let $(u_n) \in \overline{G}$ and let T_{m_n} be any subsequence of (T_n). If $u_n \rightharpoonup u$ in X and*

$T_{m_n}(u_n) \rightharpoonup w$ *in* X^* *and* $\limsup_{n\to\infty}\langle T_{m_n}(u_n), u_n\rangle \leq \langle w, u\rangle$, *then* $\langle T_{m_n}(u_n), u_n\rangle \to \langle w, u\rangle$, *and if* $u \in \overline{G}$, *then also* $u \in D(T)$ *and* $w \in T(u)$.

Definition 5.1.3 *T is of class $A_G(QM)$ if there is an approximating sequence (T_n) in $F_G(QM)$ satisfying (A_1), (A_2) and the following condition:*

- (A_5) *Let $(u_n) \in \overline{G}$ and let T_{m_n} be any subsequence of (T_n). If $u_n \rightharpoonup u$ in X and $T_{m_n}(u_n) \rightharpoonup w$ in X^*, then* $\liminf_{n\to\infty}\langle T_{m_n}(u_n), u_n\rangle \geq \langle w, u\rangle$. *If $u_n \to u$ in X and $T_{m_n}(u_n) \rightharpoonup w$ in X^*, then $u \in D(T)$ and $w \in T(u)$.*

If $T \in F_G(S_+)$, we choose $T_n = T$ for all $n \in N$. Then (T_n) satisfies (A_1), (A_2) and (A_3); hence $T \in A_G(S_+)$ and thus $F_G(S_+) \subset A_G(S_+)$. By a similar argument $F_G(PM) \subset A_G(PM)$ and $F_G(QM) \subset A_G(QM)$.
In what follows all subsequences of $\{u_n\}$ or $\{x_n\}$ will still be denoted by $\{u_n\}, \{x_n\}$.

Lemma 5.1.4 *Let G be an open subset of X. If $S \in F_G(S_+)$ and $T \in A_G(QM)$, then $S + T \in A_G(S_+)$.*

Proof. Let (T_n) be an approximating sequence of T and let $R_n = S + T_n$. We show that $R_n \in F_G(S_+)$. Since S and T_n are bounded and demicontinuous so is R_n. Next suppose that $(u_n) \in \overline{G}$ such that $u_n \rightharpoonup u$ in X and $\limsup_{n\to\infty}\langle R_n(u_n), u_n - u\rangle \leq 0$. Then

$$\begin{aligned} \limsup_{n\to\infty}\langle S(u_n), u_n - u\rangle &\leq \limsup_{n\to\infty}\langle R_n(u_n), u_n - u\rangle - \liminf_{n\to\infty}\langle T_n(u_n), u_n - u\rangle \\ &\leq 0. \end{aligned}$$

We have used the fact that, since T_n is demicontinuous and quasimonotone it implies

$$\liminf_{n\to\infty}\langle T_n(u_n), u_n - u\rangle \geq 0.$$

Since S is of type (S_+), we have $u_n \to u$. Hence $R_n \in F_G(S_+)$.
Since S is bounded and T_n satisfies (A_1) so R_n satisfies (A_1). Let $(t_n) \subset [0, 1]$,

$(u_n) \in \overline{G}$ and R_{m_n} be a subsequence of (R_n). Then by definition of R_n

$$t_n T_{m_n}(u_n) = t_n R_{m_n}(u_n) - t_n S(u_n).$$

If $t_n \to 0$, $u_n \to u$ in X $t_n R_{m_n}(u_n) \rightharpoonup z$ in X^*, then by the boundedness of S we have $t_n T_{m_n}(u_n) \rightharpoonup z$ in X^*. Since (T_n) satisfies (A_2), $z = 0$.

To show that (R_n) satisfies (A_3), let $(u_n) \in \overline{G}$ and let R_{m_n} be any subsequence of (R_n). Assume that $u_n \rightharpoonup u$ in X and $R_{m_n}(u_n) \rightharpoonup w$ in X^*. Since S is bounded, we may assume that for a subsequence of u_n, also denoted by u_n, $S(u_n) \rightharpoonup y$ in X^* to some $y \in X^*$ and $\limsup_{n\to\infty}\langle R_{m_n}(u_n), u_n\rangle \leq \langle w, u\rangle$. Then $T_{m_n}(u_n) = R_{m_n}(u_n) - S(u_n) \rightharpoonup w - y$ in X^*. Since $T \in A_G(QM)$, we have $\liminf_{n\to\infty}\langle T_{m_n}(u_n), u_n\rangle \geq \langle w - y, u\rangle$ and hence

$$\begin{aligned}
\limsup_{n\to\infty}\langle S(u_n), u_n - u\rangle &\leq \limsup_{n\to\infty}\langle R_{m_n}(u_n), u_n - u\rangle - \liminf_{n\to\infty}\langle T_{m_n}(u_n), u_n - u\rangle \\
&\leq 0.
\end{aligned}$$

Since S is of type (S_+) we have $u_n \to u$ in X and $S(u_n) \rightharpoonup S(u) = y$ in X^*. Hence $u \in D(T)$ and $w - S(u) \in T(u)$, i.e, $u \in D(S+T)$ and $w \in (S+T)(u)$.

Remark 5.1.5 *It is easily seen that condition (A_3) implies (A_4) and that (A_4) implies (A_5). Thus the inclusions*

$$A_G(S_+) \subset A_G(PM) \subset A_G(QM)$$

are valid.

5.2 The Degree for Operators of Class $A_G(S_+)$

Definition 5.2.1 *Let G be a bounded open subset of X. A family $\{T_t | 0 \leq t \leq 1\}$ of mappings of $\overline{G}$ into X^* is an (S_+)-homotopy if for any sequence $(u_n) \in \overline{G}$ and*

$(t_n) \subset [0,1]$ *with* $t_n \to t$, $u_n \rightharpoonup u$ *in* X *and*

$$\limsup_{n\to\infty} \langle T_{t_n}(u_n), u_n - u\rangle \leq 0,$$

we have $u_n \to u$ *in* X *and* $T_{t_n}(u_n) \rightharpoonup T_t(u)$.

Definition 5.2.2 *A family* $\{H_t | 0 \leq t \leq 1\}$ *of mappings* $H_t : D(H_t) \to 2^{X^*}$ *is called a* a homotopy of class $HA_G(S_+)$ *if there exists a sequence* $(H_{n,t})$ *of bounded* (S_+)*-homotopies* $H_{n,t} : [0,1] \times \overline{G} \to X^*$ *such that* $(H_{n,t})$ *satisfies conditions* (A_1) *and* (A_2) *for each fixed* $t \in [0,1]$ *and the following condition:*

- (A_6) *If* $(t_n) \subset [0,1]$, $t_n \to t$, $(u_n) \in \overline{G}$, $u_n \rightharpoonup u$ *in* X, $H_{m_n.t_n}(u_n) \rightharpoonup w$ *in* X^* *and* $\limsup_{n\to\infty}\langle H_{m_n.t_n}(u_n), u_n\rangle \leq \langle w, u\rangle$, *then* $u_n \to u$ *in* X, $u \in D(H_t)$ *and* $w \in H_t(u)$.

The following lemma can be found in [23] which we state it without proof.

Lemma 5.2.3 *Let* $S : \overline{G} \to X^*$ *be a bounded demicontinuous* (S_+)*-mapping and let* $T \in A_G(S_+)$. *Then the affine homotopy* $H_t = (1-t)S + tT$ *is of class* $HA_G(S_+)$.

Definition 5.2.4 *Let* G *be an open bounded subset of* X *and let* $T \in A_G(S_+)$. *If* $y \notin T(\partial G \cap D(T))$, *we define* $D(T, G, y)$, *the degree of* T *over* G *with respect to* y, *by*
$D(T, G, y) = \{k \in \overline{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty, +\infty\} | k$ *is a cluster point of the sequence*
$d_{S_+}(T_n, G, y)$ *for some approximating sequence* (T_n) *of* $T\}$.

Here the degree d_{S_+} is the degree for bounded demicontinuous (S_+) mapping defined by Skypnik in [28]. It is obvious that $D(T, G, y)$ is always a nonempty subset on $\overline{\mathbb{Z}}$. If $D(T, G, y)$ contains only one element k, we shortly write $D(T, G, y) = k$ instead of the precise notation $D(T, G, y) = \{k\}$. This degree is the generalization of the d_{S_+} since if $T \in F_G(S_+)$, then $D(T, G, y) = d_{S_+}(T, G, y)$
We state the following theorem without proof. For the details of the proof we refer

the reader to [23].

Theorem 5.2.5 *Let G be an open bounded subset of X and $T \in A_G(S_+)$. Then:*

(a) *If $D(T, G, y) \neq 0$, then there exists $u \in G \cap D(T)$ such that $y \in T(u)$.*

(b) *Let G_1 and G_2 be disjoint open subset of G. If $y \notin T(\overline{G} \backslash (G_1 \cup G_2))$, then*

$$D(T, G, y) \subset D(T, G_1, y) + D(T, G_2, y).$$

(c) *Let $H_t \in HA_G(S_+)$ and let $\{y(t) : 0 \leq t \leq 1\}$ be a continuous curve in X^*. If $y(t) \notin H_t(\partial G)$, then $D(H_t, G, y(t))$ is constant in for all $t \in [0, 1]$.*

(d) *Let $J : X \to X^*$ be the duality mapping of X. Then*

$$D(J, G, y) = \begin{cases} 1 & \text{if } y \in J(G); \\ 0 & \text{if } y \notin J(G). \end{cases}$$

Lemma 5.2.6 *Let G be an open bounded subset of X such that $0 \in G$. If $T \in A_G(S_+)$ and*

$$\langle w, u \rangle > -\|w\| \|u\| \tag{4.1}$$

for all $u \in \partial G \cap D(T)$ and $w \in T(u)$, then $D(T, G, 0) = 1$ and there exists $u_0 \in G \cap D(T)$ such that $0 \in T(u_0)$.

Proof. We consider the homotopy $H_t = (1-t)J + tT \in HA_G(S_+)$, by Lemma 4.2.3 for all $u \in \overline{G}$. Then $H_0(u) = J(u) \neq 0$ for all $u \in \partial G$ and $0 \notin T(u) = H_1(u)$ for all $u \in \partial G \cap D(T)$ by (4.1). If

$$(1-t)J(u) + tw = 0 \tag{4.2}$$

for some $t \in (0, 1)$, $u \in \partial G \cap D(T)$ and $w \in T(u)$, then

$$\langle w, u \rangle = \frac{-(1-t)}{t} \langle J(u), u \rangle = \frac{-(1-t)}{t} \|u\|^2.$$

By (4.2)

$$t\|w\| = (1-t)\|J(u)\| = (1-t)\|u\|.$$

Thus,

$$\begin{aligned}\langle w, u\rangle &= -\frac{t\|w\|}{t}\|u\| \\ &= -\|w\|u\|.\end{aligned}$$

This contradicts (4.1). Hence $0 \notin H_t(u)$ for all $t \in [0,1]$ and $u \in \partial G \cap D(T)$. Therefore

$$D(T, G, 0) = D(J, G, 0) = d_{S_+}(J, G, 0) = 1$$

and hence the conclusion of the lemma follows.

Corollary 5.2.7 *Let G be an open bounded subset of X with $0 \in G$. Let $T \in A_G(QM)$ satisfying (4.1) for all $u \in (\partial G \cap D(T))$ and $w \in Tu$. Let ϵ be a positive number. Then the degree $D(T + \epsilon J, G, 0) = 1$*

Proof. Since $T \in A_G(QM)$ and $J \in F_G(S_+)$ then by Lemma 4.1.4 and for $\epsilon > 0$, $T + \epsilon J \in A_G(S_+)$. Let $w \in T(u)$ for $u \in \partial G \cap D(T)$. Then

$$\begin{aligned}\langle w + \epsilon J(u), u\rangle &= \langle w, u\rangle + \epsilon\langle J(u), u\rangle \\ &> -\|w\|\|u\| + \epsilon\|J(u)\|\|u\| \\ &= -(\|w\| - \epsilon\|J(u)\|)\|u\| \\ &> -(\|w + \epsilon J(u)\|)\|u\|.\end{aligned}$$

This shows that $w + \epsilon J(u)$ satisfies the hypothesis of Lemma 4.2.6. Hence if we consider the homotopy $H_t = (1-t)J + t(T + \epsilon J)$, then $H_t \in HA_G(S_+)$ by Lemma 4.2.3. It follows that for all $t \in [0,1]$ and $u \in \partial G \cap D(T)$, we have $0 \notin H_t(u)$ by the same argument in lemma 4.2.6. Therefore $D(T + \epsilon J, G, 0) = d_{S_+}(J, G, 0) = 1$.

We also note here that, the conclusion of this corollary implies, the inclusion $Tx +$

$\epsilon Jx \ni$ has no solution in $D(T) \cap \partial G$ for every $\epsilon > 0$.

5.3 Statement of the Eigenvalue Problem

Theorem 5.3.1 *Let G be open and bounded subset of X with $0 \in G$. Let $S : \overline{G} \to X^*$ be a mapping of class $F_G(S_+)$. Let $T : X \supset D(T) \to 2^{X^*}$ be of class $A_G(QM)$ satisfying the condition (4.1). Let ϵ_0, ϵ be positive numbers. Assume that*

(P) *there exists $\lambda \in (0, \Lambda]$ such that the inclusion*

$$Tx + \lambda Sx + \epsilon Jx \ni 0 \tag{4.3}$$

has no solution $x \in D(T) \cap G$.

Then

(i)there exists $(\lambda_0, x_0) \in (0, \Lambda] \times (D(T) \cap \partial G)$ such that

$$Tx_0 + \lambda_0 Sx_0 + \epsilon Jx_0 \ni 0; \tag{4.4}$$

(ii) If $0 \notin T(D(T) \cap \partial G)$, T_n satisfy condition S_q on the ∂G and property (P) *is satisfied for every $\epsilon \in (0, \epsilon_0]$, then there exists $(\lambda_0, x_0) \in (0, \Lambda] \times (D(T) \cap \partial G)$ such that $Tx_0 + \lambda_0 Sx_0 \ni 0$.*

Proof. Assume (4.4) is not true. Then for every $\lambda \in (0, \Lambda]$, the inclusion

$$Tx + \lambda Sx + \epsilon Jx \ni 0 \tag{4.5}$$

has no solution $x \in D(T) \cap \partial G$. Consider the homotopy inclusion

$$H(t, x) \equiv Tx + t\lambda Sx + \epsilon Jx \ni 0 \tag{4.6}$$

for all $t \in [0,1]$. We are going to show first that $H_t(x) = H(t,x)$ is a homotopy of class $HA_G(S_+)$. Define

$$H_{n,t}(x) = T_n(x) + t\lambda Sx + \epsilon Jx$$

where T_n is an approximating sequence of T. Clearly $H_{n,t}$ is bounded for all $t \in [0,1]$ and $n \in \mathbb{N}$. We next show that it is an (S_+)-homotopy. For any sequence $(t_j) \in [0,1]$ and $(x_j) \in \overline{G}$, consider the sequence

$$H_{n,t_j}(x_j) = T_n(x_j) + t_j\lambda Sx_j + \epsilon Jx_j \tag{4.7}$$

for each $n \in \mathbb{N}$. Passing to subsequence if necessary, we may assume that $t_j \to t$, $x_j \rightharpoonup x$ in X and

$$\limsup_{j\to\infty}\langle H_{n,t_j}(x_j), x_j - x\rangle \leq 0.$$

Then, it follows that

$$\begin{aligned}
\limsup_{j\to\infty}\langle \epsilon Jx_j, x_j - x\rangle &= \limsup_{j\to\infty}\langle H_{n,t_j}(x_j), x_j - x\rangle - \liminf_{j\to\infty}\langle T_n(x_j), x_j - x\rangle \\
&- t\lambda \liminf_{j\to\infty}\langle S(x_j), x_j - x\rangle \\
&\leq 0.
\end{aligned}$$

We have used the fact that for each $n \in \mathbb{N}$, $T_n \in F_G(QM)$ and that $S \in F_G(S_+)$. Since J is of class (S_+) we have $x_j \to x$. Hence $J(x_j) \to J(x)$, $T_n(x_j) \rightharpoonup T_n(x)$ and $S(x_j) \rightharpoonup S(x)$. Hence $H_{n,t_j}(x_j) \rightharpoonup H_{n,t}(x)$.

Next we show that $(H_{n,t})$ satisfies conditions (A_1), (A_2) for each fixed $t \in [0,1]$ as well as (A_6). By definition of $H_{n,t}$,

$$\begin{aligned}
\langle H_{n,t}(x), x\rangle &= \langle T_n(x), x\rangle + \lambda t\langle Sx, x\rangle + \epsilon\langle Jx, x\rangle \\
&\geq -K + \lambda t\langle Sx, x\rangle + \epsilon\|x\|^2 \\
&\geq -K + \lambda t\langle Sx, x\rangle \\
&\geq -M
\end{aligned}$$

for some $M > 0$. Here we have used the fact that S is bounded and T satisfies (A_1). To show (A_2), let $s_n \subset [0,1]$, $(x_n) \in \overline{G}$ and $H_{m_n,t}$ be a subsequence of $H_{n,t}$. Suppose $s_n \to 0$, $x_n \to x$ in X and $s_n H_{m_n,t} \rightharpoonup z$ in X^*. We show that $z = 0$. By the definition of $H_{n,t}$,

$$s_n T_{m_n}(x_n) = s_n H_{m_n,t}(x_n) - \lambda t s_n S x_n - \epsilon s_n J x_n.$$

Since J and S are bounded, $s_n T_{m_n} \rightharpoonup z$. But T satisfies (A_2). Hence $z = 0$. Next we show (A_6) is also satisfied. To this end, let $\{t_n\} \subset [0,1]$, $(x_n) \in \overline{G}$ be such that $t_n \to t$, $x_n \rightharpoonup x$ in X and $H_{m_n,t_n}(x_n) \rightharpoonup w$ in X^* with

$$\limsup_{n\to\infty} \langle H_{m_n,t_n}(x_n), x_n \rangle \leq \langle w, x \rangle,$$

which implies

$$\limsup_{n\to\infty} \langle H_{m_n,t_n}(x_n), x_n - x \rangle \leq 0.$$

For a subsequence we assume that $J(x_n) \rightharpoonup j^*$ and $S(x_n) \rightharpoonup y$. Now

$$T_{m_n}(x_n) = H_{m_n,t_n}(x_n) - \lambda t_n S x_n - \epsilon J x_n \rightharpoonup w - \epsilon j^* - \lambda t y.$$

Since $T \in A_G(QM)$ we have

$$\liminf_{n\to\infty} \langle T_{m_n}(x_n), x_n \rangle \geq \langle w - \epsilon j^* - \lambda t y, x \rangle.$$

Hence

$$\liminf_{n\to\infty} \langle T_{m_n}(x_n), x_n - x \rangle \geq 0.$$

From (4.7), we have

$$\langle \epsilon J x_n, x_n - x \rangle = \langle H_{m_n,t_n}(x_n), x_n - x \rangle - \lambda t_n \langle S x_n, x_n - x \rangle - \langle T_{m_n}(x_n), x_n - x \rangle.$$

Therefore

$$\limsup_{n\to\infty} \langle \epsilon J x_n, x_n - x \rangle \leq 0.$$

Since J is of type (S_+), we have that $x_n \to x$ and $x \in \overline{G}$. Hence it follows that $J(x_n) \to J(x) = j^*$ and $S(x_n) \rightharpoonup S(x) = y$. Since $T \in A_G(QM)$, we also have $x \in D(T)$ and $w - \epsilon J(x) - \lambda t S(x) \in T(x)$.

Thus, we have $w \in T(x) + \lambda t S(x) + \epsilon J(x) = H_t(x)$ and $x \in D(T) \cap \overline{G}$.

Next we show that $\forall t \in [0,1]$, the inclusion $H_t(x) \ni 0$ is not solvable for all $x \in D(T) \cap \partial G$. Suppose this is false. Then for an approximating sequence T_n of T with $(t_n) \in [0,1]$ and $(x_n) \in \partial G$, we have

$$T_n(x_n) + \lambda t_n S x_n + \epsilon J x_n = 0.$$

If $t_n = 0$, we have $T_n(x_n) + \epsilon J x_n = 0$. Passing to subsequence if necessary, we assume that $x_n \rightharpoonup x$ in X. Since J is bounded, $J(x_n) \rightharpoonup j^*$. Hence $T_{m_n}(x_n) \rightharpoonup -\epsilon j^*$. Since $T \in A_G(QM)$ we conclude that

$$\liminf_{n\to\infty} \langle T_{m_n}(x_n), x_n \rangle \geq \langle -\epsilon j^*, x \rangle,$$

from which follows

$$\liminf_{n\to\infty} \langle T_{m_n}(x_n), x_n - x \rangle \geq 0.$$

Hence

$$\limsup_{n\to\infty} \langle \epsilon J x_n, x_n - x \rangle \leq - \liminf_{n\to\infty} \langle T_{m_n}(x_n), x_n - x \rangle \leq 0.$$

Since J is of type (S_+), we have that $x_n \to x$ and $x \in \partial G$. Therefore $J(x_n) \to J(x) = j^*$. Again since $T \in A_G(QM)$, we have $x \in D(T)$ and $-\epsilon J(x) \in T(x)$ or $Tx + \epsilon J(x) \ni 0$ for $x \in D(T) \cap \partial G$, i.e., a contradiction to our assumption of the theorem. Therefore $t_n \neq 0$. Hence $t_n > 0$. Assume that $t_n \to t = 0$.

Then $T_{m_n}(x_n) = -\lambda t_n S x_n - \epsilon J(x_n) \rightharpoonup -\epsilon j^*$, since S is bounded. By the same argument above, we get $0 \in (T + \epsilon J)(D(T) \cap \partial G)$, i.e., a contradiction. It follows that $t > 0$. Since S is bounded, we may assume $S(x_n) \rightharpoonup y$. Then we have

$$T_{m_n}(x_n) = -\lambda t_n S x_n - \epsilon J(x_n) \rightharpoonup -\lambda t y - \epsilon j^*.$$

Again, since $T \in A_G(QM)$,

$$\liminf_{n\to\infty} \langle T_{m_n}(x_n), x_n \rangle \geq \langle -\lambda t y - \epsilon j^*, x \rangle,$$

which implies

$$\liminf_{n\to\infty} \langle T_{m_n}(x_n), x_n - x \rangle \geq 0.$$

Since $S \in F_G(S_+)$,

$$\limsup_{n\to\infty} \langle \epsilon J x_n, x_n - x \rangle \leq -\liminf_{n\to\infty} \langle T_{m_n}(x_n), x_n - x \rangle - \lambda t \liminf_{n\to\infty} \langle S x_n, x_n - x \rangle \leq 0.$$

Since J is of type (S_+), $x_n \to x$ and $x \in \partial G$. Therefore $J(x_n) \to J(x) = j^*$ and $S(x_n) \rightharpoonup S(x) = y$. Also $T_{m_n}(x_n) \rightharpoonup -\lambda t S(x) - \epsilon J(x)$,
and hence $x \in D(T)$ and $-\lambda t S(x) - \epsilon J(x) \in T(x)$ or $Tx + \lambda t S(x) + \epsilon J(x) \ni 0$ for $x \in D(T) \cap \partial G$. This is a contradiction to our assumption that the inclusion

$$Tx + \lambda t S(x) + \epsilon J(x) \ni 0$$

has no solution for $x \in D(T) \cap \partial G$. Therefore we have shown that $H(t, x)$ is an admissible homotopy for which $0 \notin H_t(D(T) \cap \partial G)$ for all $t \in [0, 1]$. Hence the degree $D(H_t, G, 0)$ is fixed for all $t \in [0, 1]$. Therefore

$$D(T + \lambda S + \epsilon J, G, 0) = D(H(1, .), G, 0) = D(H(0, .), G, 0) = D(T + \epsilon J, G, 0) = 1.$$

This says that the inclusion

$$Tx + \lambda S(x) + \epsilon J(x) \ni 0$$

has a solution $x \in D(T) \cap G$ for $\lambda \in (0, \Lambda]$. This is a contradiction to (P) and the proof of (i) is complete.
(ii) Let $(x_n) \in D(T) \cap \partial G$, T_n an approximating sequence of T, $\lambda_n \in (0, \Lambda]$ be such

that

$$T_n(x_n) + \lambda_n S x_n + \frac{1}{n} J x_n = 0.$$

We assume that $\lambda_n \to \lambda$, $x_n \rightharpoonup x$, $Sx_n \rightharpoonup y$ and $Jx_n \rightharpoonup j^*$. We consider the follow two cases. (a) $\lambda = 0$ and (b) $\lambda > 0$. For case (a), we have for a subsequence T_{m_n} of T_n,

$$T_{m_n}(x_n) = -\lambda_n S x_n - \frac{1}{n} J(x_n) \to 0. \tag{4.8}$$

Since T_n satisfies condition S_q on the ∂G, we have $x_n \to x$ and hence $x \in \partial G$.
Also by (4.8), we have

$$\liminf_{n\to\infty} \langle T_{m_n}(x_n), x_n \rangle = 0.$$

It follows that $x \in D(T)$ and $0 \in T(x)$. This implies that $0 \in T(D(T) \cap \partial G)$, i.e, a contradiction which indicates that case (a) is impossible.
For $\lambda > 0$, we have

$$T_{m_n}(x_n) = -\lambda_n S x_n - \frac{1}{n} J(x_n) \rightharpoonup -\lambda y.$$

Since $T \in A_G(QM)$, we have

$$\liminf_{n\to\infty} \langle T_{m_n}(x_n), x_n \rangle \geq \langle -\lambda y, x \rangle,$$

from which follows

$$\liminf_{n\to\infty} \langle T_{m_n}(x_n), x_n - x \rangle \geq 0.$$

Now by (4.8)

$$\langle \lambda_n S x_n, x_n - x \rangle = -\langle \frac{1}{n} J(x_n), x_n - x \rangle - \langle T_{m_n}(x_n), x_n - x \rangle$$

Therefore

$$\limsup_{n\to\infty} \langle S x_n, x_n - x \rangle \leq 0.$$

Since S is of class (S_+), $x_n \to x \in \partial G$ and $S(x_n) \rightharpoonup S(x) = y$. Hence $T_{m_n}(x_n) \rightharpoonup -\lambda S(x)$. It follows that $x \in D(T)$ and $-\lambda S(x) \in T(x)$. We conclude that $Tx + \lambda S(x) \ni 0$ for $x \in D(T) \cap \partial G$ and $\lambda \in (0, \Lambda]$.

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