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Analysis of quasiconformal maps in R^n

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Analysis of Quasiconformal Maps in \mathbb{R}^n

by

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A dissertation submitted in partial fulfillment
of the requirements for the degree of
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ANALYSIS OF QUASICONFORMAL MAPS IN \mathbb{R}^n

ANDREW PURCELL

ABSTRACT

In this thesis, we examine quasiconformal mappings in \mathbb{R}^n . We begin by proving basic properties of the modulus of curve families. We then give the geometric, analytic, and metric space definitions of quasiconformal maps and show their equivalence. We conclude with several computational Examples.

1 BACKGROUND AND MOTIVATION

1.1 Introduction and Motivation

When studying mappings from \mathbb{R}^n into \mathbb{R}^n , it is desirable to consider maps that do not distort the geometry of the domain. The most natural choice of such mappings is those functions that map circles into circles. Such maps, called conformal maps, turn out to be too restrictive in their properties. A theorem by [L] shows that such mappings reduce to restrictions of translations, dilations or rotations.

Because of this rigidity, we relax our geometric condition and consider those maps whose images of circles are elliptical with bounded distortion. In other words, the ratio between the major and minor axis of the image is controlled by a fixed finite constant. These maps, called quasiconformal maps, provide an interesting course of study and have been considered in a variety of metric spaces. In this thesis, we will focus on quasiconformal mappings in \mathbb{R}^n and discuss various equivalent definitions. In addition, we examine various properties and computationally explore some Examples in \mathbb{R}^3 , enabling a deeper understanding of the mechanics of quasiconformal mappings. The material in this thesis provides a basis for exploration of quasiconformal maps in general metric spaces.

1.2 Preliminaries

1.2.1 Möbius Space

Let \mathbb{R} denote the field of real numbers. For $n \geq 2$, $\mathbb{R}^n = \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ times}}$ and $x \in \mathbb{R}^n$ is given by $x = (x_1, x_2, \dots, x_n)$ where each $x_i \in \mathbb{R}$. With the usual addition and scalar

multiplication, \mathbb{R}^n becomes a n -dimensional vector space over \mathbb{R} . The vectors

$$\begin{aligned} e_1 &= (1, 0, 0, \dots, 0), \\ e_2 &= (0, 1, 0, \dots, 0), \\ &\vdots \\ \text{and } e_n &= (0, 0, \dots, 0, 1) \end{aligned}$$

form a basis for \mathbb{R}^n .

Now for $x, y \in \mathbb{R}^n$ we define the Euclidean inner product of x and y as

$$\langle x, y \rangle = x_1y_1 + x_2y_2 + \dots + x_ny_n$$

and the related Euclidean norm as

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}.$$

In addition, we have the following notation:

$$\begin{aligned} B^n(x, r) &= \{y \in \mathbb{R}^n : |y - x| < r\}, \\ \overline{B^n}(x, r) &= \{y \in \mathbb{R}^n : |y - x| \leq r\}, \\ \text{and } S^{n-1}(x, r) &= \{y \in \mathbb{R}^n : |y - x| = r\}. \end{aligned}$$

In this thesis we will work in the Möbius space $\widehat{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$ which is the one-point compactification of \mathbb{R}^n .

1.2.2 Linear Algebra

In this section we will detail needed ideas from Linear Algebra, as found in [HK].

Let $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear transformation, that is for all $\alpha, \beta \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$

$$A(\alpha x + \beta y) = \alpha A(x) + \beta A(y).$$

Define $L(A) = |A| = \max_{\|x\|=1} |Ax|$ and $\ell(A) = \min_{\|x\|=1} |Ax|$. These quantities

are called the maximal and minimal stretchings of A .

We denote the composition of linear transforms A and B by AB . For a linear transform $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $B : \mathbb{R}^m \rightarrow \mathbb{R}^p$ it is clear from the definitions that

$$L(AB) \leq L(A)L(B) \text{ and } \ell(AB) \geq \ell(A)\ell(B).$$

A linear transform $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is said to be non-singular if and only if $\ell(A) > 0$, which leads to the relations:

$$L(A^{-1}) = \ell(A)^{-1} \text{ and } \ell(A^{-1}) = L(A)^{-1}.$$

Recall from Modern Algebra the non-singular linear transforms of \mathbb{R}^n form a group with composition denoted by $GL(n)$ [R][Chapter 2].

A linear transform $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is said to be an orthogonal transform if $|Ax| = \|x\|$ for all $x \in \mathbb{R}^n$ or equivalently if $\langle Ax, Ay \rangle = \langle x, y \rangle$ for all $x, y \in \mathbb{R}^n$. Moreover, the orthogonal transforms of \mathbb{R}^n form a subgroup of $GL(n)$ denoted by $O(n)$.

For every linear transform A , there exists a unique linear transform $A^* : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $\langle Ax, y \rangle = \langle x, A^*y \rangle$ for all $x, y \in \mathbb{R}^n$; we call the linear transform A^* the adjoint of A . Note that if $A \in GL(n)$, then $A \in O(n)$ if and only if $A^{-1} = A^*$.

Recall that if $A \in O(n)$, then $L(A) = 1 = \ell(A)$ and that $\det(A) = \pm 1$. If $\det(A) = 1$, then $A \in SO(n)$ (the special orthogonal group) which is a subgroup of $O(n)$. [R] Noting that if $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transform, and B and C are orthogonal transforms, then

$$L(CAB) = L(A) \text{ and } \ell(CAB) = \ell(A).$$

When a linear transform $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$ has the property that $S = S^*$ we say that S is symmetric or self-adjoint. Note that if $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an arbitrary transform both A^*A and AA^* are symmetric. The following theorem concerning symmetric matrices can be found in [HK].

Theorem 1.2.1 *If S is a symmetric linear transform, then there exists an $A \in O(n)$*

such that $D = A^{-1}SA$, where $Dx = (\lambda_1x_1, \lambda_2x_2, \dots, \lambda_nx_n)$. Moreover, the eigenvalues of S , denoted $\lambda_1, \dots, \lambda_n$, are real.

The transform D in Theorem 1.2.1 is said to be a diagonal transform and from Linear Algebra we have

$$\min_{1 \leq i \leq n} |\lambda_i| \leq |Dx| = \sqrt{\lambda_1^2x_1^2 + \dots + \lambda_n^2x_n^2}$$

when $\|x\| = 1$.

Hence,

$$L(D) = \max |\lambda_i|, \ell(D) = \min |\lambda_i|, \text{ and } \det(D) = \lambda_1\lambda_2 \cdots \lambda_n.$$

By an appropriate choice of A in Theorem 1.2.1 we always assume $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$.

We say that a linear transform $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is positive semi-definite (or positive definite) if $\langle x, Ax \rangle \geq 0$ (or if $\langle x, Ax \rangle > 0$) for all $0 \neq x \in \mathbb{R}^n$. Moreover, for any linear transform $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$, AA^* and A^*A are always positive semi-definite.

As a consequence of Theorem 1.2.1 and the facts above, we have that if $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a linear transform such that A is positive semi-definite and symmetric, the eigenvalues of A can be written as $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$. We then conclude $L(A) = L(T^{-1}AT) = L(D) = \lambda_1$ and similarly $\ell(A) = \lambda_n$, where $T \in O(n)$ such that $T^{-1}AT = D$ as in Theorem 1.2.1.

Theorem 1.2.2 *Any linear transform $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ may be factored into $A = PB$, where $B \in O(n)$ and P is both symmetric and positive semi-definite.*

When considering the transform in Theorem 1.2.2 we see P is uniquely determined by A and we have:

$$AA^* = (PB)(PB)^* = PBB^*P^* = PIP = P^2,$$

where I is the $n \times n$ identity matrix. Recall $BB^* = I$ since $B \in O(n)$.

We call P the unique symmetric, positive semi-definite square root of AA^* .

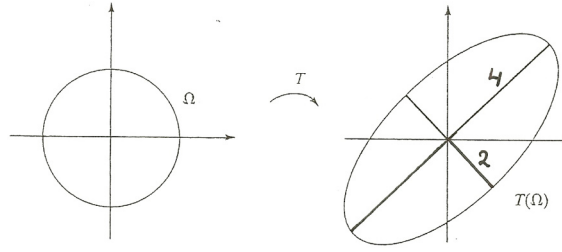


Figure 1.1: The dilatation ellipse

Therefore we may say that the eigenvalues of P are $\lambda_1^{\frac{1}{2}}, \dots, \lambda_n^{\frac{1}{2}}$, where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$ and each λ_i is an eigenvalue of AA^* .

From a geometric perspective the transform A maps the unit sphere S^{n-1} to an ellipsoid E , denoted by $E(A)$ [V].

Example 1.2.3 Let

$$T = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}.$$

The transform T maps S^1 onto the ellipse $E(T)$ as shown in Figure 1.1. The transform T has eigenvalues 2 and 4, which correspond to eigenvectors $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

We conclude with an extension of Theorem 1.2.1.

Theorem 1.2.4 *Let $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear transform and let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of AA^* such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$, then there exists $U, V \in O(n)$ such that $VAU = D$ where $D(x) = (\lambda_1^{\frac{1}{2}}x_1, \dots, \lambda_n^{\frac{1}{2}}x_n)$.*

1.2.3 Partial Derivatives

Define a function $f : A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$, and let $\partial_i f$ denote the partial derivative of f with respect to the i^{th} coordinate. That is,

$$\partial_i f(x) = \lim_{t \rightarrow 0} \frac{f(x + te_i) - f(x)}{t}$$

for all $x \in A^\circ$, where A° denotes the interior of A , for which the limit exists in \mathbb{R}^m . Writing $f = (f_1, f_2, \dots, f_m)$ we have $\partial_i f(x)$ exists if and only if $\partial_i f_j(x)$ exists for $j = 1, 2, \dots, m$. In particular, $\partial_i f(x) = (\partial_i f_1(x), \dots, \partial_i f_m(x))$.

1.2.4 Differentiability

Definition 1.2.5 Let $f : A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$. We say f is differentiable at a point x , if $x \in A^\circ$ and there exists a linear transform $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that

$$f(y) = f(x) + T(y - x) + |y - x|\varepsilon(y) \quad (1.2.1)$$

for all $y \in A$ with $\lim_{y \rightarrow x} \varepsilon(y) = 0$.

If such a transform T exists, it is unique for each f and for all $h \in \mathbb{R}^n$

$$T(h) = \lim_{h \rightarrow 0} \frac{f(x + th) - f(x)}{t}. \quad (1.2.2)$$

T is called the (Frechet) derivative of f at x and denoted by $f'(x)$ or $Df(x)$.

Now, we let $f = (f_1, \dots, f_m)$ and consider the individual components of the function in Equation (1.2.1). We see that f is differentiable at x if and only if each f_i is differentiable at x and that if f is differentiable then f is continuous.

If $f'(x)$ exists, it implies the existence of the partial derivatives of f . Moreover, Equation (1.2.2) implies that $\partial_i f(x) = f'(x)e_i$. Hence, the matrix $f'(x)$ with respect to the standard basis for \mathbb{R}^n and \mathbb{R}^m is :

$$\begin{vmatrix} \partial_1 f_1(x) & \partial_2 f_1(x) & \cdots & \partial_n f_1(x) \\ \partial_1 f_2(x) & \partial_2 f_2(x) & \cdots & \partial_n f_2(x) \\ \vdots & \vdots & \ddots & \cdots \\ \partial_1 f_m(x) & \partial_2 f_m(x) & \cdots & \partial_n f_m(x) \end{vmatrix}.$$

The existence of each $\partial_i f(x)$ does not imply the existence of $f'(x)$ [S]. However, if each $\partial_i f$ is defined in some neighborhood of x and each is continuous at x then f is differentiable at x .

Next, consider $A \subseteq \mathbb{R}^n$ and let $x \in A^\circ$ we define the maximal stretching $L_f(x)$ and the minimal stretching $l_f(x)$ at x of a function $f : A \rightarrow \mathbb{R}^n$ by:

$$L_f(x) = \limsup_{h \rightarrow 0} \frac{|f(x+h) - f(x)|}{|h|},$$

$$l_f(x) = \liminf_{h \rightarrow 0} \frac{|f(x+h) - f(x)|}{|h|}.$$

Theorem 1.2.6 *If f is differentiable at x , then $L_f(x) = L(f'(x))$ and $l_f(x) = l(f'(x))$.*

Proof. Recall that if $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transform then

$$L(A) = \max_{\|x\|=1} |Ax| \text{ and } \ell(A) = \min_{\|x\|=1} |Ax|.$$

“ \leq ” Now, h close to 0 implies $x+h \in A$ and Equation (1.2.1) yields:

$$|f(x+h) - f(x)| = |f'(x)h + |h|\varepsilon(h)| \leq |h|L(f'(x)) + |h||\varepsilon(h)|.$$

Hence,

$$L_f(x) \leq \limsup_{h \rightarrow 0} L(f'(x)) + |\varepsilon(h)| = L(f'(x)).$$

“ \geq ” We may pick $h \in S^{n-1}$ such that $L(f'(x)) = |f'(x)h|$. Therefore by Equation (1.2.2),

$$L_f(x) \geq \lim_{t \rightarrow 0} \frac{|f(x+th) - f(x)|}{t} = |f'(x)h| = L(f'(x)).$$

Similarly, $l_f(x) = l(f'(x))$. ■

Corollary 1.2.7 *If f is differentiable at x , then $L_f(x) = |f'(x)|$.*

Theorem 1.2.8 *Let $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$. If $f : A \rightarrow \mathbb{R}^m$ is differentiable at x and if $g : B \rightarrow \mathbb{R}^p$ is differentiable at $y = f(x)$, then the composition $g \circ f : A \cap f^{-1}(B) \rightarrow \mathbb{R}^p$*

is differentiable at x . Moreover

$$(g \circ f)'(x) = g'(y)f'(x).$$

2 THE MODULUS OF A CURVE FAMILY

In this chapter we will explore the notion of the modulus of a curve family. This will be the main tool we employ when discussing the properties of quasiconformal maps.

2.1 The Geometry of Paths

2.1.1 Paths

Definition 2.1.1 A path in $\widehat{\mathbb{R}^n}$ is a continuous map $\alpha : \Delta \rightarrow \widehat{\mathbb{R}^n}$, where Δ is an interval in \mathbb{R} .

The path is said to be open or closed depending on whether Δ is open or closed. The locus $|\alpha|$ of a path $\alpha : \Delta \rightarrow \widehat{\mathbb{R}^n}$ is the set $\{\alpha\Delta\} \subset \widehat{\mathbb{R}^n}$. A subpath of $\alpha : \Delta \rightarrow \widehat{\mathbb{R}^n}$ is the restriction of α to a continuous subinterval of Δ .

Consider a partition of $[a, b]$ such that $a = t_0 \leq t_1 \leq \dots \leq t_n = b$. We denote the length of $\alpha : [a, b] \rightarrow \widehat{\mathbb{R}^n}$ by $\ell(\alpha)$ such that

$$\ell(\alpha) = \sup\left\{\sum_{i=1}^n |\alpha(t_i) - \alpha(t_{i-1})|\right\}. \quad (2.1.1)$$

Hence, $0 \leq \ell(\alpha) \leq \infty$ for all $\alpha \subset \widehat{\mathbb{R}^n}$. Clearly $\ell(\alpha) = 0$ if and only if α is a constant path.

Definition 2.1.2 We say the path α is rectifiable if $\ell(\alpha) < \infty$.

Theorem 2.1.3 If $\alpha : [a, b] \rightarrow \widehat{\mathbb{R}^n}$ is a rectifiable path, and if $a = t_0 \leq t_1 \leq \dots \leq t_n = b$ is a partition of $[a, b]$, then every restriction $\alpha|_{[t_{i-1}, t_i]}$ is rectifiable.

Moreover,

$$\ell(\alpha) = \sum_{i=1}^n \ell(\alpha|_{[t_{i-1}, t_i]}).$$

Let $\alpha : [a, b] \rightarrow \widehat{\mathbb{R}^n}$ be a rectifiable path. For all $t \in [a, b]$ we set $s_\alpha(t) = \ell(\alpha|_{[a, t]})$, sometimes denoted by $s(t)$. We say the function $s_\alpha : [a, b] \rightarrow \mathbb{R}$ is the length function of α .

2.1.2 Change of Parameter

Definition 2.1.4 A path $\alpha : [a, b] \rightarrow \widehat{\mathbb{R}^n}$ is obtained from a path $\beta : [c, d] \rightarrow \widehat{\mathbb{R}^n}$ by an increasing/decreasing change of parameter if there exists an increasing/decreasing continuous map $h : [a, b] \rightarrow [c, d]$ such that $\alpha = \beta \circ h$.

In particular, if $-\infty < a \leq b < \infty$, the inverse of a path $\alpha : [a, b] \rightarrow \widehat{\mathbb{R}^n}$ is the path $\bar{\alpha} : [a, b] \rightarrow \widehat{\mathbb{R}^n}$ defined by $\bar{\alpha}(t) = \alpha(a + b - t)$.

Theorem 2.1.5 If α is obtained from β by a change of parameter, then $\ell(\alpha) = \ell(\beta)$.

Theorem 2.1.6 If $\alpha : [a, b] \rightarrow \widehat{\mathbb{R}^n}$ is a rectifiable path, then there exists a unique path $\alpha_0 : [0, c] \rightarrow \widehat{\mathbb{R}^n}$ that satisfies the following properties:

- 1) α is obtained from α_0 by an increasing change of parameter;
- 2) $\ell(\alpha_0|_{[0, t]}) = t$ for $0 \leq t \leq c$, i.e., $s_{\alpha_0}(t) = t$.

Moreover, $c = \ell(\alpha)$ and $\alpha = \alpha_0 \circ s_\alpha$.

Proof. Take α_0 to be a path that satisfies conditions 1) and 2). Then $\alpha = \alpha_0 \circ h$ where $h : [a, b] \rightarrow [0, c]$ is increasing. Now if $a \leq t \leq b$, Theorem 2.1.5 implies that $\ell(\alpha|_{[a, t]}) = \ell(\alpha_0|_{[0, h(t)]}) = h(t)$. Therefore, $h = s_\alpha$ and so α_0 is unique. Now, if $s_\alpha(t_1) = s_\alpha(t_2)$, then $\alpha|_{[t_1, t_2]}$ is constant. Therefore there exists a well-defined mapping $\alpha_0 : [0, \ell(\alpha)] \rightarrow \widehat{\mathbb{R}^n}$ such that $\alpha = \alpha_0 \circ s_\alpha$. ■

Definition 2.1.7 The path α_0 in Theorem 2.1.6 is called the arc-length parametrization of α .

Definition 2.1.8 A path $\alpha : \Delta \rightarrow \widehat{\mathbb{R}^n}$ is locally rectifiable if every compact subpath of α is rectifiable. We define $\ell(\alpha) = \sup\{\ell(\beta)\}$, over all compact subpaths β of α .

We note that on closed paths, the definitions of length are equivalent. Through the next theorem, we may extend certain paths to closed paths without changing length.

Theorem 2.1.9 If $\alpha : (a, b) \rightarrow \mathbb{R}^n$ is a rectifiable open path, then there exists a unique extension to a closed path $\alpha_* : [a, b] \rightarrow \mathbb{R}^n$. Moreover, $\ell(\alpha) = \ell(\alpha_*)$.

Proof. Let $t \in (a, b)$. Suppose to the contrary, $\lim_{t \rightarrow b} \alpha(t)$ does not exist. We then can find a positive number r and a sequence of numbers

$$t_1 < u_1 < t_2 < u_2 < \dots < t_j < u_j < \dots < b$$

such that $|\alpha(u_j) - \alpha(t_j)| > r$, for all $j \in \mathbb{N}$.

Hence,

$$\ell(\alpha|_{[t_1, u_k]}) \geq \sum_{j=1}^k |\alpha(u_j) - \alpha(t_j)| > kr$$

for every k . This leads to a contradiction, since α is a rectifiable path. The last assertion follows directly from the existence of the extension. ■

We conclude our discussion of length with the following theorem.

Theorem 2.1.10 Let $\alpha : (a, b) \rightarrow \mathbb{R}^n$ be an open path such that α is absolutely continuous on every closed subinterval of (a, b) , then α is locally rectifiable and

$$\ell(\alpha) = \int_a^b |\alpha'(t)| dt.$$

2.1.3 Line Integrals

Henceforth, we will assume $A \subset \widehat{\mathbb{R}^n}$ is a Borel set and $\rho : A \rightarrow \mathbb{R} \cup \{\infty\}$ is a non-negative Borel function. Let $\alpha : [a, b] \rightarrow A$ be a closed rectifiable path. We define the

line integral of ρ over α by:

$$\int_{\alpha} \rho \, ds = \int_0^{\ell(\alpha)} \rho(\alpha_0(t)) \, dt,$$

where α_0 is reparameterization of α from Theorem 2.1.6. We note that the integral on the left-hand side of the equation exists, since $\rho \circ \alpha_0$ is a non-negative Borel function and the integral is the usual Lebesgue integral.

We have the following important result concerning line integrals of the image of a path.

Theorem 2.1.11 *Let U be an open set in \mathbb{R}^n and $f : U \rightarrow \mathbb{R}^m$ be continuous. Also, let $\alpha : \Delta \rightarrow U$ be a locally rectifiable path such that f is absolutely on every closed subpath of α . Then $f \circ \alpha$ is locally rectifiable. If $\rho : |f \circ \alpha| \rightarrow \mathbb{R} \cup \{\infty\}$ is a non-negative Borel function, then*

$$\int_{f \circ \alpha} \rho \, ds \leq \int_{\alpha} \rho(f(x)) L_f(x) \, |dx|.$$

2.1.4 Conformal Maps

While the heart of this discussion lies in the theory of quasiconformal maps, we need to first introduce the notion of a conformal map.

Definition 2.1.12 *Let D, D' be domains in $\widehat{\mathbb{R}^n}$. A homeomorphism $f : D \rightarrow D'$ is conformal if $f \in C^1$ and if $|f'(x)h| = |f'(x)||h|$ for all $x \in D$ and $h \in \mathbb{R}^n$.*

We note that if D, D' are domains in $\widehat{\mathbb{R}^n}$; a homeomorphism $f : D \rightarrow D'$ is conformal when $f|_{D \setminus \{\infty, f^{-1}(\infty)\}}$ is conformal. Recall also that a C^1 homeomorphism f is conformal if and only if $|f'(x)|^n = |J(x, f)|$ for all $x \in D$.

A theorem of Liouville [L] says that for $n \geq 3$ every conformal map is a Möbius transformation, where a Möbius transformation is a mapping $f : \widehat{\mathbb{R}^n} \rightarrow \widehat{\mathbb{R}^n}$ such that f is a composition of a finite number of the following transformations:

- 1) Translations: $f(x) = x + a$; for a fixed $a \in \mathbb{R}$.
- 2) Stretchings: $f(x) = rx$; where $0 < r \in \mathbb{R}$.
- 3) Orthogonal mappings: f is linear and $|f(x)| = |x|$ for all $x \in \mathbb{R}$.

4) Inversion in a sphere $S(a, r)$: $f(x) = a + \frac{r^2(x-a)}{|x-a|^2}$.

2.2 The p -Modulus

In this section we will present the modulus of a curve family in $\widehat{\mathbb{R}^n}$. By a curve family Γ , we mean the elements of Γ are curves in $\widehat{\mathbb{R}^n}$.

2.2.1 Definitions and Properties

Let $F(\Gamma)$ be the set of all non-negative Borel functions $\rho : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$ such that

$$\int_{\gamma} \rho \, ds \geq 1$$

for all locally rectifiable curves $\gamma \in \Gamma$. For each $p \geq 1$ we define the p -modulus of Γ as:

$$M_p(\Gamma) = \inf_{\rho \in F(\Gamma)} \int_{\mathbb{R}^n} \rho^p \, dm$$

and if $F(\Gamma) = \emptyset$ we set $M_p(\Gamma) = \infty$. From the definition of the p -modulus we can see that $0 \leq M_p(\Gamma) \leq \infty$.

Lemma 2.2.1 $F(\Gamma) = \emptyset$ if only if Γ contains a constant path.

Proof. Define $\gamma : [a, b] \rightarrow \widehat{\mathbb{R}^n} \subset \Gamma$ to be a constant path. By a change of variable technique

$$\int_{\gamma} \rho \, ds = \int_a^b (\rho \circ \gamma) \cdot |\gamma'(t)| \, dt = 0$$

so the set of admissible functions of Γ is empty. On the other hand, if Γ has no non-constant paths then $\rho \equiv \infty$ will always be in $F(\Gamma)$. In particular, for all locally rectifiable $\gamma \in \Gamma$

$$\int_{\gamma} \rho \, ds = \infty \cdot \ell(\gamma) = \infty.$$

■

For the remainder of this discussion if $p = n$ we will write $M_p(\Gamma) = M(\Gamma)$ and say that $M(\Gamma)$ is the *modulus* of Γ .

Theorem 2.2.2 M_p is an outer measure in the space of all curves in $\widehat{\mathbb{R}^n}$. That is,

- 1) $M_p(\emptyset) = 0$.
- 2) $\Gamma_1 \subset \Gamma_2 \Rightarrow M_p(\Gamma_1) \leq M_p(\Gamma_2)$.
- 3) $M_p\left(\bigcup_{i=1}^{\infty} \Gamma_i\right) \leq \sum_{i=1}^{\infty} M_p(\Gamma_i)$.

Proof. 1) Note that $\rho \equiv 0$ belongs to $F(\emptyset)$. Therefore, $M_p(\emptyset) = 0$.

2) If $\Gamma_1 \subset \Gamma_2$, then $F(\Gamma_2) \subset F(\Gamma_1)$. This implies $M_p(\Gamma_1) \leq M_p(\Gamma_2)$.

3) If $\sum_i M_p(\Gamma_i) = \infty$, the result is clear. Therefore, take $\sum_i M_p(\Gamma_i) < \infty$. Now, given $\varepsilon > 0$, for each i pick ρ_i such that $\int_{\mathbb{R}^n} \rho_i^p dm \leq M_p(\Gamma_i) + \varepsilon 2^{-i}$.

Let $\rho = \left(\sum_{i=1}^{\infty} \rho_i^p\right)^{\frac{1}{p}}$. Given the curve $\gamma \in \bigcup_i \Gamma_i$, we have $\gamma \in \Gamma_{i_0}$ for some Γ_{i_0} . Next, we see that

$$\int_{\gamma} \rho ds = \int_{\gamma} \left(\sum_{i=1}^{\infty} \rho_i^p\right)^{\frac{1}{p}} \geq \int_{\gamma} \rho_{i_0} ds \geq 1.$$

Thus $\rho \in F\left(\bigcup_i \Gamma_i\right)$, and we compute using the Monotone Convergence Theorem

$$\begin{aligned} M_p\left(\bigcup_i \Gamma_i\right) &\leq \int_{\mathbb{R}^n} \rho^p = \int_{\mathbb{R}^n} \sum_{i=1}^{\infty} \rho_i^p \leq \sum_{i=1}^{\infty} \int_{\mathbb{R}^n} \rho_i^p \leq \sum_{i=1}^{\infty} M_p(\Gamma_i) + \sum_{i=1}^{\infty} \varepsilon 2^{-i} \\ &= \sum_{i=1}^{\infty} M_p(\Gamma_i) + \varepsilon. \end{aligned}$$

■

Definition 2.2.3 Let Γ_1 and Γ_2 be curve families in $\widehat{\mathbb{R}^n}$ such that for each $\gamma \in \Gamma_2$, γ has a subcurve belonging to Γ_1 . In this case we say that Γ_2 minorizes Γ_1 and denote it $\Gamma_1 \prec \Gamma_2$.

Theorem 2.2.4 If $\Gamma_1 \prec \Gamma_2$, then $M_p(\Gamma_1) \geq M_p(\Gamma_2)$.

Proof. If $M_p(\Gamma_1) = \infty$ the result is clear. Therefore, take $M_p(\Gamma_1) < \infty$, and pick $\rho \in F(\Gamma_1)$. If $\gamma \in \Gamma_2$ is locally rectifiable and if $\beta \in \Gamma_1$ is picked such that $\beta \prec \gamma$, then

$$\int_{\gamma} \rho \, ds \geq \int_{\beta} \rho \, ds \geq 1.$$

Hence, $\rho \in F(\Gamma_2)$ and so $M_p(\Gamma_2) \leq \int_{\mathbb{R}^n} \rho^p \, dm$. Therefore $M_p(\Gamma_2) \leq M_p(\Gamma_1)$ by taking the infimum over all $\rho \in F(\Gamma_1)$. ■

We note that the the p -modulus of a curve family Γ is large if there are many curves in Γ or if the curves in Γ are short.

Definition 2.2.5 *The curve families $\Gamma_1, \Gamma_2, \dots$ are called separate if there exists disjoint Borel sets $E_i \in \mathbb{R}^n$ such that if $\gamma \in \Gamma_i$ is locally rectifiable, then*

$$\int_{\gamma} \chi_{E_i^c} \, ds = 0.$$

Lemma 2.2.6 *If $\Gamma_1, \Gamma_2, \dots$ are separate, then*

$$M_p\left(\bigcup_{i=1}^{\infty} \Gamma_i\right) = \sum_{i=1}^{\infty} M_p(\Gamma_i).$$

Proof. It suffices to show that $\sum_i M_p(\Gamma_i) \leq M_p(\Gamma)$, where $\Gamma = \bigcup_i \Gamma_i$, since M_p is an outer measure. If $M_p(\Gamma) = \infty$, then the result is clear. Therefore, take $M_p(\Gamma) < \infty$. Let $\rho \in F(\Gamma)$ and define $\rho_i = \rho \cdot \chi_{E_i}$, where $\{E_i\}$ is the collection of Borel sets separating Γ_i . Hence, for locally rectifiable $\gamma \in \Gamma_i$ we have

$$\int_{\gamma} \chi_{E_i^c} \, ds = 0.$$

Therefore, by the Monotone Convergence Theorem:

$$0 \leq \int_{\gamma} \rho \cdot \chi_{E_i^c} = \lim_{k \rightarrow \infty} \int_{\gamma} \min\{\rho, k\} \cdot \chi_{E_i^c} \leq \lim_{k \rightarrow \infty} k \int \chi_{E_i^c} \, ds = 0.$$

Hence for all γ ,

$$\int_{\gamma} \rho \cdot \chi_{E_i^c} = 0.$$

We then have

$$\begin{aligned} 1 \leq \int_{\gamma} \rho \, ds &= \int_{\gamma} \rho(\chi_{E_i} + \chi_{E_i^c}) \, ds \\ &= \int_{\gamma} \rho_i \, ds + \int_{\gamma} \rho \cdot \chi_{E_i^c} \, ds \\ &= \int_{\gamma} \rho_i \, ds. \end{aligned}$$

Therefore, $\rho_i \in F(\Gamma_i)$.

Hence, using disjointness

$$\begin{aligned} \sum_i M_p(\Gamma_i) &\leq \sum_i \int_{\mathbb{R}^n} \rho_i^p \, dm = \sum_i \int_{E_i} \rho^p \, dm \\ &= \int_{\bigcup_i E_i} \rho^p \, dm \leq \int_{\mathbb{R}^n} \rho^p \, dm. \end{aligned}$$

Taking the infimum over all $\gamma \in F(\Gamma)$ implies

$$\sum_i M_p(\Gamma_i) \leq M_p(\Gamma).$$

■

Theorem 2.2.7 *If $\Gamma_1, \Gamma_2, \dots$ are separate and if $\Gamma \prec \Gamma_i$ for all i , then*

$$M_p(\Gamma) \geq \sum_i M_p(\Gamma_i).$$

Proof. By Lemma 2.2.6 and the fact that $\Gamma \prec \bigcup_i \Gamma_i$ the result is clear.

■

2.2.2 Influence of Non-Rectifiable Curves

From the definition of the p -modulus, the curves which are not locally rectifiable have no impact on the p -modulus. That is, the p -modulus of curves that are not locally rectifiable is zero. Hence, $M_p(\Gamma) = M_p(\Gamma_0)$, where Γ_0 is the family of all locally rectifiable curves in Γ . This next theorem shows when $p \geq n$, we may restrict ourselves to rectifiable curves. We define the following notation: If Γ is a curve family in $\widehat{\mathbb{R}^n}$, we denote the $F_r(\Gamma)$ to be the family of all non-negative Borel functions $\rho : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$ such that for all rectifiable $\gamma \in \Gamma$

$$\int_{\gamma} \rho \, ds \geq 1.$$

In general $F(\Gamma) \subset F_r(\Gamma)$.

Theorem 2.2.8 *If Γ is a curve family in $\widehat{\mathbb{R}^n}$, then for $p \geq n$,*

$$M_p(\Gamma) = \inf_{\rho \in F_r(\Gamma)} \int \rho^n \, dm = M_p(\Gamma_r).$$

Proof. By the fact that $F(\Gamma) \subset F_r(\Gamma)$, we have $M_p(\Gamma) \geq M_p(\Gamma_r)$. Now to show the reverse inequality: Fix $p \geq n$, and define the auxiliary function $\rho_0 : \widehat{\mathbb{R}^n} \rightarrow [0, \infty]$ by

$$\rho_0(x) = \begin{cases} (|x|^{\frac{n}{p}} \ln |x|)^{-1} & |x| \geq 2, \\ 1 & |x| < 2. \end{cases}$$

We first compute the space integral.

$$\begin{aligned}
\int_{\mathbb{R}^n} \rho_0^p dm &= \int_{|x|<2} 1^p dm + \int_{|x|\geq 2} \left(\frac{1}{|x|^{\frac{n}{p}} \ln |x|} \right)^p dm \\
&= 2^n \cdot \Omega_n + \int \int_2^\infty \left(\frac{1}{|x|^{\frac{n}{p}} \ln |x|} \right)^p \cdot r^{n-1} dr \frac{d\theta}{n-1} \\
&= 2^n \cdot \Omega_n + \omega_{n-1} \int_2^\infty r^{-1} (\ln r)^{-p} dr \\
&= 2^n \cdot \Omega_n + \omega_{n-1} \cdot \frac{(\ln r)^{1-p}}{1-p} \Big|_2^\infty \\
&= 2^n \cdot \Omega_n + \omega_{n-1} \cdot \frac{(\ln 2)^{1-p}}{1-p} < \infty.
\end{aligned}$$

Recall that Ω_n is the volume of the unit ball in \mathbb{R}^n and ω_n is the surface area of the sphere in \mathbb{R}^n .

We now wish to show $\int_\gamma \rho_0 ds = \infty$, for all locally rectifiable $\gamma \in \Gamma$ that are not rectifiable.

Case I: γ is bounded.

Let $a = \inf_{x \in |\gamma|} \{\rho_0(x)\} > 0$. Hence,

$$\int_\gamma \rho_0 ds \geq \int_\gamma a ds = a \cdot \ell(\gamma) = \infty.$$

Case II: γ is unbounded.

Pick $x \in |\gamma|$ such that $|x| \geq 2$. We then have

$$\begin{aligned}
\int_\gamma \rho_0 ds &\geq \int_{|x|}^\infty \frac{1}{t^{\frac{n}{p}} \ln t} dt \geq \int_{|x|}^\infty \frac{1}{t \ln t} dt \\
&= \ln \ln t \Big|_{|x|}^\infty = \infty.
\end{aligned}$$

Now let $\rho \in F_r(\Gamma)$. For all $\varepsilon > 0$, we set $\rho_\varepsilon = (\rho^p + \varepsilon^p \rho_0^p)^{\frac{1}{p}}$. We want to show that $\rho_\varepsilon \in F(\Gamma)$.

Case A: γ is rectifiable.

$$\int_{\gamma} \rho_{\varepsilon} ds = \int_{\gamma} (\rho^p + \varepsilon^p \rho_0^p)^{\frac{1}{p}} ds > \int_{\gamma} \rho \geq 1.$$

Case B: γ is not rectifiable.

$$\int_{\gamma} \rho_{\varepsilon} ds \geq \int_{\gamma} \varepsilon \cdot \rho_0 ds = \infty.$$

Hence, $\rho_{\varepsilon} \in F(\Gamma)$, which leads us to:

$$M_p(\Gamma) \leq \int_{\mathbb{R}^n} \rho_{\varepsilon}^p dm = \int_{\mathbb{R}^n} \rho^p dm + \varepsilon^p \cdot \int_{\mathbb{R}^n} \rho_0^p dm.$$

Letting $\varepsilon \rightarrow 0$, we obtain

$$M_p(\Gamma) \leq \int_{\mathbb{R}^n} \rho^p dm.$$

The theorem follows by taking the infimum over all $\rho \in F_r(\Gamma)$. ■

Corollary 2.2.9 *If Γ_r is the family of all rectifiable curves in Γ , then $M_p(\Gamma \setminus \Gamma_r) = 0$, when $p \geq n$.*

2.2.3 Upper and Lower Bounds for the p -Modulus

Given a curve family Γ in $\widehat{\mathbb{R}^n}$, because we are taking the infimum, generally it is only possible to find an upper bound. Precisely, picking $\rho \in F(\Gamma)$ produces

$$M_p(\Gamma) \leq \int \rho^p dm$$

as demonstrated by the next Example.

Theorem 2.2.10 *Let $p \geq 0$. Let Γ be a curve family such that if $\gamma \in \Gamma$, then γ is contained in a Borel set $G \subset \widehat{\mathbb{R}^n}$ and $\ell(\gamma) \geq r > 0$ for all $\gamma \in \Gamma$, where γ is locally*

rectifiable. Then,

$$M_p(\Gamma) \leq \frac{m(G)}{r^p}.$$

Proof. Define $\rho : \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$\rho(x) = \begin{cases} \frac{1}{r} & x \in G, \\ 0 & x \notin G. \end{cases}$$

We compute that

$$\int_{\gamma} \rho \, ds = \frac{1}{r} \ell(\gamma) \geq \frac{1}{r} \cdot r = 1.$$

Hence $\rho \in F(\Gamma)$, which leads to

$$\int_{\mathbb{R}^n} \rho^p = \int_G \frac{1}{r^p} = \frac{m(G)}{r^p} \geq M_p(\Gamma) \geq 0.$$

■

In Theorem 2.2.10 we established an upper bound for the p -modulus of Γ , for a specific curve family Γ . The task of finding a lower bound is unfortunately much more difficult. To find a lower bound, we must choose an arbitrary $\rho \in F(\Gamma)$ and for all $\gamma \in \Gamma$, find a fixed constant K so that $\int_{\gamma} \rho^p \geq K$. This is usually done by applying Hölder's inequality and Fubini's Theorem. We use this technique in the next three Examples.

Example 2.2.11 The Cylinder

Let E be a Borel set in \mathbb{R}^{n-1} and let $h > 0$ and $p \geq 0$. Define

$$G = \{x \in \mathbb{R}^n \mid (x_1, \dots, x_{n-1}) \in E \text{ and } 0 < x_n < h\}.$$

Hence, G is a cylinder with bases E and $F = E + he_n$. Define the curve family $\Gamma = \{\gamma \subset G : \text{closed } \gamma \text{ joins } E \text{ to } F\}$. For all $\gamma \in \Gamma$ we have $\ell(\gamma) \geq h$. Now by Theorem 2.2.10 we have

$$M_p(\Gamma) \leq \frac{m(G)}{h^p}.$$

For each $y \in E$ let $\gamma_y : [0, h] \rightarrow G$ be a vertical path such that

$$\gamma_y(t) = y + te_n$$

which implies $\gamma_y \in \Gamma$.

Let $\rho \in F(\Gamma)$, implying for all locally rectifiable $\gamma \in \Gamma$

$$\int_{\gamma} \rho \, ds \geq 1.$$

If $p > 1$ we have

$$\left(\int_{\gamma} \rho \, ds \right)^p \geq 1$$

and using Hölder's inequality we have,

$$1 \leq \int_{\gamma} \rho \, ds = \int_{\gamma} \rho \cdot 1 \, ds \leq \left(\int_{\gamma} \rho^p \, ds \right)^{\frac{1}{p}} \left(\int_{\gamma} 1^{\frac{p}{p-1}} \, ds \right)^{\frac{p-1}{p}}.$$

Therefore, for $1 \leq p$ we have

$$\begin{aligned} 1 &\leq \left(\int_{\gamma} \rho^p \, ds \right) \left(\int_{\gamma} ds \right)^{p-1} \\ &= \int_{\gamma} \rho^p \, ds \cdot h^{p-1} \\ &= \left(\int_0^h \rho(y + te_n)^p \, dt \right) \cdot h^{p-1}, \end{aligned} \tag{2.2.2}$$

since for each $y \in E$, we have $\gamma_y : [0, h] \rightarrow G$, which sends each $t \in [0, h]$ to $y + te_n$.

Now, integrate both sides of Equation (2.2.2) over E . Hence, we obtain

$$\int_E 1 \, dm_{n-1} \leq \int_E h^{p-1} \int_0^h \rho(y + te_n)^p \, dt \, dm_{n-1},$$

which is equivalent to

$$m_{n-1}(E) \leq h^{p-1} \int_E \int_0^h \rho(y + te_n)^p \, dt \, dm_{n-1},$$

which is again equivalent to

$$\begin{aligned}
\frac{m_{n-1}(E)}{h^{p-1}} &\leq \int_E \int_0^h \rho(y + te_n)^p dt dm_{n-1} \\
&= \int_G \rho^p dm_n \quad (\text{Fubini's Theorem}) \\
&\leq \int_{\mathbb{R}^n} \rho^p dm_n.
\end{aligned}$$

Taking the infimum over all $\rho \in F(\Gamma)$, we obtain

$$\frac{m_{n-1}(E)}{h^{p-1}} \leq M_p(\Gamma).$$

Hence,

$$\frac{m_{n-1}(E)}{h^{p-1}} \leq M_p(\Gamma) \leq \frac{m_n(G)}{h^p} = \frac{m_{n-1}(E)}{h^{p-1}}.$$

Example 2.2.12 The Spherical Ring

Let $0 < a < b < \infty$. Define the domain $A = B^n(b) \setminus \overline{B^n(a)}$. We will call A a spherical ring. Set $E = S^{n-1}(a)$, $F = S^{n-1}(b)$, and $\Gamma_A = \{\gamma \subset A : \text{closed } \gamma \text{ joins } E \text{ to } F\}$.

We choose $\rho \in F(\Gamma_A)$. For each unit vector $y \in S^{n-1}$, define the radial function $\gamma_y : [a, b] \rightarrow \mathbb{R}^n$ by $\gamma_y(t) = ty$. We note that γ_y is a locally rectifiable absolutely continuous curve in Γ_A and since $\rho \in F(\Gamma_A)$, we have

$$\int_{\gamma_y} \rho ds \geq 1.$$

For $1 < p \leq n$ we obtain,

$$\begin{aligned}
1 &\leq \left(\int_{\gamma_y} \rho ds \right)^p = \left(\int_a^b \rho(ty) dt \right)^p \\
&= \left(\int_a^b \rho(ty) t^{\frac{n-1}{p}} t^{\frac{1-n}{p}} dt \right)^p \\
&\leq \left(\int_a^b \rho^p(ty) t^{n-1} dt \right) \cdot \left(\int_a^b t^{\frac{1-n}{p-1}} dt \right)^{p-1}.
\end{aligned}$$

Letting $q = \frac{p-n}{p-1}$ and $c = \int_a^b t^{q-1} dt$, we have

$$\begin{aligned} 1 &\leq \left(\int_a^b (\rho^p(ty)t^{n-1} dt) \right) \cdot \left(\int_a^b t^{q-1} dt \right)^{p-1} \\ &= c^{p-1} \cdot \int_a^b (\rho^p(ty))t^{n-1} dt. \end{aligned} \quad (2.2.3)$$

Therefore,

$$1 \leq c^{p-1} \cdot \int_a^b \rho^p(ty)t^{n-1} dt.$$

Note:

$$c = \begin{cases} \int_a^b t^{\frac{1-n}{p-1}} dt & p \neq n, \\ \left(\ln \frac{b}{a} \right) & p = n \end{cases}$$

and so when $p \neq n$ then we have

$$\begin{aligned} c &= \int_a^b t^{\frac{1-n}{p-1}} dt = \left(\frac{p-1}{p-n} \right) t^{\frac{p-n}{p-1}} \Big|_a^b \\ &= \frac{p-1}{p-n} \left(b^{\frac{p-n}{p-1}} - a^{\frac{p-n}{p-1}} \right). \end{aligned} \quad (2.2.4)$$

Integrating Equation (2.2.3) over $y \in S^{n-1}$ with respect to surface area yields

$$\int_{y \in S^{n-1}} 1 dm_{n-1} \leq \int_{y \in S^{n-1}} c^{p-1} \int_a^b \rho^p(ty)t^{n-1} dt dm_{n-1}.$$

We obtain

$$\begin{aligned} \omega_{n-1} &\leq c^{p-1} \cdot \int_{S^{n-1}} \int_a^b \rho^p(ty)t^{n-1} dt dm_{n-1} \\ &= c^{p-1} \cdot \int_a^b \int_{S^{n-1}} \rho^p(ty)t^{n-1} dt dm_{n-1} \\ &= c^{p-1} \cdot \int_A \rho^p dm \leq c^{p-1} \cdot \int_{\mathbb{R}^n} \rho^p dm. \end{aligned}$$

Taking the infimum implies

$$\omega_{n-1} \leq c^{p-1} \cdot M_p(\Gamma_A).$$

Now we will show the reverse inequality. Define $\rho \in F(\Gamma_A)$ by

$$\rho(x) = \begin{cases} c^{-1}|x|^{q-1} & x \in A \\ 0 & x \notin A. \end{cases}$$

Now for all locally rectifiable $\gamma \in \Gamma_A$

$$\int_{\gamma} \rho \, ds \geq \int_a^b \rho(t) \, dt = \frac{1}{c} \int_a^b t^{q-1} dt = 1.$$

Therefore, $\rho \in F(\Gamma_A)$. Now consider,

$$\begin{aligned} M_p(\Gamma_A) &\leq \int_{\mathbb{R}^n} \rho^p \, dm = \int_a^b \left(\int_{S^{n-1}} \rho^p(ty) \, dm_{n-1} \right) t^{n-1} \, dt \\ &= c^{-p} \int_a^b \left(\int_{S^{n-1}} t^{p(q-1)} \, dm_{n-1} \right) t^{n-1} \, dt \\ &= c^{-p} \cdot \omega_{n-1} \int_a^b t^{pq-p+n-1} \, dt \\ &= c^{-p} \cdot \omega_{n-1} \int_a^b t^{q-1} \, dt \\ &= c^{1-p} \cdot \omega_{n-1}. \end{aligned}$$

We then conclude

$$\omega_{n-1} \cdot c^{1-p} = M_p(\Gamma_A).$$

Example 2.2.13 The Degenerate Ring

Let $0 < a < b < \infty$ and $\Gamma = \{\gamma \subset G : \text{closed } \gamma \text{ joins } E \text{ to } F\}$, where

$$\begin{aligned} E &= \{0\} \\ F &= S^{n-1}(b) \\ A &= B^n(b) \setminus \{0\}. \end{aligned}$$

Also, define Γ_A as we did in Example 2.2.12. We know that $\Gamma_A \prec \Gamma$, therefore by Example 2.2.12 and Theorem 2.2.4

$$\begin{aligned} 0 \leq M(\Gamma) \leq M(\Gamma_A) &= \omega_{n-1} \cdot \left(\ln \frac{b}{a}\right)^{1-n} && \text{if } p = n, \\ 0 \leq M_p(\Gamma) \leq M_p(\Gamma_A) &= \omega_{n-1} \left(\int_a^b t^{\frac{1-n}{p-1}} dt\right)^{1-p} && \text{if } p \neq n. \\ &= \omega_{n-1} \left[\frac{p-1}{p-n} \left(b^{\frac{p-n}{p-1}} - a^{\frac{p-n}{p-1}}\right)\right]^{1-p}. \end{aligned}$$

We note that $p - n < 0$, therefore letting a approach 0 in either case leads to $M_p(\Gamma) = 0$.

Example 2.2.14 Paths through a point

Let $1 < p \leq n$. Let $x_0 \in \widehat{\mathbb{R}}^n$ and let Γ be the family of all non-constant paths γ such that $x_0 \in |\gamma|$. If $x_0 = \infty$ then $M_p(\Gamma) = 0$, since every ρ that is not identically zero satisfies $\rho \in F(\Gamma)$. Now if $x_0 \neq \infty$, define the path family $\Gamma_k = \{\gamma \in \Gamma : S(x_0, k) \cap |\gamma| \neq \emptyset\}$. Hence $M_p(\Gamma_k) = 0$ by Example 2.2.13 and Theorem 2.2.4. Now consider $\Gamma = \bigcup_k \Gamma_k$. Since the p -modulus is an outer measure this implies that

$$0 \leq M_p(\Gamma) \leq \sum_k M_p(\Gamma_k) = 0.$$

Therefore for any given $x_0 \in \widehat{\mathbb{R}}^n$ almost every non-constant path avoids x_0 .

As a consequence of the previous Example, we have the following theorem, which states that the p -modulus is unchanged when considering open or closed paths.

Theorem 2.2.15 *Let $p \geq n$ and let E, F and G be Borel sets.*

$$M_p(\{\gamma \subset G : \text{open } \gamma \text{ joins } E \text{ to } F\}) = M_p(\{\gamma \subset G : \text{closed } \gamma \text{ joins } E \text{ to } F\}).$$

Proof. Let $\Gamma = \{\gamma \subset G : \text{open } \gamma \text{ joins } E \text{ to } F\}$ and $\Gamma_0 = \{\gamma \subset G : \text{closed } \gamma \text{ joins } E \text{ to } F\}$. Since, $\Gamma_0 \prec \Gamma$ we have by Theorem 2.2.4 $M_p(\Gamma_0) \geq M_p(\Gamma)$. To show the reverse inequality, it suffices to show $F(\Gamma) \subset F_r(\Gamma_0)$ since

$$M_p(\Gamma) = \inf_{\rho \in F_r(\Gamma)} \int \rho^p dm.$$

Let $\rho \in F(\Gamma)$, let γ be a rectifiable path in Γ_0 , and let γ_* be the closed extension of γ given by Theorem 2.1.9. Hence, $|\gamma_*| = |\bar{\gamma}|$ touches both E and F . Therefore, there exists t_1 and t_2 such that $\gamma_*(t_1) \in E$ and $\gamma_*(t_2) \in F$; we can take $t_1 \leq t_2$. Now define $\beta = \gamma_*|_{(t_1, t_2)} \in \Gamma$. Hence,

$$\int_{\gamma} \rho ds = \int_{\gamma_*} \rho ds \geq \int_{\beta} \rho ds \geq 1.$$

Hence, $\rho \in F_r(\Gamma_0)$. ■

2.2.4 The Modulus of Conformal Mappings

Let $A \in \widehat{\mathbb{R}^n}$ and suppose $f : A \rightarrow \widehat{\mathbb{R}^n}$ is a continuous map. If Γ is a family of paths in A , then we set $\Gamma' = \{f \circ \gamma : \gamma \in \Gamma\}$ and call Γ' the image of Γ under f .

Theorem 2.2.16 *If $f : D \rightarrow D'$ is conformal, then $M(\Gamma') = M(\Gamma)$, for all $\Gamma \subset D$.*

Proof. Let $\rho' \in F(\Gamma')$ and define $\rho(x) = \rho'(f(x)) \cdot |f'(x)|$. We have $\rho \in F(\Gamma)$, since for all locally rectifiable $\gamma \in \Gamma$

$$\begin{aligned} \int_{\gamma} \rho ds &= \int_{\gamma} \rho'(f(x)) \cdot |f'(x)| ds \\ &= \int_{f \circ \gamma} \rho'(x) ds \geq 1. \end{aligned}$$

Now since $\rho \in F(\Gamma)$, this leads us to:

$$\begin{aligned}
M(\Gamma) &\leq \int_D \rho^n \, dm \\
&= \int_D (\rho'(f(x)) \cdot |f'(x)|)^n \, dm \\
&= \int_D \rho'(f(x))^n \cdot |f'(x)|^n \, dm \quad (\text{since } f \text{ is conformal}) \\
&= \int_D \rho'(f(x))^n \cdot J(x, f) \, dm \\
&= \int_{D'} \rho'(f(x))^n \, dm \leq \int_{\mathbb{R}^n} \rho'^n \, dm.
\end{aligned}$$

Now taking the infimum, we obtain $M(\Gamma) \leq M(\Gamma')$.

We recall that f^{-1} is conformal if f is conformal. Hence $M(\Gamma') \leq M(\Gamma)$ and so,

$$M(\Gamma') = M(\Gamma).$$

■

It should be noted that the p -modulus is not a conformal invariant if $p \neq n$. In particular, we introduce a formula that shows what will happen to the p -modulus in a conformal linear map.

Theorem 2.2.17 *Let $c > 0$ and define $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $f(x) = cx$. Denote the image of a path family $\Gamma \in \mathbb{R}^n$ under f by $c\Gamma$. Then*

$$M_p(c\Gamma) = c^{n-p} M_p(\Gamma)$$

Proof. $f(x) = cx$ implies that $|f'(x)| = c$ and since f is a conformal linear map we note that

$$\ell_f(x) = L_f(x) = |f'(x)| = c.$$

Then, $|J(x, f)| = c^n$ for all $x \in \mathbb{R}^n$. By Theorem 2.2.16 it suffices to show

$$c^{n-p}M_p(\Gamma) \leq M_p(c\Gamma) \quad \text{or} \quad M_p(\Gamma) \leq c^{p-n}M_p(c\Gamma).$$

Let $\hat{\rho} \in F(c\Gamma)$ and define $\rho(x) = c\hat{\rho}(f(x))$. Hence,

$$1 \leq \int_{f \circ \gamma} \hat{\rho} \, ds \leq \int_{\gamma} (\hat{\rho} \circ f) \cdot |f'(x)| \, ds = \int_{\gamma} (\hat{\rho} \circ f) c \, ds = \int_{\gamma} \rho \, ds.$$

This implies that $\rho \in F(\Gamma)$, and so

$$\begin{aligned} M_p(\Gamma) &\leq \int_{\mathbb{R}^n} \rho^p \, dm = \int_{\mathbb{R}^n} c^p (\hat{\rho} \circ f)^p \, dm \\ &= c^{p-n} \int_{\mathbb{R}^n} (\hat{\rho} \circ f)^p c^n \, dm = c^{p-n} \int_{\mathbb{R}^n} (\hat{\rho} \circ f)^p \cdot |J(x, f)| \, dm \\ &= c^{p-n} \int_{\mathbb{R}^n} \hat{\rho}^p \, dm. \end{aligned}$$

Now by taking the infimum over all $\rho \in F(\Gamma)$, we obtain:

$$M_p(\Gamma) \leq c^{p-n}M_p(c\Gamma).$$

■

3 QUASICONFORMAL MAPPINGS

In this chapter we will give three different definitions of quasiconformal maps discussed by J. Väisälä [V]. We will then explore different properties of quasiconformal maps. Lastly, we will demonstrate the equivalence of the three definitions.

3.1 The Geometric Definition of Quasiconformality and Properties

3.1.1 The Dilatation of a Homeomorphism

Let $f : D \rightarrow D'$ be a homeomorphism and let Γ be a curve family in D . Now define the quantities:

$$K_I(f) = \sup \frac{M(\Gamma')}{M(\Gamma)} \quad \text{and} \quad K_O(f) = \sup \frac{M(\Gamma)}{M(\Gamma')}$$

where $\Gamma' = \{f \circ \gamma : \gamma \in \Gamma\}$ and suprema are taken over all $\gamma \subset \Gamma$ such that $M(\Gamma)$ and $M(\Gamma')$ are not both 0 or ∞ . We say $K_I(f)$ is the inner dilatation of f , $K_O(f)$ is the outer dilatation of f , and $K(f) = \max\{K_O(f), K_I(f)\}$ is the maximal dilatation of f . We note that $K_I \geq 1$ or $K_O \geq 1$, hence $K \geq 1$.

Definition 3.1.1 *If $K(f) = K < \infty$, we say f is K -quasiconformal (or K -qc). The map f is K -qc if and only if*

$$\frac{M(\Gamma)}{K} \leq M(\Gamma') \leq KM(\Gamma)$$

for all $\Gamma \subset D$.

We note by Theorem 2.2.16, f is conformal if and only if $M(\Gamma) = M(\Gamma')$, which implies that $K=1$, in Definition 3.1.1.

Theorem 3.1.2 *Let $f : D \rightarrow D'$ be a homeomorphism. The following properties hold for all $x \in D$:*

$$1) \quad K_I(f) = K_O(f^{-1}), \quad (3.1.1)$$

$$2) \quad K_O(f) = K_I(f^{-1}), \quad (3.1.2)$$

$$3) \quad K(f^{-1}) = K(f), \quad (3.1.3)$$

$$4) \quad K_I(f \circ g) \leq K_I(f) \cdot K_I(g), \quad (3.1.4)$$

$$5) \quad K_O(f \circ g) \leq K_O(f) \cdot K_O(g), \quad (3.1.5)$$

$$6) \quad K(f \circ g) \leq K(f)K(g). \quad (3.1.6)$$

Proof. Let Γ be a family of paths in D . By the definition of Γ' , the results are clear if any of the dilatations are infinite. Hence, we will assume K_O and K_I to be finite. Recall: $f : D \rightarrow D'$, $\Gamma \subset D$, and $\Gamma' = f\Gamma \subset D'$. To show Relation (3.1.1) we note

$$\frac{M(\Gamma')}{M(\Gamma)} = \frac{M(\Gamma')}{M(f^{-1}\Gamma')} \leq K_O(f^{-1})$$

and by taking the supremum over all Γ we obtain $K_I(f) \leq K_O(f^{-1})$. Interchanging the roles of Γ and Γ' we obtain Relation (3.1.2).

Relation (3.1.3) follows from Relations (3.1.1) and (3.1.2). Now looking at $K_I(f \circ g)$ we see that:

$$\frac{M(\Gamma')}{M(\Gamma)} = \frac{M((f \circ g)\Gamma)}{M(\Gamma)} = \frac{M((f \circ g)\Gamma)}{M(g\Gamma)} \cdot \frac{M(g\Gamma)}{M(\Gamma)} \leq K_I(f)K_I(g).$$

Taking the supremum over all Γ gives us Relation (3.1.4). Relation (3.1.5) follows in a similar fashion. Relations (3.1.4) and (3.1.5) together gives us

$$K_I(f \circ g) \leq K_I(f) \max\{K_O(g), K_I(g)\},$$

hence, $K_I(f \circ g) \leq K_I(f)K(g)$. Similarly, $K_O(f \circ g) \leq K_O(f)K(g)$. Therefore

$$\begin{aligned} K_O(f \circ g) &\leq K(f)K(g), \\ K_I(f \circ g) &\leq K(f)K(g), \end{aligned}$$

and so $K(f \circ g) \leq K(f)K(g)$. ■

Corollary 3.1.3 *If f is K -qc, then f^{-1} is K -qc.*

Corollary 3.1.4 *If $h = f \circ g$, where f is K_1 -qc and g is K_2 -qc, then h is K_1K_2 -qc.*

3.1.2 The Dilatation of a Linear Map

Let $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear bijection, we define the following quantities:

$$H_I(A) = \frac{|\det A|}{\ell(A)^n}, \quad H_O(A) = \frac{|A|^n}{|\det A|}, \quad \text{and} \quad H(A) = \frac{|A|}{\ell(A)}. \quad (3.1.7)$$

We say the quantities H_I , H_O , and H are the *inner*, *outer*, and *linear dilatations* of A , respectively. In the next section we will show that $H_I(A) = K_I(A)$ and $H_O(A) = K_O(A)$, which explains the terminology.

In geometric terms, $H(A)$ measures the “out of roundness” or eccentricity of the ellipsoid $E(A)$, see Figure 3.1 [IM], while $H_I(A)$ and $H_O(A)$ relate the volume of $E(B^n)$ to the volumes of the inscribed and circumscribed balls centered about $E(A)$.

Observe that

$$H_I(A) = \frac{m(E(A))}{m(B_I(A))}, \quad H_O(A) = \frac{m(B_O(A))}{m(E(A))}$$

and $H(A)$ is the ratio of the greatest and smallest semi-axis of $E(A)$. See Figure 3.2 [IM]. [Here $a_1 \geq a_2 \geq \dots \geq a_n$ are the semi-axis of $E(A)$.]

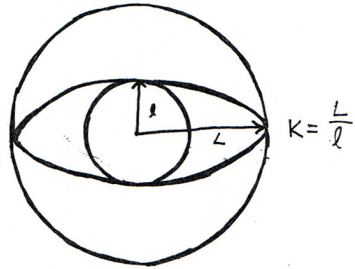


Figure 3.1: Eccentricity of the linear dilatation

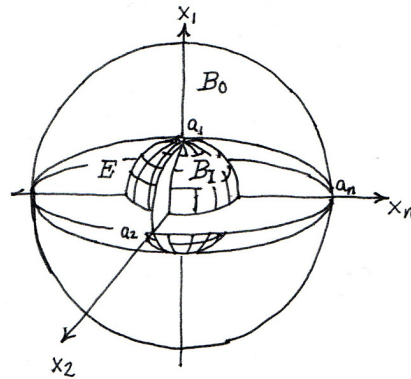


Figure 3.2: Inscribed and circumscribed balls of $E(A)$

We recall that the numbers a_i are the positive square roots of the eigenvalues of A^*A , where A^* is the adjoint of A . We also recall

$$a_1 = |A|, \quad a_n = \ell(A), \quad \text{and} \quad \det(A) = a_1 a_2 \cdots a_n.$$

By the definition of the dilatations we can write

$$H_O(A) = \frac{a_1^{n-1}}{a_2 \cdots a_n}, \quad H_I(A) = \frac{a_1 \cdots a_{n-1}}{a_n^{n-1}}, \quad \text{and} \quad H(A) = \frac{a_1}{a_n}. \quad (3.1.8)$$

Theorem 3.1.5 *If $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a linear bijection, then*

- 1) $H_I(A) \leq H_O(A)^{n-1}$,
- 2) $H_O(A) \leq H_I(A)^{n-1}$,
- 3) $H(A)^n = H_I(A) \cdot H_O(A)$,
- 4) $H(A) \leq \min\{H_I(A), H_O(A)\}$
 $\leq H(A)^{\frac{n}{2}} \leq \max\{H_I(A), H_O(A)\} \leq H(A)^{n-1}$.

Proof. Let $a_1 \geq a_2 \geq \cdots \geq a_n$ be the semi-axis of $E(A)$.

1) We have $a_1 \cdots a_{n-1} \leq a_1^{n-1}$. From this we obtain:

$$\begin{aligned} a_1 \cdots a_{n-1} &\leq a_1^{n-1} \\ a_2 \cdots a_{n-1} &\leq a_1^{n-2} \\ (a_2 \cdots a_{n-1})^n &\leq (a_1^{n-2})^n = a_1^{n^2-2n} \\ a_1(a_2 \cdots a_{n-1})^n a_n^{n-1} &\leq a_1 \cdot a_1^{n^2-2n} \cdot a_n^{n-1} \\ a_1(a_2 \cdots a_{n-1})^n a_n^{n-1} &\leq a_1^{n^2-2n+1} a_n^{n-1} = (a_1^{n-1})^{n-1} \cdot a_n^{n-1} \\ (a_1 \cdots a_{n-1})(a_2 \cdots a_n)^{n-1} &\leq (a_1^{n-1})^{n-1} \cdot a_n^{n-1} \\ \frac{a_1 \cdots a_{n-1}}{a_n^{n-1}} &\leq \left(\frac{a_1^{n-1}}{a_2 \cdots a_n} \right)^{n-1} \\ H_I(A) &\leq H_O(A)^{n-1}. \end{aligned}$$

2) We have $a_n^{n-1} \leq a_2 \cdots a_n$. From this we obtain:

$$\begin{aligned}
a_n^{n-2} &\leq a_2 \cdots a_{n-1} \\
(a_n^{n-2})^n &\leq (a_2 \cdots a_{n-1})^n \\
a_1^{n-1} (a_n^{n^2-2n}) a_n &\leq a_1^{n-1} (a_2 \cdots a_{n-1})^n a_n \\
a_1^{n-1} \cdot a_n^{n^2-2n+1} &\leq (a_2 \cdots a_n) (a_1 \cdots a_{n-1})^{n-1} \\
a_1^{n-1} (a_n^{n-1})^{n-1} &\leq (a_2 \cdots a_n) (a_1 \cdots a_{n-1})^{n-1} \\
\frac{a_1^{n-1}}{a_2 \cdots a_n} &\leq \frac{(a_1 \cdots a_{n-1})^{n-1}}{(a_n^{n-1})^{n-1}} \\
\frac{a_1^{n-1}}{a_2 \cdots a_n} &\leq \left(\frac{a_1 \cdots a_{n-1}}{a_n^{n-1}} \right)^{n-1} \\
H_O(A) &\leq H_I(A)^{n-1}.
\end{aligned}$$

3)

$$\begin{aligned}
H(A)^n &= \left(\frac{a_1}{a_n} \right)^n = \frac{a_1^n}{a_n^n} \\
&= \frac{a_1^n}{a_n^n} \cdot \left(\frac{a_2 \cdots a_{n-1}}{a_2 \cdots a_{n-1}} \right) \\
&= \frac{(a_1 \cdots a_{n-1})}{a_n^{n-1}} \cdot \frac{a_1^{n-1}}{a_2 \cdots a_n} \\
&= H_I(A) \cdot H_O(A).
\end{aligned}$$

4) $H(A) \leq \min\{H_I(A), H_O(A)\}$.

We will proceed in parts:

Case I: $H(A) \leq H_I(A)$.

We have

$$\begin{aligned}
a_n^{n-1} &\leq a_2 \cdots a_n \\
a_1 \cdot a_n^{n-1} &\leq a_1 \cdots a_n = (a_1 \cdots a_{n-1}) a_n \\
\frac{a_1}{a_n} &\leq \frac{a_1 \cdots a_{n-1}}{a_n^{n-1}} \\
H(A) &\leq H_I(A).
\end{aligned}$$

Case II: $H(A) \leq H_O(A)$.

We have

$$\begin{aligned} a_1 \cdots a_{n-1} &\leq a_1^{n-1} \\ a_1 \cdots a_{n-1} \cdot a_n &\leq a_1^{n-1} \cdot a_n \\ \frac{a_1}{a_n} &\leq \frac{a_1^{n-1}}{a_2 \cdots a_n} \\ H(A) &\leq H_O(A). \end{aligned}$$

We then conclude $H(A) \leq \min\{H_I(A), H_O(A)\}$.

We next show $\max\{H_I(A), H_O(A)\} \leq H(A)^{n-1}$.

Case I: $H_I(A) \leq H(A)^{n-1}$.

$$\begin{aligned} a_1 \cdots a_{n-1} &\leq a_1^{n-1} \\ \frac{a_1 \cdots a_{n-1}}{a_n^{n-1}} &\leq \frac{a_1^{n-1}}{a_n^{n-1}} = \left(\frac{a_1}{a_n}\right)^{n-1} \\ H_I(A) &\leq H(A)^{n-1}. \end{aligned}$$

Case II: $H_O(A) \leq H(A)^{n-1}$.

$$\begin{aligned} a_n^{n-1} &\leq a_2 \cdots a_n \\ a_1^{n-1} a_n^{n-1} &\leq a_1^{n-1} a_2 \cdots a_n \\ \frac{a_1^{n-1}}{a_2 \cdots a_n} &\leq \frac{a_1^{n-1}}{a_n^{n-1}} = \left(\frac{a_1}{a_n}\right)^{n-1} \\ H_O(A) &\leq H(A)^{n-1}. \end{aligned}$$

We then conclude $\max\{H_I(A), H_O(A)\} \leq H(A)^{n-1}$.

It remains to show

$$\min\{H_I(A), H_O(A)\} \leq H(A)^{\frac{n}{2}} \leq \max\{H_I(A), H_O(A)\}.$$

This follows easily from part 3, which implies

$$\min\{H_I(A), H_O(A)\}^2 \leq H(A)^n \leq \max\{H_I(A), H_O(A)\}^2.$$

■

Example 3.1.6 If the linear transform A is given by a matrix, it is usually a difficult task to compute the dilatations of A . However when $n = 2$, this is accomplished by a easily obtained formula. Let $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear bijection, now using complex notation in \mathbb{R}^2 define the map $Az = ax + by + i(cx + dy)$, where a, b, c, d are real numbers such that $ad - bc \neq 0$. Since $n = 2$, $H(A) = H_I(A) = H_O(A)$. Let $H = H(A)$, we now compute H in terms of a, b, c, d . Let

$$A = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

be the matrix representation of A . Then

$$A^T = \begin{vmatrix} a & c \\ b & d \end{vmatrix}, \text{ and } A^T A = \begin{vmatrix} a & c \\ b & d \end{vmatrix} \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{vmatrix}.$$

Now

$$\begin{aligned} \det(\lambda I - A^T A) &= \det \begin{vmatrix} \lambda - (a^2 + c^2) & -ab - cd \\ -ab - cd & \lambda - (b^2 + d^2) \end{vmatrix} \\ &= (\lambda - (a^2 + c^2))(\lambda - (b^2 + d^2)) - (-ab - cd)(-ab - cd) \\ &= \lambda^2 - \lambda(a^2 + b^2 + c^2 + d^2) + (ad - bc)^2. \end{aligned}$$

Now let λ_1 and λ_2 be solutions to $\det(\lambda I - A^T A) = 0$. Hence,

$$\begin{aligned}\lambda_1 + \lambda_2 &= a^2 + b^2 + c^2 + d^2, \\ \lambda_1 \lambda_2 &= (ad - bc)^2, \\ H &= \sqrt{\frac{\lambda_1}{\lambda_2}},\end{aligned}$$

and

$$H + \frac{1}{H} = \sqrt{\frac{\lambda_1}{\lambda_2}} + \sqrt{\frac{\lambda_2}{\lambda_1}} = \frac{\lambda_1 + \lambda_2}{\sqrt{\lambda_1 \lambda_2}} = \frac{a^2 + b^2 + c^2 + d^2}{|ad - bc|}.$$

Therefore H is the greater root of this equation, the other root is H^{-1} .

3.1.3 Quasiconformal Diffeomorphisms

In this section we will show that the dilatations of a diffeomorphism can be derived in terms of its derivative. Consider a diffeomorphism $f : D \rightarrow D'$, where D and D' are domains in \mathbb{R}^n . We recall that a diffeomorphism is a C^1 homeomorphism, whose Jacobian does not vanish. That is, $J(x, f) \neq 0$ for all $x \in D$. We note that if f is a diffeomorphism, by Equation (3.1.7) we have:

$$H_O(f'(x)) = \frac{|f'(x)|^n}{|J(x, f)|} \quad \text{and} \quad H_I(f'(x)) = \frac{|J(x, f)|}{\ell(f'(x))^n}.$$

Theorem 3.1.7 *Let $f : D \rightarrow D'$ be a homeomorphism. If f is differentiable at the point $a \in D$ and if $K_O(f) < \infty$, then*

$$|f'(a)|^n \leq K_O(f) |J(a, f)|.$$

Proof. Without loss of generality we may take $a = 0 = f(a)$, since if not, we may consider the map $g : \{x \in \mathbb{R}^n : x + a \in D\} \rightarrow D'$ defined by $g(x) = f(x + a) - f(a)$. Therefore, $g'(0) = f'(a)$. So by Theorem 3.1.2, we have $K_O(f(a)) = K_O(g(0))$. We

may also take $f'(0)$ to be of the form

$$f'(0)x = (a_1x_1, \dots, a_nx_n)^T,$$

where $a_1 \geq a_2 \geq \dots \geq a_n$. Hence,

$$|f'(0)| = a_1, \quad \text{and } |J(0, f)| = a_1 \cdots a_n.$$

We then need to verify

$$a_1^n \leq K_O(f) \cdot a_1 \cdots a_n.$$

The result is clear when $a_1 = 0$, hence we take $a_1 > 0$. Now, for all $x \in D$ we can write

$$f(x) = f'(0)x + |x|\varepsilon(x)$$

where $\lim_{x \rightarrow 0} \varepsilon(x) = 0$. Now, let $\varepsilon \in (0, \frac{a_1}{2})$ and choose $\delta > 0$ such that the n -interval

$$Q = [0, \delta] \times [0, \delta] \times \cdots \times [0, \delta]$$

is contained D and such that $|\varepsilon(x)| \leq n^{-\frac{1}{2}}\varepsilon$ holds for all $x \in Q$. So,

$$|f(x) - f'(0)x| = |x| \cdot |\varepsilon(x)| < \varepsilon\delta. \tag{3.1.9}$$

Now, let E and F be the faces of Q on which $x_0 = 0$ and $x_1 = \delta$. Define the curve family

$$\Gamma = \{\gamma \subset Q^\circ : \gamma \text{ joins } E \text{ and } F\}.$$

By Example 2.2.11, $M(\Gamma) = 1$. We next estimate $M(\Gamma')$. By Equation (3.1.9) we have

$$f(Q) \subset G = \{x : -\varepsilon\delta \leq x_i \leq (a_i + \varepsilon)\delta\}$$

and by Theorem 2.2.10

$$M(\Gamma') \leq (a_1 + 2\varepsilon) \cdots (a_n + 2\varepsilon)(a_1 - 2\varepsilon)^{-n}.$$

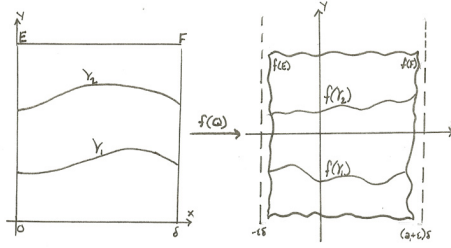


Figure 3.3: The cube

Combining the estimates of $M(\Gamma)$ and $M(\Gamma')$, we have

$$(a_1 - 2\varepsilon)^n \leq \frac{M(\Gamma)}{M(\Gamma')} (a_1 + 2\varepsilon) \cdots (a_n + 2\varepsilon).$$

Letting $\varepsilon \rightarrow 0$ yields

$$a_1^n \leq \frac{M(\Gamma)}{M(\Gamma')} (a_1 \cdots a_n),$$

or

$$a_1^n \leq K_O(f) a_1 \cdots a_n.$$

■

An application of Theorem 3.1.7 is the following:

Theorem 3.1.8 *Suppose $f'(a) = 0$ whenever $J(a, f) = 0$. If $f : D \rightarrow D'$ is a diffeomorphism, then*

$$K_I(f) = \sup_{x \in D} H_I(f'(x)), \quad \text{and} \quad K_O(f) = \sup_{x \in D} H_O(f'(x)).$$

Proof. The first equation follows from the second by applying the inverse mapping f^{-1} , hence it suffices to only show the second equation.

Let $\sup H_O(f'(x)) = K < \infty$. We must show $K_O(f) = \frac{M(\Gamma)}{M(\Gamma')} \leq K$ for an arbitrary path family Γ in D . Let $\rho' \in F(\Gamma')$. Now, define $\rho : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$ by

$$\rho(x) = \begin{cases} \rho'(f(x))|f'(x)| & x \in D \\ 0 & x \notin D. \end{cases}$$

Suppose that γ is a locally rectifiable path in Γ . By [V] [Corollary 5.4] we obtain:

$$\int_{\gamma} \rho \, ds \geq \int_{f \circ \gamma} \rho' \, ds \geq 1.$$

Therefore, $\rho \in F(\Gamma)$. This leads us to:

$$\begin{aligned} M(\Gamma) &\leq \int_D \rho^n \, dm = \int_D \rho'(f(x))^n |f'(x)|^n \, dm \\ &\leq K \int_D \rho'(f(x))^n |J(x, f)| \, dm = K \int_{f(D)} \rho'^n \, dm \leq K \int_{\mathbb{R}^n} (\rho')^n \, dm. \end{aligned}$$

Taking the infimum we obtain $M(\Gamma) \leq KM(\Gamma')$. The reverse inequality is a consequence of Theorem 3.1.7. ■

Corollary 3.1.9 *The function $f : D \rightarrow D'$ is a K -qc diffeomorphism if and only if*

$$\frac{|f'(x)|^n}{K} \leq |J(x, f)| \leq K \ell(f'(x))^n$$

for all $x \in D$. Moreover,

$$1 \leq K_I(f) \leq K_O(f)^{n-1} \quad \text{and} \quad 1 \leq K_O(f) \leq K_I(f)^{n-1}.$$

As a direct consequence of Theorem 3.1.5 and Theorem 3.1.8 we have the following corollary.

Corollary 3.1.10 *If a qc-mapping f is differentiable at a point a , then either $f'(a) = 0$ or $J(a, f) \neq 0$.*

3.2 Metric and Analytic Definitions and Properties

3.2.1 The Linear Dilatation

In this section we will define the linear dilatation of a homeomorphism.

Let D and D' be domains in $\widehat{\mathbb{R}^n}$, and define a homeomorphism $f : D \rightarrow D'$. For each $x \in D$ with $f(x) \neq \infty$, and for each $r > 0$ such that $S^{n-1}(x, r) \subset D$ we define the linear dilatations as follows:

Definition 3.2.1 *Set the quantities*

$$\begin{aligned} L(x, f, r) &= \max_{|y-x|=r} |f(y) - f(x)|, \\ \text{and } l(x, f, r) &= \min_{|y-x|=r} |f(y) - f(x)|. \end{aligned}$$

The linear dilatation of f at a point $x \in D$ is

$$H(x, f) = \limsup_{r \rightarrow 0} \frac{L(x, f, r)}{l(x, f, r)}.$$

If $x = \infty$, such that $f(x) \neq \infty$, we set $H(x, f) = H(0, f \circ u)$, where u is the inversion map $u(x) = \frac{x}{\|x\|^2}$. Also, if $f(x) = \infty$ we define $H(x, f) = (x, u \circ f)$.

Since $l(x, f, r) \leq L(x, f, r)$, we have

$$1 \leq H(x, f) \leq \infty.$$

Now if A is a bijective linear map, we note that $H(x, A) = H(A)$ for all $x \in \mathbb{R}^n$, where $H(A)$ is defined as in Equation (3.1.7). We also note by Theorem 1.2.6, that if f is differentiable at x and if $|J(x, f)| \neq 0$, then $H(x, f) = H(f'(x))$.

We omit the very technical proof of the following theorem. For details see [V] [pp. 78-80].

Theorem 3.2.2 *Let $f : D \rightarrow D'$ be a homeomorphism such that for some $K < \infty$ either $K_O(f) \leq K$ or $K_I(f) \leq K$. Then $H(x, f)$ is bounded by a constant which depends only on n and K .*

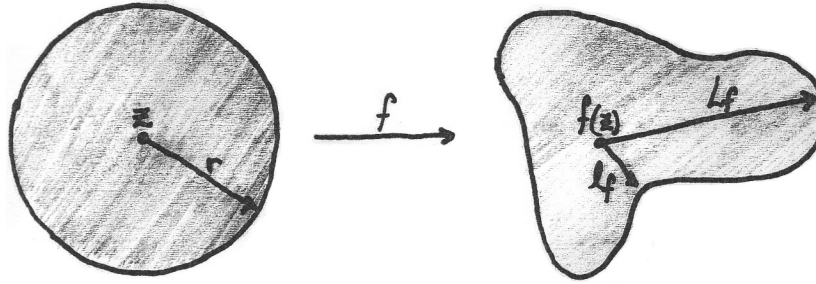


Figure 3.4: The linear dilatation

Corollary 3.2.3 *If $f : D \rightarrow D'$ is qc, then $H(x, f)$ is bounded.*

3.2.2 The ACL Property

This subsection is modeled after Sections 22 and 31 of [V].

We set $\mathbb{R}_i^{n-1} = \{x \in \mathbb{R}^n : x_i = 0\}$. In particular P_i is the projection of \mathbb{R}^n onto \mathbb{R}_i^{n-1} . That is, $P_i(x) = x - x_i e_i$.

Definition 3.2.4 *Let $Q = \{x \in \mathbb{R}^n : x \in [a_i, b_i]\}$ be a closed n -interval. A mapping $f : Q \rightarrow \mathbb{R}^m$ is said to be absolutely continuous on lines (ACL) if f is continuous and f is absolutely continuous on almost every line segment in Q parallel to the coordinate axes.*

We note that if E_i is the set of all $x \in P_i Q$ such that $x \mapsto f(x + t e_i)$ is not absolutely continuous on $[a_i, b_i]$, then $m_{n-1}(E_i) = 0$ for $1 \leq i \leq n$. We also say if U is an open set in \mathbb{R}^n , then the mapping $f : U \rightarrow \mathbb{R}^m$ is ACL if the map f restricted to Q is ACL for all closed intervals $Q \subset U$.

Definition 3.2.5 *An ACL mapping $f : U \rightarrow \mathbb{R}^m$ is said to be ACL^p , for $p \geq 1$, if $\frac{\partial f}{\partial x_i} \in L_{loc}^p(U)$ for all $1 \leq i \leq n$.*

It is well known that $ACL^p = W^{1,p}$, see [Z][2.1.4].

Theorem 3.2.6 [V] [Theorem 31.2] *Let $f : D \rightarrow D'$ be a homeomorphism such that $H(x, f)$ is bounded, then f is ACL.*

Corollary 3.2.7 *Every qc map is ACL.*

Theorem 3.2.8 [V] [Theorem 32.1] *Let $f : D \rightarrow D'$ be a homeomorphism such that $H(x, f)$ is bounded. Then f is differentiable almost everywhere.*

Corollary 3.2.9 *A qc map is differentiable almost everywhere.*

Theorem 3.2.10 [V] [Theorem 32.3] *Let $f : D \subset \widehat{\mathbb{R}}^n \rightarrow D'$ be a homeomorphism. If $1 \leq K \leq \infty$, then the following are equivalent:*

- 1) $K_O(f) \leq K$.
- 2) f is ACL, almost everywhere differentiable, and $|f'(x)|^n \leq K|J(x, f)|$ a.e.

Moreover, f is ACL^n whenever (1) and (2) hold.

Proof. Assume $K_O(f) \leq K$. By Theorem 3.2.2, $H(x, f)$ is bounded. By Theorems 3.2.6 and 3.2.8, f is ACL and a.e. differentiable. The inequality in 2) follows from Theorem 3.1.7.

Next, choose E to be a compact set in $D \setminus \{\infty, f^{-1}(\infty)\}$. We see that

$$\int_E |f'(x)|^n dm \leq K \int_E |J(x, f)| dm \leq K \cdot m(fE) < \infty.$$

Noting that $|\partial_i f(x)| \leq |f'(x)|$ at every point x of differentiability, we see $\partial_i f \in L^n$, and thus we have f is ACL^n .

Now assume the conditions in 2) are satisfied. Let Γ_0 be the family of all locally rectifiable paths $\gamma \in \Gamma \subset D$ such that f is absolutely continuous on every closed subpath of γ and let $\Gamma_\infty = \{\gamma \in \Gamma : \infty \in f \circ \gamma\}$. Hence by Example 2.2.13 $M(\Gamma_\infty) = 0$. Since f is ACL^n , Fuglede's Theorem [F] implies $M(\Gamma \setminus (\Gamma_\infty \setminus \Gamma_0)) = 0$. Therefore $M(\Gamma_0) = M(\Gamma)$, and it suffices to show that $M(\Gamma_0) \leq K \cdot M(\Gamma')$. Recall

$$L_f(x) = \limsup_{h \rightarrow 0} \frac{|f(x+h) - f(x)|}{|h|}.$$

Let $\rho' \in F(\Gamma')$, and for $x \in D$ define $\rho : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$ by

$$\rho(x) = \begin{cases} \rho'(x) \cdot L_f(x) & x \in D \\ 0 & x \notin D. \end{cases}$$

If $\gamma \in \Gamma_0$, Theorem 2.1.11 gives us

$$\int_{\gamma} \rho \, ds \geq \int_{f \circ \gamma} \rho' \, ds \geq 1.$$

Hence $\rho \in F(\Gamma_0)$, which implies

$$\begin{aligned} M(\Gamma_0) &\leq \int_D \rho^n \, dm = \int_D \rho'(f(x))^n L_f(x)^n \, dm \\ &= \int_D \rho'(f(x))^n |f'(x)|^n \, dm \leq K \int_D \rho'(f(x))^n |J(x, f)| \, dm \\ &\leq K \int_{\mathbb{R}^n} (\rho')^n \, dm. \end{aligned}$$

Since this holds for all $\rho' \in F(\Gamma')$, this implies that $M(\Gamma_0) \leq KM(\Gamma')$. ■

Corollary 3.2.11 *A qc map is ACLⁿ.*

3.2.3 The Metric and Analytic Definitions of Quasiconformality

Lemma 3.2.12 *If $f : D \rightarrow D'$ is a homeomorphism, such that $H(x, f)$ is bounded by a constant C , then*

$$|f'(x)|^n \leq C^{n-1} |J(x, f)|.$$

Proof. By Theorem 3.2.6, f is ACL, and by Theorem 3.2.8 f is differentiable almost everywhere. Using Corollary 3.1.10 and Theorem 1.2.6 we have that for $x \in D$ either $f'(x) = 0$ or

$$0 < H(f'(x)) = H(x, f) = C.$$

If $f'(x) = 0$, then the result is clear, so we take $0 < H(f'(x)) = H(x, f) = C$.

Hence,

$$\frac{|f'(x)|^n}{|J(x, f)|} = H_O(f'(x)) \leq \max\{H_I(f'), H_O(f')\} \leq H(f')^{n-1} = C^{n-1},$$

or

$$|f'(x)|^n \leq C^{n-1}|J(x, f)|.$$

■

Theorem 3.2.13 [*The Metric Definition of Quasiconformality*]

A homeomorphism $f : D \rightarrow D'$ is qc if and only if $H(x, f)$ is bounded.

Proof. Suppose f is qc. Then $H(x, f)$ is bounded by Corollary 3.2.3. Now suppose for all $x \in D, H(x, f) = C < \infty$. Now, by Theorem 3.2.6 f is ACL, and by Theorem 3.2.8 f is differentiable almost everywhere.

So, for $x \in D, f'(x) = 0$ or $0 < H(f'(x)) = H(x, f) = C$ by Corollary 3.1.10 and Theorem 1.2.6.

Lemma 3.2.12 leads to $K_O(f) \leq C^{n-1}$ by letting $C^{n-1} = K$ in Theorem 3.2.10. Since $K_I(f^{-1}) = K_O(f)$, Equation 3.2.2 gives us $H(y, f^{-1}) < \infty$ for all $y \in D'$. By a similar argument $H(x, f) < \infty$ for all $x \in D$. Hence $K_I(f) = K_O(f^{-1}) < \infty$, and so f is qc.

■

Theorem 3.2.14 *A homeomorphism $f : D \rightarrow D'$ is qc if and only if one of the dilatations $K_I(f)$ or $K_O(f)$ is finite.*

Proof. Suppose f is qc. By Theorem 3.2.13 $H(x, f)$ is bounded and at least one of the dilatations is finite. Now suppose $K_I(f)$ or $K_O(f)$ is finite. By Theorem 3.2.2 $H(x, f)$ is bounded, then by Theorem 3.2.13 we are done.

■

Theorem 3.2.15 [*The Analytic Definition of Quasiconformality*] *Let $f : D \subset \widehat{\mathbb{R}^n} \rightarrow$*

D' be a homeomorphism. Then f is K -qc if and only if the following are satisfied:

- 1) f is ACL,
- 2) f is differentiable almost everywhere,
- 3) For almost every $x \in D$

$$\frac{|f'(x)|^n}{K} \leq |J(x, f)| \leq K\ell(f'(x))^n.$$

Proof. Suppose f is K -qc. By Corollary 3.2.7 f is ACL, and by Corollary 3.2.9 f is differentiable almost everywhere. Hence conditions (1) and (2) hold true. Now, f is K -qc implies that

$$\frac{|f'(x)|^n}{|J(x, f)|} \leq K$$

for all $x \in D$ such that f is differentiable at x . We may also take $|J(x, f)| \neq 0$. The inverse $g = f^{-1}$ is also K -qc and has a derivative at $y = f(x)$. Hence,

$$|J(x, f)| = |J(y, g)|^{-1} \leq K|g'(y)|^n = K\ell(f'(x))^n.$$

Now assume the three conditions to hold true. That is, f is an ACL map, differentiable almost everywhere, and at almost all $x \in D$ we have:

$$\frac{|f'(x)|^n}{K} \leq |J(x, f)| \leq K\ell(f'(x))^n.$$

We take note that $K_O(f) < K$ implies by Theorem 3.2.10 $K_O(f)$ is finite, and so by Theorem 3.2.14 f is qc. Therefore, $|J(x, f)| \neq 0$ almost everywhere, which implies $K_I(f) \leq K$. ■

3.3 Equivalence of the Definitions

Theorem 3.3.1 *Let $f : D \subset \widehat{\mathbb{R}^n} \rightarrow D'$ be a homeomorphism. For all $\Gamma \subset D$ the following are equivalent:*

- 1) $\frac{1}{K}M(\Gamma) \leq M(\Gamma') \leq KM(\Gamma)$.
- 2) f is ACL, almost everywhere differentiable, and
$$\frac{|f'(x)|^n}{K} \leq |J(x, f)| \leq K\ell(f'(x))^n.$$
- 3) $H(x, f)$ is bounded.

Proof. By Theorems 3.2.13 and 3.2.15 the result is clear. ■

4 COMPUTATIONAL EXPLORATIONS

In this section we give some Examples of quasiconformal maps.

4.1 Radial Mappings

Example 4.1.1 Let $0 \neq a \in \mathbb{R}$, and let $f : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}^n \setminus \{0\}$ be a diffeomorphism defined by $f(x) = |x|^{a-1}x$. We can extend f to a homeomorphism $f^* : \widehat{\mathbb{R}}^n \rightarrow \widehat{\mathbb{R}}^n$ by defining $f^*(0) = 0$, $f^*(\infty) = \infty$ if $a > 0$, and $f^*(0) = \infty$, $f^*(\infty) = 0$ if $a < 0$.

Note if $a = 1$, f is the identity map. If $a = -1$, f is conformal and f is the inversion in the unit sphere S^{n-1} .

We recall that

$$H_O(f'(x)) = \frac{|f'(x)|^n}{|J(x, f)|} = \frac{a_1^{n-1}}{a_2 a_3 \cdots a_n}$$

and

$$H_I(f'(x)) = \frac{|J(x, f)|}{\ell(f'(x))^n} = \frac{a_1 a_2 \cdots a_{n-1}}{a_n^{n-1}},$$

where a_i are the positive square roots of the eigenvalues of $A^T A$. Hence if

$$f(x) = |x|^{a-1}x = (|x|^{a-1}x_1, |x|^{a-1}x_2, \dots, |x|^{a-1}x_n).$$

Then we compute,

$$(f')_{ij} = \frac{\partial}{\partial x_j}(f_i) = ((a-1)|x|^{a-3}x_j x_i + |x|^{a-1}\delta_{ij}).$$

Hence $[(f')(f')^T]_{ij} =$

$$\sum_{k=1}^n [((a-1)|x|^{a-3}x_{ki} + |x|^{a-1}\delta_{ik}) \cdot ((a-1)|x|^{a-3}x_{ik} + |x|^{a-1}\delta_{ki})],$$

and the matrix has eigenvalues $|a||x|^{a-1}, \underbrace{|x|^{a-1}, \dots, |x|^{a-1}}_{n-1}$.

Now if $|a| > 1$ we have

$$K_I(f) = |a|, \text{ and } K_O(f) = |a|^{n-1}$$

and if $|a| < 1$ we have

$$K_I(f) = |a|^{1-n}, \text{ and } K_O(f) = |a|^{-1}.$$

Using Example 2.2.14 the following relations hold, since the modulus through a arbitrary point is zero.

$$\begin{aligned} K_I(f) &= K_I(f^*), \\ K_O(f) &= K_O(f^*). \end{aligned}$$

We illustrate this Example by considering the case when $n = 3$. Hence,

$$\begin{aligned} f(x) &= |x|^{a-1}x = (|x|^{a-1}x_1, |x|^{a-1}x_2, |x|^{a-1}x_3) \\ &= ((x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}(a-1)}x_1, (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}(a-1)}x_2, (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}(a-1)}x_3). \end{aligned}$$

This implies that

$$f'(x) = \begin{vmatrix} (a-1)|x|^{a-3}x_1^2 + |x|^{a-1} & (a-1)|x|^{a-3}x_2x_1 & (a-1)|x|^{a-3}x_3x_1 \\ (a-1)|x|^{a-3}x_1x_2 & (a-1)|x|^{a-3}x_2^2 + |x|^{a-1} & (a-1)|x|^{a-3}x_3x_2 \\ (a-1)|x|^{a-3}x_1x_3 & (a-1)|x|^{a-3}x_2x_3 & (a-1)|x|^{a-3}x_3^2 + |x|^{a-1} \end{vmatrix},$$

and that

$$(f'(x))^T = \begin{vmatrix} (a-1)|x|^{a-3}x_1^2 + |x|^{a-1} & (a-1)|x|^{a-3}x_1x_2 & (a-1)|x|^{a-3}x_1x_3 \\ (a-1)|x|^{a-3}x_2x_1 & (a-1)|x|^{a-3}x_2^2 + |x|^{a-1} & (a-1)|x|^{a-3}x_2x_3 \\ (a-1)|x|^{a-3}x_3x_1 & (a-1)|x|^{a-3}x_3x_2 & (a-1)|x|^{a-3}x_3^2 + |x|^{a-1} \end{vmatrix}$$

With the aid of a symbolic computation program we find the positive square roots of the eigenvalues of $(f') \cdot (f')^T$ to be

$$|x|^{a-1}, |x|^{a-1}, a|x|^{a-3}x_1^2 - |x|^{a-3}x_1^2 + a|x|^{a-3}x_2^2 - |x|^{a-3}x_2^2 + a|x|^{a-3}x_3^2 - |x|^{a-3}x_3^2 + |x|^{a-1}.$$

The last eigenvalue clearly simplifies to $a|x|^{a-1}$. Therefore the image of the unit sphere under the map f is an ellipsoid with semi-axis $a|x|^{a-1}, |x|^{a-1}, |x|^{a-1}$.

4.2 Folding

Example 4.2.1 Let (r, ϕ, z) be the cylindrical coordinates of a point $(x, y, z) \in \mathbb{R}^3$. This implies that $r \geq 0, 0 \leq \phi \leq 2\pi, z \in \mathbb{R}$, with $x = r \cos \phi, y = r \sin \phi$, and $z = z$. Using x, y, z we define the map $C : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ by $C(r, \phi, z) = (x, y, z)$. The domain D_α defined by $0 < \phi < \alpha$, is called a wedge of angle $\alpha, 0 < \alpha \leq 2\pi$. We consider two wedges D_α and D_β with $\alpha \leq \beta$. Define a homeomorphism $f : D_\alpha \rightarrow D_\beta$ by $f(r, \phi, z) = (r, \frac{\beta\phi}{\alpha}, z)$; this mapping is called a folding. We wish to compute the values of the semi-axis of the dilatation ellipsoid a_1, a_2, a_3 . Hence we need to find the eigenvalues of DF as detailed below.

$$\begin{array}{ccc} (x, y, z) & \xrightarrow{C^{-1}} & (r, \phi, z) \\ F \downarrow & & f \downarrow \\ (\hat{x}, \hat{y}, \hat{z}) & \xleftarrow{C} & (r, \frac{\beta\phi}{\alpha}, z) \end{array}$$

We note that $F = C \circ f \circ C^{-1}$. Hence $DF = DC \cdot Df \cdot DC^{-1}$, where

$$DC^{-1} = \begin{vmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{vmatrix},$$

$$Df = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \frac{\beta}{\alpha} & 0 \\ 0 & 0 & 1 \end{vmatrix},$$

and

$$DC = \begin{vmatrix} \cos \frac{\beta\phi}{\alpha} & \sin \frac{\beta\phi}{\alpha} & 0 \\ -r \sin \frac{\beta\phi}{\alpha} & r \cos \frac{\beta\phi}{\alpha} & 0 \\ 0 & 0 & 1 \end{vmatrix}.$$

Hence with the aid of a symbolic computation program we find

$$DF = \begin{vmatrix} \cos \phi \cos \frac{\beta\phi}{\alpha} + \frac{\beta}{\alpha} \sin \phi \sin \frac{\beta\phi}{\alpha} & r^{-1} \left(-\cos \frac{\beta\phi}{\alpha} \sin \phi + \frac{\beta}{\alpha} \cos \phi \sin \frac{\beta\phi}{\alpha} \right) & 0 \\ r \left(\frac{\beta}{\alpha} \cos \frac{\beta\phi}{\alpha} \sin \phi - \cos \phi \sin \frac{\beta\phi}{\alpha} \right) & \frac{\beta}{\alpha} \cos \phi \cos \frac{\beta\phi}{\alpha} + \sin \phi \sin \frac{\beta\phi}{\alpha} & 0 \\ 0 & 0 & 1 \end{vmatrix}.$$

The characteristic polynomial of DF is

$$-(x-1) \left(\frac{\beta}{\alpha} + x^2 - \left(1 + \frac{\beta}{\alpha} \right) \left(\cos \left(\phi - \frac{\beta\phi}{\alpha} \right) \right) \right) x.$$

Therefore we have the following eigenvalues:

$$1, \frac{1}{2} \left(1 + \frac{\beta}{\alpha} \right) \left(\cos \left(\phi - \frac{\beta\phi}{\alpha} \right) \right) \pm \frac{1}{2} \sqrt{\left(1 + \frac{\beta}{\alpha} \right)^2 \left(\cos \left(\phi - \frac{\beta\phi}{\alpha} \right) \right)^2 - 4 \frac{\beta}{\alpha}}.$$

The maximum of the largest eigenvalue occurs when we allow $\cos \left(\phi - \frac{\beta\phi}{\alpha} \right)$ to be 1, therefore we obtain the eigenvalues:

$$\frac{\beta}{\alpha}, 1, 1.$$

Now according to Equation (3.1.8) we have

$$K_O(f) = \frac{\beta}{\alpha} \quad \text{and} \quad K_I(f) = \left(\frac{\beta}{\alpha}\right)^2.$$

4.3 Cones

Example 4.3.1 Let (R, ϕ, θ) be the spherical coordinates of a point $(x, y, z) \in \mathbb{R}^3$. This implies that $R \geq 0$, $0 \leq \phi < 2\pi$, $0 \leq \theta \leq \pi$, with $x = R \sin \theta \cos \phi$, $y = R \sin \theta \sin \phi$, and $z = R \cos \theta$. Using x, y, z we define the map $S : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ by $S(R, \phi, \theta) = (x, y, z)$. The domain C_α defined by $\theta < \alpha$, is called a cone of angle α . When $\alpha \leq \beta$, define a homeomorphism $f : C_\alpha \rightarrow C_\beta$ by $f(R, \phi, \theta) = (R, \phi, \frac{\beta\theta}{\alpha})$. We wish to compute the values of the semi-axis of the dilatation ellipsoid a_1, a_2, a_3 . Hence we need to find the eigenvalues of DF as detailed below.

$$\begin{array}{ccc} (x, y, z) & \xrightarrow{S^{-1}} & (R, \phi, \theta) \\ F \downarrow & & f \downarrow \\ (\hat{x}, \hat{y}, \hat{z}) & \xleftarrow{S} & (R, \phi, \frac{\beta\theta}{\alpha}) \end{array}$$

We note that $F = S \circ f \circ S^{-1}$. Hence $DF = DS \cdot Df \cdot DS^{-1}$,

where

$$DS^{-1} = \begin{vmatrix} -\cos \phi \sin \theta & R^{-1} \sin \phi \csc \theta & -R^{-1} \cos \phi \cos \theta \\ -\sin \phi \sin \theta & R^{-1} \cos \phi \csc \theta & -R^{-1} \cos \theta \sin \phi \\ \cos \theta & 0 & -R^{-1} \sin \theta \end{vmatrix},$$

$$Df = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{\beta}{\alpha} \end{vmatrix},$$

and

$$DS = \begin{vmatrix} \cos \phi \sin \frac{\beta\theta}{\alpha} & \sin \phi \sin \frac{\beta\theta}{\alpha} & \cos \frac{\beta\theta}{\alpha} \\ -R \sin \phi \sin \frac{\beta\theta}{\alpha} & R \cos \phi \sin \frac{\beta\theta}{\alpha} & 0 \\ R \cos \phi \cos \frac{\beta\theta}{\alpha} & R \cos \frac{\beta\theta}{\alpha} \sin \phi & -R \sin \frac{\beta\theta}{\alpha} \end{vmatrix}.$$

Hence with the aid of a symbolic computation program we find

$$DF = \begin{vmatrix} \frac{\beta}{\alpha} \cos \theta \cos \frac{\beta\theta}{\alpha} + \sin \theta \sin \frac{\beta\theta}{\alpha} & 0 & \frac{\beta}{\alpha} \cos \frac{\beta\theta}{\alpha} \sin \theta + \cos \theta \sin \frac{\beta\theta}{\alpha} \\ 0 & \csc \theta \sin \frac{\beta\theta}{\alpha} & 0 \\ R(\cos \frac{\beta\theta}{\alpha} \sin \theta - \frac{\beta}{\alpha} \cos \theta \sin \frac{\beta\theta}{\alpha}) & 0 & \cos \theta \cos \frac{\beta\theta}{\alpha} + \frac{\beta}{\alpha} \sin \theta \sin \frac{\beta\theta}{\alpha} \end{vmatrix}.$$

The characteristic polynomial of DF is

$$\left[-x^2 + \left(1 - \frac{\beta}{\alpha}\right)\left(\cos\left(\theta - \frac{\beta\theta}{\alpha}\right)\right)x - \frac{\beta}{\alpha}\right]\left(x - \csc \theta \sin \frac{\beta\theta}{\alpha}\right).$$

Therefore we have the following eigenvalues:

$$\csc \theta \sin \frac{\beta\theta}{\alpha}, \frac{1}{2}\left[\left(1 + \frac{\beta}{\alpha}\right)\cos\left(\theta - \frac{\beta\theta}{\alpha}\right) \pm \sqrt{\left(1 + \frac{\beta}{\alpha}\right)^2 \cos^2\left(\theta - \frac{\beta\theta}{\alpha}\right) - 4\frac{\beta}{\alpha}}\right].$$

We note that for each $0 \leq x \leq 1$, $\frac{\sin mx}{\sin m}$ is increasing in m , when $0 \leq m \leq \pi$. Thus if $x = \frac{\theta}{\alpha}$, we have

$$\frac{\sin \beta}{\sin \alpha} \leq \frac{\sin \frac{\beta\theta}{\alpha}}{\sin \theta}. \quad (4.3.1)$$

The maximum of the last two eigenvalues will occur when we allow $\cos\left(\theta - \frac{\beta\theta}{\alpha}\right) = 1$, therefore we obtain the eigenvalues:

$$\csc \theta \sin \frac{\beta\theta}{\alpha}, \frac{\beta}{\alpha}, 1.$$

We recall from Calculus that $\frac{\sin x}{x}$ is always decreasing. Hence for $\alpha \leq \beta$ we obtain the relationships $\frac{\beta}{\alpha} \geq \frac{\sin \beta}{\sin \alpha}$ and $\frac{\beta}{\alpha} \geq 1$. Now according to Equation 3.1.8 and using

Equation 4.3.1, we obtain

$$K_I(f) = \max\left\{\frac{\beta^2}{\alpha^2}, \frac{\beta \sin^2 \alpha}{\alpha \sin^2 \beta}\right\},$$

and $K_O(f) = \frac{\beta^2 \sin \alpha}{\alpha^2 \sin \beta}$.

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